# Demand Theory for Poverty and Affluence: A Contribution to Utility Theory

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# DEMAND THEORY FOR POVERTY AND AFFLUENCE: A CONTRIBUTION TO UTILITY THEORY[1]

**by**
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**ABSTRACT:**

Van Praag (1968) developed a multiplicative utility function, based on 'leaning S-shaped, bounded cardinal utilities', comprising *increasing* marginal utility (MU) initially, representing 'deprivation', leading to diminishing MU representing 'sufficiency'. A new separability rule, based on the satisfaction of human needs, suggests when to multiply and when to add utilities.

A functional form is derived to explore the theoretical effects of *adding* two S-shaped bounded utilities, yielding both convex- and concave-to-the-origin indifference curves, the latter defining 'dysfunctional poverty'. The convex-to-the-origin indifference curves potentially provide all of superior-normal, inferior-normal and Giffen *responses*. Each derived structural form, including labour supply, manifests a discontinuity, an envelope curve and high elasticities associated with deprivation.

This provides an integrating framework for analysing utility and demand where the emphasis is on people and the satisfaction of needs, with applications in: housing, health services, education, wellbeing; poverty and inequality studies; tax and benefit policy analysis; and behavioural economics.

(150 words)

**KEYWORDS**: Increasing marginal utility, additive utilities, absolute poverty line, Giffen good, reservation wage.

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**DEMAND THEORY FOR POVERTY AND AFFLUENCE:**
**A CONTRIBUTION TO UTILITY THEORY**
**by**

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## TWO-PAGE SUMMARY

### PROPOSITION 1. THE SHAPE OF THE UTILITY FUNCTION.

An individual's experience of consumption, $X_i$, of a commodity i, (good, service or event), $X_i \geq 0$, for i = 1, 2, …, m, can be represented by a continuous, smooth, single-valued, utility function, $U_i = \mathrm{U}(X_i)$, $0 \leq U_i \leq 1$, shaped like a leaning '*S*' curve, bounded below and above, where marginal utility, $U_i'$, is always less than infinity.

| | | | | | | |
|---|---|---|---|---|---|---|
| $U_i''$ | $> 0$ | $> 0$ | $= 0$ | $< 0$ | $< 0$ | $< 0$ |
| $U_i'$ | $= 0$ | $0 < U_i < \infty$ | $0 < U_i < \infty$ | $0 < U_i < \infty$ | $= 0$ | $0 > U_i$ |
| $U_i$ | $= 0$ | $0 > U_i > 1$ | $0 > U_i > 1$ | $0 > U_i > 1$ | $= 1$ | $0 < U_i < 1$ |
| $X_i$ | $X_i = 0$ | $0 < X_i < \mu_i$ | $X_i = \mu_i$ | $\mu_i < X_i < \mathrm{sat}_i$ | $X_i = \mathrm{sat}_i$ | $X_i >$ finite $\mathrm{sat}_i$ |
| Individual experience: | minimum: | deprived | inflection: subsistence | sufficiency | maximum: satiated | surfeit |

A point of inflection occurs at $X_i = \mu_i$, (where $\mu_i$ is a subsistence parameter comparable to the survival parameters in some econometric demand models). The consumer is sated in $X_i$ when utility reaches a maximum of 1 at $X_i = \mathrm{sat}_i$. If $\mathrm{sat}_i < \infty$, as $X_i$ increases further, utility decreases and is diminishing. This is based on the seminal work of B M S Van Praag (1968).

This theory is based on bounded cardinal utility. The steepness of the slope, $U_i'$, around the point of inflection, represented by $\sigma_i$, represents intensity-of-need for the $i^{th}$ commodity. The smaller is $\sigma_i$, the more intense is the need.

### PROPOSITION 2. THE SEPARABILITY OF COMMODITIES.

The utilities of a group of commodities that satisfy the same need are multiplicatively related (with or without dependence), and the utilities of groups of satisfiers, each group satisfying a different need, are additively related.
It is assumed that there is a finite number of fundamental human needs, which are universal and a-historic. The fulfilment of needs cannot be observed except by the effect of their satisfiers. Needs are satisfied by an infinite diversity of culturally-determined satisfiers.

The following may be noted from the **INDIFFERENCE CURVE MAPS** created from an additive utility function, (see Figures 2 and 5):

* The subsistence parameters, at **$X_1 = \mu_1$ and $X_2 = \mu_2$, create borders** on the left and lower parts of the indifference map, representing deprivations in each commodity.

* **In Figure 2, straight-line indifference curve, AB**, has slope $-\sigma_2/\sigma_1$, and separates convex-to-the-origin indifference curves from the concave-to-the-origin ones in the non-solution 'dysfunctional poverty' space, OAB. In figure 5, for the leisure-consumption choice, the comparable line, CD, can be regarded as part of an *Absolute* Poverty Line. Convex and concave curves together lead to discontinuities.

* In the convex-to-the-origin part of the indifference curve map, each commodity can provide **ultra-superior, superior-normal, inferior-normal and inferior-Giffen** *experiences*. These are marked for commodity 1 on Figures 2 and 5. The experience of a good as 'inferior' can be regarded as 'functional poverty', where the consumer is sufficient in only one commodity.

For the **LEISURE-CONSUMPTION** choice in Figure 5, (with consumption, $X_2 \geq 0$, on the vertical axis), leisure, ($0 \leq X_1 \leq 168$ hours pw, on the horizontal axis), is experienced as both inferior-normal and inferior-Giffen when $X_1 \geq \mu_1$ for $X_2 \leq \mu_2$, (ie. the consumer is not deprived of $X_1$, but is deprived of $X_2$). The straight-line indifference curve in Figure 5 is labelled CD.

* In Figure 5, the point ($\mu_1$, $\mu_2$) is labelled E, and EF is the locus of points on the indifference curves where $dX_2/dX_1 = -\sigma_2/\sigma_1$. The convex-to-the-origin indifference curves part of the map can now be divided into **4 areas, labelled as L, M, N and R**.

Let us assume that the utility function is maximised subject to a linear budget.
The **LINEAR BUDGET** is expressed as $Z_1.p_1 + Z_2.p_2$, where $Z_1$ is an endowment of time in a given period, (eg. 168 hours pw), priced at $p_1$, and $Z_2$ is an endowment of material goods, priced at $p_2$; $Z_2.p_2$ is unearned income. $(Z_1 - X_1)$ measures hours of paid work; $(Z_1 - X_1).p_1/p_2$ is real earnings; $X_2 = (Z_1 - X_1).p_1/p_2 + Z_2$.

**Backward-bending LABOUR SUPPLY (Ls) CURVES** are derived, see Figure 6; with real wage rate, $p_1/p_2$ on the vertical axis; parameter $\sigma_2/\sigma_1$ can be interpreted as a 'natural wage-rate'; and labour hours, $(Z_1 - X_1)$, with parameter $(Z_1 - \mu_1)$, is on the horizontal axis.

* **Note**: The **four areas, L, M, N and R** from the indifference map can be identified with the corresponding areas on the Ls curves diagram. Area R leads to elastic *backward-sloping curves* for high-wages, deprived of leisure; Areas N and M yield inelastic, high- and low-waged Ls curves respectively, and neither is deprived of either leisure or consumption; Area L leads to elastic, low-waged, Ls curves, deprived of income but not of leisure.

* **An envelope curve** bounds the lower limit of the Ls curves, associated with the change from inferior to superior characteristics, representing the boundary at $X_2 = \mu_2$, for $X_1 \geq \mu_1$.

* **The intercepts of the Ls curves on the $p_1/p_2$ axis represent the 'reservation wage'**, the consumer's minimum acceptable wage-rate. It can be shown that the reservation wage is a U-shaped function of unearned consumption, $Z_2$, being highest when $Z_2 = 0$, flattening out between C and F, with a minimum when $Z_2 = \mu_2$, and increasing again for $Z_2 > F$.

* **POLICY IMPLICATIONS**. A low endowment of material goods**,** such that $0 < Z_2 < C$, (via a state benefit, for instance), leads to a polarised outcome in terms of consumption. Faced with a high wage, ($p_1/p_2 > \sigma_2/\sigma_1$), the consumer's solution could be in area R of the indifference curve map, deprived of leisure. Faced with a low wage, the consumer remains unemployed, and deprived of consumption, as a non-tangential corner solution. An individual 'chooses' his/her cheaper deprivation. This may explain how governments can spend a lot of money, while still keeping people well below subsistence.

* For $C < Z_2 < \mu_2$, the low-waged individual, working part-time and responding to a change in wage rates, would still be deprived of consumption. A National Minimum Wage (NMW) could provide an incentive for an individual to work longer hours.

* For $Z_2 \geq \mu_2$, the consumer, who could be either low- or high-waged, would be deprived of neither consumption nor leisure.

# DEMAND THEORY FOR POVERTY AND AFFLUENCE: A CONTRIBUTION TO UTILITY THEORY

> ‘How can we convince a sceptic that this “law of demand’ is really true of all consumers, all times, all commodities? Not by a few (4 or 4,000) selected examples, surely. Not by a rigorous theoretical proof, for none exists – it is an empirical rule. Not by stating, what is true, that economists believe it, for we could be wrong. Perhaps as persuasive proof as is readily summarised is this: if an economist were to demonstrate its failure in a particular market at a particular time, he would be assured of immortality, professionally speaking, and a rapid promotion. Since most economists would not dislike either reward, we may assume that the total absence of exceptions is not from lack of trying to find them.’
>
> (Stigler, 1966: p.24)

## I. INTRODUCTION

The first purpose of this paper is to present two propositions about utility, the first suggesting that its most likely shape is that of a leaning S-shaped bounded cardinal utility function and the second providing a decision rule about when to multiply utilities and when to add them. The second purpose of the paper, using a functional form derived in the appendix, is to explore the outcomes resulting from *adding* S-shaped bounded cardinal utilities, both for the general case and for the leisure-consumption choice.

