A role for confidence: volition regimes and news

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Abstract

Economic literature exhibits a variety of empirical structural impulse response function (SIRF) patterns in real consumption and real output due to changes in confidence or sentiment, with particular regard to the USA and the EA. This work replicates them in the orbit of a neo-Keynesian dynamic stochastic general equilibrium (NK-DSGE) model especially characterised by macroeconomic agents and derived from start to end. Confidence is specifically modelled as an endogenous variable characterised by a coalescence of three processes regulated by a degree of volition, the processes being permanent technology, transitory technology and noise technology. The first two processes affect real production technology with a delay of one lag, while the third does not at all. Short run responses to changes in confidence are displayed whenever the degree of volition allow confidence to shift real consumption and aggregate labour, thereby being non-negligible. Whenever the degree of volition were by contrast negligible exogenous shocks in noise technology would cause no fluctuations in real consumption and real output whatsoever.

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MSC codes: 91B51; 91B55; 91B62; 91B84; 91C99.
Keywords: aggregate labour; confidence; volition; permanent technology; transitory technology; noise technology; real economic activity; real consumption; real output.

Contents

1. Introduction and contributions

*saccal.alessandro@gmail.com. Disclaimer: the author has no declaration of interest related to this research; all views and errors in this research are the author's. Note: two former versions of this article titled "Volition regimes and news" and "A role for confidence" and all variants thereof are hereby supplanted.

1.1 Introduction and scientific literature. Defining the relationship between confidence and real economic activity is a complex task, for despite being regarded as vital confidence's nature is rather elusive. Indeed, reciprocity characterises the two: confidence is said to influence real economic activity and real economic activity is said to impact confidence in turn.

Economic literature nonetheless provides two main justifications. The first conceives of confidence as waves of pure sentiment, demand or noise, dating back to Keynes [14] and having been more recently expanded by Akerlof and Shiller [2], Lorenzoni [15], Angeletos and La'O [3] and Angeletos *et alii* [4].

The second regards it as a proxy for news and noise shocks in economic fundamentals, dating back to Pigou [16]; related contemporary works comprise those of Cochrane [11], Beaudry and Portier [6], Barsky and Sims [5], Sims [19], Blanchard *et alii* [7], Chahrour and Jurado [10] and Saccal [17]. Independent as they are, this work hinges on both.

1.2 Notional and methodological contributions. This work's notional contribution is the theoretical explanation of all types of empirical SIRFs in real consumption and real output to exogenous shocks in news and noise processes, normally proxied by economic sentiment or confidence.

The empirical SIRFs need not have all been observed by the pertinent economic literature, but Barsky and Sims [5] and Saccal [17] effectively did, at least substantially. Such authors globally construct trivariate structural vector auto-regressions (SVARs) of order 4 featuring confidence, real consumption and real output in log-levels for the United States of America (USA), the Euro Area (EA) and other European nations and presented a variety of empirical SIRFs in real consumption and real output given changes in confidence.

Formally: $x_t = \Pi_1 x_{t-1} + \ldots + \Pi_4 x_{t-4} + w_t$, in which observable vector $x_t = [s_t\ c_t\ y_t]^\top$ and w_t is a vector of white noises. Such a $VAR(4)$ is rewritten as an $SVAR(1)$: $z_t = \Gamma z_{t-1} + \varepsilon_t$, in which observable vector $z_t = [x_t\ x_{t-1}\ x_{t-2}\ x_{t-3}\ x_{t-4}]^\top$, companion matrix $\Gamma = [(\Pi_1\ \Pi_2\ \Pi_3\ \Pi_4\ 0)\ (I\ 0\ 0\ 0\ 0)\ (0\ I\ 0\ 0\ 0)\ (0\ 0\ I\ 0\ 0)\ (0\ 0\ 0\ I\ 0)]^\top$ and white noise vector $[w_t\ 0\ 0\ 0\ 0]^\top = \varepsilon_t = D\eta_t$, D being a (5×5) lower triangular matrix such that expectations $\mathbb{E}_t\left[\varepsilon_t\ \varepsilon_t^\top\right] = DD^\top$ and $\mathbb{E}_t\left[\eta_t\ \eta_t^\top\right] = I$.

Therefrom causality triggers a Structural Vector Moving Average (SVMA) of infinite order: $z_t = \sum_{j=0}^{\infty} \Gamma^j D\eta_{t-j}$, for SIRFs $\sum_{j=0}^{\infty} \Gamma^j D$, in which coefficients and errors are estimated by means of ordinary least squares (OLS); data are treated in log-levels for purposes of co-integration robustness. The empirical SIRFs globally exhibited patterns of (i) immediate irreversibility, (ii) delayed irreversibility and (iii) (immediate or delayed) reversibility, hereby reproduced by means of theory.

Table 1: Empirical SIRFs

Pattern		SIRF	
		Short run	Long run
Reversibility		Response	No response
Irreversibility	Delayed	No response	Response
	Immediate	Response	Response

Note. Empirical SIRF patterns in real consumption and real output at a 40 period horizon. For any time period taken from integers, the short run is redefined to range from period 0 to period 29 and the long run is redefined to range from period 30 to period 40 : $\forall t \in \mathbb{Z}$, the short run is such that $t \in [0,\ 30)$ and the long run is such that $t \in [30,\ 40]$. Irreversibility is accordingly differentiated between delayed irreversibility and immediate irreversibility, while reversibility is not, although it may. Immediate reversibility could feature responses formally spanning $t \in [0,\ 10)$ and no responses therefrom; delayed reversibility could feature responses formally spanning $t \in [10,\ 30)$ and no responses before or afterwards.

The reason for which confidence is normally chosen as an empirical proxy for news and noise processes is that the latter are unobservable, both empirically and theoretically, as explained by Sims [19] in relation to Blanchard *et alii* [7].

Confidence can thus act as their proxy both in models and in data, so that theoretical and empirical SIRFs in real consumption and real output given changes in confidence reveal the effective nature of the exogenous shocks. Such is also in line with the contribution adduced by Chahrour and Jurado [10], who showed that news and noise proxies are equivalent representations of news and noise processes.

The theoretical explanation of all types of empirical SIRFs in question is developed by means of a minimalistic NK-DSGE model in discrete time and is as such the work's methodological contribution.

In such a model confidence Υ_t is an endogenous variable and figures as a coalescence of two technology processes pt_t and t_t, permanent and transitory, and one noise process n_t, which are all endogenous variables as well; coalescence $pt_t t_t n_t$ is especially regulated by a volition parameter γ endowed with the potential to dampen the three processes' propagation.

The NK-DSGE is minimalistic in the sense that the substantial extensions relative to a real business cycle (RBC) model are merely those of rigid prices and monetary policy. Whether the theoretical explanation of all types of empirical SIRFs in question may work in a mere RBC model as well is an issue reserved for future research.

1.3 Other contributions. Another notional distinction, relative to ordinary DSGE models, is that the economy is not delineated by representative agents, but by macroeconomic agents, thereby eluding the fallacies stressed by the "Anything goes"[1] theorem by which the conceptual aggregation of microeconomic agents need not guarantee the functional properties exhibited by representative agents, particularly the canonical laws of supply and demand.

Consequently, this economy is to feature the canonical laws of supply and demand by construction, as well as the functional properties otherwise pertinent to representative agents. Aggregation in this economy, whenever present, is to be therefore understood as merely pertaining to macroeconomic agents, not to homogenous microeconomic ones, that is, to no more than parts of the macroeconomy.

If representative agents were alternatively understood as macroeconomic ones, as opposed to homogenous microeconomic ones in aggregation, then the fallacies stressed by the "Anything goes" theorem would clearly not apply.

Another methodological advantage of this work is the complete derivation and resolution of its NK-DSGE model, until the conduction of policy analysis, encyclopaedically omitting no passage whatsoever and thereby benefitting all those readers in search of a comprehensive, applied guide to (such a kind of) DSGE models.

2. Confidence

2.1 Construction. Confidence Υ_t is an endogenous variable and is to be modelled as follows. First of all, any exogenous shock is a normally distributed white noise, thereby featuring a 0 mean and a finite variance: $\varepsilon_t \sim \mathcal{N}\left(0,\ \sigma_\varepsilon^2\right)$, in which $\sigma_\varepsilon^2 \in (0,\ \infty) \subset \mathbb{R}_{++}$.

Real production technology a_t then equals its amnesic lagged value $\rho_a a_{t-1}$, which is in turn augmented by (i) an exponentiated real population mean μ, ideally modelling a quarterly technological growth rate, (ii) lagged permanent technology pt_{t-1} and (iii) lagged transitory technology t_{t-1} : $a_t = e^{\mu}\rho_a a_{t-1} pt_{t-1} t_{t-1}$, in which coefficient $\rho_a \in [0,\ 1) \subset \mathbb{R}_+$ and $\mu \in \mathbb{R}$, the equation in question being a law of motion for real production technology a_t. Present (i.e. surprise) exogenous shocks in real production technology are thus excluded.

The fact that lagged permanent technology pt_{t-1} and transitory technology t_{t-1} augment real production technology a_t models exogenous news shocks, one speaking to news regarding exogenous shocks in permanent technology and the other to news regarding exogenous shocks in transitory technology. News shocks broadly referenced can thus be understood as rational anticipations of exogenous shocks in technology at large.

Permanent technology pt_t equals its mnemonic lagged value pt_{t-1}, which is in turn augmented by an exponentiated exogenous shock ε_{ptt} weighted at its own standard deviation $\sigma_{\varepsilon_{pt}}$: $pt_t = pt_{t-1} e^{\sigma_{\varepsilon_{pt}} \varepsilon_{ptt}}$, in which $\varepsilon_{ptt} \sim \mathcal{N}\left(0,\ \sigma_{\varepsilon_{pt}}^2\right)$, the equation in question being a law of motion for permanent technology pt_t. It is thus a random walk process: $pt_t \sim RW$.

[1] https://en.wikipedia.org

In addition, the expected value of lead permanent technology $\mathbb{E}_t pt_{t+1}$ (i.e. process population mean) is non-zero such that its deviation from the steady state $\mathbb{E}_t \hat{pt}_{t+1}$ is non-zero too: $\mathbb{E}_t pt_{t+1} \neq 0$ such that $\mathbb{E}_t \hat{pt}_{t+1} \neq 0$.

Transitory technology t_t and noise technology n_t respectively equal their amnesic lagged values $\rho_t t_{t-1}$ and $\rho_n n_{t-1}$, which are in turn augmented by exponentiations of their respective exogenous shocks ε_{tt} and ε_{nt} weighted at their respective standard deviations $\sigma_{\varepsilon_{tt}}$ and $\sigma_{\varepsilon_{nt}}$: $x_t = \rho_x x_{t-1} e^{\sigma_{\varepsilon_x} \varepsilon_{xt}}$, in which coefficient $\rho_x \in [0,\ 1) \subset \mathbb{R}_+$, $\varepsilon_{xt} \sim \mathcal{N}\left(0,\ \sigma^2_{\varepsilon_x}\right)$ and $x = t,\ n$, being laws of motion for transitory technology t_t and noise technology n_t. They are therefore auto-regressive processes of order 1 : $x_t \sim AR(1)$, *ceteris paribus.*

The expected values of both processes' lead terms $\mathbb{E}_t x_{t+1}$ (i.e. process population means) are accordingly non-zero such that their deviations from the steady state $\mathbb{E}_t \hat{x}_{t+1}$ are non-zero too: $\mathbb{E}_t x_{t+1} \neq 0$ such that $\mathbb{E}_t \hat{x}_{t+1} \neq 0$.

Confidence Υ_t specifically equals the product of permanent technology pt_t, transitory technology t_t and noise technology n_t risen to volition parameter γ, which lies in a semi-open real interval between 0 and 1 : $\Upsilon_t = (pt_t t_t n_t)^\gamma$, in which $\gamma \in (0,\ 1] \subset \mathbb{R}_{++}$, the equation in question being a law of motion for confidence Υ_t.

The expected value of lead confidence $\mathbb{E}_t \Upsilon_{t+1}$ (i.e. population mean) equals 0 such that its deviation from the steady state $\mathbb{E}_t \hat{\Upsilon}_{t+1}$ equals 0 too: $\mathbb{E}_t \Upsilon_{t+1} = 0$ such that $\mathbb{E}_t \hat{\Upsilon}_{t+1} = 0$.

The methodological and theoretical consequence is that the non-nullity of the expected value of lead permanent technology $\mathbb{E}_t pt_{t+1}$ is balanced out by that pertaining to the expected values of transitory technology and noise technology $\mathbb{E}_t x_{t+1}$, especially applying at the steady state as well: $\mathbb{E}_t pt_{t+1} \neq 0$ and $\mathbb{E}_t \hat{pt}_{t+1} \neq 0$ are balanced out by $\mathbb{E}_t x_{t+1} \neq 0$ and $\mathbb{E}_t \hat{x}_{t+1} \neq 0$.

2.2 Discussion. Confidence Υ_t is to be introduced as a shifter of real consumption C_t and of aggregate labour l_t, which are endogenous variables, so that whenever volition γ lie at an infinitesimal distance from 0 confidence Υ_t is almost neutralised, either to unity (i.e. non-linearly) or to nullity (i.e. linearly).

Otherwise stated: the higher the value of volition γ the greater the enthusiasm in real consumption C_t and the effort in aggregate labour l_t; accordingly, for infinitesimal values of volition γ the impact exerted by confidence Υ_t upon real economic activity is also infinitesimal.

Consequently, while a change in confidence Υ_t be itself exogenous the extent to which macroeconomic agents may react to it is endogenous. The econometrician, theoretically and empirically, observes confidence Υ_t alone, for its constituents are unobservable; yet, he is capable of identifying both the nature of the exogenous shock and the regime of volition γ underlying a change in confidence Υ_t, particularly empirically.

Table 2: Volition regimes

Volition regime	Economic region
γ_H	1
γ_M	0.5
γ_L	0.0001

Note. Prospected calibration of volition regimes γ for an economic region formalised by means of a NK-DSGE model as outlined above. H, M and L stand for high, medium and low, respectively.

In the case of an exogenous shock in noise technology n_t unless volition γ were infinitesimal an SIRF pattern of immediate reversibility (i.e. "boom and bust" cycle) would be unavoidable, owing to the presence of a short turn response precisely triggered by a non-negligible value of volition γ as well as the absence of noise technology at any time period in real production technology a_t, thereby giving rise to an expansionary deviation from the steady state on account of noise, demand or pure sentiment (i.e. animal spirits).

In the case of an exogenous shock in permanent technology pt_t and a non-negligible value of volition γ there would correspondingly arise an SIRF pattern of immediate irreversibility (i.e. endogenous growth), whereas a negligible value of volition γ would catalyse an SIRF pattern of delayed irreversibility, owing to the sole activity of permanent technology pt_{t-1}, thereby failing to capitalise upon a positive permanent variation in the selfsame steady state.

In the case of an exogenous shock in transitory technology t_t and a non-negligible value of volition γ there would analogously arise an SIRF pattern of delayed reversibility, whereas a negligible value of volition γ would catalyse an SIRF pattern of postponed delayed reversibility, owing to the sole activity of transitory technology t_{t-1}, thereby failing to capitalise upon a positive transitory variation in the selfsame steady state. Table 2 predisposes the formalisation of all such cases.

Saccal [17] wrote the following: "Delayed reversibility suggests a noise shock driven by firm effort and household enthusiasm.". According to the potential differentiation of immediate reversibility from delayed reversibility presented in Table 1, confidence Υ_t as hereby modelled refines such an affirmation by tying exogenous shocks in noise technology n_t to patterns of immediate reversibility, for non-negligible values of volition γ, and exogenous shocks in transitory technology t_t to patters of delayed reversibility, in the presence of all feasible values of volition γ.

One can thus expect four principal scenarios: (i) immediate irreversibility, $\varepsilon_{ptt} \wedge (\gamma \gg 0)$; (ii) delayed irreversibility, $\varepsilon_{ptt} \wedge (\gamma \approx 0)$; (iii) immediate reversibility, $\varepsilon_{nt} \wedge (\gamma \gg 0)$; (iv) delayed reversibility, $\varepsilon_{tt} \wedge (\gamma \gg 0)$.

This work consequently merges the Keynesian view of confidence Υ_t with the Pigovian view one, whereby long run responses in real economic activity to changes in confidence Υ_t are indicative of news shocks in economic fundamentals and short run ones are indicative of shifts in real consumption C_t and aggregate labour l_t due to confidence Υ_t itself, which is a composite signal of technology processes regulated by a degree of volition γ (i.e. pure sentiment).

3. Household

3.1 Utility function. As per standard DSGE models, the expectation of the transfinite sum of household periodic utilities $\mathbb{E}_t \sum_{t=0}^{\infty} u(C_t,\ l_t)$ is weighted at discount factor periodic product β^t (i.e. recursively), thereby representing its present or constant value: $U(C_t,\ l_t) = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t u(C_t,\ l_t)$, in which discount factor $\beta \in (0,\ 1) \subset \mathbb{R}_{++}$.

In even greater detail household periodic utility $u(C_t,\ l_t)$ is modelled as an iso-elastic utility function[2] in which confidence Υ_t shifts real consumption C_t, itself subjected to inter-temporal inseparability (i.e. habit formation): $u(C_t,\ l_t) = \frac{\Upsilon_t (C_t - hC_{t-1})^{1-\sigma_c} - 1}{1-\sigma_c} - \frac{l_t^{1+\sigma_l}}{1+\sigma_l}$, in which consumption habit, inter-temporal substitution inverse elasticity and labour inverse elasticity $h,\ \sigma_c,\ \sigma_l \in \mathbb{R}_{++}$. Real consumption C_t and labour l_t respectively produce utility and disutility.

3.2 Household constraints. The macroeconomic household's nominal budget constraint is the equality between household nominal demand and household nominal supply. In detail, household nominal demand is the sum of real consumption C_t, real government bond b_t, real taxation tx_t and aggregate capital utilisation $\Psi(u_t) K_{t-1}$, all weighted at price $P_t :\ P_t C_t + P_t b_t + P_t tx_t + P_t \Psi(u_t) K_{t-1}$, in which aggregate capital utilisation function $\Psi(\cdot)$ is such that $\Psi(1) = 0$ and $\Psi''(\cdot) \geq 0$.

Household nominal supply is the sum of aggregate labour l_t, utilised aggregate capital $u_t K_{t-1}$, lagged real government bond return $rn_{t-1} b_{t-1}$, household real profit Π_{2t} and real transfers tf_t, respectively weighted at nominal wage Wn_t, nominal capital return Rk_t, lagged price P_{t-1} and twice price P_t : $Wn_t l_t + Rk_t u_t K_{t-1} + rn_{t-1} P_{t-1} b_{t-1} + P_t \Pi_{2t} + P_t tf_t$, in which endogenous variable rn_{t-1} is the lagged nominal interest rate.

The macroeconomic household's nominal budget constraint can therefore be written as follows: $P_t C_t + P_t b_t + P_t tx_t + P_t \Psi(u_t) K_{t-1} = Wn_t l_t + Rk_t u_t K_{t-1} + rn_{t-1} P_{t-1} b_{t-1} + P_t \Pi_{2t} + P_t tf_t \longrightarrow P_t C_t + B_t + TX_t = Wn_t l_t + [Rk_t u_t K_{t-1} - P_t \Psi(u_t) K_{t-1}] + rn_{t-1} B_{t-1} + P_t \Pi_{2t} + TF_t$.

On division by price P_t, the macroeconomic household's real budget constraint can be accordingly written as follows: $C_t + \frac{B_t}{P_t} + \frac{TX_t}{P_t} = W_t l_t + [rk_t u_t K_{t-1} - \Psi(u_t) K_{t-1}] + rn_{t-1} \frac{B_{t-1}}{P_t} + \Pi_{2t} + \frac{TF_t}{P_t} \longrightarrow C_t + b_t + tx_t = W_t l_t + [rk_t u_t K_{t-1} - \Psi(u_t) K_{t-1}] + rn_{t-1} \pi_t^{-1} b_{t-1} + \Pi_{2t} + tf_t$, in which endogenous variables $\pi_t = P_{t-1}^{-1} P_t,\ W_t = P_t^{-1} Wn_t$ and $rk_t = P_t^{-1} Rk_t$ are inflation, real wage and real capital return, respectively.

[2] https://en.wikipedia.org

Aggregate capital K_t equals the sum of lagged aggregate capital K_{t-1} weighted at $1-\delta$ and investment parameter $i:\ K_t = (1-\delta) K_{t-1} + i$, in which $i \in \mathbb{R}_{++}$ and capital depreciation rate $\delta \in (0,\ 1) \subset \mathbb{R}_{++}$, the equation in question being a law of motion for aggregate capital K_t.

The solvency supply constraint is such that the temporal limit of aggregate capital K_t and nominal government bond B_t weighted at discount factor periodic product β^t and real shadow price λ_{1t} is non-negative, that is, their supply to the macroeconomic household exacts that their priced present value be non-negative: $\lim_{t\to\infty} \mathbb{E}_t \beta^t \lambda_{1t} X_{t+1} \geq 0$, in which $\lambda_{1t} \in \mathbb{R}$ and $X = K,\ B$.

The insolvency demand constraint is analogously such that the temporal limit of aggregate capital K_t and nominal government bond B_t weighted at discount factor periodic product β^t and shadow price λ_{1t} is non-positive, that is, their demand by the macroeconomic household exacts that their priced present value be non-positive: $\lim_{t\to\infty} \mathbb{E}_t \beta^t \lambda_{1t} X_{t+1} \leq 0$, *ceteris paribus.*

By anti-symmetry said two constraints are such that the temporal limit of aggregate capital K_t and nominal government bond B_t weighted at discount factor periodic product β^t and shadow price λ_{1t} is 0, that is, the transversality condition: $\lim_{t\to\infty} \mathbb{E}_t \beta^t \lambda_{1t} X_{t+1} = 0$, *ceteris paribus.*

3.3 Household optimisation problem. For non-negative arguments relative to the objective function, the macroeconomic household's optimisation problem is thus the maximisation of the macroeconomic household's utility function $U(C_t,\ l_t)$ subject to the macroeconomic household's (i) real budget constraint and (ii) the transversality condition:

$$\underset{\{C_t,\ l_t,\ u_t,\ b_t\}_{t=0}^{\infty}}{max} U(C_t,\ l_t) = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t u(C_t,\ l_t) = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left\{ \frac{\Upsilon_t (C_t - hC_{t-1})^{1-\sigma_c} - 1}{1-\sigma_c} - \frac{l_t^{1+\sigma_l}}{1+\sigma_l} \right\} \text{ s.t.}$$
$$C_t + b_t + tx_t = W_t l_t + [rk_t u_t K_{t-1} - \Psi(u_t) K_{t-1}] + rn_{t-1}\pi_t^{-1} b_{t-1} + \Pi_{2t} + tf_t,$$
$$\lim_{t\to\infty} \mathbb{E}_t \beta^t \lambda_{1t} X_{t+1} = 0,\ \forall X = K,\ B,$$
$$C_t,\ l_t,\ u_t,\ b_t \geq 0.$$

A necessary condition for optimal solutions is the invertibility of the objective function's arguments, being hereby met by construction. A sufficient condition for optimal solutions is convexity of the objective function, being hereby translated into concavity of the macroeconomic household's utility function $U(C_t,\ l_t)$, met by construction too, since the convexity requirement relative to the negative minimisation of a negative objective function corresponds to a concavity requirement relative to the positive maximisation of a positive objective function: $-min[-U(C_t,\ l_t)] = max\ U(C_t,\ l_t)$.

In detail, the macroeconomic household's utility function is iso-elastic or one of constant relative risk aversion (CRRA) and is as such homogeneous of first degree, continuous, increasing in consumption, decreasing in labour and concave in both.

Such conditions speak to the renowned "Karush Kuhn Tucker (KKT) conditions"[3] for the optimisation of standard non-linear programming problems.

