A role for confidence: volition regimes and news

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A role for confidence: volition regimes and news

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Abstract

Economic literature exhibits a variety of empirical structural impulse response function (SIRF) patterns in real consumption and real output due to changes in confidence or sentiment, with particular regard to the USA and the EA. This work replicates them in the orbit of a neo-Keynesian dynamic stochastic general equilibrium (NK-DSGE) model especially characterised by macroeconomic agents and derived from start to end. Confidence is specifically modelled as an endogenous variable characterised by a coalescence of three processes regulated by a degree of volition, the processes being permanent technology, transitory technology and noise technology. The first two processes affect real production technology with a delay of one lag, while the third does not at all. Short run responses to changes in confidence are displayed whenever the degree of volition allow confidence to shift real consumption and aggregate labour, thereby being non-negligible. Whenever the degree of volition were by contrast negligible exogenous shocks in noise technology would cause no fluctuations in real consumption and real output whatsoever.

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MSC codes: 91B51; 91B55; 91B62; 91B84; 91C99.
Keywords: aggregate labour; confidence; volition; permanent technology; transitory technology; noise technology; real economic activity; real consumption; real output.

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1. Introduction and contributions

*saccal.alessandro@gmail.com. Disclaimer: the author has no declaration of interest related to this research; all views and errors in this research are the author’s. Note: two former versions of this article titled “Volition regimes and news” and “A role for confidence” and all variants thereof are hereby supplanted. ©Copyright 2023 Alessandro Saccal
1.1 Introduction and scientific literature. Defining the relationship between confidence and real economic activity is a complex task, for despite being regarded as vital confidence’s nature is rather elusive. Indeed, reciprocity characterises the two: confidence is said to influence real economic activity and real economic activity is said to impact confidence in turn.

Economic literature nonetheless provides two main justifications. The first conceives of confidence as waves of pure sentiment, demand or noise, dating back to Keynes [14] and having been more recently expanded by Akerlof and Shiller [2], Lorenzoni [15], Angeletos and La‘O [3] and Angeletos et alii [4].

The second regards it as a proxy for news and noise shocks in economic fundamentals, dating back to Pigou [16]; related contemporary works comprise those of Cochrane [11], Beaudry and Portier [6], Barsky and Sims [5], Sims [19], Blanchard et alii [7], Chahrour and Jurado [10] and Saccal [17]. Independent as they are, this work hinges on both.

1.2 Notional and methodological contributions. This work’s notional contribution is the theoretical explanation of all types of empirical SIRFs in real consumption and real output to exogenous shocks in news and noise processes, normally proxied by economic sentiment or confidence.

The empirical SIRFs need not have all been observed by the pertinent economic literature, but Barsky and Sims [5] and Saccal [17] effectively did, at least substantially. Such authors globally construct trivariate structural vector auto-regressions (SVARs) of order 4 featuring confidence, real consumption and real output in log-levels for the United States of America (USA), the Euro Area (EA) and other European nations and presented a variety of empirical SIRFs in real consumption and real output given changes in news and noise processes, normally proxied by economic sentiment or confidence.

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The reason for which confidence is normally chosen as an empirical proxy for news and noise processes is that the latter are unobservable, both empirically and theoretically, as explained by Sims [19] in relation to Blanchard et alii [7].
Confidence can thus act as their proxy both in models and in data, so that theoretical and empirical SIRFs in real consumption and real output given changes in confidence reveal the effective nature of the exogenous shocks. Such is also in line with the contribution adduced by Chahrour and Jurado [10], who showed that news and noise proxies are equivalent representations of news and noise processes.

The theoretical explanation of all types of empirical SIRFs in question is developed by means of a minimalistic NK-DSGE model in discrete time and is as such the work’s methodological contribution.

In such a model confidence $\Upsilon_t$ is an endogenous variable and figures as a coalescence of two technology processes $p_t$ and $t_t$, permanent and transitory, and one noise process $n_t$, which are all endogenous variables as well: coalescence $p_t t_t n_t$ is especially regulated by a volition parameter $\gamma$ endowed with the potential to dampen the three processes’ propagation.

The NK-DSGE is minimalistic in the sense that the substantial extensions relative to a real business cycle (RBC) model are merely those of rigid prices and monetary policy. Whether the theoretical explanation of all types of empirical SIRFs in question may work in a mere RBC model as well is an issue reserved for future research.

1.3 Other contributions. Another notional distinction, relative to ordinary DSGE models, is that the economy is not delineated by representative agents, but by macroeconomic agents, thereby eluding the fallacies stressed by the “Anything goes” theorem by which the conceptual aggregation of microeconomic agents need not guarantee the functional properties exhibited by representative agents, particularly the canonical laws of supply and demand.

Consequently, this economy is to feature the canonical laws of supply and demand by construction, as well as the functional properties otherwise pertinent to representative agents. Aggregation in this economy, whenever present, is to be therefore understood as merely pertaining to macroeconomic agents, not to homogenous microeconomic ones, that is, to no more than parts of the macroeconomy.

If representative agents were alternatively understood as macroeconomic ones, as opposed to homogenous microeconomic ones in aggregation, then the fallacies stressed by the “Anything goes” theorem would clearly not apply.

Another methodological advantage of this work is the complete derivation and resolution of its NK-DSGE model, until the conduction of policy analysis, encyclopaedically omitting no passage whatsoever and thereby benefitting all those readers in search of a comprehensive, applied guide to (such a kind of) DSGE models.

2. Confidence

2.1 Construction. Confidence $\Upsilon_t$ is an endogenous variable and is to be modelled as follows. First of all, any exogenous shock is a normally distributed white noise, thereby featuring a 0 mean and a finite variance: $\varepsilon_t \sim \mathcal{N}(0, \sigma^2_{\varepsilon_t})$, in which $\sigma^2_{\varepsilon_t} \in (0, \infty) \subset \mathbb{R}^+$. Real production technology $a_t$ then equals its amnesic lagged value $\rho_a a_{t-1}$, which is in turn augmented by (i) an exponentiated real population mean $\mu$, ideally modelling a quarterly technological growth rate, (ii) lagged permanent technology $p_{t-1}$ and (iii) lagged transitory technology $t_{t-1}: a_t = e^\mu \rho_a a_{t-1} p_{t-1} t_{t-1}$, in which coefficient $\rho_a \in [0, 1) \subset \mathbb{R}^+$ and $\mu \in \mathbb{R}$, the equation in question being a law of motion for real production technology $a_t$. Present (i.e. surprise) exogenous shocks in real production technology are thus excluded.

The fact that lagged permanent technology $p_{t-1}$ and transitory technology $t_{t-1}$ augment real production technology $a_t$ models exogenous news shocks, one speaking to news regarding exogenous shocks in permanent technology and the other to news regarding exogenous shocks in transitory technology. News shocks broadly referenced can thus be understood as rational anticipations of exogenous shocks in technology at large.

Permanent technology $p_t$ equals its mnemonic lagged value $p_{t-1}$, which is in turn augmented by an exponentiated exogenous shock $\varepsilon_{p_{t}}$ weighted at its own standard deviation $\sigma_{\varepsilon_{p_{t}}} = p_{t} = p_{t-1} e^{\sigma_{\varepsilon_{p_{t}}} \varepsilon_{p_{t}}}$, in which $\varepsilon_{p_{t}} \sim \mathcal{N}(0, \sigma^2_{\varepsilon_{p_{t}}})$, the equation in question being a law of motion for permanent technology $p_t$. It is thus a random walk process: $p_t \sim RW$.

1 https://en.wikipedia.org
In addition, the expected value of lead permanent technology \( E_t p_{t+1} \) (i.e. process population mean) is non-zero such that its deviation from the steady state \( E_t \hat{p}_{t+1} \) is non-zero too: \( E_t p_{t+1} \neq 0 \) such that \( E_t \hat{p}_{t+1} \neq 0 \).

Transitory technology \( t_t \) and noise technology \( n_t \) respectively equal their amnesic lagged values \( \rho_t t_{t-1} \) and \( \rho_n n_{t-1} \), which are in turn augmented by exponentiations of their respective exogenous shocks \( \varepsilon_{t_t} \) and \( \varepsilon_{n_t} \) weighted at their respective standard deviations \( \sigma_{t_t} \) and \( \sigma_{n_t} \): \( x_t = \rho_t x_{t-1} e^{\sigma_{t_t} \varepsilon_{t_t}} \), in which coefficient \( \rho_x \in (0, 1) \subset \mathbb{R}_+ \), \( \varepsilon_{xt} \sim \mathcal{N}(0, \sqrt{\sigma}^2) \) and \( x = t, n \), being laws of motion for transitory technology \( t_t \) and noise technology \( n_t \). They are therefore auto-regressive processes of order 1: \( x_t \sim AR(1) \), ceteris paribus.

The expected values of both processes’ lead terms \( E_t x_{t+1} \) (i.e. process population means) are accordingly non-zero such that their deviations from the steady state \( E_t \hat{x}_{t+1} \) are non-zero too: \( E_t \hat{x}_{t+1} \neq 0 \) such that \( E_t \hat{x}_{t+1} \neq 0 \).

Confidence \( \Upsilon_t \) specifically equals the product of permanent technology \( p_{t_t} \), transitory technology \( t_t \) and noise technology \( n_t \) risen to volition parameter \( \gamma \), which lies in a semi-open real interval between 0 and 1 : \( \Upsilon_t = (p_{t_t} t_{t_t} n_{t_t})^{\gamma} \), in which \( \gamma \in (0, 1) \subset \mathbb{R}_+ \), the equation in question being a law of motion for confidence \( \Upsilon_t \).

The expected value of lead confidence \( E_t \Upsilon_{t+1} \) (i.e. population mean) equals 0 such that its deviation from the steady state \( E_t \hat{\Upsilon}_{t+1} \) equals 0 too: \( E_t \hat{\Upsilon}_{t+1} = 0 \) such that \( E_t \hat{\Upsilon}_{t+1} = 0 \).

The methodological and theoretical consequence is that the non-nullity of the expected value of lead permanent technology \( E_t p_{t+1} \) is balanced out by that pertaining to the expected values of transitory technology and noise technology \( E_t x_{t+1} \), especially applying at the steady state as well: \( E_t p_{t+1} \neq 0 \) and \( E_t \hat{p}_{t+1} \neq 0 \) are balanced out by \( E_t x_{t+1} \neq 0 \) and \( E_t \hat{x}_{t+1} \neq 0 \).

### 2.2 Discussion

Confidence \( \Upsilon_t \) is to be introduced as a shifter of real consumption \( C_t \) and of aggregate labour \( l_t \), which are endogenous variables, so that whenever volition \( \gamma \) lie at an infinitesimal distance from 0 confidence \( \Upsilon_t \) is almost neutralised, either to unity (i.e. non-linearly) or to nullity (i.e. linearly).

Otherwise stated: the higher the value of volition \( \gamma \) the greater the enthusiasm in real consumption \( C_t \) and the effort in aggregate labour \( l_t \); accordingly, for infinitesimal values of volition \( \gamma \) the impact exerted by confidence \( \Upsilon_t \) upon real economic activity is also infinitesimal.

Consequently, while a change in confidence \( \Upsilon_t \) be itself exogenous the extent to which macroeconomic agents may react to it is endogenous. The econometrician, theoretically and empirically, observes confidence \( \Upsilon_t \) alone, for its constituents are unobservable; yet, he is capable of identifying both the nature of the exogenous shock and the regime of volition \( \gamma \) underlying a change in confidence \( \Upsilon_t \), particularly empirically.

<table>
<thead>
<tr>
<th>Volition regime</th>
<th>Economic region</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma H )</td>
<td>1</td>
</tr>
<tr>
<td>( \gamma M )</td>
<td>0.5</td>
</tr>
<tr>
<td>( \gamma L )</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

Note: Prospected calibration of volition regimes \( \gamma \) for an economic region formalised by means of a NK-DSGE model as outlined above. H, M and L stand for high, medium and low, respectively.

In the case of an exogenous shock in noise technology \( n_t \) unless volition \( \gamma \) were infinitesimal an SIRF pattern of immediate reversibility (i.e. “boom and bust” cycle) would be unavoidable, owing to the presence of a short turn response precisely triggered by a non-negligible value of volition \( \gamma \) as well as the absence of noise technology at any time period in real production technology \( a_t \), thereby giving rise to an expansionary deviation from the steady state on account of noise, demand or pure sentiment (i.e. animal spirits).

In the case of an exogenous shock in permanent technology \( p_t \) and a non-negligible value of volition \( \gamma \) there would correspondingly arise an SIRF pattern of immediate irreversibility (i.e. endogenous growth), whereas a negligible value of volition \( \gamma \) would catalyse an SIRF pattern of delayed irreversibility, owing to
the sole activity of permanent technology \( pt\), thereby failing to capitalise upon a positive permanent variation in the selfsame steady state.

In the case of an exogenous shock in transitory technology \( t\) and a non-negligible value of volition \( \gamma\) there would analogously arise an SIRF pattern of delayed reversibility, whereas a negligible value of volition \( \gamma\) would catalyse an SIRF pattern of postponed delayed reversibility, owing to the sole activity of transitory technology \( t\), thereby failing to capitalise upon a positive transitory variation in the selfsame steady state. Table 2 predisposes the formalisation of all such cases.

Saccal [17] wrote the following: “Delayed reversibility suggests a noise shock driven by firm effort and household enthusiasm." According to the potential differentiation of immediate reversibility from delayed reversibility presented in Table 1, confidence \( \Upsilon\) as hereby modelled refines such an affirmation by tying exogenous shocks in noise technology \( n\) to patterns of immediate reversibility, for non-negligible values of volition \( \gamma\), and exogenous shocks in transitory technology \( t\) to patterns of delayed reversibility, in the presence of all feasible values of volition \( \gamma\).

One can thus expect four principal scenarios: (i) immediate irreversibility, \( \varepsilon_{pt} \wedge (\gamma \gg 0)\); (ii) delayed irreversibility, \( \varepsilon_{pt} \wedge (\gamma \approx 0)\); (iii) immediate reversibility, \( \varepsilon_{nt} \wedge (\gamma \gg 0)\); (iv) delayed reversibility, \( \varepsilon_{nt} \wedge (\gamma \gg 0)\).

This work consequently merges the Keynesian view of confidence \( \Upsilon\) with the Pigovian view one, whereby long run responses in real economic activity to changes in confidence \( \Upsilon\) are indicative of shocks in economic fundamentals and short run ones are indicative of shifts in real consumption \( C\) and aggregate labour \( l\) due to confidence \( \Upsilon\) itself, which is a composite signal of technology processes regulated by a degree of volition \( \gamma\) (i.e. pure sentiment).

3. Household

3.1 Utility function. As per standard DSGE models, the expectation of the transfinite sum of household periodic utilities \( \mathbb{E}_t \sum_{t=0}^{\infty} u(C_t, l_t)\) is weighted at discount factor periodic product \( \beta^t\) (i.e. recursively), thereby representing its present or constant value: \( U(C_t, l_t) = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t u(C_t, l_t)\), in which discount factor \( \beta \in (0, 1) \subset \mathbb{R}_++\).

In even greater detail household periodic utility \( u(C_t, l_t)\) is modelled as an iso-elastic utility function in which confidence \( \Upsilon\) shifts real consumption \( C\), itself subjected to inter-temporal inseparability (i.e. habit formation): \( u(C_t, l_t) = \frac{\Psi(C_t-b(C_{t-1})^{1-\sigma_c})}{1-\sigma_c} - \frac{\Delta \pi^t}{1+\sigma_c}\), in which consumption habit, inter-temporal substitution inverse elasticity and labour inverse elasticity \( h, \sigma_c, \sigma_l \in \mathbb{R}_++\). Real consumption \( C\) and labour \( l\) respectively produce utility and disutility.

3.2 Household constraints. The macroeconomic household’s nominal budget constraint is the equality between household nominal demand and household nominal supply. In detail, household nominal demand is the sum of real consumption \( C\), real government bond \( b\), real taxation \( t\) and aggregate capital utilisation \( \Psi(u)K_{t-1}\), all weighted at price \( P_t\); \( PC_t + P_tb + Pttx + \Psi(u)K_{t-1}\), in which aggregate capital utilisation function \( \Psi(\cdot)\) is such that \( \Psi(1) = 0 \) and \( \Psi''(\cdot) \geq 0\).

Household nominal supply is the sum of aggregate labour \( l\), utilised aggregate capital \( uK_{t-1}\), lagged real government bond return \( r_{t-1}b_{t-1}\), household real profit \( \Pi_{2t}\) and real transfers \( tf_t\), respectively weighted at nominal wage \( \bar{W}_n\), nominal capital return \( \bar{R}_k\), lagged price \( P_{t-1}\) and twice price \( P_t\); \( W_nl_t + R_kuK_{t-1} + r_{t-1}b_{t-1} + \Pi_{2t} + tf_{t}\), in which endogenous variable \( r_{t-1}\) is the lagged nominal interest rate.

The macroeconomic household’s nominal budget constraint can therefore be written as follows: \( PC_t + P_tb + Pttx + \Psi(u)K_{t-1} = \bar{W}_nl_t + R_kuK_{t-1} + r_{t-1}b_{t-1} + \Pi_{2t} + tf_{t}\) → \( PC_t + B_t + TX_t = W_nl_t + [R_kuK_{t-1} - \Psi(u)K_{t-1}] + r_{t-1}b_{t-1} + \Pi_{2t} + TF_t\).

On division by price \( P_t\), the macroeconomic household’s real budget constraint can be accordingly written as follows: \( C_t + \frac{B_t}{P_t} + \frac{TX}{P_t} = W_il_t + [r_kuK_{t-1} - \Psi(u)K_{t-1}] + r_{t-1}b_{t-1} + \Pi_{2t} + \frac{TF_t}{P_t}\) → \( C_t + b_t + t = W_il_t + [r_kuK_{t-1} - \Psi(u)K_{t-1}] + r_{t-1}b_{t-1} + \Pi_{2t} + tf_{t}\), in which endogenous variables \( \pi_t = P_t^{-1}P_{t-1}\) are inflation, real wage and real capital return, respectively.

2https://en.wikipedia.org
Aggregate capital $K_t$ equals the sum of lagged aggregate capital $K_{t-1}$ weighted at $1 - \delta$ and investment parameter $i: K_t = (1 - \delta)K_{t-1} + i$, in which $i \in \mathbb{R}_{++}$ and capital depreciation rate $\delta \in (0, 1) \subset \mathbb{R}_{++}$, the equation in question being a law of motion for aggregate capital $K_t$.

The solvency supply constraint is such that the temporal limit of aggregate capital $K_t$ and nominal government bond $B_t$ weighted at discount factor periodic product $\beta^t$ and real shadow price $\lambda_{1t}$ is non-negative, that is, their supply to the macroeconomic household exacts that their priced present value be non-negative: $\lim_{t \to \infty} \mathbb{E}_t \beta^t \lambda_{1t} X_{t+1} \geq 0$, in which $\lambda_{1t} \in \mathbb{R}$ and $X = K, B$.

The insolvency demand constraint is analogously such that the temporal limit of aggregate capital $K_t$ and nominal government bond $B_t$ weighted at discount factor periodic product $\beta^t$ and shadow price $\lambda_{1t}$ is non-positive, that is, their demand by the macroeconomic household exacts that their priced present value be non-positive: $\lim_{t \to \infty} \mathbb{E}_t \beta^t \lambda_{1t} X_{t+1} \leq 0$, ceteris paribus.

By anti-symmetry said two constraints are such that the temporal limit of aggregate capital $K_t$ and nominal government bond $B_t$ weighted at discount factor periodic product $\beta^t$ is non-positive, that is, the transversality condition: $\lim_{t \to \infty} \mathbb{E}_t \beta^t \lambda_{1t} X_{t+1} = 0$, ceteris paribus.

3.3 Household optimisation problem. For non-negative arguments relative to the objective function, the macroeconomic household’s optimisation problem is thus the maximisation of the macroeconomic household’s utility function $U(C_t, l_t)$ subject to the macroeconomic household’s (i) real budget constraint and (ii) the transversality condition:

$$\max_{\{C_t, l_t, w_t, h_t\}} \sum_{t=0}^{\infty} \beta^t U(C_t, l_t) = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left\{ \frac{Y_t (C_t - hC_{t-1})^{1-\sigma_c} - 1}{1 - \sigma_c} - \frac{l_{t+1}^{1+\sigma_l}}{1 + \sigma_l} \right\}$$

s.t.

