Dynamics of Economic Efficiency in Tariff and Trade

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Abstract

When the efficiency losses or gains as a result of an ad valorem import tariff are accounted for, the exisiting literature compares the equilibrium states before and after the tariff. However, after the imposition of an ad valorem tariff, the cost of the foreign producer to sell in the domestic market jumps upwards by the extent of the ad valorem tariff. This affects the quantity of imports, and the market is no longer in the initial equilibrium. The market then adjusts and after some efficiency loss, a new equilibrium state is arrived at. The mechanism of price adjustment has a basis of lack of coordination among buyers and sellers at the exisiting prices. The economic efficiency loss when the market is out of equilibrium is not taken into consideration in the literature while deriving an optimal ad valorem tariff rate. In this paper, an optimal ad valorem tariff schedule has been derived. From optimality, it should be construed that the economic efficiency losses get minimized when the market is adjusting and also during the equilibrium. A revenue constraint has to be met in addition. (JEL F10, F11, H20, H21, H30)

Keywords: Ad valorem Import Tariff Path, Tariff Revenue, Dynamic Efficiency, Price Adjustment Path, Equilibrium

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1 Introduction

A tariff is a tax imposed at the import or export stage, when the goods get transported across the border, and just like a tax, it can either be quantity based, value based, etc. An ad valorem tariff is value based, and levied as a fraction of the value of the goods crossing the border. Robert Torrens depicted in Torrens (1844), Robbins (1959), O'Brien (1975) and Viner (2016) that there is a possibility of increase in the national welfare of a country if it is sufficiently large in an international market. Mill (1844) showed that different elasticities could affect the terms of trade. An optimum tariff model was constructed by Sidgwick (1887). Three points were particularly stressed upon by Sidgwick. The first one among those is the monopsony power, which could help achieve an improvement in terms of trade. The second channel through which the terms of trade get affected by tariff is reciprocal demands. Third, the elasticity of offer curve of a country, say B affects the effectiveness of the other country's tariff, say country A. A key insight on when the imposition of a tariff by a country could be done, was provided by Edgeworth (1894).

There have been some innovations contributed to optimum tariff theory by Bickerdike (1906) and Bickerdike (1907). A welfare gain when the trade is restricted was specified by C. F. Bickerdike. The net benefit of an import tariff was defined by him as the government revenue minus the deadweight loss in consumer surplus due to a reduction in imports, as an alternative to Edgeworth's indifference curve measure. Kaldor (1940) demonstrated that a tariff can benefit the levying country if certain conditions are met, i.e., the tariff is small, the country has some monopoly power in the international arena, and there is no retaliation from other countries in terms of tariff. Kahn (1947), Graaff (1949) and Johnson (2013) developed the classic optimum tariff formula. Their endogenous trade policy model is extended by Grossman and Helpman (1995) to the case of a large country, i.e., where the export supply is not infinitely elastic. The two good result regarding the optimal trade policy, i.e., an import tariff, is generalized to higher dimensions by Bond (1990). The optimal tariff problem in a country with some market power is analyzed by Ogawa (2007). There have been diverse views by theorists, with some arguing that it provides the underlying motive regarding the international trade, e.g., (Bagwell and Staiger 1999), while others consider it not to have much practical worth, except for the large countries, e.g., (Krugman and Obstfeld (1997)). **Wang, Wang and Lee (2010) re-examine the important tariff ranking issue under a linear mixed oligopoly model with foreign competitors and asymmetric costs. An intra-industry trade model in network goods is constructed by Fujiwara (2011). It considers the network externalities implications and the welfare effects of bilateral import duty reductions.** The optimal tariff in general equilibrium for a one sector economy with firm heterogeneity and monopolistic competition is characterized analytically by Felbermayr, Jung and Larch (2013). Cato (2017) examines the relationship between the optimal tariff structure and the degree of penetration.

Although, there is a huge literature available on tariffs and the associated economic

efficiency, however, none of the models available consider the economic efficiency loss on the dynamic adjustment path from one equilibrium to the other. *When the efficiency losses or gains as a result of an ad valorem import tariff are under consideration, the exisiting literature compares the equilibrium states before and after the tariff. However, when an ad valorem tariff is imposed, the cost of the foreign producer to sell in the domestic market jumps upwards by the extent of the ad valorem tariff. This affects the quantity of imports, and the market is no longer in the initial equilibrium. The market then adjusts and after some efficiency loss, a new equilibrium state is arrived at. The mechanism of price adjustment has a basis of lack of coordination among buyers and sellers at the exisiting prices. The economic efficiency loss when the market is out of equilibrium is not taken into consideration in the literature while deriving an optimal ad valorem tariff rate. In this paper, an optimal ad valorem tariff schedule has been derived. From optimality, it should be construed that the economic efficiency losses get minimized when the market is adjusting and also during the equilibrium. A revenue constraint has to be met in addition.*

The remainder of this paper is organized as follows: Section 2 devises a dynamic model by introducing the individual problems of market agents. Section 3 solves the model with an ad valorem import tariff imposed. In Section 4, an optimal ad valorem tariff schedule has been derived which minimizes the efficiency losses subject to a revenue constraint. Section 5 presents the static versus dynamic efficiency loss. In Section 6, a summary of findings and conclusion has been presented.

2 The Model

The set up of the model is for a perfectly competitive market for a single good, which is initially in equilibrium. There are two types of producers, i.e., a domestic producer, and a foreign producer of the same identical good. There is a middleman in the market, who sells the product to the final consumer after buying it from the producers. The consumer consumes the good after buying it from the middleman, and the government imposes the ad valorem tariff. The producers are price takers, and when an exogenous shock hits the market, the middleman changes the price during the adjustment process drifting the market to the next equilibrium. As the demand and supply are not equal at all points in time, a stock or inventory of goods is held by the middleman, which he/ she purchases from the producers for subsequently selling those to the consumer. The producers have an objective of maximizing their profits; the middlemen also maximize their profits, whereas the consumer maximizes his/ her utility subject to their respective constraints.

The basis of price adjustment is a lack of coordination between buyer and seller at the exisiting prices. The working of this market can be illustrated by the following example: Consider a market in equilibrium. An equilibrium stock of inventory is held by the middleman. An exogenous supply contraction decreases the inventory stock, as the consumers' demand at the current price does not match with the supply by the producers. The middleman increases the price in his/ her own benefit and the producers also find it optimal to produce a lower quantity than before. The next equilibrium with a lower quantity and a higher price is then reached. The equilibrium in this market is defined as follows:

(i) The middleman and both the producers maximize profits, and the consumer maximizes utility subject to their respective constraints.

(ii) When the market is in equilibrium, the inventory does not change as the quantity consumed by the consumer equals the quantity supplied by the producers.

Section 3 lists the conditions for the existence of an equilibrium in mathematical terms, i.e., the Routh–Hurwitz stability criterion as a necessary and sufficient condition for a stable dynamical linear system.

When the market is perfectly competitive, the middleman is also a price taker in equilibrium state of the market. When the market is adjusting, the middleman has an incentive to change the price. When the new equilibrium arrives, the middleman takes the price as given as far as the market remains in equilibrium. An ad valorem tariff is imposed by the government, and as soon as the tariff gets implemented, the market does not instantaneously settle at the new equilibrium, and rather the price starts adjusting over time until the market is again in equilibrium. The adjustment of price is based on the endogenous decision making of all the market agents.

Suppose a perishable good is produced by the producers in a market. They sell the product to the middleman, who subsequently sells it to the consumer in a community. The quantity bought by the consumer and the middleman equals the quantity produced by the producers, and the market is in equilibrium. If the cost of one of the producers increases due to imposition of a tariff, which reduces the supply in the market, the partial demand of the consumers will not be met by the end of the time period. We assume that there is no production friction, and the producers can change the production without delay. Similarly, there is no price rigidity in the market, and the middleman can change the price immediately after realization of the need to do so. If the middleman had perfect information about the new demand and supply pattern at various prices, he could easily have picked up the next equilibrium price, which would invite the equilibrium production by the producers, and hence the market would clear without any delay, however, this information is missing, and therefore, the middleman increases the price as per his/ her conjecture about the new supply and demand on the basis of the change in inventory. This drives the market toward the final equilibrium. The producers change their production according to their own objective. The total economic efficiency loss resulting from a tariff comprises of the deadweight loss in the final equilibrium plus the waste of resources during the adjustment of the market, which could not be used in the production of the output.