Neoclassical economists have tended to assume that marginal utility (MU) always diminishes as a consumer increases his/her consumption of a commodity (good, service or event). This may be true for an economist, who is likely to enjoy a comfortable life-style. But, would this also have been true if that same economist had been very poor, experiencing deprivation? Suppose that all individuals experience increasing MU for low consumption until it reaches subsistence, after which MU diminishes as consumption increases further. This gives rise to the leaning S-shaped utility function, as explored by Bernard Van Praag in 1968. The change from one state to the other at subsistence must surely represent a significant experience, estimable via a parameter.

Van Praag went further and recognised the possibility of an intermediate state between cardinal and ordinal utility in the form of bounded cardinal utility. He realised that if utility were bounded below and above, leading to a minimum and a maximum (satiation) level of utility (for either finite or infinite consumption), then interpersonal welfare comparisons become feasible. When neoclassical economists rejected the concept of cardinal utility because it cannot be measured, in favour of formal axiomatic demand theory, they also rejected a host of potential information contained in the *shape* of the utility function. This first proposition is based on Van Praag’s ground-breaking work, which has since been further developed and applied by The Leyden School.

The second proposition arose from the question of whether the S-shaped utility functions should be added or multiplied, and the realisation that it is not a question of either/or, but when should S-shaped bounded cardinal utility functions be added, and when multiplied?

These two propositions about utility, one about the shape of a bounded cardinal utility function and the other specifying their separability conditions, together offer an integrating framework for analysing utility and demand, and would seem to present an additional perspective on demand theory.

The plan of the rest of the paper is as follows.

Section II sets out the two propositions about utility described above, providing the foundation for this extension of utility theory. Van Praag explored the effects of *multiplying* leaning S-shaped bounded cardinal utility functions. The theoretical effects of *adding* two such utilities are explored in the rest of this paper, using diagrams which were created using a functional form with meaningful, estimable parameters, which is derived in the appendix. The functional form is introduced in section III, and is followed by an examination of its indifference curve map, which reveals both convex- and concave-to-the-origin indifference curves, the latter space defining 'dysfunctional poverty'. The convex-to-the-origin indifference curves are found to exhibit all of ultra-superior, superior-normal, inferior-normal and inferior-Giffen responses for each good, depending on the combination of its consumption with that of the other satisfier. In section IV, the introduction of a linear budget yields diagrams of Engels and demand curves, each displaying discontinuities, high elasticities associated with deprivation, and an envelope curve.

The leisure-consumption choice is explored in section V. The locus of points dividing the concave- from the convex-to-the-origin indifference curves can be identified as an *absolute* poverty line. The labour supply curves exhibit similar results to those observed for the other derived structural forms. Suggestions are offered for testing the theory empirically. Section VI summarises the results, indicating some areas for further theoretical exploration and policy applications. It concludes that where the focus is on products and markets, multiplicative utility functions continue to be the most relevant. But, where the emphasis shifts to people and the fulfilment of their needs, additive utilities could provide new insights for the relevant theories and their applications.

The theory presented here starts with plausible psychological assumptions and attempts to predict their consequences. It is found to encompass the traditional neoclassical demand theory as a special case where marginal utility diminishes. It integrates many current piecemeal results, explains some of the anomalies that arise with the traditional theory, and offers some further insights and novel predictions of its own.

## II. THE TWO PROPOSITIONS ABOUT UTILITY

Proposition 1 emphasises that diminishing MU is only part of the consumption experience, as Hicks (1939) pointed out, and though it may be the most frequently occurring in a prosperous society, (especially for most economists), *increasing* MU, representing deprivation, also yields some interesting and important phenomena for examination.

**Proposition 1. The Leaning S-shaped Bounded Cardinal Utility Function.**

The first proposition states:

> 'An individual's experience of consumption, $X_i$, of a commodity i, (good, service or event), $X_i \geq 0$, for i = 1, 2, …, m, can be represented by a continuous, smooth, single-valued, utility function, $U_i = \mathrm{U}(X_i)$, $0 \leq U_i \leq 1$, that has the shape of a leaning

'S' curve, bounded below and above, and where marginal utility, $U_i'$, is always less than infinity.'

The fact that MU has a minimum and a maximum implies that utility is cardinal, but bounded above and below, as in the seminal work of Van Praag (1968). Bounded cardinal utility functions enable interpersonal welfare comparisons to be made. It is assumed that MU is always less than infinity in order for the utility function to be single-valued.

Satiation can occur in two ways:
* at finite consumption, with over-consumption being accompanied by a reduction in utility, $U_i' < 0$, which could be a solution for a negative price, such as a wager, where the consumer is effectively being paid to consume more;
* at infinite consumption.

If consumers were rational, with perfect knowledge about the outcomes, then over-consumption would not take place, and it could be difficult to estimate the satiation parameter. But, consumers are not rational, and there is plenty of evidence revealing that many individuals over-indulge in both food and alcohol, for example, posing risks to health, evidence that finite satiation is a reality in some dimensions of utility. Also, over-consumption can occur as a result of other factors, apart from price.

Satiation for finite consumption is often rejected in the traditional ordinal theory on the grounds that, for positive prices, it would never be chosen. This argument fails to distinguish between the possibility of satiation existing though rarely manifested, and of it not even existing.

The shape of the utility function, as given in Proposition 1, is associated with different experiences. These are summarised in Table 1 and illustrated in Figure 1 below.

**TABLE 1. SIGNS ASSOCIATED WITH THE LEANING S-SHAPED UTILITY FUNCTION, $U_i = U(X_i)$.**

| | | | | | | |
|---|---|---|---|---|---|---|
| $U_i''$ | $> 0$ | $> 0$ | $= 0$ | $< 0$ | $< 0$ | $< 0$ |
| $U_i'$ | $= 0$ | $0 < U_i < \infty$ | $0 < U_i < \infty$ | $0 < U_i < \infty$ | $= 0$ | $< 0$ |
| $U_i$ | $= 0$ | $> 0$ | $> 0$ | $> 0$ | $= 1$ | $1 > U_i > 0$ |
| $X_i$ | $X_i = 0$ | $0 < X_i < \mu_i$ | $X_i = \mu_i$ | $\mu_i < X_i < \text{sat}_i$ | $X_i = \text{sat}_i$ | $X_i >$ finite $\text{sat}_i$ |
| Individual experience: | minimum: | deprivation | inflection: subsistence | sufficiency | maximum: satiation | surfeit |

The shape of the utility function is associated with different experiences. A point of inflection occurs at $X_i = \mu_i$, representing a 'subsistence' threshold comparable to the committed consumption level, or survival parameter, in the Stone-Geary utility function from which the Linear Expenditure System (LES) is derived. Consuming less than this, where MU is increasing, implies 'deprivation'. Consumption greater than the subsistence threshold, where MU is positive but diminishing, may be labelled 'sufficiency'. The point at which maximum utility occurs yields 'satiation' in that particular commodity, while consumption greater than a finite satiation point can be called a 'surfeit'. Obviously, for satiation at infinite consumption there would be no surfeit experience.

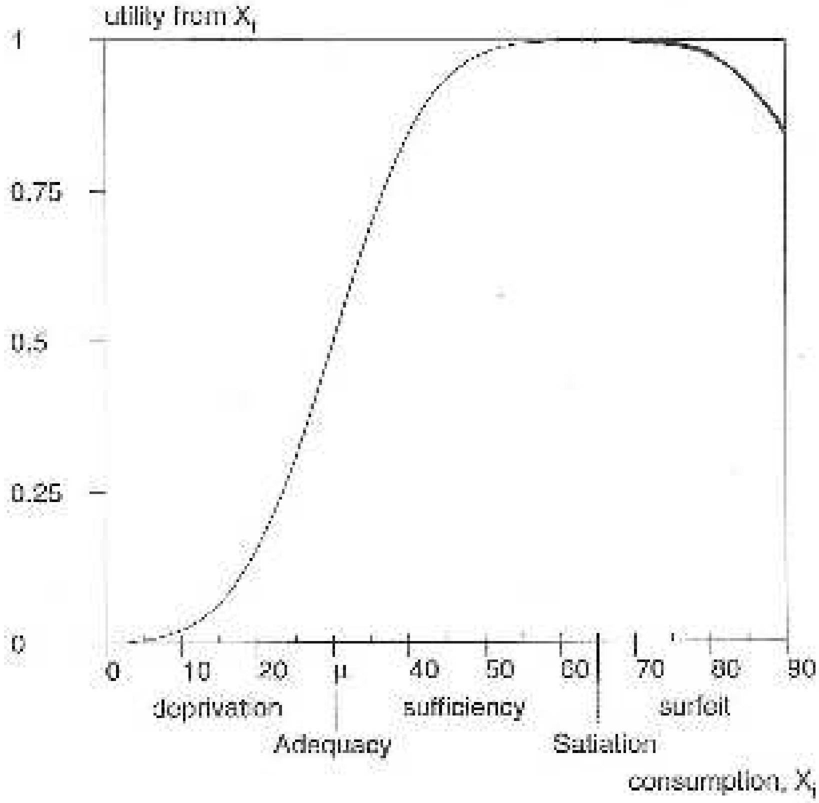


Fig. 1.- The learning S shaped utility function, μ = 30, σ = 10.

