The Anomalies Of The Wold-Juréen (1953) Functional Form In Overview

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Abstract: Both the Wold-Juréen (1953) utility function and the Wold-Juréen (1953) production function have played a central role in the modelling and the analysis of the Giffen behavior. Using an amalgam of these two functions, this paper defines the Wold-Juréen (1953) functional form, and then compares its properties to the properties of the arbitrary functional form, in an effort to provide a global perspective on the unique nature of the Wold-Juréen (1953) functional form. This paper then reports: (a) that the domain of the Wold-Juréen (1953) functional form (or the first functional form) is a subset of the domain of the arbitrary functional form (or the second functional form), (b) that the signs of the second derivatives of the Wold-Juréen (1953) functional form (or the first functional form) are the opposite of the signs of the second derivatives of the arbitrary functional form (or the second functional form), and (c) that (within the context of the unconstrained-maximization problem) the Wold-Juréen (1953) functional form (or the first functional form) is not concave, whereas the arbitrary functional form (or the second functional form) is concave.

Keywords: Wold-Juréen (1953) functional form, Arbitrary functional form, Giffen behavior, Consumer theory, Producer theory

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Highlights

- Both the Wold-Juréen (1953) utility function and the Wold-Juréen (1953) production function have played a central role in the modelling and the analysis of the Giffen behavior.
- Using an amalgam of these two functions, this paper defines the Wold-Juréen (1953) functional form, and then compares its properties to the properties of the arbitrary functional form, in an effort to provide a global perspective on the unique nature of the Wold-Juréen (1953) functional form
- This paper reports: (a) that the domain of the Wold-Juréen (1953) functional form (or the first functional form) is a subset of the domain of the arbitrary functional form (or the second functional form), (b) that the signs of the second derivatives of the Wold-Juréen (1953) functional form (or the first functional form) are the opposite of the signs of the second derivatives of the arbitrary functional form (or the second functional form), and (c) that (within the context of the unconstrained-maximization problem) the Wold-Juréen (1953) functional form (or the first functional form) is not concave, whereas the arbitrary functional form (or the second functional form) is concave.

1. Introduction

In the article entitled "Slutsky's 1915 Article: How It Came to Be Found and Interpreted", which appeared in History of Political Economy in 2003, the authors, John Chipman and Jean-Sébastien Lenfant, state:

> "In 1915 the Russian statistician and economist Eugen Slutsky sent off from Kiev an article to the Giornale degli economisti that was translated and published in the July issue of that journal: 'Sulla teoria del bi lancio del consumatore.' This article passed unnoticed. As is now well known, Slutsky's article is one of the most famous examples of those neglected and ignored works whose originality and importance are recognized only after similar results have been obtained by others."

In their history of the Slutsky equation, Chipman and Lenfant also note that one functional form has the potential to yield a Giffen good,[3] that functional form is due to Wold (1948), and to Wold and Juréen (1953, page 102),[4] and that same functional form is the primary focus of the present paper.

The Wold-Juréen (1953) utility function is seen as the progenitor of other functional forms that issued from a multi-decade search for functional forms which define a Giffen good [Sproule (2020)]. For example, Moffatt (2011, page 127) stated that: "(e)ver since Wold and Juréen's attempt to illustrate the Giffen paradox by specifying a particular direct utility function, there has been a stream of contributions from authors pursing similar objectives".

Paradoxically, it was not until the appearance of Weber (1997) that the research literature began to deeply assess the properties of the Wold-Juréen (1953) utility function. For example, Weber showed that the Giffenity of Good 1 (the inferior good) is dependent on the relative magnitudes of the decision maker's (DM) income and the price of Good 2 (the normal good). Weber wrote:

[3] Simply put, a good is said to be a Giffen good when the market price of the said good and the quantity of the said good demanded by the decision maker (DM) move in the same direction [Stigler (1947), and Jensen and Miller (2008)].

[4] The Wold-Juréen (1953) utility function is referred to in footnote 47 on page 579 of Chipman and Lenfant (2002).

> "Giffen behavior is more likely for higher … incomes" and that the Giffenity of Good 1 is more likely at lower values of the price for Good 2 [Weber (1997, page 40)].