The dynamic Lagrangian equation of said optimisation problem is such that discount factor periodic product β^t weights the expectation of the constrained transfinite sum of household periodic utilities $\mathbb{E}_t \sum_{t=0}^{\infty} [u(C_t,\ l_t) + \lambda_{1t}(\cdot)]$, in which shadow price λ_{1t} weights the macroeconomic household's real budget constraint in turn:

$$\mathcal{L}_{1t} = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left\{ \left[\frac{\Upsilon_t (C_t - hC_{t-1})^{1-\sigma_c} - 1}{1-\sigma_c} - \frac{l_t^{1+\sigma_l}}{1+\sigma_l} \right] + \right.$$
$$\left. +\lambda_{1t} \left[W_t l_t + rk_t u_t K_{t-1} - \Psi(u_t) K_{t-1} + rn_{t-1}\pi_t^{-1} b_{t-1} + \Pi_{2t} + tf_t - (C_t + b_t + tx_t) \right] \right\}.$$

First order conditions (FOCs) are:

[3] https://en.wikipedia.org

$$\frac{\partial \mathcal{L}_{1t}}{\partial C_t} = 0 \longleftrightarrow \beta^t \left[\frac{\Upsilon_t (1 - \sigma_c) (C_t - hC_{t-1})^{-\sigma_c} (1)}{1 - \sigma_c} - \lambda_{1t} (1) \right] = 0 \longrightarrow \Upsilon_t (C_t - hC_{t-1})^{-\sigma_c} = \lambda_{1t};$$
$$\frac{\partial \mathcal{L}_{1t}}{\partial l_t} = 0 \longleftrightarrow \beta^t \left[\frac{-(1 + \sigma_l) l_t^{\sigma_l}}{1 + \sigma_l} + \lambda_{1t} W_t \right] = 0 \longrightarrow \lambda_{1t} W_t = l_t^{\sigma_l};$$
$$\frac{\partial \mathcal{L}_{1t}}{\partial u_t} = 0 \longleftrightarrow \beta^t \lambda_{1t} [rk_t (1) K_{t-1} - \Psi' (u_t) (1) K_{t-1}] = 0 \longrightarrow rk_t = \Psi' (u_t);$$
$$\frac{\partial \mathcal{L}_{1t}}{\partial b_t} = 0 \longleftrightarrow \beta^t \lambda_{1t}(-1) + \mathbb{E}_t \beta^{t+1} \lambda_{1t+1} rn_t \pi_{t+1}^{-1}(1) = 0 \longrightarrow -\lambda_{1t} + \mathbb{E}_t \beta \lambda_{1t+1} rn_t \pi_{t+1}^{-1} = 0 \longrightarrow \mathbb{E}_t \beta \lambda_{1t+1} rn_t \pi_{t+1}^{-1} = \lambda_{1t},$$

recalling that $\mathbb{E}_t x_t = x_t$, in which x is any endogenous variable.

3.4 Household laws of motion. As a consequence, there firstly arises an indirect equation for stochastic discount factor $\mathbb{E}_t \beta^j \lambda_{1t}^{-1} \lambda_{1t+j}$:

$$\mathbb{E}_t \beta \lambda_{1t+1} rn_t \pi_{t+1}^{-1} = \lambda_{1t} \longrightarrow \mathbb{E}_t \pi_{t+1} = \mathbb{E}_t \beta \lambda_{1t}^{-1} \lambda_{1t+1} rn_t \longrightarrow \mathbb{E}_t \pi_{t+j} = \mathbb{E}_t \beta^j \lambda_{1t}^{-1} \lambda_{1t+j} rn_t.$$

There subsequently arise the following laws of motion:

$$\Upsilon_t (C_t - hC_{t-1})^{-\sigma_c} = \lambda_{1t} \text{ and } \lambda_{1t} W_t = l_t^{\sigma_l} \longrightarrow \Upsilon_t (C_t - hC_{t-1})^{-\sigma_c} W_t = l_t^{\sigma_l} \longrightarrow$$
$$\longrightarrow W_t = \Upsilon_t^{-1} (C_t - hC_{t-1})^{\sigma_c} l_t^{\sigma_l} \text{ (real wage or aggregate labour supply);}$$
$$\Upsilon_t (C_t - hC_{t-1})^{-\sigma_c} = \lambda_{1t} \text{ and } \mathbb{E}_t \beta \lambda_{1t+1} rn_t \pi_{t+1}^{-1} = \lambda_{1t} \longrightarrow$$
$$\longrightarrow \Upsilon_t (C_t - hC_{t-1})^{-\sigma_c} = \mathbb{E}_t \beta \left[\Upsilon_{t+1} (C_{t+1} - hC_t)^{-\sigma_c} \right] rn_t \pi_{t+1}^{-1} \text{ (real consumption or consumption Euler equation);}$$
$$rk_t = \Psi' (u_t) \text{ (real capital return).}$$

4. Retailer

4.1 Retail nominal profit. As per standard NK-DSGE models, nominal profit $P_t \Pi_{1t}$ proper to the macroeconomic retailer or final goods or services macroeconomic producer equals the difference between retail nominal marginal revenue $P_t Y_t$ and retail nominal marginal cost $\int_0^1 P_{it} Y_{it} di$, being a continuum of priced wholesale real outputs in relation to their macroeconomic producers: $P_t \Pi_{1t} = P_t Y_t - \int_0^1 P_{it} Y_{it} di$.

Wholesale aggregate real output Y_t equals a continuum of wholesale real outputs $\int_0^1 Y_{it} di$ exhibiting constant elasticity of substitution[4] (CES): $Y_t = \left(\int_0^1 Y_{it}^{\frac{1}{\theta}} di \right)^{\theta}$, in which macroeconomic producer $i \in [0,\ 1] \subset \mathbb{R}_+$ and substitution elasticity $\theta \in (-\infty,\ 1] \subset \mathbb{R}$ such that

$$\theta \begin{cases} = 1, \text{ perfect substitutes} \\ = 0, \text{ imperfect complements} \\ \to -\infty, \text{ perfect complements} \end{cases},$$

relative to the continuum of wholesale real outputs $\int_0^1 Y_{it} di$.

4.2 Retail optimisation problem. For non-negative arguments relative to the objective function, the optimisation problem of the macroeconomic retailer is thus the maximisation of retail nominal profit $P_t \Pi_{1t}$ subject to wholesale aggregate real output Y_t :

[4] https://en.wikipedia.org

$$\begin{aligned}&\underset{\{Y_{it}\}_{t=0}^{\infty}}{max}\ P_t\Pi_{1t} = P_tY_t - \int_0^1 P_{it}Y_{it}di \text{ s.t.}\\&Y_t = \left(\int_0^1 Y_{it}^{\frac{1}{\theta}}di\right)^{\theta},\\&Y_{it} \geq 0.\end{aligned}$$

The necessary condition of objective function argument invertibility and the sufficient condition of objective function convexity for optimal solutions are analogously met by construction. The Lagrangian equation of said optimisation problem is such that retail nominal profit $P_t\Pi_{1t}$ is optimised seeking retail optimal input or wholesale optimal real output Y_{it} in the face of perfect competition:

$$\begin{aligned}\mathcal{L}_{2t} &= P_t\left(\int_0^1 Y_{it}^{\frac{1}{\theta}}di\right)^{\theta} - \int_0^1 P_{it}Y_{it}di = P_t\left(\int_0^1 Y_{it}^{\frac{1}{\theta}}di\right)^{\theta} - iP_{it}Y_{it}|_0^1 =\\&= P_t\left(\int_0^1 Y_{it}^{\frac{1}{\theta}}di\right)^{\theta} - (1-0)\,P_{it}Y_{it} = P_t\left(\int_0^1 Y_{it}^{\frac{1}{\theta}}di\right)^{\theta} - P_{it}Y_{it}.\end{aligned}$$

The FOC is

$$\begin{aligned}&\frac{\partial\mathcal{L}_{2t}}{\partial Y_{it}} = 0 \longleftrightarrow P_t\theta\left(\int_0^1 Y_{it}^{\frac{1}{\theta}}di\right)^{\theta-1}\theta^{-1}\left(Y_{it}^{\frac{1}{\theta}-1}\right) - P_{it} = 0 \longrightarrow\\&\longrightarrow P_t\left(\int_0^1 Y_{it}^{\frac{1}{\theta}}di\right)^{\theta-1} Y_{it}^{\frac{1-\theta}{\theta}} = P_{it} \longrightarrow\\&\longrightarrow P_t^{-1}P_{it} = \left(\int_0^1 Y_{it}^{\frac{1}{\theta}}di\right)^{\theta-1} Y_{it}^{\frac{1-\theta}{\theta}}\end{aligned}$$

and since $Y_t^{\frac{1}{\theta}} = \left[\left(\int_0^1 Y_{it}^{\frac{1}{\theta}}di\right)^{\theta}\right]^{\frac{1}{\theta}} = \int_0^1 Y_{it}^{\frac{1}{\theta}}di$ it follows that

$$\begin{aligned}&P_t^{-1}P_{it} = \left(Y_t^{\frac{1}{\theta}}\right)^{\theta-1} Y_{it}^{\frac{1-\theta}{\theta}} \longrightarrow \left(P_t^{-1}P_{it}\right)^{\frac{\theta}{1-\theta}} = Y_t^{\frac{\theta-1}{\theta}\left(\frac{\theta}{1-\theta}\right)}Y_{it} \longrightarrow\\&\longrightarrow \left(P_t^{-1}P_{it}\right)^{\frac{\theta}{1-\theta}} = Y_t^{\frac{\theta-1}{1-\theta}}Y_{it} \longrightarrow Y_{it} = Y_t^{\frac{1-\theta}{1-\theta}}\left(P_t^{-1}P_{it}\right)^{\frac{\theta}{1-\theta}} = Y_t\left(P_t^{-1}P_{it}\right)^{\frac{\theta}{1-\theta}} \text{ (wholesale real output demand).}\end{aligned}$$

In addition,

$$\begin{aligned}&Y_t = \left(\int_0^1 Y_{it}^{\frac{1}{\theta}}di\right)^{\theta} \longrightarrow Y_t = \left\{\int_0^1 \left[Y_t\left(P_t^{-1}P_{it}\right)^{\frac{\theta}{1-\theta}}\right]^{\frac{1}{\theta}}di\right\}^{\theta} \longrightarrow\\&\longrightarrow Y_t^{\frac{1}{\theta}} = \left\{\left\{\int_0^1 \left[Y_t\left(P_t^{-1}P_{it}\right)^{\frac{\theta}{1-\theta}}\right]^{\frac{1}{\theta}}di\right\}^{\theta}\right\}^{\frac{1}{\theta}} \longrightarrow Y_t^{\frac{-1}{\theta}}Y_t^{\frac{1}{\theta}} = \int_0^1 \left[\left(P_t^{-1}P_{it}\right)^{\frac{\theta}{1-\theta}}\right]^{\frac{1}{\theta}}di \longrightarrow\\&\longrightarrow 1 = \int_0^1 \left(P_t^{-1}P_{it}\right)^{\frac{1}{1-\theta}}di \longrightarrow 1 = P_t^{\frac{-1}{1-\theta}}\int_0^1 P_{it}^{\frac{1}{1-\theta}}di \longrightarrow\\&\longrightarrow P_t^{\frac{1}{1-\theta}} = \int_0^1 P_{it}^{\frac{1}{1-\theta}}di \longrightarrow P_t = \left(\int_0^1 P_{it}^{\frac{1}{1-\theta}}di\right)^{1-\theta} \text{ (wholesale real output aggregate price).}\end{aligned}$$

5. Wholesaler

5.1 Price rigidity. As per Calvo [9], in period t a random ξ fraction of macroeconomic wholesalers or intermediate goods or services macroeconomic producers fails to adjust wholesale real output price P_{it}, indexing it to lagged inflation π_{t-1} at parameter $\tau:\ P_{it} = \pi_{t-1}^{\tau} P_{it-1}$.

Said random ξ fraction lies in an open real interval between 0 and 1; accordingly, inflation indexation τ lies in a closed real interval between 0 and $1:\ \xi \in (0,\ 1) \subset \mathbb{R}_{++};\ \tau \in [0,\ 1] \subset \mathbb{R}_{+}$. In period t the other $1-\xi$ fraction of macroeconomic wholesalers adjusts wholesale real output price P_{it} with success: $P_{it} = P_{it}^{*}$. In detail,

$$P_t = \left(\int_0^1 P_{it}^{\frac{1}{1-\theta}} di\right)^{1-\theta} = \left[\int_0^{1-\xi} \left(P_t^{*}\right)^{\frac{1}{1-\theta}} di + \int_{1-\xi}^{1} \left(P_{it}\right)^{\frac{1}{1-\theta}} di\right]^{1-\theta} =$$
$$= \left[\int_0^{1-\xi} \left(P_t^{*}\right)^{\frac{1}{1-\theta}} di + \int_{1-\xi}^{1} \left(\pi_{t-1}^{\tau} P_{it-1}\right)^{\frac{1}{1-\theta}} di\right]^{1-\theta} = \left[\int_0^{1-\xi} \left(P_t^{*}\right)^{\frac{1}{1-\theta}} di + \xi \left(\pi_{t-1}^{\tau} \int_0^1 P_{it-1} di\right)^{\frac{1}{1-\theta}}\right]^{1-\theta} =$$
$$= \left[\int_0^{1-\xi} \left(P_t^{*}\right)^{\frac{1}{1-\theta}} di + \xi \left(\pi_{t-1}^{\tau} P_{t-1}\right)^{\frac{1}{1-\theta}}\right]^{1-\theta} \longrightarrow$$
$$\longrightarrow P_t = \left[i \left(P_t^{*}\right)^{\frac{1}{1-\theta}} |_0^{1-\xi} + \xi \left(\pi_{t-1}^{\tau} P_{t-1}\right)^{\frac{1}{1-\theta}}\right]^{1-\theta} = \left[\left(1-\xi-0\right) \left(P_t^{*}\right)^{\frac{1}{1-\theta}} + \xi \left(\pi_{t-1}^{\tau} P_{t-1}\right)^{\frac{1}{1-\theta}}\right]^{1-\theta} =$$
$$= \left[\left(1-\xi\right) \left(P_t^{*}\right)^{\frac{1}{1-\theta}} + \xi \left(\pi_{t-1}^{\tau} P_{t-1}\right)^{\frac{1}{1-\theta}}\right]^{1-\theta} \text{ (aggregate price or wholesale real output aggregate price with rigidity),}$$

in which (i) $\int_{1-\xi}^{1} \left(\pi_{t-1}^{\tau} P_{it-1}\right)^{\frac{1}{1-\theta}} di = \xi \left(\pi_{t-1}^{\tau} \int_0^1 P_{it-1} di\right)^{\frac{1}{1-\theta}}$ on account of random wholesale real output price adjustment and a continuum of wholesalers and (ii) $P_{t-1}^{\frac{1}{1-\theta}} = \int_0^1 P_{it-1}^{\frac{1}{1-\theta}} di$ on account of $P_t^{\frac{1}{1-\theta}} = \int_0^1 P_{it}^{\frac{1}{1-\theta}} di$, the equation in question being a law of motion for aggregate price P_t. It follows that a macroeconomic wholesaler which adjusts its price P_{it} in period t and which cannot adjust it until period $t+j$, for any positive natural j, sets it throughout as follows: $\forall j \in \mathbb{N}_{+}$,

$$\begin{aligned}
&\mathbb{E}_t P_{it+1} = \pi_t^{\tau} P_{it}^{*},\\
&\mathbb{E}_t P_{it+2} = \mathbb{E}_t \pi_{t+1}^{\tau} P_{it+1} = \mathbb{E}_t \pi_{t+1}^{\tau} \pi_t^{\tau} P_{it}^{*},\\
&\vdots\\
&\mathbb{E}_t P_{it+j} = \mathbb{E}_t \pi_{t+j-1}^{\tau} \cdots \pi_t^{\tau} P_{it}^{*} = \mathbb{E}_t \prod_{k=0}^{j-1} \pi_{t+k}^{\tau} P_{it}^{*}.
\end{aligned}$$

5.2 Wholesale optimisation problem. Wholesale nominal profit $P_t \Pi_{3t}$ equals the difference between wholesale nominal marginal revenue $P_{it} Y_{it}$ and wholesale nominal marginal cost $\Phi_t:\ P_t \Pi_{3t} = \left(P_{it} - \Phi_t\right) Y_{it}$.

On division by price P_t, there consequently follow wholesale real profit $\Pi_{3t} = \left(P_{it} - \Phi_t\right) P_t^{-1} Y_{it}$ and future wholesale real sub-profit $\mathbb{E}_t \sum_{j=0}^{\infty} (P_{it+j} - \Phi_{t+j}) P_{t+j}^{-1} Y_{it+j}$, which on being weighted at stochastic discount factor $\mathbb{E}_t \beta^j \lambda_{1t}^{-1} \lambda_{1t+j}$ and fraction periodic product ξ^j, on account of the price adjustment failure throughout j periods on the part of the random ξ fraction of macroeconomic wholesalers, gives rise to future wholesale real profit $\Pi_{3t} = \mathbb{E}_t \sum_{j=0}^{\infty} \left(\xi\beta\right)^j \left(\lambda_{1t}^{-1} \lambda_{1t+j}\right) \left(P_{it+j} - \Phi_{t+j}\right) P_{t+j}^{-1} Y_{it+j}$.

For non-negative arguments relative to the objective function, the optimisation problem of the macroeconomic wholesaler is thus the maximisation of future wholesale real sub-profit $\mathbb{E}_t \sum_{j=0}^{\infty} \left(P_{it+j} - \Phi_{t+j}\right) P_{t+j}^{-1} Y_{it+j}$ weighted at stochastic discount factor $\mathbb{E}_t \beta^j \lambda_{1t}^{-1} \lambda_{1t+j}$ subject to future wholesale real output demand $\mathbb{E}_t Y_{it+j}$, provided wholesale real output price P_{it} have not been adjusted for j periods, by means of fraction periodic product ξ^j :

$$\underset{\{P_{it}^*\}_{t=0}^{\infty}}{max} \Pi_{3t} = \mathbb{E}_t \sum_{j=0}^{\infty} (\xi\beta)^j \left(\lambda_{1t}^{-1}\lambda_{1t+j}\right) (P_{it+j} - \Phi_{t+j}) P_{t+j}^{-1} Y_{it+j} =$$

$$= \mathbb{E}_t \sum_{j=0}^{\infty} (\xi\beta)^j \left(\lambda_{1t}^{-1}\lambda_{1t+j}\right) \left[P_{t+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^{\tau} P_{it}^* - \phi_{t+j}\right] Y_{it+j} \text{ s.t.}$$

$$\mathbb{E}_t Y_{it+j} = \mathbb{E}_t Y_{t+j} \left(P_{t+j}^{-1} P_{it+j}\right)^{\frac{\theta}{1-\theta}} = \mathbb{E}_t Y_{t+j} \left(P_{t+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^{\tau} P_{it}^*\right)^{\frac{\theta}{1-\theta}},$$

$$P_{it}^* \geq 0,$$

in which (i) nominal marginal cost $\Phi_t = P_t\phi_t$, (ii) $\mathbb{E}_t Y_{it+j} = \mathbb{E}_t Y_{t+j} \left(P_{t+j}^{-1} P_{it+j}\right)^{\frac{\theta}{1-\theta}}$ on account of $Y_{it} = Y_t \left(P_t^{-1} P_{it}\right)^{\frac{\theta}{1-\theta}}$ and (iii) $\mathbb{E}_t P_{it+j} = \mathbb{E}_t \prod_{k=0}^{j-1} \pi_{t+k}^{\tau} P_{it}^*$.

The necessary and sufficient conditions for optimal solutions are again met by construction. The dynamic Lagrangian equation of said optimisation problem is such that wholesale real profit Π_{3t} is optimised seeking optimal wholesale real output price P_{it}^* in the face of monopolistic competition:

$$\mathcal{L}_{3t} = \mathbb{E}_t \sum_{j=0}^{\infty} (\xi\beta)^j \left(\lambda_{1t}^{-1}\lambda_{1t+j}\right) \left[P_{t+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^{\tau} P_{it}^* - \phi_{t+j}\right] Y_{t+j} \left(P_{t+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^{\tau} P_{it}^*\right)^{\frac{\theta}{1-\theta}} =$$

$$= \mathbb{E}_t \sum_{j=0}^{\infty} (\xi\beta)^j \left(\lambda_{1t}^{-1}\lambda_{1t+j}\right) \left[\left(P_{t+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^{\tau} P_{it}^*\right)^{\frac{1}{1-\theta}} - \phi_{t+j} \left(P_{t+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^{\tau} P_{it}^*\right)^{\frac{\theta}{1-\theta}}\right] Y_{t+j}.$$

The FOC is

$$\frac{\partial \mathcal{L}_{3t}}{\partial P_{it}^*} = 0 \longleftrightarrow \mathbb{E}_t \sum_{j=0}^{\infty} (\xi\beta)^j \left(\lambda_{1t}^{-1}\lambda_{1t+j}\right) \left[\left(\frac{1}{1-\theta}\right) (P_{it}^*)^{\frac{\theta}{1-\theta}} \left(P_{t+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^{\tau}\right)^{\frac{1}{1-\theta}} + \right.$$

$$\left. -\phi_{t+j} \left(\frac{\theta}{1-\theta}\right) (P_{it}^*)^{\frac{\theta}{1-\theta}-1} \left(P_{t+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^{\tau}\right)^{\frac{\theta}{1-\theta}}\right] Y_{t+j} = 0 \longrightarrow$$

$$\longrightarrow \mathbb{E}_t \sum_{j=0}^{\infty} (\xi\beta)^j \left(\lambda_{1t}^{-1}\lambda_{1t+j}\right) \left(\frac{1}{1-\theta}\right) (P_{it}^*)^{\frac{\theta}{1-\theta}} \left(P_{t+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^{\tau}\right)^{\frac{1}{1-\theta}} Y_{t+j} =$$

$$= \mathbb{E}_t \sum_{j=0}^{\infty} (\xi\beta)^j \left(\lambda_{1t}^{-1}\lambda_{1t+j}\right) \phi_{t+j} \left(\frac{\theta}{1-\theta}\right) (P_{it}^*)^{\frac{\theta}{1-\theta}-1} \left(P_{t+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^{\tau}\right)^{\frac{\theta}{1-\theta}} Y_{t+j} \longrightarrow$$

$$\longrightarrow \mathbb{E}_t \sum_{j=0}^{\infty} (\xi\beta)^j \left(\lambda_{1t}^{-1}\lambda_{1t+j}\right) \left(\frac{1}{1-\theta}\right) (P_{it}^*)^{\frac{-\theta+\theta}{1-\theta}} \left(P_{t+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^{\tau}\right)^{\frac{1}{1-\theta}} Y_{t+j} =$$

$$= \mathbb{E}_t \sum_{j=0}^{\infty} (\xi\beta)^j \left(\lambda_{1t}^{-1}\lambda_{1t+j}\right) \phi_{t+j} \left(\frac{\theta}{1-\theta}\right) (P_{it}^*)^{\frac{-\theta+\theta}{1-\theta}-1} \left(P_{t+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^{\tau}\right)^{\frac{\theta}{1-\theta}} Y_{t+j} \longrightarrow$$

$$\longrightarrow \mathbb{E}_t \sum_{j=0}^{\infty} (\xi\beta)^j \left(\lambda_{1t}^{-1}\lambda_{1t+j}\right) \left(\frac{1}{1-\theta}\right) \left(P_{t+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^{\tau}\right)^{\frac{1}{1-\theta}} Y_{t+j} =$$

$$= \mathbb{E}_t \sum_{j=0}^{\infty} (\xi\beta)^j \left(\lambda_{1t}^{-1}\lambda_{1t+j}\right) \phi_{t+j} \left(\frac{\theta}{1-\theta}\right) (P_{it}^{*})^{-1} \left(P_{t+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^{\tau}\right)^{\frac{\theta}{1-\theta}} Y_{t+j} \longrightarrow$$

$$\longrightarrow P_{it}^{*} = \frac{\mathbb{E}_t \sum_{j=0}^{\infty} (\xi\beta)^j \left(\lambda_{1t}^{-1}\lambda_{1t+j}\right) \phi_{t+j} \left(\frac{\theta}{1-\theta}\right) \left(P_{t+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^{\tau}\right)^{\frac{\theta}{1-\theta}} Y_{t+j}}{\mathbb{E}_t \sum_{j=0}^{\infty} (\xi\beta)^j \left(\lambda_{1t}^{-1}\lambda_{1t+j}\right) \left(\frac{1}{1-\theta}\right) \left(P_{t+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^{\tau}\right)^{\frac{1}{1-\theta}} Y_{t+j}} \longrightarrow$$

$$\longrightarrow P_t^{-1} P_t P_{it}^{*} = \frac{\mathbb{E}_t \sum_{j=0}^{\infty} (\xi\beta)^j \left(\lambda_{1t}^{-1}\lambda_{1t+j}\right) \phi_{t+j} \left(\frac{\theta}{1-\theta}\right) \left(P_{t+k+1}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^{\tau} P_t^{-1} P_t\right)^{\frac{\theta}{1-\theta}} Y_{t+j}}{\mathbb{E}_t \sum_{j=0}^{\infty} (\xi\beta)^j \left(\lambda_{1t}^{-1}\lambda_{1t+j}\right) \left(\frac{1}{1-\theta}\right) \left(P_{t+k+1}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^{\tau} P_t^{-1} P_t\right)^{\frac{1}{1-\theta}} Y_{t+j}} \longrightarrow$$