The assumptions that the utility function $U_i = U(X_i)$ reaches a minimum, $U_i = 0$, at $X_i = 0$ and a maximum, $U_i = 1$, at $X_i = sat_i$ where $sat_i$ is either finite or ∞, are necessary conditions for utility to be bounded below and above. It is difficult to observe either satiation or zero consumption directly, (because we receive free satisfiers from our environment – for instance, warmth from the sun reduces the amount of fuel we might otherwise have consumed). However, these assumptions allow for the possibility of standardising utility over a range of $0 \leq U_i \leq 1$, say, for comparing the utility attained in satisfying one fundamental human need, or over the range $0 \leq \sum Ui \leq 1$ for k needs, permitting interpersonal comparisons of welfare (utility). Evaluations of individual welfare functions of income, based on the shape of the Distribution Function of a lognormal distribution, have already been carried out very successfully by members of The Leyden School, (see Van Praag and Kapteyn, 1994, for instance), based on the seminal work of Van Praag (1968).

Each individual might be able to experience satiation, but it is not assumed that the level of utility experienced as satiation is the same for each person. However, it is assumed that it is possible to compare each individual's utility with his/her maximum attainable.

Parameter $\sigma_i$ in Figure 1 is a measure of the intensity-of-need for commodity i. The range of $X_i$ covered by $\mu_i \pm 1.96.\sigma_i$ indicates where MU is experienced most intensely. The smaller is $\sigma_i$, the steeper is the slope of the $U(X_i)$ function around the parameter $\mu_i$, and the more intense is the need. "Commodities with a large variance are commodities for which satisfaction comes rather slowly … Commodities with a small variance are commodities … of which one is quickly satisfied. For instance, life necessities have presumably a small variance." (Van Praag, 1968, p.34). This raises an interesting question, which could be explored using this functional form, 'Would a commodity, or group of commodities, to which an individual is addicted, have an even smaller variance than life necessities?'

Each parameter can vary over time for each individual, and between different groups of people, according to demographic variables and other experiences.

**Proposition 2. Separability.**

Proposition 2 states that

> 'the utilities of a group of commodities that satisfy the same need are multiplicatively related (with or without dependence), and the utilities of groups of satisfiers, each group satisfying a different need, are additively related'.

The discussion of 'separability' and 'the grouping of commodities' in the economics literature (Green, 1976; Deaton and Muellbauer, 1980) often comes across as though they are secondary after-thoughts, unconnected with the axioms of demand theory. Discussion centres on whether the utilities gained from the consumption of different commodities are additive or multiplicative. However, it is not a question of either/or, but rather 'when are utilities additive, and when are they multiplicative?'

It is assumed here, following Mallman and Nudlar (1986), and to a lesser extent Maslow (Lutz and Lux, 1979), that there is a finite number of fundamental human needs and that these are universal and a-historic. Needs cannot be observed directly, but only through the effects of their satisfiers, or lack thereof. Needs are satisfied by an infinite diversity of culturally-determined satisfiers. The nine, finite, fundamental human needs are: for permanence (or subsistence); for protection; for affection; for understanding (one's environment); for participation (in one's community); for leisure; for creation; for identity (or meaning); and for freedom, (Max-Neef, 1986, p.49).

If needs can be defined and identified in terms of their additivity, then two needs could be confirmed as such by comparing the results of additive and multiplicative versions of the same utility function.

## III. THE PREDICTIONS OF THE TWO PROPOSITIONS

### A Functional Form for Two Added Leaning S-shaped Bounded Cardinal Utility Functions

The most obvious choice for the utility function lies between a distribution function (DF) or a frequency function, (but they do not have any probabilistic connotations in this context). Proposition 1 allows for satiation to occur at either finite or infinite consumption. A distribution function (DF) allows for satiation at infinity, while a frequency function would allow for finite satiation and over consumption.

The steeper the slope around the subsistence threshold, the more intense is the desire for, and the satisfaction gained from consuming, that commodity. If it is important that the intensity of need for the commodity should be fully experienced before finite satiation occurs, then this places some restrictions on the relationship between the subsistence threshold, $\mu_i$, the level of consumption associated with satiation, $sat_i$, and the measure of intensity of need, $\sigma_i$. This latter condition suggests that a DF provides a better representation of the main part of the S-shaped function than would part of a frequency function.

There is also the choice between the Normal distribution (for which $-\infty \leq X_i \leq +\infty$, implying that consumption could take negative values) or the Log Normal distribution (for which $0 \leq X_i \leq +\infty$). That $X_i$ may take negative values could be explained by ‘free satisfiers’. That is, some fulfilment of a need may be provided by natural circumstances, such as where a warm climate can heat a home before the consumption of fuel is required.

This present work builds on the ground-breaking work of Van Praag and The Leyden School. Van Praag (1968) gave a persuasive argument for choosing the DF of the lognormal distribution, (LN-DF), confirmed empirically by Van Herwaarden and Kapteyn (1981). He concentrated on the outcomes of an n-variable, *multiplicative*, lognormal distribution function, bounded cardinal utility function, (n.Mult.LN-DF), representing S-shaped utility, satiated at infinity.

The functional form used here, derived in the appendix, and used to produce figures 2-7, is based on the DF of the normal distribution, (N-DF). The N-DF was chosen for pragmatic reasons, because it is fairly tractable, and is useful for illustrating many aspects of the theory, providing a reasonable approximation for the part of the leaning S-shape around the subsistence threshold. Further, it has the added advantage that its two parameters, $\mu_i$ and $\sigma_i$, have important economic interpretations, and are potentially estimable. The parameter $\mu_i$ is the survival level or subsistence threshold. The functional form used here, is based on a 2-variable, *additive*, normal DF, bounded cardinal utility function, (2.Add.N-DF), also representing S-shaped utility, satiated at infinity.

Thus, there are two ways in which this present work differs from that of Van Praag.

- Proposition 2 distinguishes between situations when utilities are additive and when multiplicative, although only the case of additive utilities is explored here. Van Praag assumes that the relationship between the utilities from commodities is multiplicative, both without and with dependence (substitutes and complements).

- He makes a case for using the log-normal DF as his functional form (pp.81, 86, 119), whereas for additive utilities, the normal DF is much more tractable.

**Indifference Curves**

The two propositions together define the utility function for a vector of commodities, and its associated indifference curves. The continuity, smoothness, and single-valued nature of proposition 1, (dependent on the restriction that MU is less than infinity), preserve the continuity, smoothness and transitivity properties of the indifference curves.

The separability proposition gives rise to two very different types of indifference curve maps. The multiplicative one is similar to the familiar representative convex-to-the-origin indifference curves found in textbooks, (some sample diagrams of which can be seen in Van Praag, 1968, p.88) and will not be discussed here.

Additive utilities provide more interest. This is illustrated in Figure 2[2], for the consumption, $X_1$ and $X_2$, of two commodities satisfying needs 1 and 2 respectively.

**The first feature** to note is that the two subsistence thresholds, where $X_1 = \mu_1$ and $X_2 = \mu_2$, divide the indifference map into a left-hand border and a lower border, separating deprivation from sufficiency in each need.

**The second feature** to note is that the indifference curves close to the origin are concave-to-the-origin or (quasi-concave). However, their shapes change so that those further from the origin become less concave, until there is a straight-line indifference curve, which divides the concave- from the convex-to-the-origin indifference curves lying further away from the origin. Thus, the straight-line indifference curve, labelled AB in Figure 2, divides the indifference curve map into a triangular area OAB (of which it forms the hypotenuse), from the more familiar type of indifference curve map. The triangle OAB is comparable with the border found in the Stone-Geary utility function from which the Linear Expenditure System (LES) model is derived. It is as though the inner axes of the Stone-Geary indifference curve map have been prised open in order to form the straight-line AB. The presence of both concave and convex indifference curves, divided here by the straight-line indifference curve, is the source of a discontinuity in the derived structural forms.

The triangle OAB represents a non-solution space, except for corner solutions on the axes. These are non-tangential ‘choices’, representing extreme deprivation in one or other dimension of need, and can be interpreted as ‘dysfunctional poverty’. Poverty is a multi-dimensional function of all needs, (as is affluence).

For complex additive functional forms, a natural **‘poverty line’** (PL) could be defined as a locus of points dividing concave- from convex-to-the-origin indifference curves, that would be neither a straight line, nor co-incidental with any one indifference curve. For the special 2-variable case examined here, derived from the DF for a *symmetric* frequency function, the

[2] Figures 2 - 6 were created using Seppo Mustonen’s SURVO software.