The problem with Weber's precondition is that it is vague and amorphous, and therefore it lacks intuitive appeal. This critique was overcome by Sproule's (2020) precondition. In particular, Sproule showed that if the price of the inferior good (Good 1) is greater than or equal to the price of the normal good (Good 2), then Good 1 is a Giffen good. Clearly, Sproule's precondition is more intuitively appealing than Weber's precondition in that: (a) Sproule's precondition accords with a basic tenet of microeconomics, which is that economic decision-making is predicated on relative prices or on a change in relative prices, whereas (b) Weber's precondition does not. It must be noted here that Sproule's precondition is also used: (a) in a second paper on the Wold-Juréen (1953) utility function [see Sproule and Karras (2022)], and (b) in a paper on the Wold-Juréen (1953) production function [see Sproule (2022)].

The present paper offers a new synthesis of much of the above: it offers the first-ever comparison of the properties of the Wold-Juréen (1953) functional form with the properties of the generic or arbitrary functional form. In doing so, we show that the Wold-Juréen (1953) functional form has three anomalous properties worthy of note. These anomalies are as follows:

- **Anomaly 1:** The domain of the arbitrary functional form is coterminous with the first quadrant of the Cartesian plane, whereas the domain of the Wold-Juréen (1953) functional form is a subset of the first quadrant.
- **Anomaly 2:** The arbitrary function exhibits negative second derivatives in both of the two independent variables, x and y. whereas the Wold-Juréen (1953) function exhibits non-negative second derivatives in x and y
- **Anomaly 3:** On one hand, both the arbitrary function and the Wold-Juréen (1953) function exhibit concavity when the objective function is the constrained maximization problem (as would be the case for the consumer's constrained utility-maximization problem or the producer's constrained output-maximization problem). On the other, only the arbitrary functional form exhibits concavity when the objective function is the unconstrained maximization problem (as would be the case for the consumer's

unconstrained utility-maximization problem or the producer's unconstrained output-maximization problem), whereas the Wold-Juréen (1953) function is non-concave when the objective function is the unconstrained maximization problem.

This paper is organized as follows. In Section 2, we define our benchmark model -- the arbitrary functional form, and its core properties. In Section 3, we define the Wold-Juréen (1953) functional form, and compare its core properties with those for the arbitrary functional form. Concluding remarks are offered in Section 4.

2. The Arbitrary Functional Form As Our Benchmark Model

Let $f=f(x,y)$ denote an arbitrary function, where the domain of $f=f(x,y)$ is $\left\{\left(x,y\right)\in R^2_{++}\right\}$ or the first quadrant in the Cartesian plane. Suppose too that this function is at least two times differentiable, and that it is increasing and concave in both x and y, viz., it has derivatives with the following signs: $f_x > 0$, $f_y > 0$, $f_{xx} < 0$, and $f_{yy} < 0$. Finally, note that this functional form contains two special cases in microeconomics: the case of the consumer problem with an arbitrary utility function or the case of the producer problem with an arbitrary production function.

2.1. The Constrained-Maximization Problem: Consider the following constrained-maximization problem:

$$\max_{x,y,\lambda} z = f(x,y) + \lambda\, g(x,y) \qquad (1)$$

where z and λ denote the Lagrange function and the Lagrange multiplier respectively Now $g(x,y)$ is an implicit function $g(x,y) = 0$, and that it denotes a linear constraint such that it is binding [that is, $\lambda \neq 0$], it is at least two times differentiable, and it is both increasing and linear in x and y [that is, $g_x > 0$, $g_y > 0$, $g_{xx} = 0$, and $g_{yy} = 0$].

A unique solution to Equation (1) exists if the Lagrange function, $z = f(x,y) + \lambda\, g(x,y)$, is concave in x and y, where the concavity of z in x and y requires that the second-order total differential of $z = f(x,y) + \lambda\, g(x,y)$ be negative; that is:

$$d^2z = -\left(-f_{xx}.\left(g_y\right)^2 + 2f_{xy}.g_x.g_y - f_{yy}.\left(g_x\right)^2 \right)\left(\frac{dy}{g_x}\right)^2 < 0 \tag{2}$$

[See Chiang and Wainwright (2005, pages 356-359), Hands (1991, pages 342-345), and.Roberts and Schulze (1976, pages 160-161).] Now Equation (2) can be rewritten as:

$$\begin{aligned} d^2z &= -\left(-f_{xx}.\left(g_y\right)^2 + 2f_{xy}.g_x.g_y - f_{yy}.\left(g_x\right)^2 \right)\left(\frac{dy}{g_x}\right)^2 \\ &= -\left(\left|H_1^B\right|\right)\left(\frac{dy}{g_x}\right)^2 < 0 \end{aligned} \tag{3}$$

where $\left|H_1^B\right| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & f_{xx} & f_{xy} \\ g_y & f_{xy} & f_{yy} \end{vmatrix} > 0$ denotes both a bordered Hessian determinant and the SOC for Equation (1).