For the illustration of the model in mathematical terms, the objective of each one of the market agents is maximized subject to the constraints, and the outcome equations are solved simultaneously in order to capture the collective impact of agents on the market. To justify the linearization of demand and supply curves, we assume that the post-tariff equilibrium is not too off from the pre-tariff equilibrium.

2.1 Middleman

The middleman sells the goods to the final consumer after purchasing those from the producers, and maintains an inventory as a difference of his purchases and sales. The difference of supply and demand in the market is reflected in the inventory, therefore the inventory is an important intermediary stage between supply and demand. When the supply and demand are the same, the inventory also remains the same in the market. However, when the inventory changes, it implies either supply, demand or both are changing at different rates.

The following explanation describes the connection among inventory, demand, supply and price. When supply in a market shifts to the right, while demand does not change, the inventory piles up due to excessive supply in the market, and the price goes down in the new equilibrium. Similarly, if the demand shifts to the right, while supply remains the same, the inventory level goes down in the market and the price goes up in the new equilibrium. Therefore, a change in price is inversely related to a change in inventory, ceteris peribus. If both demand and supply shocks shift the demand and supply curves in such a manner that there is no change in inventory, there will be no change in price either. The channel of both demand and supply shocks regarding influencing the market is the same, as they are both affecting inventory, and hence both kinds of shocks can be called as an, "inventory shock." Now let us discuss the mechanism of a price change with an inventory change, which are both inversely related. Let us assume a perfectly competitive market, where the inventories are held, and some costs incurred for mainitaining those by the middlemen. The cost and size of an inventory are positively related, i.e., it is more costly to hold a bigger inventory, and vice versa. If the market is in equilibrium and no exogenous shock happens, the price stays the same in the market. Figure 1 describes the inventory, demand, supply and price link.

Now suppose that a positive supply shock happens in the market, which enhances the supply, while demand stays the same. There is a pile up in the market inventory due to excessive supply. The piled up inventories can be held either by the producers or the middlemen, however, the key point is that this scenario will not be sustainable in the long run, and hence there has to be some way out of this scanario. Assuming, that the piled up inventroies are with the middlemen, the middlemen will have to decrease the price to increase the demand along the demand curve in order to have a sustainable storage of inventories with them. The price will finally equal the new marginal cost, as the market is perfectly competitive, however, the adjustment path of the market will be determined by the response of the middleman regarding changing the price to bring the new equilibrium. Mathematically, the middleman's problem is described below:

2.1.1 Static Problem

The middleman's short run (in discrete analog, one time period) problem is as follows:

$$\Pi = pq(p) - \varsigma(m(p, e)), \tag{1}$$

where
Π = profit,
p = market price,
$q(p)$ = quantity sold at price p,
m = inventory (total number of goods held by the *middleman*),
e = other factors which influence inventory other than the market price including the middleman's purchase price from the producer,
$\varsigma(m(p, e))$ = cost as a function of inventory (increasing in inventory).
Taking the derivative of eq. (1) with respect to price, we get:

$$pq'(p) + q(p) - \varsigma'(m(p, e))m_1'(p, e) = 0, \tag{2}$$

As the market is perfectly competitive, therefore, all the market agents have to be price takers when the market is in equilibrium, and the middleman has a benefit in changing the price only during the adjustment process. The middleman has no more incentive to change the price after the equilibrium is already arrived at, and rather will be losing business by deviating from the equilbrium price; whereas, when the market is adjusting, the demand and supply differ, and the price must be changed by an economic agent in his/ her own benefit to bridge the gap to bring the new equilibrium, therefore, a price change by the middleman on the adjustment path is in fact a market force, unlike when there is an equilibrium in a perfectly competitive market, and the demand is infinitely elastic as follows:

$$pq'(p) + q(p) = \varsigma'(m(p, e))m_1'(p, e),$$
$$p\left[1 + \frac{1}{\textit{demand elasticity}}\right] = \varsigma'(m(p, e))\frac{m_1'(p, e)}{q'(p)}.$$

In the above expression, the price is equal to the marginal cost (the expression on the right hand side), when the elasticity of demand is infinite. Suppose, that a supply shock happens in the market as a result of which, the marginal cost of production decreases, and the supply shifts to the right. The supply will expand, no matter, whether the marginal cost reduction happens for the domestic supplier, the foreign supplier or both. As the supply and demand are no longer equal, the market is out of equilibrium. The supply and demand will adjust as a result of a price change in the market, however, price cannot jump on its own to bring the final equilibrium, and the middleman will realize about the supply shock after his/ her inventory begins piling up. Before that, he/ she will continue charging the previous price, which is higher than the exisiting marginal cost. After, the inventory

starts building up, he/ she will reduce the price to maximize the profits. If the supply shock leads to a shrinkage in production, the price will go up in the new equilibrium. The middleman will not change the price, which is now lower than the new marginal cost until he/ she gets a signal of shrinkage in the production through his/ her depeleted inventory. In this scenario, the consumer will be a beneficiary regarding paying a price, which is now lower than the new marginal cost until the price increases in the final equilibrium. When the market arrives at the new equilibrium, no market agent is a beneficiary anymore, it is only during the adjustment process that the gains are reaped by some of them depending on a case by case basis.
For a mathematical demonstration, suppose, as a result of a positive supply shock, such as a reduced marginal cost as a result of some technological innovation, the marginal cost for the middleman to hold another unit in his/ her inventory, i.e., the term $\varsigma'(m(p,e))\frac{m_1'(p,e)}{q'(p)}$ is bigger at the existing price, on account of the fact that the term $\varsigma'(m(p,e))$ is higher at the existing price. The reason is intuitive: as the storage capacity approaches its potential, storing the goods become more and more costly due to enhanced demand of the storage houses, godowns, warehouses, etc. The term, $\frac{m_1'(p,e)}{q'(p)}$ is a function of price, and has not changed as the price is the same as before. This is due to the reason that the purchase price for the middleman has not changed yet, on account of the fact that the producer is a price taker throughout, i.e., when the market is adjusting as well as in equilibrium; and the middleman is charged a fixed fraction of the market price by the producer. A discrete analog can help clarify the above scenario as follows: the middleman maximizes his/ her profits in each time period, where they take the purchase price as given and only chooses the sale/ market price. The middleman does not take into account the profits in furture time periods as his/ her problem is myopic. At the exisiting price, the middleman now faces the following inequality

$$\frac{\partial \Pi}{\partial p} = pq'(p) + q(p) - \varsigma'(m(p,e))m_1'(p,e) < 0, \tag{3}$$

The above inequality implies that the middleman must reduce the price to have another unit of inventory and maximize profits after the supply shock. In this example, the short term benefits accrue to the producer, as the producer's marginal cost has decreased but he/ she keeps receiving the same price from the middleman until the middleman reduces the price. A plot of various profit maximizing combinations of prices versus inventories is a downward sloping *inventory curve* with price on the y-axis and the inventory on the x-axis. The concept is analogous to the *demand* and *supply curves* for the utility maximizing consumers and profit maximizing producers respectively.