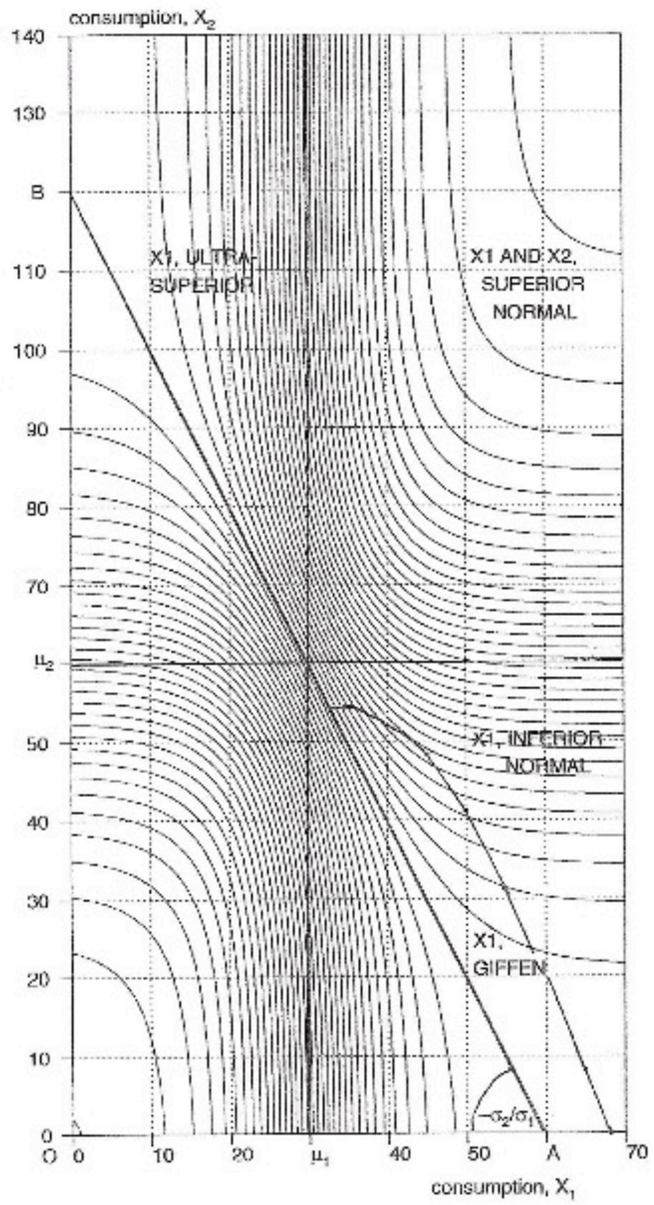


Fig. 2. Indifference curves. $\mu_1 = 30$, $\sigma_1 = 10$.
$\mu_2 = 60$, $\sigma_2 = 20$.
U takes values from 0.005 to 0.995 in steps of 0.015.

poverty line is the straight-line indifference curve and it can be represented by PL = $\mu_1/\sigma_1 + \mu_2/\sigma_2$. If the straight-line indifference curve were found to be a reasonable approximation of the poverty line for *all* functional forms, then, by extension, for a finite, k-need system, this approximation for the poverty line is PL = $\sum \mu_i / \sigma_i$ for all i = 1, 2, … k. The intercept of this *plane* on each axis, is $\sigma_i$.PL.

The slope of the straight-line indifference curve is $-\sigma_2/\sigma_1$, and it provides a measure of the consumer's relative intensities-of-need or relative preferences between commodities satisfying those needs. The smaller the value of $\sigma_1$, the greater the slope of the straight-line indifference curve (measured at corner A), the greater the intensity-of-need for commodity 1 compared with commodity 2. Estimates of these ratios might enable one to rank needs, and to test Maslow's hypothesis (Lutz and Lux, 1979) that there is a hierarchy of needs, without resorting to assumptions about lexicographic orderings of preferences.

**The third feature** refers to the characteristics of the two commodities in the convex-to-the-origin part of the indifference curve map. It can be shown that in the top right-hand quadrant of Figure 2, both commodities are experienced as superior normal goods, (additivity and positive diminishing marginal utilities always yield superior normal characteristics). The indifference curve map may be further divided by a locus of points where the slope of the convex-to-the-origin indifference curves is equal to the slope of the straight-line indifference curve, $\sigma_2/\sigma_1$. This locus is a straight-line from the point ($\mu_1$, $\mu_2$), and it is symmetric about $X_1 = \mu_1$ with the upper left-hand part of AB. It divides that part of the indifference curve map where both commodities act as superior goods into two areas. In the area above and to the left of the new line, $p_1/p_2 > \sigma_2/\sigma_1$, (commodity 1 is relatively high-priced), and for the area below and to the right of the new line, $p_1/p_2 < \sigma_2/\sigma_1$, (commodity 1 is relatively low-priced).

In all of that part of the lower border where the indifference curves are convex-to-the-origin (to the right of line AB), the individual has sufficient of commodity 1, but it is experienced as an inferior good. The consumer's second need is not being satisfied and the deprivation is manifest as increasing MU in response to the consumption $X_2$ of commodity 2. This joint experience might be thought of as 'functional poverty'. Further, in the small area adjacent to the straight-line indifference curve, indicated in Figure 2, $X_1$ is experienced as a Giffen good (Dougan, 1982; Silberberg *et al*, 1984). This confirms that the Giffen experience is one in which the consumer is able to fulfil that need sufficiently (need 1 in this example), as anticipated by Berg (1987), but the satisfier $X_1$ has low unit value, and the consumer is deprived in the second dimension of need (need 2). That the Giffen experience is associated with a straight-line indifference curve, adjacent to a triangular non-solution space, was anticipated by Davies (1994).

In that part of the left-hand border where the indifference curves are convex-to-the-origin, the consumer experiences commodity 1 as a deprivation, (with increasing MU), and, following Hirschleifer's terminology (1976, chap.4), $X_1$ is here termed an ultra-superior good. Kohli (1985) calls this experience an 'anti-Giffen good', but 'anti-inferior' would be more accurate.

This outcome emphasises that a *commodity* should not be categorised as only one of superior, inferior-normal or a Giffen good, because it could be all of these, depending on its combination with another good. Rather than categorising the commodity, it is the consumer's *experience of, and response to, the satisfaction of that need* that should be categorised as ultra-superior, superior, inferior-normal or Giffen, according to his/her relative prices and income circumstances. This would appear to confirm Spiegel's belief 'that Giffen goods are far more pervasive than is generally believed' (1994, p.137). That the challenge of formulating a utility function for the elusive 'Giffen *good*' (as opposed to the pervasive

Giffen *experience*) continues to engage economists is evidenced by Sørensen (2007), Jensen *et al* (2008), Moffatt (2012), Haagsma (2012) and Biederman (2015).

The theory based on these two propositions, encompassing the traditional static, axiomatic, ordinal theory of demand as a special case, predicts some already well-established facts. It challenges the assumption that MU is everywhere diminishing and thus the 'convexity everywhere' axioms. The convexity assumption of neoclassical demand theory seems to be based on the statement that 'maximising utility will usually yield a solution for an indifference curve that is convex to the origin (for positive prices), and thus indifference curves must be everywhere convex to the origin'. This is true for all *multiplicative* utility functions, and, in fact, the indifference curves for *both* additive and multiplicative utility functions that exclude the possibility of increasing MU are, indeed, everywhere convex to the origin.

## IV. PREDICTIONS ABOUT THE DERIVED STRUCTURAL FORMS

### Linear Budget and Engels Curves

Given prices, $p_1$ and $p_2$, a linear budget constraint can be expressed in terms of its allocation of income, $Y = X_1.p_1 + X_2.p_2$.

A key parameter associated with the consumption of $X_i$ is $\mu_i$, and similarly a key concept associated with income is the 'subsistence income' or 'survival income', defined as $\mu_1.p_1 + \mu_2.p_2$, where the consumer is just able to fulfil his/her subsistence levels of consumption.

Any budget constraint that passes through the co-ordinate ($\mu_1$, $\mu_2$) is a subsistence or survival budget, including the budget that is co-incidental with the straight-line indifference curve, AB.

The surplus of the consumer's budget, over his/her survival budget is called supernumerary income, as in the LES, where

$$(X_1 - \mu_1).p_1 + (X_2 - \mu_2).p_2 = Y - \mu_1.p_1 - \mu_2.p_2.$$

### Novel phenomena

Each of the structural forms derived for commodity $X_1$ from the utility function will display three features:
1) a discontinuity occurs, associated with the straight-line indifference curve, caused by the combination of concave- and convex-to-the-origin indifference curves;
2) the deprivation of a need is associated with high elasticity; and
3) an envelope curve is identified.

It is relatively easy to map from the quadrants of the indifference curve map to the corresponding quadrants in the diagrams for the derived structural forms.

### Engels Curves

Figure 3 illustrates the Engels curves, plotting consumption $X_1$ on the vertical axis against income on the horizontal one, for given levels of price, $p_1$. A discontinuity can be identified for income equal to survival income, $Y - \mu_1.p_1 - \mu_2.p_2 = 0$, on account of the straight-line indifference curve.

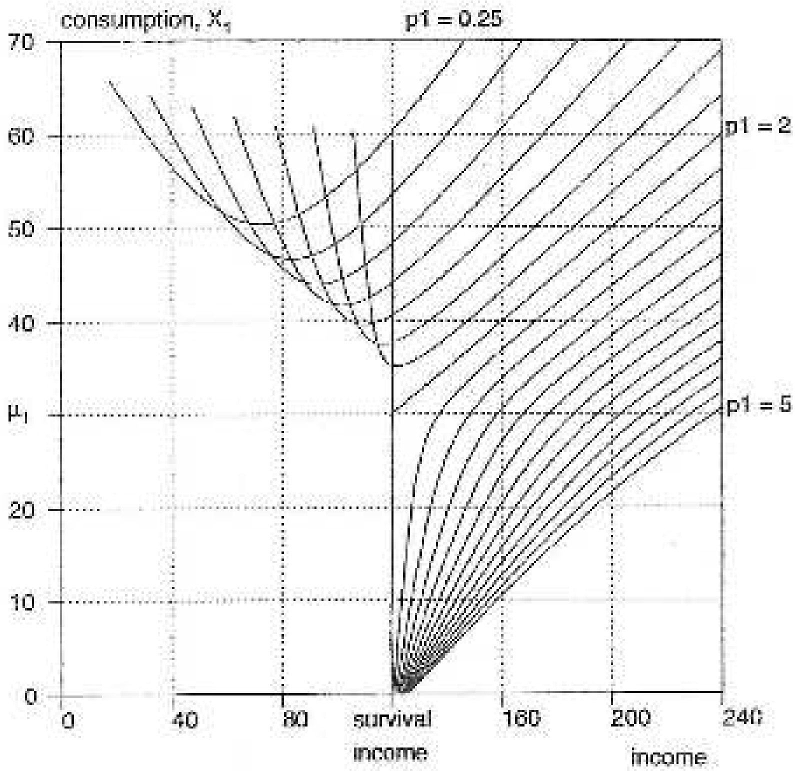


Fig. 3.- Engels curves, generated by equation (4). $\mu_1 = 30$, $\sigma_1 = 10$, $\mu_2 = 60$, $\sigma_2 = 20$, $p_2 = 1$.
p1 takes values from 0.25 to 5.00 in steps of 0.25.

