An Analytic Solution for Valuing Guaranteed Equity Securities

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ABSTRACT

Equity-linked securities with a guaranteed amount have some specific interesting features for investors, like downside protection and capital appreciation. The contract has a guaranteed return plus a payment linked to the performance of a basket of equities or indices averaged over a certain period. This article presents an analytical model for valuing equity-linked notes and computing the corresponding hedge ratios. The model appears to be accurate over a wide range of valuation parameters based on numerical studies. Finally, we use the model to value a segregated fund with a guarantee amount at maturity.

Key Words: Equity-linked securities, segregated fund, asset pricing, derivative valuation, hedge ratio.

JEL Classification: E44, G21, G12, G24, G32, G33, G18, G28

1 Introduction

Equity-linked securities with guaranteed return typically can be public or private, and the equity sensitivity can be to a single stock, a single index, a basket of stocks, or a basket of indices.

The products enable investors with comparatively low budget and knowledge to invest indirectly into derivatives, whereas a direct investment would often not be feasible for them.

There is a vast literature studying equity-linked securities. Kotadia (2021) provides a brief background on the pricing of equity-linked structured products and discusses issues around valuation of these products.

Tiong (2000) studies the pricing of the embedded financial option in such contracts with the asset price following a geometric Brownian motion. Gerber, et al. (2012) carry out the analysis in a risk-neutral framework through Esscher transforms and develop explicit pricing results within that framework.

Rieger (2012) analyzes the reason for retail investors buying structured products and concludes the probability mis-estimation and behavioral biases play dominant role in investors mind while buying such products.

Zhang et al. (2020) utilize the exponential Lévy process for modeling the stock price process to analyze the equity-linked pricing problem. By using the Fast Fourier Transform, they derive the price of the structured products and obtain the price for various payoffs.

Wang et al. (2021) analyze the valuation problem of equity-linked instruments with regime-switching jump diffusion models. Their method of Fourier expansion and Fourier transform has been used to derive closed expressions for some contracts. Their method's effectiveness is demonstrated by numerical values that confirm its efficiency.

Kirkby and Nguyen (2021) focus their work on determining the payoff of equity-linked products and are able to derive a closed form of the price of such products when the risky index process follows the exponential Lévy process.

This paper presents a new model for pricing equity-linked security with a guaranteed return. The structured contract can be viewed as a fixed return plus an embedded Asian style option on the basket of indices.

Numerically we compare the model against both a Monte Carlo benchmark and the Levy model. The numerical results show close agreement with the Monte Carlo benchmark over a wide range of pricing parameters, indicating the model is quite accurate.

We use the model to price a segregated fund whose value at maturity is guaranteed to be greater than the starting invested principal. The fund holder incurs a protection fee towards the fund's guaranteed minimum value at maturity

The rest of this article is organized as follows: First we elaborate equity-linked security with guaranteed return. Second, we present a new model for valuing the security. Third, we provide numerical results. Next, we use the model to price a segregated fund. Finally, the conclusions are provided.

2 Equity-Linked Security with Guaranteed Amount

The objectives for an equity-linked security are, first, to provide equity returns whose performance is linked in some fashion to the equity market. Second, we'd like to get favorable capital treatment. We can define the product more rigorously as follows:

We consider a security whose payoff depends on the return from a finite number, M, of equity-linked indices. Let I_t^j, for $j = 1,\ldots,M$, denote the price of the j^{th} index at time equal to t, and let ω_j denote a fixed, positive weight corresponding to this index.

Next let T denote the security's maturity. Furthermore let $\{t_1,\ldots,t_N\}$, where $N > 0$ and $0 < t_1 < \ldots < t_N \le T$, be a finite set of observation times. Finally let P denote an amount guaranteed to the security holder at maturity.