2.1.2 Dynamic Problem

In this sub-section, a dynamic/ long term problem of the middleman has been considered. The present discounted value of future stream of profits is maximized by the middleman, and the following expression represents his/ her present value at rime zero:

$$V(0) = \int_0^\infty [pq(p) - \varsigma(m(p,e))] \, e^{-rt} dt, \tag{4}$$

r is the discount rate. $p(t)$ is the *control variable* and $m(t)$ the *state variable*. The middleman's maximization problem is framed below:

$$\underset{\{p(t)\}}{Max} V(0) = \int_0^\infty [pq(p) - \varsigma(m(p,e))] \, e^{-rt} dt,$$

subject to the constraints that
$\dot{m}(t) = m_1'(p(t), e(p(t), z))\dot{p}(t) + m_2'(p(t), e(p(t), z))e_1'(p(t), z)\dot{p}(t)$ (state equation, describing how the state variable changes with time; z being an exogenous factor),
$m(0) = m_s$ (initial condition),
$m(t) \geq 0$ (non-negativity constraint on state variable),
$m(\infty)$ free (terminal condition).
The current-value Hamiltonian for this case is

$$\widetilde{H} = p(t)q(p(t)) - \varsigma(m(p(t), e(p(t), z))) + \mu(t)\dot{p}(t) \left[\begin{array}{c} m_1'(p(t), e(p(t), z)) + m_2'(p(t), e(p(t), z)) * \\ e_1'(p(t), z) \end{array} \right]. \tag{5}$$

The maximizing conditions are
(i) $p^*(t)$ maximizes $\widetilde{H}$ for all t: $\frac{\partial \widetilde{H}}{\partial p} = 0$,
(ii) $\dot{\mu} - r\mu = -\frac{\partial \widetilde{H}}{\partial m}$,
(iii) $\dot{m}^* = \frac{\partial \widetilde{H}}{\partial \mu}$ (this just gives back the state equation),
(iv) $\lim_{t \to \infty} \mu(t)m(t)e^{-rt} = 0$ (the transversality condition).
The following are the first two conditions:

$$\frac{\partial \widetilde{H}}{\partial p} = 0, \tag{6}$$

and

$$\dot{\mu} - r\mu = -\frac{\partial \widetilde{H}}{\partial m} = \varsigma'(m(p(t), e(p(t), z))). \tag{7}$$

In equilibrium, $\dot{p}(t) = 0$, and the expression $\frac{\partial \widetilde{H}}{\partial p}$ boils down to the following:

$$p(t) \left[1 + \frac{1}{demand\ elasticity} \right] = \varsigma'(m(p(t), e(p(t), z))) \left\{ \frac{m_1'(p(t), e(p(t), z))}{q'(p(t))} + \frac{m_2'(p(t), e(p(t), z))e_1'(p(t), z)}{q'(p(t))} \right\},$$

that is, the price equals the marginal cost for an infinitely elastic demand. The right hand side of the above expression is the marginal cost, and not the same as in the static/ short term problem, due to the fact that in the long run, the middleman also considers the market price effect on his future purchase price from the producer.

Now, in case the middleman would like to add another unit to inventory, his/ her marginal cost to have an extra unit will be higher, as the term $\varsigma'(m(p(t), e(p(t), z)))$ will be higher at the existing price at that time. In the above marginal cost expression, the term in parentheses, i.e., $\frac{m_1'(p(t),e(p(t),z))}{q'(p(t))} + \frac{m_2'(p(t),e(p(t),z))e_1'(p(t),z)}{q'(p(t))}$ depends on price, therefore has the same value at the exisiting price. Now, the middleman faces the following:

$$\frac{\partial \widetilde{H}}{\partial p} < 0.$$

Therefore, for maximization of profit in a dynamic set up after some positive supply shock, the middleman must reduce the price to have an extra unit of inventory, and hence an inverse relationship between inventory and price. An inventory in a goods market is a unifying factor between supply and demand. When the market is in equilibrium, the inventory stays the same as the supply and demand rates are the same. If either the supply or the demand rate changes due to some external shock, and no other counter shock happens, the inventory, and the price will continue changing until the saturation point of the market arrives. The following formulation is a mathematical depiction of the above explanation:

$$\begin{aligned}
&\textit{Price change} \propto \textit{change in market inventory.}\\
&P = \textit{price change.}\\
&M = m - m_s = \textit{change in inventory in the market,}\\
&m = \textit{inventory at time } t,\\
&m_s = \textit{inventory in steady state equilibrium.}\\
&\textit{Input} - \textit{output} = \frac{dm}{dt} = \frac{d(m - m_s)}{dt} = \frac{dM}{dt},\\
&\text{or } M = \int (\textit{input} - \textit{output})\, dt.\\
&\textit{Price change} \propto \int (\textit{supply rate} - \textit{demand rate})\, dt, \text{ or}\\
&P = -K_m \int (\textit{supply rate} - \textit{demand rate})\, dt,
\end{aligned}$$

where K_m is a positive proportionality constant. The multiplying negative sign suggests that when the difference of *supply rate*, and the *demand rate* is positive, P is negative (i.e., the price decreases).

Rearranging the above expression gives:

$$\int (supply\ rate - demand\ rate)\, dt = -\frac{P}{K_m},\ \text{or}$$

$$\int (w_i - w_0)\, dt = -\frac{P}{K_m}, \tag{8}$$

$$w_i = supply\ rate,$$

$$w_0 = demand\ rate,$$

$$K_m = dimensional\ constant.$$

Suppose that our initial condition is the initial market equilibrium, i.e., at time $t = 0$, *supply rate* = *demand rate*, and eq. (8) becomes

$$\int (w_{is} - w_{0s})\, dt = 0. \tag{9}$$

The subscript s denotes the initial equilibrium (steady state), and $P = 0$, when the market is in equilibrium. Subtracting eq. (9) from eq. (8) gives:

$$\int (w_i - w_{is})\, dt - \int (w_0 - w_{0s})\, dt = -\frac{P}{K_m},\ \text{or}$$

$$\int (W_i - W_0)\, dt = -\frac{P}{K_m}, \tag{10}$$

where $w_i - w_{is} = W_i = change\ in\ supply\ rate,$

$$w_0 - w_{0s} = W_0 = change\ in\ demand\ rate.$$

P, W_i and W_0 are the variables of deviation from the initial equilibrium, with zero initial values. Rearranging eq. (10), we get:

$$P = -K_m \int W\, dt = -K_m M, \tag{11}$$

where $W = W_i - W_0$. If price changes due to some factor other than inventory, an input can be added which modifies eq. (11) as follows:

$$P = -K_m \int W dt + J = -K_m M + J. \tag{11a}$$

There could also be some input or external shock, other than the price feedback for inventory.

2.2 Producers

There are two types of producers, i.e., a domestic producer and a foreign producer. Their objective is to maximize the profits in a dynamic setting. Furthermore, their objective is identical, and hence not considered separately. They maximize the present discounted value of future stream of profits, and the following expression represents their present value at rime zero:

$$V(0) = \int_0^\infty \left[\alpha p(t) F\left(K\left(t\right), L\left(t\right)\right) - w(t) L\left(t\right) - \Re(t) I(t)\right] e^{-rt} dt, \tag{12}$$

The fraction of the market price charged by the producers to the middleman is α. r represents the discount rate. $L(t)$ (labor) and $I(t)$ (level of investment) are the *control variables* and $K(t)$ the *state variable*. The producer's maximization problem is framed below:

$$\underset{\{L(t),I(t)\}}{Max} V(0) = \int_0^\infty \left[\alpha p(t) F\left(K\left(t\right), L\left(t\right)\right) - w(t) L\left(t\right) - \Re(t) I(t)\right] e^{-rt} dt,$$

subject to the constraints that
$\dot{K}(t) = I(t) - \delta K(t)$ (state equation, describing how the state variable changes with time),
$K(0) = K_0$ (initial condition),
$K(t) \geq 0$ (non-negativity constraint on state variable),
$K(\infty)$ free (terminal condition).
The current-value Hamiltonian for this case is

$$\widetilde{H} = \alpha p(t) F\left(K\left(t\right), L\left(t\right)\right) - w(t) L\left(t\right) - \Re(t) I(t) + \mu(t)\left[I(t) - \delta K(t)\right]. \tag{13}$$

The maximizing conditions are as follows:
(*i*) $L^*(t)$ and $I^*(t)$ maximize $\widetilde{H}$ for all t: $\frac{\partial \widetilde{H}}{\partial L} = 0$ and $\frac{\partial \widetilde{H}}{\partial I} = 0$,
(*ii*) $\dot{\mu} - r\mu = -\frac{\partial \widetilde{H}}{\partial K}$,
(*iii*) $\dot{K}^* = \frac{\partial \widetilde{H}}{\partial \mu}$ (this just gives back the state equation),
(*iv*) $\lim_{t\to\infty} \mu(t) K(t) e^{-rt} = 0$ (the transversality condition).
The first two conditions are given below:

$$\frac{\partial \widetilde{H}}{\partial L} = \alpha p(t) F_2'\left(K\left(t\right), L\left(t\right)\right) - w(t) = 0, \tag{14}$$

$$\frac{\partial \widetilde{H}}{\partial I} = -\Re(t) + \mu(t) = 0, \tag{15}$$

and

$$\dot{\mu} - r\mu = -\frac{\partial \widetilde{H}}{\partial K} = -\left[\alpha p(t) F_1'\left(K\left(t\right), L\left(t\right)\right) - \delta\mu(t)\right]. \tag{16}$$