Surprisingly, perhaps, the Engels curves in the lower right-hand quadrant of Figure 3, corresponding to the consumer being deprived of commodity 1 but having an income greater than survival level, when commodity 1 is being experienced as an ultra-superior good, the income elasticity of demand for commodity 1 is greater than 1. Perversely, commodity 1 is labelled a 'luxury good' in this circumstance, (although to someone who is deprived of it, the experience of increased consumption of that commodity, due to an increase in income, must feel luxurious). As income increases and consumption reaches sufficiency, these Engels curves extend and display the response of a high-priced superior good in the top right-hand quadrant of Figure 3, and have positive income elasticities of demand that are less than 1, (commodity 1 becomes a 'necessity').

The U-shaped Engels curves for commodity 1 in the top left-hand quadrant of Figure 3 representing inferior experience are obvious. The consumer is not deprived of commodity 1, but has less than survival income, and so while s/he is able to consume sufficient of it, when it has a relatively low price, ($p_1/p_2 < \sigma_2/\sigma_1$), s/he is deprived of commodity 2. The Engels curves in the top-left-hand quadrant are elastic. As income increases to greater than survival income, the Engel Curves extend into the top-right-hand quadrant of Figure 3, with responses becoming those of a low-priced superior good with income elasticities of demand that are less than 1.

The envelope curve on the Engels curves can be shown to coincide with the boundary between the consumer's inferior-normal and the inferior-Giffen experience of commodity 1.

**Demand curves**

The demand curves for commodity 1 are illustrated in Figure 4. Consumption $X_1$ on the horizontal axis is plotted against relative price, $p_1/p_2$, on the vertical axis, for different levels of income. A few economists at various times in the past (Stonier and Hague, 1980, p.77; Hirschleifer, 1976, pp. 98 and 114) have tried to draw a series of demand curves for a commodity as it transforms from superior to inferior-normal (or from an inferior-normal to an inferior-Giffen good). The two key parameters are again $X_1 = \mu_1$, and the relative intensity-of-need parameters, $\sigma_2/\sigma_1$, which again divide Figure 4 into quadrants.

The convex-to-the-origin demand curves underlying Figure 4 will seem relatively familiar. The top left-hand quadrant corresponds to the top left-hand quadrant of the indifference curve map. The demand curves are the responses of people deprived of high-priced commodity 1. As the relative price of commodity 1 decreases, the demand curves display high-priced superior responses in the top-right-hand quadrant in Figure 4, (corresponding to the high-priced section of the top right-hand quadrant of the indifference curve map), before responding as low-priced ($p_1/p_2 < \sigma_2/\sigma_1$) superior goods in the lower right-hand quadrant of Figure 4. However, some other demand curves in the lower right-hand quadrant of Figure 4 shift to the right as income increases, having a normal slope at relatively low prices, representing inferior responses, and suddenly bend forward as relative price, $p_1/p_2$, rises, when commodity 1 is experienced as a 'Giffen good'. An envelope curve occurs in the lower right-hand quadrant on the boundary between superior and inferior responses. Figure 4 also indicates that part of some of the demand curves can be slightly concave to the origin.

The locus of points dividing the concave- and convex-to-the-origin indifference curves in Figure 2, represented in this example by the straight-line indifference curve, leads to a discontinuity in the demand function at $p_1/p_2 = \sigma_2/\sigma_1$. This instability can lead to an apparent instability in behaviour; that is, large reactions can occur in response to small changes in relative prices. If prices were to waver slightly around $\sigma_2/\sigma_1$, then behaviour could appear to oscillate markedly. In some sense, $\sigma_2/\sigma_1$ can be regarded as a ‘natural price’.

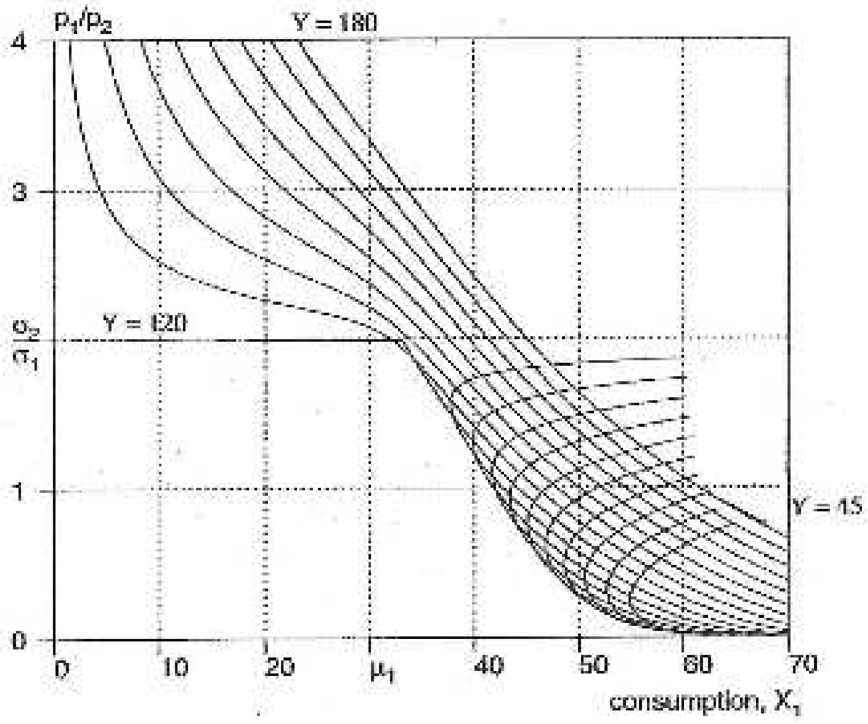


Fig. 4. Demand curves, generated by equation (4): $\mu_1 = 30$, $\sigma_1 = 10$, $\mu_2 = 60$, $\sigma_2 = 20$, $p_2 = 1$. Y takes values from 45 to 180 in steps of 7.5 units.

With additive utilities, the two goods are net substitutes for each other.

**Envelope Curves**

An unexpected novel phenomenon predicted by the two propositions is the presence of envelope curves, each providing minimum bounds (for tangential solutions), one to the demand curves on the boundary between the inferior and superior experience, and the other to the Engels curves on the boundary between the inferior-normal and inferior-Giffen experience. With hindsight, it should have been intuitively obvious that there might be an envelope curve on demand curves, if one assumes that the demand curves slope down from left to right. If they shift to the right as income increases for superior goods, and they also shift to the right as income decreases for inferior goods, then the envelope must occur on the boundary between a good being inferior and its being superior.

A second method of demonstrating this is through the process of deriving the envelope curve. $X_1$ in the demand equation is differentiated with respect to income, *Y*, and the derivative is set equal to 0, (and then expressed in terms of *Y*). The expression for *Y* is then substituted into the demand equation for $X_1$ to obtain the envelope curve. However, this is the condition for the threshold between commodity 1 being superior and its being inferior, ie coincidental with the boundary $X_2 = \mu_2$, for $X_1 > \mu_1$.

Using similar reasoning, it is easy to prove that the envelope curve for the Engels curve is coincidental with the threshold between commodity 1 being inferior-normal and its being inferior-Giffen.

## V. APPLICATION TO LEISURE-CONSUMPTION CHOICE.

Empirical evidence has rejected the assumption of additive utilities for consumption and leisure within the Stone-Geary (LES) model (Blundell, 1988). This is to be expected, since it is based only on diminishing marginal utilities and it is therefore impossible to distinguish between multiplicative and additive separabilities. However, the *ex ante* predictions for labour supply based on the two propositions presented in section II above suggest otherwise.

In Figure 5, the indifference curve map for the consumption-leisure choice is almost the same as for any other pair of additive utilities. $X_1$ now becomes leisure and $X_2$ becomes consumption.

