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Free Licensing in a Differentiated Duopoly

Abstract: We construct a differentiated duopoly model to study whether free licensing can be profitable without network externalities and demand shift effect. The efficient firm possesses a superior input-saving technology and sells inputs to the backward firm. However, the optimal input price can be constrained or unconstrained in equilibrium depending on the constellation of parameters. We have shown that free licensing can be profitable if the innovation size is small and the transferee's input production cost is sufficiently large. But free licensing is never profitable if products are homogeneous. An increase in market size also reduces the possibility of free licensing. We have also derived an implication of free licensing in the context of pollution problem.

Keywords: Transferred technology, free licensing, product differentiation, input pricing.

JEL Classification: D43; D45; L13; L24.

1. Introduction

Technology licensing is a common phenomenon in industries. A firm owning a superior production technology licenses its technology to another firm initially holding relatively backward technologies. Technology transfer reduces the transferee's cost of production or improves the quality of the product. If the transferor is a non-producing firm, or if it does not compete in the transferee's market after technology transfer takes place, then such a transfer is always feasible to the extent that it enhances the transferee's profit. In such a situation the problem of the transferor is to design a contract so as to extract the maximum possible surplus. On the other hand, if the licensor and the licensee compete in the same market place, technology licensing reduces the operational profit of the licensor. Then, technology licensing may take place provided that the licensee can fully compensate for the loss of payoff of the licensor. The literature on technology licensing discusses, among other things, the optimal licensing contracts.[1] Typically, licensing takes place against a payment by the licensee in the form of fixed fee and/or royalty. There is also a literature that shows that sometimes technology licensing results in an upward shift of the market demand. This further counters the negative effect of technology transfer on the transferor's profit. For instance, in Shepard (1987), licensing induces quality competition, and this acts as quality commitment that increases industry demand. Similarly, in

[1] For a subset of the literarture on optimal licensing contracts one may look at Kabiraj (2004), Sen and Tauman (2007), Erutku and Richelle (2007), Lee and Kabiraj (2011), Poddar and Sinha (2010), Lu and Poddar (2014), Sinha (2016), Banerjee, Mukherjee and Poddar (2015), Kabiraj and Kabiraj (2017), and Mukherjee and Tsai (2023).

Stamatopoulos and Tauman (2008), licensing of a quality-improving innovation directly affects consumers' preferences and their willingness to pay for the product. They have studied the problem in a logit demand framework and have shown that the optimal licensing contract depends on whether the market is fully covered or not, and whether the consumers' heterogeneity is sufficiently large.

In the present paper our concern is the following. Suppose for some reasons or the other the firm owning a superior production technology cannot charge a fee or royalty to overcompensate the loss of profit due to technology transfer. Such a situation is theoretically possible. For example, suppose that intellectual property right protection is very poor and that once the technology is transferred, the transferee can immediately imitate the transferred technology. In such a situation, royalty licensing is just not possible. Even when fee licensing is possible, it may not be profitable. This is the case, for example, in a Cournot duopoly fee licensing is not profitable unless unit cost reduction under superior technology is large enough and/or goods are less substitutes (Marjit (1990), Wang (1998, 2002)). In any case, in the present paper we are not going to discuss licensing under the optimal licensing contact. Our simple concern here is to examine theoretically whether free licensing can at all be profitable, but without questioning whether it is optimal or not.

We often observe free licensing in the context of music industry. It is shown by Peitz and Waelbroeck (2006) that the music industry may benefit by allowing free downloading to music consumers. This is indeed the case when consumers' taste heterogeneity and product diversity are sufficiently large. Here the industry gains from file-sharing networks. Free licensing of innovations in the computer software industry is also common (e.g., see Vetter (2006)). However, free and open source software appears to be close to sharing of knowledge and cooperation in research. In the context of global pollution problem the developed countries often complain that developing countries, which generally produce goods and services using backward technologies, generate pollution to such an extent that not only the country of origin is adversely affected but, in fact, all other countries are similarly affected by the cross-border movement of pollutants. The developing countries, in turn, ask the developed countries to transfer their superior technologies freely (and the associated inputs at a subsidized price) so that the overall pollution generation becomes least, and this in turn benefits the developed countries as well.

Therefore, it is argued that cooperation among the countries should include, among other things, the right of access to environment friendly technologies by all countries (e.g., see Jeffery (1992) and West (2020)).

In the present paper we construct a differentiated duopoly model with quantity competition when initially one firm holds a superior input production technology which, if transferred, will save resources of the transferee, hence reduce the unit cost of production. Further, in our set up the licensor sells common inputs to the licensee because it is optimal for the licensee to buy inputs from the licensor. Although by sharing its production knowledge the licensor creates competition from the licensee, but the former's loss of profit due to competition may be outweighed by the revenue it earns from the sale of inputs to the competitor, hence free licensing of a superior technology may be mutually beneficial. Before we go to the model and results, we first briefly discuss the existing literature on free licensing. This will help the readers place our paper properly in the literature.

There are only a few works, in the literature, that discuss the possibility of free licensing, that is, transfer of technology for free; still it is a rational decision on the part of the transferor which wants to maximize its overall payoff. The existing literature focuses mostly on network externalities and the shift of market demand for the product. For instance, Conner (1995) analyzes the benefits of market expansion from licensing and derives conditions for which it is profitable for an incumbent to license its technology for free to an entrant who uses the licensed technology to introduce its own product and compete in the incumbent's market. The paper suggests that innovator's best strategy may be to encourage clones of its products when a network externality is present. Then Boivin and Langinier (2005) extend the analysis to examine whether free licensing can be profitable even in a homogeneous good duopoly setting. The paper explicitly assumes that the structure of the market influences the market demand. In particular, consumers' willingness to pay for a product is larger when it is duopoly than when it is monopoly. Thus, the paper assumes that licensing results in an upward shift of the market demand function. The increase in demand resulting from licensing can be large enough to induce the incumbent to share its technology. Free revealing of the detailed workings of newly innovated products and services, to others, is also studied in von Hippel and von Krogh (2006).

In contrast, in our paper there is neither network externality nor market demand-shift effect. Moreover, the pre-licensing situation is also a duopoly. So, licensing of a superior production technology to a rival will intensify competition in the product market and reduce the licensor’s payoff from market competition depending on the degree of substitution between the products sold by them. Since we assume that the licensor can supply the relevant input at a cheaper rate, therefore the rival will buy the input from the licensor. So, when the superior production technology is transferred, revenue from input sale must have to outweigh the loss from product market competition. It may be noted that in our model the input demand of the licensee may not necessarily increase in the post-licensing situation. The reason is that under the transferred technology input requirement per unit of output is reduced. Therefore, whether the revenue from input sale will go up sufficiently or not depends on the transferred technology, the degree of product differentiability and the input price to be charged by the transferor for selling inputs. Since the licensor is constrained to charge an input price above the input production cost of the transferee, the optimal input price will, therefore, be constrained or unconstrained in equilibrium. We have shown that if the products of two firms are perfect substitutes, free technology transfer will never be profitable. The fierce competition in the product market will reduce the transferor's operational profit to the extent that the overall profit will fall. However, when the products of two firms are less than perfect substitutes, free technology transfer will be profitable provided that the transferred technology is not that much of superior and the constrained input price is above a critical level. The reason is that transfer of a much superior technology will reduce the licensor’s operating profit to a significant amount, and unless the input price is high enough, additional revenue from input sale will not be sufficient to compensate for the loss of profit.

Consider the extreme case of product differentiability. If the products of two firms are independent, the licensor will have no loss of profit from market operation. But this does not mean that free licensing will be profitable without any restriction. As superior technology is transferred, the post-licensing output of the licensee becomes larger, but this does not necessarily mean than input demand of the licensee will be large enough; so the corresponding loss of profit can be compensated only if an input price be charged above a critical level. This dimension of the problem had not been discussed earlier in the literature.