$\begin{align*}
C_t + b_t + tx_t &= W_t l_t + [rk_t u_t K_{t-1} - \Psi(u_t) K_{t-1}] + r n_{t-1} \pi^{-1} b_{t-1} + \Pi_2 t + tf_t, \\
\lim_{t \to \infty} \mathbb{E}_t \beta^t \lambda_{1t} X_{t+1} &= 0, \forall X = K, B, \\
C_t, l_t, u_t, b_t &\geq 0.
\end{align*}$

A necessary condition for optimal solutions is the invertibility of the objective function’s arguments, being hereby met by construction. A sufficient condition for optimal solutions is convexity of the objective function, being hereby translated into concavity of the macroeconomic household’s utility function $U(C_t, l_t)$, met by construction too, since the convexity requirement relative to the negative minimisation of a negative objective function corresponds to a concavity requirement relative to the positive maximisation of a positive objective function: $-\min [-U(C_t, l_t)] = \max U(C_t, l_t)$.

In detail, the macroeconomic household’s utility function is iso-elastic or one of constant relative risk aversion (CRRA) and is as such homogeneous of first degree, continuous, increasing in consumption, decreasing in labour and concave in both.

Such conditions speak to the renowned “Karush Kuhn Tucker (KKT) conditions” for the optimisation of standard non-linear programming problems.

The dynamic Lagrangian equation of said optimisation problem is such that discount factor periodic product $\beta^t$ weights the expectation of the constrained transfinite sum of household periodic utilities $\mathbb{E}_t \sum_{t=0}^{\infty} [u(C_t, l_t) + \lambda_{1t} (\cdot)]$, in which shadow price $\lambda_{1t}$ weights the macroeconomic household’s real budget constraint in turn:

$$L_{1t} = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left\{ \left[ \frac{Y_t (C_t - hC_{t-1})^{1-\sigma_c} - 1}{1 - \sigma_c} - \frac{l_{t+1}^{1+\sigma_l}}{1 + \sigma_l} \right] + \lambda_{1t} \left[ W_t l_t + r k_t u_t K_{t-1} - \Psi(u_t) K_{t-1} + r n_{t-1} \pi^{-1} b_{t-1} + \Pi_2 t + tf_t - (C_t + b_t + tx_t) \right] \right\}.$$

First order conditions (FOCs) are:

https://en.wikipedia.org
\[
\frac{\partial C_t}{\partial l_t} = 0 \overset{\beta}{\leftarrow} \beta^{t} \left[ \frac{\Gamma_t (1-\sigma_c) (C_t - hC_{t-1})^{-\sigma_c} (1)}{1-\sigma_c} - \lambda_{tt} (1) \right] = 0 \rightarrow \Upsilon_t (C_t - hC_{t-1})^{-\sigma_c} = \lambda_{tt};
\]

\[
\frac{\partial L_t}{\partial l_t} = 0 \overset{\beta}{\leftarrow} \beta^{t} \left[ \frac{- (1 + \sigma_t) \Gamma_t l_t}{1 + \sigma_t} + \lambda_{tt} W_t \right] = 0 \rightarrow \lambda_{tt} W_t = l_t^{\sigma_t};
\]

\[
\frac{\partial L_t}{\partial u_t} = 0 \overset{\beta'}{\leftarrow} \beta' \lambda_{tt} [r_k (1) K_{t-1} - \Psi' (u_t) (1) K_{t-1}] = 0 \rightarrow r_k = \Psi' (u_t);
\]

\[
\frac{\partial L_t}{\partial b_t} = 0 \overset{\beta'}{\leftarrow} \beta' \lambda_{tt} (-1) + E_t \beta^{t+1} \lambda_{tt+1} r_n \pi_{t+1}^{-1} (1) = 0 \rightarrow -\lambda_{tt} + E_t \beta \lambda_{tt+1} r_n \pi_{t+1}^{-1} = 0 \rightarrow E_t \beta \lambda_{tt+1} r_n \pi_{t+1}^{-1} = \lambda_{tt},
\]

recalling that \( E_t x_t = x_t \), in which \( x \) is any endogenous variable.

**3.4 Household laws of motion.** As a consequence, there firstly arises an indirect equation for stochastic discount factor \( E_t \beta \lambda_{tt}^{-1} \lambda_{tt+j} \):

\[
E_t \beta \lambda_{tt+1} r_n \pi_{t+1}^{-1} = \lambda_{tt} \rightarrow E_t \pi_{t+1} = E_t \lambda_{tt}^{-1} \lambda_{tt+1} r_n \rightarrow E_t \pi_{t+j} = E_t \beta^{j} \lambda_{tt}^{-1} \lambda_{tt+j} r_n.
\]

There subsequently arise the following laws of motion:

\[
\Upsilon_t (C_t - hC_{t-1})^{-\sigma_c} = \lambda_{tt} \text{ and } \lambda_{tt} W_t = l_t^{\sigma_t} \rightarrow \Upsilon_t (C_t - hC_{t-1})^{-\sigma_c} W_t = l_t^{\sigma_t}; \rightarrow W_t = \Upsilon_t^{-1} (C_t - hC_{t-1})^{\sigma_c} l_t^{\sigma_t} \text{ (real wage or aggregate labour supply)};
\]

\[
\Upsilon_t (C_t - hC_{t-1})^{-\sigma_c} = \lambda_{tt} \text{ and } E_t \beta \lambda_{tt+1} r_n \pi_{t+1}^{-1} = \lambda_{tt} \rightarrow \Upsilon_t (C_t - hC_{t-1})^{-\sigma_c} = E_t \beta \left[ \Upsilon_{t+1} (C_{t+1} - hC_{t+1})^{-\sigma_c} \right] r_n \pi_{t+1}^{-1} \text{ (real consumption or consumption Euler equation)};
\]

\( r_k = \Psi' (u_t) \) (real capital return).

**4. Retailer**

**4.1 Retail nominal profit.** As per standard NK-DSGE models, nominal profit \( P_t \Pi_{tt} \) proper to the macroeconomic retailer or final goods or services macroeconomic producer equals the difference between retail nominal marginal revenue \( P_t Y_t \) and retail nominal marginal cost \( \int_0^t P_t Y_t di \), being a continuum of priced wholesale real outputs in relation to their macroeconomic producers: \( P_t \Pi_{tt} = P_t Y_t - \int_0^t P_t Y_t di \).

Wholesale aggregate real output \( Y_t \) equals a continuum of wholesale real outputs \( \int_0^t Y_t di \) exhibiting constant elasticity of substitution\(^4\) (CES): \( Y_t = \left( \int_0^t Y_t^\theta di \right)^{\frac{1}{\theta}} \), in which macroeconomic producer \( i \in [0, 1] \subset \mathbb{R}_+ \) and substitution elasticity \( \theta \in (-\infty, 1] \subset \mathbb{R} \) such that

\[
\theta \begin{cases} 
1, & \text{perfect substitutes} \\
0, & \text{imperfect complements} \\
\rightarrow -\infty, & \text{perfect complements}
\end{cases}
\]

relative to the continuum of wholesale real outputs \( \int_0^t Y_t di \).

**4.2 Retail optimisation problem.** For non-negative arguments relative to the objective function, the optimisation problem of the macroeconomic retailer is thus the maximisation of retail nominal profit \( P_t \Pi_{tt} \) subject to wholesale aggregate real output \( Y_t \):

\(^4\)https://en.wikipedia.org
The necessary condition of objective function argument invertibility and the sufficient condition of objective function convexity for optimal solutions are analogously met by construction. The Lagrangian equation of said optimisation problem is such that retail nominal profit $P_t \Pi_{it}$ is optimised seeking retail optimal input or wholesale optimal real output $Y_{it}$ in the face of perfect competition:

$$\max_{\{Y_{it}\}_{t=0}^\infty} P_t \Pi_{it} = P_t Y_t - \int_0^1 P_t Y_{it} di \text{ s.t.}$$

$$Y_t = \left(\int_0^1 Y_{it}^\frac{1}{\theta} di\right)^\theta,$$

$$Y_{it} \geq 0.$$
5. Wholesaler

5.1 Price rigidity. As per Calvo [9], in period \( t \) a random \( \xi \) fraction of macroeconomic wholesalers or intermediate goods or services macroeconomic producers fails to adjust wholesale real output price \( P_{it} \), indexing it to lagged inflation \( \pi_{i-1} \) at parameter \( \tau : P_{it} = \pi_{i-1}^{\tau} P_{it-1} \).

Said random \( \xi \) fraction lies in an open real interval between 0 and 1; accordingly, inflation indexation \( \tau \) lies in a closed real interval between 0 and 1 : \( \xi \in (0, 1) \subset \mathbb{R}^+ ; \tau \in [0, 1] \subset \mathbb{R}^+ \). In period \( t \) the other \( 1 - \xi \) fraction of macroeconomic wholesalers adjusts wholesale real output price \( P_{it} \) with success: \( P_{it} = P_{it}^* \). In detail,

\[
P_t = \left( \int_0^1 P_{it}^{\frac{1}{1-\xi}} di \right)^{1-\theta} = \left[ \int_0^{1-\xi} (P_t^*)^{\frac{1}{1-\xi}} di + \int_{1-\xi}^1 (P_{it})^{\frac{1}{1-\xi}} di \right]^{1-\theta} = \left[ \int_0^{1-\xi} (P_t^*)^{\frac{1}{1-\xi}} di + \int_{1-\xi}^1 (\pi_{i-1}^{\tau} P_{it-1})^{\frac{1}{1-\xi}} di \right]^{1-\theta} = \left[ \int_0^{1-\xi} (P_t^*)^{\frac{1}{1-\xi}} di + \xi (\pi_{i-1}^{\tau} P_{it-1})^{\frac{1}{1-\xi}} \right]^{1-\theta} \rightarrow P_t = \left[ i (P_t^*)^{\frac{1}{1-\theta}} \int_0^{1-\xi} (\pi_{i-1}^{\tau} P_{it-1})^{\frac{1}{1-\theta}} di + \xi (\pi_{i-1}^{\tau} P_{it-1})^{\frac{1}{1-\theta}} \right]^{1-\theta} = \left[ (1 - \xi) (P_t^*)^{\frac{1}{1-\theta}} + \xi (\pi_{i-1}^{\tau} P_{it-1})^{\frac{1}{1-\theta}} \right]^{1-\theta}
\]

in which (i) \( \int_0^{1-\xi} (\pi_{i-1}^{\tau} P_{it-1})^{\frac{1}{1-\theta}} di = \xi \left( \pi_{i-1}^{\tau} \int_0^1 P_{it-1} di \right)^{\frac{1}{1-\theta}} \) on account of random wholesale real output price adjustment and a continuum of wholesalers and (ii) \( P_{it}^{\frac{1}{1-\xi}} = \int_0^1 P_{it}^{\frac{1}{1-\xi}} di \) on account of \( P_{it}^{\frac{1}{1-\xi}} = \int_0^1 P_{it}^{\frac{1}{1-\xi}} di \), the equation in question being a law of motion for aggregate price \( P_t \). It follows that a macroeconomic wholesaler which adjusts its price \( P_{it} \) in period \( t \) and which cannot adjust it until period \( t+j \), for any positive natural \( j \), sets it throughout as follows: \( \forall j \in \mathbb{N}_+ \),

\[
E_t P_{it+1} = \pi_i^t P_{it}^*, \\
E_t P_{it+2} = E_t \pi_{i+1}^t P_{it+1} = E_t \pi_{i+1}^t \pi_i^t P_{it}^*, \\
\vdots \\
E_t P_{it+j} = E_t \pi_{i+j-1}^t \pi_{i+j-2}^t \cdots \pi_i^t P_{it}^* = E_t \prod_{k=0}^{j-1} \pi_{i+k}^t P_{it}^*.
\]

5.2 Wholesale optimisation problem. Wholesale nominal profit \( P_{it} \Pi_{it} \) equals the difference between wholesale nominal marginal revenue \( P_{it} Y_{it} \) and wholesale nominal marginal cost \( \Phi_t \) : \( P_{it} \Pi_{it} = (P_{it} - \Phi_t) Y_{it} \).

On division by price \( P_t \), there consequently follow wholesale real profit \( \Pi_{it}^r = (P_{it} - \Phi_t) P_{it}^{-1} Y_{it} \) and future wholesale real sub-profit \( E_t \sum_{j=0}^\infty (P_{it+j} - \Phi_{it+j}) P_{it+j}^{-1} Y_{it+j} \), which on being weighted at stochastic discount factor \( E_t \beta^j \lambda_{it+j} \lambda_{it+j}^{-1} \) and fraction periodic product \( \xi^j \), on account of the price adjustment failure throughout \( j \) periods on the part of the random \( \xi \) fraction of macroeconomic wholesalers, gives rise to future wholesale real profit \( \Pi_{it}^r = E_t \sum_{j=0}^\infty (\xi \beta^j \lambda_{it+j} \lambda_{it+j}^{-1}) (P_{it+j} - \Phi_{it+j}) P_{it+j}^{-1} Y_{it+j} \).

For non-negative arguments relative to the objective function, the optimisation problem of the macroeconomic wholesaler is thus the maximisation of future wholesale real sub-profit \( E_t \sum_{j=0}^\infty (P_{it+j} - \Phi_{it+j}) P_{it+j}^{-1} Y_{it+j} \), weighted at stochastic discount factor \( E_t \beta^j \lambda_{it+j} \lambda_{it+j}^{-1} \) subject to future wholesale real output demand \( E_t Y_{it+j} \), provided wholesale real output price \( P_{it} \) have not been adjusted for \( j \) periods, by means of fraction periodic product \( \xi^j \) :
\[
\max_{\{P_{it}\}_{t=0}^\infty} \Pi_{it} = \mathbb{E}_t \sum_{j=0}^\infty (\xi \beta)^j (\lambda_{1t}^{-1} \lambda_{1t+j}) (P_{it+j} - \Phi_{it+j}) P_{it+j}^{-1} Y_{it+j} = \\
= \mathbb{E}_t \sum_{j=0}^\infty (\xi \beta)^j (\lambda_{1t}^{-1} \lambda_{1t+j}) \left[ P_{it+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^t P_{it}^* - \phi_{it+j} \right] Y_{it+j} \text{ s.t.} \\
\mathbb{E}_t Y_{it+j} = \mathbb{E}_t Y_{it+j} \left( P_{it+j}^{-1} P_{it+j}^\alpha \right)^{\frac{\theta}{1-\theta}} = \mathbb{E}_t Y_{it+j} \left( P_{it+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^t P_{it}^* \right)^{\frac{\theta}{1-\theta}}, \\
P_{it}^* \geq 0,
\]

in which (i) nominal marginal cost \( \Phi_t = P_t \phi_t \), (ii) \( \mathbb{E}_t Y_{it+j} = \mathbb{E}_t Y_{it+j} \left( P_{it+j}^{-1} P_{it+j}^\alpha \right)^{\frac{\theta}{1-\theta}} \) on account of \( Y_{it} = Y_t \left( P_{it}^{-1} P_{it}^\alpha \right)^{\frac{\theta}{1-\theta}} \) and (iii) \( \mathbb{E}_t Y_{it+j} = \mathbb{E}_t \prod_{k=0}^{j-1} \pi_{t+k}^t P_{it}^* \).

The necessary and sufficient conditions for optimal solutions are again met by construction. The dynamic Lagrangian equation of said optimisation problem is such that wholesale real profit \( \Pi_{it} \) is optimised seeking optimal wholesale real output price \( P_{it}^* \) in the face of monopolistic competition:

\[
\mathcal{L}_{it} = \mathbb{E}_t \sum_{j=0}^\infty (\xi \beta)^j (\lambda_{1t}^{-1} \lambda_{1t+j}) \left[ P_{it+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^t P_{it}^* - \phi_{it+j} \right] Y_{it+j} \left( P_{it+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^t P_{it}^* \right)^{\frac{\theta}{1-\theta}} = \\
= \mathbb{E}_t \sum_{j=0}^\infty (\xi \beta)^j (\lambda_{1t}^{-1} \lambda_{1t+j}) \left[ \left( P_{it+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^t P_{it}^* \right)^{\frac{\theta}{1-\theta}} \right] Y_{it+j} = \\
\text{The FOC is}
\]

\[
\frac{\partial \mathcal{L}_{it}}{\partial P_{it}^*} = 0 \iff \mathbb{E}_t \sum_{j=0}^\infty (\xi \beta)^j (\lambda_{1t}^{-1} \lambda_{1t+j}) \left( \frac{1}{1-\theta} \right) \left( P_{it}^* \right)^{\frac{\theta}{1-\theta}} \left( P_{it+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^t P_{it}^* \right)^{\frac{1}{1-\theta}} \\
- \phi_{it+j} \left( \frac{\theta}{1-\theta} \right) \left( P_{it}^* \right)^{\frac{\theta}{1-\theta}-1} \left( P_{it+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^t P_{it}^* \right)^{\frac{\theta}{1-\theta}} Y_{it+j} = 0 \iff \\
\mathbb{E}_t \sum_{j=0}^\infty (\xi \beta)^j (\lambda_{1t}^{-1} \lambda_{1t+j}) \left( \frac{1}{1-\theta} \right) \left( P_{it}^* \right)^{\frac{\theta}{1-\theta}} \left( P_{it+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^t P_{it}^* \right)^{\frac{1}{1-\theta}} Y_{it+j} = \\
= \mathbb{E}_t \sum_{j=0}^\infty (\xi \beta)^j (\lambda_{1t}^{-1} \lambda_{1t+j}) \phi_{it+j} \left( \frac{\theta}{1-\theta} \right) \left( P_{it}^* \right)^{\frac{\theta}{1-\theta}-1} \left( P_{it+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^t P_{it}^* \right)^{\frac{\theta}{1-\theta}} Y_{it+j} \rightarrow \\
= \mathbb{E}_t \sum_{j=0}^\infty (\xi \beta)^j (\lambda_{1t}^{-1} \lambda_{1t+j}) \phi_{it+j} \left( \frac{\theta}{1-\theta} \right) \left( P_{it}^* \right)^{\frac{\theta}{1-\theta}-1} \left( P_{it+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^t P_{it}^* \right)^{\frac{\theta}{1-\theta}} Y_{it+j} = \\
= \mathbb{E}_t \sum_{j=0}^\infty (\xi \beta)^j (\lambda_{1t}^{-1} \lambda_{1t+j}) \phi_{it+j} \left( \frac{\theta}{1-\theta} \right) \left( P_{it}^* \right)^{\frac{\theta}{1-\theta}-1} \left( P_{it+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^t P_{it}^* \right)^{\frac{\theta}{1-\theta}} Y_{it+j} \rightarrow
\]
\[ \rightarrow E_t \sum_{j=0}^{\infty} (\xi \beta)^j (\lambda_{t_{j+1}}^{-1} \lambda_{t_{j+2}}) \left( \frac{1}{1-\theta} \right) \left( P_{t+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^r \right) \frac{\sigma}{\phi} Y_{t+j} = \]

\[ = E_t \sum_{j=0}^{\infty} (\xi \beta)^j (\lambda_{t_{j+1}}^{-1} \lambda_{t_{j+2}}) \phi_{t+j} \left( \frac{\theta}{1-\theta} \right) \left( P_{t+j}^{-1} \prod_{k=0}^{j-1} \pi_{t+k}^r \right) \frac{\sigma}{\phi} Y_{t+j} \rightarrow \]

\[ \rightarrow P_{it}^{-1} P_{it} = \frac{E_t \sum_{j=0}^{\infty} (\xi \beta)^j (\lambda_{t_{j+1}}^{-1} \lambda_{t_{j+2}}) \phi_{t+j} \left( \frac{\theta}{1-\theta} \right) \left( \prod_{k=0}^{j-1} \pi_{t+k+1}^{-1} \pi_{t+k+1}^r \right) \frac{\sigma}{\phi} Y_{t+j} \rightarrow \]

\[ \rightarrow P_{it}^{-1} P_{it} = \frac{E_t \sum_{j=0}^{\infty} (\xi \beta)^j (\lambda_{t_{j+1}}^{-1} \lambda_{t_{j+2}}) \phi_{t+j} \left( \frac{\theta}{1-\theta} \right) \left( \prod_{k=0}^{j-1} \pi_{t+k+1}^{-1} \pi_{t+k+1}^r \right) \frac{\sigma}{\phi} Y_{t+j} \rightarrow \]

\[ \rightarrow P_t = \left[ (P_{t}^{-1}) \frac{\sigma}{\phi} \right]^{-1} \left( P_{t}^{-1} \right) \frac{\sigma}{\phi} = P_t \frac{1}{1-\theta} P_t \frac{\sigma}{\phi}. \] It follows that optimal adjusted price \( p_{it}^* \) is