$$\longrightarrow P_t p_{it}^{*} = \frac{\mathbb{E}_t \sum_{j=0}^{\infty} (\xi\beta)^j \left(\lambda_{1t}^{-1}\lambda_{1t+j}\right) \phi_{t+j} \left(\frac{\theta}{1-\theta}\right) \left(\prod_{k=0}^{j-1} \pi_{t+k+1}^{-1} \pi_{t+k}^{\tau} P_t^{-1}\right)^{\frac{\theta}{1-\theta}} Y_{t+j}}{\mathbb{E}_t \sum_{j=0}^{\infty} (\xi\beta)^j \left(\lambda_{1t}^{-1}\lambda_{1t+j}\right) \left(\frac{1}{1-\theta}\right) \left(\prod_{k=0}^{j-1} \pi_{t+k+1}^{-1} \pi_{t+k}^{\tau} P_t^{-1}\right)^{\frac{1}{1-\theta}} Y_{t+j}} \longrightarrow$$

$$\longrightarrow p_{it}^{*} = \frac{\mathbb{E}_t \sum_{j=0}^{\infty} (\xi\beta)^j \left(\lambda_{1t}^{-1}\lambda_{1t+j}\right) \phi_{t+j} \left(\frac{\theta}{1-\theta}\right) \left(\prod_{k=0}^{j-1} \pi_{t+k+1}^{-1} \pi_{t+k}^{\tau}\right)^{\frac{\theta}{1-\theta}} Y_{t+j}}{\mathbb{E}_t \sum_{j=0}^{\infty} (\xi\beta)^j \left(\lambda_{1t}^{-1}\lambda_{1t+j}\right) \left(\frac{1}{1-\theta}\right) \left(\prod_{k=0}^{j-1} \pi_{t+k+1}^{-1} \pi_{t+k}^{\tau}\right)^{\frac{1}{1-\theta}} Y_{t+j}} \longrightarrow$$

$$\longrightarrow p_{it}^{*} = \frac{\mathbb{E}_t \sum_{j=0}^{\infty} (\xi\beta)^j \lambda_{1t+j} \phi_{t+j} \theta \left(\prod_{k=0}^{j-1} \pi_{t+k+1}^{-1} \pi_{t+k}^{\tau}\right)^{\frac{\theta}{1-\theta}} Y_{t+j}}{\mathbb{E}_t \sum_{j=0}^{\infty} (\xi\beta)^j \lambda_{1t+j} \left(\prod_{k=0}^{j-1} \pi_{t+k+1}^{-1} \pi_{t+k}^{\tau}\right)^{\frac{1}{1-\theta}} Y_{t+j}},$$

in which (i) optimal adjusted price or optimal wholesale real output price $p_{it}^{*} = \frac{P_{it}^{*}}{P_t}$, (ii-a) led inflation $\mathbb{E}_t \pi_{t+j} := \mathbb{E}_t P_t^{-1} P_{t+j}$, (ii-b) $k = j-1 \longrightarrow j = k+1$ and thus (ii-c) $\mathbb{E}_t \pi_{t+j} = \mathbb{E}_t \pi_{t+k+1}$ and (iii) $P_t = \left[\left(P_t^{-1}\right)^{\frac{1}{1-\theta}}\right]^{-1} \left(P_t^{-1}\right)^{\frac{\theta}{1-\theta}} = P_t^{\frac{1}{1-\theta}} P_t^{\frac{-\theta}{1-\theta}}$. It follows that optimal adjusted price p_{it}^{*} is

$$p_{it}^{*} = \frac{A_t}{B_t},$$

in which $A_t = \lambda_{1t}\phi_t\theta Y_t + \xi\beta\mathbb{E}_t \left(\frac{\pi_t^{\tau}}{\pi_{t+1}}\right)^{\frac{\theta}{1-\theta}} A_{t+1}$ and $B_t = \lambda_{1t} Y_t + \xi\beta\mathbb{E}_t \left(\frac{\pi_t^{\tau}}{\pi_{t+1}}\right)^{\frac{1}{1-\theta}} B_{t+1}$ such that if $\xi = 0$ then $p_{it}^{*} = \theta\phi_t$. In detail,

$$A_t = \mathbb{E}_t \sum_{j=0}^{\infty} (\xi\beta)^j \lambda_{1t+j} \phi_{t+j} \theta \left(\prod_{k=0}^{j-1} \pi_{t+k+1}^{-1} \pi_{t+k}^{\tau}\right)^{\frac{\theta}{1-\theta}} Y_{t+j} =$$

$$= \mathbb{E}_t \sum_{j=0}^{\infty} (\xi\beta)^j \lambda_{1t+j} \phi_{t+j} \theta \left[\left(\pi_{t+1}^{-1}\pi_t^{\tau}\right) \cdots \left(\pi_{t+j}^{-1}\pi_{t+j-1}^{\tau}\right)\right]^{\frac{\theta}{1-\theta}} Y_{t+j} =$$

$$= \lambda_{1t}\phi_t\theta Y_t + \mathbb{E}_t \xi\beta\lambda_{1t+1}\phi_{t+1}\theta \left(\pi_{t+1}^{-1}\pi_t^{\tau}\right)^{\frac{\theta}{1-\theta}} Y_{t+1} + \mathbb{E}_t (\xi\beta)^2 \lambda_{1t+2}\phi_{t+2}\theta \left[\left(\pi_{t+1}^{-1}\pi_t^{\tau}\right)\left(\pi_{t+2}^{-1}\pi_{t+1}^{\tau}\right)\right]^{\frac{\theta}{1-\theta}} Y_{t+2} + \ldots$$

matches

$$A_t = \lambda_{1t}\phi_t\theta Y_t + \xi\beta\mathbb{E}_t\left(\frac{\pi_t^\tau}{\pi_{t+1}}\right)^{\frac{\theta}{1-\theta}} A_{t+1} = \lambda_{1t}\phi_t\theta Y_t + \xi\beta\mathbb{E}_t\left(\frac{\pi_t^\tau}{\pi_{t+1}}\right)^{\frac{\theta}{1-\theta}}\left[\lambda_{1t+1}\phi_{t+1}\theta Y_{t+1} + \xi\beta\mathbb{E}_t\left(\frac{\pi_{t+1}^\tau}{\pi_{t+2}}\right)^{\frac{\theta}{1-\theta}} A_{t+2}\right] =$$

$$= \lambda_{1t}\phi_t\theta Y_t + \xi\beta\mathbb{E}_t\left(\frac{\pi_t^\tau}{\pi_{t+1}}\right)^{\frac{\theta}{1-\theta}}\left\{\lambda_{1t+1}\phi_{t+1}\theta Y_{t+1} + \xi\beta\mathbb{E}_t\left(\frac{\pi_{t+1}^\tau}{\pi_{t+2}}\right)^{\frac{\theta}{1-\theta}}\left[\lambda_{1t+2}\phi_{t+2}\theta Y_{t+2} + \xi\beta\mathbb{E}_t\left(\frac{\pi_{t+2}^\tau}{\pi_{t+3}}\right)^{\frac{\theta}{1-\theta}} A_{t+3}\right]\right\}$$

and

$$B_t = \mathbb{E}_t\sum_{j=0}^{\infty}(\xi\beta)^j\lambda_{1t+j}\left(\prod_{k=0}^{j-1}\pi_{t+k+1}^{-1}\pi_{t+k}^\tau\right)^{\frac{1}{1-\theta}} Y_{t+j} =$$

$$= \mathbb{E}_t\sum_{j=0}^{\infty}(\xi\beta)^j\lambda_{1t+j}\left[\left(\pi_{t+1}^{-1}\pi_t^\tau\right)\cdots\left(\pi_{t+j}^{-1}\pi_{t+j-1}^\tau\right)\right]^{\frac{1}{1-\theta}} Y_{t+j} =$$

$$= \lambda_{1t}Y_t + \mathbb{E}_t\xi\beta\lambda_{1t+1}\left(\pi_{t+1}^{-1}\pi_t^\tau\right)^{\frac{1}{1-\theta}} Y_{t+1} + \mathbb{E}_t(\xi\beta)^2\lambda_{1t+2}\left[\left(\pi_{t+1}^{-1}\pi_t^\tau\right)\left(\pi_{t+2}^{-1}\pi_{t+1}^\tau\right)\right]^{\frac{1}{1-\theta}} Y_{t+2} + \ldots$$

matches

$$B_t = \lambda_{1t}Y_t + \xi\beta\mathbb{E}_t\left(\frac{\pi_t^\tau}{\pi_{t+1}}\right)^{\frac{1}{1-\theta}} B_{t+1} = \lambda_{1t}Y_t + \xi\beta\mathbb{E}_t\left(\frac{\pi_t^\tau}{\pi_{t+1}}\right)^{\frac{1}{1-\theta}}\left[\lambda_{1t+1}Y_{t+1} + \xi\beta\mathbb{E}_t\left(\frac{\pi_{t+1}^\tau}{\pi_{t+2}}\right)^{\frac{1}{1-\theta}} B_{t+3}\right] =$$

$$= \lambda_{1t}Y_t + \xi\beta\mathbb{E}_t\left(\frac{\pi_t^\tau}{\pi_{t+1}}\right)^{\frac{1}{1-\theta}}\left\{\lambda_{1t+1}Y_{t+1} + \xi\beta\mathbb{E}_t\left(\frac{\pi_{t+1}^\tau}{\pi_{t+2}}\right)^{\frac{1}{1-\theta}}\left[\lambda_{1t+2}Y_{t+2} + \xi\beta\mathbb{E}_t\left(\frac{\pi_{t+2}^\tau}{\pi_{t+3}}\right)^{\frac{1}{1-\theta}} B_{t+3}\right]\right\},$$

since if $j = 0$ then $\prod_{k=0}^{j-1}\pi_{t+k+1}^{-1}\pi_{t+k}^\tau = \prod_{k=0}^{-1}\pi_{t+k+1}^{-1}\pi_{t+k}^\tau = \left(\pi_{t+1}^{-1}\pi_t^\tau\right)\left(\pi_t^{-1}\pi_{t-1}^\tau\right) = 0$ and if $j = 1$ then $\prod_{k=0}^{j-1}\pi_{t+k+1}^{-1}\pi_{t+k}^\tau = \prod_{k=0}^{0}\pi_{t+k+1}^{-1}\pi_{t+k}^\tau = \pi_{t+1}^{-1}\pi_t^\tau$.

6. Real production

6.1 Production function and real production cost. Along the lines of a neo-classical growth model with stochastic technology, wholesale real output Y_{it} is equal to a CES production function of imperfect complements[5], being utilised capital $\tilde{K}_{it-1}$ and labour l_{it}, shifted by real production technology a_t :

$$Y_{it} = a_t\left(u_tK_{it-1}\right)^\alpha\left(\Upsilon_t l_{it}\right)^{1-\alpha} = a_t\tilde{K}_{it-1}^\alpha\left(\Upsilon_t l_{it}\right)^{1-\alpha} \text{ (real production or production function),}$$

in which (i) $\tilde{K}_{it-1} = u_tK_{it-1}$ and (ii) capital in output share $\alpha \in [0,\ 1] \subset \mathbb{R}_+$. Real production technology a_t can thus be said to be factor augmenting or "Hicks neutral"[6]. Confidence Υ_t shifts labour l_{it} and can similarly be said to be labour augmenting or "Harrod neutral".

Real production cost Γ_{1t} equals the sum of labour l_{it} weighted at real wage W_t and utilised capital $\tilde{K}_{it-1}$ weighted at real capital return rk_t : $\Gamma_{1t} = W_tl_{it} + rk_t\tilde{K}_{it-1}$.

6.2 Real production optimisation problem. For non-negative arguments relative to the objective function, the second optimisation problem of the macroeconomic wholesaler is thus the minimisation of real production cost Γ_{1t} subject to real production Y_{it} :

[5] https://en.wikipedia.org
[6] https://en.wikipedia.org

$$\min_{\{l_{it},\ \tilde{K}_{it-1}\}_{t=0}^{\infty}} \Gamma_{1t} = W_t l_{it} + rk_t \tilde{K}_{it-1} \text{ s.t.}$$
$$Y_{it} = a_t \tilde{K}_{it-1}^{\alpha} \left(\Upsilon_t l_{it}\right)^{1-\alpha},$$
$$l_{it},\ \tilde{K}_{it-1} \geq 0.$$

The necessary and sufficient conditions for optimal solutions are afresh met by construction. The Lagrangian equation of said optimisation problem is such that real production cost Γ_{1t} is optimised seeking wholesale real production optimal inputs, being labour l_{it} and utilised capital $\tilde{K}_{it-1}$, in the face of perfect competition, in which real marginal cost ϕ_t weights real production Y_{it} :

$$\mathcal{L}_{4t} = \left[W_t l_{it} + rk_t \tilde{K}_{it-1}\right] + \phi_t \left[Y_{it} - a_t \tilde{K}_{it-1}^{\alpha} \left(\Upsilon_t l_{it}\right)^{1-\alpha}\right].$$

FOCs are:

$$\frac{\partial \mathcal{L}_{4t}}{\partial l_{it}} = 0 \longleftrightarrow W_t + \phi_t \left[-a_t \tilde{K}_{it-1}^{\alpha} \left(1-\alpha\right) \Upsilon_t \left(\Upsilon_t l_{it}\right)^{-\alpha}\right] = 0 \longrightarrow$$
$$\longrightarrow W_t = \phi_t a_t \tilde{K}_{it-1}^{\alpha} \left(1-\alpha\right) \Upsilon_t \left(\Upsilon_t l_{it}\right)^{-\alpha};$$
$$\frac{\partial \mathcal{L}_{4t}}{\partial \tilde{K}_{it-1}} = 0 \longleftrightarrow rk_t + \phi_t \left[-a_t \alpha \tilde{K}_{it-1}^{\alpha-1} \left(\Upsilon_t l_{it}\right)^{1-\alpha}\right] = 0 \longrightarrow$$
$$\longrightarrow rk_t = \phi_t a_t \alpha \tilde{K}_{it-1}^{\alpha-1} \left(\Upsilon_t l_{it}\right)^{1-\alpha},$$

from which there follow

$$rk_t^{-1} W_t = \left[\phi_t a_t \alpha \tilde{K}_{it-1}^{\alpha-1} \left(\Upsilon_t l_{it}\right)^{1-\alpha}\right]^{-1} \left[\phi_t a_t \tilde{K}_{it-1}^{\alpha} \left(1-\alpha\right) \Upsilon_t \left(\Upsilon_t l_{it}\right)^{-\alpha}\right] = \left(\alpha l_{it}\right)^{-1} \left(1-\alpha\right) \tilde{K}_{it-1} \longrightarrow$$
$$\longrightarrow l_{it} = \left(W_t \alpha\right)^{-1} \left(1-\alpha\right) rk_t \tilde{K}_{it-1} = \alpha^{-1} \left(1-\alpha\right) W_t^{-1} rk_t u_t K_{it-1} \text{ (labour or labour demand)},$$
$$W_t = \left(1-\alpha\right) a_t \tilde{K}_{it-1}^{\alpha} \left(\Upsilon_t l_{it}\right)^{-\alpha} \phi_t \Upsilon_t \left(\Upsilon_t l_{it}\right)^{-1} \left(\Upsilon_t l_{it}\right) = \left(1-\alpha\right) \left(\Upsilon_t l_{it}\right)^{-1} Y_{it} \phi_t \Upsilon_t \text{ (average wage) and}$$
$$rk_t = \alpha a_t \tilde{K}_{it-1}^{\alpha-1} \left(\Upsilon_t l_{it}\right)^{1-\alpha} \phi_t \tilde{K}_{it-1}^{-1} \tilde{K}_{it-1} = \alpha \tilde{K}_{it-1}^{-1} Y_{it} \phi_t \text{ (average capital return)},$$

in turn implying

$$\Upsilon_t l_{it} = \left(1-\alpha\right) W_t^{-1} Y_{it} \phi_t \Upsilon_t \text{ and}$$
$$\tilde{K}_{it-1} = \alpha rk_t^{-1} Y_{it} \phi_t$$

such that

$$Y_{it} = a_t \tilde{K}_{it-1}^{\alpha} \left(\Upsilon_t l_{it}\right)^{1-\alpha} = a_t \left(\alpha rk_t^{-1} Y_{it} \phi_t\right)^{\alpha} \left[\left(1-\alpha\right) W_t^{-1} Y_{it} \phi_t \Upsilon_t\right]^{1-\alpha} =$$
$$= \alpha^{\alpha} \left(1-\alpha\right)^{1-\alpha} a_t Y_{it} \phi_t \Upsilon_t^{1-\alpha} rk_t^{-\alpha} W_t^{\alpha-1} \longrightarrow$$
$$\longrightarrow \phi_t = \alpha^{-\alpha} \left(1-\alpha\right)^{\alpha-1} a_t^{-1} \Upsilon_t^{\alpha-1} rk_t^{\alpha} W_t^{1-\alpha},$$

being a law of motion for real marginal cost ϕ_t.

7. Central bank and treasury

7.1 Nominal interest rate. Nominal interest rate rn_t is set according to a "Taylor rule"[7]:

[7] https://en.wikipedia.org

$$\left(\frac{rn_t}{rn}\right) = \left(\frac{rn_{t-1}}{rn}\right)^{\rho_{rn}} \left[\left(\frac{\pi_t/\pi}{\pi_T/\pi}\right)^{\phi_\pi} \left(\frac{\pi_t/\pi}{\pi_{t-1}/\pi}\right)^{\phi_{\pi_g}} \left(\frac{Y_t}{Y}\right)^{\phi_y} \left(\frac{Y_t/Y}{Y_{t-1}/Y}\right)^{\phi_{y_g}}\right]^{1-\rho_{rn}} e^{\varphi},$$

in which interest rate persistence, inflation coefficient, inflation gap coefficient, output coefficient and output gap coefficient $\rho_{rn},\ \phi_\pi,\ \phi_{\pi_g},\ \phi_y,\ \phi_{y_g} \in \mathbb{R}_{++}$ and monetary policy parameter $\varphi \in \mathbb{R}$.

All endogenous variables are divided by their values at the steady state, thereby representing non-linear deviations from it. The deviation of nominal interest rate rn_t from its steady state is therefore the weighted product of (i) exponentiated monetary policy parameter e^{φ}, (ii) the deviation of lagged nominal interest rate rn_{t-1} from its steady state and (iii) that of a product of inflation π_t, aggregate real output Y_t and their respective gaps $\pi_{t-1}^{-1}\pi_t$ and $Y_{t-1}^{-1}Y_t$. Such a "Taylor rule" is a law of motion for nominal interest rate rn_t.

7.2 Public finance. The treasury's nominal budget constraint is the equality between its nominal demand and its nominal supply. In detail, the treasury's nominal demand is the sum of real fiscal policy or government expenditure parameter g weighted at price P_t, lagged real government bond return $rn_{t-1}b_{t-1}$ weighted at lagged price P_{t-1} and real transfers tf_t weighted at price $P_t:\ P_tg + rn_{t-1}P_{t-1}b_{t-1} + P_ttf_t$, in which $g \in \mathbb{R}_{++}$.

The treasury's nominal supply is the sum of real government bond return b_t and real taxation tx_t, both weighted at price $P_t:\ P_tb_t + P_ttx_t$. The treasury's nominal budget constraint can therefore be written as follows: $P_tg + rn_{t-1}P_{t-1}b_{t-1} + P_ttf_t = P_tb_t + P_ttx_t \longrightarrow P_tg + rn_{t-1}B_{t-1} + TF_t = B_t + TX_t$.

On division by price P_t, the treasury's real budget constraint can be accordingly written as follows: $g + \frac{rn_{t-1}B_{t-1}}{P_t} + \frac{TF_t}{P_t} = \frac{B_t}{P_t} + \frac{TX_t}{P_t} \longrightarrow g + rn_{t-1}\pi_t^{-1}b_{t-1} + tf_t = b_t + tx_t$. Such in turn implies an equation for real government bond $b_t:\ b_t = g + rn_{t-1}\pi_t^{-1}b_{t-1} + tf_t - tx_t$.

8. Aggregation

8.1 Household real profit. Owing to market clearing, aggregate labour l_t equals a continuum of labour $\int_0^1 l_{it}di$ and utilised aggregate capital $\tilde{K}_{t-1}$ equals a continuum of utilised capital $\int_0^1 \tilde{K}_{it-1}di:\ l_t = \int_0^1 l_{it}di$ and $\tilde{K}_{t-1} = \int_0^1 \tilde{K}_{it-1}di$. Such implies the following aggregation:

$$\int_0^1 l_{it}di = \alpha^{-1}(1-\alpha)W_t^{-1}rk_tu_t\int_0^1 K_{it-1}di \longrightarrow$$
$$\longrightarrow l_t = \alpha^{-1}(1-\alpha)W_t^{-1}rk_tu_tK_{t-1},$$

being a law of motion for aggregate labour or aggregate labour demand l_t. Household nominal profit $P_t\Pi_{2t}$ is consequently aggregated as follows:

$$P_t\Pi_{2t} = \int_0^1 \left[P_{it}Y_{it} - Wn_tl_{it} - Rk_t\tilde{K}_{it-1}\right]di \longrightarrow$$
$$\longrightarrow \Pi_{2t} = \int_0^1 \left[P_t^{-1}P_{it}Y_{it} - W_tl_{it} - rk_t\tilde{K}_{it-1}\right]di \longrightarrow$$
$$\longrightarrow \Pi_{2t} = \int_0^1 \left[P_t^{-1}P_{it}Y_t\left(P_t^{-1}P_{it}\right)^{\frac{\theta}{1-\theta}} - W_tl_{it} - rk_t\tilde{K}_{it-1}\right]di =$$
$$= \int_0^1 \left[P_t^{\frac{-(1-\theta)-\theta}{1-\theta}}P_{it}^{\frac{(1-\theta)+\theta}{1-\theta}}Y_t - W_tl_{it} - rk_t\tilde{K}_{it-1}\right]di \longrightarrow$$
$$\longrightarrow \Pi_{2t} = P_t^{\frac{-1}{1-\theta}}Y_t\int_0^1 P_{it}^{\frac{1}{1-\theta}}di - W_t\int_0^1 l_{it}di - rk_t\int_0^1 \tilde{K}_{it-1}di \longrightarrow$$
$$\longrightarrow \Pi_{2t} = P_t^{\frac{-1}{1-\theta}}Y_tP_t^{\frac{1}{1-\theta}} - W_tl_t - rk_t\tilde{K}_{t-1} \longrightarrow$$
$$\longrightarrow \Pi_{2t} = Y_t - W_tl_t - rk_t\tilde{K}_{t-1} = Y_t - W_tl_t - rk_tu_tK_{t-1} \text{ (household real profit)},$$

in which (i) $Y_{it} = Y_t \left(P_t^{-1} P_{it}\right)^{\frac{\theta}{1-\theta}}$, (ii) $l_t = \int_0^1 l_{it} di$, (iii) $\tilde{K}_{t-1} = \int_0^1 \tilde{K}_{it-1} di$ and (iv) $P_t^{\frac{1}{1-\theta}} = \int_0^1 P_{it}^{\frac{1}{1-\theta}} di$.

8.2 Aggregate capital utilisation. The substitution of real government bond b_t and household real profit Π_{2t} into the macroeconomic household's real budget constraint gives rise to a law of motion for aggregate capital utilisation $\Psi(u_t)K_{t-1}$ or aggregate resources Y_t :

$$
\begin{aligned}
& C_t + b_t + tx_t = W_t l_t + [rk_t u_t K_{t-1} - \Psi(u_t)K_{t-1}] + rn_{t-1}\pi_t^{-1} b_{t-1} + \Pi_{2t} + tf_t \longrightarrow \\
& \longrightarrow C_t + (g + rn_{t-1}\pi_t^{-1} b_{t-1} + tf_t - tx_t) + tx_t = \\
& = W_t l_t + [rk_t u_t K_{t-1} - \Psi(u_t)K_{t-1}] + rn_{t-1}\pi_t^{-1} b_{t-1} + (Y_t - W_t l_t - rk_t u_t K_{t-1}) + tf_t \longrightarrow \\
& \longrightarrow C_t + g = -\Psi(u_t)K_{t-1} + Y_t \longrightarrow \\
& \longrightarrow Y_t = C_t + g + \Psi(u_t)K_{t-1},
\end{aligned}
$$

in which (i) $b_t = g + rn_{t-1}\pi_t^{-1} b_{t-1} + tf_t - tx_t$ and (ii) $\Pi_{2t} = Y_t - W_t l_t - rk_t u_t K_{t-1}$.