However, leisure is different from other commodities because it is an endowment constrained to a time limit in a given time period, such as 168 hours per week. The other difference is that, normally, the linear budget is given by the expenditure on the commodities for a given amount of money income, $Y = X_1.p_1 + X_2.p_2$. In the leisure-consumption choice, *Y* represents a set of endowments, $(Z_1, Z_2)$, where $Z_1$ is the maximum time available for leisure in a given time period, and $Z_2$ is a flow of material endowments. These are valued at $p_1$ and $p_2$ respectively, such that $Z_2.p_2$ represents unearned income. The value of the endowments is allocated between leisure, $X_1$ and consumption $X_2$, Now the linear budget becomes:

$$Z_1.p_1 + Z_2.p_2 = X_1.p_1 + X_2.p_2$$

or

$X_2 = (Z_1 - X_1).p_1/p_2 + Z_2$

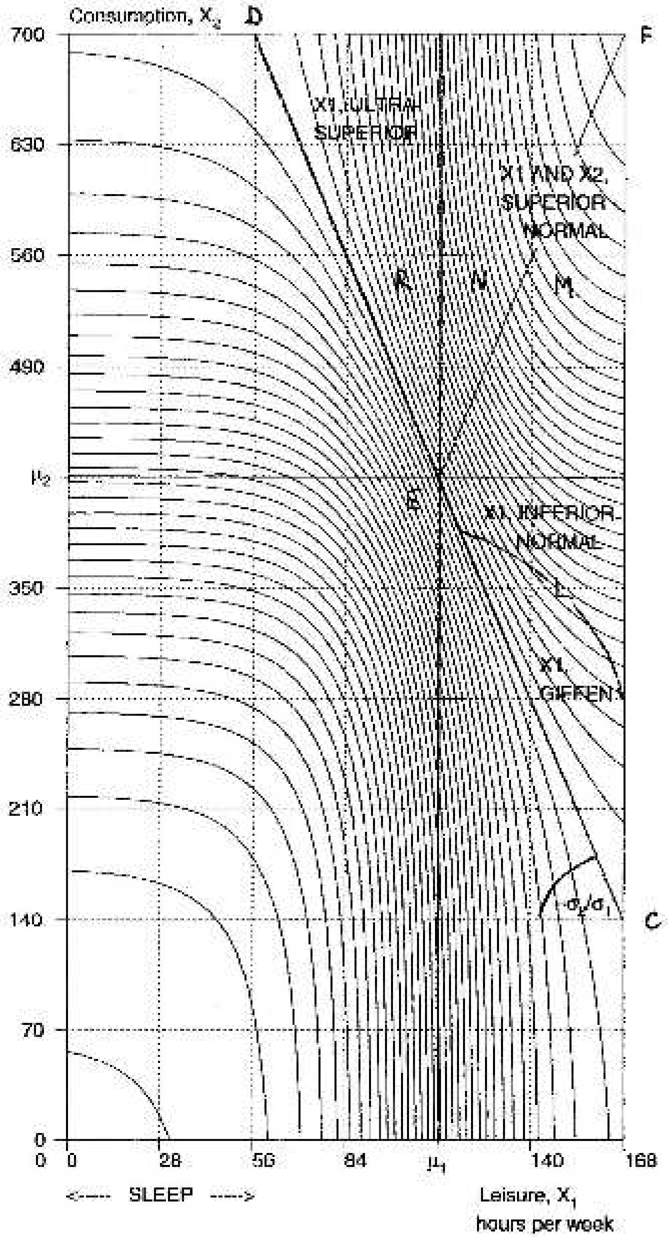


Fig. 5. Indifference curves for Consumption Leisure choice.
$\mu_1 = 112$, $\sigma_1 = 28$.
$\mu_2 = 420$, $\sigma_2 = 140$,
U takes values from 0.005 to 0.995 in steps of 0.015.

The time not spent on leisure, $(Z_1 - X_1)$, is assumed to be spent working for pay, $p_1$. $(Z_1 - X_1).p_1$ is earnings.

The parameters, $\mu_1$ and $\mu_2$ are the subsistence parameters, committed leisure and subsistence consumption respectively. The slope of the straight-line indifference curve is $-\sigma_2/\sigma_1$, and it provides a measure of the consumer's 'relative want intensity' or relative preferences between commodities, in this case of leisure over consumption. The lower the value of $\sigma_1$, the more highly valued is that need. The greater the slope of the straight-line indifference curve (measured at corner C), the greater the intensity-of-need for leisure compared with that of consumption.

The main difference now is that $X_1 = Z_1$ provides a constraint, or right-hand axis, which, in this example, cuts the straight-line indifference curve at point C. The shape defined by the two axes, and the intercept $Z_2$ = C on the right-hand axis at $X_1 = Z_1$, together with the straight-line indifference curve, (of which only a part, CD, is shown), represents this special case of 'dysfunctional poverty'. The straight-line indifference curve, dividing the concave- from the convex-to-the origin indifference curves for these two special commodities, can be identified as an **absolute poverty line, APL**.

The characteristics of the commodities noted on the indifference curve map in Figure 2 above apply equally to consumption and leisure. Each can display ultra-superior, superior-normal, inferior-normal and inferior-Giffen responses. The subsistence parameters, $X_1 = \mu_1$ and $X_2 = \mu_2$, divide the map into four parts, and the left hand and lower borders represent deprivations of leisure and consumption respectively.

The indifference curve map may be further divided by a locus of points where the slope of the convex-to-the-origin indifference curves is equal to the slope of the straight-line indifference curve, $\sigma_2/\sigma_1$. This locus is a straight-line labelled EF, where E is point ($\mu_1$, $\mu_2$). For this functional form, EF and ED are symmetric about $X_1 = \mu_1$, and EF and EC are symmetric around $X_2 = \mu_2$. Thus, EF divides that part of the indifference curve map where leisure acts as a superior good, into two areas. The map containing convex-to-the origin indifference curves is now divided into four parts, which are labelled, L, M, N and R in Figure 5.

In area L, leisure acts as an inferior good and represents low-wage solutions, $p_1/p_2 < \sigma_2/\sigma_1$.
In area M, leisure responds as a superior good but also represents low-wage solutions.
In area N, leisure acts as a superior good and represents high-wage solutions, $p_1/p_2 > \sigma_2/\sigma_1$.
In area R, leisure responds as an ultra-superior good, and also represents high-wage solutions.

Figure 6 illustrates the set of labour supply curves that are derived from this type of additive utility function (with a linear budget). The horizontal axis denotes hours-worked-for-pay, labelled ($Z_1 - X_1$), with a parameter ($Z_1 - \mu_1$). The vertical axis measures real wage–rates, $p_1/p_2$, with parameter $\sigma_2/\sigma_1$. This latter may be interpreted as a 'natural wage-rate'. Each labour supply curve is associated with a different level of unearned consumption, $Z_2$. The discontinuity associated with the straight-line indifference curve, at $p_1/p_2 = \sigma_2/\sigma_1$, is also apparent for labour supply curves.

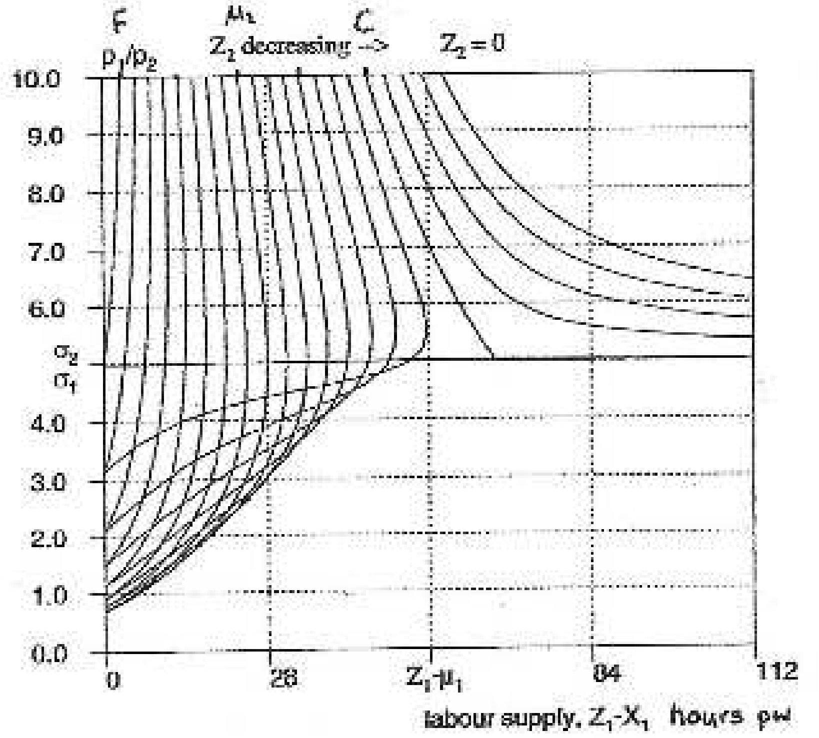


Fig. 6.- Labour supply curves, generated by equation (4a): $\mu_1 = 112$, $Z_1 - \mu_1 = 56$, $\sigma_1 = 28$. $\mu_2 = 420$, $\sigma_2 = 140$, $p_2 = 1$.
Unearned income, Z2, takes values from 0 to 700 in steps of 35 units.

The labour supply curves for unearned consumption C < $Z_2$ < $\mu_2$, are *backward-bending curves*, but different sections of them can be identified as solutions associated with areas N, M and L on the indifference curve map.

Area L on the indifference curve map leads to the *highly-elastic* labour supply curves in response to wage-rate changes facing a worker, who is low-paid ($p_1/p_2 < \sigma_2/\sigma_1$), and 'part-time', ($Z_1 - X_1$) < ($Z_1$ - $\mu_1$).

Area M leads to the mainly *inelastic* curves in response to wage-rate changes for a low-waged ($p_1/p_2 < \sigma_2/\sigma_1$), part-time worker, $(Z_1 - X_1) < (Z_1 - \mu_1)$. Areas L and M on the indifference curve map lead to solutions in the same lower left-hand quadrant of the labour supply curve diagram.

Area N leads to the *inelastic* curves for higher real wage rates, $p_1/p_2 > \sigma_2/\sigma_1$, where hours are also $(Z_1 - X_1) < (Z_1 - \mu_1)$.