\[ p_{it}^* = \frac{A_t}{B_t}, \]

in which \( A_t = \lambda_{t_{j+1}} \phi_{t+j} Y_t + \xi \beta E_t \left( \frac{\pi_{t}}{\pi_{t+1}} \right) \frac{\sigma}{\phi} A_{t+1} \) and \( B_t = \lambda_{t_{j+1}} Y_t + \xi \beta E_t \left( \frac{\pi_{t}}{\pi_{t+1}} \right) \frac{\sigma}{\phi} B_{t+1} \) such that if \( \xi = 0 \) then \( p_{it}^* = \theta \phi_t \). In detail,

\[ A_t = E_t \sum_{j=0}^{\infty} (\xi \beta)^j \lambda_{t_{j+1}} \phi_{t+j} \theta \left( \prod_{k=0}^{j-1} \pi_{t+k+1}^{-1} \pi_{t+k}^r \right) \frac{\sigma}{\phi} Y_{t+j} = \]

\[ = E_t \sum_{j=0}^{\infty} (\xi \beta)^j \lambda_{t_{j+1}} \phi_{t+j} \theta \left[ (\pi_{t+1}^{-1} \pi_{t}) \cdots (\pi_{t+j-1}^{-1} \pi_{t+j-1}) \right] \frac{\sigma}{\phi} Y_{t+j} = \]

\[ = \lambda_{t_{j+1}} \phi_{t+j} Y_t + E_t \xi \beta \lambda_{t_{j+1}} \phi_{t+1} \theta (\pi_{t+1}^{-1} \pi_{t}) \frac{\sigma}{\phi} Y_{t+1} + E_t (\xi \beta)^2 \lambda_{t_{j+1}} \phi_{t+1} \theta (\pi_{t+1}^{-1} \pi_{t}) \frac{\sigma}{\phi} Y_{t+2} + \ldots \]

matches
\[ A_t = \lambda_{1t} \phi_t y_t + \xi \beta e_t \left( \frac{\pi_t^l}{\pi_{t+1}} \right)^{\frac{\phi}{\eta}} A_{t+1} = \lambda_{1t} \phi_t y_t + \xi \beta e_t \left( \frac{\pi_t^l}{\pi_{t+1}} \right)^{\frac{\phi}{\eta}} \left[ \lambda_{1t+1} \phi_{t+1} y_{t+1} + \xi \beta e_t \left( \frac{\pi_{t+1}^l}{\pi_{t+2}} \right)^{\frac{\phi}{\eta}} A_{t+2} \right] = \lambda_{1t} \phi_t y_t + \xi \beta e_t \left( \frac{\pi_t^l}{\pi_{t+1}} \right)^{\frac{\phi}{\eta}} \left\{ \lambda_{1t+1} \phi_{t+1} y_{t+1} + \xi \beta e_t \left( \frac{\pi_{t+1}^l}{\pi_{t+2}} \right)^{\frac{\phi}{\eta}} A_{t+3} \right\} \]

and

\[ B_t = E_t \sum_{j=0}^{\infty} (\xi \beta)^j \lambda_{1t+j} \left( \prod_{k=0}^{j-1} \frac{\pi_t^{-1}}{\pi_{t+k}} \right)^{\frac{1}{\lambda}} Y_{t+j} = E_t \sum_{j=0}^{\infty} (\xi \beta)^j \lambda_{1t+j} \left[ \left( \frac{\pi_t^{-1}}{\pi_{t+1}} \right) \cdots \left( \frac{\pi_{t+j}^{-1}}{\pi_{t+j-1}} \right) \right]^{\frac{1}{\lambda}} Y_{t+j} = \lambda_{1t} Y_t + E_t \xi \beta \lambda_{1t+1} \left( \frac{\pi_t^{-1}}{\pi_{t+1}} \right)^{\frac{1}{\lambda}} Y_{t+1} + E_t (\xi \beta)^2 \lambda_{1t+2} \left[ \left( \frac{\pi_t^{-1}}{\pi_{t+1}} \right) \left( \frac{\pi_{t+1}^{-1}}{\pi_{t+2}} \right)^{\frac{1}{\lambda}} Y_{t+2} + \ldots \right] \]

matches

\[ B_t = \lambda_{1t} Y_t + \xi \beta e_t \left( \frac{\pi_t^l}{\pi_{t+1}} \right)^{\frac{1}{\lambda}} B_{t+1} = \lambda_{1t} Y_t + \xi \beta e_t \left( \frac{\pi_t^l}{\pi_{t+1}} \right)^{\frac{1}{\lambda}} \left[ \lambda_{1t+1} Y_{t+1} + \xi \beta e_t \left( \frac{\pi_{t+1}^l}{\pi_{t+2}} \right)^{\frac{1}{\lambda}} B_{t+2} \right] = \lambda_{1t} Y_t + \xi \beta e_t \left( \frac{\pi_t^l}{\pi_{t+1}} \right)^{\frac{1}{\lambda}} \left\{ \lambda_{1t+1} Y_{t+1} + \xi \beta e_t \left( \frac{\pi_{t+1}^l}{\pi_{t+2}} \right)^{\frac{1}{\lambda}} B_{t+3} \right\}, \]

since if \( j = 0 \) then \( \prod_{k=0}^{j-1} \pi_t^{-1} \pi_{t+k} = \prod_{k=0}^{j-1} \pi_{t+k} \pi_{t+k}^{-1} = \pi_t^{-1} \pi_{t-1} = \pi_t^{-1} \) and if \( j = 1 \) then

\[ \prod_{k=0}^{j-1} \pi_t^{-1} \pi_{t+k} = \prod_{k=0}^{j-1} \pi_{t+k} \pi_{t+k}^{-1} = \pi_t^{-1} \pi_t. \]

6. Real production

6.1 Production function and real production cost. Along the lines of a neo-classical growth model with stochastic technology, wholesale real output \( Y_t \) is equal to a CES production function of imperfect complements\(^5\), being utilised capital \( K_{it-1} \) and labour \( l_{it} \), shifted by real production technology \( a_t \):

\[ Y_{it} = a_t (u_t K_{it-1})^{\alpha} (Y_{it} l_{it})^{1-\alpha} = a_t K_{it-1}^{\alpha} (Y_{it} l_{it})^{1-\alpha} \ 	ext{(real production or production function)}, \]

in which (i) \( K_{it-1} = u_t K_{it-1} \) and (ii) capital in output share \( \alpha \in [0, 1] \subset \mathbb{R}_+ \). Real production technology \( a_t \) thus be said to be factor augmenting or “Hicks neutral”\(^6\). Confidence \( Y_t \) shifts labour \( l_{it} \) and can similarly be said to be labour augmenting or “Harrod neutral”.

Real production cost \( \Gamma_{it} \) equals the sum of labour \( l_{it} \) weighted at real wage \( W_t \) and utilised capital \( K_{it-1} \) weighted at real capital return \( r_k_t \) : \( \Gamma_{it} = W_t l_{it} + r_k_t K_{it-1} \).

6.2 Real production optimisation problem. For non-negative arguments relative to the objective function, the second optimisation problem of the macroeconomic wholesaler is thus the minimisation of real production cost \( \Gamma_{it} \) subject to real production \( Y_{it} \):

---

\(^5\)https://en.wikipedia.org
\(^6\)https://en.wikipedia.org
\[
\begin{align*}
\min_{\{l_t, K_{it-1}\}} \Gamma_t &= W_t l_t + r_k K_{it-1} \quad \text{s.t.} \\
Y_t &= a_t K_{it-1}^{\alpha} (Y_{it})^{1-\alpha}, \\
l_t, K_{it-1} &\geq 0.
\end{align*}
\]

The necessary and sufficient conditions for optimal solutions are afresh met by construction. The Lagrangian equation of said optimisation problem is such that real production cost \(\Gamma_t\) is optimised seeking wholesale real production optimal inputs, being labour \(l_t\) and utilised capital \(K_{it-1}\), in the face of perfect competition, in which real marginal cost \(\phi_t\) weights real production \(Y_t\):

\[
L_t = [W_t l_t + r_k K_{it-1}] + \phi_t \left[ Y_t - a_t K_{it-1}^{\alpha} (Y_{it})^{1-\alpha} \right].
\]

FOCs are:

\[
\frac{\partial L_t}{\partial l_t} = 0 \iff W_t + \phi_t \left[ -a_t K_{it-1}^{\alpha} (1 - \alpha) Y_t (Y_{it})^{-\alpha} \right] = 0 \implies \]

\[
W_t = \phi_t a_t K_{it-1}^{\alpha} (1 - \alpha) Y_t (Y_{it})^{-\alpha};
\]

\[
\frac{\partial L_t}{\partial K_{it-1}} = 0 \iff r_k + \phi_t \left[ -a_t \alpha K_{it-1}^{\alpha} (1 - \alpha) Y_t (Y_{it})^{-\alpha} \right] = 0 \implies \]

\[
r_k = \phi_t a_t K_{it-1}^{\alpha} (1 - \alpha) Y_t (Y_{it})^{-\alpha},
\]

from which there follow

\[
\begin{align*}
r_k^{-1} W_t &= \left[ \phi_t a_t K_{it-1}^{\alpha} (1 - \alpha) Y_t (Y_{it})^{-\alpha} \right]^{-1} \phi_t a_t K_{it-1}^{\alpha} (1 - \alpha) Y_t (Y_{it})^{-\alpha} = (a_l l_t)^{-1} (1 - \alpha) K_{it-1}^{-1} \\

t_t &= (W_t \alpha)^{-1} (1 - \alpha) r_k K_{it-1}^{-1} \ (\text{labour or labour demand}), \\
\end{align*}
\]

\[
W_t = (1 - \alpha) a_t K_{it-1}^{\alpha} (1 - \alpha) Y_t (Y_{it})^{-\alpha} Y_t (Y_{it})^{-1} (Y_{it})^{-1} \ (\text{average wage}) \text{ and}
\]

\[
r_k = \alpha a_t K_{it-1}^{\alpha} (1 - \alpha) Y_t (Y_{it})^{-\alpha} K_{it-1}^{-1} = \alpha K_{it-1}^{\alpha} Y_t \phi_t (\text{average capital return}),
\]

in turn implying

\[
Y_{it} = (1 - \alpha) W_t^{-1} Y_t \phi_t Y_t \text{ and}
\]

\[
K_{it-1} = \alpha r_k^{-1} Y_t \phi_t
\]

such that

\[
Y_{it} = a_t K_{it-1}^{\alpha} (Y_{it})^{1-\alpha} = a_t (\alpha r_k^{-1} Y_t \phi_t)^{\alpha} [(1 - \alpha) W_t^{-1} Y_t \phi_t Y_t]^{1-\alpha} = \]

\[
= \alpha \ (1 - \alpha)^{1-\alpha} a_t Y_t \phi_t Y_t^{1-\alpha} r_k^{-\alpha} W_t^{-1} \implies \\
\]

\[
\phi_t = \alpha^{-\alpha} (1 - \alpha)^{-1} a_t Y_t^{1-\alpha} r_k^{-\alpha} W_t^{-1},
\]

being a law of motion for real marginal cost \(\phi_t\).

7. Central Bank and Treasury

7.1 Nominal interest rate. Nominal interest rate \(r_n\) is set according to a “Taylor rule”\(^7\):

\[^7\text{https://en.wikipedia.org}\]
\[
\frac{\left( r_{n} \right)}{r_{n}} = \left( \frac{r_{n} - 1}{r_{n}} \right)^{\rho_{n}} \left[ \left( \frac{\pi_{1}/\pi}{\pi_{T}/\pi} \right)^{\phi_{r}} \left( \frac{\pi_{1}/\pi}{\pi_{T}/\pi} \right)^{\phi_{s}} \left( \frac{Y_{t}/Y}{Y_{t-1}/Y} \right)^{\phi_{b}} \right]^{1-\rho_{n}} e^{\varphi},
\]

in which interest rate persistence, inflation coefficient, inflation gap coefficient, output coefficient and output gap coefficient \(\rho_{n}, \phi_{r}, \phi_{s}, \phi_{b}, \phi_{g} \in \mathbb{R}_{+} \) and monetary policy parameter \(\varphi \in \mathbb{R}\).

All endogenous variables are divided by their values at the steady state, thereby representing non-linear deviations from it. The deviation of nominal interest rate \(r_{n}\) from its steady state is therefore the weighted product of (i) exponentiated monetary policy parameter \(e^{\varphi}\), (ii) the deviation of lagged nominal interest rate \(r_{n-1}\) from its steady state and (iii) that of a product of inflation \(\pi_{t}\), aggregate real output \(Y_{t}\) and their respective gaps \(\pi_{t-1}\) and \(Y_{t-1}\). Such a “Taylor rule” is a law of motion for nominal interest rate \(r_{n}\).

7.2 Public finance. The treasury’s nominal budget constraint is the equality between its nominal demand and its nominal supply. In detail, the treasury’s nominal demand is the sum of real fiscal policy or government expenditure parameter \(g\) weighted at price \(P_{t}\), lagged real government bond return \(r_{n-1}b_{t-1}\) weighted at lagged price \(P_{t-1}\) and real transfers \(tf\) weighted at price \(P_{t}\) : \(P_{tg} + r_{n-1}P_{t-1}b_{t-1} + Pt tf\), in which \(g \in \mathbb{R}_{++}\).

The treasury’s nominal supply is the sum of real government bond return \(b_{t}\) and real taxation \(tx\), both weighted at price \(P_{t}\) : \(P_{tb_{t}} + P_{t}tx\). The treasury’s nominal budget constraint can therefore be written as follows: \(P_{tg} + r_{n-1}P_{t-1}b_{t-1} + P_{t}tf = P_{tb_{t}} + P_{t}tx\).

On division by price \(P_{t}\), the treasury’s real budget constraint can be accordingly written as follows: \(g + \frac{r_{n-1}P_{t-1}b_{t-1}}{P_{t}} + \frac{P_{t}tf}{P_{t}} = \frac{P_{tb_{t}}}{P_{t}} + \frac{P_{t}tx}{P_{t}}\). Such in turn implies an equation for real government bond \(b_{t} : b_{t} = g + r_{n-1}P_{t-1}^{-1}b_{t-1} + tf_{t} - tx_{t}\).

8. Aggregation

8.1 Household real profit. Owing to market clearing, aggregate labour \(l_{t}\) equals a continuum of labour \(\int_{0}^{1} l_{it} di\) and utilised aggregate capital \(K_{t-1}\) equals a continuum of utilised capital \(\int_{0}^{1} K_{it-1} di\) : \(l_{t} = \int_{0}^{1} l_{it} di\) and \(K_{t-1} = \int_{0}^{1} K_{it-1} di\). Such implies the following aggregation:

\[
\int_{0}^{1} l_{it} di = \alpha^{-1} (1 - \alpha) W_{t}^{-1} rk_{it} l_{i} \int_{0}^{1} K_{it-1} di \rightarrow
\]

\[
l_{t} = \alpha^{-1} (1 - \alpha) W_{t}^{-1} rk_{it} K_{i-1},
\]

being a law of motion for aggregate labour or aggregate labour demand \(l_{t}\). Household nominal profit \(P_{t} \Pi_{2t}\) is consequently aggregated as follows:

\[
P_{t} \Pi_{2t} = \int_{0}^{1} \left[ P_{t}Y_{it} - W_{n}l_{it} - Rk_{i}K_{it-1} \right] di \rightarrow
\]

\[
\Pi_{2t} = \int_{0}^{1} \left[ P_{t}^{-1} P_{it} Y_{it} - W_{it}l_{it} - rk_{it} K_{it-1} \right] di \rightarrow
\]

\[
\Pi_{2t} = \int_{0}^{1} \left[ P_{t}^{-1} P_{it} Y_{it} \left( P_{t}^{-1} P_{it} \right)^{\gamma} - W_{it}l_{it} - rk_{it} K_{it-1} \right] di =
\]

\[
= \int_{0}^{1} \left[ P_{t}^{-1} \left( \gamma + \gamma \right) P_{it}^{-1} Y_{it} - W_{it}l_{it} - rk_{it} K_{it-1} \right] di \rightarrow
\]

\[
\Pi_{2t} = P_{t}^{-1} Y_{it} \int_{0}^{1} P_{it}^{-1} di - W_{it} \int_{0}^{1} l_{it} di - rk_{it} K_{it-1} \rightarrow
\]

\[
\Pi_{2t} = P_{t}^{-1} Y_{it} P_{t}^{-1} - W_{it}l_{it} - rk_{it} K_{it-1} \rightarrow
\]

\[
\Pi_{2t} = Y_{it} - W_{it}l_{it} - rk_{it} K_{it-1} = Y_{it} - W_{it}l_{it} - rk_{it} u_{kt-1} (\text{household real profit}),
\]
in which (i) $Y_{it} = Y_t \left( P_t^{-1} P_{it} \right)^{\frac{\theta}{\theta - 1}}$, (ii) $l_t = \int_0^1 l_{it} \, dt$, (iii) $\dot{K}_{it-1} = \int_0^1 \dot{K}_{it-1} \, dt$ and (iv) $P_{it}^{\frac{\theta}{\theta - 1}} = \int_0^1 P_{it}^{\frac{\theta}{\theta - 1}} \, dt$.

8.2 Aggregate capital utilisation. The substitution of real government bond $b_t$ and household real profit $\Pi_{2t}$ into the macroeconomic household’s real budget constraint gives rise to a law of motion for aggregate capital utilisation $\Psi(u_t)K_{t-1}$ or aggregate resources $Y_t$:

$$C_t + b_t + tx_t = W_t l_t \left[ r_k(u_t)K_{t-1} + r_{n_{it}} - \Psi(u_t)K_{t-1} \right] + (Y_t - W_t l_t - r_k u_t K_{t-1}) + t_f \rightarrow$$
$$\rightarrow C_t + g + \left[ r_n - \Psi(u_t)K_{t-1} \right] + \left( Y_t - W_t l_t - r_k u_t K_{t-1} \right) + t_f \rightarrow$$

$$Y_t = C_t + g + \Psi(u_t)K_{t-1},$$

in which (i) $b_t = g + r_{n_{it}} - \Psi(u_t)K_{t-1} + t_f - t x_t$ and (ii) $\Pi_{2t} = Y_t - W_t l_t - r_k u_t K_{t-1}$.

8.3 Aggregate real production. Real production $Y_{it}$ is analogously aggregated in the following manner:

$$Y_{it} = a_t \dot{K}_{it-1} \left( Y_{it} l_{it} \right)^{1-\alpha} \rightarrow$$
$$\rightarrow \int_0^1 Y_{it} \, dt = \int_0^1 a_t \dot{K}_{it-1} \left( Y_{it} l_{it} \right)^{1-\alpha} \, dt \rightarrow$$
$$\rightarrow \int_0^1 Y_t \left( P_t^{-1} P_{it} \right)^{\frac{\theta}{\theta - 1}} \, dt = a_t Y_t^{1-\alpha} \int_0^1 \dot{K}_{it-1} l_{it}^{1-\alpha} \, dt \rightarrow$$
$$\rightarrow Y_t \int_0^1 \left( P_t^{-1} P_{it} \right)^{\frac{\theta}{\theta - 1}} \, dt = Y_t p_{di} = a_t \dot{K}_{it-1} \left( Y_{it} l_{it} \right)^{1-\alpha} \rightarrow$$

$$Y_t = p_{di}^{-1} a_t \dot{K}_{it-1} \left( Y_{it} l_{it} \right)^{1-\alpha},$$

in which (i) $Y_{it} = Y_t \left( P_t^{-1} P_{it} \right)^{\frac{\theta}{\theta - 1}}$, (ii) $l_t = \int_0^1 l_{it} \, dt$, (iii) $\dot{K}_{it-1} = \int_0^1 \dot{K}_{it-1} \, dt$ and (iv) price dispersion $p_{di} = \int_0^1 \left( P_t^{-1} P_{it} \right)^{\frac{\theta}{\theta - 1}} \, dt$, the equation in question being a law of motion for aggregate real production or aggregate production function $Y_t$.