There are a number of papers that discuss licensing in a differentiated good model (e.g., see Wang and Yang (1999), Mukherjee and Balasubramanian (2001), Wang (2002), and Bagchi and Mukherjee (2014)). Our paper, however, differs from those in various respects. First, those papers consider technology transfer under a fee or royalty contract (or under two-part tariff contracts), whereas we consider the possibility of licensing completely free of cost. Second, in our paper the licensor has two sources of profits, viz., profits from product market operation and revenue from input sale. Third, in those papers in the post-licensing situation the licensor and the licensee have generally symmetric costs of production. In our paper superior technology reduces input requirements per unit of final good production, but even after licensing the cost asymmetry between the firms prevails, although the gap is reduced. Generally, a larger market size increases the possibility of licensing. Interestingly, in our paper the effect of an increase in market size may reduce the possibility of licensing.

There are some works (e.g., Mukherjee (2019) and Mukherjee et al. (2009)) which although do not discuss free licensing but can be implicated. These papers show that there are situations when entry of a new firm in the industry can benefit the incumbent(s). The papers show that entry in the product market has its effect on the input market, that results in lowering of input price for the incumbent(s). Contrarily, in our paper both pre- and post-licensing market structure is duopoly, and licensing has no effect on its input cost. However, under licensing revenue from input sale may go up sufficiently .

Finally, we have derived, heuristically, an implication of free licensing in the context of the pollution problem in the production process. In particular, we have shown that the level of pollution will be lowered under specific conditions. Further, since consumers and producers gain under free licensing, the overall welfare goes up unambiguously.

The layout of the paper is the following. Section 2 defines the structure of the model. Section 3 studies free liecnsing vs. no-licensing and examines the conditions under which free liecnsing can be profitable. Section 4 derives an implication of free licensing in the context of pollution problem. Section 5 concludes the paper. All major proofs are relegated in the appendix.

2. Model

Consider a differentiated duopoly. Two firms produce differentiated products and compete in quantities. Let q_1 and q_2 be the quantities produced by firm 1 and firm 2, respectively. The market demand as faced by firm i is given by

$$p_i = a - q_i - bq_j, \quad i \neq j; i, j = 1, 2$$

where p_i is price of the product i, and $a > 0$ is the common demand parameter, representing market size. Finally, b measures the degree of substitution between the products such that $b = 0$ means two products are independent (that is, zero substitution), and $b = 1$ means two products are perfectly homogeneous; hence we assume $b \in (0, 1)$.

In our paper, a production technology defines the input requirement per unit of final goods. Assume that the firms hold asymmetric technologies. In particular, firm 1 possesses a superior technology. To make the analysis simple, we further assume that both the products require one common input, and no other inputs are required. Now given the technologies, assume that firm 2 (here the backward firm) requires one unit of the relevant input to produce one unit of the final good. Further, it can produce inputs at a cost r per unit; $0 < r < a$. On the other hand, firm 1's input requirement per unit of the final product is $m, 0 < m < 1$, and it can produce inputs at a lower cost, c, per unit; $0 < c < r$. Hence the initial restrictions on the parameters are:

(A1) $$a > r > c > 0, 0 < b < 1 \text{ and } 0 < m < 1$$

Note that firm 1's input production cost is lower than that of firm 2. Hence firm 1 can always sell inputs to firm 2 at an input price acceptable to firm 2. In fact, if firm 1 charges a price w ($c < w \leq r$), then, instead of producing inputs for itself, firm 2 will always buy inputs from firm 1. Let the unconstrained (i.e., without caring for firm 2's input price r) optimal input price be $\tilde{w}$. Now, if $r < \tilde{w}$, then it is optimal for firm 1 to charge an input price $w = r$. This is the case of constrained input pricing. On the other hand, if $r \geq \tilde{w}$, firm 1 will charge the unconstrained optimal input price $w = \tilde{w}$.

Now consider that firm 1's production technology is transferable. Hence, firm 1 has an option to license its superior production technology to firm 2. In this paper we consider the possibility of *free* licensing if licensing is to occur, and free licensing will occur if it is profitable. Then given the technological specifications, under no-licensing situation firm 1's unit cost of final production will be mc and that of firm 2 will be w. However, if licensing occurs, their unit costs of production will be, respectively, mc and mw. Moves of the players are given in the following game tree.

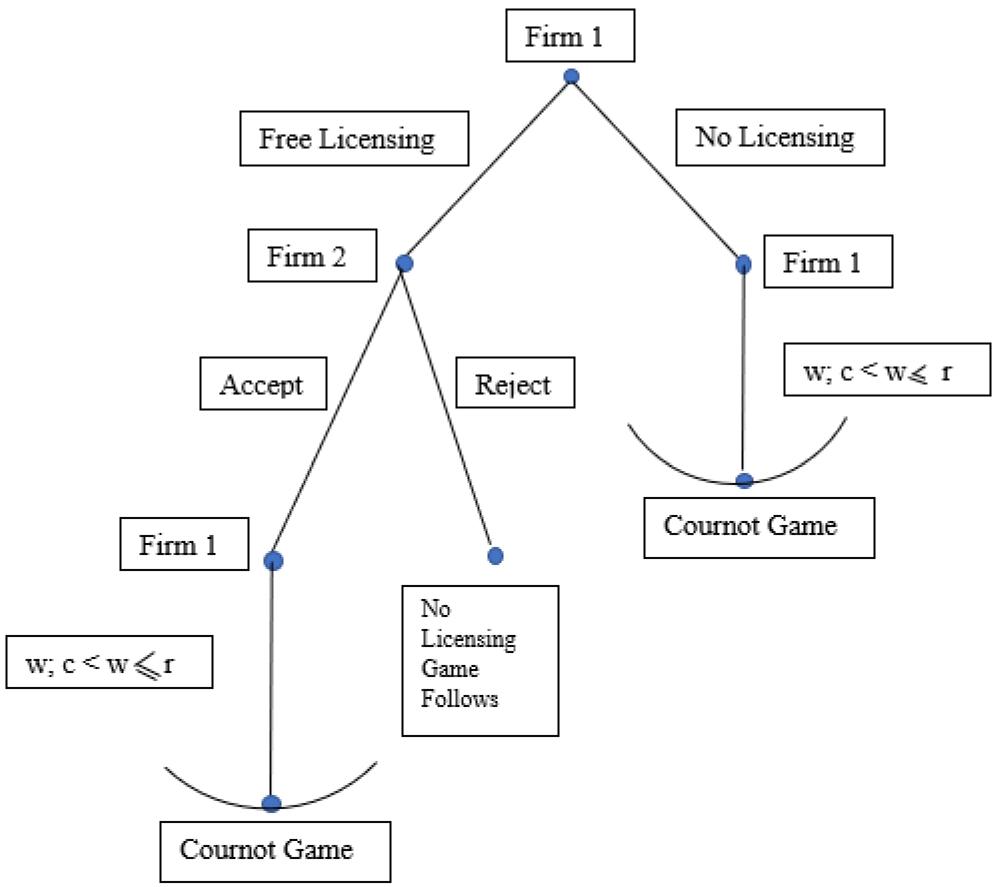

Fig. 1: Game tree showing sequence of moves

Firm 1 first decides whether it will offer free license its superior technology to firm 2; firm 2 decides whether to accept or reject the offer. Then firm 1 decides input price optimally at which it will sell inputs to firm 2. Firm 2 will accept the input price if it is not higher than its input production cost. Finally, they play a Cournot game. The game is solved in a backwards induction fashion. We first solve the final stage problem, then determine the optimal input price at the preceding stage for each of no-licensing and free licensing equilibrium. Finally, we solve the first stage problem to decide whether free licensing will occur or not. We assume that free licensing will occur if it is profitable over no-licensing.