8.3 Aggregate real production. Real production Y_{it} is analogously aggregated in the following manner:

$$
\begin{aligned}
& Y_{it} = a_t \tilde{K}_{it-1}^{\alpha} \left(\Upsilon_t l_{it}\right)^{1-\alpha} \longrightarrow \\
& \longrightarrow \int_0^1 Y_{it} di = \int_0^1 a_t \tilde{K}_{it-1}^{\alpha} \left(\Upsilon_t l_{it}\right)^{1-\alpha} di \longrightarrow \\
& \longrightarrow \int_0^1 Y_t \left(P_t^{-1} P_{it}\right)^{\frac{\theta}{1-\theta}} di = a_t \Upsilon_t^{1-\alpha} \int_0^1 \tilde{K}_{it-1}^{\alpha} l_{it}^{1-\alpha} di \longrightarrow \\
& \longrightarrow Y_t \int_0^1 \left(P_t^{-1} P_{it}\right)^{\frac{\theta}{1-\theta}} di = Y_t pd_t = a_t \tilde{K}_{t-1}^{\alpha} \left(\Upsilon_t l_t\right)^{1-\alpha} \longrightarrow \\
& \longrightarrow Y_t = pd_t^{-1} a_t \tilde{K}_{t-1}^{\alpha} \left(\Upsilon_t l_t\right)^{1-\alpha},
\end{aligned}
$$

in which (i) $Y_{it} = Y_t \left(P_t^{-1} P_{it}\right)^{\frac{\theta}{1-\theta}}$, (ii) $l_t = \int_0^1 l_{it} di$, (iii) $\tilde{K}_{t-1} = \int_0^1 \tilde{K}_{it-1} di$ and (iv) price dispersion $pd_t = \int_0^1 \left(P_t^{-1} P_{it}\right)^{\frac{\theta}{1-\theta}} di$, the equation in question being a law of motion for aggregate real production or aggregate production function Y_t.

8.4 Price dispersion and optimal adjusted aggregate price. Price dispersion pd_t naturally follows aggregate price P_t :

$$
\begin{aligned}
& P_t = \left(\int_0^1 P_{it}^{\frac{1}{1-\theta}} di\right)^{1-\theta} = \left[\int_0^{1-\xi} \left(P_t^*\right)^{\frac{1}{1-\theta}} di + \int_{1-\xi}^1 \left(P_{it}\right)^{\frac{1}{1-\theta}} di\right]^{1-\theta} \longrightarrow \\
& \longrightarrow pd_t = \int_0^1 \left(P_t^{-1} P_{it}\right)^{\frac{\theta}{1-\theta}} di = \int_0^{1-\xi} \left(P_t^{-1} P_{it}^*\right)^{\frac{\theta}{1-\theta}} di + \int_{1-\xi}^1 \left(P_t^{-1} P_{it}\right)^{\frac{\theta}{1-\theta}} di = \\
& = \int_0^{1-\xi} \left(P_t^{-1} P_{it}^*\right)^{\frac{\theta}{1-\theta}} di + \int_{1-\xi}^1 \left(P_t^{-1} \pi_{t-1}^{\tau} P_{it-1}\right)^{\frac{\theta}{1-\theta}} di = \int_0^{1-\xi} \left(P_t^{-1} P_{it}^*\right)^{\frac{\theta}{1-\theta}} di + \xi \left(P_t^{-1} \pi_{t-1}^{\tau} \int_0^1 P_{it-1} di\right)^{\frac{\theta}{1-\theta}} = \\
& = \int_0^{1-\xi} \left(P_{t-1}^{-1} P_{t-1}\right)^{\frac{\theta}{1-\theta}} \left(P_t^{-1} P_{it}^*\right)^{\frac{\theta}{1-\theta}} di + \left(P_{t-1}^{-1} P_{t-1}\right)^{\frac{\theta}{1-\theta}} \xi \left(P_t^{-1} \pi_{t-1}^{\tau} \int_0^1 P_{it-1} di\right)^{\frac{\theta}{1-\theta}} = \\
& = \int_0^{1-\xi} \left(\pi_t^{-1} \pi_{it}^*\right)^{\frac{\theta}{1-\theta}} di + \xi \left(P_{t-1}^{-1} \pi_t^{-1} \pi_{t-1}^{\tau} \int_0^1 P_{it-1} di\right)^{\frac{\theta}{1-\theta}} =
\end{aligned}
$$

$$= i\left(\pi_t^{-1}\pi_{it}^*\right)^{\frac{\theta}{1-\theta}}|_0^{1-\xi} + \xi\int_0^1\left(P_{t-1}^{-1}P_{it-1}\right)^{\frac{\theta}{1-\theta}}di\left(\pi_t^{-1}\pi_{t-1}^{\tau}\right)^{\frac{\theta}{1-\theta}} = (1-\xi-0)\left(\pi_t^{-1}\pi_{it}^*\right)^{\frac{\theta}{1-\theta}} + \xi pd_{t-1}\left(\pi_t^{-1}\pi_{t-1}^{\tau}\right)^{\frac{\theta}{1-\theta}} =$$

$$= (1-\xi)\left(\pi_t^{-1}\pi_{it}^*\right)^{\frac{\theta}{1-\theta}} + \xi pd_{t-1}\left(\pi_t^{-1}\pi_{t-1}^{\tau}\right)^{\frac{\theta}{1-\theta}} = \xi pd_{t-1}\left(\frac{\pi_{t-1}^{\tau}}{\pi_t}\right)^{\frac{\theta}{1-\theta}} + (1-\xi)\left(\frac{\pi_t^*}{\pi_t}\right)^{\frac{\theta}{1-\theta}},$$

in which (i) $\int_{1-\xi}^1\left(P_t^{-1}\pi_{t-1}^{\tau}P_{it-1}\right)^{\frac{\theta}{1-\theta}}di = \xi\left(P_t^{-1}\pi_{t-1}^{\tau}\int_0^1 P_{it-1}di\right)^{\frac{\theta}{1-\theta}}$ on account of random wholesale real output price adjustment and a continuum of wholesalers and (ii) $\pi_t^{\frac{-\theta}{1-\theta}} = \left(P_t^{-1}P_{t-1}\right)^{\frac{-\theta}{1-\theta}}$ on account of $\pi_t^{-1} = P_t^{-1}P_{t-1}$, the equation in question being a law of motion for price dispersion pd_t.

Owing to market clearing, optimal adjusted aggregate price p_t^* accordingly equals a continuum of optimal adjusted prices $\int_0^1 p_{it}^* di:\ p_t^* = \int_0^1 p_{it}^* di = \int_0^1 \frac{A_t}{B_t}di = i\frac{A_t}{B_t}|_0^1 = (1-0)\frac{A_t}{B_t} = \frac{A_t}{B_t}$, *ceteris paribus.* It follows that the law of motion for optimal adjusted aggregate price p_t^* is

$$p_t^* = \frac{A_t}{B_t},$$

all else equal.

9. Equilibrium

9.1 Price equilibrium with transfers. A price equilibrium with transfers is a pair of feasible allocation $\{U(C_t,\ l_t),\ \Upsilon_t,\ C_t,\ l_t,\ u_t,\ K_t,\ \Psi(u_t),\ b_t,\ \Pi_{2t},\ \Pi_{1t},\ Y_t,\ Y_{it},\ \Pi_{3t},\ \Gamma_{1t},\ l_{it},\ K_{it}\}_{t=0}^{\infty}$ and prices $\{\lambda_{1t},\ W_t,\ rk_t,\ \pi_t,\ P_t,\ P_{it},\ P_{it}^*,\ \phi_t,\ pd_t,\ \pi_t^*\}_{t=0}^{\infty}$ such that retail nominal profit $P_t\Pi_{1t}$ and wholesale real profit Π_{3t}, real production cost Γ_{1t} and household utility $U(C_t,\ l_t)$ (i.e. preferences) are optimal and markets clear: *ceteris paribus,*

(household utility optimisation)

$$\max_{\{C_t,\ l_t,\ u_t,\ b_t\}_{t=0}^{\infty}} U(C_t,\ l_t) = \mathbb{E}_t\sum_{t=0}^{\infty}\beta^t\left\{\frac{\Upsilon_t(C_t - hC_{t-1})^{1-\sigma_c} - 1}{1-\sigma_c} - \frac{l_t^{1+\sigma_l}}{1+\sigma_l}\right\} \text{ s.t.}$$

$$C_t + b_t + tx_t = W_t l_t + [rk_t u_t K_{t-1} - \Psi(u_t)K_{t-1}] + rn_{t-1}\pi_t^{-1}b_{t-1} + \Pi_{2t} + tf_t,$$

$$K_t = (1-\delta)K_{t-1} + i,$$

$$\lim_{t\to\infty}\mathbb{E}_t\beta^t\lambda_{1t}X_{t+1} = 0,\ \forall X = K,\ B,$$

$$C_t,\ l_t,\ u_t,\ b_t \geq 0;$$

(retail nominal profit optimisation)

$$\max_{\{Y_{it}\}_{t=0}^{\infty}} P_t\Pi_{1t} = P_tY_t - \int_0^1 P_{it}Y_{it}di \text{ s.t.}$$

$$Y_t = \left(\int_0^1 Y_{it}^{\frac{1}{\theta}}di\right)^{\theta},$$

$$Y_{it} \geq 0;$$

(wholesale real profit optimisation)

$$\max_{\{P_{it}^*\}_{t=0}^{\infty}} \Pi_{3t} = \mathbb{E}_t\sum_{j=0}^{\infty}(\xi\beta)^j\left(\lambda_{1t}^{-1}\lambda_{1t+j}\right)\left[P_{t+j}^{-1}\prod_{k=0}^{j-1}\pi_{t+k}^{\tau}P_{it}^* - \phi_{t+j}\right]Y_{it+j} \text{ s.t.}$$

$$Y_{it+j} = Y_{t+j}\left(P_{t+j}^{-1}P_{it+j}\right)^{\frac{\theta}{1-\theta}} = Y_{t+j}\left(P_{t+j}^{-1}\prod_{k=0}^{j-1}\pi_{t+k}^{\tau}P_{it}^*\right)^{\frac{\theta}{1-\theta}},$$

$$P_{it}^* \geq 0;$$

(real production cost optimisation)

$$\underset{\{l_{it},\ \tilde{K}_{it-1}\}_{t=0}^{\infty}}{min} \Gamma_{1t} = W_t l_{it} + rk_t \tilde{K}_{it-1} \text{ s.t.}$$

$Y_{it} = a_t \tilde{K}_{it-1}^{\alpha} \left(\Upsilon_t l_{it}\right)^{1-\alpha}$,

$l_{it},\ \tilde{K}_{it-1} \geq 0;$

(aggregate price) $P_t = \left[\xi(\pi_{t-1}^{\tau} P_{t-1})^{\frac{1}{1-\theta}} + (1-\xi)(P_t^*)^{\frac{1}{1-\theta}}\right]^{1-\theta}$;

(price dispersion) $pd_t = \xi pd_{t-1}\left(\frac{\pi_{t-1}^{\tau}}{\pi_t}\right)^{\frac{\theta}{1-\theta}} + (1-\xi)\left(\frac{\pi_t^*}{\pi_t}\right)^{\frac{\theta}{1-\theta}}$;

(confidence) $\Upsilon_t = \left(pt_t t_t n_t\right)^{\gamma}$;

(permanent technology) $pt_t = pt_{t-1} e^{\sigma_{\varepsilon_{pt}} \varepsilon_{ptt}}$;

(transitory technology and noise technology) $x_t = \rho_x x_{t-1} e^{\sigma_{\varepsilon_x} \varepsilon_{xt}},\ \forall x = t,\ n$;

(real production technology) $a_t = e^{\mu} \rho_a a_{t-1} pt_{t-1} t_{t-1}$;

(nominal interest rate) $\left(\frac{rn_t}{rn}\right) = \left(\frac{rn_{t-1}}{rn}\right)^{\rho_{rn}} \left[\left(\frac{\pi_t/\pi}{\pi_T/\pi}\right)^{\phi_\pi} \left(\frac{\pi_t/\pi}{\pi_{t-1}/\pi}\right)^{\phi_{\pi_g}} \left(\frac{Y_t}{Y}\right)^{\phi_y} \left(\frac{Y_t/Y}{Y_{t-1}/Y}\right)^{\phi_{y_g}}\right]^{1-\rho_{rn}} e^{\varphi}$;

(aggregate capital utilisation) $Y_t = C_t + g + \Psi\left(u_t\right) K_{t-1}$.

9.2 Feasible Pareto efficient allocation. Endogenous variables can be subdivided as follows. Consumption, endowment and production endogenous variables: $\left\{U\left(C_t,\ l_t\right),\ \Upsilon_t,\ C_t,\ l_t,\ u_t,\ K_t,\ \Psi\left(u_t\right),\ b_t,\ \Pi_{2t},\ \Pi_{1t},\ Y_t,\ Y_{it},\ \Pi_{3t},\ \Gamma_{1t},\ l_{it},\ K_{it}\right\}_{t=0}^{\infty}$.

Price endogenous variables: $\{\lambda_{1t},\ W_t,\ rk_t,\ \pi_t,\ P_t,\ P_{it},\ P_{it}^*,\ \phi_t,\ pd_t,\ \pi_t^*\}_{t=0}^{\infty}$. Technology endogenous variables: $\{pt_t,\ n_t,\ t_t,\ a_t\}_{t=0}^{\infty}$. Policy endogenous variables: $\{rn_t,\ tf_t,\ tx_t\}_{t=0}^{\infty}$.

Parameters are $\left\{\beta,\ h,\ \sigma_c,\ \sigma_l,\ \delta,\ i,\ \theta,\ \xi,\ \tau,\ \alpha,\ \gamma,\ \sigma_{pt},\ \rho_n,\ \sigma_n,\ \rho_t,\ \sigma_t,\ \mu,\ \rho_a,\ \rho_{rn},\ \phi_\pi,\ \phi_{\pi_g},\ \phi_y,\ \phi_{y_g},\ \varphi,\ g\right\}$.

The feasible allocation is characterised by consumption, endowment and production endogenous variables. Prices are characterised by price endogenous variables. Technology endogenous variables should be feasible allocation endogenous variables, but are recorded separately for scopes of clarity.

Policy endogenous variables should be both feasible allocation and price endogenous variables, but are recorded separately for identical scopes. Strictly speaking, in fact, endowment endogenous variables should have to be transfers tf_t and taxation tx_t alone, being there none in the feasible allocation.

A feasible allocation is Pareto efficient if and only if there exists no other feasible allocation such that almost all agents prefer it to the given one and at least one agent strictly prefers it to the given one. By construction, markets are complete. The first fundamental theorem of welfare economics consequently applies, whereby a price equilibrium with transfers in a complete market system is a feasible Pareto efficient allocation.

10. Laws of motion and normalisation

10.1 Laws of motion. There thus emerge the following laws of motion:

$pt_t = pt_{t-1} e^{\sigma_{\varepsilon_{pt}} \varepsilon_{ptt}}$ (permanent technology);

$x_t = \rho_x x_{t-1} e^{\sigma_{\varepsilon_x} \varepsilon_{xt}},\ \forall x = t,\ n$ (transitory technology and noise technology);

$a_t = e^{\mu} \rho_a a_{t-1} pt_{t-1} t_{t-1}$ (real production technology);

$rk_t = \Psi'(u_t)$ (real capital return);

$P_t = \left[\xi(\pi_{t-1}^{\tau} P_{t-1})^{\frac{1}{1-\theta}} + (1-\xi)(P_t^*)^{\frac{1}{1-\theta}}\right]^{1-\theta}$ (aggregate price);

$p_t^* = \frac{A_t}{B_t}$ (optimal adjusted aggregate price), for

$A_t = \lambda_{1t}\phi_t\theta Y_t + \xi\beta\mathbb{E}_t\left(\frac{\pi_t^{\tau}}{\pi_{t+1}}\right)^{\frac{\theta}{1-\theta}} A_{t+1}$ and

$B_t = \lambda_{1t}Y_t + \xi\beta\mathbb{E}_t\left(\frac{\pi_t^{\tau}}{\pi_{t+1}}\right)^{\frac{1}{1-\theta}} B_{t+1}$;

$pd_t = \xi pd_{t-1}\left(\frac{\pi_{t-1}^{\tau}}{\pi_t}\right)^{\frac{\theta}{1-\theta}} + (1-\xi)\left(\frac{\pi_t^{*}}{\pi_t}\right)^{\frac{\theta}{1-\theta}}$ (price dispersion);

$W_t = \Upsilon_t^{-1}(C_t - hC_{t-1})^{\sigma_c} l_t^{\sigma_l}$ (real wage);

$l_t = \alpha^{-1}(1-\alpha)W_t^{-1} rk_t u_t k_{t-1}$ (aggregate labour);

$\Upsilon_t(C_t - hC_{t-1})^{-\sigma_c} = \mathbb{E}_t\beta\left[\frac{\Upsilon_{t+1}(C_{t+1} - hC_t)^{-\sigma_c} rn_t}{\pi_{t+1}}\right]$ (real consumption);

$\left(\frac{rn_t}{rn}\right) = \left(\frac{rn_{t-1}}{rn}\right)^{\rho_{rn}}\left[\left(\frac{\pi_t/\pi}{\pi_T/\pi}\right)^{\phi_\pi}\left(\frac{\pi_t/\pi}{\pi_{t-1}/\pi}\right)^{\phi_{\pi_g}}\left(\frac{Y_t}{Y}\right)^{\phi_y}\left(\frac{Y_t/Y}{Y_{t-1}/Y}\right)^{\phi_{y_g}}\right]^{1-\rho_{rn}} e^{\varphi}$ (nominal interest rate);

$Y_t = pd_t^{-1} a_t \tilde{K}_{t-1}^{\alpha}(\Upsilon_t l_t)^{1-\alpha}$ (aggregate real production);

$\Upsilon_t = (pt_t t_t n_t)^{\gamma}$ (confidence);

$\phi_t = \alpha^{-\alpha}(1-\alpha)^{\alpha-1} a_t^{-1}\Upsilon_t^{\alpha-1} rk_t^{\alpha} W_t^{1-\alpha}$ (real marginal cost);

$K_t = (1-\delta)K_{t-1} + i$ (aggregate capital);

$Y_t = C_t + g + \Psi(u_t)K_{t-1}$ (aggregate capital utilisation).

10.2 Normalisation. Certain endogenous variables, being real consumption C_t, aggregate capital K_t, aggregate real output Y_t and real wage W_t, abide by the permanent changes to the steady state dictated by permanent technology pt_t and are normalised thus: $X_t = x_t pt_t$, in which $X = C,\ K,\ Y,\ W$.

11. Log-linearisation

The DSGE model at hand is solved by resorting to a first order linear approximation whereby its laws of motion are log-linearised about the steady state of each endogenous variable. Specifically, a first order Taylor expansion is conducted about the logarithmic form of each law of motion: $f(x) = f(a) + \frac{f'(a)}{1!}(x-a) = f(a) + f'(a)\bar{x}$, in which a is the endogenous variable's steady state and $\bar{x}$ is its deviation therefrom.

Permanent technology:

$$pt_t = pt_{t-1}e^{\sigma_{\varepsilon_{pt}}\varepsilon_{ptt}};$$
$$pt = pte^{\sigma_{\varepsilon_{pt}}\varepsilon_{pt}} \text{ (steady state);}$$
$$pt = pt \text{ (steady state, in which } \varepsilon_{pt} = 0\text{, admitting } pt = 1\text{);}$$
$$lnpt = lnpt + \sigma_{\varepsilon_{pt}}\varepsilon_{pt};$$
$$lnpt + \frac{\bar{pt}_t}{pt} = lnpt + \frac{\bar{pt}_{t-1}}{pt} + \sigma_{\varepsilon_{pt}}\varepsilon_{pt} + \sigma_{\varepsilon_{pt}}(\varepsilon_{ptt} - \varepsilon_{pt}) \longrightarrow$$
$$\longrightarrow \hat{pt}_t = \hat{pt}_{t-1} + \sigma_{\varepsilon_{pt}}\varepsilon_{ptt} \longrightarrow$$
$$\longrightarrow \hat{pt}_t = \hat{pt}_{t-1} + \varepsilon_{ptt} \text{ (imposing } \sigma_{\varepsilon_{pt}} = 1\text{).}$$

Transitory technology and noise technology:

$$x = t,\ n;$$
$$x_t = \rho_x x_{t-1}e^{\sigma_{\varepsilon_x}\varepsilon_{xt}};$$
$$x = \rho_x xe^{\sigma_{\varepsilon_x}\varepsilon_x} \text{ (steady state);}$$

$x = \rho_x x$ (steady state, in which $\varepsilon_x = 0$, admitting $x = 1$);
$1 = \rho_x,$ but $\rho_x < 1,$ so $1 = \rho_x e^{\sigma_{\varepsilon_x}\varepsilon_x} \longrightarrow \rho_x = e^{-\sigma_{\varepsilon_x}\varepsilon_x} < 1 \longrightarrow$
$\longrightarrow -\sigma_{\varepsilon_x}\varepsilon_x lne < ln1 \longrightarrow -\sigma_{\varepsilon_x}\varepsilon_x < 0 \longrightarrow \varepsilon_x > 0$ (steady state, being there an attendant shock);
$lnx = \rho_x lnx + \sigma_{\varepsilon_x}\varepsilon_x;$

$$lnx + \frac{\bar{x}_t}{x} = \rho_x lnx + \frac{\rho_x \bar{x}_{t-1}}{x} + \sigma_{\varepsilon_x}\varepsilon_x + \sigma_{\varepsilon_x}(\varepsilon_{xt} - \varepsilon_x) \longrightarrow$$

$$\longrightarrow \hat{x}_t = \rho_x \hat{x}_{t-1} + \sigma_{\varepsilon_x}\varepsilon_{xt} \longrightarrow$$

$$\longrightarrow \hat{x}_t = \rho_x \hat{x}_{t-1} + \varepsilon_{xt} \text{ (imposing } \sigma_{\varepsilon_x} = 1\text{).}$$

Real production technology:

$$a_t = e^{\mu}\rho_a a_{t-1} pt_{t-1} t_{t-1};$$
$$a = e^{\mu}\rho_a aptt \text{ (steady state, admitting } a = 1\text{);}$$
$$\rho_a^{-1} = e^{\mu} \text{ (steady state, in which } pt = t = 1\text{);}$$
$$-ln\rho_a = \mu lne \longrightarrow \mu = -ln\rho_a \text{ (steady state);}$$
$$lna = \mu + \rho_a lna + lnpt + lnt;$$
$$lna + \frac{\bar{a}_t}{a} = \mu + \rho_a lna + \frac{\rho_a \bar{a}_{t-1}}{a} + lnpt + \frac{\bar{pt}_{t-1}}{pt} + lnt + \frac{\bar{t}_{t-1}}{t} \longrightarrow$$
$$\longrightarrow \hat{a}_t = \rho_a \hat{a}_{t-1} + \hat{pt}_{t-1} + \hat{t}_{t-1}.$$

Real capital return:

$$rk_t = \Psi'(u_t);$$
$$rk = \Psi'(u) \text{ (steady state);}$$
$$rk = \Psi'(1) \text{ (steady state, imposing } u = 1\text{);}$$
$$lnrk = ln\Psi'(u);$$
$$lnrk + \frac{\bar{rk}_t}{rk} = ln\Psi'(u) + \frac{\Psi''(u)(1)\bar{u}_t}{\Psi'(u)} \longrightarrow$$
$$\longrightarrow \frac{\bar{rk}_t}{rk} = \frac{\Psi''(u)(1)\bar{u}_t u}{\Psi'(u)u} \longrightarrow$$
$$\longrightarrow \hat{rk}_t = \frac{\Psi''(u)\hat{u}_t}{\Psi'(u)} \longrightarrow$$
$$\longrightarrow \hat{rk}_t = \omega^{-1}\hat{u}_t \left(\omega = \frac{\Psi'(u)}{\Psi''(u)}\right) \longrightarrow$$
$$\longrightarrow \hat{u}_t = \omega \hat{rk}_t.$$

One notices that parameter ω models capital utilisation adjustment cost inverse elasticity, being a positive real number: $\omega \in \mathbb{R}_{++}$.