Finally, we note the following:

Proposition 1: If both the FOCs and the SOC for Equation (1) hold, then there exists a unique solution set for Equation (1), namely there exists $(x^*_{\text{Equation (1)}},\ y^*_{\text{Equation (1)}},\ \lambda^*_{\text{Equation (1)}})$.

Corollary 1: If the solution set for the consumer's constrained utility-maximization problem is unique (Proposition 1), then there exists related Marshallian demand functions.

Corollary 2 : If the solution set for the producer's constrained output-maximization problem is unique (Proposition 1), then there exists related conditional factor demand functions.[5]

[5] It is useful to note that a case-specific example of Proposition 1, Corollary 1, and Corollary 2, can be found in the square-root functional form due to Sproule (2015), viz., $f\left(x,y\right) = \sqrt{x} + \sqrt{y}$

2.2. The Unconstrained-Maximization Problem: Now suppose that the linear constraint in Equation (1) is non-binding; that is, suppose that $\lambda = 0$. In this case, Equation (1) collapses to the following unconstrained-maximization problem:

$$\max_{x,y} z = f(x,y) \tag{4}$$

where $f(x,y)$ denotes the said objective function, and where all of the properties of $f(x,y)$ remain as before. The objective function, $z = f(x,y)$, is concave if the second-order total derivative of $z = f(x,y)$ is negative; that is, $z = f(x,y)$ is concave if

$$d^2z = f_{xx}\left(dx + \frac{(f_{xy}).dy}{f_{xx}}\right)^2 + \left(\frac{f_{xx}.f_{yy} - (f_{xy})^2}{f_{xx}}\right)(dy)^2 < 0 \tag{5}$$

From Equation (5), it is clear that two sufficient conditions consistent with $d^2z < 0$ are $f_{xx} < 0$ and $f_{xx}.f_{yy} - (f_{xy})^2 > 0$. These two conditions serve as the SOCs for Equation (4) **[**See Baldani et al. (2005, page 181), Chiang and Wainwright (2005, page 298), Hands (1991, page 134) , and.Roberts and Schulze (1976, page 153).**]**

Proposition 2: If the FOCs and the SOCs for Equation (4) hold, then there exists a unique solution set for Equation (4), namely there exists $(x^*_{\text{Equation (4)}}, y^*_{\text{Equation (4)}})$

Corollary 3: If the solution set for the consumer's unconstrained utility-maximization problem is unique (Proposition 3), then there exists the related Frischian demand functions. [6]

Corollary 4 : If the solution set for the producer's unconstrained output-maximization problem is unique (Proposition 3), then there exists the related unconditional factor demand functions. [7]

[6] For details on the Frischian demand function, see Cornes (1992, pages 163-165), and Sproule (2013 and 2015).
[7] It is useful to note that a case-specific example of Proposition 2, Corollary 3, and Corollary 4, can be found in the square-root functional form due to Sproule (2015), viz., $f(x,y) = \sqrt{x} + \sqrt{y}$

3. The Wold-Juréen (1953) Functional Form, and Its Anomalous Properties

The Wold-Juréen (1953) functional form is defined as $f(x,y) = (x\text{-}1)(y\text{-}2)^{-2}$, where the domain of $f(x,y) = (x\text{-}1)(y\text{-}2)^{-2}$ is $\{(x,y) \in R^2_{++} : x>1, 0<y<2\}$ [see Vives (1987, page 99), Weber (1997, page 39), and Sproule (2020 and 2022)]. This functional form contains two special cases of particular interest in microeconomics: (a) the Wold-Juréen (1953) utility function [Sproule (2020)] and (b) the Wold-Juréen (1953) production function [Sproule (2022)].

Two additional preliminary remarks about the Wold-Juréen (1953) functional form are warranted.