2.1 No licensing equilibrium

Given the game structure, under no-licensing situation for any input price w $(> c)$ charged by firm 1, the unit costs of production of firm 1 and firm 2 are respectively mc and w. Then the profit functions of firm 1 and firm 2 are respectively,

$$\Pi_{1n} = [(a - mc) - q_1 - bq_2]q_1 + (w - c)q_2 \quad (1a)$$

$$\Pi_{2n} = [(a - w) - q_2 - bq_1]q_2 \quad (1b)$$

where the subscript n denotes no-licensing. The profit maximizing outputs at the third stage can be solved from the two first order conditions, $\frac{\partial \Pi_i}{\partial q_i} = 0, i = 1, 2$. The unique equilibrium outputs of the firms will be,[2]

$$q_1 = \frac{(2-b)a - 2mc + bw}{4 - b^2} \equiv q_{1n}(\mathrm{w}) \quad (2a)$$

$$q_2 = \frac{(2-b)a - 2w + bmc}{4 - b^2} \equiv q_{2n}(w) \quad (2b)$$

Since we assume an initial duopoly, therefore for any w to be charged by firm 1, we need to assume that $q_{2n}(w) > 0$, i.e.,

(A2) $\quad w < \frac{(2-b)a + bmc}{2} \equiv \overline{w}(m)$.[3]

Again, when firm 1 is going to set the input price for firm 2, it must satisfy the constraint $w \leq r$. Hence, moving backward, the optimal input price under no licensing is solved from the following problem:

$$\max_{w} \Pi_{1n} \quad \text{s.t.} \quad w \leq r \quad (3)$$

The optimal solution (w_n^0) to the problem is given by

[2] Here both the second order and the uniqueness conditions are satisfied.
[3] It can be easily checked that $\overline{w}(m) < a$ necessarily holds for all $m \in [0, 1]$ and $0 < b < 1$.

$$w_n^o = \begin{cases} r & if\ r < w_n^*(m) \\ w_n^*(m) & if\ r \geq w_n^*(m) \end{cases} \quad (4)$$

where

$$w_n^*(m) = \frac{(4+2b-b^2)(2-b)a+2(4-b^2)c-b^3mc}{2(8-3b^2)} \quad (5)$$

is the unconstrained optimization solution.

Note that $q_{2n}(w)$ is falling in w (see 2(b)), therefore for $q_{2n}(w_n^*(m)) > 0$ we need to satisfy $\overline{w}(m) > w_n^*(m)$, that is, $a(1-b) - c(1-bm) > 0$.[4] We restrict to the assumption that the condition has to be satisfied for all m including $m = 0$; hence we assume

(A2.1) $a(1-b) > c$

In fact, (A2.1) is the sufficient condition for satisfying (A2).[5]

Therefore, in equilibrium the no-licensing payoffs of the firms are

$$\Pi_{1n}(r,m) = q_{1n}^2(w_n^o) + (w_n^o - c)q_{2n}(w_n^o) \quad (6a)$$

$$\Pi_{2n}(r,m) = q_{2n}^2(w_n^o) \quad (6b)$$

where w_n^o is given by (4).

2.2 Free Licensing equilibrium

Under licensing both the firms use the same (superior) technology represented by the parameter $m\ (< 1)$. Then for any input price $w\ (> c)$ to be charged by firm 1 to firm 2, the unit costs of production of firm 1 and 2 are respectively mc and mw.[6] The corresponding profit expressions of the firms are:

$$\Pi_{1f} = [(a - mc) - q_1 - bq_2]q_1 + (w - c)mq_2 \quad (7a)$$

[4] Here, $q_{2n}(w_n^*) = \frac{2[(a(1-b)-c(1-bm)]}{8-3b^2}$. Then $\overline{w}(m) - w_n^*(m) = \frac{(4-b^2)[(1-b)a-c+bcm]}{(8-3b^2)} = \frac{4-b^2}{2} q_{2n}(w_n^*)$.

[5] Note that assumption (A2) may hold even when (A2.1) does not hold. This may be the case when $w = r < w_n^*$. Later we shall derive implications when (A2.1) is not satisfied but (A2) is satisfied..

[6] Note that in our formulation firm 2's unit production cost even after licensing is higher than that of firm 1.

$$\Pi_{2f} = [(a - mw) - q_2 - bq_1]q_2 \tag{7b}$$

where the subscript f denotes free licensing. The profit maximizing solutions in the production stage are given by

$$q_{1f}(w) = \frac{(2-b)a - 2mc + bmw}{4-b^2} \tag{8a}$$

$$q_{2f}(w) = \frac{(2-b)a - 2mw + bmc}{4-b^2} \tag{8b}$$

Note that $q_{2f}(w) > q_{2n}(w)$ for all w. Then in the preceding stage input price is determined from the problem:

$$\max_{w} \Pi_{1f} \quad \text{s.t. } w \leq r \tag{9}$$

The optimal solution to the problem is given by

$$w_f^o = \begin{cases} r & if\ r < w_f^*(m) \\ w_f^*(m) & if\ r \geq w_f^*(m) \end{cases} \tag{10}$$

where

$$w_f^* = \frac{\left(4+2b-b^2\right)(2-b)a + 2\left(4-b^2\right)mc - b^3mc}{2(8-3b^2)m} \tag{11}$$

is the unconstrained input price under free licensing. Hence the payoff under free licensing equilibrium are

$$\Pi_{1f}(r,m) = q_{1f}^2(w_f^o) + \left(w_f^o - c\right)m\, q_{2f}(w_f^o) \tag{12a}$$

$$\Pi_{2f}(r,m) = q_{2f}^2(w_f^o) \tag{12b}$$

3. Free licensing vs. No-licensing

In the previous section we have derived (free) licensing and no-licensing payoffs of the firms assuming that either free licensing or no-licensing occurs. However, free licensing can occur only if $\Pi_{1f} > \Pi_{1n}$, that is, free licensing payoff of the licensor is strictly larger than the no-licensing payoff. We assume that if the free licensing payoff of the licensor is strictly larger, free licensing will occur. In this section we examine the situations when in equilibrium free licensing will occur. However, input pricing can be constrained or unconstrained depending on the constellation of the parameters.

In our analysis we shall suppress the role of the parameters a, b and c, but focus on the efficiency of the transferred technology (m) and the constrained input price parameter (r). In particular, we show that *for free licensing to be profitable, the transferred technology must not be too superior and the backward firm's input production cost must be high above.* To be more precise, for free licensing to be profitable each of these two parameters will have to belong to an interval. Before we prove the main result, we shall prove the following lemmas under the assumption (A2.1).

Lemma 1: $w_f^*(m) > w_n^*(m)$ for $m < 1$ and $w_f^*(m) = w_n^*(m)$ for $m = 1$.

See *Appendix 1* for the proof. This means, if unconstrained input pricing occurs under both licensing and no-licesing, then under licensing, the licensor will charge a higher input price to extract more surplus due to transferred technology.

Lemma 2: $\Pi_{1f}\big(w_f^*(m), m\big) > \Pi_{1n}(w_n^*(m), m) \forall m < 1$.

The proof is given in *Appendix 2*. The implication of this result is that if the constrained input price is very large, that is, $r \geq w_f^*(m)$, [hence $r \geq \max\{w_n^*(m), w_f^*(m)\}$], then in equilibrium, unconstrained input pricing will occur under both licensing and no-licensing situations, and the corresponding payoffs of firm 1 and firm 2 are $\Pi_{1f}(w_f^*(m), m)$ and $\Pi_{1n}(w_n^*(m), m)$, respectively. Then Lemma 2 says that for such a large r, free licensing is always profitable.[7] Note that $q_{2n}(w) = 0$ for all $w \geq \overline{w}(m)$, and the licensor's per unit input cost is c, therefore the optimal input price will lie in between c and $\overline{w}(m)$. This also means that the admissible interval of r in our analysis is $(c, \overline{w}(m))$. In the following lemma we evaluate the payoffs at $r = c$.

Lemma 3: $\Pi_{1f}\big|_{r=c} - \Pi_{1n}\big|_{r=c} = \Pi_{1f}(c, m) - \Pi_{1n}(c, m) < 0$.