If we plug the value of $\dot{\mu}$ and μ from eq. (15) into eq. (16), we obtain:

$$\alpha p(t) F_1'\left(K\left(t\right), L\left(t\right)\right) - (r+\delta)\Re(t) + \dot{\Re}(t) = 0.$$

If there is a price increase, i.e., the value of $p(t)$ goes up, the producer faces the following inequalities at the exisiting level of investment and labor

$$\alpha p(t) F_2'\left(K\left(t\right), L\left(t\right)\right) - w(t) > 0,$$

$$\alpha p(t) F_1'\left(K\left(t\right), L\left(t\right)\right) - (r+\delta)\Re(t) + \dot{\Re}(t) > 0.$$

Therefore, for maximization of profit in a dynamic set up, the producer must increase the production after an increase in the price. Let p be the market price; c, a reference price, including the production cost, and the profits of the producers and the middlemen, determining the feasibility of the business, and with respect to which the variation in the market price is considered by the producers for enhancing or reducing the production levels.

$$W_m = \textit{Change in production due to change in price},$$

A higher value of $(p-c)$ provides an incentive for a higher level of production to the producer, i.e.,

$$W_m \propto \alpha(p-c), \text{ or}$$

$$W_m = K_s(p-c). \tag{17}$$

In equilibrium, $W_m = 0$, or

$$0 = K_s(p_s - c_s). \tag{18}$$

K_s is a positive proportionality constant. p_s and c_s denote the initial equilibrium values. Subtracting eq. (18) from eq. (17) gives:

$$W_m = K_s\left[(p-p_s) - (c-c_s)\right] = -K_s\left(C-P\right) = -K_s\varepsilon, \tag{19}$$

where W_m, C and P are the variables of deviation from the initial steady state equilibrium.

2.3 Consumer

The present discounted value of future stream of utilities is maximized by the consumer, and the following expression represents his/ her present value at rime zero:

$$V(0) = \int_0^\infty U(x(t))e^{-\rho t}dt, \tag{20}$$

ρ is the discount rate, and $x(t)$, the *control variable.* The consumer's maximization problem is framed below:

$$\underset{\{x(t)\}}{Max} V(0) = \int_0^\infty U(x(t))e^{-\rho t}dt,$$

subject to the constraints that

$\dot{a}(t) = R(t)a(t) + w(t) - p(t)x(t)$ (state equation, describing how the state variable changes with time). $a(t)$ is asset holdings (a *state variable*) and $w(t)$ and $R(t)$ are exogenous time path of wages and return on assets.

$a(0) = a_s$ (initial condition),

$a(t) \geq 0$ (non-negativity constraint on state variable),

$a(\infty)$ free (terminal condition).

The current-value Hamiltonian for this case is

$$\widetilde{H} = U(x(t)) + \mu(t)\left[R\left(t\right)a\left(t\right) + w\left(t\right) - p\left(t\right)x\left(t\right)\right]. \tag{21}$$

The maximizing conditions are as given below:

(i) $x^*(t)$ maximizes $\widetilde{H}$ for all t: $\frac{\partial \widetilde{H}}{\partial x} = 0$,

(ii) $\dot{\mu} - \rho\mu = -\frac{\partial \widetilde{H}}{\partial a}$,

(iii) $\dot{a}^* = \frac{\partial \widetilde{H}}{\partial \mu}$ (this just gives back the state equation),

(iv) $\lim_{t\to\infty} \mu(t)a(t)e^{-\rho t} = 0$ (the transversality condition).

The first two conditions are as follows:

$$\frac{\partial \widetilde{H}}{\partial x} = U'\left(x\left(t\right)\right) - \mu(t)p(t) = 0, \tag{22}$$

and

$$\dot{\mu} - \rho\mu = -\frac{\partial \widetilde{H}}{\partial a} = -\mu(t)R(t). \tag{23}$$

If the price of good x increases, then the consumer faces the following inequality at the existing level of consumption:

$$\frac{\partial \widetilde{H}}{\partial x} = U'\left(x\left(t\right)\right) - \mu(t)p(t) < 0.$$

Therefore, for maximization of utility in a dynamic set up, the consumer must decrease the consumption after an increase in the price. Let the change in demand as a result of a price change, is proportional to P, i.e., a price change with respect to the initial equilibrium, i.e.,

$$\textit{Change in demand} \propto P, \text{ or}$$

$$W_d = -K_d P. \tag{24}$$

W_d and P are variables of deviation from the initial equilibrium, and when P is positive, W_d is negative.

3 Solution of the Model with an Ad Valorem Import Tariff

The domestic market's total supply, i.e., imports as well as the domestic production is $W_m(t)$. We can bifurcate the domestic supply and the imports as follows:

$$W_m(t) = -K_{sd}\left[C_d(t) - P(t)\right] - K_{se}\left[C_e(t) - P(t)\right], \tag{25}$$

The subscript d represents the domestic producer, and e denotes the exporter, i.e., the foreign producer. After solving the model, we get the following expression:

$$\frac{dP(t)}{dt} + K_m(K_{sd} + K_{se} + K_d)P(t) = K_m\left[K_{sd}C_d(t) + K_{se}C_e(t)\right]. \tag{26}$$

If $C_e(t) = T\alpha p(t)$, and $C_d(t) = 0$, i.e., an ad valorem tariff is imposed by the government on the imports at $t = 0$, then the above differential equation can be written as:

$$\frac{dP(t)}{dt} + K_m\left\{K_{sd} + K_{se}(1 - T\alpha) + K_d\right\}P(t) = K_m K_{se} T\alpha p_s. \tag{27}$$

A necessary and sufficient stability condition for a dynamical (linear) system is given by Routh–Hurwitz, which is the following in this case: $K_m\left\{K_{sd} + K_{se}(1 - T\alpha) + K_d\right\} > 0$. This guarantees the existence of an equilibrium the domestic market settles at, after an economic shock. The above differential equation's solution can be written as:

$$P(t) = C_1 + C_2 e^{-[K_m\{K_{sd}+K_{se}(1-T\alpha)+K_d\}]t}. \tag{28}$$

After we plug the values of C_1 and C_2 in eq. (28):

$$P(t) = \frac{K_{se}T\alpha p_s}{\{K_{sd} + K_{se}(1 - T\alpha) + K_d\}} - \frac{K_{se}T\alpha p_s}{\{K_{sd} + K_{se}(1 - T\alpha) + K_d\}} e^{-[K_m\{K_{sd}+K_{se}(1-T\alpha)+K_d\}]t}. \tag{29}$$

The initial condition is that $P(0) = 0$, when $t = 0$. The final condition or the final equilibrium value is $P(\infty) = \frac{K_{se}T\alpha p_s}{\{K_{sd}+K_{se}(1-T\alpha)+K_d\}}$, when $t = \infty$. The dynamics of price after an ad valorem tariff is levied by the government depends on the values of the parameters K_{sd}, K_{se}, K_d, K_m, T and α. The demand must equal the supply in the final equilibrium, which holds.