Firstly, as indicated above, the original interest in the Wold-Juréen (1953) functional form arises from the fact that it provides an analytical framework for the definition and the study of "Giffen behavior" or "Giffenity" [see Weber (1997), Heijman and van Mouche (2011), Moffatt (2002 and 2011), Sproule (2020 and 2022), and Sproule and Karras (2022)]. The second preliminary remark is found in Lemma 1 …

Lemma 1 [Weber (1997, page 39)]: The properties of the Wold-Juréen (1953) functional form include these derivatives and their signs:

$$f_x = (y-2)^{-2} > 0 \tag{6}$$

$$f_y = -2(x\text{-}1)(y\text{-}2)^{-3} > 0 \tag{7}$$

$$f_{xx} = 0 \tag{8}$$

$$f_{yy} = 6(x\text{-}1)(y\text{-}2)^{-4} > 0 \tag{9}$$

$$f_{xy} = -2(y\text{-}2)^{-3} > 0 \tag{10}$$

$$f_{xx}.f_{yy} - (f_{xy})^2 = -(f_{xy})^2 = -(\text{-}2(y\text{-}2)^{-3})^2 < 0 \tag{11}$$

3.1. The Constrained-Maximization Problem: Next consider the following constrained-maximization problem:

$$\max_{x,y,\lambda'} z = \left(x\text{-}1\right)\left(y\text{-}2\right)^{-2} + \lambda' \, g(x,y) \tag{12}$$

where $x>1$ and $0<y<2$, The SOC for this constrained-maximization problem is met in that:

Lemma 2 [Weber (1997, page 39)]: Weber (1997) has shown that:

$$|H_2^B| = \begin{vmatrix} 0 & f_x & f_y \\ f_x & f_{xx} & f_{xy} \\ f_y & f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & (y\text{-}2)^{-2} & -2(x\text{-}1)(y\text{-}2)^{-3} \\ (y\text{-}2)^{-2} & 0 & -2(y\text{-}2)^{-3} \\ -2(x\text{-}1)(y\text{-}2)^{-3} & -2(y\text{-}2)^{-3} & 6(x\text{-}1)(y\text{-}2)^{-4} \end{vmatrix}$$

$$= 2(x - 1)(y - 2)^{-8} > 0$$

since $x>1$ and $0<y<2$ [by assumption].

Lemma 3 [Chiang and Wainwright (2005, page 372)]: From Lemma 2, it follows that:

$$|H_2^B| = \left(\lambda'\right)^2 |H_1^B| = \left(\lambda'\right)^2 \begin{vmatrix} 0 & g_x & g_y \\ g_x & f_{xx} & f_{xy} \\ g_y & f_{xy} & f_{yy} \end{vmatrix}$$

Lemma 4: $|H_1^B| > 0$ and $d^2z < 0$

Proof: Since $\text{sign}\left(|H_1^B|\right) = \text{sign}\left(|H_2^B|\right)$ [Lemma 3], and since $|H_2^B| > 0$ [Lemma 2], therefore $|H_1^B| > 0$ And since $|H_1^B| > 0$, therefore $d^2z < 0$ [Equation (3)].•

Proposition 3: Since both the FOCs and the SOC for Equation (12) hold, then there exists a unique solution set for Equation (102), ($x^*_{\text{Equation (10)}}$, $y^*_{\text{Equation (10)}}$, $\lambda^*_{\text{Equation (10)}}$)

Corollary 5 [Wold and Juréen (1953, page 102), Vives (1987, page 99), Weber (1997, pages 39-40), Chipman and Lenfant (2002, page 579, footnote 47), Sproule (2020), and Sproule and Karras (2022)]: Since the solution set for the consumer's constrained utility-maximization problem is unique (Proposition 3), then there exists related Marshallian demand functions, and these are defined as:

$$x^{*}_{\text{Equation (10)}} = x^{*}_{\text{Equation (10)}}(p_x, p_y, m) = 2 + \frac{2p_y - m}{p_x}. \tag{13}$$

$$y^{*}_{\text{Equation (10)}} = y^{*}_{\text{Equation (10)}}(p_x, p_y, m) = 2\left(\frac{m - p_x}{p_y} - 1\right) \tag{14}$$

where m denotes an exogenous level of consumer income, p_x denotes the exogenous price of the inferior good (or Good x), and p_y denotes the exogenous price of the normal good (or Good y). [8]

Corollary 6 [Sproule (2022)]: Since the solution set for the producer's constrained output-maximization problem is unique (Proposition 3), then there exists related conditional factor demand functions, and these are defined as:

$$x^{*}_{\text{Equation (10)}} = x^{*}_{\text{Equation (10)}}(p_x, p_y, c) = 2 + \frac{2p_y - c}{p_x}. \tag{15}$$

$$y^{*}_{\text{Equation (10)}} = y^{*}_{\text{Equation (10)}}(p_x, p_y, c) = 2\left(\frac{c - p_x}{p_y} - 1\right) \tag{16}$$

where c denotes an exogenous level of producer cost, p_x denotes the exogenous price of the inferior factor (or Factor x) , and p_y denotes the exogenous price of the normal factor (or Factor y). [9]

3.2. The Unconstrained-Maximization Problem: Now suppose that the linear constraint in Equation (12) is non-binding; that is, suppose that $\lambda' = 0$. In this case, Equation (12) collapses to the unconstrained-maximization problem:

[8] For details on the Hicksian demand functions when the utility function is the Wold-Juréen (1953) utility function, see Sproule (2020).