The proof is given in *Appendix 3*. Now combining Lemma 2 and Lemma 3, and given the nature of the functions $\Pi_{1f}(r, m)$ and $\Pi_{1n}(r, m)$, we shall derive an important result. First note that

[7] Note that when $m = 1$, both the functions coincide with each other.

both the functions $\Pi_{1f}(r,m)$ and $\Pi_{1n}(r,m)$ (as defined in (12a) and (6a)) are concave and each has a unique maximum. Each function is first increasing in r, then reaches its maximum, thereafter remains constant at higher r. Thus the function $\Pi_{1f}(r,m)$ reaches maximum at $r = w_f^*(m)$ and then remains constant at $\Pi_{1f}(w_f^*(m),m)$ for all $r \geq w_f^*(m)$; and the function $\Pi_{1n}(r,m)$ reaches its maximum at $r = w_n^*(m)$ and then remains constant at $\Pi_{1n}(w_n^*(m),m)$ for all r $\geq w_n^*(m)$.

Second, for any $m < 1$, we have $w_f^*(m) > w_n^*(m)$ and $\Pi_{1f}\left(w_f^*(m),m\right) > \Pi_{1n}(w_n^*(m),m)$, and as $m \to 1$, we have $w_f^*(m) = w_n^*(m)$ and $\Pi_{1f}\left(w_f^*(m),m\right) = \Pi_{1n}(w_n^*(m),m)$.[8] Therefore, given $m < 1$, if these two functions, $\Pi_{1f}(r,m)$ and $\Pi_{1n}(r,m)$, intersect each other, they can intersect only once. Then, given Lemma 2 and 3, immediately we have the following lemma.

Lemma 4: $\forall m \in (0,1)\ \exists r^*(m) \in (c, w_f^*(m))$ *such that* $r \gtreqless r^*(m) \Leftrightarrow \Pi_{1f}(r,m) \gtreqless \Pi_{1n}(r,m)$.

Lemma 4 clearly shows that not only r^*exists, but it is unique. Then for free licensing to be profitable it is necessary that $r > r^*(m)$, given m . However, whether $r^*(m) \gtreqless w_n^*(m)$, depends on the size of m. In particular, for large m we shall have $r^*(m) < w_n^*(m)$, and for small m, we shall have $r^*(m) < w_n^*(m)$ (see *Apendix* 5 for the formal proof). For instance, suppose: $a = 10, b = 0.5, c = 4, r \in (4,10)$ and $m = 0.95$. Then we must have $r^* < w_n^*$ where, $r^* = 6.296$ (approx.) and $w_n^* = 7.459$ (approx.). On the other hand, if in this example $m = 0.95$ is replaced by $m = 0.05$, then we shall have $r^* = 10.718$ (approx.) and $\bar{w} = 7.55$; hence $r^* > w_n^*$, since by assumption (A2.1) $\bar{w}(m) > w_n^*(m) \forall m$. The functions $\Pi_{1f}(r;\ m)$ and $\Pi_{1n}(r;\ m)$ are drawn in *Figure 2*. Next we prove the following lemma.

[8] We have $\frac{d}{dm}\left[w_f^*(m) - w_n^*(m)\right] = -\frac{8a-4ab^2+ab^3-b^3m^2c}{2m^2(8-3b^2)} < 0$ and $\frac{d}{dm}\left[\Pi_{1f}(w_f^*(m)) - \Pi_{1n}(w_n^*(m))\right] = -\frac{2c(2bcm-cm-bc-ab+a)}{(8-3b^2)} < 0$.

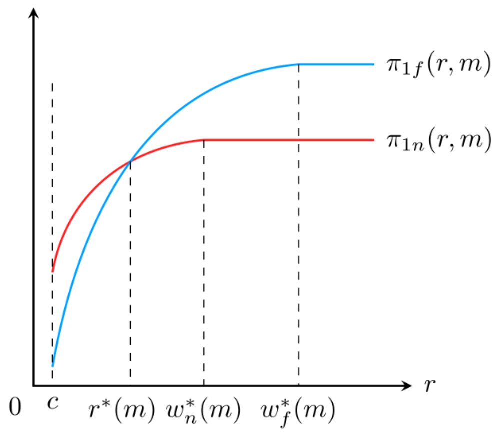

Fig. 2(a): Payoff functions for large m.

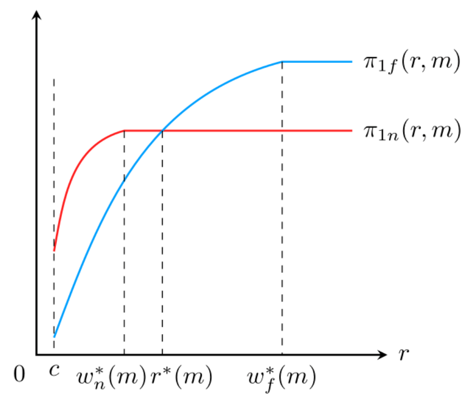

Fig. 2(b): Payoff functions for small m.

Lemma 5: : $\frac{\partial r^*}{\partial m} < 0 \;\; \forall m \in (0,1)$.

The proof is given in *Appendix 4*. The implication is that for m close to 1, we have $r^* < w_n^*$, but then as m falls, r^* increases, goes beyond w_n^* and tends to w_f^*.

Now given that $\overline{w}(m)$ is defined in (A2), we have $\frac{\partial \overline{w}}{\partial m} > 0$ for all $m < 1$. Then this result along with Lemma 5 leads to the following important result.

Lemma 6: $\exists m = m^*$ such that for $m \gtreqless m^*$, $r^*(m) \lesseqgtr \overline{w}(m)$ and m^* is unique.

In *Appendix 5* we have shown that when m close to 1, we have $r^*(m) < \overline{w}(m)$, and when m is close to 0, we have $r^*(m) > \overline{w}(m)$. Then the result follows from the continuity property of the functions. Here m^* is the critical technology level that corresponds to $r^*(m) = \overline{w}(m)$.

From the above it also follows that:

$$\exists\; m = m_n^* \text{ for which } r^*(m) = w_n^*(m)$$

and m_n^* is unique. Finally, given assumption (A2.1), we have $\overline{w}(m) > w_n^*(m)$; hence the admissible interval of r considered here is $(\text{c}, \overline{w}(m))$.

We are now in a position to write the main result of our paper.

Proposition 1: *Given any (a, b, c) and the functions $r^*(m)$ and $\overline{w}(m)$, hence m^*, free licensing is profitable for all (m, r) such that $m \in (m^*,1)$ and $r \in (r^*(m), \overline{w}(m))$.*

The proposition, therefore, states that (i) if $m > m^*$(so that $\overline{w} > r^*$), then $\Pi_{1f} > \Pi_{1n}$ $\forall r \in (r^*, \overline{w})$, but $\Pi_{1n} > \Pi_{1f}$ for r $\in$ (c,r^*), and (ii) if $m < m^*$ (so that $\overline{w} < r^*$), then $\Pi_{1n} > \Pi_{1f}$ $\forall r \in$ (c,$\overline{w}$). This means, for free licensing to occur, it is required that the transferred technology is relatively backward and the constrained input production cost is above a critical level.

It is not difficult to give an intuition of the result. Under licensing, firm 1's operational profit always falls due to competition from the licensee. Now if m is very small (that is, transferred technology is much superior), then under licensing, loss of firm1's operational profit will be large enough so that it cannot be overcompensated by the gain from input sale. On the other hand, if r is is not sufficiently high, increase in revenue from input sale will not be large enough for free licensing to be profitable.

To further clarify and elaborate the results, consider *Figure* 3 which portrays the functions $w_n^*(m)$, $w_f^*(m)$, $\overline{w}(m)$ and $r^*(m)$. Note that $w_n^*(m)$, $w_f^*(m)$ and $r^*(m)$ are falling in m and $\overline{w}(m)$ is increasing in m, with the following properties:

(i) $w_n^*(0) < \overline{w}(0) < \lim_{m\to 0} r^*(m) = \lim_{m\to 0} w_f^*(m)$;

(ii) $\overline{w}(1) > w_f^*(1) = w_n^*(1) > r^*(1) > c$;

(iii) $r^*(m) < w_f^*(m)$ $\forall m \in (0, 1]$

(iv) $w_n^*(m) < w_f^*(m)$ $\forall m \in (0,1)$ by Lemma 1;

(v) $r^*(m) \lesseqgtr \overline{w}(m)$ according as m $\gtreqless m^*$ (see Lemma 6);

(vi) $\exists\, m = m_f < 1 \mid w_f^*(m) = \overline{w}(m)$; m_f is unique and $m_f > m^*$.