Aggregate price:

$$P_t = \left[\xi\left(\pi_{t-1}^{\tau}P_{t-1}\right)^{\frac{1}{1-\theta}} + (1-\xi)(P_t^*)^{\frac{1}{1-\theta}}\right]^{1-\theta} \longrightarrow$$
$$\longrightarrow P_t^{\frac{1}{1-\theta}} = \xi\left(\pi_{t-1}^{\tau}P_{t-1}\right)^{\frac{1}{1-\theta}} + (1-\xi)(P_t^*)^{\frac{1}{1-\theta}};$$

$$\left(\frac{P_t}{P_t}\right)^{\frac{1}{1-\theta}} = \xi\left(\frac{\pi_{t-1}^{\tau}P_{t-1}}{P_t}\right)^{\frac{1}{1-\theta}} + (1-\xi)\left(\frac{P_t^*}{P_t}\right)^{\frac{1}{1-\theta}} \longleftrightarrow$$

$$\longleftrightarrow 1 = \xi\left(\frac{\pi_{t-1}^{\tau}}{\pi_t}\right)^{\frac{1}{1-\theta}} + (1-\xi)\left(p_t^*\right)^{\frac{1}{1-\theta}} \text{ (standardisation);}$$

$$1 = \xi\left(\frac{\pi^{\tau}}{\pi}\right)^{\frac{1}{1-\theta}} + (1-\xi)\left(p^*\right)^{\frac{1}{1-\theta}} \text{ (steady state);}$$

$$1 = \xi + (1-\xi)\left(p^*\right)^{\frac{1}{1-\theta}} \text{ (steady state, imposing } \pi = 1\text{);}$$

$$1-\xi = (1-\xi)\left(p^*\right)^{\frac{1}{1-\theta}} \longrightarrow$$

$$\longrightarrow 1 = \left(p^*\right)^{\frac{1}{1-\theta}} \longrightarrow$$

$$\longrightarrow 1^{1-\theta} = \left[\left(p^*\right)^{\frac{1}{1-\theta}}\right]^{1-\theta} \longrightarrow$$

$$\longrightarrow 1 = p^* \text{ (steady state);}$$

$$ln1 = ln\left[\xi\left(\frac{\pi^{\tau}}{\pi}\right)^{\frac{1}{1-\theta}} + (1-\xi)\left(p^*\right)^{\frac{1}{1-\theta}}\right];$$

$$ln1 = ln\left[\xi\left(\frac{\pi^{\tau}}{\pi}\right)^{\frac{1}{1-\theta}} + (1-\xi)\left(p^*\right)^{\frac{1}{1-\theta}}\right] + \frac{(1-\xi)(1-\theta)^{-1}\left(p^*\right)^{\frac{1}{1-\theta}-1}\bar{p}_t^*}{1} +$$

$$+ \frac{\xi\tau(1-\theta)^{-1}\pi^{\frac{\tau}{1-\theta}-1}\pi^{\frac{-1}{1-\theta}}\bar{\pi}_{t-1}}{1} + \frac{-\xi(1-\theta)^{-1}\pi^{\frac{\tau}{1-\theta}}\pi^{\frac{-1}{1-\theta}-1}\bar{\pi}_t}{1} \longrightarrow$$

$$\longrightarrow 0 = \frac{(1-\xi)1^{\frac{1-1+\theta}{1-\theta}}p^*\bar{p}_t^*}{(1-\theta)p^*} + \frac{\xi\tau 1^{\frac{\tau-1+\theta}{1-\theta}}1^{\frac{-1}{1-\theta}}\pi\bar{\pi}_{t-1}}{(1-\theta)\pi} - \frac{\xi 1^{\frac{\tau}{1-\theta}}1^{\frac{-1-1+\theta}{1-\theta}}\pi\bar{\pi}_t}{(1-\theta)\pi} \text{ (in which } \pi = p^* = 1\text{)} \longrightarrow$$

$$\longrightarrow 0 = \frac{(1-\xi)\hat{p}_t^*}{(1-\theta)} + \frac{\xi\tau\hat{\pi}_{t-1}}{(1-\theta)} - \frac{\xi\hat{\pi}_t}{(1-\theta)} \text{ (in which } \pi = p^* = 1\text{)} \longrightarrow$$

$$\longrightarrow (1-\xi)\hat{p}_t^* = \xi\left(\hat{\pi}_t - \tau\hat{\pi}_{t-1}\right) \longrightarrow$$

$$\longrightarrow \hat{p}_t^* = \frac{\xi\left(\hat{\pi}_t - \tau\hat{\pi}_{t-1}\right)}{(1-\xi)}.$$

Optimal adjusted aggregate price:
Numerator:

$$A_t = \theta\phi_t + \xi\beta\mathbb{E}_t\left(\frac{\pi_t^{\tau}}{\pi_{t+1}}\right)^{\frac{\theta}{1-\theta}} A_{t+1};$$

$$A = \theta\phi + \xi\beta\left(\frac{\pi^{\tau}}{\pi}\right)^{\frac{\theta}{1-\theta}} A \text{ (steady state);}$$

$$A = \theta\phi + \xi\beta A \text{ (steady state, in which } \pi = 1\text{);}$$

$$(1-\xi\beta)A = \theta\phi \longrightarrow$$

$$\longrightarrow A = (1-\xi\beta)^{-1}\theta\phi \text{ (steady state);}$$

$$lnA = ln\left[\theta\phi + \xi\beta\left(\frac{\pi^{\tau}}{\pi}\right)^{\frac{\theta}{1-\theta}} A\right];$$

$$lnA + \frac{\bar{A}_t}{A} = ln\left[\theta\phi + \xi\beta\left(\frac{\pi^{\tau}}{\pi}\right)^{\frac{\theta}{1-\theta}} A\right] + \frac{\theta\bar{\phi}_t}{\theta\phi + \xi\beta A} +$$

$$+\frac{\xi\beta\theta\tau\left(1-\theta\right)^{-1}\pi^{\frac{\tau\theta}{1-\theta}-1}\pi^{\frac{-\theta}{1-\theta}}\bar{\pi}_t}{\theta\phi+\xi\beta A}+\frac{-\xi\beta\theta\left(1-\theta\right)^{-1}\pi^{\frac{\tau\theta}{1-\theta}}\pi^{\frac{-\theta}{1-\theta}-1}\mathbb{E}_t\bar{\pi}_{t+1}}{\theta\phi+\xi\beta A}+\frac{\xi\beta\pi^{\frac{\tau\theta}{1-\theta}}\pi^{\frac{-\theta}{1-\theta}}\mathbb{E}_t\bar{A}_{t+1}}{\theta\phi+\xi\beta A}\longrightarrow$$

$$\longrightarrow\hat{A}_t=\frac{\theta\phi\bar{\phi}_t}{A\phi}+\frac{\xi\beta\theta\tau\left(1-\theta\right)^{-1}\pi\bar{\pi}_t}{A\pi}-\frac{\xi\beta\theta\left(1-\theta\right)^{-1}\pi\mathbb{E}_t\bar{\pi}_{t+1}}{A\pi}+\xi\beta\mathbb{E}_t\hat{A}_{t+1}\text{ (in which }\pi=1)\longrightarrow$$

$$\longrightarrow\hat{A}_t=\frac{\theta\phi\hat{\phi}_t}{A}+\frac{\xi\beta\theta\left(\tau\hat{\pi}_t-\mathbb{E}_t\hat{\pi}_{t+1}\right)}{A\left(1-\theta\right)}+\xi\beta\mathbb{E}_t\hat{A}_{t+1}.$$

Denominator:

$$B_t=1+\xi\beta\mathbb{E}_t\left(\frac{\pi_t^{\tau}}{\pi_{t+1}}\right)^{\frac{1}{1-\theta}}B_{t+1};$$

$$B=1+\xi\beta\left(\frac{\pi^{\tau}}{\pi}\right)^{\frac{1}{1-\theta}}B\text{ (steady state)};$$

$$B=1+\xi\beta B\text{ (steady state, in which }\pi=1);$$

$$\left(1-\xi\beta\right)B=1\longrightarrow$$

$$\longrightarrow B=\left(1-\xi\beta\right)^{-1}\text{ (steady state)};$$

$$lnB=ln\left[1+\xi\beta\left(\frac{\pi^{\tau}}{\pi}\right)^{\frac{1}{1-\theta}}B\right];$$

$$lnB+\frac{\bar{B}_t}{B}=ln\left[1+\xi\beta\left(\frac{\pi^{\tau}}{\pi}\right)^{\frac{1}{1-\theta}}B\right]+\frac{\xi\beta\tau\left(1-\theta\right)^{-1}\pi^{\frac{\tau}{1-\theta}-1}\pi^{\frac{-1}{1-\theta}}\bar{\pi}_t}{1+\xi\beta B}+$$

$$+\frac{\xi\beta\left(1-\theta\right)^{-1}\pi^{\frac{\tau}{1-\theta}}\pi^{\frac{-1}{1-\theta}-1}\mathbb{E}_t\bar{\pi}_{t+1}}{1+\xi\beta B}+\frac{\xi\beta\pi^{\frac{\tau}{1-\theta}}\pi^{\frac{-1}{1-\theta}}\mathbb{E}_t\bar{B}_{t+1}}{1+\xi\beta B}\longrightarrow$$

$$\longrightarrow\hat{B}_t=\frac{\xi\beta\tau\left(1-\theta\right)^{-1}\pi\bar{\pi}_t}{B\pi}+\frac{\xi\beta\left(1-\theta\right)^{-1}\pi\mathbb{E}_t\bar{\pi}_{t+1}}{B\pi}+\xi\beta\mathbb{E}_t\hat{B}_{t+1}\text{ (in which }\pi=1)\longrightarrow$$

$$\longrightarrow\hat{B}_t=\frac{\xi\beta\left(\tau\hat{\pi}-\mathbb{E}_t\hat{\pi}_{t+1}\right)}{\left(1-\theta\right)B}+\xi\beta\mathbb{E}_t\hat{B}_{t+1}.$$

Fraction:

$$p_t^*=\frac{A_t}{B_t};$$

$$p^*=\frac{A}{B}\text{ (steady state)};$$

$$1=\frac{A}{B}\text{ (steady state, in which }p^*=1\text{, admitting }A=B=1);$$

$$lnp^*=lnA-lnB;$$

$$lnp^*+\frac{\bar{p}_t^*}{p^*}=lnA+\frac{\bar{A}_t}{A}-lnB-\frac{\bar{B}_t}{B}\longrightarrow$$

$$\longrightarrow\hat{p}_t^*=\hat{A}_t-\hat{B}_t\longrightarrow$$

$$\longrightarrow\hat{p}_t^*=\left[\frac{\theta\phi\hat{\phi}_t}{A}+\frac{\xi\beta\theta\left(\tau\hat{\pi}_t-\mathbb{E}_t\hat{\pi}_{t+1}\right)}{A\left(1-\theta\right)}+\xi\beta\mathbb{E}_t\hat{A}_{t+1}\right]-\left[\frac{\xi\beta\left(\tau\hat{\pi}-\mathbb{E}_t\hat{\pi}_{t+1}\right)}{\left(1-\theta\right)B}+\xi\beta\mathbb{E}_t\hat{B}_{t+1}\right];$$

$$\hat{p}_t^*=\frac{\theta\phi\hat{\phi}_t}{A}+\frac{\xi\beta\left(1-\theta\right)\mathbb{E}_t\hat{\pi}_{t+1}}{A\left(1-\theta\right)}-\frac{\xi\beta\left(1-\theta\right)\tau\hat{\pi}_t}{\left(1-\theta\right)B}+\xi\beta\mathbb{E}_t\hat{p}_{t+1}^*\text{ (in which }\mathbb{E}_t\hat{p}_{t+1}^*=\mathbb{E}_t\hat{A}_{t+1}-\mathbb{E}_t\hat{B}_{t+1})\longrightarrow$$

$$\longrightarrow \frac{\xi\left(\hat{\pi}_t - \tau\hat{\pi}_{t-1}\right)}{(1-\xi)} = \frac{\theta\phi\hat{\phi}_t}{A} + \frac{\xi\beta\mathbb{E}_t\hat{\pi}_{t+1}}{A} - \frac{\xi\beta\tau\hat{\pi}_t}{B} + \xi\beta\frac{\xi\left(\mathbb{E}_t\hat{\pi}_{t+1} - \tau\hat{\pi}_t\right)}{(1-\xi)} \longrightarrow$$
$$\longrightarrow \hat{\pi}_t - \tau\hat{\pi}_{t-1} = \frac{(1-\xi)\,\theta\phi\hat{\phi}_t}{\xi A} + \frac{(1-\xi)\,\beta\mathbb{E}_t\hat{\pi}_{t+1}}{A} - \frac{(1-\xi)\,\beta\tau\hat{\pi}_t}{B} + \xi\beta\left(\mathbb{E}_t\hat{\pi}_{t+1} - \tau\hat{\pi}_t\right) \longrightarrow$$
$$\longrightarrow \hat{\pi}_t - \tau\hat{\pi}_{t-1} = \frac{(1-\xi)\,\theta\phi\hat{\phi}_t}{\xi A} + (1-\xi)\,\beta\left(A^{-1}\mathbb{E}_t\hat{\pi}_{t+1} - B^{-1}\tau\hat{\pi}_t\right) + \xi\beta\left(\mathbb{E}_t\hat{\pi}_{t+1} - \tau\hat{\pi}_t\right) \longrightarrow$$
$$\longrightarrow \hat{\pi}_t - \tau\hat{\pi}_{t-1} = \frac{(1-\xi)\,\theta\phi\hat{\phi}_t}{\xi} + \beta\left(\mathbb{E}_t\hat{\pi}_{t+1} - \tau\hat{\pi}_t\right) \text{ (in which } A = B = 1) \longrightarrow$$
$$\longrightarrow (1+\beta\tau)\,\hat{\pi}_t = \frac{(1-\xi)\,\theta\phi\hat{\phi}_t}{\xi} + \beta\mathbb{E}_t\hat{\pi}_{t+1} + \tau\hat{\pi}_{t-1} \longrightarrow$$
$$\longrightarrow \hat{\pi}_t = \frac{(1-\xi)\,\theta\phi\hat{\phi}_t}{(1+\beta\tau)\,\xi} + \frac{\beta\mathbb{E}_t\hat{\pi}_{t+1}}{(1+\beta\tau)} + \frac{\tau\hat{\pi}_{t-1}}{(1+\beta\tau)} \longrightarrow$$
$$\longrightarrow \hat{\pi}_t = \frac{(1-\xi)\,(1-\xi\beta)\,\phi\hat{\phi}_t}{(1+\beta\tau)\,\xi\phi} + \frac{\beta\mathbb{E}_t\hat{\pi}_{t+1}}{(1+\beta\tau)} + \frac{\tau\hat{\pi}_{t-1}}{(1+\beta\tau)}$$
$$\left[\text{in which } A = (1-\xi\beta)^{-1}\theta\phi \longrightarrow 1 = (1-\xi\beta)^{-1}\theta\phi \longrightarrow \theta = \phi^{-1}(1-\xi\beta)\right] \longrightarrow$$
$$\longrightarrow \hat{\pi}_t = \frac{(1-\xi)\,(1-\xi\beta)\,\hat{\phi}_t}{(1+\beta\tau)\,\xi} + \frac{\beta\mathbb{E}_t\hat{\pi}_{t+1}}{(1+\beta\tau)} + \frac{\tau\hat{\pi}_{t-1}}{(1+\beta\tau)}.$$

Price dispersion:

$$pd_t = \xi pd_{t-1}\left(\frac{\pi_{t-1}^{\tau}}{\pi_t}\right)^{\frac{\theta}{1-\theta}} + (1-\xi)\left(\frac{\pi_t^*}{\pi_t}\right)^{\frac{\theta}{1-\theta}}$$
$$pd = \xi pd\left(\frac{\pi^{\tau}}{\pi}\right)^{\frac{\theta}{1-\theta}} + (1-\xi)\left(\frac{\pi^*}{\pi}\right)^{\frac{\theta}{1-\theta}} \text{ (steady state);}$$
$$pd = \xi pd + (1-\xi) \text{ (steady state, in which } \pi = \pi^* = 1);$$
$$(1-\xi)\,pd = (1-\xi) \longrightarrow$$
$$\longrightarrow pd = 1 \text{ (steady state);}$$
$$lnpd = ln\left[\xi pd\left(\frac{\pi^{\tau}}{\pi}\right)^{\frac{\theta}{1-\theta}} + (1-\xi)\left(\frac{\pi^*}{\pi}\right)^{\frac{\theta}{1-\theta}}\right];$$
$$lnpd + \frac{\bar{pd}_t}{pd} = ln\left[\xi pd\left(\frac{\pi^{\tau}}{\pi}\right)^{\frac{\theta}{1-\theta}} + (1-\xi)\left(\frac{\pi^*}{\pi}\right)^{\frac{\theta}{1-\theta}}\right] + \frac{\xi\pi^{\frac{\tau\theta}{1-\theta}}\pi^{\frac{-\theta}{1-\theta}}\bar{pd}_{t-1}}{pd} + \frac{\xi\tau\theta\,(1-\theta)^{-1}\,pd\pi^{\frac{\tau\theta}{1-\theta}-1}\pi^{\frac{-\theta}{1-\theta}}\bar{\pi}_{t-1}}{\pi} +$$
$$+ \frac{-\xi\theta\,(1-\theta)^{-1}\,pd\pi^{\frac{\tau\theta}{1-\theta}}\pi^{\frac{-\theta}{1-\theta}-1}\bar{\pi}_t}{\pi} + \frac{(1-\xi)\,\theta\,(1-\theta)^{-1}\,(\pi^*)^{\frac{\theta}{1-\theta}-1}\,\pi^{\frac{-\theta}{1-\theta}}\bar{\pi}_t^*}{\pi} + \frac{-(1-\xi)\,\theta\,(1-\theta)^{-1}\,(\pi^*)^{\frac{\theta}{1-\theta}}\,\pi^{\frac{-\theta}{1-\theta}-1}\bar{\pi}_t}{\pi} \longrightarrow$$
$$\longrightarrow \hat{pd}_t = \xi\hat{pd}_{t-1} + \xi\tau\theta\,(1-\theta)^{-1}\,\hat{\pi}_{t-1} - \xi\theta\,(1-\theta)^{-1}\,\hat{\pi}_t + (1-\xi)\,\theta\,(1-\theta)^{-1}\,\hat{\pi}_t^* - (1-\xi)\,\theta\,(1-\theta)^{-1}\,\hat{\pi}_t$$
$$\text{(in which } pd = \pi = \pi^* = 1) \longrightarrow$$
$$\longrightarrow \hat{pd}_t = \xi\hat{pd}_{t-1} + \xi\tau\theta\,(1-\theta)^{-1}\,\hat{\pi}_{t-1} + (1-\xi)\,\theta\,(1-\theta)^{-1}\,\hat{\pi}_t^* - \theta\,(1-\theta)^{-1}\,\hat{\pi}_t \longrightarrow$$
$$\longrightarrow \hat{pd}_t = \xi\hat{pd}_{t-1} + \theta\,(1-\theta)^{-1}\left[\xi\tau\hat{\pi}_{t-1} + (1-\xi)\,\hat{\pi}_t^* - \hat{\pi}_t\right].$$

Since $(1-\xi)\,\hat{p}_t^* = \xi\left(\hat{\pi}_t - \tau\hat{\pi}_{t-1}\right) = \xi\hat{\pi}_t - \xi\tau\hat{\pi}_{t-1} \longrightarrow \xi\tau\hat{\pi}_{t-1} = \xi\hat{\pi}_t - (1-\xi)\,\hat{p}_t^*$

$$\hat{pd}_t = \xi\hat{pd}_{t-1} + \theta\,(1-\theta)^{-1}\left[\xi\hat{\pi}_t - (1-\xi)\,\hat{p}_t^* + (1-\xi)\,\hat{\pi}_t^* - \hat{\pi}_t\right] \longrightarrow$$
$$\longrightarrow \hat{pd}_t = \xi\hat{pd}_{t-1} + \theta\,(1-\theta)^{-1}\left[(\xi-1)\,\hat{\pi}_t - (1-\xi)\,\hat{p}_t^* + (1-\xi)\,\hat{\pi}_t^*\right].$$

Since $\hat{\pi}_t^* = \hat{P}_t^* - \hat{P}_{t-1}$ and $\hat{p}_t^* = \hat{P}_t^* - \hat{P}_t$

$$\hat{pd}_t = \xi \hat{pd}_{t-1} + \theta (1-\theta)^{-1} \left[(\xi - 1) \hat{\pi}_t - (1-\xi) \left(\hat{P}_t^* - \hat{P}_t \right) + (1-\xi) \left(\hat{P}_t^* - \hat{P}_{t-1} \right) \right] \longrightarrow$$
$$\longrightarrow \hat{pd}_t = \xi \hat{pd}_{t-1} + \theta (1-\theta)^{-1} \left[(\xi - 1) \hat{\pi}_t + (1-\xi) \left(\hat{P}_t - \hat{P}_{t-1} \right) \right].$$

Since $\hat{\pi}_t = \hat{P}_t - \hat{P}_{t-1}$

$$\hat{pd}_t = \xi \hat{pd}_{t-1} + \theta (1-\theta)^{-1} \left[(\xi - 1) \hat{\pi}_t + (1-\xi) \hat{\pi}_t \right] \longrightarrow$$
$$\longrightarrow \hat{pd}_t = \xi \hat{pd}_{t-1} + \theta (1-\theta)^{-1} (1-\xi) (\hat{\pi}_t - \hat{\pi}_t) \longrightarrow$$
$$\longrightarrow \hat{pd}_t = \xi \hat{pd}_{t-1}.$$

Since $\hat{pd}_{t-1} = 0$ (zero inflation steady state)

$$\hat{pd}_t = 0.$$

Real wage:

$$W_t = \Upsilon_t^{-1} (C_t - hC_{t-1})^{\sigma_c} l_t^{\sigma_l};$$
$$w_t pt_t = \Upsilon_t^{-1} (c_t pt_t - hc_{t-1} pt_{t-1})^{\sigma_c} l_t^{\sigma_l} \text{ (normalisation)};$$
$$wpt = \Upsilon^{-1} (cpt - hcpt)^{\sigma_c} l^{\sigma_l} \text{ (steady state)};$$
$$w = (c - hc)^{\sigma_c} l^{\sigma_l} \text{ (steady state, in which } \Upsilon = pt = 1);$$
$$lnw + lnpt = -ln\Upsilon + \sigma_c ln (cpt - hcpt) + \sigma_l lnl;$$
$$lnw + \frac{\bar{w}_t}{w} + lnpt + \frac{\bar{pt}_t}{pt} = -ln\Upsilon - \frac{\bar{\Upsilon}_t}{\Upsilon} + \sigma_c ln (cpt - hcpt) + \frac{\sigma_c pt \bar{c}_t}{(cpt - hcpt)} +$$
$$+ \frac{\sigma_c c\bar{pt}_t}{(cpt - hcpt)} - \frac{\sigma_c hpt \bar{c}_{t-1}}{(cpt - hcpt)} - \frac{\sigma_c hc\bar{pt}_{t-1}}{(cpt - hcpt)} + \sigma_l lnl + \frac{\sigma_l \bar{l}_t}{l} \longrightarrow$$
$$\longrightarrow \frac{\bar{w}_t}{w} + \frac{\bar{pt}_t}{pt} = -\frac{\bar{\Upsilon}_t}{\Upsilon} + \frac{\sigma_c pt \bar{c}_t}{cpt (1-h)} + \frac{\sigma_c c\bar{pt}_t}{cpt (1-h)} - \frac{\sigma_c hpt \bar{c}_{t-1}}{cpt (1-h)} - \frac{\sigma_c hc\bar{pt}_{t-1}}{cpt (1-h)} + \frac{\sigma_l \bar{l}_t}{l} \longrightarrow$$
$$\longrightarrow \hat{w}_t + \hat{pt}_t = -\hat{\Upsilon}_t + \frac{\sigma_c \hat{c}_t}{(1-h)} + \frac{\sigma_c \hat{pt}_t}{(1-h)} - \frac{\sigma_c h \hat{c}_{t-1}}{(1-h)} - \frac{\sigma_c h \hat{pt}_{t-1}}{(1-h)} + \sigma_l \hat{l}_t \longrightarrow$$
$$\longrightarrow \hat{w}_t = \sigma_l \hat{l}_t + \frac{\sigma_c \left(\hat{c}_t + \hat{pt}_t \right)}{(1-h)} - \frac{\sigma_c h \left(\hat{c}_{t-1} - \hat{pt}_{t-1} \right)}{(1-h)} - \hat{\Upsilon}_t - \hat{pt}_t.$$

Aggregate labour:

$$l_t = \alpha^{-1} (1-\alpha) W_t^{-1} rk_t u_t k_{t-1};$$
$$l_t = \alpha^{-1} (1-\alpha) (w_t pt_t)^{-1} rk_t u_t k_{t-1} pt_{t-1} \text{ (normalisation)};$$
$$l = \alpha^{-1} (1-\alpha) (wpt)^{-1} rkukpt \text{ (steady state)};$$
$$l = \alpha^{-1} (1-\alpha) (w)^{-1} rkuk \text{ (steady state, in which } pt = 1);$$
$$lnl = -ln\alpha + ln (1-\alpha) - lnw + lnrk + lnu + lnk + lnpt - lnpt;$$
$$lnl + \frac{\bar{l}_t}{l} = -ln\alpha + ln (1-\alpha) - lnw - \frac{\bar{w}_t}{w} + lnrk + \frac{\bar{rk}_t}{rk} + lnu + \frac{\bar{u}_t}{u} + lnk + \frac{\bar{k}_{t-1}}{k} + lnpt + \frac{\bar{pt}_{t-1}}{pt} - lnpt - \frac{\bar{pt}_t}{pt} \longrightarrow$$

$$\longrightarrow \frac{\bar{l}_t}{l} = -\frac{\bar{w}_t}{w} + \frac{\bar{rk}_t}{rk} + \frac{\bar{u}_t}{u} + \frac{\bar{k}_{t-1}}{k} + \frac{\bar{pt}_{t-1}}{pt} - \frac{\bar{pt}_t}{pt} \longrightarrow$$

$$\longrightarrow \hat{l}_t = -\hat{w}_t + \hat{rk}_t + \hat{u}_t + \hat{k}_{t-1} - \hat{pt}_{t-1} - \hat{pt}_t \longrightarrow$$

$$\longrightarrow \hat{l}_t = \hat{rk}_t + \omega\hat{rk}_t + \hat{k}_{t-1} - \hat{pt}_{t-1} - \hat{w}_t - \hat{pt}_t \longrightarrow$$

$$\longrightarrow \hat{l}_t = (1+\omega)\,\hat{rk}_t + \hat{k}_{t-1} - \hat{pt}_{t-1} - \hat{w}_t - \hat{pt}_t.$$