The payoff at maturity depends on the weighted sum, over each index, of the relative change in the arithmetic average of the index's price, with respect to the set observation points above, from the index's initial level. Formally the payoff at maturity is given by

$$\max\left(P + \sum_{i=1}^{M} \omega_i \frac{\frac{1}{N}\sum_{j=1}^{N} I_{t_j}^i - I_0^i}{I_0^i},\ P \right) = P + \max\left(\sum_{i=1}^{M} \omega_i \frac{\frac{1}{N}\sum_{j=1}^{N} I_{t_j}^i - I_0^i}{I_0^i},\ 0 \right). \qquad (2.1)$$

Next let

$$Z_t = \sum_{j=1}^{M} \alpha_j I_t^j$$

denote the price at time t of a basket of the equity-linked indices above; here $\alpha_j = \frac{\omega_j}{I_0^j}$ is the ratio of the j^{th} index's weight over the index's initial level.

Then the payoff (2.1) is equivalent to

$$P + \max\left(\frac{1}{N}\sum_{i=1}^{N} Z_{t_i} - \sum_{i=1}^{M} \omega_i, 0 \right),$$

which is the sum of the payoff from an Asian style option sampled at the discrete points above plus the guaranteed component.

3 Valuation Model

In this section we present our model for pricing an Asian style option with payoff at maturity, T, of the form

$$\max\left(\frac{1}{N}\sum_{i=1}^{N} Z_{t_i} - \sum_{i=1}^{M}\omega_i, 0\right). \tag{3.1}$$

Here we assume that each index follows geometric Brownian motion with drift under its respective risk-neutral probability measure. Each index is then expressed under the domestic risk-neutral probability measure by a corresponding change of measure. Observe that, under these assumptions, the random variable

$$Y = \frac{1}{N}\sum_{i=1}^{N} Z_{t_i} \tag{3.2}$$

is not log-normally distributed. This, then, makes it mathematically difficult to value the payoff (3.1) using analytical techniques.

The standard Levy approach (see Levy (1992)) towards valuing the payoff (3.1) is to approximate Y in (3.2) by a log-normally distributed random variable. Here the defining parameters for the log-normal random variable are uniquely determined by matching its first two moments with those of Y.

The option value is then given from an analytical formula by taking the expected value of the payoff (3.1), but where the underlying security value, Y, is replaced by that of the log-normally distributed random variable.

Our valuation approach aims to match more moments, and can be viewed as an extension of Levy's. Specifically, we approximate Y in (3.2) by a shifted log-normal random variable, of the form

$$a + e^{b+c\varepsilon}, \tag{3.3}$$

where a and b are constants, c is a positive constant, and ε is a standard, normally distributed random variable. Here a, b and c are uniquely determined, with analytical form, by matching the first three moments of Y with those of (3.3).

An analytical, approximate option pricing formula is then derived by taking the expected value of the payoff (3.1), but where the underlying security's value is replaced by that of the shifted log-normal random variable.

Assume that, under the domestic risk neutral probability measure, the process $\left\{ I_t^j \;\middle|\; t \geq 0 \right\}$, for $j = 1,\ldots,M$, satisfies a stochastic differential equation of the form

$$dI_t^j = I_t^j \left(\mu_j dt + \sigma_j dW_t^j \right)$$

where μ_j is a constant drift parameter, σ_j is a constant volatility parameter, and $\left\{ W_t^j \middle| t \geq 0 \right\}$ is a standard Brownian motion.

Suppose also that the Brownian motions $\left\{ W_t^j \middle| t \geq 0 \right\}$ and $\left\{ W_t^k \middle| t \geq 0 \right\}$, for $j,k \in \{1,\ldots, M\}$, have a constant instantaneous correlation coefficient, ρ_{jk}. The first moment of Y then equals

$$E(Y)=\frac{1}{N}\sum_{k=1}^{N}\sum_{i=1}^{M}\alpha_i I_0^i e^{\mu_i t_k} \ .$$

Furthermore, the second moment of Y is given by

$$E(Y^2)=\frac{1}{N^2}\sum_{k,l\in\{1,\dots,N\}} E\left(Z_{t_k} Z_{t_l}\right),$$

where

$$E\left(Z_{t_k} Z_{t_l}\right)=\sum_{m,n\in\{1,\dots,M\}}\alpha_m\alpha_n I_0^m I_0^n e^{\mu_m t_k+\mu_n t_l+\sigma_m\sigma_n\rho_{mn}\min(t_k,t_l)}.$$