(vii) Finally, $\exists\, m = m_n^*$ for which $r^*(m) = w_n^*(m)$; however, $m_n^* \lesseqgtr m_f$ depending on the parameters.[9]

[9] To illustrate, consider: (i) $a = 10, b = 0.5$ and $c = 4$. Then, $m_n^* \cong 0.7313$ and $m_f \cong 0.7860$. (ii) $a = 10, b = 0.1$ and $c = 4$. Then, $m_n^* \cong 0.6546$ and $m_f \cong 0.5018$.

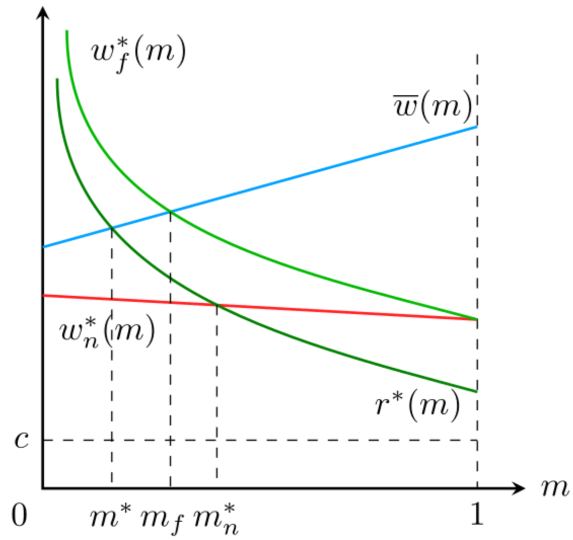

Fig. 3: Possibility of free licensing for $b \in (0, 1)$

The above figure is drawn for the case when $m_n^* > m_f$. We have then three possible cases underlying Proposition 1.

Case (i) : $\boldsymbol{m \in (m_f, 1)}$

Consider that m is very large, i.e., $m \in (m_f, 1)$. Then, for such an m, $\overline{w}(m) \left(> \overline{w}(m_f)\right) > w_f^*(m) > r^*(m)$. Hence we have the following result.

***Result 1**: Given any* (a, b, c)*, and* $m \in (m_f, 1)$*, free licensing will occur for all* $r \in (r^*(m), \overline{w}(m))$ *and no licensing otherwise.*

This contains the following sub-cases.

(i) If $r \geq w_f^*(m)$, free licensing occurs for $r \in (w_f^*(m), \overline{w}(m))$; it is unconstrained input pricing under each of licensing and no-licensing.

(ii) If $w_n^*(m) \leq r^*(m) < r < w_f^*(m)$, then free licensing for $r \in (r^*(m), w_f^*(m))$, but it is constrained input pricing under licensing, but unconstrained input pricing under no-licensing.

(iii) If $r^*(m) < r < w_n^*(m)$, then free licensing for $r \in (r^*(m), w_n^*(m))$, but constrained input pricing under both licensing and no-licensing.

(iv) If $r \in (c, r^*(m))$, then no-licensing with constrained input pricing is the outcome.

Case (ii) : $\boldsymbol{m} \in (\boldsymbol{m}^*, \boldsymbol{m_f})$

Here m is in an intermediate level. For this m, we have $r^*(m) < \bar{w}(m) < w_f^*(m)$. We have the following result.

Result 2: *Given any* (a, b, c) *and* $m \in (m^*, m_f)$*, free licensing will occur for all* $r \in (r^*(m),$ $\bar{w}(m))$*, and no licensing otherwise.*

Clearly, when $m \in (m^*, m_f)$ and $r > r^*(m)$, free licensing occurs with constrained input pricing whereas it is unconstrained input pricing under no-licensing. But if $r < r^*(m)$, free licensing is not profitable, so no-licensing will occur. However, whether it is constrained or unconstrained input pricing under no licensing situation depends on whether $r < w_n^*(m)$ or $r > w_n^*(m)$ depending on m.

Case (iii) : $\boldsymbol{m} \in (\boldsymbol{0}, \boldsymbol{m}^*)$

This is the case when m is small, that is superior technology is superior enough. Under this situation $\bar{w}(m) < r^*(m)$.

Result 3: *Given any* (a, b, c) *and* $m < m^*$*, we have* $\Pi_{1n}(r, m) > \Pi_{1f}(r, m)$ *for all* $r \in (c, \bar{w})$*, hence free licensing will never occur.*

In this case if $r \in$ (c, $w_n^*(m)$), in no-licensing equilibrium it will be constrained input pricing, but for $r \in (w_n^*(m), \bar{w}(m))$, it is unconstrained input pricing.

To complete the analysis, consider the situation when assumption (A2.1) does not hold but (A2) holds, therefore $\bar{w}(0) \leq w_n^*(0)$, hence $\bar{w}(m) \leq w_n^*(m)$ for small m. This means, for relatively large m for which $r^*(m) < \bar{w}(m)$, free licensing can occur only for $r \in (r^*(m), \bar{w}(m))$; otherwise no licensing will occur.

3.1 Two Special Cases

We have so far assumed $b \in (0, 1)$, that is, goods are substitutes to each other. Consider now two special cases, viz., $b = 1$ and $b = 0$.

Case 1: $b = 1$

This is the case when goods are perfect substitutes. Immediately we can see that assumption (A2.1) will never be satisfied. This means, $\bar{w}(m) \leq w_n^*(m)\ \forall m \in (0, 1)$. However, we continue to assume (A2). Now, if $r < \bar{w}(m)$, only constrained input pricing is possible and the optimal input price will be r under both licensing and no-licensing situation. The corresponding payoffs are $\Pi_{1f}(r, m)$ and $\Pi_{1n}(r, m)$. We have shown in *Appendix 6* that $\Pi_{1f}(r, m) < \Pi_{1n}(r, m)$ for $b = 1$, given any tuple (a, c, m, r). Hence we have the following proposition.

Proposition 2: *Free licensing is never profitable if the products are perfect substitutes to each other.*

The intuition of the result is the following. If the products are homogeneous, and licensing would occur, the fierce competition in the product market would reduce firm 1's operational profit substantially making the overall profit to fall.

Case 2: $b = 0$

Here the products of two firms are independent, and given $b = 0$ means that (A2.1) is always satisfied. In this case firm 1 will retain its monopoly both in the pre- and post-licensing situations; hence its market operated profits will be the same in both the situations. Then free licensing can occur if and only if its revenue from input sale under licensing is larger than that under no-licensing. This requires that:

$$R_f(w_f^o) \equiv (w_f^o - c)mq_{2f}(w_f^o) > (w_n^o - c)q_{2n}(w_n^o) \equiv R_n(w_n^o) \qquad (13)$$

where w_n^o and w_f^o are optimal input prices charged by firm 1 under no-licensing and licensing, given any r. Now, given $b = 0$, we have:

$$w_n^*(m) = \frac{a+c}{2},\ w_f^*(m) = \frac{a+mc}{2m} \text{ and } \bar{w}(m) = a \qquad (14)$$

Hence, firm 2's equilibrium outputs will be

$$\mathrm{q}_{2\mathrm{n}}(r)=\begin{cases}\frac{a-r}{2} & if\ r<w_n^*(m)\\ \frac{a-c}{4} & if\ r\geq w_n^*(m)\end{cases}\quad \text{and}\quad \mathrm{q}_{2\mathrm{f}}(r)=\begin{cases}\frac{a-mr}{2} & if\ r<w_f^*(m)\\ \frac{a-mc}{4} & if\ r\geq w_f^*(m)\end{cases}\qquad (15)$$

This corresponds:

$$R_n(r)=\begin{cases}\frac{(r-c)(a-r)}{2} & if\ r<w_n^*(m)\\ \frac{(a-c)^2}{8} & if\ r\geq w_n^*(m)\end{cases}\quad \text{and}\quad R_f(r)=\begin{cases}\frac{(r-c)m(a-mr)}{2} & if\ r<w_f^*(m)\\ \frac{(a-mc)^2}{8} & if\ r\geq w_f^*(m)\end{cases}\qquad (16)$$