Lastly, area R leads to the *backward-sloping* curves of a worker facing high real wage-rates, $p_1/p_2 > \sigma_2/\sigma_1$, and excessive hours, $(Z_1 - X_1) > (Z_1 - \mu_1)$, worked by self-employed people, for instance, and others, that are initially inelastic but become *very elastic* as wage rates decrease and hours-worked-for-pay increase.

There is an envelope curve below the labour supply curves, for $(Z_1 - X_1) < (Z_1 - \mu_1)$, that differentiates between the inferior and superior characteristics of leisure. It can be shown that this occurs where $X_2 = \mu_2$, for $X_1 > \mu_1$. Once a worker has attained his/her subsistence level of consumption, his/her behaviour changes, from elastic responses to wage-rate changes to inelastic responses. The envelope curve illustrates why a government has to exert (sometimes brutal) conditionality if it wishes to coerce people to work for longer hours than their natural limits.

A second novel prediction is that the minimum acceptable wage (reservation wage, RW), below which it is not worth someone working for pay, and which is indicated by the intercept of a labour supply curve on the real-wage axis, is a U-shaped function of unearned consumption, $Z_2$. As shown in Figure 7, for unearned consumption $Z_2 = 0$, the reservation wage starts high and then decreases as $Z_2$ increases, until RW equals $\sigma_2/\sigma_1$ when $Z_2$ is at the point labelled C on the indifference curve map. RW then flattens out between points C and F as $Z_2$ increases, reaching a minimum when $Z_2 = \mu_2$, At the point labelled F on the indifference curve map, RW takes the value of $\sigma_2/\sigma_1$ again. It rises steeply above $\sigma_2/\sigma_1$ for both $Z_2 \leq$ C and $Z_2 \geq$ F.

The reservation wage, participation decisions, voluntary and involuntary unemployment, all arise naturally from the context of the theory. The functional form is capable of illustrating a wide variety of observed facts associated with labour supply economics. Similarly, the roles of the parameters (committed leisure, subsistence consumption, and the consumer's consumption-leisure preference parameters) are unambiguous and estimable for a group of similar individuals. The theory has interesting implications for tax and benefit policy.

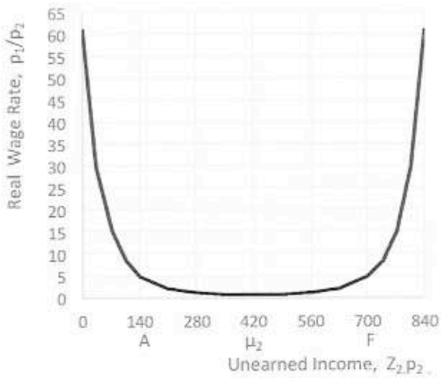


Fig.7. The Reservation Wage is a U-shaped function of unearned income, $Z_2.p_2$

$\mu_1 = 112$, $sigma_1 = 28$
$\mu_2 = 420$, $sigma_2 = 140$
$sigma_2/sigma_1 = 5$, $p_2 = 1$

The two fundamental propositions could be tested empirically as follows:

* The very-non-linear derived structural forms[3] derived in the Appendix are fitted to the data for any two appropriate variables[4], (the obvious examples are leisure and consumption)[5], and compared with the same data fitted to other direct explicit functional forms[6] based on diminishing MU and/or multiplicative utilities, and also compared with flexible functional forms (Deaton *et al*, 1980).
* The parameters, which have realistic psychological interpretations, are also estimated. Both the survival levels (subsistence parameters), $\mu_1$, $\mu_2$, and the intensity-of-need parameters, $\sigma_2/\sigma_1$ and $\sigma_2$, could be estimated for different groups of people in the dataset.

## VI. CONCLUSION

The two propositions presented here, taken together, encompassing the shape and the separability of utilities, offer another perspective on utility and demand theory. Together they extend the range of consumption in traditional demand theory to that of predicting the responses of individuals when experiencing *deprivation with respect to the satisfaction of a need*. The theory predicts many already well-established facts and former anomalies, including 'Giffen goods' and the very elastic labour supply curves observed for both high-waged and low-waged workers, that are now revealed as aspects of behaviour when consumers suffer from deprivation in at least one dimension of need. The theory provides a definition for an absolute poverty line, reveals a discontinuity in the derived Engels, demand and labour supply curves, and identifies an envelope curve associated with each.

When the focus is on products and their markets, multiplicative utilities with indifference curves that are convex-to-the-origin everywhere, will continue to be the most relevant. However, when the emphasis shifts to people and the satisfaction of their needs, added leaning S-shaped bounded cardinal utilities could provide useful insights with respect to those theories and their applications, such as housing, health services, education, the allocation of time to different types of activities, labour supply, individual welfare functions, wellbeing and happiness, social choice, public economics and economic development. It could also contribute to the growing field of behavioural economics. This theory has important policy implications, for instance, predicting the differential effects of tax and benefit proposals for a population with a wide variety of wage rates and unearned incomes, and in policies to reduce poverty and inequality.

Clearly, there is also scope for theoretical explorations of various aspects of this theory, including the following:

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[3] Ideally, the separability assumption would be tested for pairs, or two groups, of commodities by using both additive and multiplicative versions of the same functional forms, and testing between them, when suitable functional forms have been developed.
[4] With additive utilities, the shapes of the indifference curves for any two needs are independent of other needs. Thus, a pair of commodities satisfying two different needs could be studied independently of commodities satisfying other needs.
[5] It could also test whether housing and/or insurance, (representing satisfiers of the need for protection and security), is additive with other forms of consumption. Or, are different types of addictions additively or multiplicatively separable? Test Maslow's 'Hierarchy of Needs'. How many 'needs' can be identified?
[6] It is possible that estimates from the Linear Expenditure System could severely underestimate $\mu_i$.

* What are the implications of the two propositions, if any, for general equilibrium analysis or optimal taxation theory?
* What are the properties of the contract curves derived from Edgeworth boxes, when one or both parties are deprived of one or other of the needs for which commodities are being traded? Could it be used to define exploitation?

Table 2 shows how the two propositions provide a framework for the traditional neoclassical and ordinal theories, for The Leyden School and for additive utilities. The theory does not refute the traditional neoclassical theory of demand, but encompasses it as a special case of ‘diminishing MU’, while challenging some of its more restrictive assumptions. Introducing increasing MU and combining it with additive utilities challenges the ‘convexity everywhere’ axiom. Van Praag’s theory, which is applied extensively on Continental Europe by The Leyden School (Van Herwaarden *et al*, 1981; Hagenaars, 1986) is encompassed within the multiplicative utilities row.

**Table 2. PROPOSITIONS FOR UTILITY THEORY**

<table>
<tr><td>1. SHAPE of UTILITY →<br><br>2. SEPARABILITY:</td><td>Increasing MU<br>(deprivation of need.)</td><td colspan="2">Diminishing MU<br>(sufficiency)</td></tr>
<tr><td>Added utilities</td><td>dysfunctional poverty</td><td colspan="2">. affluence. .</td></tr>
<tr><td>Multiplicative utilities</td><td colspan="2">. The Leyden School. .</td><td>Traditional neoclassical and ordinal theories</td></tr>
</table>

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**APPENDIX: Specification of a functional form for additive utilities.**

This functional form, derived by adding two utilities, each representing the consumption of a commodity, each satisfying a different need, was developed in Miller (1985). It is based on the normal distribution function and will be referred to as the '2-variable, additive, normal-distribution function, utility function', ('2 Add N-DF', for short). It does not fit the first proposition completely, because utility from the consumption of the commodities is not zero when consumption is zero. This could represent the fact that sometimes free satisfiers may fulfil a need before measured ones are used, for instance, the warmth of the sun heating one's home during daytime, but needing to consume fuel in the evening. Nor does it feature satiation at finite consumption levels. However, it is useful because it is tractable, and because it contains meaningful, estimable parameters for subsistence or adequacy, $\mu_i$, and intensity-of-need, $\sigma_i$, for both commodities. It also fulfils the other conditions for proposition 1 and for additive utilities. It was used to create figures 2, 3 and 4 above. Clearly, it would be desirable to develop further tractable functional forms, which will embody both propositions fully.

The additive N-DF utility function is defined as the sum of the distribution functions for the normal distribution (which have no statistical connotations in the present context), of consumption levels $X_i$, $-\infty < X_i < +\infty$, where the i'th commodity fulfils the i'th need. The sum is scaled such that utility, U, lies between 0 and 1.

The '2 Add N-DF' utility function fits the 'two needs, two commodities' case.

$$U(X_1, X_2) = \tfrac{1}{2} F_1(X_1) + \tfrac{1}{2} F_2(X_2)$$

$$(1) \qquad U(X_1, X_2) = \tfrac{1}{2}\int_{-\infty}^{X_1} \frac{e^{-(R_1-\mu_1)^2/2\sigma_1^2}}{\sigma_1.\sqrt{2\pi}} .dR_1 + \tfrac{1}{2}\int_{-\infty}^{X_2} \frac{e^{-(R_2-\mu_2)^2/2\sigma_2^2}}{\sigma_2.\sqrt{2\pi}} .dR_2$$

where U, $0 \le U \le 1$, is utility,

$\mu_1, \mu_2 \geq 0$ are subsistence parameters representing 'adequacy' thresholds, and

$\sigma_1, \sigma_2 > 0$ are parameters representing intensity of need for commodities 1 and 2.

The utility function, together with the budget constraint, represents the structural form of the model.