8.4 Price dispersion and optimal adjusted aggregate price. Price dispersion $p_{di}$ naturally follows aggregate price $P_t$:

$$P_t = \left( \int_0^1 P_{it}^{\frac{\theta}{\theta - 1}} \, dt \right)^{\frac{1}{1-\theta}} = \left[ \int_{1-\xi}^{1-\xi} \left( P_t^{-1} \right)^{\frac{\theta}{\theta - 1}} \, dt + \int_{1-\xi}^{1-\xi} \left( P_{it} \right)^{\frac{\theta}{\theta - 1}} \, dt \right]^{\frac{1}{1-\theta}} \rightarrow$$
$$\rightarrow p_{di} = \int_0^1 \left( P_t^{-1} P_{it} \right)^{\frac{\theta}{\theta - 1}} \, dt = \int_{1-\xi}^{1-\xi} \left( P_t^{-1} P_{it} \right)^{\frac{\theta}{\theta - 1}} \, dt + \int_{1-\xi}^{1-\xi} \left( P_t^{-1} P_{it} \right)^{\frac{\theta}{\theta - 1}} \, dt =$$
$$= \int_{1-\xi}^{1-\xi} \left( P_t^{-1} P_{it} \right)^{\frac{\theta}{\theta - 1}} \, dt + \int_{1-\xi}^{1-\xi} \left( P_t^{-1} P_{it} \right)^{\frac{\theta}{\theta - 1}} \, dt + \int_{1-\xi}^{1-\xi} \left( P_t^{-1} P_{it} \right)^{\frac{\theta}{\theta - 1}} \, dt + \xi \left( P_t^{-1} P_{it} \right)^{\frac{\theta}{\theta - 1}} \int_0^1 P_{it} \, dt \right) =$$
$$= \int_{1-\xi}^{1-\xi} \left( P_t^{-1} P_{it} \right)^{\frac{\theta}{\theta - 1}} \, dt + \int_{1-\xi}^{1-\xi} \left( P_t^{-1} P_{it} \right)^{\frac{\theta}{\theta - 1}} \, dt \left( P_t^{-1} P_{it} \right)^{\frac{\theta}{\theta - 1}} \int_0^1 P_{it} \, dt \right)^{\frac{\theta}{\theta - 1}} =$$
$$= \int_{1-\xi}^{1-\xi} \left( P_t^{-1} P_{it} \right)^{\frac{\theta}{\theta - 1}} \, dt + \xi \int_{1-\xi}^{1-\xi} \left( P_t^{-1} P_{it} \right)^{\frac{\theta}{\theta - 1}} \, dt \left( P_t^{-1} P_{it} \right)^{\frac{\theta}{\theta - 1}} \int_0^1 P_{it} \, dt \right)^{\frac{\theta}{\theta - 1}} =$$

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Following that the law of motion for optimal adjusted aggregate price $p_\tau$ follows in account of random wholesale real output price adjustment and a continuum of wholesalers and (ii) $\pi^{-1}_t\pi^{-1}_\tau$ on account of $\pi^{-1}_t = P^{-1}_t P_{t-1}$, the equation in question being a law of motion for price dispersion $p_d$. Owing to market clearing, optimal adjusted aggregate price $p^*_t$ accordingly equals a continuum of optimal adjusted prices $\int p^*_t di : p^*_t = \int p^*_t di = \int A^t_{t-1} di = i A^t_{t-1} = (1 - 0) A^t_{t-1} = A^t_{t-1}$, ceteris paribus. It follows that the law of motion for optimal adjusted aggregate price $p^*_t$ is

$$p^*_t = A^t_{t-1},$$

all else equal.

9. Equilibrium

9.1 Price equilibrium with transfers. A price equilibrium with transfers is a pair of feasible allocation $\{U(C_t, l_t), Y_t, C_t, l_t, u_t, K_t, \Psi(u_t), b_t, \Pi_{2t}, \Pi_{kt}, Y_t, Y_{kt}, \Gamma_{kt}, l_{kt}, K_{kt}\}_{t=0}^{\infty}$ and prices $\{\lambda_{kt}, W_t, r_{kt}, \pi_t, \Gamma_{kt}, \Pi_{kt}, P_t, P_{kt}, \phi_t, p_{kt}, \pi^{-1}_t\}_{t=0}^{\infty}$ such that retail nominal profit $P_t \Pi_{kt}$ and wholesale real profit $\Pi_{kt}$, real production cost $\Gamma_{kt}$ and household utility $U(C_t, l_t)$ (i.e. preferences) are optimal and markets clear: ceteris paribus,
(real production cost optimisation)
\[
\min_{\{l_t, K_{t-1}\}_{t=0}^{\infty}} \Gamma_{it} = W_l l_t + r k_t \tilde{K}_{it-1} \text{ s.t.}
\]
\[
Y_{it} = a_t \tilde{K}_{it-1}^\alpha (Y_{it-1})^{1-\alpha},
\]
\[
l_{it}, \tilde{K}_{it-1} \geq 0;
\]

(aggregate price) \[
P_t = \left[ \xi(\pi_{t-1}^* P_{t-1}) \right]^{\frac{1}{1-\theta}} + (1 - \xi) \left[ (P_t^*) \right]^{\frac{1}{1-\theta}} \right]^{1-\theta};
\]

(price dispersion) \[
\rho_d = \xi \rho_d \left( \frac{\pi_{t-1}^*}{\pi_t} \right) \left[ (1 - \xi) \left( \frac{\pi_t^*}{\pi_t} \right) \right]^{1-\theta};
\]

(confidence) \[
\Upsilon_t = (p_t u_t)^{\gamma};
\]

(permanent technology) \[
p_t = p_{t-1} e^{\sigma_x e_{xt}};
\]

(transitory technology and noise technology) \[
x_t = \rho_x x_{t-1} e^{\sigma_x e_{xt}}, \ \forall x = t, n;
\]

(real production technology) \[
a_t = e^{\mu} d_{t-1} p_{t-1} (1 - \lambda_t);
\]

(nominal interest rate) \[
\left( \frac{r_{n+1}}{r_n} \right)^{\phi_{\rho}} = \left( \frac{p_{t-1}}{P_t} \right)^{\phi_{\rho}} \left( \frac{\pi_{t-1}^*/\pi_t^*}{\pi_{t-1}/\pi_t} \right)^{\phi_{\rho}} \left( \frac{Y_t^*/Y_t}{Y_{t-1}^*/Y_{t-1}} \right)^{1-\phi_{\rho}} e^\phi;
\]

(aggregate capital utilisation) \[
Y_t = C_t + g + \Psi(u_t) K_{t-1}.
\]

9.2 Feasible Pareto efficient allocation. Endogenous variables can be subdivided as follows. Consumption, endowment and production endogenous variables:
\[
\{U(C_t, l_t), Y_t, C_t, l_t, u_t, K_t, \Psi(u_t), b_t, \Pi_t, \Pi_{t-1}, Y_t, Y_{t-1}, \Pi_{t-1}, \Gamma_{it}, l_{it}, K_{it}\}_{t=0}^{\infty}.
\]

Price endogenous variables: \{\lambda_{lt}, W_t, k_t, \pi_t, P_t, P_{t-1}, \pi_{t-1}, \phi_t, pd_t, \pi_t^*\}_{t=0}^{\infty}. Technology endogenous variables: \{p_{t-1}, n_t, a_t\}_{t=0}^{\infty}. Policy endogenous variables: \{r_n, t_f, t_x\}_{t=0}^{\infty}.

Parameters are \{\beta, \theta, \lambda, \sigma, \alpha, \sigma, \xi, \tau, \rho, \sigma, \rho, \sigma, \mu, \rho, \sigma, \psi, \phi, \phi, \varphi, \gamma, g\}.

The feasible allocation is characterised by consumption, endowment and production endogenous variables. Prices are characterised by price endogenous variables. Technology endogenous variables should be feasible allocation endogenous variables, but are recorded separately for scopes of clarity.

Policy endogenous variables should be both feasible allocation and price endogenous variables, but are recorded separately for identical scopes. Strictly speaking, in fact, endowment endogenous variables should have to be transfers \text{f}\text{r} and taxation \text{t}\text{r} alone, being there none in the feasible allocation.

A feasible allocation is Pareto efficient if and only if there exists no other feasible allocation such that almost all agents prefer it to the given one and at least one agent strictly prefers it to the given one. By construction, markets are complete. The first fundamental theorem of welfare economics consequently applies, whereby a price equilibrium with transfers in a complete market system is a feasible Pareto efficient allocation.

10. Laws of motion and normalisation

10.1 Laws of motion. There thus emerge the following laws of motion:

\[
\begin{align*}
pt_t &= pt_{t-1} e^{\sigma_x e_{xt}} \text{ (permanent technology);} \\
x_t &= \rho_x x_{t-1} e^{\sigma_x e_{xt}}, \ \forall x = t, n \text{ (transitory technology and noise technology);} \\
a_t &= e^{\mu} p_{t-1} pt_{t-1} (1 - \lambda_t) \text{ (real production technology);} \\
rk_t &= \Psi'(u_t) \text{ (real capital return);} \\
P_t &= \left[ \xi(\pi_{t-1}^* P_{t-1}) \right]^{\frac{1}{1-\theta}} + (1 - \xi) \left[ (P_t^*) \right]^{\frac{1}{1-\theta}} \right]^{1-\theta} \text{ (aggregate price);} \\
p_t^* &= \frac{A_t}{B_t} \text{ (optimal adjusted aggregate price), for}
\end{align*}
\]
\[ A_t = \lambda_1 \phi_1 \theta Y_t + \xi \beta \mathbb{E}_t \left( \frac{\pi_t}{\pi_{t+1}} \right)^{\frac{\phi_Y}{\phi}} A_{t+1} \text{ and} \]
\[ B_t = \lambda_1 Y_t + \xi \beta \mathbb{E}_t \left( \frac{\pi_t}{\pi_{t+1}} \right)^{\frac{\phi_Y}{\phi}} B_{t+1}; \]
\[ pd_t = \xi pd_{t-1} \left( \frac{\pi_{t-1}}{\pi_t} \right)^{\frac{\phi_Y}{\phi}} + (1 - \xi) \left( \frac{\pi_t}{\pi_t} \right)^{\frac{\phi_Y}{\phi}} \text{ (price dispersion);} \]
\[ W_t = \Upsilon_t^{-1}(C_t - hC_{t-1})^{\sigma_c} t_{t-1}^{\eta_t} \text{ (real wage);} \]
\[ l_t = \alpha^{-1} (1 - \alpha) W_t^{-1} r k_t u_t k_{t-1} \text{ (aggregate labour);} \]
\[ \Upsilon_t (C_t - hC_{t-1})^{-\sigma_c} = \mathbb{E}_t \left[ \frac{\Upsilon_{t+1}(C_{t+1} - hC_{t+1})^{-\sigma_c} r n_t}{\pi_{t+1}} \right] \text{ (real consumption);} \]
\[ \left( \frac{\pi_t}{\pi_{t-1}} \right)^{\phi} \left( \frac{\pi_t}{\pi_{t-1}} \right) \left( \frac{Y_t}{Y_{t-1}} \right) \left( \frac{Y_t}{Y_{t-1}} \right)^{\phi} e^{\pi} \text{ (nominal interest rate);} \]
\[ Y_t = pd_{t-1} a_t \hat{K}_{t-1}^{-1}(\Upsilon_t l_t)^{1-\alpha} \text{ (aggregate real production);} \]
\[ \Upsilon_t = (pt_t n_t)^{\gamma} \text{ (confidence);} \]
\[ \phi_t = \alpha^{-\alpha}(1 - \alpha) \alpha^{-1} a_t^{-1} \Upsilon_t^{\alpha-1} r k_t^{\alpha} W_t^{1-\alpha} \text{ (real marginal cost);} \]
\[ K_t = (1 - \delta) K_{t-1} + i \text{ (aggregate capital);} \]
\[ Y_t = C_t + g + \Psi(u_t) K_{t-1} \text{ (aggregate capital utilisation).} \]

10.2 **Normalisation.** Certain endogenous variables, being real consumption \( C_t \), aggregate capital \( K_t \), aggregate real output \( Y_t \) and real wage \( W_t \), abide by the permanent changes to the steady state dictated by permanent technology \( pt_t \) and are normalised thus: \( X_t = x_t pt_t \), in which \( X = C, K, Y, W \).

11. **Log-linearisation**

The DSGE model at hand is solved by resorting to a first order linear approximation whereby its laws of motion are log-linearised about the steady state of each endogenous variable. Specifically, a first order Taylor expansion is conducted about the logarithmic form of each law of motion: \( f(x) = f(a) + f'(a)(x - a) = f(a) + f'(a)\bar{x}, \) in which \( a \) is the endogenous variable’s steady state and \( \bar{x} \) is its deviation therefrom.

Permanent technology:

\[ pt_t = pt_{t-1} e^{\sigma_{pt} \varepsilon_{ptt}}; \]
\[ pt = pt e^{\sigma_{pt} \varepsilon_{pt}} \text{ (steady state);} \]
\[ pt = pt \text{ (steady state, in which } \varepsilon_{pt} = 0, \text{ admitting } pt = 1); \]
\[ \ln pt = \ln pt + \sigma_{\varepsilon_{pt}} \varepsilon_{pt}; \]
\[ \ln pt + \frac{\hat{p}_{t-1}^t}{pt} = \ln pt + \hat{p}_{t-1} + \sigma_{\varepsilon_{pt}} \varepsilon_{pt} + \sigma_{\varepsilon_{pt}} (\varepsilon_{ptt} - \varepsilon_{pt}) \longrightarrow \hat{p}_{t} = \hat{p}_{t-1} + \sigma_{\varepsilon_{pt}} \varepsilon_{ptt} \longrightarrow \hat{p}_{t} = \hat{p}_{t-1} + \varepsilon_{ptt} \text{ (imposing } \sigma_{\varepsilon_{pt}} = 1). \]

Transitory technology and noise technology:

\[ x = t, n; \]
\[ x_t = \rho x x_{t-1} e^{\sigma_{xz} \varepsilon_{xzt}}; \]
\[ x = \rho x e^{\sigma_{xz} \varepsilon_{x}} \text{ (steady state);} \]
\[ x = \rho x \text{ (steady state, in which } \varepsilon_x = 0, \text{ admitting } x = 1); \]
\[ 1 = \rho x, \text{ but } \rho < 1, \text{ so } 1 = \rho x e^{\sigma x} e_x \rightarrow \rho x = e^{-\sigma x} e_x < 1 \rightarrow \]
\[ -\sigma x \varepsilon_x \ln e < \ln 1 \rightarrow -\sigma x \varepsilon_x < 0 \rightarrow \varepsilon_x > 0 \text{ (steady state, being there an attendant shock);} \]
\[ \ln x = \rho x \ln x + \sigma x \varepsilon_x; \]
\[ \ln x + \frac{\ddot{x}}{x} = \rho x \ln x + \frac{\rho x \ddot{x} \! - \! 1}{x} + \sigma x \varepsilon_x + \sigma x (\varepsilon_x - \varepsilon_x) \rightarrow \]
\[ \rightarrow \dot{x} = \rho x \ddot{x} \! - \! 1 + \sigma x \varepsilon_{xt} \rightarrow \]
\[ \rightarrow \dot{x} = \rho x \ddot{x} \! - \! 1 + \varepsilon_{xt} \text{ (imposing } \sigma x = 1). \]

Real production technology:

\begin{align*}
    a_t &= e^\rho a_t \! - \! 1 pt \! - \! 1 t_{t-1}; \\
    a &= e^\rho aaptt \text{ (steady state, admitting } a = 1); \\
    \rho a^{-1} &= e^\rho \text{ (steady state, in which } pt = t = 1); \\
    - \ln \rho a &= \mu \ln e \rightarrow \mu = -\ln \rho a \text{ (steady state);} \\
    \ln a &= \mu + \rho a \ln a + \ln pt + \ln t; \\
    \ln a + \frac{\ddot{a}}{a} &= \mu + \rho a \ln a + \frac{\rho a \ddot{a} \! - \! 1}{a} + \ln pt + \frac{\ddot{p}t}{pt} + \ln t + \frac{\ddot{t}}{t} \rightarrow \]
\end{align*}
\[ \rightarrow \ddot{a} = \rho a \ddot{a} \! - \! 1 + \ddot{p}t \! - \! 1 + \ddot{t} \! - \! 1. \]

Real capital return:

\begin{align*}
    r k_t &= \Psi' (u_t); \\
    r k &= \Psi' (u) \text{ (steady state);} \\
    r k &= \Psi' (1) \text{ (steady state, imposing } u = 1); \\
    \ln r k &= \ln \Psi' (u); \\
    \ln r k + \frac{\ddot{r} k_t}{r k} &= \ln \Psi' (u) + \frac{\Psi'' (u) (1) \ddot{u}_t}{\Psi' (u)} \rightarrow \]
\end{align*}
\[ \rightarrow \frac{\ddot{r} k_t}{r k} = \frac{\Psi'' (u) (1) \ddot{u}_t u}{\Psi' (u) u} \rightarrow \]
\[ \rightarrow \dot{r} k_t = \frac{\Psi'' (u) \ddot{u}_t}{\Psi' (u)} \rightarrow \]
\[ \rightarrow \dot{r} k_t = \omega^{-1} \ddot{u}_t \left( \omega = \frac{\Psi' (u)}{\Psi'' (u)} \right) \rightarrow \]
\[ \ddot{u}_t = \omega r k_t. \]

One notices that parameter \( \omega \) models capital utilisation adjustment cost inverse elasticity, being a positive real number: \( \omega \in \mathbb{R}_{++} \).

Aggregate price:

\[ P_t = \left[ \xi (\pi^{r \! - \! 1}_t P_t \! - \! 1) \right]^{\frac{1}{r}} + (1 - \xi) (P^r_t) \frac{1}{r} \]
\[ \rightarrow P_t^{\frac{1}{r}} = \xi (\pi^{r \! - \! 1}_t P_t \! - \! 1) \frac{1}{r} + (1 - \xi) (P^r_t) \frac{1}{r}; \]
\[
\left( \frac{P_t}{P_{t-1}} \right)^{\frac{\pi_t}{\pi}} = \xi \left( \frac{\pi_{t-1} P_{t-1}}{P_t} \right)^{\frac{\pi_t}{\pi}} + (1 - \xi) \left( \frac{P_t}{P_t} \right)^{\frac{\pi_t}{\pi}} \iff \\
\rightarrow 1 = \xi \left( \frac{\pi_t}{\pi} \right)^{\frac{\pi_t}{\pi}} + (1 - \xi) \left( p_t^* \right)^{\frac{\pi_t}{\pi}} \quad (\text{standardisation}); \\
1 = \xi + (1 - \xi) \left( p_t^* \right)^{\frac{\pi_t}{\pi}} \quad (\text{steady state}); \\
1 - \xi = (1 - \xi) \left( p_t^* \right)^{\frac{\pi_t}{\pi}} \rightarrow \\
\rightarrow 1 = (p_t^*)^{\frac{\pi_t}{\pi}} \rightarrow \\
\rightarrow 1^{1 - \theta} = \left( (p_t^*)^{\frac{\pi_t}{\pi}} \right)^{1 - \theta} \rightarrow \\
\rightarrow 1 = p^* \quad (\text{steady state}); \\
ln 1 = ln \left[ \xi \left( \frac{\pi_t}{\pi} \right)^{\frac{\pi_t}{\pi}} + (1 - \xi) \left( p_t^* \right)^{\frac{\pi_t}{\pi}} \right]; \\
ln 1 = ln \left[ \xi \left( \frac{\pi_t}{\pi} \right)^{\frac{\pi_t}{\pi}} + (1 - \xi) \left( p_t^* \right)^{\frac{\pi_t}{\pi}} \right] + \frac{(1 - \xi)(1 - \theta)^{-1}(p_t^*)^{\frac{\pi_t}{\pi}}}{1} + \\
\frac{\xi (1 - \theta)^{-1} \pi^{1 - \theta} p_t^* \hat{\pi}_t^*}{1} + \frac{\xi \tau (1 - \theta)^{-1} \pi^{1 - \theta} \pi \hat{\pi}_{t-1}}{1} + \frac{-\xi (1 - \theta)^{-1} \pi^{1 - \theta} \pi \hat{\pi}_t}{1} + \\
\rightarrow 0 = \frac{(1 - \xi)(1 - \theta)^{-1} \pi^{1 - \theta} p_t^* \hat{\pi}_t^*}{1} + \frac{\xi \tau (1 - \theta)^{-1} \pi^{1 - \theta} \pi \hat{\pi}_{t-1}}{1} - \frac{\xi \tau (1 - \theta)^{-1} \pi^{1 - \theta} \pi \hat{\pi}_t}{1} \quad (\text{in which } \pi = p^* = 1) \rightarrow \\
\rightarrow 0 = \frac{(1 - \xi) \hat{\pi}_t^*}{(1 - \theta)} + \frac{\xi \tau \hat{\pi}_{t-1}}{(1 - \theta)} - \frac{\xi \hat{\pi}_t}{(1 - \theta)} \quad (\text{in which } \pi = p^* = 1) \rightarrow \\
\rightarrow (1 - \xi) \hat{\pi}_t^* = \xi (\hat{\pi}_t - \tau \hat{\pi}_{t-1}) \rightarrow \\
\rightarrow \hat{\pi}_t^* = \frac{\xi (\hat{\pi}_t - \tau \hat{\pi}_{t-1})}{(1 - \xi)}. \\
\]