4 An Optimal Import Tariff Path

Generally, when the efficiency losses are quantified in the economics literature, only the equilibrium states before and after a specific measure are compared. However, there are some efficiency losses during the adjustment of the market to the new equilibrium. When an ad valorem tariff is adopted by a government, the import cost increases, and hence affects the quantity of imports. The demand and the supply in the domestic market adjust over time as a feedback of price adjustment, and the market settles at a new equilibrium. The supply in the market decreases right after the imposition of an ad valorem tariff, whereas the demand has not yet changed, therefore, the inventory gets a negative jump at time zero. Now the market is already out of equilibrium, and the adjustment process starts to drift the market toward a new equilibrium. The price adjusts and the inventory changes as a feedback of price change. The quantum of supply and demand determine the quantum of inventory. The inventory piles up if the supply is higher than the demand and vice versa. When the market is in equilibrium, the supply and demand are the same, and there are no efficiency losses. If the market is out of equilibrium, and the supply is different from demand, either the output and/ or consumption gets wasted. This waste is the efficiency loss. If this waste is summed up, the total efficiency loss during the adjustment process can be quantified. The total efficiency losses, including the losses during the adjustment process are given below:

$$\mathbf{EL} = -\int_0^\infty \mathbf{W}_m(\mathbf{t})\mathbf{dt} = |M(t)| - \int_0^\infty \mathbf{W}_d(\mathbf{t})\mathbf{dt}. \tag{30}$$

The above expression can be decomposed into the welfare gain of the domestic government, a decrease in foreign welfare, and an increase in local producer welfare as follows:

$$EG_g = \int_0^\infty \xi(t) \{w_{ime}(0) + W_{me}(t)\} \, dt, \tag{31}$$

$$EL_e = \int_0^\infty - [\theta(t) |W_{me}(t)| + [\xi(t) \{1 - \psi(t)\} \{w_{ime}(0) + W_{me}(t)\}]] \, dt. \tag{32}$$

$$EG_d = \int_0^\infty [|W_{md}(t)| \, dt + \{1 - \theta(t)\} |W_{me}(t)|] \, dt, \tag{33}$$

where $\xi(t)$ *is the fraction of the quantity of the imported goods as the domestic government revenue,* $\psi(t)$ *is the fraction of the consumer surplus gone into the government welfare, and* $\theta(t)$ *is the share of surplus of producer and the factors of production in the foreign country*. **As a result of an ad valorem tariff, by eq.** (25), **the total domestic supply changes as follows:**

$$W_m(0) = -K_{sd} [C_d(0) - P(0)] - K_{se} [C_e(0) - P(0)] = -K_{se} T \alpha p_s,$$

as $P(0) = 0$.

The inventory goes down by $K_{se} T \alpha p_s$ **at time zero (as the demand has not yet changed). The market forces start drifting the market toward a new equilibrium as soon as the market gets out of equilibrium. The middleman starts changing the market price, which affects the inventory through feedback. From eq.** $(11a)$:

$$\mathbf{P(t)} = -\mathbf{K}_m \mathbf{M(t)} + \mathbf{J}.$$

Through imposition of the initial conditions, the value of J **can be found as follows:**

$$P(0) = -K_m M(0) + J,$$

$$0 = K_m K_{se} T \alpha p_s + J,$$

$$J = -K_m K_{se} T \alpha p_s.$$

Plugging the value of J **in eq.** $(11a)$ **leads to**

$$P(t) = -K_m M(t) - K_m K_{se} T\alpha p_s, \text{ or}$$
$$M(t) = -\frac{1}{K_m}\left[P(t) + K_m K_{se} T\alpha p_s\right].$$

With Tariff Revenue Constraint:

Our problem is from the perspective of generating a target amount of revenue in a specific time period by imposing an ad valorem tariff at the import of a specific commodity. In this scenario, an ad valorem tariff has to be imposed by a small country as the objective is revenue collection, however, an optimal tariff can minimize the efficiency losses subject to this constraint. **The tariff revenue (TR) expression for the government is given below:**

$$\mathbf{TR} = \mathbf{T}\alpha\mathbf{p(t)}\left[w_{ime}(0) - K_{se}\left\{T\alpha p(t) - P(t)\right\}\right]. \tag{34}$$

Our problem of minimizing the efficiency losses subject to the tariff revenue constraint, i.e., the revenue collection must be at least of the amount of G, in a specific time period is given below:

$$\min_T \mathbf{EL} \quad \text{s.t.} \quad \mathbf{TR} \geq \mathbf{G}.$$

The choice variable is the tariff rate, and the constraint is binding. There is no closed form solution, therefore, we use the Perturbation Method to obtain general solution of the first order approximation of the equilibrium system as follows:

Suppose that the government wants to generate a tariff revenue equal to the amount of \$1000. For K_m, K_d, K_{se} and K_{sd}, each one equal to 1, $\alpha = 0.75$, $w_{ime}(0) = 100$, and $p_s = 10$, the expression for the tariff revenue can be written as

$$0.75T\left[P(t) + 10\right]\left[100 - (0.75T - 1)P(t) - 7.5T\right] = 1000.$$

For $t = 0$, $P(0) = 0$, the above expression can be written as

$$T^2 - 13.333T + 17.777 = 0.$$

We write the problem into a perturbation problem indexed by a small parameter φ as follows:

$$T^2 - (13 + \varphi)T + 53.387\varphi = 0,$$

where $\varphi \equiv 0.333$. We index the solution as a function of the perturbation parameter $T = f(\varphi)$:

$$f(\varphi)^2 - (13+\varphi)f(\varphi) + 53.387\varphi = 0,$$

and assume each of this solution is smooth (this can be shown to be the case for our particular example). For $\varphi = 0$*, we have*

$$f(0)^2 - 13f(0) = 0,$$

$$f(0) = 0, 13.$$

Since we require that the efficiency loss must be minimized, we take $f(0) = 0$*. By Taylor Theorem:*

$$T = f(\varphi) \mid_{\varphi=0} = f(0) + \sum_{n=1}^{\infty} \frac{f^n(0)}{n!}\varphi^n.$$

Let us take the derivative of $f(\varphi)^2 - (13+\varphi)f(\varphi) + 53.387\varphi = 0$*, with respect to* φ*:*

$$2f(\varphi)f'(\varphi) - f(\varphi) - (13+\varphi)f'(\varphi) + 53.387 = 0,$$

$$f'(\varphi) = 4.1.$$

By Taylor:

$$T = f(\varphi) \mid_{\varphi=0} \simeq f(0) + \frac{f^{'}(0)}{1!}\varphi^1,$$

or $T \simeq 4.1\varphi$.

We can also find an infinite number of values of T *against time, and through regression techniques, can get an approximate relationship of* T *and time, that will approximate the functional form over time.*

In this case, we can also solve the problem numerically through finding the roots of the following equation:

$$T^2 - 13.333T + 17.777 = 0.$$

This implies that

$$\mathbf{T = 1.502689, 11.830644}.$$

The absolute value of EL **in expression** (52) **is minimized for** $T = 1.502689$. **Similarly, for** $t = \infty$, **the expression for tariff revenue can be written as**

$$\mathbf{2.5875T^2 - 11.25T + 9 = 0}.$$

This implies that

$$\mathbf{T = 1.056936,\ 3.29089}.$$

The efficiency loss is minimized for $T = 1.056936$**.**

5 Static Versus Dynamic Efficiency Loss

A change in demand as a feedback of price change by eq. (24), can be written as

$$W_d(t) = -K_d P(t).$$

The change in demand in the final equilibrium is:

$$\begin{aligned} W_d(\infty) &= -K_d P(\infty), \text{ or} \\ W_d(\infty) &= \frac{-K_d K_{se} T \alpha p_s}{\{K_{sd} + K_{se}(1 - T\alpha) + K_d\}}. \end{aligned} \tag{35}$$

Eq. (35) gives the quantity of total consumption lost as a result of imposition of an ad valorem import tariff T in the final equilibrium as compared to that in the initial equilibrium or the static efficiency loss. The dynamic efficiency loss is given by eq. (30) as follows:

$$EL = \int_0^{t_e} \left[K_{se} T \alpha p_s - \frac{K_{se} T \alpha p_s (K_{sd} + K_{se})}{\{K_{sd} + K_{se}(1 - T\alpha) + K_d\}} + \frac{K_{se} T \alpha p_s (K_{sd} + K_{se})}{\{K_{sd} + K_{se}(1 - T\alpha) + K_d\}} e^{-[K_m\{K_{sd} + K_{se}(1 - T\alpha) + K_d\}]t} \right] dt, \tag{36}$$

where the new equilibrium arrives in t_e time.

6 Conclusion

When an ad valorem tariff is imposed by a government, the cost of the foreign producer gets an upward jump to the extent of costs enhanced due to tariff policy. This affects the quantity of imports and the market gets out of equilibrium. The market supply and demand of the commodity for which the tariff policy is adopted, gets adjusted over time until the post-policy equilibrium arrives. In the existing literature, the efficiency losses during the time the market is adjusting are not accounted for, when the costs and benefits of a tariff policy are computed. As during the adjustment process, the market is out of equilibrium and there are some efficiency losses, it is worthwhile to consider those losses while designing an optimal ad valorem tariff. Section 4 calculates an optimal ad valorem import tariff path over time, which satisfies the cost constraint at all points in time while taking into consideration the adjustment of the supply and demand over time. An optimal ad valorem tariff policy is a function of the demand, supply and the inventory curves' slopes as well as the initial equilibrium quantity. In contrast to this, in a static model which ignores the dynamic dimension of the problem, the government would only be taking into consideration the efficiency in the equilibrium. Based upon that, the government would decide to impose an inefficient tariff.