One can easily check the following:

(i) $w_f^*(m)>w_n^*(m)$ for $m<1$

(ii) $\overline{w}(m)>w_n^*(m)\forall m$

(iii) $\overline{w}(m)>w_f^*(m)\Leftrightarrow m\gtreqless\frac{\mathrm{a}}{2\mathrm{a}-\mathrm{c}}\equiv m_f$

Finally, to characterize the $r^*(m)$ function, first consider the case that $r<w_n^*(m)$. Under this case, $\Pi_{1f}(r;\ m)>\Pi_{1n}(r;\ m)$ iff $R_f(r;m)>R_n(r;m)$, that is, $\frac{(r-c)m(a-mr)}{2}>\frac{(r-c)(a-r)}{2}$, or $r>r^*(m)\equiv\frac{a}{1+m}$. Then $r^*(m)<w_n^*(m)$ requires that $m>\frac{\mathrm{a-c}}{\mathrm{a+c}}\equiv m_n^*$. Now consider the case when $w_n^*(m)<r<w_f^*(m)$. Under this case, $R_f(r;m)>R_n(w_n^*(m);m)$ iff $r>r^*(m)\equiv\frac{(a+mc)-\sqrt{c(1-m)(2a-c(1+m))}}{2m}$. Then $r>w_n^*(m)$ requires $m<\frac{\mathrm{a-c}}{\mathrm{a+c}}\equiv m_n^*$. Note that $\frac{a}{1+m}=\frac{(a+mc)-\sqrt{c(1-m)(2a-c(1+m))}}{2m}$ at $m=m_n^*$. Finally, one can check that $m_n^*\lesseqgtr m_f$ according as $(a-c)^2\lesseqgtr 2ac$. Thus $r^*(m)$ function is falling in m with $r^*(0)=a$, $r^*(1)=\frac{a}{2}$, and intersecting the function $w_n^*(m)$ at $m=m_n^*$. Note that the sufficient condition for $r^*>c$ is $a>2c$.[10] *Figure 3* is transformed into *Figure 4* for the case $b=0$. However, we have the figure for the case when $m_n^*<m_f$.

[10] Note that when $a<2c$, r^* cannot reach $\frac{a}{2}$ because r^* cannot fall below c.

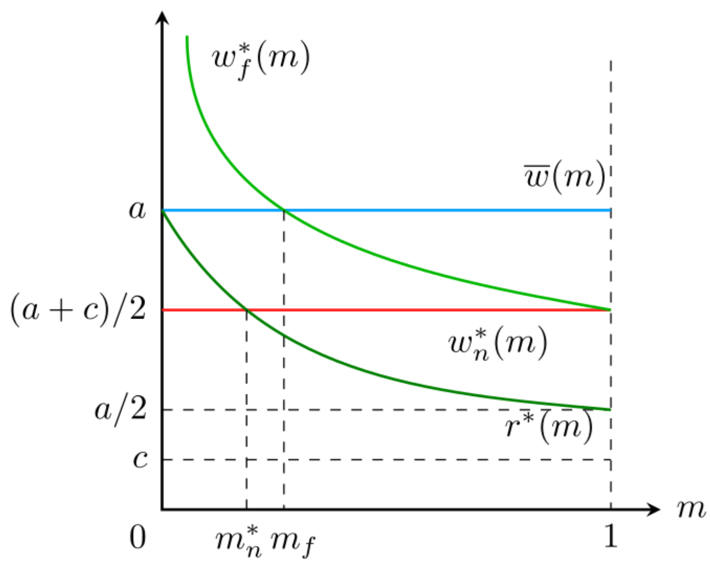

Fig. 4: Possibility of free licensing when $b = 0$

One can easily check that we have exactly the similar result as before. But one important difference is that here $r^*(m) < \overline{w}(m)\ \forall m > 0$ (hence $m^* = 0$ here). Hence, for any $m \in (0,1)$, free licensing will occur for all $r \in (r^*(m), \overline{w}(m))$ where $r^*(m) = \frac{a}{1+m}$ for $m \leq m_n^*$, $r^*(m) = \frac{(a+mc)-\sqrt{c(1-m)(2a-c(1+m))}}{2m}$ for $m \geq m_n^*$, and$\overline{w}(m) = a$.

3.2 Market Size and Free Licensing

In this section we shall study the effect of an increase in market size on the possibility of free licensing. First consider the special case $b = 0$. In this case, given any m, if a increases, both $r^*(m)$ and $\overline{w}(m)$ will rise. This means, for free licensing to be profitable, a larger input price (r) is needed. Since the condition becomes stringent, the possibility of free licensing will fall.

Now consider the general case $0 < b < 1$. We can prove the following lemma.

Lemma 7: Given any $m \in (0, 1)$, $\frac{\partial r^*(m)}{\partial a} > 0$ and $\frac{\partial \overline{w}(m)}{\partial a} > 0$.

Proof is given in *Appendix 7.* Here we have similar result as in the case of $b = 0$. Moreover, since both the functions $r^*(m)$ and $\overline{w}(m)$ shift up, the effect on m^* is ambiguous in general.

This means if m^* increases, the interval of m for profitable free licensing gets reduced, implying that the possibility of free licensing will further fall if the market size goes up.

4. Implication of Free Licensing in the Context of Pollution Problem

In this section we provide a heuristic example to show that free licensing can reduce overall pollution and increase welfare by simply sharing superior technologies by the firms,. After all, production by backward technologies damage the atmosphere, air space and water-body by means of emitting carbon, gaseous pollutants and other obnoxious chemicals and particles. In our simple construct we restrict to the scenario of constrained input pricing under both licensing and no licensing.

Let s_0 be the pollution per unit of output generated for using old (backward) technologies, and s_1 be the same for new or superior technologies. It is then reasonable to assume that $s_0 > s_1 > 0$. Then under duopoly, total pollution generated from production in the pre-licensing and post- free licensing situations will be respectively,

$$L_0 = s_1 q_{1n} + s_0 q_{2n}$$

$$L_1 = s_1 q_{1f} + s_1 q_{2f}$$

Then free licensing will reduce global pollution level if and only if $L_1 < L_0$, that is,

$$[s_1 q_{2f} - s_0 q_{2n}] + s_1[q_{1f} - q_{1n}] < 0 \qquad (17)$$

Since $q_{2f} > q_{2n}$, $q_{1f} < q_{1n}$ and $s_1 < s_0$, therefore the sufficient condition that (17) will be satisfied is:

$$\frac{s_0}{s_1} > \frac{q_{2f}}{q_{2n}} \qquad (18)$$

Condition (18) requires that s_0 will be sufficiently larger than s_1. Note that $q_{2f} > q_{2n}$, but under free licensing m has to be greater than m^*; hence q_{2f} cannot be much larger than q_{2n}.

We conclude the section by noting the welfare implication of our result. Under free licensing each firm's profit is going up, so industry profit increases. Industry output also increases, because $q_{1f} + q_{2f} > q_{1n} + q_{2n}$. This means that the consumers' surplus also increases unambiguously. Moreover, when (18) holds, overall environment becomes less polluted. This, further increases welfare. Hence under this situation, the overall welfare under free licensing must increase unambiguously.

5. Conclusion

In the present paper we have studied the possibility of free licensing, and that too without any network externality or demand shift effect. We have constructed a stylized differentiated duopoly model with quantity competition, where the transferred technology reduces input requirements of the licensee. In our model the efficient firm also produces inputs at a lower cost, hence it is optimal for the other firm to buy inputs from the efficient firm. Then possibility of free licensing arises because under licensing the licensor's loss of profit due to competition can be outweighed under some conditions by the (net) revenue it earns from the sale of inputs to the competitor. We have shown that given the market size and the degree of product differentiability, free licensing is profitable provided that the transferred technology is not much superior, and at the same time the input price at which the transferor sells inputs to the transferee is above a critical level. Even when firms' products are independent, so that the licensor faces no competition in the product market, a profitable free licensing to take place requires restrictions on the input price. On the other hand, if the products are perfect substitutes to each other, free licensing can never be profitable. Interestingly, an increase in market size also reduces the possibility of free licensing. The other important feature of our model is that cost asymmetry prevails even in the post-licensing situation, and if free licensing occurs, both consumers and producers gain, hence total surplus goes up unambiguously.