Optimal adjusted aggregate price:
Numerator:

\[
A_t = \theta \phi_t + \xi \beta E_t \left( \frac{\pi_t}{\pi_{t+1}} \right)^{\frac{\pi_t}{\pi}} A_{t+1}; \\
A = \theta \phi + \xi \beta \left( \frac{\pi_t}{\pi} \right)^{\frac{\pi_t}{\pi}} A \quad (\text{steady state}); \\
A = \theta \phi + \xi \beta A \quad (\text{steady state}, \text{in which } \pi = 1); \\
(1 - \xi \beta) A = \theta \phi \rightarrow \\
\rightarrow A = (1 - \xi \beta)^{-1} \theta \phi \quad (\text{steady state}); \\
ln A = ln \left[ \theta \phi + \xi \beta \left( \frac{\pi_t}{\pi} \right)^{\frac{\pi_t}{\pi}} A \right]; \\
ln A + \frac{\bar{A}_t}{A} = ln \left[ \theta \phi + \xi \beta \left( \frac{\pi_t}{\pi} \right)^{\frac{\pi_t}{\pi}} A \right] + \frac{\theta \bar{\phi}_t}{\theta \phi + \xi \beta A} +
\]
\[
\begin{align*}
&+ \frac{\xi \beta \theta \tau (1 - \theta)^{-1} \pi^{\pi_{t+1}} - \pi^{\pi_{t}}}{\theta \phi + \xi \beta A} + \frac{-\xi \beta \theta (1 - \theta)^{-1} \pi^{\pi_{t+1}} - \pi^{\pi_{t}}}{\theta \phi + \xi \beta A} + \frac{\xi \beta \pi^{\pi_{t} - \pi_{t+1}} E_{t} \hat{A}_{t+1}}{\theta \phi + \xi \beta A} \\
\rightarrow \hat{A}_{t} &= \frac{\theta \phi \phi_{t}}{A \phi} + \frac{\xi \beta \theta \tau (1 - \theta)^{-1} \pi_{t}}{A \pi} - \frac{-\xi \beta \theta (1 - \theta)^{-1} \pi_{t} E_{t} \hat{A}_{t+1}}{A \pi} + \frac{\xi \beta E_{t} \hat{A}_{t+1}}{A (1 - \theta)} + \xi \beta E_{t} \hat{A}_{t+1}.
\end{align*}
\]

Denominator:

\[
B_{t} = 1 + \xi \beta E_{t} \left( \frac{\pi_{t}}{\pi_{t+1}} \right)^{1 - \theta} B_{t+1};
\]

\[
B = 1 + \xi \beta \left( \frac{\pi}{\pi} \right)^{1 - \theta} B \text{ (steady state)};
\]

\[
B = 1 + \xi \beta B \text{ (steady state, in which } \pi = 1);\]

\[
(1 - \xi \beta) B = 1 \rightarrow\]

\[
B = (1 - \xi \beta)^{-1} \text{ (steady state)};
\]

\[
\ln B = \ln \left[ 1 + \xi \beta \left( \frac{\pi}{\pi} \right)^{1 - \theta} B \right];
\]

\[
\ln B + \frac{B_{t}}{B} = \ln \left[ 1 + \xi \beta \left( \frac{\pi_{t}}{\pi_{t+1}} \right)^{1 - \theta} B \right] + \frac{\xi \beta \theta (1 - \theta)^{-1} \pi^{\pi_{t+1}} - \pi^{\pi_{t}}}{1 + \xi \beta B} + \frac{\xi \beta \pi^{\pi_{t+1}} - \pi^{\pi_{t}} E_{t} \hat{A}_{t+1}}{1 + \xi \beta B} \rightarrow\]

\[
\hat{B}_{t} = \frac{\xi \beta \theta (1 - \theta)^{-1} \pi_{t} E_{t} \hat{A}_{t+1}}{B \pi} + \frac{\xi \beta (1 - \theta)^{-1} \pi_{t} E_{t} \hat{A}_{t+1}}{B \pi} + \frac{\xi \beta E_{t} \hat{B}_{t+1}}{B \pi} \text{ (in which } \pi = 1) \rightarrow\]

\[
\hat{B}_{t} = \frac{\xi \beta (\tau \pi - \pi_{t+1})}{(1 - \theta) B} + \xi \beta E_{t} \hat{B}_{t+1}.
\]

Fraction:

\[
p_{t}^{*} = \frac{A_{t}}{\hat{B}_{t}};
\]

\[
p^{*} = \frac{A}{B} \text{ (steady state)};
\]

\[
1 = \frac{A}{B} \text{ (steady state, in which } p^{*} = 1, \text{ admitting } A = B = 1);
\]

\[
\ln p^{*} = \ln A - \ln B;
\]

\[
\ln p^{*} + \frac{\ddot{p}_{t}^{*}}{p^{*}} = \ln A + \frac{\ddot{A}_{t}}{A} - \ln B - \frac{\ddot{B}_{t}}{B} \rightarrow\]

\[
\ddot{p}_{t}^{*} = \ddot{A}_{t} - \ddot{B}_{t} \rightarrow\]

\[
\ddot{p}_{t}^{*} = \left[ \frac{\theta \phi \phi_{t}}{A} + \frac{\xi \beta \theta (1 - \theta) \pi_{t} E_{t} \hat{A}_{t+1}}{A (1 - \theta)} + \xi \beta E_{t} \hat{A}_{t+1} \right] - \left[ \frac{\xi \beta (\tau \pi - \pi_{t+1})}{(1 - \theta) B} + \xi \beta E_{t} \hat{B}_{t+1} \right];
\]

\[
\ddot{p}_{t}^{*} = \frac{\theta \phi \phi_{t}}{A} + \frac{\xi \beta (1 - \theta) \pi_{t} E_{t} \hat{A}_{t+1}}{A (1 - \theta)} - \frac{\xi \beta (1 - \theta) \pi_{t} E_{t} \hat{A}_{t+1}}{A (1 - \theta) B} + \xi \beta E_{t} \hat{p}_{t+1}^{*} \text{ (in which } E_{t} \ddot{p}_{t+1}^{*} = E_{t} \ddot{A}_{t+1} - E_{t} \ddot{B}_{t+1}) \rightarrow
\]

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\[
\begin{align*}
\dot{\xi} (\hat{\tau}_t - \tau \hat{\tau}_{t-1}) = & \quad \frac{\theta \phi \hat{\dot{\delta}}_t}{A} + \frac{\xi \beta \mathbb{E}_t \hat{\pi}_{t+1} - \xi \beta \tau \hat{\pi}_t}{B} + \xi \beta \xi (\mathbb{E}_t \hat{\pi}_{t+1} - \tau \hat{\pi}_t) \\
\dot{\hat{\pi}}_t - \tau \hat{\pi}_{t-1} = & \quad \left(1 - \xi\right) \theta \phi \hat{\dot{\delta}}_t + \frac{(1 - \xi) \beta \mathbb{E}_t \hat{\pi}_{t+1}}{A} - \left(1 - \xi\right) \beta \tau \hat{\pi}_t + \xi \beta (\mathbb{E}_t \hat{\pi}_{t+1} - \tau \hat{\pi}_t) \\
\dot{\hat{\pi}}_t - \tau \hat{\pi}_{t-1} = & \quad \frac{(1 - \xi) \theta \phi \hat{\dot{\delta}}_t}{\xi A} + (1 - \xi) \beta (A^{-1} \mathbb{E}_t \hat{\pi}_{t+1} - B^{-1} \tau \hat{\pi}_t) + \beta (\mathbb{E}_t \hat{\pi}_{t+1} - \tau \hat{\pi}_t) \\
\dot{\hat{\pi}}_t - \tau \hat{\pi}_{t-1} = & \quad \left(1 - \xi\right) \theta \phi \hat{\dot{\delta}}_t + \beta (\mathbb{E}_t \hat{\pi}_{t+1} - \tau \hat{\pi}_t) \quad \text{(in which } A = B = 1) \\
\dot{\hat{\pi}}_t = & \quad \frac{(1 - \xi) \theta \phi \hat{\dot{\delta}}_t}{\xi} +\beta \mathbb{E}_t \hat{\pi}_{t+1} + \tau \hat{\pi}_{t-1} \\
\dot{\hat{\pi}}_t = & \quad \frac{(1 - \xi) \theta \phi \hat{\dot{\delta}}_t}{(1 + \beta \tau) \xi} + \frac{\beta \mathbb{E}_t \hat{\pi}_{t+1}}{(1 + \beta \tau)} + \frac{\tau \hat{\pi}_{t-1}}{(1 + \beta \tau)} \\
\text{[in which } A = (1 - \xi) \beta^{-1} \theta \phi \rightarrow 1 = (1 - \xi) \beta^{-1} \theta \phi \rightarrow \theta = \phi^{-1} (1 - \xi \beta)] \\
\dot{\hat{\pi}}_t = & \quad \frac{(1 - \xi) (1 - \xi) \theta \phi \hat{\dot{\delta}}_t}{(1 + \beta \tau) \xi} + \frac{\beta \mathbb{E}_t \hat{\pi}_{t+1}}{(1 + \beta \tau)} + \frac{\tau \hat{\pi}_{t-1}}{(1 + \beta \tau)}.
\end{align*}
\]

Price dispersion:

\[
\begin{align*}
pd_t = & \quad \xi pd_{t-1} \left(\frac{\pi^*_{t-1}}{\pi_t}\right) + (1 - \xi) \left(\frac{\pi^*_t}{\pi_t}\right) \left(\frac{\theta}{\pi}\right) \\
\frac{pd}{pd} = & \quad \xi pd \left(\frac{\pi^*}{\pi}\right) + (1 - \xi) \left(\frac{\pi^*}{\pi}\right) \left(\frac{\theta}{\pi}\right) \quad \text{(steady state);} \\
pd = & \quad \xi pd + (1 - \xi) \quad \text{(steady state, in which } \pi = \pi^* = 1) \\
\rightarrow pd = & \quad 1 \quad \text{(steady state)} \\
lnpd = & \quad \ln \left[\xi pd \left(\frac{\pi^*}{\pi}\right) + (1 - \xi) \left(\frac{\pi^*}{\pi}\right) \left(\frac{\theta}{\pi}\right)\right]: \\
lnpd + \frac{\ddot{pd}}{pd} = & \quad \ln \left[\xi pd \left(\frac{\pi^*}{\pi}\right) + (1 - \xi) \left(\frac{\pi^*}{\pi}\right) \left(\frac{\theta}{\pi}\right)\right] + \frac{\xi \pi \frac{\theta}{\pi} \pi \frac{\theta}{\pi} \mathbb{E}_t \hat{\pi}_{t-1} + \xi \theta (1 - \theta)^{-1} \frac{pd \pi \frac{\theta}{\pi} \pi \frac{\theta}{\pi} \hat{\pi}_{t-1}}{\pi} + \frac{pd \pi \frac{\theta}{\pi} \pi \frac{\theta}{\pi} \hat{\pi}_{t-1}}{\pi} + \frac{pd \pi \frac{\theta}{\pi} \pi \frac{\theta}{\pi} \hat{\pi}_{t-1}}{\pi} + (1 - \xi) \theta (1 - \theta)^{-1} \frac{\hat{\pi}_{t-1}}{\pi} + (1 - \xi) \theta (1 - \theta)^{-1} \frac{\hat{\pi}_{t-1}}{\pi} + (1 - \xi) \theta (1 - \theta)^{-1} \frac{\hat{\pi}_{t-1}}{\pi} \\
\rightarrow \ddot{pd}_t = & \quad \xi pd_{t-1} + \xi \theta (1 - \theta)^{-1} \frac{\pi \frac{\theta}{\pi} \pi \frac{\theta}{\pi} \hat{\pi}_{t-1}}{\pi} + (1 - \xi) \theta (1 - \theta)^{-1} \frac{\hat{\pi}_{t-1}}{\pi} + (1 - \xi) \theta (1 - \theta)^{-1} \frac{\hat{\pi}_{t-1}}{\pi} + (1 - \xi) \theta (1 - \theta)^{-1} \frac{\hat{\pi}_{t-1}}{\pi} \\
\text{(in which } pd = \pi^* = 1) \\
\rightarrow \ddot{pd}_t = & \quad \xi pd_{t-1} + \xi \theta (1 - \theta)^{-1} \frac{\pi \frac{\theta}{\pi} \pi \frac{\theta}{\pi} \hat{\pi}_{t-1}}{\pi} + (1 - \xi) \theta (1 - \theta)^{-1} \frac{\hat{\pi}_{t-1}}{\pi} + (1 - \xi) \theta (1 - \theta)^{-1} \frac{\hat{\pi}_{t-1}}{\pi} \rightarrow \\
\ddot{pd}_t = & \quad \xi pd_{t-1} + \theta (1 - \theta)^{-1} [\xi \pi \frac{\theta}{\pi} \pi \frac{\theta}{\pi} \hat{\pi}_{t-1} + (1 - \xi) \theta (1 - \theta)^{-1} \frac{\hat{\pi}_{t-1}}{\pi} - \theta (1 - \theta)^{-1} \frac{\hat{\pi}_{t-1}}{\pi}] \\
\text{Since } (1 - \xi) \hat{\pi}^*_t = & \quad \xi (\hat{\pi}_t - \tau \hat{\pi}_{t-1}) = \xi \hat{\pi}_t - \xi \tau \hat{\pi}_{t-1} \rightarrow \xi \tau \hat{\pi}_{t-1} = \xi \hat{\pi}_t - (1 - \xi) \hat{\pi}^*_t \\
\ddot{pd}_t = & \quad \xi pd_{t-1} + \theta (1 - \theta)^{-1} [\xi \hat{\pi}_t - (1 - \xi) \hat{\pi}^* + (1 - \xi) \hat{\pi}^*_t - \hat{\pi}_t] \\
\rightarrow \ddot{pd}_t = & \quad \xi pd_{t-1} + \theta (1 - \theta)^{-1} [\xi \hat{\pi}_t - (1 - \xi) \hat{\pi}^* + (1 - \xi) \hat{\pi}^*_t + (1 - \xi) \hat{\pi}^*_t] \\
\end{align*}
\]
Since $\hat{\pi}_t = \hat{\Pi}_t - \hat{\Pi}_{t-1}$ and $\hat{p}_t = \hat{P}_t - \hat{P}_t$

$$\hat{p}_t = \xi \hat{p}_t + \theta (1 - \theta)^{-1} \left[ (\xi - 1) \hat{\pi}_t - (1 - \xi) \left( \hat{\Pi}_t - \hat{\Pi}_{t-1} \right) \right] \rightarrow$$

$$\hat{p}_t = \xi \hat{p}_t + \theta (1 - \theta)^{-1} \left[ (\xi - 1) \hat{\pi}_t + (1 - \xi) \left( \hat{\Pi}_t - \hat{\Pi}_{t-1} \right) \right].$$

Since $\hat{\pi}_t = \hat{\Pi}_t - \hat{\Pi}_{t-1}$

$$\hat{p}_t = \xi \hat{p}_t + \theta (1 - \theta)^{-1} \left[ (\xi - 1) \hat{\pi}_t + (1 - \xi) (\hat{\pi}_t - \hat{\pi}_{t-1}) \right] \rightarrow$$

$$\hat{p}_t = \xi \hat{p}_t.$$

Since $\hat{p}_t = 0$ (zero inflation steady state)

$$\hat{p}_t = 0.$$
\[
\frac{\dot{I}}{I} = -\frac{\dot{\omega}_t}{w} + \frac{\dot{r}k_t}{r} + \frac{\dot{u}_t}{u} + \frac{\dot{k}_{t-1}}{k} + \frac{\dot{p}_{t-1}}{\dot{p}_t} - \frac{\dot{p}_t}{\dot{p}_t} \\
\dot{I} = -\dot{\omega}_t + \dot{r}k_t + \dot{u}_t + \dot{k}_{t-1} - \dot{p}_{t-1} - \dot{p}_t \\
\dot{I} = \dot{r}k_t + \omega r k_t + \dot{k}_{t-1} - \dot{p}_{t-1} - \dot{w}_t - \dot{p}_t \\
\dot{I} = (1 + \omega) \dot{r}k_t + \dot{k}_{t-1} - \dot{p}_{t-1} - \dot{w}_t - \dot{p}_t.
\]

Real consumption:

\[
\Upsilon_t (C_t - hC_{t-1})^{-\sigmac} = \mathbb{E}_t \beta \left[ \Upsilon_{t+1} (C_{t+1} - hC_t)^{-\sigmac} \frac{rn_t}{\pi_{t+1}} \right] ;
\]

\[
\Upsilon_t (c_t pt_t - hc_{t-1} pt_{t-1})^{-\sigmac} = \mathbb{E}_t \beta \left[ \Upsilon_{t+1} (c_{t+1} pt_{t+1} - hc_{t+1} pt_{t+1})^{-\sigmac} \frac{rn_t}{\pi_{t+1}} \right] (normalisation);
\]

\[
\Upsilon (cpt - hcpt)^{-\sigmac} = \beta \left[ \frac{\Upsilon (cpt - hcpt)^{-\sigmac} \frac{rn}{\pi}}{\pi} \right] (steady state);
\]

\[
(c - hc)^{-\sigmac} = \beta (c - hc)^{-\sigmac} \frac{rn}{\pi} (steady state, in which \Upsilon = pt = \pi = 1);
\]

\[
1 = \beta rn \rightarrow \frac{rn}{\pi} = \beta^{-1} (steady state);
\]

\[
ln \Upsilon - \sigma c ln (cpt - hcpt) = ln \beta + ln \Upsilon - \sigma c ln (cpt - hcpt) + ln rn - ln \pi;
\]

\[
ln \Upsilon + \frac{\dot{\Upsilon}}{\Upsilon} - \sigma c ln (cpt - hcpt) - \frac{\sigma_c pt \dot{\hat{c}}_t}{(cpt - hcpt)} - \frac{\sigma_c cpt \dot{\hat{c}}_t}{(cpt - hcpt)} + \frac{\sigma_c hpt \dot{\hat{c}}_{t-1}}{(cpt - hcpt)} + \frac{\sigma_c hcpt \dot{\hat{c}}_{t-1}}{(cpt - hcpt)} = ln \beta +
\]

\[
+ \frac{\sigma_c hpt \dot{\hat{c}}_t}{(cpt - hcpt)} + \frac{\sigma_c hcpt \dot{\hat{c}}_{t-1}}{(cpt - hcpt)} + \frac{\sigma_c hpt \dot{\hat{c}}_t}{(cpt - hcpt)} + ln rn + \frac{\dot{r}u_t}{rn} - ln \pi - \frac{\dot{r}u_t}{\pi} \rightarrow
\]

\[
\frac{\dot{\Upsilon}}{\Upsilon} - \frac{\sigma_c hpt \dot{\hat{c}}_t}{cpt (1-h)} - \frac{\sigma_c hpt \dot{\hat{c}}_{t-1}}{cpt (1-h)} + \frac{\sigma_c hpt \dot{\hat{c}}_t}{cpt (1-h)} + \frac{\sigma_c hpt \dot{\hat{c}}_{t-1}}{cpt (1-h)} =
\]

\[
= \frac{E_t \dot{\Upsilon}_{t+1}}{\Upsilon} - \frac{E_t \sigma_c hpt \dot{\hat{c}}_{t+1}}{cpt (1-h)} - \frac{E_t \sigma_c hpt \dot{\hat{c}}_{t+1}}{cpt (1-h)} + \frac{\sigma_c hpt \dot{\hat{c}}_{t+1}}{cpt (1-h)} + \frac{\sigma_c hpt \dot{\hat{c}}_{t+1}}{cpt (1-h)} \rightarrow
\]

\[
- \frac{\sigma c (\dot{c}_t + \dot{p}_t)}{cpt (1-h)} + \frac{\sigma c h (\dot{c}_{t-1} + \dot{p}_{t-1})}{cpt (1-h)} = \frac{E_t \sigma c (\dot{c}_{t+1} + \dot{p}_{t+1})}{cpt (1-h)} + \frac{\sigma c h (\dot{c}_t + \dot{p}_t)}{cpt (1-h)} + \frac{\dot{r}u_t}{rn} - E_t \dot{\Upsilon}_{t+1} \rightarrow
\]

\[
(\text{in which } E_t \dot{\Upsilon}_{t+1} = 0, \text{ but } E_t \dot{p}_{t+1} \neq 0) \rightarrow
\]

\[
- (\dot{c}_t + \dot{p}_t) + h (\dot{c}_{t-1} + \dot{p}_{t-1}) = -E_t \dot{c}_t + E_t \dot{p}_{t+1} + h (\dot{c}_t + \dot{p}_t) + \frac{(1-h) (\dot{r}u_t - E_t \dot{\Upsilon}_{t+1} - \dot{\Upsilon}_t)}{\sigma_c} \rightarrow
\]

\[
- (1 + h) (\dot{c}_t + \dot{p}_t) = \frac{(1-h) (\dot{r}u_t - E_t \dot{\Upsilon}_{t+1} - \dot{\Upsilon}_t)}{\sigma_c} - h (\dot{c}_{t-1} + \dot{p}_{t-1}) - (E_t \dot{c}_{t+1} + E_t \dot{p}_{t+1}) \rightarrow
\]
\[ \dot{c}_t + \dot{p}_t = \frac{(1 - h) \left( \dot{Y}_t + \mathbb{E}_t \dot{\pi}_{t+1} - \dot{r}_n t \right)}{\sigma_e (1 + h)} + \frac{h (\dot{c}_{t-1} + \dot{p}_{t-1})}{(1 + h)} + \frac{\mathbb{E}_t \dot{c}_{t+1} + \mathbb{E}_t \dot{p}_{t+1}}{(1 + h)} \rightarrow \]

\[ \dot{c}_t = \frac{(1 - h) \left( \dot{Y}_t + \mathbb{E}_t \dot{\pi}_{t+1} - \dot{r}_n t \right)}{\sigma_e (1 + h)} + \frac{h (\dot{c}_{t-1} + \dot{p}_{t-1})}{(1 + h)} + \frac{\mathbb{E}_t \dot{c}_{t+1} + \mathbb{E}_t \dot{p}_{t+1}}{(1 + h)} - \dot{p}_t. \]