7 Appendix:

7.1 Dynamic Problem of the Middleman

The present discounted value of future stream of profits is maximized by the middleman, and the following expression represents his/ her present value at rime zero:

$$V(0) = \int_0^\infty \left[pq(p) - \varsigma(m(p,e))\right] e^{-rt} dt, \tag{37}$$

r is the discount rate. $p(t)$ is the *control variable* and $m(t)$ the *state variable*. The middleman's maximization problem is framed below:

$$\underset{\{p(t)\}}{Max} V(0) = \int_0^\infty \left[pq(p) - \varsigma(m(p,e))\right] e^{-rt} dt,$$

subject to the constraints that
$\dot{m}(t) = m_1'(p(t), e(p(t), z))\dot{p}(t) + m_2'(p(t), e(p(t), z))e_1'(p(t), z)\dot{p}(t)$ (state equation, describing how the state variable changes with time; z are exogenous factors),
$m(0) = m_s$ (initial condition),
$m(t) \geq 0$ (non-negativity constraint on state variable),
$m(\infty)$ free (terminal condition).
The current-value Hamiltonian for this case is

$$\widetilde{H} = p(t)q(p(t)) - \varsigma(m(p(t), e(p(t), z))) + \mu(t)\dot{p}(t) \begin{bmatrix} m_1'(p(t), e(p(t), z)) + m_2'(p(t), e(p(t), z)) * \\ e_1'(p(t), z) \end{bmatrix}. \tag{38}$$

The maximizing conditions are
(i) $p^*(t)$ maximizes $\widetilde{H}$ for all t: $\frac{\partial \widetilde{H}}{\partial p} = 0$,
(ii) $\dot{\mu} - r\mu = -\frac{\partial \widetilde{H}}{\partial m}$,

(iii) $\dot{m}^* = \frac{\partial \widetilde{H}}{\partial \mu}$ (this just gives back the state equation),
(iv) $\lim_{t\to\infty} \mu(t)m(t)e^{-rt} = 0$ (the transversality condition).
The first two conditions are as follows:

$$\frac{\partial \widetilde{H}}{\partial p} = q(p(t)) + p(t)q'(p(t)) - \varsigma'(m(p(t), e(p(t), z))) \left\{ \begin{array}{c} m'_1(p(t), e(p(t), z)) + m'_2(p(t), e(p(t), z)) * \\ e'_1(p(t), z) \end{array} \right\}$$
$$+ \mu(t)\dot{p}(t) * \left[\begin{array}{c} m''_{11}(p(t), e(p(t), z)) + m''_{12}(p(t), e(p(t), z))e'_1(p(t), z) + \\ m''_{21}(p(t), e(p(t), z))e'_1(p(t), z) + m''_{22}(p(t), e(p(t), z))e'^2_1(p(t), z) + \\ m'_2(p(t), e(p(t), z))e''_{11}(p(t), z) \end{array} \right]$$
$$= 0, \tag{39}$$

and

$$\dot{\mu} - r\mu = -\frac{\partial \widetilde{H}}{\partial m} = \varsigma'(m(p(t), e(p(t), z))). \tag{40}$$

In equilibrium, $\dot{p}(t) = 0$, and the expression $\frac{\partial \widetilde{H}}{\partial p}$ boils down to the following:

$$q(p(t)) + p(t)q'(p(t)) - \varsigma'(m(p(t), e(p(t), z))) \left\{ \begin{array}{c} m'_1(p(t), e(p(t), z)) + m'_2(p(t), e(p(t), z)) * \\ e'_1(p(t), z) \end{array} \right\}$$
$$= 0,$$

$$p(t)q'(p(t)) + q(p(t)) = \varsigma'(m(p(t), e(p(t), z))) \left\{ \begin{array}{c} m'_1(p(t), e(p(t), z)) + m'_2(p(t), e(p(t), z)) * \\ e'_1(p(t), z) \end{array} \right\},$$
$$p(t)\left[1 + \frac{1}{demand\ elasticity}\right] = \varsigma'(m(p(t), e(p(t), z))) \left\{ \frac{m'_1(p(t), e(p(t), z))}{q'(p(t))} + \frac{m'_2(p(t), e(p(t), z))e'_1(p(t), z)}{q'(p(t))} \right\},$$

that is, the price equals the marginal cost for an infinitely elastic demand. The right hand side of the above expression is the marginal cost, and not the same as in the static/ short term problem, due to the fact that in the long run, the middleman also considers the market price effect on his future purchase price from the producer.
Now, in case the middleman would like to add another unit to inventory, his/ her marginal cost to have an extra unit will be higher, as the term $\varsigma'(m(p(t), e(p(t), z)))$ will be higher at the existing price at that time. In the above marginal cost expression, the term in parentheses, i.e.,

$\frac{m_1'(p(t),e(p(t),z))}{q'(p(t))} + \frac{m_2'(p(t),e(p(t),z))e_1'(p(t),z)}{q'(p(t))}$ depends on price, therefore has the same value at the exisiting price. Now, the middleman faces the following:

$$\frac{\partial \widetilde{H}}{\partial p} = q(p(t)) + p(t)q'(p(t)) - \varsigma'(m(p(t),e(p(t),z))) \left\{ \begin{array}{c} m_1'(p(t),e(p(t),z)) + m_2'(p(t),e(p(t),z))* \\ e_1'(p(t),z) \end{array} \right\}$$
$$+ \mu(t)\dot{p}(t) * \left[\begin{array}{c} m_{11}''(p(t),e(p(t),z)) + m_{12}''(p(t),e(p(t),z))e_1'(p(t),z)+ \\ m_{21}''(p(t),e(p(t),z))e_1'(p(t),z) + m_{22}''(p(t),e(p(t),z))e_1'^2(p(t),z)+ \\ m_2'(p(t),e(p(t),z))e_{11}''(p(t),z) \end{array} \right]$$
$$< 0.$$

Therefore, for maximization of profit in a dynamic set up after some positive supply shock, the middleman must reduce the price to have an extra unit of inventory, and hence an inverse relationship between inventory and price. An inventory in a goods market is a unifying factor between supply and demand. When the market is in equilibrium, the inventory stays the same as the supply and demand rates are the same. If either the supply or the demand rate changes due to some external shock, and no other counter shock happens, the inventory, and the price will continue changing until the saturation point of the market arrives. The following formulation is a mathematical depiction of the above explanation:

$$\textit{Price change} \propto \textit{change in market inventory.}$$
$$P = \textit{price change.}$$
$$M = m - m_s = \textit{change in inventory in the market,}$$
$$m = \textit{inventory at time } t\textit{,}$$
$$m_s = \textit{inventory in steady state equilibrium.}$$
$$\textit{Input} - \textit{output} = \frac{dm}{dt} = \frac{d(m-m_s)}{dt} = \frac{dM}{dt},$$
$$\text{or } M = \int (\textit{input} - \textit{output})\, dt.$$
$$\textit{Price change} \propto \int (\textit{supply rate} - \textit{demand rate})\, dt, \text{ or}$$
$$P = -K_m \int (\textit{supply rate} - \textit{demand rate})\, dt,$$

where K_m is a positive proportionality constant. The multiplying negative sign suggests that when the difference of *supply rate*, and the *demand rate* is positive, P is negative (i.e., the price decreases). Rearranging the above expression gives:

$$\int (supply\ rate - demand\ rate)\, dt = -\frac{P}{K_m}, \text{ or}$$

$$\int (w_i - w_0)\, dt = -\frac{P}{K_m}, \tag{41}$$

$$w_i = supply\ rate,$$

$$w_0 = demand\ rate,$$

$$K_m = dimensional\ constant.$$

Suppose that our initial condition is the initial market equilibrium, i.e., at time $t = 0$, *supply rate* = *demand rate*, and eq. (41) becomes

$$\int (w_{is} - w_{0s})\, dt = 0. \tag{42}$$

The subscript s denotes the initial equilibrium (steady state), and $P = 0$, when the market is in equilibrium. Subtracting eq. (42) from eq. (41) gives:

$$\int (w_i - w_{is})\, dt - \int (w_0 - w_{0s})\, dt = -\frac{P}{K_m}, \text{ or}$$

$$\int (W_i - W_0)\, dt = -\frac{P}{K_m}, \tag{43}$$

$$\text{where } w_i - w_{is} = W_i = change\ in\ supply\ rate,$$

$$w_0 - w_{0s} = W_0 = change\ in\ demand\ rate.$$

P, W_i and W_0 are the variables of deviation from the initial equilibrium, with zero initial values. Eq. (43) can also be written as:

$$P = -K_m \int W dt = -K_m M, \tag{44}$$

where $W = W_i - W_0$. If price changes due to some factor other than inventory, an input can be added which modifies eq. (44) as follows:

$$P = -K_m \int W dt + J = -K_m M + J. \tag{47a}$$

There could also be some input or external shock, other than the price feedback for inventory.