Real consumption:

$$\Upsilon_t\left(C_t - hC_{t-1}\right)^{-\sigma_c} = \mathbb{E}_t\beta\left[\frac{\Upsilon_{t+1}\left(C_{t+1} - hC_t\right)^{-\sigma_c} rn_t}{\pi_{t+1}}\right];$$

$$\Upsilon_t\left(c_t pt_t - hc_{t-1}pt_{t-1}\right)^{-\sigma_c} = \mathbb{E}_t\beta\left[\frac{\Upsilon_{t+1}\left(c_{t+1}pt_{t+1} - hc_t pt_t\right)^{-\sigma_c} rn_t}{\pi_{t+1}}\right] \text{ (normalisation)};$$

$$\Upsilon\left(cpt - hcpt\right)^{-\sigma_c} = \beta\left[\frac{\Upsilon\left(cpt - hcpt\right)^{-\sigma_c} rn}{\pi}\right] \text{ (steady state)};$$

$$(c - hc)^{-\sigma_c} = \beta\,(c - hc)^{-\sigma_c}\, rn \text{ (steady state, in which } \Upsilon = pt = \pi = 1);$$

$$1 = \beta rn \longrightarrow$$

$$\longrightarrow rn = \beta^{-1} \text{ (steady state)};$$

$$ln\Upsilon - \sigma_c ln\left(cpt - hcpt\right) = ln\beta + ln\Upsilon - \sigma_c ln\left(cpt - hcpt\right) + lnrn - ln\pi;$$

$$ln\Upsilon + \frac{\bar{\Upsilon}_t}{\Upsilon} - \sigma_c ln\left(cpt - hcpt\right) - \frac{\sigma_c pt\bar{c}_t}{(cpt - hcpt)} - \frac{\sigma_c c\bar{pt}_t}{(cpt - hcpt)} + \frac{\sigma_c hpt\bar{c}_{t-1}}{(cpt - hcpt)} + \frac{\sigma_c hc\bar{pt}_{t-1}}{(cpt - hcpt)} = ln\beta +$$

$$+ ln\Upsilon + \frac{\mathbb{E}_t\bar{\Upsilon}_{t+1}}{\Upsilon} - \sigma_c ln\left(cpt - hcpt\right) - \frac{\mathbb{E}_t\sigma_c pt\bar{c}_{t+1}}{(cpt - hcpt)} - \frac{\mathbb{E}_t\sigma_c c\bar{pt}_{t+1}}{(cpt - hcpt)} +$$

$$+ \frac{\sigma_c hpt\bar{c}_t}{(cpt - hcpt)} + \frac{\sigma_c hc\bar{pt}_t}{(cpt - hcpt)} + lnrn + \frac{\bar{rn}_t}{rn} - ln\pi - \frac{\mathbb{E}_t\bar{\pi}_{t+1}}{\pi} \longrightarrow$$

$$\longrightarrow \frac{\bar{\Upsilon}_t}{\Upsilon} - \frac{\sigma_c pt\bar{c}_t}{cpt\,(1-h)} - \frac{\sigma_c c\bar{pt}_t}{cpt\,(1-h)} + \frac{\sigma_c hpt\bar{c}_{t-1}}{cpt\,(1-h)} + \frac{\sigma_c hc\bar{pt}_{t-1}}{cpt\,(1-h)} =$$

$$= \frac{\mathbb{E}_t\bar{\Upsilon}_{t+1}}{\Upsilon} - \frac{\mathbb{E}_t\sigma_c pt\bar{c}_{t+1}}{cpt\,(1-h)} - \frac{\mathbb{E}_t\sigma_c c\bar{pt}_{t+1}}{cpt\,(1-h)} + \frac{\sigma_c hpt\bar{c}_t}{cpt\,(1-h)} + \frac{\sigma_c hc\bar{pt}_t}{cpt\,(1-h)} + \frac{\bar{rn}_t}{rn} - \frac{\mathbb{E}_t\bar{\pi}_{t+1}}{\pi} \longrightarrow$$

$$\longrightarrow \hat{\Upsilon}_t - \frac{\sigma_c\hat{c}_t}{(1-h)} - \frac{\sigma_c\hat{pt}_t}{(1-h)} + \frac{\sigma_c h\hat{c}_{t-1}}{(1-h)} + \frac{\sigma_c h\hat{pt}_{t-1}}{(1-h)} =$$

$$= \mathbb{E}_t\hat{\Upsilon}_{t+1} - \frac{\mathbb{E}_t\sigma_c\hat{c}_{t+1}}{(1-h)} - \frac{\mathbb{E}_t\sigma_c\hat{pt}_{t+1}}{(1-h)} + \frac{\sigma_c h\hat{c}_t}{(1-h)} + \frac{\sigma_c h\hat{pt}_t}{(1-h)} + \hat{rn}_t - \mathbb{E}_t\hat{\pi}_{t+1} \longrightarrow$$

$$\longrightarrow -\frac{\sigma_c\left(\hat{c}_t + \hat{pt}_t\right)}{(1-h)} + \frac{\sigma_c h\left(\hat{c}_{t-1} + \hat{pt}_{t-1}\right)}{(1-h)} = -\frac{\mathbb{E}_t\sigma_c\left(\hat{c}_{t+1} + \hat{pt}_{t+1}\right)}{(1-h)} + \frac{\sigma_c h\left(\hat{c}_t + \hat{pt}_t\right)}{(1-h)} + \hat{rn}_t - \mathbb{E}_t\hat{\pi}_{t+1} - \hat{\Upsilon}_t$$

(in which $\mathbb{E}_t\hat{\Upsilon}_{t+1} = 0$, but $\mathbb{E}_t\hat{pt}_{t+1} \neq 0$) $\longrightarrow$

$$\longrightarrow -\left(\hat{c}_t + \hat{pt}_t\right) + h\left(\hat{c}_{t-1} + \hat{pt}_{t-1}\right) = -\mathbb{E}_t\hat{c}_{t+1} - \mathbb{E}_t\hat{pt}_{t+1} + h\left(\hat{c}_t + \hat{pt}_t\right) + \frac{(1-h)\left(\hat{rn}_t - \mathbb{E}_t\hat{\pi}_{t+1} - \hat{\Upsilon}_t\right)}{\sigma_c} \longrightarrow$$

$$\longrightarrow -(1+h)\left(\hat{c}_t + \hat{pt}_t\right) = \frac{(1-h)\left(\hat{rn}_t - \mathbb{E}_t\hat{\pi}_{t+1} - \hat{\Upsilon}_t\right)}{\sigma_c} - h\left(\hat{c}_{t-1} + \hat{pt}_{t-1}\right) - \left(\mathbb{E}_t\hat{c}_{t+1} + \mathbb{E}_t\hat{pt}_{t+1}\right) \longrightarrow$$

$$\longrightarrow \hat{c}_t + \hat{pt}_t = \frac{(1-h)\left(\hat{\Upsilon}_t + \mathbb{E}_t\hat{\pi}_{t+1} - \hat{rn}_t\right)}{\sigma_c(1+h)} + \frac{h\left(\hat{c}_{t-1} + \hat{pt}_{t-1}\right)}{(1+h)} + \frac{\mathbb{E}_t\hat{c}_{t+1} + \mathbb{E}_t\hat{pt}_{t+1}}{(1+h)} \longrightarrow$$

$$\longrightarrow \hat{c}_t = \frac{(1-h)\left(\hat{\Upsilon}_t + \mathbb{E}_t\hat{\pi}_{t+1} - \hat{rn}_t\right)}{\sigma_c(1+h)} + \frac{h\left(\hat{c}_{t-1} + \hat{pt}_{t-1}\right)}{(1+h)} + \frac{\mathbb{E}_t\hat{c}_{t+1} + \mathbb{E}_t\hat{pt}_{t+1}}{(1+h)} - \hat{pt}_t.$$

As mentioned above, one notices the following: $\mathbb{E}_t\hat{\Upsilon}_{t+1} = 0$ and $\mathbb{E}_t\hat{pt}_{t+1} \neq 0$ imply $\mathbb{E}_t\hat{x}_{t+1} \neq 0$, whereby, $\forall\gamma \in (0,\ 1] \subset \mathbb{R}_{++}$, $\mathbb{E}_t\gamma\hat{t}_{t+1} + \mathbb{E}_t\gamma\hat{n}_{t+1} = \gamma\left(\mathbb{E}_t\hat{t}_{t+1} + \mathbb{E}_t\hat{n}_{t+1}\right) = -\mathbb{E}_t\gamma\hat{pt}_{t+1} = -\gamma\mathbb{E}_t\hat{pt}_{t+1}$, since $\mathbb{E}_t\hat{\Upsilon}_{t+1} = \mathbb{E}_t\left[\gamma\left(\hat{pt}_{t+1} + \hat{t}_{t+1} + \hat{n}_{t+1}\right)\right] = \mathbb{E}_t\gamma\hat{pt}_{t+1} + \mathbb{E}_t\gamma\hat{t}_{t+1} + \mathbb{E}_t\gamma\hat{n}_{t+1} = \gamma\left(\mathbb{E}_t\hat{pt}_{t+1} + \mathbb{E}_t\hat{t}_{t+1} + \mathbb{E}_t\hat{n}_{t+1}\right).$

Nominal interest rate:

$$\left(\frac{rn_t}{rn}\right) = \left(\frac{rn_{t-1}}{rn}\right)^{\rho_{rn}} \left[\left(\frac{\pi_t/\pi}{\pi_T/\pi}\right)^{\phi_\pi} \left(\frac{\pi_t/\pi}{\pi_{t-1}/\pi}\right)^{\phi_{\pi_g}} \left(\frac{Y_t}{Y}\right)^{\phi_y} \left(\frac{Y_t/Y}{Y_{t-1}/Y}\right)^{\phi_{y_g}}\right]^{1-\rho_{rn}} e^{\varphi};$$

$$\left(\frac{rn_t}{rn}\right) = \left(\frac{rn_{t-1}}{rn}\right)^{\rho_{rn}} \left[\left(\frac{\pi_t/\pi}{\pi_T/\pi}\right)^{\phi_\pi} \left(\frac{\pi_t/\pi}{\pi_{t-1}/\pi}\right)^{\phi_{\pi_g}} \left(\frac{y_t pt_t}{ypt}\right)^{\phi_y} \left(\frac{y_t pt_t/ypt}{y_{t-1}pt_{t-1}/ypt}\right)^{\phi_{y_g}}\right]^{1-\rho_{rn}} e^{\varphi} \text{ (normalisation)};$$

$$\left(\frac{rn}{rn}\right) = \left(\frac{rn}{rn}\right)^{\rho_{rn}} \left[\left(\frac{\pi/\pi}{\pi/\pi}\right)^{\phi_\pi} \left(\frac{\pi/\pi}{\pi/\pi}\right)^{\phi_{\pi_g}} \left(\frac{ypt}{ypt}\right)^{\phi_y} \left(\frac{ypt/ypt}{ypt/ypt}\right)^{\phi_{y_g}}\right]^{1-\rho_{rn}} e^{\varphi} \text{ (steady state)};$$

$$1 = e^{\varphi} \longrightarrow$$

$$\longrightarrow 0 = \varphi \text{ (steady state)};$$

$$lnrn - lnrn = \rho_{rn}(lnrn - lnrn) + (1-\rho_{rn})\left[\phi_\pi(ln\pi - ln\pi + ln\pi - ln\pi) + \phi_{\pi_g}(ln\pi - ln\pi + ln\pi - ln\pi) + \right.$$
$$\left. + \phi_y(lny + lnpt - lny - lnpt) + \phi_{y_g}(lny + lnpt - lny - lnpt + lny + lnpt - lny - lnpt)\right] + \varphi \longrightarrow$$

$$\longrightarrow lnrn + \frac{\bar{rn}_t}{rn} - lnrn = \rho_{rn}(lnrn - lnrn) + \frac{\rho_{rn}\bar{rn}_{t-1}}{rn} +$$
$$+ (1-\rho_{rn})\left[\phi_\pi(ln\pi - ln\pi + ln\pi - ln\pi) + \frac{\phi_\pi\bar{\pi}_t}{\pi} + \phi_{\pi_g}(ln\pi - ln\pi + ln\pi - ln\pi) + \frac{\phi_{\pi_g}\bar{\pi}_t}{\pi} - \frac{\phi_{\pi_g}\bar{\pi}_{t-1}}{\pi} + \right.$$
$$+ \phi_y(lny + lnpt - lny - lnpt) + \frac{\phi_y\bar{y}_t}{y} + \frac{\phi_y\bar{pt}_t}{pt} + \phi_{y_g}(lny + lnpt - lny - lnpt + lny + lnpt - lny - lnpt) +$$
$$\left. + \frac{\phi_{y_g}\bar{y}_t}{y} + \frac{\phi_{y_g}\bar{pt}_t}{pt} - \frac{\phi_{y_g}\bar{y}_{t-1}}{y} - + \frac{\phi_{y_g}\bar{pt}_{t-1}}{pt}\right] + \varphi \longrightarrow$$

$$\longrightarrow \hat{rn}_t = \rho_{rn}\hat{rn}_{t-1} + (1-\rho_{rn})\left[\phi_\pi\hat{\pi}_t + \phi_{\pi_g}(\hat{\pi}_t - \hat{\pi}_{t-1}) + \phi_y\left(\hat{y}_t + \hat{pt}_t\right) + \phi_{y_g}\left(\hat{y}_t + \hat{pt}_t - \hat{y}_{t-1} - \hat{pt}_{t-1}\right)\right].$$

Aggregate real production:

$$Y_t = pd_t^{-1} a_t \tilde{K}_{t-1}^{\alpha} (\Upsilon_t l_t)^{1-\alpha} \longrightarrow$$
$$\longrightarrow Y_t = pd_t^{-1} a_t (u_t k_{t-1})^{\alpha} (\Upsilon_t l_t)^{1-\alpha};$$
$$y_t pt_t = pd_t^{-1} a_t (u_t k_{t-1} pt_{t-1})^{\alpha} (\Upsilon_t l_t)^{1-\alpha} \text{ (normalisation)};$$
$$ypt = pd^{-1} a (ukpt)^{\alpha} (\Upsilon l)^{1-\alpha} \text{ (steady state)};$$
$$y = a(uk)^{\alpha} l^{1-\alpha} \text{ (steady state, in which } \Upsilon = pt = pd = 1);$$
$$lny + lnpt = -lnpd + lna + \alpha(lnu + lnk + lnpt) + (1-\alpha)(ln\Upsilon + lnl);$$
$$lny + \frac{\bar{y}_t}{y} + lnpt + \frac{\bar{pt}_t}{pt} = -lnpd - \frac{\bar{pd}_t}{pd} + lna + \frac{\bar{a}_t}{a} +$$

$$+\alpha\left(lnu+lnk+lnpt\right)+\frac{\alpha\bar{u}_t}{u}+\frac{\alpha\bar{k}_{t-1}}{k}+\frac{\alpha\bar{pt}_{t-1}}{pt}+\left(1-\alpha\right)\left(ln\Upsilon+lnl\right)+\frac{\left(1-\alpha\right)\bar{\Upsilon}_t}{\Upsilon}+\frac{\left(1-\alpha\right)\bar{l}_t}{l}\longrightarrow$$
$$\longrightarrow\hat{y}_t+\hat{pt}_t=-\hat{pd}_t+\hat{a}_t+\alpha\left(\hat{u}_t+\hat{k}_{t-1}+\hat{pt}_{t-1}\right)+\left(1-\alpha\right)\left(\hat{\Upsilon}_t+\hat{l}_t\right)\longrightarrow$$
$$\longrightarrow\hat{y}_t=\hat{a}_t+\alpha\left(\hat{u}_t+\hat{k}_{t-1}+\hat{pt}_{t-1}\right)+\left(1-\alpha\right)\left(\hat{\Upsilon}_t+\hat{l}_t\right)-\hat{pd}_t-\hat{pt}_t\longrightarrow$$
$$\longrightarrow\hat{y}_t=\hat{a}_t+\alpha\omega\hat{rk}_t+\alpha\left(\hat{k}_{t-1}+\hat{pt}_{t-1}\right)+\left(1-\alpha\right)\left(\hat{\Upsilon}_t+\hat{l}_t\right)-\hat{pt}_t\text{ (in which }\hat{pd}_t=0).$$

Confidence:

$$\Upsilon_t=\left(pt_tt_tn_t\right)^{\gamma};$$
$$\Upsilon=\left(pttn\right)^{\gamma}\text{ (steady state);}$$
$$1=1\text{ (steady state, in which }pt=t=n=1);$$
$$ln\Upsilon=\gamma\left(lnpt+lnt+lnn\right);$$
$$ln\Upsilon+\frac{\bar{\Upsilon}_t}{\Upsilon}=\gamma\left(lnpt+lnt+lnn\right)+\frac{\gamma\bar{pt}_t}{pt}+\frac{\gamma\bar{t}_t}{t}+\frac{\gamma\bar{n}_t}{n}\longrightarrow$$
$$\longrightarrow\hat{\Upsilon}_t=\gamma\left(\hat{pt}_t+\hat{t}_t+\hat{n}_t\right).$$

Real marginal cost:

$$\phi_t=\alpha^{-\alpha}\left(1-\alpha\right)^{\alpha-1}a_t^{-1}\Upsilon_t^{\alpha-1}rk_t^{\alpha}W_t^{1-\alpha};$$
$$\phi_t=\alpha^{-\alpha}\left(1-\alpha\right)^{\alpha-1}a_t^{-1}\Upsilon_t^{\alpha-1}rk_t^{\alpha}\left(w_tpt_t\right)^{1-\alpha}\text{ (normalisation);}$$
$$\phi=\alpha^{-\alpha}\left(1-\alpha\right)^{\alpha-1}a^{-1}\Upsilon^{\alpha-1}rk^{\alpha}\left(wpt\right)^{1-\alpha}\text{ (steady state);}$$
$$\phi=\alpha^{-\alpha}\left(1-\alpha\right)^{\alpha-1}rk^{\alpha}w^{1-\alpha}\text{ (steady state, in which }\Upsilon=pt=a=1);$$
$$ln\phi=-\alpha ln\alpha+\left(\alpha-1\right)ln\left(1-\alpha\right)-lna+\left(\alpha-1\right)ln\Upsilon+\alpha lnrk+\left(1-\alpha\right)\left(lnw+lnpt\right);$$
$$ln\phi+\frac{\bar{\phi}_t}{\phi}=-\alpha ln\alpha+\left(\alpha-1\right)ln\left(1-\alpha\right)-lna-\frac{\bar{a}_t}{a}+\left(\alpha-1\right)ln\Upsilon+$$
$$+\frac{\left(\alpha-1\right)\bar{\Upsilon}_t}{\Upsilon}+\alpha lnrk+\frac{\alpha\bar{rk}_t}{rk}+\left(1-\alpha\right)\left(lnw+lnpt\right)+\frac{\left(1-\alpha\right)\bar{w}_t}{w}+\frac{\left(1-\alpha\right)\bar{pt}_t}{pt}\longrightarrow$$
$$\longrightarrow\hat{\phi}_t=-\hat{a}_t+\left(\alpha-1\right)\hat{\Upsilon}_t+\alpha\hat{rk}_t+\left(1-\alpha\right)\hat{w}_t+\left(1-\alpha\right)\hat{pt}_t\longrightarrow$$
$$\longrightarrow\hat{\phi}_t=\alpha\hat{rk}_t+\left(1-\alpha\right)\left(\hat{w}_t+\hat{pt}_t-\hat{\Upsilon}_t\right)-\hat{a}_t.$$

Aggregate capital:

$$K_t=\left(1-\delta\right)K_{t-1}+i;$$
$$k_tpt_t=\left(1-\delta\right)k_{t-1}pt_{t-1}+i\text{ (normalisation);}$$
$$kpt=\left(1-\delta\right)kpt+i\text{ (steady state);}$$
$$\delta k=i\text{ (steady state, in which }pt=1)\longrightarrow$$
$$\longrightarrow k=\delta^{-1}i\text{ (steady state);}$$
$$lnk+lnpt=ln\left[\left(1-\delta\right)kpt+i\right];$$
$$lnk+\frac{\bar{k}_t}{k}+lnpt+\frac{\bar{pt}_t}{pt}=ln\left[\left(1-\delta\right)kpt+i\right]+\frac{\left(1-\delta\right)pt\bar{k}_{t-1}}{\left(1-\delta\right)kpt+i}+\frac{\left(1-\delta\right)k\bar{pt}_{t-1}}{\left(1-\delta\right)kpt+i}\longrightarrow$$

$$\longrightarrow \frac{\bar{k}_t}{k} + \frac{\bar{pt}_t}{pt} = \frac{(1-\delta)\, pt\bar{k}_{t-1}}{kpt} + \frac{(1-\delta)\, k\bar{pt}_{t-1}}{kpt} \longrightarrow$$
$$\longrightarrow \hat{k}_t + \hat{pt}_t = (1-\delta)\left(\hat{k}_{t-1} + \hat{pt}_{t-1}\right) \longrightarrow$$
$$\longrightarrow \hat{k}_t = (1-\delta)\left(\hat{k}_{t-1} + \hat{pt}_{t-1}\right) - \hat{pt}_t.$$

Aggregate capital utilisation:

$Y_t = C_t + g + \Psi(u_t) K_{t-1}$;

$y_t pt_t = c_t pt_t + g + \Psi(u_t) k_{t-1} pt_{t-1}$ (normalisation);

$ypt = cpt + g + \Psi(u)\, kpt$ (steady state);

$y = c + g + \Psi(1)\, k$ (steady state, in which $pt = 1$);

$y = c + g$ [steady state, in which $\Psi(1) = 0$];

$lny + lnpt = ln\left[cpt + g + \Psi(u)\, kpt\right]$;

$$lny + \frac{\bar{y}_t}{y} + lnpt + \frac{\bar{pt}_t}{pt} = ln\left[cpt + g + \Psi(u)\, kpt\right] + \frac{pt\bar{c}_t}{cpt + g + \Psi(u)\, kpt} + \frac{c\bar{pt}_t}{cpt + g + \Psi(u)\, kpt} +$$
$$+ \frac{kpt\Psi'(u)(1)\,\bar{u}_t}{cpt + g + \Psi(u)\, kpt} + \frac{\Psi(u)\, pt\bar{k}_{t-1}}{cpt + g + \Psi(u)\, kpt} + \frac{\Psi(u)\, k\bar{pt}_{t-1}}{cpt + g + \Psi(u)\, kpt} \longrightarrow$$
$$\longrightarrow \frac{\bar{y}_t}{y} + \frac{\bar{pt}_t}{pt} = \frac{\bar{c}_t}{ypt} + \frac{c\bar{pt}_t}{ypt} + \frac{k\Psi'(1)\,\bar{u}_t}{ypt} + \frac{\Psi(1)\,\bar{k}_{t-1}}{ypt} + \frac{\Psi(1)\, k\bar{pt}_{t-1}}{ypt} \longrightarrow$$
$$\longrightarrow \hat{y}_t + \hat{pt}_t = \frac{c\bar{c}_t}{yc} + \frac{c\hat{pt}_t}{y} + \frac{krku\bar{u}_t}{yu} + \frac{\Psi(1)\, k\bar{k}_{t-1}}{yk} + \frac{\Psi(1)\, k\hat{pt}_{t-1}}{y} \longrightarrow$$
$$\longrightarrow \hat{y}_t + \hat{pt}_t = \left(\frac{c}{y}\right)\left(\hat{c}_t + \hat{pt}_t\right) + \left(\frac{k}{y}\right)\left[rk\hat{u}_t + \Psi(1)\left(\hat{k}_{t-1} + \hat{pt}_{t-1}\right)\right] \longrightarrow$$
$$\longrightarrow \hat{y}_t = \left(\frac{c}{y}\right)\left(\hat{c}_t + \hat{pt}_t\right) + \left(\frac{k}{y}\right) rk\omega\hat{rk}_t - \hat{pt}_t.$$

One notices that parameters $rk,\ y^{-1}c$ and $y^{-1}k$ respectively model steady state capital return, consumption to output ratio and capital to output ratio, being positive real numbers: $rk,\ y^{-1}c,\ y^{-1}k \in \mathbb{R}_{++}$.