As mentioned above, one notices the following: \( \mathbb{E}_t \dot{Y}_{t+1} = 0 \) and \( \mathbb{E}_t \dot{p}_{t+1} \neq 0 \) imply \( \mathbb{E}_t \dot{x}_{t+1} \neq 0 \), whereby, \( \forall \gamma \in (0, 1] \subset \mathbb{R}_+ \), \( \mathbb{E}_t \dot{\gamma}_{t+1} + \mathbb{E}_t \dot{\gamma}_{t+1} = \gamma (\mathbb{E}_t \dot{Y}_{t+1} + \mathbb{E}_t \dot{\pi}_{t+1}) = -\mathbb{E}_t \dot{\gamma} \mathbb{E}_t \dot{p}_{t+1} = -\gamma \mathbb{E}_t \dot{p}_{t+1} \), since \( \mathbb{E}_t \dot{Y}_{t+1} = \mathbb{E}_t [\gamma (\dot{p}_{t+1} + \dot{t}_{t+1} + \dot{n}_{t+1})] = \mathbb{E}_t \dot{\gamma} \mathbb{E}_t \dot{p}_{t+1} + \mathbb{E}_t \dot{\gamma} \dot{t}_{t+1} + \mathbb{E}_t \dot{\gamma} \dot{n}_{t+1} = \gamma (\mathbb{E}_t \dot{p}_{t+1} + \mathbb{E}_t \dot{t}_{t+1} + \mathbb{E}_t \dot{n}_{t+1}) \).

Nominal interest rate:

\[
\left( \frac{r_n t}{r_n} \right) = \left( \frac{r_n t-1}{r_n} \right)^{\rho_{r_n}} \left[ \left( \frac{\pi t}{\pi} \right)^{\phi_\pi} \left( \frac{\pi t-1}{\pi} \right)^{\phi_{\pi \gamma}} \left( \frac{Y_t}{Y} \right)^{\phi_y} \left( \frac{Y_{t-1}/Y}{Y} \right)^{\phi_y} \right] 1^{-\rho_{r_n}} e^{\rho};
\]

\[
\left( \frac{r_n t}{r_n} \right) = \left( \frac{r_n t-1}{r_n} \right)^{\rho_{r_n}} \left[ \left( \frac{\pi t}{\pi} \right)^{\phi_\pi} \left( \frac{\pi t-1}{\pi} \right)^{\phi_{\pi \gamma}} \left( \frac{y/p_t}{ypt} \right)^{\phi_y} \left( \frac{y/p_{t-1}/ypt}{y_{t-1}/p_{t-1}/ypt} \right)^{\phi_y} \right] 1^{-\rho_{r_n}} e^{\rho} \text{ (normalisation)};
\]

\[
\left( \frac{r_n t}{r_n} \right) = \left( \frac{r_n t}{r_n} \right)^{\rho_{r_n}} \left[ \left( \frac{\pi t}{\pi} \right)^{\phi_\pi} \left( \frac{\pi t-1}{\pi} \right)^{\phi_{\pi \gamma}} \left( \frac{y/p_t}{ypt} \right)^{\phi_y} \left( \frac{y/p_{t-1}/ypt}{y_{t-1}/p_{t-1}/ypt} \right) \right] 1^{-\rho_{r_n}} e^{\rho} \text{ (steady state)};
\]

\[ 1 = e^{\rho} \rightarrow \]

\[ \rightarrow 0 = \varphi \text{ (steady state)}; \]

\[ \ln r_n - \ln r_n = \rho_{r_n} (\ln r_n - \ln r_n) + (1 - \rho_{r_n}) \left[ \phi_\pi (\ln \pi - \ln \pi + \ln \pi - \ln \pi) + \phi_{\pi \gamma} (\ln \pi - \ln \pi + \ln \pi - \ln \pi) + \phi_y (\ln y + \ln pt - \ln y - \ln pt) + \phi_{y \gamma} (\ln y + \ln pt - \ln y - \ln pt - \ln y - \ln pt) \right] + \varphi \rightarrow \]

\[ \ln r_n + \frac{\dot{r}_n t}{r_n} - \ln r_n = \rho_{r_n} (\ln r_n - \ln r_n) + \frac{\rho_{r_n} \dot{r}_n t_{t-1}}{r_n} + \]

\[ + (1 - \rho_{r_n}) \left[ \phi_\pi (\ln \pi - \ln \pi + \ln \pi - \ln \pi) + \phi_{\pi \gamma} (\ln \pi - \ln \pi + \ln \pi - \ln \pi) + \phi_y (\ln y + \ln pt - \ln y - \ln pt) + \phi_{y \gamma} (\ln y + \ln pt - \ln y - \ln pt - \ln y - \ln pt) + \right. \]

\[ + \phi_{y \gamma} \frac{\dot{y}_t}{y} + \frac{\phi_{y \gamma} \dot{p}_t}{p_t} + \frac{\phi_{y \gamma} \dot{y}_{t-1}}{y} + \left. \frac{\phi_{y \gamma} \dot{p}_{t-1}}{p_t} \right] + \varphi \rightarrow \]

\[ \dot{r}_n t = \rho_{r_n} \dot{r}_n t_{t-1} + (1 - \rho_{r_n}) \left[ \phi_\pi \dot{Y}_t + \phi_{\pi \gamma} (\dot{\pi}_t - \dot{\pi}_{t-1}) + \phi_y (\dot{y}_t + \dot{p}_t) + \phi_{y \gamma} (\dot{y}_t + \dot{p}_t - \dot{y}_{t-1} - \dot{p}_{t-1}) \right]. \]

Aggregate real production:

\[ Y_t = p d_{l-1} a_t (Y_{l-1})^{1-\alpha} \rightarrow \]

\[ Y_t = p d_{l-1} a_t (u_k t_{l-1})^{\alpha} (Y_{l-1})^{1-\alpha}; \]

\[ y/p_t = p d_{l-1} a_t (u_k t_{l-1} t_{l-1} t_{l-1})^{\alpha} (Y_{l-1})^{1-\alpha} \text{ (normalisation);} \]

\[ y/p_t = p d_{l-1} a_t (u_k t_{l-1} t_{l-1} t_{l-1})^{\alpha} (Y_{l-1})^{1-\alpha} \text{ (steady state);} \]

\[ \ln y + \ln pt = -\ln pd + \ln a + \alpha (\ln u + \ln k + \ln pt) + (1 - \alpha) (\ln Y + \ln l); \]

\[ \ln y + \frac{\dot{y}_t}{y} + \ln pt + \frac{\dot{p}_t}{p_t} = -\ln pd + \frac{\dot{p}_t}{p_t} + \ln a + \frac{\dot{a}_t}{a} + \]
\[ + \alpha (lnu + lnk + lnpt) + \frac{\alpha \bar{u}_t}{u} + \frac{\alpha k_{t-1}}{k} + \frac{\alpha \bar{p}_{t-1}}{pt} + (1 - \alpha) (lnY + lnl) + \frac{(1 - \alpha) \bar{\Upsilon}_t}{Y} + \frac{(1 - \alpha) \bar{l}_t}{l} \rightarrow \]
\[ \rightarrow \bar{y}_t + \bar{p}_t = -\bar{p}_t + \bar{a}_t + \alpha \left( \bar{u}_t + \bar{k}_{t-1} + \bar{p}_{t-1} \right) + (1 - \alpha) \left( \bar{\Upsilon}_t + \bar{l}_t \right) \rightarrow \]
\[ \rightarrow \bar{y}_t = \bar{a}_t + \alpha \left( \bar{u}_t + \bar{k}_{t-1} + \bar{p}_{t-1} \right) + (1 - \alpha) \left( \bar{\Upsilon}_t + \bar{l}_t \right) - \bar{p}_t - \bar{p}_t \rightarrow \]
\[ \rightarrow \bar{y}_t = \bar{a}_t + \alpha \omega \bar{r}k_{t-1} + \alpha \left( \bar{k}_{t-1} + \bar{p}_{t-1} \right) + (1 - \alpha) \left( \bar{\Upsilon}_t + \bar{l}_t \right) - \bar{p}_t \] (in which \( \bar{p}_t = 0 \)).

Confidence:

\[ \bar{\Upsilon}_t = (pt_t t_t n_t)^\gamma; \]
\[ \bar{\Upsilon}_t = (pttn)^\gamma \] (steady state);
\[ 1 = 1 \] (steady state, in which \( pt = t = n = 1 \));
\[ ln \bar{\Upsilon} = \gamma (lnpt + lnt + lnn); \]
\[ ln \bar{\Upsilon} + \frac{\bar{\Upsilon}_t}{\bar{\Upsilon}} = \gamma (lnpt + lnt + lnn) + \frac{\gamma \bar{p}_t}{pt} + \frac{\gamma \bar{l}_t}{l} + \frac{\gamma \bar{n}_t}{n} \rightarrow \]
\[ \rightarrow \bar{\Upsilon}_t = \gamma \left( \bar{p}_t + \bar{l}_t + \bar{a}_t \right). \]

Real marginal cost:

\[ \phi_t = \alpha^{-\alpha} (1 - \alpha)^{\alpha - 1} a_{t-1}^{\alpha - 1} \bar{Y}_t^{-\alpha} r_k^{\alpha} w_{t}^{1-\alpha}; \]
\[ \phi_t = \alpha^{-\alpha} (1 - \alpha)^{\alpha - 1} a_{t}^{\alpha - 1} \bar{Y}_t^{-\alpha} r_k^{\alpha} (w_t pt_t)^1(1-\alpha) \] (normalisation);
\[ \phi = \alpha^{-\alpha} (1 - \alpha)^{\alpha - 1} a_{t}^{\alpha - 1} \bar{Y}_t^{-\alpha} r_k^{\alpha} (wpt)^1(1-\alpha) \] (steady state);
\[ \phi = \alpha^{-\alpha} (1 - \alpha)^{\alpha - 1} r_k^{\alpha} w_1(1-\alpha) \] (steady state, in which \( \bar{\Upsilon} = pt = a = 1 \));
\[ ln \phi = -a \ln a + (\alpha - 1) ln (1 - \alpha) - lna + (\alpha - 1) ln \bar{\Upsilon} + \alpha ln \bar{r}k + (1 - \alpha) (lnw + lnpt); \]
\[ ln \phi + \frac{\bar{\phi}_t}{\phi} = -a \ln a + (\alpha - 1) ln (1 - \alpha) - lna - \frac{\bar{a}_t}{a} + (\alpha - 1) ln \bar{\Upsilon} + \]
\[ + \frac{(\alpha - 1) \bar{\Upsilon}_t}{\bar{\Upsilon}} + \alpha ln \bar{r}k + \frac{\alpha r_k}{r_k} + (1 - \alpha) (lnw + lnpt) + \frac{(1 - \alpha) \bar{w}_t}{w} + \frac{(1 - \alpha) \bar{p}_t}{pt} \rightarrow \]
\[ \rightarrow \bar{\phi}_t = -\bar{a}_t + (\alpha - 1) \bar{\Upsilon}_t + \alpha r_k + (1 - \alpha) \bar{w}_t + (1 - \alpha) \bar{p}_t \rightarrow \]
\[ \rightarrow \bar{\phi}_t = \alpha r_k + (1 - \alpha) \left( \bar{w}_t + \bar{p}_t - \bar{\Upsilon}_t \right) - \bar{a}_t. \]

Aggregate capital:

\[ K_t = (1 - \delta) K_{t-1} + i; \]
\[ k_{i} pt_{i} = (1 - \delta) k_{i-1} pt_{i-1} + i \] (normalisation);
\[ k pt = (1 - \delta) kpt + i \] (steady state);
\[ \delta k = i \] (steady state, in which \( pt = 1 \)) \[ \rightarrow \]
\[ k = \delta^{-1} i \] (steady state);
\[ lnk + lnpt = ln [(1 - \delta) kpt + i]; \]
\[ lnk + \frac{\bar{k}_t}{k} + lnpt + \frac{\bar{p}_t}{pt} = ln [(1 - \delta) kpt + i] + \frac{(1 - \delta) pt \bar{k}_{t-1}}{(1 - \delta) kpt + i} + \frac{(1 - \delta) \bar{p}_t}{(1 - \delta) kpt + i} \rightarrow \]
\[
\begin{align*}
\dot{k}_t + \frac{\dot{p}_t}{pt} &= \frac{(1-\delta)pt\dot{k}_{t-1}}{kpt} + \frac{(1-\delta)k\dot{p}_{t-1}}{kpt} \\
\dot{c}_t + \dot{\pi}_t &= (1-\delta)\left(\dot{k}_{t-1} + \dot{p}_{t-1}\right) \\
\dot{k}_t &= (1-\delta)\left(\dot{k}_{t-1} + \dot{p}_{t-1}\right) - \dot{p}_t.
\end{align*}
\]

Aggregate capital utilisation:

\[Y_t = C_t + g + \Psi (u_t) K_{t-1};\]
\[ypt = c + g + \Psi (u_t) k_{t-1} \quad \text{(normalisation)};\]
\[ypt = c + g + \Psi (u) k pt \quad \text{(steady state)};\]
\[y = c + g + \Psi (1) k \quad \text{(steady state, in which } pt = 1);\]
\[y = c + g \quad \text{[steady state, in which } \Psi (1) = 0];\]
\[lny + lnpt = \ln [c + g + \Psi (u) k pt] ;\]
\[\dot{y}_t + \dot{p}_t = \frac{\dot{c}_t}{ypt} + \frac{\dot{p}_t}{ypt} + \frac{k\Psi'(u)(1)\tilde{u}_t}{ypt} + \frac{\Psi(u)\tilde{k}_{t-1}}{ypt} + \frac{\Psi(1)\tilde{p}_{t-1}}{ypt} \rightarrow \]
\[\dot{y}_t + \dot{p}_t = \left(\frac{\dot{c}_t + \dot{p}_t}{ypt}\right) + \left(\frac{k}{y}\right) \left[ r\tilde{u}_t + \Psi(1) \left(\tilde{k}_{t-1} + \tilde{p}_{t-1}\right) \right] \rightarrow \]
\[\dot{y}_t = \left(\frac{c}{y}\right) (\dot{c}_t + \dot{p}_t) + \left(\frac{k}{y}\right) r\tilde{u}_t r\dot{k}_t - \dot{p}_t.\]

One notices that parameters \(rk\), \(y^{-1}c\) and \(y^{-1}k\) respectively model steady state capital return, consumption to output ratio and capital to output ratio, being positive real numbers: \(rk\), \(y^{-1}c\), \(y^{-1}k \in \mathbb{R}_+^+\).

### 12. Parametrisation and solution

#### 12.1 Calibration.
The log-linearised laws of motion of the economy are consequently the following:

\[\hat{p}_t = \hat{p}_{t-1} + \varepsilon_{ptt} \quad \text{(permanent technology)};\]
\[\hat{n}_t = \rho_n \hat{n}_{t-1} + \varepsilon_{nt} \quad \text{(noise technology)};\]
\[\hat{\ell}_t = \rho_t \hat{\ell}_{t-1} + \varepsilon_{\ell t} \quad \text{(transitory technology)};\]
\[\hat{a}_t = \rho_a \hat{a}_{t-1} + \hat{p}_{t-1} + \hat{\ell}_{t-1} \quad \text{(real production technology)};\]
\[\hat{\pi}_t = \frac{(1-\xi)(1-\xi\beta)\hat{\ell}_t}{(1+\beta\tau)\xi} + \beta \xi \hat{\pi}_{t+1} + \frac{\tau \hat{\pi}_{t-1}}{(1+\beta\tau)} \quad \text{(inflation)};\]
\[\hat{w}_t = \sigma_t \hat{\ell}_t + \sigma_t \left(\hat{c}_t + \hat{\pi}_t\right) - \sigma_t h \left(\hat{c}_{t-1} + \hat{p}_{t-1}\right) \rightarrow \hat{\ell}_t - \hat{p}_t \quad \text{(real wage)};\]
\[\hat{\ell}_t = (1+\omega) r\dot{k}_t + \hat{k}_{t-1} - \hat{p}_{t-1} - \hat{w}_t - \hat{p}_t \quad \text{(aggregate labour)};\]
\[
\hat{c}_t = \frac{(1 - h) (\hat{Y}_t + E_t \hat{r}_{t+1} - r_{n_t})}{\sigma_c (1 + h)} + h (\hat{c}_{t-1} + \hat{p}_{t-1}) + \frac{E_t \hat{c}_{t+1} + E_t \hat{p}_{t+1}}{(1 + h)} - \hat{p}_t \) (real consumption);
\]
\[
\hat{r}_{n_t} = \rho_{rn} \hat{r}_{n_{t-1}} + (1 - \rho_{rn}) \left[ \phi_r \hat{n}_t + \phi_{\pi} (\hat{\pi}_t - \hat{\pi}_{t-1}) + \phi_y (\hat{y}_t + \hat{p}_t - \hat{y}_{t-1} - \hat{p}_{t-1}) \right] \) (nominal interest rate);
\]
\[
\hat{y}_t = \hat{a}_t + \alpha \omega \hat{k}_t + \alpha (\hat{k}_{t-1} + \hat{p}_{t-1}) + (1 - \alpha) (\hat{Y}_t + \hat{\epsilon}_t) - \hat{p}_t \) (aggregate real production);
\]
\[
\hat{Y}_t = \gamma (\hat{p}_t + \hat{\epsilon}_t + \hat{n}_t) \) (confidence);
\]
\[
\hat{\phi}_t = \alpha \hat{k}_t + (1 - \alpha) \left( \hat{w}_t + \hat{p}_t - \hat{Y}_t \right) - \hat{a}_t \) (real marginal cost);
\]
\[
\hat{k}_t = (1 - \delta) (\hat{k}_{t-1} + \hat{p}_{t-1}) - \hat{p}_t \) (aggregate capital);
\]
\[
\hat{y}_t = \left( \frac{c}{y} \right) (\hat{c}_t + \hat{p}_t) + \left( \frac{k}{y} \right) r_k \omega \hat{k}_t - \hat{p}_t \) (aggregate capital utilisation).
\]