Finally, we have provided a simple structure to derive implications of our results in the context of pollution problem in the production process. We have shown that if the superior technology generates relatively less pollution per unit of output, free licensing is likely to reduce overall pollution.

Appendices

Appendix 1: Proof of Lemma 1

We have

$$w_f^*(m) - w_n^*(m) = \frac{(1-m)}{2(8-3b^2)m}\left[\left(4+2b-b^2\right)(2-b)a - b^3mc\right]$$

$$= \frac{(1-m)}{2(8-3b^2)m}\left[4a(2-b^2) + b^3(a-mc)\right] > 0 \text{ for } m < 1$$

and $w_f^*(m) = w_n^*(m)$ for $m = 1$. □

Appendix 2: Proof of Lemma 2

$\Pi_{1f}(w_f^*(m), m) - \Pi_{1f}(w_n^*(m), m)$

$$= \frac{c(1-m)(2a-2ab-c-cm+2bmc)}{(8-3b^2)}$$

$$= \frac{c(1-m)(2a(1-b)-c(1+m)+2bmc)}{(8-3b^2)}$$

$$> \frac{c(1-m)(2c-c(1+m)+2bmc)}{(8-3b^2)}$$

(since $a(1-b) > c$ by assumption A2.1)

$$= \frac{c(1-m)(c(1-m)+2bmc)}{(8-3b^2)} > 0$$

Hence the proof. □

Appendix 3: Proof of Lemma 3

First we can check that

$$w_n^*(m) - c = \frac{4(2-b^2)(a-c) + b^3(a-mc)}{2(8-3b^2)} > 0$$

Therefore for all $r < w_n^*(m) < w_f^*(m)$, there will be constrained input pricing under both free licensing and no licensing and the optimal input price under both cases will be r. Then for any such r, we have

$$\Pi_{1f}(r,m) - \Pi_{1n}(r,m) = (r-c)\left(mq_{2f}(r) - q_{2n}(r)\right) - (q_{1n}^2(r) - q_{1f}^2(r))$$

Now, since for any arbitrary r, we have $q_{1n}(.) > q_{1f}(.)$, therefore,

$\Pi_{1f}(c,m) - \Pi_{1n}(c,m) = -\left(q_{1n}^2(c) - q_{1f}^2(c)\right) < 0.$

Appendix 4: Proof of Lemma 5

Here we shall prove that $\frac{\partial r^*}{\partial m} < 0 \forall m \in (0,1)$. Note that we have already proved that $r^* \in (c,$ $w_f^*)$. Further, (A2.1) is assumed.

Case 1: First, consider the scenario $r^* < w_n^*$.This is the case when m is very large. This means,r^* is the solution of r to the equation $\pi_{1n}(a,b,c,m,r) - \pi_{1f}(a,b,c,m,r) = 0$. Now

$$\pi_{1n}(a,b,c,m,r) - \pi_{1f}(a,b,c,m,r) = \frac{1-m}{4-b^2}[\alpha_1(m)r^2 + \beta_1(m)r + \gamma_1(m)]$$

where

$\alpha_1(m) = [3b^2m - 8m + 3b^2 - 8] < 0;$

$\beta_1(m) = [-b^3cm - 2b^2cm + 8cm - 2b^2c + 8c + ab^3 - 4ab^2 + 8a] > 0;$

$\gamma_1(m) = [b^3c^2m - 4bc^2m - ab^3c + 2ab^2c + 4abc - 8ac] < 0.$

Hence r^*must satisfy the quadratic equation

$$\alpha_1(m)r^{*2} + \beta_1(m)r^* + \gamma_1(m) = 0. \qquad (\clubsuit)$$

Differentiating with respect to m gives

$$\frac{\partial r^*}{\partial m} = -\frac{\frac{\partial \alpha_1}{\partial m} r^{*2} + \frac{\partial \beta_1}{\partial m} r^* + \frac{\partial \gamma_1}{\partial m}}{2\alpha_1 r^* + \beta_1}$$

Given the nature of the payoff functions, r^* must be the smaller solution of the quadratic; therefore,

$$r^* = \frac{-\beta_1 - \sqrt{\beta_1^2 - 4\alpha_1\gamma_1}}{2\alpha_1}$$

This gives,

$$2\alpha_1 r^* + \beta_1 = -\sqrt{\beta_1^2 - 4\alpha_1\gamma_1} < 0$$

Define

$$\Lambda_1(r|a,b,c) = \frac{\partial \alpha_1}{\partial m} r^2 + \frac{\partial \beta_1}{\partial m} r + \frac{\partial \gamma_1}{\partial m}.$$

Then,

$\Lambda_1\big((c|a,b,c\big) = -(4-b)bc^2 < 0$, and

$\frac{\partial \Lambda_1}{\partial r} = 6b^2 r - 16r - b^3 c - 2b^2 c + 8c < 0$ for all $r \geq c$.

Therefore,

$\frac{\partial \alpha_1}{\partial m} r^{*2} + \frac{\partial \beta_1}{\partial m} r^* + \frac{\partial \gamma_1}{\partial m} < 0$ as $r^* > c$.

This completes the proof that if $r^* < w_n^*$ then $\frac{\partial r^*}{\partial m} < 0$.

Case 2: Now consider the scenario $r^* > w_n^*$; hence r^* will be the solution of the equation

$$\pi_{1n}(a,b,c,m,w_n) = \pi_{1f}(a,b,c,m,r).$$

This will lead to

$$r^* = \frac{\left(-b^3-2b^2+8\right)cm + ab^3 - 4ab^2 + 8a + \left(2b^2-8\right)\Lambda(a,b,c,m)}{(16-6b^2)m} \quad (\heartsuit)$$

where

$$\Lambda(a,b,c,m) \coloneqq \sqrt{(1-2b)c^2m^2 + (2bc^2 + (2ab-2a)c)m - c^2 + (2a-2ab)c}\ .$$

Note that

$$(1-2b)c^2m^2 + (2bc^2+(2ab-2a)c)m - c^2 + (2a-2ab)c$$
$$= c(1-m)(2bcm - cm - c - 2ab + 2a)$$
$$= (1-m)c(2(1-b)a + 2bmc - 2c + (1-m)c)$$
$$\geq (1-m)^2c^2$$

Therefore,$\Lambda(a,b,c,m) \geq 0$. Let us define

$$z(a,b,c,m) := \frac{(-b^3-2b^2+8)cm + ab^3 - 4ab^2 + 8a + (2b^2-8)\Lambda(a,b,c,m)}{(16-6b^2)m}$$

for all $m \in [0,1]$.Now to show that$\frac{dz}{dm} < 0$ when $z > w_n^*$, note that

$$\frac{\partial z}{\partial m} = \frac{\Phi(a,b,c,m)}{\Omega(a,b,c,m)}$$

where

$$\Phi(a,b,c,m) = (ab^3 - 4ab^2 + 8a)\Lambda(a,b,c,m) + \big((2b^3-8b)c^2 + (2ab^3 - 2ab^2 - 8ab + 8a)c\big)m + (8-2b^2)c^2 + (-4ab^3 + 4ab^2 + 16ab - 16a)c$$

$$\Omega(a,b,c,m) = \Lambda(a,b,c,m)m^2(6b^2-16) \leq 0$$

Then $\frac{dz}{dm} = 0$ is a quadratic equation of m; so z has at most two interior extrema in $m \in [0,1]$.

Now, $\Omega(a,b,c,0) = \Omega(a,b,c,1) = 0$ and $\Omega(a,b,c,m) < 0$ when $m \in (0,1)$. This means z is vertical at $m = 0$ and $m = 1$. Further,$\Phi(a,b,c,1) = 2(b-2)(b-1)(b+2)c(c-a) < 0$. This means z is upward sloping near $m = 1$.