Also, the third moment of Y equals

$$E\left(Y^3\right)=\frac{1}{N^3}\sum_{l,m,n\in\{1,\dots,N\}} E\left(Z_{t_l} Z_{t_m} Z_{t_n}\right),$$

where

$$E\left(Z_{t_l} Z_{t_m} Z_{t_n}\right)=\sum_{i,j,k,\in\{1,\dots,M\}}\alpha_i\alpha_j\alpha_k I_0^i I_0^j I_0^k e^{\mu_i t_l+\mu_j t_m+\mu_k t_n+\sigma_i\sigma_j\rho_{ij}\min(t_l,t_m)+\sigma_i\sigma_k\rho_{ik}\min(t_l,t_n)+\sigma_j\sigma_k\rho_{jk}\min(t_m,t_n)}.$$

By matching the first three moments of Y with those of the shifted log-normal random variable (3.3), we obtain the system of nonlinear equations

$$E(Y) = a + e^{b+\frac{c^2}{2}}, \tag{3.4a}$$

$$E(Y^2) = a^2 + 2ae^{b+\frac{c^2}{2}} + e^{2b+2c^2}, \tag{3.4b}$$

$$E(Y^3) = a^3 + 3a^2 e^{b+\frac{c^2}{2}} + 3ae^{2b+2c^2} + e^{3b+\frac{9}{2}c^2}, \tag{3.4c}$$

for the unknowns a, b and c. It can be shown that, under certain conditions, the nonlinear system of equations above has a closed-form, unique, real solution.

Let r denote the risk-free interest rate (see https://finpricing.com/lib/IrCurveIntroduction.html) for a term equal to the option maturity, T. The Asian style option with payoff (3.1) then has value

$$\Omega = e^{-rT} E\left(\max\left(\frac{1}{N}\sum_{i=1}^{N} Z_{t_i} - \sum_{i=1}^{M} \omega_i, 0 \right) \right), \tag{3.7a}$$

which we approximate by

$$\tilde{\Omega} = e^{-rT} E\left(\max\left(a + e^{b+c\varepsilon} - \sum_{i=1}^{M} \omega_i, 0 \right) \right). \tag{3.7b}$$

Let X denote $\sum_{i=1}^{M} \omega_i$. If $X > a$, then (3.7b) equals

$$e^{-rT} \int_{\frac{\log(X-a)-b}{c}}^{+\infty} \left(a + e^{b+cy} - X \right) f(y)dy$$
$$= e^{-rT}\left((a - X)\, n\left(\frac{b - \log(X-a)}{c} \right) + e^{b+\frac{c^2}{2}}\, n\left(c + \frac{b - \log(X-a)}{c} \right) \right),$$

where $f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$ is the probability density function for a standard, normally distributed random variable and $n(z) = \int_{-\infty}^{z} f(y)dy$ is the corresponding cumulative distribution function.

If $X \le a$, then (3.7b) equals

$$e^{-rT}\left(a - X + e^{b+\frac{c^2}{2}}\right).$$

We may be interested in the option delta, $\frac{\partial \Omega}{\partial I_0^j}$, and the option Vega, $\frac{\partial \Omega}{\partial \sigma_j}$, for $j = 1,\ldots,M$. These hedge ratios are respectively approximated by $\frac{\partial \tilde{\Omega}}{\partial I_0^j}$ and $\frac{\partial \tilde{\Omega}}{\partial \sigma_j}$, for $j = 1,\ldots,M$, and are obtained from direct differentiation of $\tilde{\Omega}$ using the chain rule.

Let the floor level, F, be given such that $F \le -1$. Furthermore let $N = 0$ be the number of price returns to be capped. The combined price return is then of the form

$$\begin{aligned}\sum_{i=1}^{M} \max\left(\frac{I_i}{I_{i-1}} - 1, F\right) &= \sum_{i=1}^{M}\left(\frac{I_i}{I_{i-1}} - 1\right), \\ &= -M + \sum_{i=1}^{M} \frac{I_i}{I_{i-1}}, \\ &= -M + \sum_{i=1}^{M} e^{\left(r-q-\frac{\sigma^2}{2}\right)(t_i - t_{i-1}) + \sigma\left(W_{t_i} - W_{t_{i-1}}\right)},\end{aligned}$$

since $\frac{I_i}{I_{i-1}} > 0$, for $i \in \{1,.., M\}$. The value of the combined price return then equals

$$E\left(\frac{-M+\sum_{i=1}^{M} e^{\left(r-q-\frac{\sigma^2}{2}\right)(t_i-t_{i-1})+\sigma\left(W_{t_i}-W_{t_{i-1}}\right)}}{e^{rT_M}}\right)=e^{-rT_M}\left(-M+\sum_{i=1}^{M} e^{(r-q)(t_i-t_{i-1})}\right).$$