Endogenous variables, exogenous shocks and parameters can be thus collected. Endogenous variables:
\[
\{ \hat{Y}_t, \hat{c}_t, \hat{\epsilon}_t, \hat{k}_t, \hat{\epsilon}_t, \hat{w}_t, \hat{r}_k, \hat{\pi}_t, \hat{\phi}_t, \hat{\phi}_y, \hat{n}_t, \hat{\epsilon}_t, \hat{r}_{n_t} \}_{t=0}^{\infty}.
\]
Exogenous shocks:
\[
\{ \varepsilon_{pt}, \varepsilon_{nt}, \varepsilon_{\ell t} \}_{t=0}^{\infty}.
\]
Parameters:
\[
\Theta = \{ \beta, h, \sigma_c, \sigma_t, \delta, \xi, \tau, \alpha, \gamma, \rho_n, \rho_t, \rho_a, \rho_{rn}, \phi_\pi, \phi_{\pi y}, \phi_y, \phi_{y y}, \omega, r_k, y^{-1}c, y^{-1}k \} \in \mathbb{R}_{++}.
\]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>USA</th>
<th>EA</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.99</td>
<td>0.99</td>
<td>Discount factor</td>
</tr>
<tr>
<td>$h$</td>
<td>0.69</td>
<td>0.573</td>
<td>Consumption habit</td>
</tr>
<tr>
<td>$\sigma_c$</td>
<td>1.62</td>
<td>1.353</td>
<td>Inter-temporal substitution inverse elasticity</td>
</tr>
<tr>
<td>$\sigma_t$</td>
<td>2.45</td>
<td>2.4</td>
<td>Labour inverse elasticity</td>
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<tr>
<td>$\delta$</td>
<td>0.025</td>
<td>0.025</td>
<td>Capital depreciation rate</td>
</tr>
<tr>
<td>$\xi$</td>
<td>0.87</td>
<td>0.908</td>
<td>Price adjustment failure fraction</td>
</tr>
<tr>
<td>$\tau$</td>
<td>0.66</td>
<td>0.469</td>
<td>Inflation indexation</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.24</td>
<td>0.3</td>
<td>Capital in output share</td>
</tr>
<tr>
<td>$\gamma_i$</td>
<td>$i = H, M, L$</td>
<td>$i = H, M, L$</td>
<td>Volition regime</td>
</tr>
<tr>
<td>$\rho_n$</td>
<td>0.65</td>
<td>0.65</td>
<td>Noise technology shock persistence</td>
</tr>
<tr>
<td>$\rho_t$</td>
<td>0.95</td>
<td>0.95</td>
<td>Transitory technology shock persistence</td>
</tr>
<tr>
<td>$\rho_a$</td>
<td>0.822</td>
<td>0.823</td>
<td>Production technology shock persistence</td>
</tr>
<tr>
<td>$\rho_{rn}$</td>
<td>0.88</td>
<td>0.961</td>
<td>Interest rate persistence</td>
</tr>
<tr>
<td>$\phi_\pi$</td>
<td>1.48</td>
<td>1.684</td>
<td>Inflation coefficient</td>
</tr>
<tr>
<td>$\phi_{\pi y}$</td>
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<td>0.14</td>
<td>Inflation gap coefficient</td>
</tr>
<tr>
<td>$\phi_y$</td>
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<td>0.099</td>
<td>Output coefficient</td>
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<td>$\phi_{y y}$</td>
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<td>0.159</td>
<td>Output gap coefficient</td>
</tr>
<tr>
<td>$\omega$</td>
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<td>5.917</td>
<td>Capital utilisation adjustment cost inverse elasticity</td>
</tr>
<tr>
<td>$r_k$</td>
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<td>0.0351</td>
<td>Steady state capital return</td>
</tr>
<tr>
<td>$y^{-1}c$</td>
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<td>0.6</td>
<td>Consumption to output ratio</td>
</tr>
<tr>
<td>$y^{-1}k$</td>
<td>6.8</td>
<td>6.8</td>
<td>Capital to output ratio</td>
</tr>
</tbody>
</table>

Table 3: Calibration

Note. Calibration of parameters for the USA and the EA, in which volition regimes $\gamma_i$ are calibrated as outlined in Table 2.

Such laws of motion can be cast into a linear rational expectations (LRE) model:
\[
Q(\Theta) x_t = R(\Theta) x_{t-1} + S \varepsilon_t,
\]
in which endogenous variables $x_t \in \mathbb{R}^{n_x}$, exogenous shocks $\varepsilon_t \in \mathbb{R}^{n_x}$, companion matrices $Q(\Theta)$, $R(\Theta) \in \mathbb{R}^{n_x \times n_x}$ and exogenous shock matrix $S \in \mathbb{R}^{n_x \times n_x}$, being composed of zeros and ones.
In the spirit of the Lucas critique, whereby space-time independence is necessary for policy robustness, parametrisation follows calibration over maximum likelihood or Bayesian estimation, and is according to the aforementioned parameter specifics and common sense at large; its exploitation of Smets and Wouters’s Bayesian estimation is thus only auxiliary and subordinated to the aforementioned parameter specifics and common sense at large, as formalised by the pertinent economic literature.

Bayesian estimation of parameters also noteworthy conflicts with log-linearisation of laws of motion, for its idiosyncratic spirit would require a correspondence in non-linear laws of motion, instead lost before. An ulterior reason for which calibration is preferred to maximum likelihood or Bayesian estimation of parameters concerns the desire to merely replicate the empirical SIRF patterns in question all else remaining equal. Calibration, in the regards of the USA and the EA, is reported in Table 3.

12.2 Unique and stable solution. As per Blanchard and Kahn, the LRE model in question evolves as follows:

\[
Qx_t = Rx_{t-1} + S\varepsilon_t \quad \iff \\
\begin{bmatrix}
Q_{11} & Q_{12} \\
Q_{21} & Q_{22}
\end{bmatrix}
\begin{bmatrix}
x_{1t} \\
x_{2t}
\end{bmatrix}
= \begin{bmatrix}
R_{11} & R_{12} \\
R_{21} & R_{22}
\end{bmatrix}
\begin{bmatrix}
x_{1t-1} & \varepsilon_{t-1} \\
0 & S
\end{bmatrix}
+ \begin{bmatrix}
S_1 \\
S_2
\end{bmatrix}
\varepsilon_t,
\]

in which non-expectational or past endogenous variables \(x_{1t} = \begin{bmatrix}
\bar{y}_t & \hat{\phi}_t & \hat{\pi}_t & \hat{\rho}_t & \hat{\tau}_t & \hat{\rho}_t & \hat{\pi}_t & \hat{\tau}_t & \hat{\rho}_t & \hat{\tau}_t
\end{bmatrix}^T\), explicable or future endogenous variables \(E_t{x}_{2t+1} = [E_t\hat{\phi}_{t+1} E_t\hat{\pi}_{t+1} E_t\hat{\rho}_{t+1}]^T\) and exogenous shocks \(\varepsilon_t = [\varepsilon_{pt} \varepsilon_{nl} \varepsilon_{lt}]^T\), that is, \(x_t \in \mathbb{R}^{14+3}, \varepsilon_t \in \mathbb{R}^3, Q, R \in \mathbb{R}^{(14+3)\times(14+3)}\) and \(S \in \mathbb{R}^{(14+3)\times3}\), one notices that sub-matrices \(Q_{21}\) and \(R_{22}\) are selector matrices and sub-matrix \(Q_{22} = R_{21} = 0\), since, \(\forall i = 1, \ldots, n_{x_2}\), non-expectational endogenous variable \(x_{1lt} = E_t{x}_{2lt}\), having observed no exogenous shock or no longer being there uncertainty in period \(t\).

A generalised Schur decomposition solves the generalised eigenvalue problem \(Qv = \lambda Rv\) such that matrices \(Q = HJ_0K^T\) and \(R = HJ_1K^T\) and generalised eigenvalue \(\lambda_i = \frac{J_{22}}{J_{21}}\), matrices \(J_0\) and \(J_1\) being eigenvalues situated along the respective diagonals.

Matrices \(J_0\) and \(J_1\) are upper triangular and matrices \(HH^T = HH^{-1} = KK^T = K^{-1} = I\); in detail, matrices \(J_0, J_1 \in \mathbb{R}^{n_x \times n_x}, H, K \in \mathbb{R}^{n_x \times n_x}, K^{-1} = H^{-1} \in \mathbb{R}^{n_x \times n_x}\).

Matrices \(J_0\) and \(J_1\) are additionally reordered such that sub-matrices \(J_{Q11}\) and \(J_{R11}\) respectively contain all eigenvalues smaller than one in modulus; accordingly, sub-matrices \(J_{Q22}\) and \(J_{R22}\) are reordered to contain all eigenvalues smaller than one in modulus: modulus eigenvalues \(|\lambda_{J_0}(\lambda)| < 1\) in \(J_{Q11}\) and \(|\lambda_{J_1}(\lambda)| \geq 1\) in \(J_{Q22}\) for characteristic polynomial \(J_0(\lambda) = J_0 - \lambda I\) in determinant \(\text{det} J_0(\lambda) = 0\); modulus eigenvalues \(|\lambda_{J_1}(\lambda)| < 1\) in \(J_{R11}\) and \(|\lambda_{J_2}(\lambda)| \geq 1\) in \(J_{R22}\) for characteristic polynomial \(J_1(\lambda) = J_1 - \lambda I\) in determinant \(\text{det} J_1(\lambda) = 0\). Formally:

\[
\begin{bmatrix}
Q_{11} & Q_{12} \\
Q_{21} & Q_{22}
\end{bmatrix}
= \begin{bmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{bmatrix}
\begin{bmatrix}
J_{Q11} & J_{Q12} \\
J_{R11} & J_{R12}
\end{bmatrix}
\begin{bmatrix}
K_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix},
\]

\[
\begin{bmatrix}
R_{11} & R_{12} \\
R_{21} & R_{22}
\end{bmatrix}
= \begin{bmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{bmatrix}
\begin{bmatrix}
J_{Q11} & J_{Q12} \\
J_{R11} & J_{R12}
\end{bmatrix}
\begin{bmatrix}
K_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix}.
\]

8https://en.wikipedia.org
9https://en.wikipedia.org
10https://en.wikipedia.org
The LRE model in question thus continues to evolve as follows:

\[
Q x_t = Rx_{t-1} + S \epsilon_t \rightarrow \\
\rightarrow H J_Q K^T x_t = H J_R K^T x_{t-1} + S \epsilon_t \rightarrow \\
\rightarrow H J_Q z_t = H J_R z_{t-1} + S \epsilon_t, \text{ in which } z_t = K^T x_t \rightarrow \\
\rightarrow J_Q z_t = J_R z_{t-1} + H^T S \epsilon_t \leftarrow \\
\leftarrow \begin{bmatrix} J_{Q11} & J_{Q12} \\ 0 & J_{Q22} \end{bmatrix} \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} J_{R11} & J_{R12} \\ 0 & J_{R22} \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix} + \begin{bmatrix} \hat{H}_{11} & \hat{H}_{12} \\ \hat{H}_{21} & \hat{H}_{22} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \epsilon_t = \\
= \begin{bmatrix} J_{R11} & J_{R12} \\ 0 & J_{R22} \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix} + \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \epsilon_t, \text{ in which } U = H^T S \rightarrow \\
\rightarrow J_{Q11} z_{1t} + J_{Q12} z_{2t} = J_{R11} z_{1t-1} + J_{R12} z_{2t-1} + U_1 \epsilon_t \text{ and } \\
J_{Q22} z_{2t} = J_{R22} z_{2t-1} + U_2 \epsilon_t \rightarrow \\
\rightarrow J_{R22} z_{2t-1} = J_{Q22} z_{2t} - U_2 \epsilon_t \rightarrow \\
\rightarrow z_{2t-1} = J^{-1}_{R22} J_{Q22} z_{2t} - J^{-1}_{R22} U_2 \epsilon_t \rightarrow \\
\rightarrow z_{2t} = J^{-1}_{R22} J_{Q22} E_t z_{2t+1} - J^{-1}_{R22} U_2 E_t \epsilon_{t+1} = \\
= J^{-1}_{R22} J_{Q22} \left[ J^{-1}_{R22} J_{Q22} E_t z_{2t+1} - J^{-1}_{R22} U_2 E_t \epsilon_{t+2} \right] - J^{-1}_{R22} U_2 E_t \epsilon_{t+1} = \\
= \left( J^{-1}_{R22} J_{Q22} \right)^2 E_t z_{2t+2} - J^{-1}_{R22} J_{Q22} U_2 E_t \epsilon_{t+2} - J^{-1}_{R22} U_2 E_t \epsilon_{t+1} \rightarrow \\
\rightarrow z_{2t} = \lim_{j \to \infty} \left( J^{-1}_{R22} J_{Q22} \right)^j E_t z_{2t+j} - \sum_{j=1}^{\infty} J^{-1}_{R22} J_{Q22} U_2 E_t \epsilon_{t+j} = 0,
\]

noticing the following facts. Expectational endogenous variable stationarity: \( \lim_{j \to \infty} E_t z_{2t+j} < \infty \). Eigenvalue instability: \( \lim_{j \to \infty} J^{-1}_{R22} = 0 \). Exogenous shock zero mean (i.e. white noise): \( \sum_{j=1}^{\infty} E_t \epsilon_{t+j} = 0 \).

For clarity, matrices \( U_1 \in \mathbb{R}^{n_{x_1} \times n_{x_2}} \) and \( U_2 \in \mathbb{R}^{n_{x_2} \times n_{x_2}} \). Since matrix

\[
\begin{bmatrix} \hat{K}_{11} \\ \hat{K}_{21} \\ \hat{K}_{22} \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t+1} \\ z_{1t} \end{bmatrix} = \begin{bmatrix} z_{1t} \\ Z_{2t} \end{bmatrix},
\]

there arise the following manipulations:

\[
0 = \hat{K}_{21} x_{1t} + \hat{K}_{22} E_t x_{2t+1} \rightarrow \\
\rightarrow \hat{K}_{21} x_{1t} = -\hat{K}_{22} E_t x_{2t+1} \rightarrow \\
\rightarrow E_t x_{2t+1} = -\hat{K}_{22} \hat{K}_{21} x_{1t} = -L_2 x_{1t},
\]
in which \( L_2 = -\hat{K}_{22}^{-1} \hat{K}_{21} \), provided \( n_{\lambda_2} = n_{x_2} \), and

\[
z_{1t} = \hat{K}_{11} x_{1t} + \hat{K}_{12} E_t x_{2t+1} = \hat{K}_{11} x_{1t} + \hat{K}_{12} \left( -\hat{K}_{22}^{-1} \hat{K}_{21} x_{1t} \right) = \\
= \left( \hat{K}_{11} - \hat{K}_{12} \hat{K}_{22}^{-1} \hat{K}_{21} \right) x_{1t} = \left( \hat{K}_{11} - \hat{K}_{12} \hat{K}_{22}^{-1} \hat{K}_{21} \right) x_{1t} = L_1 x_{1t},
\]
in which \( L_1 = \hat{K}_{11} - \hat{K}_{12} \hat{K}_{22}^{-1} \hat{K}_{21} \).

In detail, condition \( n_{\lambda_2} = n_{x_2} \) signifies that the cardinality of unstable generalised eigenvalues equals that of expectational endogenous variables, to the end of a unique and stable solution. Indeed, condition \( n_{\lambda_2} < n_{x_2} \) is indicative of indeterminacy and condition \( n_{\lambda_2} > n_{x_2} \) is indicative of no solution. The LRE model in question consequently finalises its evolution thus: since \( z_{2t} = 0 \) and \( z_{1t} = L_1 x_{1t} \),

30
\( J_{Q11}z_{lt} + J_{Q12}z_{lt} = J_{R11}z_{lt-1} + J_{R12}z_{lt-1} + U_{l}\varepsilon_t \longrightarrow \)
\( J_{Q11}z_{lt} = J_{R11}z_{lt-1} + U_{l}\varepsilon_t \longrightarrow \)
\( J_{Q11}L_1x_{lt} = J_{R11}L_1x_{lt-1} + U_{l}\varepsilon_t \longrightarrow \)
\( J_{Q11}x_{lt} = J_{R11}x_{lt-1} + L^{-1}_1U_{l}\varepsilon_t \longrightarrow \)
\( x_{lt} = J_{Q11}^{-1}J_{R11}x_{lt-1} + J_{Q11}^{-1}L^{-1}_1U_{l}\varepsilon_t \) and
\( \mathbb{E}x_{2t+1} = -L_2x_{lt} = -L_2\left(J_{Q11}^{-1}J_{R11}x_{lt-1} + J_{Q11}^{-1}L^{-1}_1U_{l}\varepsilon_t \right) \longrightarrow \)
\( \begin{bmatrix} x_{lt} \\ \mathbb{E}_t x_{2t+1} \end{bmatrix} = \begin{bmatrix} J_{Q11}^{-1}J_{R11} & 0 \\ -L_2J_{Q11}^{-1}J_{R11} & 0 \end{bmatrix} \begin{bmatrix} x_{lt-1} \\ \mathbb{E}_t-1x_{2t} \end{bmatrix} + \begin{bmatrix} J_{Q11}^{-1}L^{-1}_1U_1 \\ -L_2J_{Q11}^{-1}L^{-1}_1U_1 \end{bmatrix} \varepsilon_t \longleftarrow \)
\( \begin{bmatrix} x_{lt} \\ \mathbb{E}_t x_{2t+1} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & 0 \end{bmatrix} \begin{bmatrix} x_{lt-1} \\ \mathbb{E}_t-1x_{2t} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \varepsilon_t \longleftarrow \)
\( \longleftarrow x_t = Ax_{t-1} + B\varepsilon_t. \)

13. IRFs

13.1 IRF construction. Such a solution proper to the LRE model in question, computed in MATLAB or Octave by means of CEPR EMAP [1]'s Dynare, is more specifically identified as the transition or state equation of a linear time invariant (LTI) state space representation in discrete time, being itself a first order linear heterogeneous difference equation: \( \forall t \geq 1 \), function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that, \( \forall t \in \mathbb{Z} \), \( x_t = Ax_{t-1} + B\varepsilon_t \), in which states \( x_t \in \mathbb{R}^{n_x} \), inputs \( \varepsilon_t \in \mathbb{R}^{n_x} \), companion matrix \( A \in \mathbb{R}^{n_x \times n_x} \) and input matrix \( B \in \mathbb{R}^{n_x \times n_x} \). It is also a fundamental \( VAR(1) \) process, because of companion matrix \( A\)'s stability, thereby bearing the potential to be rewritten as a causal \( VMA(\infty) \) process:

\[ x_t = Ax_{t-1} + B\varepsilon_t \quad [\text{fundamental } VAR(1)] \longrightarrow \]
\[ \rightarrow (I - AL)x_t = A(L)x_t = B\varepsilon_t, \]

in which operator \( L : \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that \( L = x_{t-1}^{-1}x_{t-1} \longrightarrow \)
\[ \rightarrow x_t = A^{-1}(L)B\varepsilon_t = \sum_{j=0}^{\infty} A^jL^jB\varepsilon_t = \sum_{j=0}^{\infty} A^jB\varepsilon_{t-j} \quad [\text{causal } VMA(\infty)], \]

since, \( \forall|s| < 1 \) and operator \( k : \mathbb{R} \rightarrow \mathbb{R} \), \( \lim_{n \rightarrow \infty} S = \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} s^j k^j = (1 - sk)^{-1} (1 - s^n k^n) = (1 - sk)^{-1} \), because \( S - Ssk = (1 - sk)S = \sum_{j=0}^{n-1} s^j k^j - \sum_{j=0}^{n-1} s^j k^{j+1} = s^0 k^0 - s^n k^n = 1 - s^n k^n \), thus, \( \sum_{j=0}^{\infty} A^jL^j = (I - AL)^{-1} = A^{-1}(L) \) if and only if modulus eigenvalues \( |\lambda_{A(\lambda)}| < 1 \) for characteristic polynomial \( A(\lambda) = A - I \) in determinant \( \det[A(\lambda)] = 0 \), which is equivalent to stating if and only if trace \( tr(AA^\top) = \sum_{i, j=1}^{n_x} a_{ij}a_{ji} = \sum_{i, j=1}^{n_x} a_{ij}^2 < \infty \), whence

\[ \frac{\partial x_t}{\partial \varepsilon_{t-j}} = A^jB \quad [\text{IRF coefficients}]. \]

In fact, first order IRFs are analytically constructed thus: \( \forall j \in \mathbb{N} \) and exogenous shock \( \varepsilon_t \sim \mathcal{N}(0, \sigma^2) \), IRF \( \mathbb{I}_{x_t} := \mathbb{E}_t x_{t+j} - \mathbb{E}_{t-1} x_{t+j} | x_t = \tilde{\varepsilon} \), in which \( \tilde{\varepsilon} \) is a realisation of \( \varepsilon_t \).