12. Parametrisation and solution

12.1 Calibration. The log-linearised laws of motion of the economy are consequently the following:

$\hat{pt}_t = \hat{pt}_{t-1} + \varepsilon_{ptt}$ (permanent technology);

$\hat{n}_t = \rho_n \hat{n}_{t-1} + \varepsilon_{nt}$ (noise technology);

$\hat{t}_t = \rho_t \hat{t}_{t-1} + \varepsilon_{tt}$ (transitory technology);

$\hat{a}_t = \rho_a \hat{a}_{t-1} + \hat{pt}_{t-1} + \hat{t}_{t-1}$ (real production technology);

$$\hat{\pi}_t = \frac{(1-\xi)(1-\xi\beta)\,\hat{\phi}_t}{(1+\beta\tau)\,\xi} + \frac{\beta\mathbb{E}_t\hat{\pi}_{t+1}}{(1+\beta\tau)} + \frac{\tau\hat{\pi}_{t-1}}{(1+\beta\tau)} \text{ (inflation);}$$

$$\hat{w}_t = \sigma_l \hat{l}_t + \frac{\sigma_c\left(\hat{c}_t + \hat{pt}_t\right)}{(1-h)} - \frac{\sigma_c h\left(\hat{c}_{t-1} - \hat{pt}_{t-1}\right)}{(1-h)} - \hat{\Upsilon}_t - \hat{pt}_t \text{ (real wage);}$$

$\hat{l}_t = (1+\omega)\,\hat{rk}_t + \hat{k}_{t-1} - \hat{pt}_{t-1} - \hat{w}_t - \hat{pt}_t$ (aggregate labour);

$$\hat{c}_t = \frac{(1-h)\left(\hat{\Upsilon}_t + \mathbb{E}_t\hat{\pi}_{t+1} - \hat{rn}_t\right)}{\sigma_c(1+h)} + \frac{h\left(\hat{c}_{t-1} + \hat{pt}_{t-1}\right)}{(1+h)} + \frac{\mathbb{E}_t\hat{c}_{t+1} + \mathbb{E}_t\hat{pt}_{t+1}}{(1+h)} - \hat{pt}_t \text{ (real consumption);}$$

$$\hat{rn}_t = \rho_{rn}\hat{rn}_{t-1} + (1-\rho_{rn})\left[\phi_\pi\hat{\pi}_t + \phi_{\pi_g}(\hat{\pi}_t - \hat{\pi}_{t-1}) + \right.$$
$$\left. +\phi_y\left(\hat{y}_t + \hat{pt}_t\right) + \phi_{y_g}\left(\hat{y}_t + \hat{pt}_t - \hat{y}_{t-1} - \hat{pt}_{t-1}\right)\right] \text{ (nominal interest rate);}$$

$$\hat{y}_t = \hat{a}_t + \alpha\omega\hat{rk}_t + \alpha\left(\hat{k}_{t-1} + \hat{pt}_{t-1}\right) + (1-\alpha)\left(\hat{\Upsilon}_t + \hat{l}_t\right) - \hat{pt}_t \text{ (aggregate real production);}$$

$$\hat{\Upsilon}_t = \gamma\left(\hat{pt}_t + \hat{t}_t + \hat{n}_t\right) \text{ (confidence);}$$

$$\hat{\phi}_t = \alpha\hat{rk}_t + (1-\alpha)\left(\hat{w}_t + \hat{pt}_t - \hat{\Upsilon}_t\right) - \hat{a}_t \text{ (real marginal cost);}$$

$$\hat{k}_t = (1-\delta)\left(\hat{k}_{t-1} + \hat{pt}_{t-1}\right) - \hat{pt}_t \text{ (aggregate capital);}$$

$$\hat{y}_t = \left(\frac{c}{y}\right)\left(\hat{c}_t + \hat{pt}_t\right) + \left(\frac{k}{y}\right) rk\omega\hat{rk}_t - \hat{pt}_t \text{ (aggregate capital utilisation).}$$

Endogenous variables, exogenous shocks and parameters can be thys collected. Endogenous variables: $\{\hat{\Upsilon}_t,\ \hat{c}_t,\ \hat{l}_t,\ \hat{k}_t,\ \hat{y}_t,\ \hat{w}_t,\ \hat{rk}_t,\ \hat{\pi}_t,\ \hat{\phi}_t,\ \hat{pt}_t,\ \hat{n}_t,\ \hat{t}_t,\ \hat{a}_t,\ \hat{rn}_t\}_{t=0}^{\infty}$.

Exogenous shocks: $\{\varepsilon_{ptt},\ \varepsilon_{nt},\ \varepsilon_{tt}\}_{t=0}^{\infty}$. Parameters: $\Theta = \{\beta,\ h,\ \sigma_c,\ \sigma_l,\ \delta,\ \xi,\ \tau,\ \alpha,\ \gamma,\ \rho_n,\ \rho_t,\ \rho_a,\ \rho_{rn},\ \phi_\pi,\ \phi_{\pi_g},\ \phi_y,\ \phi_{y_g},\ \omega,\ rk,\ y^{-1}c,\ y^{-1}k\} \in \mathbb{R}_{++}$.

Table 3: Calibration

Parameter	USA	EA	Name
β	0.99	0.99	Discount factor
h	0.69	0.573	Consumption habit
σ_c	1.62	1.353	Inter-temporal substitution inverse elasticity
σ_l	2.45	2.4	Labour inverse elasticity
δ	0.025	0.025	Capital depreciation rate
ξ	0.87	0.908	Price adjustment failure fraction
τ	0.66	0.469	Inflation indexation
α	0.24	0.3	Capital in output share
γ_i	$i = H,\ M,\ L$	$i = H,\ M,\ L$	Volition regime
ρ_n	0.65	0.65	Noise technology shock persistence
ρ_t	0.95	0.95	Transitory technology shock persistence
ρ_a	0.822	0.823	Production technology shock persistence
ρ_{rn}	0.88	0.961	Interest rate persistence
ϕ_π	1.48	1.684	Inflation coefficient
ϕ_{π_g}	0.24	0.14	Inflation gap coefficient
ϕ_y	0.08	0.099	Output coefficient
ϕ_{y_g}	0.24	0.159	Output gap coefficient
ω	3.23	5.917	Capital utilisation adjustment cost inverse elasticity
rk	0.0351	0.0351	Steady state capital return
$y^{-1}c$	0.65	0.6	Consumption to output ratio
$y^{-1}k$	6.8	6.8	Capital to output ratio

Note. Calibration of parameters for the USA and the EA, in which volition regimes γ_i are calibrated as outlined in Table 2.

Such laws of motion can be cast into a linear rational expectations (LRE) model:

$$Q(\Theta)\, x_t = R(\Theta)\, x_{t-1} + S\varepsilon_t,$$

in which endogenous variables $x_t \in \mathbb{R}^{n_x}$, exogenous shocks $\varepsilon_t \in \mathbb{R}^{n_\varepsilon}$, companion matrices $Q(\Theta),\ R(\Theta) \in \mathbb{R}^{n_x \times n_x}$ and exogenous shock matrix $S \in \mathbb{R}^{n_x \times n_\varepsilon}$, being composed of zeros and ones.

In the spirit of the Lucas critique[8], whereby space-time independence is necessary for policy robustness, parametrisation follows calibration over maximum likelihood[9] or Bayesian estimation[10] of parameters and is according to the aforementioned parameter specifics and common sense at large; its exploitation of Smets and Wouters [20]'s Bayesian estimation is thus only auxiliary and subordinated to the aforementioned parameter specifics and common sense at large, as formalised by the pertinent economic literature.

Bayesian estimation of parameters also noteworthily conflicts with log-linearisation of laws of motion, for its idiosyncratic spirit would require a correspondence in non-linear laws of motion, instead lost before. An ulterior reason for which calibration is preferred to maximum likelihood or Bayesian estimation of parameters concerns the desire to merely replicate the empirical SIRF patterns in question all else remaining equal. Calibration, in the regards of the USA and the EA, is reported in Table 3.

12.2 Unique and stable solution. As per Blanchard and Kahn [8], the LRE model in question evolves as follows:

$$Qx_t = Rx_{t-1} + S\varepsilon_t \longleftrightarrow$$
$$\longleftrightarrow \begin{bmatrix} \underset{(n_{x_1}\times n_{x_1})}{Q_{11}} & \underset{(n_{x_1}\times n_{x_2})}{Q_{12}} \\ \underset{(n_{x_2}\times n_{x_1})}{Q_{21}} & \underset{(n_{x_2}\times n_{n_2})}{Q_{22}} \end{bmatrix} \begin{bmatrix} \underset{(n_{x_1}\times 1)}{x_{1t}} \\ \underset{(n_{x_2}\times 1)}{\mathbb{E}_t x_{2t+1}} \end{bmatrix} = \begin{bmatrix} \underset{(n_{x_1}\times n_{x_1})}{R_{11}} & \underset{(n_{x_1}\times n_{x_2})}{R_{12}} \\ \underset{(n_{x_2}\times n_{x_1})}{R_{21}} & \underset{(n_{x_2}\times n_{n_2})}{R_{22}} \end{bmatrix} \begin{bmatrix} \underset{(n_{x_1}\times 1)}{x_{1t-1}} \\ \underset{(n_{x_2}\times 1)}{\mathbb{E}_{t-1} x_{2t}} \end{bmatrix} + \begin{bmatrix} \underset{(n_{x_1}\times n_{\varepsilon})}{S_1} \\ \underset{(n_{x_2}\times n_{\varepsilon})}{S_2} \end{bmatrix} \varepsilon_t,$$

in which non-expectational or past endogenous variables $x_{1t} = \left[\hat{\Upsilon}_t\ \hat{c}_t\ \hat{l}_t\ \hat{k}_t\ \hat{y}_t\ \hat{w}_t\ \hat{rk}_t\ \hat{\pi}_t\ \hat{\phi}_t\ \hat{pt}_t\ \hat{n}_t\ \hat{t}_t\ \hat{a}_t\ \hat{rn}_t\right]^\top$, expectational or future endogenous variables $\mathbb{E}_t x_{2t+1} = \left[\mathbb{E}_t\hat{c}_{t+1}\ \mathbb{E}_t\hat{\pi}_{t+1}\ \mathbb{E}_t\hat{pt}_{t+1}\right]^\top$ and exogenous shocks $\varepsilon_t = \left[\varepsilon_{ptt}\ \varepsilon_{nt}\ \varepsilon_{tt}\right]^\top$, that is, $x_t \in \mathbb{R}^{14+3}$, $\varepsilon_t \in \mathbb{R}^3$, $Q,\ R \in \mathbb{R}^{(14+3)\times(14+3)}$ and $S \in \mathbb{R}^{(14+3)\times 3}$; one notices that sub-matrices Q_{21} and R_{22} are selector matrices and sub-matrix $Q_{22} = R_{21} = 0$, since, $\forall i = 1, \ldots, n_{x_2}$, non-expectational endogenous variable $x_{1it} = \mathbb{E}_{t-1}x_{2it}$, having observed no exogenous shock or no longer being there uncertainty in period t.

A generalised Schur decomposition solves the generalised eigenvalue problem $Qv = \lambda Rv$ such that matrices $Q = HJ_QK^\top$ and $R = HJ_RK^\top$ and generalised eigenvalue $\lambda_i = \frac{J_{Rii}}{J_{Qii}}$, matrices J_Q and J_R eigenvalues being situated along the respective diagonals.

Matrices J_Q and J_R are upper triangular and matrices $HH^\top = HH^{-1} = KK^\top = KK^{-1} = I$; in detail, matrices $J_Q,\ J_R \in \mathbb{R}^{n_\lambda \times n_\lambda}$, $H,\ K \in \mathbb{R}^{n_x \times n_\lambda}$ and $K^\top,\ H^\top \in \mathbb{R}^{n_\lambda \times n_x}$.

Matrices J_Q and J_R are additionally reordered such that sub-matrices J_{Q11} and J_{R11} respectively contain all eigenvalues smaller than one in modulus; accordingly, sub-matrices J_{Q22} and J_{R22} are reordered to contain all eigenvalues no smaller than one in modulus: modulus eigenvalues $|\lambda_{J_Q(\lambda)}| < 1$ in J_{Q11} and $|\lambda_{J_Q(\lambda)}| \geq 1$ in J_{Q22} for characteristic polynomial $J_Q(\lambda) = J_Q - \lambda I$ in determinant $det\left[J_Q(\lambda)\right] = 0$; modulus eigenvalues $|\lambda_{J_R(\lambda)}| < 1$ in J_{R11} and $|\lambda_{J_R(\lambda)}| \geq 1$ in J_{R22} for characteristic polynomial $J_R(\lambda) = J_R - \lambda I$ in determinant $det\left[J_R(\lambda)\right] = 0$. Formally:

$$\begin{bmatrix} \underset{(n_{x_1}\times n_{x_1})}{Q_{11}} & \underset{(n_{x_1}\times n_{x_2})}{Q_{12}} \\ \underset{(n_{x_2}\times n_{x_1})}{Q_{21}} & \underset{(n_{x_2}\times n_{n_2})}{Q_{22}} \end{bmatrix} = \begin{bmatrix} \underset{(n_{x_1}\times n_{\lambda_1})}{H_{11}} & \underset{(n_{x_1}\times n_{\lambda_2})}{H_{12}} \\ \underset{(n_{x_2}\times n_{\lambda_1})}{H_{21}} & \underset{(n_{x_2}\times n_{\lambda_2})}{H_{22}} \end{bmatrix} \begin{bmatrix} \underset{(n_{\lambda_1}\times n_{\lambda_1})}{J_{Q11}} & \underset{(n_{\lambda_1}\times n_{\lambda_2})}{J_{Q12}} \\ 0 & \underset{(n_{\lambda_2}\times n_{\lambda_2})}{J_{Q22}} \end{bmatrix} \begin{bmatrix} \underset{(n_{\lambda_1}\times n_{x_1})}{\hat{K}_{11}} & \underset{(n_{\lambda_1}\times n_{x_2})}{\hat{K}_{12}} \\ \underset{(n_{\lambda_2}\times n_{x_1})}{\hat{K}_{21}} & \underset{(n_{\lambda_2}\times n_{x_2})}{\hat{K}_{22}} \end{bmatrix};$$

$$\begin{bmatrix} \underset{(n_{x_1}\times n_{x_1})}{R_{11}} & \underset{(n_{x_1}\times n_{x_2})}{R_{12}} \\ \underset{(n_{x_2}\times n_{x_1})}{R_{21}} & \underset{(n_{x_2}\times n_{n_2})}{R_{22}} \end{bmatrix} = \begin{bmatrix} \underset{(n_{x_1}\times n_{\lambda_1})}{H_{11}} & \underset{(n_{x_1}\times n_{\lambda_2})}{H_{12}} \\ \underset{(n_{x_2}\times n_{\lambda_1})}{H_{21}} & \underset{(n_{x_2}\times n_{\lambda_2})}{H_{22}} \end{bmatrix} \begin{bmatrix} \underset{(n_{\lambda_1}\times n_{\lambda_1})}{J_{R11}} & \underset{(n_{\lambda_1}\times n_{\lambda_2})}{J_{R12}} \\ 0 & \underset{(n_{\lambda_2}\times n_{\lambda_2})}{J_{R22}} \end{bmatrix} \begin{bmatrix} \underset{(n_{\lambda_1}\times n_{x_1})}{\hat{K}_{11}} & \underset{(n_{\lambda_1}\times n_{x_2})}{\hat{K}_{12}} \\ \underset{(n_{\lambda_2}\times n_{x_1})}{\hat{K}_{21}} & \underset{(n_{\lambda_2}\times n_{x_2})}{\hat{K}_{22}} \end{bmatrix}.$$

[8] https://en.wikipedia.org
[9] https://en.wikipedia.org
[10] https://en.wikipedia.org

The LRE model in question thus continues to evolve as follows:

$$
\begin{aligned}
& Qx_t = Rx_{t-1} + S\varepsilon_t \longrightarrow \\
& \longrightarrow HJ_QK^\top x_t = HJ_RK^\top x_{t-1} + S\varepsilon_t \longrightarrow \\
& \longrightarrow HJ_Qz_t = HJ_Rz_{t-1} + S\varepsilon_t, \text{ in which } z_t = K^\top x_t \longrightarrow \\
& \longrightarrow J_Qz_t = J_Rz_{t-1} + H^\top S\varepsilon_t \longleftrightarrow \\
& \longleftrightarrow \begin{bmatrix} J_{Q11} & J_{Q12} \\ 0 & J_{Q22} \end{bmatrix} \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} J_{R11} & J_{R12} \\ 0 & J_{R22} \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix} + \begin{bmatrix} \hat{H}_{11} & \hat{H}_{12} \\ \hat{H}_{21} & \hat{H}_{22} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \varepsilon_t = \\
& = \begin{bmatrix} J_{R11} & J_{R12} \\ 0 & J_{R22} \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix} + \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \varepsilon_t, \text{ in which } U = H^\top S \longrightarrow \\
& \longrightarrow J_{Q11}z_{1t} + J_{Q12}z_{2t} = J_{R11}z_{1t-1} + J_{R12}z_{2t-1} + U_1\varepsilon_t \text{ and} \\
& J_{Q22}z_{2t} = J_{R22}z_{2t-1} + U_2\varepsilon_t \longrightarrow \\
& \longrightarrow J_{R22}z_{2t-1} = J_{Q22}z_{2t} - U_2\varepsilon_t \longrightarrow \\
& \longrightarrow z_{2t-1} = J_{R22}^{-1}J_{Q22}z_{2t} - J_{R22}^{-1}U_2\varepsilon_t \longrightarrow \\
& \longrightarrow z_{2t} = J_{R22}^{-1}J_{Q22}\mathbb{E}_tz_{2t+1} - J_{R22}^{-1}U_2\mathbb{E}_t\varepsilon_{t+1} = \\
& = J_{R22}^{-1}J_{Q22}\left[J_{R22}^{-1}J_{Q22}\mathbb{E}_tz_{2t+2} - J_{R22}^{-1}U_2\mathbb{E}_t\varepsilon_{t+2}\right] - J_{R22}^{-1}U_2\mathbb{E}_t\varepsilon_{t+1} = \\
& = \left(J_{R22}^{-1}J_{Q22}\right)^2\mathbb{E}_tz_{2t+2} - J_{R22}^{-2}J_{Q22}U_2\mathbb{E}_t\varepsilon_{t+2} - J_{R22}^{-1}U_2\mathbb{E}_t\varepsilon_{t+1} \longrightarrow \\
& \longrightarrow z_{2t} = \lim_{j\to\infty}\left(J_{R22}^{-1}J_{Q22}\right)^j\mathbb{E}_tz_{2t+j} - \sum_{j=1}^{\infty} J_{R22}^{-j}J_{Q22}U_2\mathbb{E}_t\varepsilon_{t+j} = 0,
\end{aligned}
$$

noticing the following facts. Expectational endogenous variable stationarity: $\lim_{j\to\infty}\mathbb{E}_tz_{2t+j} < \infty$. Eigenvalue instability: $\lim_{j\to\infty} J_{R22}^{-j} = 0$. Exogenous shock zero mean (i.e. white noise): $\sum_{j=1}^{\infty}\mathbb{E}_t\varepsilon_{t+j} = 0$. For clarity, matrices $U_1 \in \mathbb{R}^{n_{\lambda_1}\times n_\varepsilon}$ and $U_2 \in \mathbb{R}^{n_{\lambda_2}\times n_\varepsilon}$. Since matrix

$$
\begin{bmatrix} \underset{(n_{\lambda_1}\times n_{x_1})}{\hat{K}_{11}} & \underset{(n_{\lambda_1}\times n_{x_2})}{\hat{K}_{12}} \\ \underset{(n_{\lambda_2}\times n_{x_1})}{\hat{K}_{21}} & \underset{(n_{\lambda_2}\times n_{x_2})}{\hat{K}_{22}} \end{bmatrix} \begin{bmatrix} \underset{(n_{x_1}\times 1)}{x_{1t}} \\ \underset{(n_{x_2}\times 1)}{\mathbb{E}_tx_{2t+1}} \end{bmatrix} = \begin{bmatrix} \underset{(n_{x_1}\times 1)}{z_{1t}} \\ \underset{(n_{x_2}\times 1)}{z_{2t}} \end{bmatrix} = \begin{bmatrix} \underset{(n_{x_1}\times 1)}{z_{1t}} \\ \underset{(n_{x_2}\times 1)}{0} \end{bmatrix},
$$

there arise the following manipulations:

$$
\begin{aligned}
& 0 = \hat{K}_{21}x_{1t} + \hat{K}_{22}\mathbb{E}_tx_{2t+1} \longrightarrow \\
& \longrightarrow \hat{K}_{21}x_{1t} = -\hat{K}_{22}\mathbb{E}_tx_{2t+1} \longrightarrow \\
& \longrightarrow \mathbb{E}_tx_{2t+1} = -\hat{K}_{22}^{-1}\hat{K}_{21}x_{1t} = -L_2x_{1t}, \\
& \text{in which } L_2 = \hat{K}_{22}^{-1}\hat{K}_{21}, \text{ provided } n_{\lambda_2} = n_{x_2}, \text{ and} \\
& z_{1t} = \hat{K}_{11}x_{1t} + \hat{K}_{12}\mathbb{E}_tx_{2t+1} = \hat{K}_{11}x_{1t} + \hat{K}_{12}\left(-\hat{K}_{22}^{-1}\hat{K}_{21}x_{1t}\right) = \\
& = \left(\hat{K}_{11} - \hat{K}_{12}\hat{K}_{22}^{-1}\hat{K}_{21}\right)x_{1t} = \left(\hat{K}_{11} - \hat{K}_{12}\hat{K}_{22}^{-1}\hat{K}_{21}\right)x_{1t} = L_1x_{1t}, \\
& \text{in which } L_1 = \hat{K}_{11} - \hat{K}_{12}\hat{K}_{22}^{-1}\hat{K}_{21}.
\end{aligned}
$$

In detail, condition $n_{\lambda_2} = n_{x_2}$ signifies that the cardinality of unstable generalised eigenvalues equals that of expectational endogenous variables, to the end of a unique and stable solution. Indeed, condition $n_{\lambda_2} < n_{x_2}$ is indicative of indeterminacy and condition $n_{\lambda_2} > n_{x_2}$ is indicative of no solution. The LRE model in question consequently finalises its evolution thus: since $z_{2t} = 0$ and $z_{1t} = L_1x_{1t}$,

$$\begin{aligned}
&J_{Q11}z_{1t} + J_{Q12}z_{2t} = J_{R11}z_{1t-1} + J_{R12}z_{2t-1} + U_1\varepsilon_t \longrightarrow \\
&\longrightarrow J_{Q11}z_{1t} = J_{R11}z_{1t-1} + U_1\varepsilon_t \longrightarrow \\
&\longrightarrow J_{Q11}L_1x_{1t} = J_{R11}L_1x_{1t-1} + U_1\varepsilon_t \longrightarrow \\
&\longrightarrow J_{Q11}x_{1t} = J_{R11}x_{1t-1} + L_1^{-1}U_1\varepsilon_t \longrightarrow \\
&\longrightarrow x_{1t} = J_{Q11}^{-1}J_{R11}x_{1t-1} + J_{Q11}^{-1}L_1^{-1}U_1\varepsilon_t \text{ and} \\
&\mathbb{E}_t x_{2t+1} = -L_2x_{1t} = -L_2\left(J_{Q11}^{-1}J_{R11}x_{1t-1} + J_{Q11}^{-1}L_1^{-1}U_1\varepsilon_t\right) \longrightarrow \\
&\longrightarrow \begin{bmatrix} x_{1t} \\ \mathbb{E}_t x_{2t+1} \end{bmatrix} = \begin{bmatrix} J_{Q11}^{-1}J_{R11} & 0 \\ -L_2J_{Q11}^{-1}J_{R11} & 0 \end{bmatrix} \begin{bmatrix} x_{1t-1} \\ \mathbb{E}_{t-1}x_{2t} \end{bmatrix} + \begin{bmatrix} J_{Q11}^{-1}L_1^{-1}U_1 \\ -L_2J_{Q11}^{-1}L_1^{-1}U_1 \end{bmatrix}\varepsilon_t \longleftrightarrow \\
&\longleftrightarrow \begin{bmatrix} x_{1t} \\ \mathbb{E}_t x_{2t+1} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & 0 \end{bmatrix} \begin{bmatrix} x_{1t-1} \\ \mathbb{E}_{t-1}x_{2t} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\varepsilon_t \longleftrightarrow \\
&\longleftrightarrow x_t = Ax_{t-1} + B\varepsilon_t.
\end{aligned}$$

13. IRFs

13.1 IRF construction. Such a solution proper to the LRE model in question, computed in MATLAB or OCTAVE by means of CEPREMAP [1]'s `dynare,` is more specifically identified as the transition or state equation of a linear time invariant (LTI) state space representation in discrete time, being itself a first order linear heterogeneous difference equation: $\forall n \geq 1$, function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that, $\forall t \in \mathbb{Z},\ x_t = Ax_{t-1} + B\varepsilon_t$, in which states $x_t \in \mathbb{R}^{n_x}$, inputs $\varepsilon_t \in \mathbb{R}^{n_\varepsilon}$, companion matrix $A \in \mathbb{R}^{n_x \times n_x}$ and input matrix $B \in \mathbb{R}^{n_x \times n_\varepsilon}$. It is also a fundamental $VAR(1)$ process, because of companion matrix A's stability, thereby bearing the potential to be rewritten as a causal $VMA(\infty)$ process:

$$\begin{aligned}
&x_t = Ax_{t-1} + B\varepsilon_t \text{ [fundamental } VAR(1)] \longrightarrow \\
&\longrightarrow (I - AL)\,x_t = A(L)\,x_t = B\varepsilon_t, \\
&\text{in which operator } L : \mathbb{R}^n \to \mathbb{R}^n \text{ such that } L = x_t^{-1}x_{t-1} \longrightarrow \\
&\longrightarrow x_t = A^{-1}(L)\,B\varepsilon_t = \sum_{j=0}^{\infty} A^jL^jB\varepsilon_t = \sum_{j=0}^{\infty} A^jB\varepsilon_{t-j} \text{ [causal } VMA(\infty)],
\end{aligned}$$

since, $\forall |s| < 1$ and operator $k : \mathbb{R} \to \mathbb{R}$, $\lim_{n\to\infty} S = \lim_{n\to\infty} \sum_{j=0}^{n-1} s^jk^j = (1 - sk)^{-1}(1 - s^nk^n) = (1 - sk)^{-1}$, because $S - Ssk = (1 - sk)\,S = \sum_{j=0}^{n-1} s^jk^j - \sum_{j=0}^{n-1} s^{j+1}k^{j+1} = s^0k^0 - s^nk^n = 1 - s^nk^n$, thus, $\sum_{j=0}^{\infty} A^jL^j = (I - AL)^{-1} = A^{-1}(L)$ if and only if modulus eigenvalues $|\lambda_{A(\lambda)}| < 1$ for characteristic polynomial $A(\lambda) = A - \lambda I$ in determinant $det[A(\lambda)] = 0$, which is equivalent to stating if and only if trace $tr\left(AA^\top\right) = \sum_{i,\,j=1}^{n_x} a_{ij}a_{ji}^\top = \sum_{i,\,j=1}^{n_x} a_{ij}^2 < \infty$, whence

$$\frac{\partial x_t}{\partial \varepsilon_{t-j}} = A^jB \text{ (IRF coefficients).}$$

In fact, first order IRFs are analytically constructed thus: $\forall j \in \mathbb{N}$ and exogenous shock $\varepsilon_t \sim \mathcal{N}\left(0,\ \sigma^2\right)$, IRF $\mathcal{I}_{x_j} := \mathbb{E}_t x_{t+j} - \mathbb{E}_{t-1}x_{t+j}|\varepsilon_t = \tilde{\varepsilon}$, in which $\tilde{\varepsilon}$ is a realisation of ε_t.