4 Numerical Results

It is interesting to compare the accuracy of our option pricing formula, as well as that of a Levy based pricing formula, against a Monte Carlo benchmark. To this end we consider the following Levy based pricing approach. Let U be a log-normally distributed random variable, of the form

$$U = e^{a+b\varepsilon},$$

where a is a constant, b is a positive constant, and ε is a standard, normally distributed random variable. We choose a and b by matching the first two moments of the basket's price at maturity with those of the log-normal random variable U. We then approximate the option's price, (3.7a), by

$$e^{-rT} E\left(\max\left(U-\sum_{i=1}^{M}\omega_i, 0\right)\right).$$

We have implemented both our pricing model, described in Section 3, and the Levy based approach above.

As an example, we consider the Asian style option arising from a security dependent on the return from a basket of five indices. Here the payoff, of the form (3.1), depends on the arithmetic average of the basket's price at twelve observation points. These points are respectively set to the last business day in each of the eleven months that precede the month in which the security matures and the business day that immediately precedes the maturity date.

Here the security was issued on July 23, 2018, and matures on July 23, 2023; the valuation date is on June 2, 2019. In Figure 4.1 we show the initial level and corresponding weight for each index, as well as the observation points.

Figure 4.1. Snapshot of basket description screen (here we display two significant digits).

Basket Components					
Index	1	2	3	4	5
Initial level	2421.04	391.64	1147.27	15944.36	2913.59
Weight	25	30	10	2.5	2.5

Observation Schedule		
1	3.25	2022-08-31
2	3.33	2022-09-30
3	3.41	2022-10-31
4	3.50	2022-11-30
5	3.58	2022-12-31
6	3.67	2023-01-31
7	3.74	2023-02-28
8	3.83	2023-03-31
9	3.91	2023-04-30
10	3.99	2023-05-31
11	4.08	2023-06-30
12	4.14	2023-07-22

In Figure 4.2 we show pricing results for various embedded options specified from the parameters in Figure 4.1. Here the Volatility Shift parameter indicates a respective relative shift to all original volatility parameter values.

The corresponding benchmark option prices, shown in Figure 4.2, are numerical values for Formula (3.7a), which were computed using crude Monte Carlo simulation based on four million sample paths with .7% standard error. We note that, for the case of zero shift to the volatilities, the parameter values for the shifted log-normal random variable,

$$a + e^{b+c\varepsilon},$$

were computed as $a = .3259$, $b = -.6821$ and $c = 0.3332$.

Figure 4.2. Numerical option pricing results (expressed as a percentage of the notional amount).

Volatility shift	Model price	MC Price	Levy Price
-50	12.59	12.61	12.61
0	13.48	13.50	13.74
50	15.10	15.13	15.80
100	16.99	17.06	18.37

In Figure 4.3 we display various hedge ratios, with respect to the first index, for the option specified from the original parameters shown in Figure 4.1. The hedge ratios based on Formula (3.7b) are from the direct, analytical differentiation of (3.7b). Benchmark hedge ratios are computed using a one-sided finite difference approximation applied to the true option pricing formula, (3.7a); here numerical values for (3.7a) were obtained using crude Monte Carlo simulation based on 4 million sample paths.

Levy based hedge ratios are based on a finite difference approximation. Observe that the relative error in the Levy based vega value from the Monte Carlo (MC) benchmark is approximately 37%, while the vega from Formula (3.7b) differs from the benchmark only by 5.8%.

Figure 4.3. Hedge ratios with respect to the first index.