Assuming that exogenous shock realisation \( \tilde{\varepsilon} = \sigma \), the following unfolds:

\[ \mathbb{E}_t x_t = \mathbb{E}_t (Ax_{t-1} + B\varepsilon_t) = \mathbb{E}_t (Ax_{t-1} + B\tilde{\varepsilon}) = \mathbb{E}_t (Ax_{t-1} + B\sigma) = Ax_{t-1} + B\sigma, \]
since \( \mathbb{E}_t x_{t-1} = x_{t-1} \) and \( \mathbb{E}_t \sigma = \sigma \) (observations),
\[ \mathbb{E}_t x_{t+1} = \mathbb{E}_t (Ax_t + B\varepsilon_{t+1}) = Ax_t = A(Ax_{t-1} + B\sigma) = A^2 x_{t-1} + AB\sigma, \]

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\[ \mathbb{E}_t x_{t+2} = \mathbb{E}_t (A x_{t+1} + B \varepsilon_{t+2}) = \mathbb{E}_t A x_{t+1} = A (A^2 x_{t-1} + AB \sigma) = A^3 x_{t-1} + A^2 B \sigma, \]

whereby \( I_{x_i} = \mathbb{E}_t x_{t+j} - \mathbb{E}_t x_{t+j} = A^{j+1} x_{t-1} + A^{j} B \sigma - A^{j+1} x_{t-1} = A^{j} B \sigma \) such that \( I_{x_0} = A^0 B \sigma = B \sigma, I_{x_2} = A^2 B \sigma, I_{x_5} = A^5 B \sigma. \)

13.2 IRF commentary. The empirical SIRF patterns presented in Table 1 are all replicated by accounting for all exogenous shocks underlying changes in confidence \( \hat{Y} \) and all volition regimes \( \gamma \). This establishes volition \( \gamma \) as the ultimate determiner of fluctuations in real economic activity in the face of changes in confidence \( \hat{Y} \). For clarity, fluctuations in real economic activity refer to its cycle component, rather its trend component, which instead refers to the economy’s balanced growth or decline path.

A pattern of immediate irreversibility is exhibited by a combination entailing an exogenous shock in permanent technology \( \varepsilon_{ptt} \) and a high or medium volition regime \( \gamma_{H,M} \). A pattern of delayed irreversibility is exhibited by a combination entailing an exogenous shock in permanent technology \( \varepsilon_{ptt} \) and a low volition regime \( \gamma_L \).

At one order of magnitude below the others, a pattern of immediate reversibility is exhibited by a combination entailing an exogenous shock in noise technology \( \varepsilon_{nt} \) and a high or medium volition regime \( \gamma_{H,M} \). A pattern of delayed reversibility is exhibited by a combination entailing an exogenous shock in transitory technology \( \varepsilon_{tt} \) and any volition regime \( \gamma \).

The regime of volition \( \gamma \) therefore gives rise to a compromise between endogenous growth and a “boom and bust” cycle. An exogenous shock in noise technology \( \varepsilon_{nt} \) gives rise to a “boom and bust” cycle whenever the regime of volition \( \gamma \) be non-negligible. On the other hand, an exogenous shock in permanent technology \( \varepsilon_{ptt} \) in the face of a non-negligible regime of volition \( \gamma \) causes endogenous growth.

Correspondingly, a “boom and bust” cycle is avoided in the face of an exogenous shock in noise technology \( \varepsilon_{nt} \) whenever the regime of volition \( \gamma \) be negligible, although avoiding endogenous growth too in the face of an exogenous shock in permanent technology \( \varepsilon_{ptt} \).
14. MINIMAL POOR MAN’S INVERTIBILITY CONDITION

14.1 Poor man’s invertibility condition. For a selector matrix suitably composed of zeros and ones there arises a measurement or observation equation proper to an LTI state space representation in discrete time: \( M x_t = M A x_{t-1} + MB \epsilon_t \implies y_t = C x_{t-1} + D \epsilon_t \), in which selector matrix \( M \in \mathbb{R}^{n_y \times n_x} \), outputs or observables \( y_t \in \mathbb{R}^{n_y} \), companion matrix \( C \in \mathbb{R}^{n_y \times n_x} \) and input matrix \( D \in \mathbb{R}^{n_y \times n_x} \). Assume input matrix \( D \) to be invertible and thus square: dimension \( n_y = n_x \leq n_x \in \mathbb{N}^+_0 \); in fact, \( y_t = \left[ \hat{y}_t \; \check{c}_t \; \hat{y}_t \right]^T \). Then,

\[
y_t = C x_{t-1} + D \epsilon_t \implies \quad \implies D \epsilon_t = y_t - C x_{t-1} \implies \quad \implies \epsilon_t = D^{-1} (y_t - C x_{t-1}) \implies \quad \implies x_t = A x_{t-1} + B D^{-1} (y_t - C x_{t-1}) \implies \quad \implies x_t = (A - B D^{-1} C) x_{t-1} + B D^{-1} y_t = F x_{t-1} + B D^{-1} y_t \implies \quad \implies x_t - F x_{t-1} = (I - F L) x_t = F (L) x_t = B D^{-1} y_t \implies \quad \implies x_t = F^{-1} (L) B D^{-1} y_t = \sum_{j=0}^{\infty} F^j L^1 B D^{-1} y_{t-j} = \sum_{j=0}^{\infty} F^j B D^{-1} y_{t-j-1} + D \epsilon_t \]

being a VAR representation of states \( x_t \) in outputs \( y_t \).

14.2 Minimality. For controllability matrix \( C = [B \cdots A^{n_x-1} B] \) and observability matrix \( O = [C \cdots CA^{n_x-1}]^T \) the LTI state space representation is minimal if and only if dimension \( n_x = r_C = r_O \). If it is non-minimal it is then discretionally reduced to minimality as follows:

(i) if dimension \( n_x > r_C \) (i.e. non-controllable) one then constructs similarity transformation matrix \( T = [C r_C v_{n_x-r_C}] \) for vector \( \vec{x} \epsilon_t = T^{-1} x_t \) and matrices \( \vec{A}_{\vec{x}} = T^{-1} A T \), \( \vec{B}_{\vec{c}} = T^{-1} B \), \( \vec{C}_{\vec{x}} = C T \), \( \vec{C}_{\vec{c}} = T^{-1} C \) and \( \vec{O}_{\vec{c}} = \vec{O} T \), in which the first \( r_C \) states are controllable, namely, vector \( \vec{x} \epsilon_t \) and matrices \( \vec{A}_{\vec{x}} \), \( \vec{B}_{\vec{c}} \), \( \vec{C}_{\vec{x}} \), \( \vec{C}_{\vec{c}} \) and \( \vec{O}_{\vec{c}} \); if dimension \( n_x = r_C \) (i.e. controllable) one then directly acknowledges vector \( \vec{x} \epsilon_{\vec{c}} \) and matrices \( \vec{A}_{\vec{x}} \), \( \vec{B}_{\vec{c}} \), \( \vec{C}_{\vec{x}} \), \( \vec{C}_{\vec{c}} \) and \( \vec{O}_{\vec{c}} \);

(ii) if dimension \( n_x > r_{\vec{c}} \) (i.e. non-observable) one then constructs similarity transformation matrix \( T = [\vec{O}_{\vec{c}} \vec{O}_{\vec{c}} v_{n_x-r_{\vec{c}}} \] for vector \( \vec{x} \epsilon_{\vec{c}} = T^{-1} x_{\vec{c}} \) and matrices \( \vec{A}_{\vec{c}} = T^{-1} A_{\vec{c}} T \), \( \vec{B}_{\vec{c}} = T^{-1} B_{\vec{c}} \), \( \vec{C}_{\vec{c}} = \vec{C}_{\vec{c}} \), \( \vec{O}_{\vec{c}} = \vec{O} T \), in which the first \( r_{\vec{c}} \) are controllable and observable (i.e. minimal), namely, vector \( \vec{x} \epsilon_{\vec{c}} = x_{\vec{c}} \) and matrices \( \vec{A}_{\vec{c}} = A_{\vec{c}} \), \( \vec{B}_{\vec{c}} = B_{\vec{c}} \), \( \vec{C}_{\vec{c}} = C_{\vec{c}} \), \( \vec{C}_{\vec{c}} = C_{\vec{c}} \) and \( \vec{O}_{\vec{c}} = \vec{O}_{\vec{c}} \); if dimension \( n_{\vec{x}} > r_{\vec{c}} \) (i.e. controllable and observable, minimal) one then directly acknowledges vector \( \vec{x} \epsilon_{\vec{c}} = x_{\vec{c}} \) and matrices \( \vec{A}_{\vec{c}} = A_{\vec{c}} \), \( \vec{B}_{\vec{c}} = B_{\vec{c}} \), \( \vec{C}_{\vec{c}} = C_{\vec{c}} \), \( \vec{C}_{\vec{c}} = C_{\vec{c}} \) and \( \vec{O}_{\vec{c}} = \vec{O}_{\vec{c}} \).

It follows that minimal transition and measurement equations

\[
x_{\vec{c}} t = A_{\vec{c}} x_{\vec{c}} t-1 + B_{\vec{c}} \epsilon_t \quad \text{and} \quad y_t = C_{\vec{c}} x_{\vec{c}} t-1 + D \epsilon_t,
\]

in which dimension \( n_{\vec{x}} = r_{\vec{c}} = r_O \), give rise to minimal fundamental \( \text{VAR}(\infty) \) \( y_t = C_m \sum_{j=0}^{\infty} F^j_m B_m D^{-1} y_{t-j-1} + D \epsilon_t \) for minimal matrix \( F_m = A_m - B_m D^{-1} C_m \).

In minimal LTI state space representations the IRFs of the transition equation and the coefficients of the VAR representation of states \( x_t \) in outputs \( y_t \) are invariant, as especially remarked by Franchi [13]: \( \forall j \in \mathbb{N}^+_0 \).
\( N_+ \), \( CA'B = C_m A'n B_m \neq 0 \), from \( x_t = \sum_{j=0}^{\infty} A' B \xi_{t-j} \rightarrow y_t = C x_{t-1} + D \xi_t = C \sum_{j=0}^{\infty} A' B \xi_{t-j} + D \xi_t \), and \( CF'B = C_m F_m B_m \neq 0 \), from \( y_t = C_m \sum_{j=0}^{\infty} F_m B_m D^{-1} y_{t-j-1} + D \xi_t. \) Absent loss of generality, one can therefore conduct suitable evaluations in terms of the minimal poor man’s invertibility condition (mPMIC).

As borne out by the attendant eigenvalues in the annexed code, the mPMICs for the USA and the EA hereby computed fail to give rise to a VAR representation of states \( x_t \) in outputs \( y_t \) across all three volition regimes \( \gamma \) except for a medium volition regime \( \gamma_M \) with regard to the EA.

### 14.3 Discussion

As implicitly shown by Sims [19], if the mPMIC fails it need not mean that states \( x_t \) may not be practically represented in outputs \( y_t \) by means of VARs, thereby recovering the nature of the underlying exogenous shocks in the observed endogenous variables under consideration all the same.

In fact, Saccal [18] showed that for any minimal transition equation the \( VMA(0) \) representation of the form \( y_t = D \xi_t \) is almost sure, in all of its empirical futility, whereby the adjunction of other VAR representations of states \( x_t \) in outputs \( y_t \) is probabilistically negligible. The recovery of the underlying exogenous shocks in the observed endogenous variables is consequently almost always tied to a structural model other than that of the transition equation.

The unique and stable solution of the first order approximation of the present NK-DSGE model is consequently salvaged by recourse to axiomatic abstraction, deeming it logically valid and its hypotheses no less than probable, which judgement appears to be confirmed by the successful replication of the empirical SIRF patterns at hand.

### 15. Conclusion

Economic literature exhibits a variety of empirical SIRF patterns in real economic activity in the face of changes in confidence or sentiment, with particular regard to the USA and the EA. This work successfully endeavoured to replicate them in the orbit of a NK-DSGE model especially characterised by macroeconomic agents and derived from start to end. Confidence \( T_t \) has been specifically modelled as an endogenous variable characterised by a coalescence of two technology processes \( p t_t \) and \( t_t \), permanent and transitory, and one noise process \( n_t \), being globally regulated by a degree of volition \( \gamma \). The first two processes affect real production technology \( a_t \) with a lag delay, while the third does not. Short run responses to changes in confidence \( T_t \) are therefore displayed whenever confidence \( T_t \) shift real consumption \( c_t \) and aggregate labour \( l_t \). In turn, confidence \( T_t \) shifts real consumption \( c_t \) and aggregate labour \( l_t \) whenever volition \( \gamma \) be not infinitesimal. Whenever volition \( \gamma \) were infinitesimal, by contrast, exogenous shocks in noise \( n_t \) would not cause fluctuations in real economic activity at all.

### References

journal of economics.


[18] Saccal A. (2023) “A finite, empirically useless and almost sure VAR representation for all minimal transition equations”, MPRA.


APPENDIX

This is the dynare code for a unique and stable solution of the first order approximation of the NK-DSGE model at hand.

```plaintext
/* A role for confidence: volition regimes and news (Alessandro Saccal) */

var w l c pt rk rn pi y phi a k Upsilon t n; // Endogenous variables
varexo e_pt e_t e_n; // Exogenous shocks
parameters sigma_l sigma_c h omega rho_rn phi_pi phi_y phi_pi_g phi_y_g beta tau xi omega rkss alpha c_y k_y rho_a rho_t rho_n gamma; // Parameters

/* USA parameters */
 rho_t=0.95; // Transitory technology persistence
 rho_n=0.65; // Noise technology persistence
 rho_a=0.822; // Production technology persistence
 gamma=1; // Volition regime; 1, 0.5, 0.0001
 beta=0.99; // Discount factor
 tau=0.469; // Inflation indexation
 xi=0.908; // Price adjustment failure fraction
 sigma_l=2.4; // Labour inverse elasticity
 h=0.573; // Consumption habit
 omega=5.917; // Capital utilisation adjustment cost inverse elasticity
 rho_rn=0.961; // Interest rate persistence
 phi_pi=1.684; // Inflation coefficient
 phi_y=0.099; // Output coefficient
 phi_pi_g=0.14; // Inflation gap coefficient
 phi_y_g=0.159; // Output gap coefficient
 A=0.025; // Capital depreciation rate
 rkss=0.0351; // Steady state capital return
 alpha=0.3; // Capital in output share
 c_y=0.6; // Consumption to output ratio
 k_y=8.8; // Capital to output ratio

/* EA parameters */
 rho_t=0.95; // Transitory technology persistence
 rho_n=0.65; // Noise technology persistence
 rho_a=0.823; // Production technology persistence
 gamma=1; // Volition regime; 1, 0.5, 0.0001
 beta=0.99; // Discount factor
 tau=0.469; // Inflation indexation
 xi=0.908; // Price adjustment failure fraction
 sigma_l=2.4; // Labour inverse elasticity
 h=0.573; // Consumption habit
 omega=5.917; // Capital utilisation adjustment cost inverse elasticity
 rho_rn=0.961; // Interest rate persistence
 phi_pi=1.684; // Inflation coefficient
 phi_y=0.099; // Output coefficient
 phi_pi_g=0.14; // Inflation gap coefficient
 phi_y_g=0.159; // Output gap coefficient
 A=0.025; // Capital depreciation rate
 rkss=0.0351; // Steady state capital return
 alpha=0.3; // Capital in output share
 c_y=0.6; // Consumption to output ratio
 k_y=8.8; // Capital to output ratio

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\sigma_l=2.45; // Labour inverse elasticity
\sigma_c=1.62; // Inter temporal substitution inverse elasticity
h=0.69; // Consumption habit
omega=3.23; // Capital utilisation adjustment cost inverse elasticity
\rho_{rn}=0.88; // Interest rate persistence
\phi_{\pi}=1.48; // Inflation coefficient
\phi_{y}=0.08; // Output coefficient
\phi_{\pi_g}=0.24; // Inflation gap coefficient
\phi_{y_g}=0.24; // Output gap coefficient
\Delta=0.025; // Capital depreciation rate
rkss=0.0351; // Steady state capital return
alpha=0.24; // Capital in output share
c_y=0.65; // Consumption to output ratio
k_y=6.8; // Capital to output ratio

\text{model(linear);} ;
\text{pt}=\text{pt}(-1)+\varepsilon_{pt}; // Permanent technology
\text{t}=\rho_{t}\text{t}(-1)+\varepsilon_{t}; // Transitory technology
\text{n}=\rho_{n}\text{n}(-1)+\varepsilon_{n}; // Noise technology
\text{a}=\rho_{a}\text{a}(-1)+\text{pt}(-1)+\varepsilon_{a}; // Production technology
\text{pi}=\left(\left(1-\xi\right)\left(1-\beta\xi\right)\right)/\left(1+\beta\tau\right)\xi\phi+\beta/(1+\beta\tau)\text{pi}(+1)+\tau/(1+\beta\tau)\text{pi}(-1); ... // Inflation
\text{w}=\sigma_{l}\text{l}+(\sigma_{c}/\left(1-h\right))\left(\text{c}+\text{pt}\right)-(\sigma_{c}/\left(1-h\right))\left(\text{c}(-1)+\text{pt}(-1)\right)-\Upsilon_{pt}; // Real wage
\text{l}=\left(1+\omega\right)\text{k}+\text{k}(-1)+\text{pt}(-1)-\varepsilon_{l}; // Aggregate labour
\text{c}=\left(1-h\right)/\left(\sigma_{c}+\left(1+h\right)\right)\Upsilon_{pt}+\text{pi}(+1)+\text{pi}(+1)\tau/(1+\beta\tau)+\text{pi}(-1)+\text{pi}(-1); ... // Real consumption
\text{rn}=\rho_{rn}\text{rn}(-1)+\left(1-\rho_{rn}\right)\phi_{\pi}\text{pi}+\phi_{\pi_g}\text{pi}(+1)+\phi_{y}\left(\text{y}+\text{pt}\right)+\phi_{y_g}\left(\text{y}+\text{pt}+\text{pt}(-1)\right)+\phi_{\pi_g}\left(\text{pi}+\text{pi}(-1)\right)+\phi_{y_g}\left(\text{y}+\text{pt}+\text{pt}(-1)\right); ... // Nominal interest rate
\text{y}=\alpha\omega\text{rk}+\alpha\left(k(-1)+\text{pt}(-1)\right)+\left(1-\alpha\right)\Upsilon_{pt}; // Aggregate real ... production
\Upsilon_{\text{c}}=\gamma\left(\text{pt}+\text{t}+\text{n}\right); // Confidence
\phi=\alpha\text{rk}+\phi_{\pi}\text{pi}+\phi_{\pi_g}\text{pi}(+1)+\phi_{y}\left(\text{y}+\text{pt}\right)+\phi_{y_g}\left(\text{y}+\text{pt}+\text{pt}(-1)\right)+\phi_{\pi_g}\left(\text{pi}+\text{pi}(-1)\right)+\phi_{y_g}\left(\text{y}+\text{pt}+\text{pt}(-1)\right)+\phi_{\pi_g}\left(\text{pi}+\text{pi}(-1)\right); ... // Real marginal cost
\text{k}=\left(1-\Delta\right)\left(k(-1)+\text{pt}(-1)\right)-\varepsilon_{k}; // Aggregate capital
\text{y}=\text{c}_{y}\left(c+\text{pt}\right)+\text{k}_{y}\text{rkss}\omega_{a}\text{rk}+\varepsilon_{k}; // Aggregate capital utilisation
\text{end};
\text{initial};
w=0; l=0; c=0; pt=0; rk=0; rn=0; pi=0; y=0; \phi=0; a=0; k=0; \Upsilon_{0}=0; t=0; n=0;
\text{end};
\text{steady};
\text{check}; // Rational expectations stable unique solution check
\text{shocks};
\text{var e_{pt}; stderr 0.01;}
\text{var e_{t}; stderr 0.01;}
\text{var e_{n}; stderr 0.01;}
\text{end};
\text{stoch_simul(irf=40, order=1) c y; // graph_format=(none) and nograph can be added to ... omit first order IRF graphs}
102 varobs Upsilon c y;
103 [result, eigenvalue_modulo, A, B, C, D]=ABCD_test(M_, options_, oo_, 0); // 0 can be ...
               changed to 1 for minimality