7.2 Solution of the Model with an Ad Valorem Import Tariff

From eqs. (11a), (19) and (24):

$$\frac{dP(t)}{dt} = -K_m W(t),$$
$$W_m(t) = -K_s \varepsilon(t),$$
$$\varepsilon(t) = C(t) - P(t),$$
$$W_d(t) = -K_d P(t),$$

and

$$W(t) = W_m(t) - W_d(t),$$

when there is no exogenous demand or supply shock. The domestic market's total supply, i.e., imports as well as the domestic production is $W_m(t)$. We can bifurcate the domestic supply and the imports as follows:

$$W_m(t) = -K_{sd}\left[C_d(t) - P(t)\right] - K_{se}\left[C_e(t) - P(t)\right], \tag{45}$$

The subscript d represents the domestic producer, and e denotes the exporter, i.e., the foreign producer. After solving the model, we get the following expression:

$$\begin{aligned}\frac{dP(t)}{dt} &= -K_m\left[W_m(t) - W_d(t)\right] \\ &= -K_m\left[-K_{sd}\left\{C_d(t) - P(t)\right\} - K_{se}\left\{C_e(t) - P(t)\right\} + K_d P(t)\right] \\ &= -K_m\left[-K_{sd}C_d(t) - K_{se}C_e(t) + (K_{sd} + K_{se} + K_d)P(t)\right].\end{aligned}$$

Rearranging the above expression gives:

$$\frac{dP(t)}{dt} + K_m(K_{sd} + K_{se} + K_d)P(t) = K_m\left[K_{sd}C_d(t) + K_{se}C_e(t)\right]. \tag{46}$$

If $C_e(t) = T\alpha p(t)$, and $C_d(t) = 0$, i.e., an ad valorem tariff is levied by the government on the imports at $t = 0$, then the above differential equation can be written as:

$$\frac{dP(t)}{dt} + K_m\left\{K_{sd} + K_{se}(1 - T\alpha) + K_d\right\}P(t) = K_m K_{se} T\alpha p_s. \tag{47}$$

The characteristic function of the differential equation is as follows:

$$x + K_m\left\{K_{sd} + K_{se}(1 - T\alpha) + K_d\right\} = 0.$$

The single root of the characteristic function is given by:

$$x = -K_m \left\{K_{sd} + K_{se}(1 - T\alpha) + K_d\right\}.$$

The complementary solution is given by:

$$P_c(t) = C_2 e^{-[K_m\{K_{sd}+K_{se}(1-T\alpha)+K_d\}]t}.$$

The form of the particular solution is as follows:

$$P_p(t) = C_1.$$

The solution, therefore, has the following form:

$$P(t) = C_1 + C_2 e^{-[K_m\{K_{sd}+K_{se}(1-T\alpha)+K_d\}]t}. \tag{48}$$

By substituting the above expression into the differential equation, the value of the constant C_1 could be found as follows:

$$-K_m \left\{K_{sd} + K_{se}(1 - T\alpha) + K_d\right\} C_2 e^{-[K_m\{K_{sd}+K_{se}(1-T\alpha)+K_d\}]t} + K_m \left\{K_{sd} + K_{se}(1 - T\alpha) + K_d\right\} C_1$$

$$+K_m \left\{K_{sd} + K_{se}(1 - T\alpha) + K_d\right\} C_2 e^{-[K_m\{K_{sd}+K_{se}(1-T\alpha)+K_d\}]t} = K_m K_{se} T\alpha p_s,$$

$$C_1 = \frac{K_{se}T\alpha p_s}{\{K_{sd} + K_{se}(1 - T\alpha) + K_d\}}.$$

The initial conditions, i.e., $P(0) = 0$, can help determine the value of C_2 as follows:

$$P(0) = \frac{K_{se}T\alpha p_s}{\{K_{sd} + K_{se}(1 - T\alpha) + K_d\}} + C_2 = 0,$$

$$C_2 = -\frac{K_{se}T\alpha p_s}{\{K_{sd} + K_{se}(1 - T\alpha) + K_d\}}.$$

After we plug the values of C_1 and C_2 in eq. (48):

$$P(t) = \frac{K_{se}T\alpha p_s}{\{K_{sd} + K_{se}(1 - T\alpha) + K_d\}} - \frac{K_{se}T\alpha p_s}{\{K_{sd} + K_{se}(1 - T\alpha) + K_d\}} e^{-[K_m\{K_{sd}+K_{se}(1-T\alpha)+K_d\}]t}. \tag{49}$$

The initial condition is that $P(0) = 0$, when $t = 0$. The final condition or the final equilibrium value is $P(\infty) = \frac{K_{se}T\alpha p_s}{\{K_{sd}+K_{se}(1-T\alpha)+K_d\}}$, when $t = \infty$. The dynamics of price after an ad valorem tariff is imposed by the government depends on the values of the parameters K_{sd}, K_{se}, K_d, K_m, T and α.

7.3 An Optimal Import Tariff Path

Generally, when the efficiency losses are quantified in the economics literature, only the equilibrium states before and after a specific measure are compared. However, there are some efficiency losses when the market is adjusting and leading to the new equilibrium. When an ad valorem tariff is adopted by a government, the import cost increases, and hence affects the quantity of imports. The demand and the supply in the domestic market adjust over time as a feedback of price adjustment, and the market settles at a new equilibrium. The supply in the market decreases right after the imposition of an ad valorem tariff, whereas the demand has not yet changed, therefore, the inventory gets a negative jump at time zero. Now the market is already out of equilibrium, and the adjustment process starts to drift the market toward a new equilibrium. The price adjusts and the inventory changes as a feedback of price change. The quantum of supply and demand determine the quantum of inventory. The inventory piles up if the supply is higher than the demand and vice versa. When the market is in equilibrium, the supply and demand are the same, and there are no efficiency losses. If the market is out of equilibrium, and the supply is different from demand, either the output and/ or consumption gets wasted. This waste is the efficiency loss. If this waste is summed up, the total efficiency loss during the adjustment process can be quantified. The total efficiency losses, including the losses during the adjustment process are given below:

$$\mathbf{EL} = -\int_0^\infty \mathbf{W}_m(\mathbf{t})\mathbf{dt} = |M(t)| - \int_0^\infty \mathbf{W}_d(\mathbf{t})\mathbf{dt}. \tag{50}$$

The above expression can be decomposed into the welfare gain of the domestic government, a decrease in foreign welfare, and an increase in local producer welfare as follows:

$$EG_g = \int_0^\infty \xi(t)\left\{w_{ime}(0) + W_{me}(t)\right\}dt, \tag{51}$$

$$EL_e = \int_0^\infty -\left[\theta(t)\left|W_{me}(t)\right| + \left[\xi(t)\left\{1-\psi(t)\right\}\left\{w_{ime}(0) + W_{me}(t)\right\}\right]\right]dt. \tag{52}$$

$$EG_d = \int_0^\infty \left[\left|W_{md}(t)\right|dt + \left\{1-\theta(t)\right\}\left|W_{me}(t)\right|\right]dt, \tag{53}$$

where $\xi(t)$ is the fraction of the quantity of the imported goods as the domestic government revenue, $\psi(t)$ is the fraction of the consumer surplus gone into the government welfare, and $\theta(t)$ is the share of surplus of producer and the factors of production in the foreign country. **As a result of an ad valorem tariff, by eq.** (25), **the total domestic supply changes as follows:**

$$W_m(0) = -K_{sd}\left[C_d(0) - P(0)\right] - K_{se}\left[C_e(0) - P(0)\right] = -K_{se}T\alpha p_s,$$

as $P(0) = 0$.