A *linear budget constraint* is expressed in the form of allocation of expenditure, Y, on the amounts, $X_1$ and $X_2$, of two commodities, at prices $p_1$ and $p_2$ respectively:

$$Y = X_1.p_1 + X_2.p_2,$$

where Y, $p_1$ and $p_2 \geq 0$.

Maximising $U = F(X_1, X_2)$ subject to the budget constraint Y, and using the Lagrangian multiplier method gives the optimality condition as

(2) $$\left(\frac{X_2-\mu_2}{\sigma_2}\right)^2 = \left(\frac{X_1-\mu_1}{\sigma_1}\right)^2 + 2.\ln\left(\frac{\sigma_1.p_1}{\sigma_2.p_2}\right)$$

Equation (2) also gives the locus of points describing the *income-consumption locus* for a given price ratio, $p_1/p_2$.

Using the following short-hand notation for elements that appear frequently

$ONE = \sigma_1.p_1$
$TWO = \sigma_2.p_2$
$H = Y - \mu_1.p_1 - \mu_2.p_2$ (supernumerary income)

and substituting for $X_2 = Y/p_2 - (p_1/p_2).X_1$ into equation (2) yields

(3) $$\left[\frac{(H-(X_1-\mu_1).p_1)}{TWO}\right]^2 = \left[\frac{(X_1-\mu_1).p_1}{ONE}\right]^2 + 2.\ln\left(\frac{ONE}{TWO}\right)$$

This 'implicit demand equation' is a quadratic equation in $(X_1 - \mu_1).p_1$, which is solved using the negative square root, yielding the *expenditure equation* for the first commodity:

(4) $$X_1.p_1 = \mu_1.p_1 + \frac{H.ONE^2 - ONE.TWO\sqrt{\left[H^2 + \left(ONE^2 - TWO^2\right).2.\ln\left(\frac{ONE}{TWO}\right)\right]}}{\left(ONE^2 - TWO^2\right)}$$

In order to accommodate the effect of constraining $X_2 \geq 0$, equation (4) must be qualified such that $0 \leq X_1 \leq Y/p_1$. Thus, if $X_1 < 0$, put $X_1 = 0$, and if $X_1 > Y/p_1$, put $X_1 = Y/p_1$. Similarly, $0 \leq X_2 \leq Y/p_2$. These give corner solutions on the axes outwith the non-solution space.

Equation (4) is the solution to a quadratic equation in $(X_1 - \mu_1).p_1$, and gives two solutions. The negative root maximises utility. The expenditure equations for $X_1$ and $X_2$ are symmetric and homogeneous of degree zero in $p_1$, $p_2$ and Y. The two demand equations represent the reduced form of the model.

There are several different ways of arranging equation (4) to simplify it, or to facilitate an intuitive understanding of it, or to work out the best way to estimate it. For instance, $X_1$ can be expressed in terms of Y, $p_1/p_2$, $p_1$ and $p_2$. It is very non-linear in every version.

When both $Y \geq H$, and the budget line is parallel to the straight line indifference curve, and thus $(\sigma_1.p_1)/(\sigma_2.p_2) = 1$, and using the negative root, the expenditure equation simplifies to

(5) $$X_1.p_1 = \mu_1.p_1 + \frac{H\ .\ ONE}{(ONE + TWO)}$$

By expressing equation (2) in terms of $p_1$, and differentiating with respect to $X_2$ and $X_1$, $dX_1/dX_2$ can be found by implicit differentiation. By setting $dX_1/dX_2 = 0$, the locus for the threshold between *$X_1$ being superior and its being inferior*, on the indifference curve map, is found to be coincidental with $X_2 = \mu_2$, for $X_1 > \mu_1$.

By differentiating $X_1$ in equation (4) with respect to Y, and setting the partial derivative equal to zero, one obtains the condition

(6) $$H = ONE \times \sqrt{\left[-2 \times \ln\left(\tfrac{ONE}{TWO}\right)\right]}, \qquad if\left(\tfrac{ONE}{TWO}\right) < 1.$$

Substituting for H from equation (6) into equation (4) gives the *envelope curve on the demand equations*, for $p_1/p_2 \leq \sigma_2/\sigma_1$.

(7) $$X_1 = \mu_1 + \frac{ONE \times \sqrt{\left[-2 \times \ln\left(\tfrac{ONE}{TWO}\right)\right]}}{p_1}.$$

In order to obtain the locus for the *boundary between the inferior-normal and inferior-Giffen experience in $X_1$*, the following procedure is adopted.

The equation for the budget line is rearranged as $p_1 = (Y - X_2.p_2)/X_1$, and $p_1$ is substituted into the optimality equation (2) above, resulting in equation (8), which, when solved numerically, gives *the price-consumption loci* for given income levels on the indifference curve map.

(8) $$e^{(X_2-\mu_2)^2/2\sigma_2^2-(X_1-\mu_1)^2/2\sigma_1^2} = \left(\frac{(Y - X_2.p_2)}{X_1.p_2}\right)\left(\frac{\sigma_1}{\sigma_2}\right)$$

Equation (8) is rearranged in terms of Y, which is then differentiated with respect to $X_1$ and $X_2$. Using implicit differentiation, $dX_1/dX_2$ is obtained and set equal to zero, eliminating Y. This gives the locus of points for the boundary:

(9) $$\left(\frac{X_1-\mu_1}{\sigma_1}\right)^2 = \left(\frac{X_2-\mu_2}{\sigma_2}\right)^2 + 2\times\ln\left(\frac{X_1}{\sigma_1}\right) + 2\times\ln\left(\frac{\mu_2 - X_2}{\sigma_2}\right), \qquad \text{for } X_2 < \mu_2.$$

Equation (9) has to be solved numerically to find the solutions for $X_1$. It gives two solutions, one each side of the straight-line indifference curve. The locus cuts the straight-line indifference curve at

$$½\left(\mu_1 + \sqrt{\left[\mu_1^2 + 4\times\sigma_1^2\right]}\right), \qquad \mu_2 - \left(\frac{\sigma_2}{2\sigma_1}\right)\times\left(\sqrt{\left[\mu_1 + 4\sigma_1^2\right]} - \mu_1\right)$$

* * * * * * * * *

**The Consumption-Leisure choice**

The 2.Add.N-DF utility-function can be applied to the choice between Consumption, ($X_2 \geq 0$), and Leisure, $X_1$, and to the Labour Supply equations, as follows:

The usual budget constraint is $Y = X_1.p_1 + X_2.p_2$, where Y is income, but now it becomes $Y = Z_1.p_1 + Z_2.p_2$, where $Z_1$ and $Z_2$ comprise a set of endowments: $Z_1$ of time, (in this case 168 hours per week), priced at $p_1$; $Z_2$, of material resources priced at $p_2$, ($Z_2.p_2$ could be unearned income)). Thus, the appropriate linear budget constraint is

$$X_2 = (Z_1 - X_1)p_1/p_2 + Z_2.$$

$(Z_1 - X_1)$ represents hours worked for pay, and $(Z_1 - X_1).p_1$ is earnings, where $0 \leq X_1 \leq Z_1$.

Supernumerary income, H, in this case becomes

$$H = (Z_1 - \mu_1).p_1 + (Z_2 - \mu_2).p_2$$

The **labour supply equation** can be obtained by subtracting $X_1.p_1$ from $Z_1.p_1$ in equation (4) and substituting for $H = (Z_1 - \mu_1)\, p_1 + (Z_2 - \mu_2)\, p_2$.

$$\text{(4a)} \quad (Z_1 - X_1).p_1 = \frac{-(Z_1 - \mu_1).p_1.\text{TWO}^2 - (Z_2 - \mu_2).p_2.\text{ONE}^2 - \text{ONE x TWO}.\sqrt{[\text{Bracket}]}}{(\text{ONE}^2 - \text{TWO}^2)}$$

where [Bracket] = $[H^2 + (\text{ONE}^2 - \text{TWO}^2).2.\ln(\text{ONE/TWO})]$. The negative root is used.

The envelope curve on the Labour Supply curves is given by

$$\text{(7a)} \quad (Z_1 - X_1).p_1 = (Z_1 - \mu_1).p_1 - \text{ONE x } \sqrt{[-2 \text{ x } \ln(\text{ONE/TWO})]}.$$

It can be shown that this locus of points representing the difference between superior and inferior characteristics is co-incidental with the line dividing the two areas on the indifference curve map, ie $X_2 = \mu_2$, for $X_1 \geq \mu_1$.

The **reservation wage** is a function of unearned endowments, $Z_2$, and the real wage rate, $p_1/p_2$. It can be obtained by setting $(Z_1 - X_1) = 0$ in equations (4a) or (7a) and rearranging it in terms of $p_1$, leading eventually to

$$\text{(10a)} \quad \left(\frac{z_2 - \mu_2}{\sigma_2}\right)^2 - \left(\frac{z_1 - \mu_1}{\sigma_1}\right)^2 = 2 \text{ x } \ln\left(\frac{\sigma_1.p_1}{\sigma_2.p_2}\right)$$

Rearranging this in terms of $p_1$ gives

$$\text{(11a)} \quad p_1 = \left(\frac{\sigma_2.p_2}{\sigma_1}\right) + \sqrt{e^{(z_2-\mu_2)^2/\sigma_2^2 - (z_1-\mu_1)^2/\sigma_2^2}}\ .$$

This quadratic in $Z_2$ can be solved using the negative root, yielding an expression for the U-shaped reservation wage, $p_1$, which is symmetric about $Z_2 = \mu_2$ for the 2.Add.N-DF model.

All the other results in equations (5) – (9) for the '2.Add.N-DF' utility function case hold for consumption and leisure, with the above qualifications.