We shall now show that z is downward sloping near $m = 0$. Note that

$$\Phi(a,b,c,0) = -(2-b)\left((ab^2-2ab-4a)\sqrt{(2a-2ab)c-c^2} + (-2b-4)c^2 + (-4ab^2-4ab+8a)c\right)$$

Further note that

$$(ab^2-2ab-4a)\sqrt{(2a-2ab)c-c^2} + (-2b-4)c^2 + (-4ab^2-4ab+8a)c$$
$$= 4a(1-b)(2+b)c - 2(2+b)c^2 - (4+2b-b^2)a\sqrt{2(1-b)ac-c^2}$$

$$= 2(2+b)\left[2(1-b)ac - c^2 - \left(1 - \frac{b^2}{4+2b}\right)a\sqrt{2(1-b)ac - c^2}\right]$$

$$= 2(2+b)\sqrt{2(1-b)ac - c^2}\left[\sqrt{2(1-b)ac - c^2} - \left(1 - \frac{b^2}{4+2b}\right)a\right]$$

$$< 2(2+b)\sqrt{2(1-b)ac - c^2}\left[\sqrt{2(1-b)ac - c^2} - (1-b)a\right] < 0$$

because

$[(1-b)a - c]^2 > 0$ (given assumption (A2.1)

$$\Leftrightarrow (1-b)^2a^2 - 2(1-b)ac + c^2 > 0$$

$$\Leftrightarrow (1-b)^2a^2 > 2(1-b)ac - c^2$$

$$\Leftrightarrow (1-b)a > \sqrt{2(1-b)ac - c^2}$$

So we have

(1) z is vertical at $m = 0$ and z is downward sloping if m is close to 0.
(2) z is vertical at $m = 1$ and z is upward sloping if m is close to 1.
(3) z has at most two extrema in $m \in [0,1]$.

Because of (2) and (3) we know that one extremum will be surely attained when $z < w_n^*$. Because of (1) and since z is continuous, we know that the other extremum will never be attained when $z > w_n^*$. Together these imply that when $z > w_n^*$, z is downward sloping. □

Appendix 5: Proof of Lemma 6

To show that a uniquem^* exists satusfying$\overline{w}(a,b,c,m^*) = r^*(a,b,c,m^*)$, note that

$$\overline{w}(a,b,c,1) - w_f^*(a,b,c,1) = \frac{(2-b)(1-b)(2+b)(a-c)}{8-3b^2} > 0$$

Since both w_f and $\overline{w}$ are continuous, we have $\overline{w} > w_f^*$ for m close to 1. So when m is close to 1 we have $\overline{w} > w_f^* > r^*$.

Now consider small m so that $r^* > w_n^*$. In this case r^* is given by (♥). Correspondingly,

$$r^*(a,b,c,m) - \bar{w}(a,b,c,m) = \frac{\Theta(a,b,c,m)}{2(8-3b^2)m}$$

where

$$\begin{aligned}\Theta(a,b,c,m) = {} & ab^3 - 4ab^2 + 8a + \big((-b^3 - 2b^2 + 8)c - 3ab^3 + 6ab^2 + 8ab - 16a\big)m \\ & + (3b^3 - 8b)cm^2 \\ & - (8 - 2b^2)\sqrt{(1-2b)c^2m^2 + (2bc^2 + (2ab - 2a)c)m - c^2 + (2a - 2ab)c}\end{aligned}$$

Note that $\Theta(a,b,c,0) = (2b^2 - 8)\sqrt{2a(1-b)c - c^2} + ab^3 - 4ab^2 + 8a$ is decreasing in c. Now since by (A2.1),$c < (1-b)a$, we have

$$\Theta(a,b,c,0) > \Theta(a,b,(1-b)a,0) = (2-b)b(4+b)a > 0$$

Therefore,

$$\lim_{m \to 0}[r^*(a,b,c,m) - \bar{w}(a,b,c,m)] \to \infty$$

This means, when m is close to 0 we have $r^* > \bar{w}$.

Since r^* and $\bar{w}$ are continuous, by IVT we get at least one m^*. Finally, since r^* is falling an $\bar{w}$ is increasing throughout,this meansm^* is also unique. □

Appendix 6: Proof of $\Pi_{1f}(r,m) - \Pi_{1n}(r,m) < 0$ for $b = 1$

When $b = 1$, Assumption (A2.1) never holds, but we assume (A2). In this case, since the optimal input price under both licensing and no-licensing is r, therefore (A2) reduces to the condition $a + mc > 2r$.From (12a) and (6a),

$$\Pi_{1f}(r,m) - \Pi_{1n}(r,m)$$

$$= \frac{(1-m)[(4-b^2)(r-c)\{2r(1+m) - (2-b)a - bmc\} - br\{2(2-b)a - 4mc + br(1+m)\}]}{(4-b^2)^2}$$

$$= \frac{(1-m)}{(4-b^2)^2} H(a,b,c,r,m) \qquad (8)$$

Where

$$H(a,b,c,r,m) = [(4-b^2)(r-c)\{2r(1+m)-(2-b)a-bmc\}$$
$$-br\{2(2-b)a- 4mc+br(1+m)\}]$$

When $b = 1$,

$H(a,1,c,r,m) = 3(r-c)(2r(1 + m) - a - mc) - r(2a - 4mc + r(1 + m))$.

Now,

$$\begin{aligned} &H(a,1,c,r,m) \\ &= 3(r - c)(2r(1 + m) - a - mc) - r(2a - 4mc + r(1 + m)) \\ &< 3(r - c)(2rm) - r(2a - 4mc + r(1 + m)) \text{ [since } a + mc > 2r] \\ &= r[6(r - c)m - (2a - 4mc + r(1 + m))] \\ &= r[6rm - 6mc - 2a + 4mc - r - rm] \\ &= r[5rm - 2mc - 2a - r] \\ &< r[5rm - 4r - r] \qquad \text{[as } a + mc > 2r] \\ &= -5r^2(1 - m) < 0\,. \end{aligned}$$

Therefore,when $b = 1$,

$$\Pi_{1f}(r,m) - \Pi_{1n}(r,m) < 0 \text{ for all } m < 1. \qquad \square$$

Appendix 7: Effect of an increae in market size (a)

When $r^* < w_n^*$, from (♣),

$$\frac{\partial r^*}{\partial a} = -\frac{\frac{\partial \alpha_1}{\partial a}r^{*2} + \frac{\partial \beta_1}{\partial a}r^* + \frac{\partial \gamma_1}{\partial a}}{2\alpha_1 r^* + \beta_1}$$

So, $2\alpha_1 r^* + \beta_1 < 0$. Define

$$\Lambda_2(r|a,b,c) = \frac{\partial \alpha_1}{\partial a}r^2 + \frac{\partial \beta_1}{\partial a}r + \frac{\partial \gamma_1}{\partial a}$$

Note that

$$\Lambda_2(c|a,b,c) = (2-b)bc > 0 \text{ and } \frac{\partial \Lambda_2}{\partial r} = (b-2)(b^2 - 2b - 4) > 0 \text{ for all } r \geq c.$$

Therefore,

$$\frac{\partial \alpha_1}{\partial a} r^{*2} + \frac{\partial \beta_1}{\partial a} r^* + \frac{\partial \gamma_1}{\partial a} > 0 \text{ as } r^* > c.$$

This completes the proof that if $r^* < w_n^*$ then $\frac{\partial r^*}{\partial a} > 0$.

Again, from (♥), we have

$$\frac{\partial r^*}{\partial a} = \frac{(2b^3 - 2b^2 - 8b + 8)cm + (-2b^3 + 2b^2 + 8b - 8)c + (b^3 - 4b^2 + 8)\Lambda(a,b,c,m)}{(16 - 6b^2)m\Lambda(a,b,c,m)}$$

$$\geq \frac{(2-b)b(b+4)c(1-m)}{(16-6b^2)m\Lambda(a,b,c,m)} > 0$$

Hence $\frac{\partial r^*}{\partial a} > 0$ for $r^* > w_n^*$. □

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