Hedge ratio	Model value	MC value	Levy value
Price Delta	0.0083	0.0083	0.0082
Vega	2.76	2.94	1.85

5 Segregated Fund Valuation

We consider a segregated fund that invests in various foreign and domestic equities and bonds. We assume that the fund provides a maturity guarantee, that is, the fund's price at maturity is assured to be greater than the original invested amount. We also assume that the fund has no dynamic lapse or reset features, and that the holder pays periodic management and protection fees.

We model the fund's value by the price of basket of representative equity and bond-linked indices; the guarantee at maturity then measures the net shortfall from the basket's constituent indices. Specifically suppose that the basket contains a fixed number, N, of indices. Furthermore let I_t^i, for $i = 1,\ldots,N$, denote the price of the i^{th} index at time t.

Next let P denote the principal amount originally invested in the fund. Assume also that at the fund's outset a percentage, v_i, of the principal is invested in the i^{th} ($i = 1,\ldots,N$) index; the initial number of units, u_i, associated with the i^{th} index then equal

$$u_i = \frac{v_i P}{I_0^i}.$$

Let T denote the fund's maturity. Assume that protection and management fees are both collected at a set of times, $\{t_1,...,t_M\}$, where $0 < t_1 < ... < t_M < T$. Suppose also that the protection and management fee are taken, at time t_i ($i = 1,\ldots,M$), as respective percentages, p_i and m_i, of the fund's price at t_i. The fund's price at maturity then equals

$$\sum_{i=1}^{N} \omega_i I_T^i \tag{5.1}$$

where $\omega_i = u_i \prod_{j=1}^{M} \left[1 - \left(m_j + p_j\right)\right]$.

Suppose that the invested principal, P, is 100% guaranteed at maturity. The payoff at maturity from this guarantee then equals

$$\max\left(P - \sum_{i=1}^{N} \omega_i I_T^i, 0\right), \tag{5.2}$$

which has the same form as that of a European style put option. Our approach towards valuing the payoff above is based on that presented in Section 3.

Specifically, we assume that the i^{th} ($i = 1,\ldots,N$) index's price process, $\{I_t^i \mid t > 0\}$, follows geometric Brownian motion with drift under the domestic risk-neutral probability measure. We then approximate the basket's price at maturity, $\sum_{i=1}^{N} \omega_i I_T^i$, by a shifted log-normal random variable, of the form

$$a + e^{b+c\varepsilon}. \tag{5.3}$$

Here the parameters a, b and c are uniquely determined, as described in Section 3, by matching the first three moments of the shifted log-normal random variable with those of the basket's price at maturity. We next approximate the payoff (5.2) by replacing the basket's value at maturity with that of the shifted log-normal random variable, that is,

$$\max\left(P - \left(a + e^{b+c\varepsilon}\right), 0\right). \tag{5.4}$$

Let r denote the constant risk-free rate for a period equal to the fund's maturity, T. The payoff (5.4) then has value

$$e^{-rT} E\left(\max\left(P - \left(a + e^{b+c\varepsilon}\right), 0\right)\right) \tag{5.5}$$

where E denotes the domestic risk-neutral probability measure. If $a < P$, then
(5.5) equals

$$e^{-rT}\left(\left(P-a\right)n\left(\frac{\log\left(P-a\right)-b}{c}\right) - e^{b+\frac{c^2}{2}}n\left(\frac{\log\left(P-a\right)-b}{c} - c\right)\right)$$

where n is the cumulative distribution function for a standard, normally distributed random variable. If $a \geq P$, then (5.5) equals zero.

6 Summary

This article presents a new model for pricing equity-linked notes with a guaranteed amount. We value the embedded option by assuming that the underlying security follows a shifted log-normal distribution.

An analytical pricing formula is then derived by taking the expected value of the payoff and modelling the underlying security's value as the shifted log-normal random variable.

The model is numerically compared against both a Monte Carlo benchmark and the Levy model. Our pricing model shows close agreement with the Monte Carlo benchmark over a wide range of option parameter values, but has much better computational performance.

We also derive analytical formulas for hedge ratios. These formulas are numerically compared against benchmark Monte Carlo based hedge ratios. Numerical results indicate that Delta hedge ratios are in close agreement with the corresponding benchmark, but that the Vega hedge ratios are slightly off.

We use the model to price a segregated fund with a guarantee at maturity. The results are consistent with market expectation.

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