The inventory goes down by $K_{se}T\alpha p_s$ at time zero (as the demand has not yet changed). The market forces start drifting the market toward a new equilibrium as soon as the market gets out of equilibrium. The middleman starts changing the market price, which affects the inventory through feedback. From eq. $(47a)$:

$$\mathbf{P(t)} = -\mathbf{K}_m\mathbf{M(t)} + \mathbf{J}.$$

Through imposition of the initial conditions, the value of J can be found as follows:

$$P(0) = -K_m M(0) + J,$$

$$0 = K_m K_{se} T\alpha p_s + J,$$

$$J = -K_m K_{se} T\alpha p_s.$$

Plugging the value of J in eq. $(47a)$ **leads to**

$$P(t) = -K_m M(t) - K_m K_{se} T\alpha p_s, \text{ or}$$

$$M(t) = -\frac{1}{K_m}\left[P(t) + K_m K_{se} T\alpha p_s\right].$$

With Tariff Revenue Constraint:

The domestic supply change by eq. (45)**, is as follows:**

$$\mathbf{W}_m(\mathbf{t}) = -\mathbf{K}_{sd}\left[C_d(t) - P(t)\right] - \mathbf{K}_{se}\left[C_e(t) - P(t)\right].$$

The component of supply from the exporter in the foreign country on which tariff is levied is $-K_{se}\left[C_e(t) - P(t)\right]$**, i.e.,**

$$W_{me}(t) = -K_{se}\left[C_e(t) - P(t)\right],$$

$$w_{nme}(t) - w_{ime}(0) = -K_{se}\left[C_e(t) - P(t)\right],$$

where $w_{ime}(0)$ is the initial import quantity and $w_{nme}(t)$ is the new import quantity after tariff. *Our problem is from the perspective of generating a target amount of revenue in a* *specific time period by imposing an ad valorem tariff at the import of a specific commodity. In this scenario, an ad valorem tariff has to be imposed by a small country as the objective is revenue collection, however, an optimal tariff can minimize the efficiency losses subject to this constraint.* **The tariff revenue (TR) expression for the government is given below:**

$$\mathbf{TR} = \mathbf{T}\alpha\mathbf{p}(\mathbf{t})\left[w_{ime}(0) - K_{se}\left\{T\alpha p(t) - P(t)\right\}\right]. \tag{54}$$

Our problem of minimizing the efficiency losses subject to the tariff revenue constraint, i.e., the revenue collection must be at least of the amount of G, in a specific time period is given below:

$$\min_{T} \mathbf{EL} \quad \text{s.t.} \quad \mathbf{TR} \geq \mathbf{G}.$$

The choice variable is the tariff rate, and the constraint is binding. There is no closed form solution, therefore, we use the Perturbation Method to obtain general solution of the first order approximation of the equilibrium system as follows:
Suppose that the government wants to generate a tariff revenue equal to the amount of \$1000. For K_m, K_d, K_{se} *and* K_{sd}, *each one equal to 1,* $\alpha = 0.75$, $w_{ime}(0) = 100$, *and* $p_s = 10$, *the expression for the tariff revenue can be written as*

$$0.75T\left[P(t) + 10\right]\left[100 - (0.75T - 1)P(t) - 7.5T\right] = 1000.$$

For $t = 0$, $P(0) = 0$, *the above expression can be written as*

$$0.75T\left[10 - 0.75T\right] = 10,$$

$$0.5625T^2 - 7.5T + 10 = 0,$$

$$T^2 - 13.333T + 17.777 = 0.$$

We write the problem into a perturbation problem indexed by a small parameter φ as follows:

$$T^2 - (13 + \varphi)T + 53.387\varphi = 0,$$

where $\varphi \equiv 0.333$. *We index the solution as a function of the perturbation parameter* $T = f(\varphi)$:

$$f(\varphi)^2 - (13 + \varphi)f(\varphi) + 53.387\varphi = 0,$$

and assume each of this solution is smooth (this can be shown to be the case for our particular example). For $\varphi = 0$, *we have*

$$f(0)^2 - 13f(0) = 0,$$

$$f(0) = 0, 13.$$

Since we require that the efficiency loss must be minimized, we take $f(0) = 0$. *By Taylor Theorem:*

$$T = f(\varphi) \mid_{\varphi=0} = f(0) + \sum_{n=1}^{\infty} \frac{f^n(0)}{n!} \varphi^n.$$

Let us take the derivative of $f(\varphi)^2 - (13+\varphi) f(\varphi) + 53.387\varphi = 0$, *with respect to* φ:

$$2f(\varphi)f'(\varphi) - f(\varphi) - (13+\varphi)f'(\varphi) + 53.387 = 0,$$

$$f'(\varphi) = 4.1.$$

By Taylor:

$$T = f(\varphi) \mid_{\varphi=0} \simeq f(0) + \frac{f^{'}(0)}{1!} \varphi^1,$$

or $T \simeq 4.1\varphi$.

In this case, we can also solve the problem numerically through finding the roots of the following equation:

$$T^2 - 13.333T + 17.777 = 0.$$

This implies that

$$\mathbf{T = 1.502689, 11.830644}.$$

The absolute value of EL **in expression** (50) **is minimized for** $T = 1.502689$. **Similarly, for** $t = \infty$, **the expression for tariff revenue can be written as**

$$\mathbf{0.75T} \left[\frac{0.75T}{(3-0.75T)} + 1 \right] \left[10 - \frac{0.75T(0.75T-1)}{(3-0.75T)} - 0.75T \right] = \mathbf{10},$$

$$\mathbf{0.75T} * \frac{3\left(30 - 7.5T - 0.5625T^2 + 0.75T - 2.25T + 0.5625T^2\right)}{(3-0.75T)^2} = \mathbf{10},$$

$$(3-0.75T)^2 = 2.25T(3-0.9T),$$

$$9 + 0.5625T^2 - 4.5T = 6.75T - 2.025T^2,$$

$$2.5875T^2 - 11.25T + 9 = 0.$$

This implies that

$$\mathbf{T = 1.056936, 3.29089}.$$

The efficiency loss is minimized for $T = 1.056936$.

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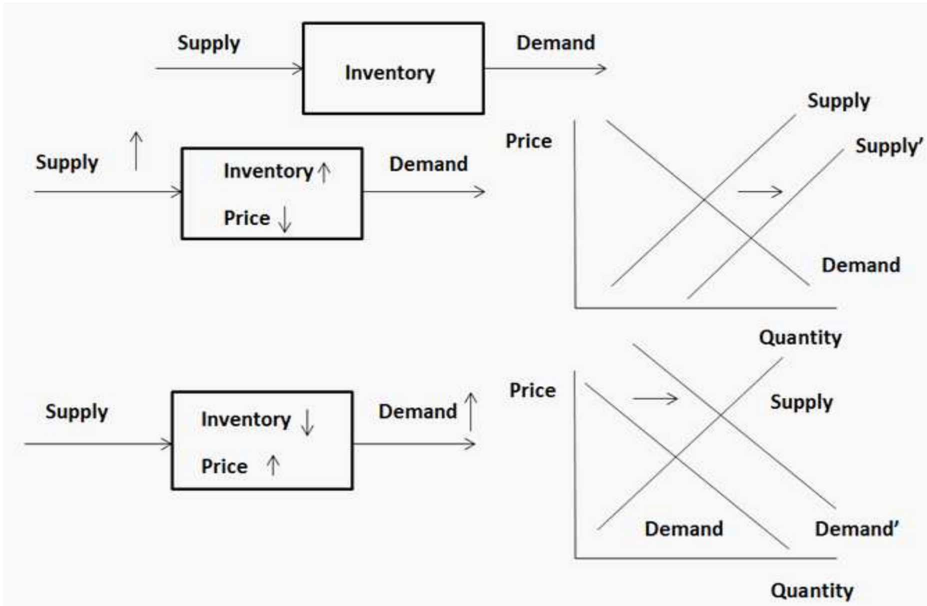

Figure 1: Movement of Price with Inventory.