Assuming that exogenous shock realisation $\tilde{\varepsilon} = \sigma$, the following unfolds:

$$\begin{aligned}
&\mathbb{E}_t x_t = \mathbb{E}_t(Ax_{t-1} + B\varepsilon_t) = \mathbb{E}_t(Ax_{t-1} + B\tilde{\varepsilon}) = \mathbb{E}_t(Ax_{t-1} + B\sigma) = Ax_{t-1} + B\sigma, \\
&\text{since } \mathbb{E}_t x_{t-1} = x_{t-1} \text{ and } \mathbb{E}_t\sigma = \sigma \text{ (observations),} \\
&\mathbb{E}_t x_{t+1} = \mathbb{E}_t(Ax_t + B\varepsilon_{t+1}) = Ax_t = A(Ax_{t-1} + B\sigma) = A^2x_{t-1} + AB\sigma,
\end{aligned}$$

$$\mathbb{E}_t x_{t+2} = \mathbb{E}_t \left(A x_{t+1} + B \varepsilon_{t+2} \right) = \mathbb{E}_t A x_{t+1} = A \left(A^2 x_{t-1} + A B \sigma \right) = A^3 x_{t-1} + A^2 B \sigma,$$

$$\vdots$$

$$\mathbb{E}_t x_{t+j} = A^{j+1} x_{t-1} + A^j B \sigma, \text{ since, } \forall j \in \mathbb{N}_{(+)}, \ \mathbb{E}_t \varepsilon_{t+j} = 0 \text{ (white noise);}$$

$$\mathbb{E}_{t-1} x_t = \mathbb{E}_{t-1} \left(A x_{t-1} + B \varepsilon_t \right) = A x_{t-1},$$

$$\mathbb{E}_{t-1} x_{t+1} = \mathbb{E}_{t-1} \left(A x_t + B \varepsilon_{t+1} \right) = A x_t = A \left(A x_{t-1} \right) = A^2 x_{t-1},$$

$$\mathbb{E}_{t-1} x_{t+2} = \mathbb{E}_{t-1} \left(A x_{t+1} + B \varepsilon_{t+2} \right) = A x_{t+1} = A \left(A^2 x_{t-1} \right) = A^3 x_{t-1},$$

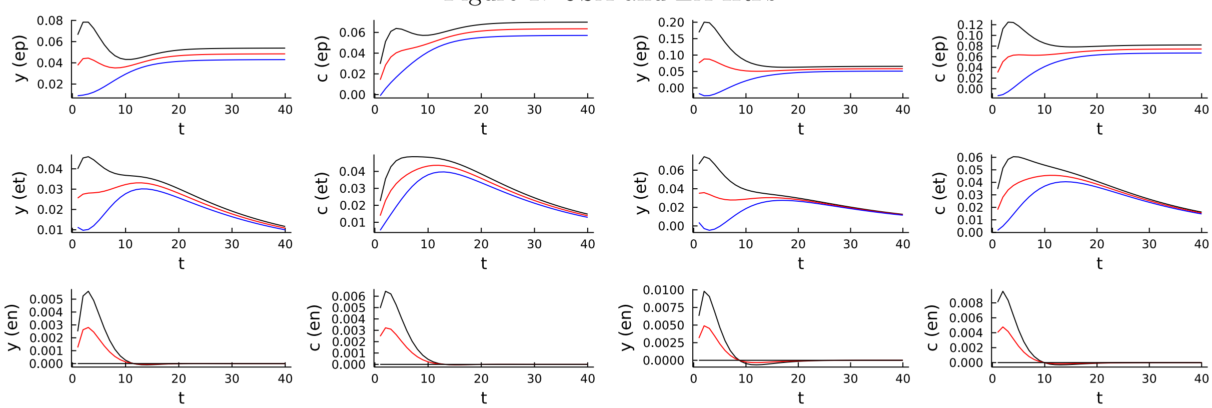
$$\vdots$$

$$\mathbb{E}_{t-1} x_{t+j} = A^{j+1} x_{t-1}, \text{ since, } \forall j \in \mathbb{N}_{(+)}, \ \mathbb{E}_{t-1} \varepsilon_{t+j} = 0 \text{ (white noise),}$$

whereby $\mathcal{I}_{x_j} = \mathbb{E}_t x_{t+j} - \mathbb{E}_{t-1} x_{t+j} = A^{j+1} x_{t-1} + A^j B \sigma - A^{j+1} x_{t-1} = A^j B \sigma$ such that $\mathcal{I}_{x_0} = A^0 B \sigma = B \sigma$, $\mathcal{I}_{x_1} = A B \sigma$, $\mathcal{I}_{x_2} = A^2 B \sigma$, $\ldots$ and $\mathcal{I}_{x_j} = A^j B \sigma$.

13.2 IRF commentary. The empirical SIRF patterns presented in Table 1 are all replicated by accounting for all exogenous shocks underlying changes in confidence $\hat{\Upsilon}_t$ and all volition regimes γ. This establishes volition γ as the ultimate determiner of fluctuations in real economic activity in the face of changes in confidence $\hat{\Upsilon}_t$. For clarity, fluctuations in real economic activity refer to its cycle component, rather its trend component, which instead refers to the economy's balanced growth or decline path.

Figure 1: USA and EA IRFs



Note. IRFs for aggregate real production $\hat{y}_t$ (y) and real consumption $\hat{c}_t$ (c), relative to the USA and the EA, given exogenous shocks at a standard deviation of 0.01 in permanent technology ε_{ptt} (ep), transitory technology ε_{tt} (et) and noise technology ε_{nt} (en), under a high (black), medium (red) and low (blue) volition regime γ.

A pattern of immediate irreversibility is exhibited by a combination entailing an exogenous shock in permanent technology ε_{ptt} and a high or medium volition regime $\gamma_{H,\,M}$. A pattern of delayed irreversibility is exhibited by a combination entailing an exogenous shock in permanent technology ε_{ptt} and a low volition regime γ_L.

At one order of magnitude below the others, a pattern of immediate reversibility is exhibited by a combination entailing an exogenous shock in noise technology ε_{nt} and a high or medium volition regime $\gamma_{H,\,M}$. A pattern of delayed reversibility is exhibited by a combination entailing an exogenous shock in transitory technology ε_{tt} and any volition regime γ.

The regime of volition γ therefore gives rise to a compromise between endogenous growth and a "boom and bust" cycle. An exogenous shock in noise technology ε_{nt} gives rise to a "boom and bust" cycle whenever the regime of volition γ be non-negligible. On the other hand, an exogenous shock in permanent technology ε_{ptt} in the face of a non-negligible regime of volition γ causes endogenous growth.

Correspondingly, a "boom and bust" cycle is avoided in the face of an exogenous shock in noise technology ε_{nt} whenever the regime of volition γ be negligible, although avoiding endogenous growth too in the face of an exogenous shock in permanent technology ε_{ptt}.

14. Minimal poor man's invertibility condition

14.1 Poor man's invertibility condition. For a selector matrix suitably composed of zeros and ones there arises a measurement or observation equation proper to an LTI state space representation in discrete time: $Mx_t = MAx_{t-1} + MB\varepsilon_t \longrightarrow y_t = Cx_{t-1} + D\varepsilon_t$, in which selector matrix $M \in \mathbb{R}^{n_y \times n_x}$, outputs or observables $y_t \in \mathbb{R}^{n_y}$, companion matrix $C \in \mathbb{R}^{n_y \times n_x}$ and input matrix $D \in \mathbb{R}^{n_y \times n_\varepsilon}$. Assume input matrix D to be invertible and thus square: dimension $n_y = n_\varepsilon \leq n_x \in \mathbb{N}_+$; in fact, $y_t = \left[\hat{\Upsilon}_t \; \hat{c}_t \; \hat{y}_t\right]^\top$. Then,

$$\begin{aligned}
& y_t = Cx_{t-1} + D\varepsilon_t \longrightarrow \\
& \longrightarrow D\varepsilon_t = y_t - Cx_{t-1} \longrightarrow \\
& \longrightarrow \varepsilon_t = D^{-1}\left(y_t - Cx_{t-1}\right) \longrightarrow \\
& \longrightarrow x_t = Ax_{t-1} + BD^{-1}\left(y_t - Cx_{t-1}\right) \longrightarrow \\
& \longrightarrow x_t = \left(A - BD^{-1}C\right)x_{t-1} + BD^{-1}y_t = Fx_{t-1} + BD^{-1}y_t \longrightarrow \\
& \longrightarrow x_t - Fx_{t-1} = \left(I - FL\right)x_t = F\left(L\right)x_t = BD^{-1}y_t \longrightarrow \\
& \longrightarrow x_t = F^{-1}\left(L\right)BD^{-1}y_t = \sum_{j=0}^{\infty} F^j L^j BD^{-1}y_t = \sum_{j=0}^{\infty} F^j BD^{-1}y_{t-j}
\end{aligned}$$

if and only if modulus eigenvalues $|\lambda_{F(\lambda)}| < 1$ for characteristic polynomial $F(\lambda) = F - \lambda I$ in determinant $det\left[F(\lambda)\right] = 0$, being Fernández-Villaverde *et alii* [12]'s poor man's invertibility condition (PMIC), which is equivalent to stating if and only if trace $tr\left(FF^\top\right) = \sum_{i,\,j=1}^{n_x} f_{ij} f_{ji}^\top = \sum_{i,\,j=1}^{n_x} f_{ij}^2 < \infty$, whence

$$y_t = Cx_{t-1} + D\varepsilon_t = C\sum_{j=0}^{\infty} F^j BD^{-1}y_{t-j-1} + D\varepsilon_t \text{ [fundamental } VAR(\infty)\text{]},$$

being a VAR representation of states x_t in outputs y_t.

14.2 Minimality. For controllability matrix $\mathcal{C} = \left[B \cdots A^{n_x-1}B\right]$ and observability matrix $\mathcal{O} = \left[C \cdots CA^{n_x-1}\right]^\top$ the LTI state space representation is minimal if and only if dimension $n_x = r_{\mathcal{C}} = r_{\mathcal{O}}$. If it is non-minimal it is then discretionally reduced to minimality as follows:

(i) if dimension $n_x > r_{\mathcal{C}}$ (i.e. non-controllable) one then constructs similarity transformation matrix $\mathcal{T} = \left[\mathcal{C}_{r_{\mathcal{C}}} v_{n_x - r_{\mathcal{C}}}\right]$ for vector $\bar{x}_{c\bar{c}t} = \mathcal{T}^{-1}x_t$ and matrices $\bar{A}_{c\bar{c}} = \mathcal{T}^{-1}A\mathcal{T},\ \bar{B}_{c\bar{c}} = \mathcal{T}^{-1}B,\ \bar{C}_{c\bar{c}} = C\mathcal{T},\ \bar{\mathcal{C}}_{c\bar{c}} = \mathcal{T}^{-1}\mathcal{C}$ and $\bar{\mathcal{O}}_{c\bar{c}} = \mathcal{O}\mathcal{T}$, in which the first $r_{\mathcal{C}}$ states are controllable, namely, vector $\bar{x}_{ct}$ and matrices $\bar{A}_c,\ \bar{B}_c,\ \bar{C}_c,\ \bar{\mathcal{C}}_c$ and $\bar{\mathcal{O}}_c$; if dimension $n_x = r_{\mathcal{C}}$ (i.e. controllable) one then directly acknowledges vector $\bar{x}_{ct}$ and matrices $\bar{A}_c,\ \bar{B}_c,\ \bar{C}_c,\ \bar{\mathcal{C}}_c$ and $\bar{\mathcal{O}}_c$;

(ii) if dimension $n_{\bar{x}_c} > r_{\bar{\mathcal{O}}_c}$ (i.e. non-observable) one then constructs similarity transformation matrix $\mathcal{T} = \left[\bar{\mathcal{O}}_{cr_{\bar{\mathcal{O}}_c}} v_{n_{\bar{x}_c} - r_{\bar{\mathcal{O}}_c}}\right]$ for vector $\bar{x}_{co\bar{o}t} = \mathcal{T}^{-1}x_{ct}$ and matrices $\bar{A}_{co\bar{o}} = \mathcal{T}^{-1}\bar{A}_c\mathcal{T},\ \bar{B}_{co\bar{o}} = \mathcal{T}^{-1}\bar{B}_c,\ \bar{C}_{co\bar{o}} = \bar{C}_c\mathcal{T},\ \bar{\mathcal{C}}_{co\bar{o}} = \mathcal{T}^{-1}\bar{\mathcal{C}}_c$ and $\bar{\mathcal{O}}_{co\bar{o}} = \bar{\mathcal{O}}_c\mathcal{T}$, in which the first $r_{\bar{\mathcal{O}}_c}$ are controllable and observable (i.e. minimal), namely, vector $\bar{x}_{cot} = x_{mt}$ and matrices $\bar{A}_{co} = A_m,\ \bar{B}_{co} = B_m,\ \bar{C}_{co} = C_m,\ \bar{\mathcal{C}}_{co} = \mathcal{C}_m$ and $\bar{\mathcal{O}}_{co} = \mathcal{O}_m$; if dimension $n_{\bar{x}_c} > r_{\bar{\mathcal{O}}_c}$ (i.e. controllable and observable, minimal) one then directly acknowledges vector $\bar{x}_{cot} = x_{mt}$ and matrices $\bar{A}_{co} = A_m,\ \bar{B}_{co} = B_m,\ \bar{C}_{co} = C_m,\ \bar{\mathcal{C}}_{co} = \mathcal{C}_m$ and $\bar{\mathcal{O}}_{co} = \mathcal{O}_m$.

It follows that minimal transition and measurement equations

$$\begin{aligned}
& x_{mt} = A_m x_{mt-1} + B_m\varepsilon_t \text{ and} \\
& y_t = C_m x_{mt-1} + D\varepsilon_t,
\end{aligned}$$

in which dimension $n_{x_m} = r_{\mathcal{C}_m} = r_{\mathcal{O}_m}$, give rise to minimal fundamental $VAR(\infty)$ $y_t = C_m\sum_{j=0}^{\infty} F_m^j B_m D^{-1}y_{t-j-1} + D\varepsilon_t$ for minimal matrix $F_m = A_m - B_m D^{-1}C_m$.

In minimal LTI state space representations the IRFs of the transition equation and the coefficients of the VAR representation of states x_t in outputs y_t are invariant, as especially remarked by Franchi [13]: $\forall j \in$

$\mathbb{N}_+$, $CA^jB = C_m A_m^j B_m \neq 0$, from $x_t = \sum_{j=0}^{\infty} A^j B \varepsilon_{t-j} \longrightarrow y_t = C x_{t-1} + D\varepsilon_t = C \sum_{j=0}^{\infty} A^j B \varepsilon_{t-j} + D\varepsilon_t$, and $CF^jB = C_m F_m^j B_m \neq 0$, from $y_t = C_m \sum_{j=0}^{\infty} F_m^j B_m D^{-1} y_{t-j-1} + D\varepsilon_t$. Absent loss of generality, one can therefore conduct suitable evaluations in terms of the minimal poor man's invertibility condition (mPMIC).

As borne out by the attendant eigenvalues in the annexed code, the mPMICs for the USA and the EA hereby computed fail to give rise to a VAR representation of states x_t in outputs y_t across all three volition regimes γ except for a medium volition regime γ_M with regard to the EA.

14.3 Discussion. As implicitly shown by Sims [19], if the mPMIC fails it need not mean that states x_t may not be practically represented in outputs y_t by means of VARs, thereby recovering the nature of the underlying exogenous shocks in the observed endogenous variables under consideration all the same.

In fact, Saccal [18] showed that for any minimal transition equation the $VMA(0)$ representation of the form $y_t = D\varepsilon_t$ is almost sure, in all of its empirical futility, whereby the adjunction of other VAR representations of states x_t in outputs y_t is probabilistically negligible. The recovery of the underlying exogenous shocks in the observed endogenous variables is consequently almost always tied to a structural model other than that of the transition equation.

The unique and stable solution of the first order approximation of the present NK-DSGE model is consequently salvaged by recourse to axiomatic abstraction, deeming it logically valid and its hypotheses no less than probable, which judgement appears to be confirmed by the successful replication of the empirical SIRF patterns at hand.

15. Conclusion

Economic literature exhibits a variety of empirical SIRF patterns in real economic activity in the face of changes in confidence or sentiment, with particular regard to the USA and the EA. This work successfully endeavoured to replicate them in the orbit of a NK-DSGE model especially characterised by macroeconomic agents and derived from start to end. Confidence Υ_t has been specifically modelled as an endogenous variable characterised by a coalescence of two technology processes pt_t and t_t, permanent and transitory, and one noise process n_t, being globally regulated by a degree of volition γ. The first two processes affect real production technology a_t with a lag delay, while the third does not. Short run responses to changes in confidence Υ_t are therefore displayed whenever confidence Υ_t shift real consumption c_t and aggregate labour l_t. In turn, confidence Υ_t shifts real consumption c_t and aggregate labour l_t whenever volition γ be not infinitesimal. Whenever volition γ were infinitesimal, by contrast, exogenous shocks in noise n_t would not cause fluctuations in real economic activity at all.

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Appendix

This is the `dynare` code for a unique and stable solution of the first order approximation of the NK-DSGE model at hand.

```
/*A role for confidence: volition regimes and news (Alessandro Saccal)*/

var w l c pt rk rn pi y phi a k Upsilon t n; // Endogenous variables

varexo e_pt e_t e_n; // Exogenous shocks

parameters sigma_l sigma_c h omega rho_rn phi_pi phi_y phi_pi_g phi_y_g beta tau xi Δ ...
    rkss alpha c_y k_y rho_a rho_t rho_n gamma; // Parameters

/*EA parameters
rho_t=0.95; // Transitory technology persistence
rho_n=0.65; // Noise technology persistence
rho_a=0.823; // Production technology persistence
gamma=1; // Volition regime; 1, 0.5, 0.0001
beta=0.99; // Discount factor
tau=0.469; // Inflation indexation
xi=0.908; // Price adjustment failure fraction
sigma_l=2.4; // Labour inverse elasticity
sigma_c=1.353; // Inter-temporal substitution inverse elasticity
h=0.573; // Consumption habit
omega=5.917; // Capital utilisation adjustment cost inverse elasticity
rho_rn=0.961; // Interest rate persistence
phi_pi=1.684; // Inflation coefficient
phi_y=0.099; // Output coefficient
phi_pi_g=0.14; // Inflation gap coefficient
phi_y_g=0.159; // Output gap coefficient
Δ=0.025; // Capital depreciation rate
rkss=0.0351; // Steady state capital return
alpha=0.3; // Capital in output share
c_y=0.6; // Consumption to output ratio
k_y=8.8; // Capital to output ratio*/

// USA parameters
rho_t=0.95; // Transitory technology persistence
rho_n=0.65; // Noise technology persistence
rho_a=0.822; // Production technology persistence
gamma=1; // Volition regime; 1, 0.5, 0.0001
beta=0.99; // Discount factor
tau=0.66; // Inflation indexation
xi=0.87; // Price adjustment failure fraction
```

```
sigma_l=2.45; // Labour inverse elasticity
sigma_c=1.62; // Inter temporal substitution inverse elasticity
h=0.69; // Consumption habit
omega=3.23; // Capital utilisation adjustment cost inverse elasticity
rho_rn=0.88; // Interest rate persistence
phi_pi=1.48; // Inflation coefficient
phi_y=0.08; // Output coefficient
phi_pi_g=0.24; // Inflation gap coefficient
phi_y_g=0.24; // Output gap coefficient
Δ=0.025; // Capital depreciation rate
rkss=0.0351; // Steady state capital return
alpha=0.24; // Capital in output share
c_y=0.65; // Consumption to output ratio
k_y=6.8; // Capital to output ratio

model(linear);

pt=pt(-1)+e_pt; // Permanent technology

t=rho_t*t(-1)+e_t; // Transitory technology

n=rho_n*n(-1)+e_n; // Noise technology

a=rho_a*a(-1)+pt(-1)+t(-1); // Production technology

pi=(((1-xi)*(1-beta*xi))/((1+beta*tau)*xi))*phi+(beta/(1+beta*tau))*pi(+1)+(tau/(1+beta*tau))*pi(-1); ...
    // Inflation

w=sigma_l*l+(sigma_c/(1-h))*(c+pt)-(sigma_c*h/(1-h))*(c(-1)+pt(-1))-Upsilon-pt; // Real wage

l=(1+omega)*rk+k(-1)+pt(-1)-w-pt; // Aggregate labour

c=((1-h)/(sigma_c*(1+h)))*(Upsilon+pi(+1)-rn)+(h/(1+h))*(c(-1)+pt(-1))+(1/(1+h))*(c(+1)+pt(+1))-pt; ...
    // Real consumption

rn=rho_rn*rn(-1)+(1-rho_rn)*(phi_pi*pi+phi_pi_g*(pi-pi(-1))+phi_y*(y+pt)+phi_y_g*(y+pt-y(-1)-pt(-1))); ...
    // Nominal interest rate

y=a+alpha*omega*rk+alpha*(k(-1)+pt(-1))+(1-alpha)*(Upsilon+l)-pt; // Aggregate real ...
    production

Upsilon=gamma*(pt+t+n); // Confidence

phi=alpha*rk+(1-alpha)*(w+pt-Upsilon)-a; // Real marginal cost

k=(1-Δ)*(k(-1)+pt(-1))-pt; // Aggregate capital

y=c_y*(c+pt)+k_y*rkss*omega*rk-pt; // Aggregate capital utilisation

end;

initval;
w=0; l=0; c=0; pt=0; rk=0; rn=0; pi=0; y=0; phi=0; a=0; k=0; Upsilon=0; t=0; n=0;
end;

steady;
check; // Rational expectations stable unique solution check

shocks;
var e_pt; stderr 0.01;
var e_t; stderr 0.01;
var e_n; stderr 0.01;
end;

stoch_simul(irf=40, order=1) c y; // graph_format=(none) and nograph can be added to ...
    omit first order IRF graphs

```

```
varobs Upsilon c y;
[result, eigenvalue_modulo, A, B, C, D]=ABCD_test(M_, options_, oo_, 0); // 0 can be ...
    changed to 1 for minimality
```