[9] For details on the producer's conditional factor demand functions when the optimization problem is the producer's constrained cost-minimization problem and the production function is the Wold-Juréen (1953) production function, see Sproule (2022).

$$\max_{x,y} z = (x\text{-}1)(y\text{-}2)^{-2} \tag{17}$$

where $x>1$ and $0<y<2$. Recall Equation (3) from our benchmark model: Equation (3) requires that

$$d^2z = f_{xx}\left(dx + \frac{(f_{xy}).dy}{f_{xx}}\right)^2 + \left(\frac{f_{xx}.f_{yy} - (f_{xy})^2}{f_{xx}}\right)(dy)^2 < 0 \tag{3}$$

Next recall several properties of the Wold-Juréen (1953) functional form reported above; that is, recall

$$f_{xx} = 0 \tag{8}$$

$$f_{yy} = 6(x\text{-}1)(y\text{-}2)^{-4} > 0 \tag{9}$$

$$f_{xy} = -\ 2(y\text{-}2)^{-3} > 0 \tag{10}$$

$$f_{xx}.f_{yy} - (f_{xy})^2 = -(f_{xy})^2 = -(\text{-}2(y\text{-}2)^{-3})^2 < 0 \tag{11}$$

since $x>1$ and $0<y<2$ [by assumption]. Note that the injection of Equations (8) to (11) into Equation (3) vitiates the sign of Equation (3); that is, the use of the Wold-Juréen (1953) functional form vitiates the SOCs for our benchmark model, or stated differently $f_{xx} = 0$ and $f_{xx}.f_{yy} - (f_{xy})^2 = -(f_{xy})^2 < 0$ are inconsistent with $f_{xx} < 0$ and $f_{xx}.f_{yy} - (f_{xy})^2 > 0$.

Proposition 4: Since the SOCs for Equation (17) do not hold, it follows that the solution set for Equation (17), $(x^*_{\text{Equation (15)}}, y^*_{\text{Equation (15)}})$, is the null set.

Corollary 7: Since the solution set for the consumer's unconstrained utility-maximization problem is the null set (Proposition 4), then the consumer's Frischian demand functions do not exist.

Corollary 8 [Sproule (2022)]: Since the solution set for the producer's unconstrained profit-maximization problem is the null set (Proposition 4), then the producer's unconditional factor demand functions do not exist.

4. Summary Remarks

The literature on the Wold-Juréen (1953) functional form has passed through three more-or-less distinct stages of development. Stage 1 spans the period 1948 to 1953, and it contains the initial definition of the Wold-Juréen (1953) functional form [see Wold (1948),, and by Wold and Juréen (1953, page 102)]. Stage 2 spans the period 1987 to 2011, and it contains the comments and analyses of Vives (1987, page 99), Weber (1997, pages 39-40), Moffatt (2002 and 2011), Chipman and Lenfant (2002 , page 579, footnote 47), and Heijman and van Mouche (2011), Finally, Stage 3 spans the period 2020 to the present, and it contains the contributions of Sproule (2020 and 2022), and Sproule and Karras (2022), What this literature presently lacks is a mathematical comparison of the Wold-Juréen (1953) functional form with the arbitrary functional form.

The present paper fills this void by offering the first-ever comprehensive comparison of the properties of the Wold-Juréen (1953) functional form with the properties of the arbitrary functional form. In doing so, this paper shows that (when compared to the arbitrary functional form) the Wold-Juréen (1953) functional form has three anomalous properties worthy of note. These anomalies are as follows:

- **Anomaly 1:** The domain of the arbitrary functional form is coterminous with the first quadrant in the Cartesian plane, whereas the domain of the Wold-Juréen (1953) functional form is a subset of the first quadrant.
- **Anomaly 2:** The arbitrary function exhibits negative second derivatives in both of the two independent variables, whereas the Wold-Juréen (1953) function exhibits non-negative second derivatives.
- **Anomaly 3:** On one hand, both the arbitrary function and the Wold-Juréen (1953) function exhibit concavity when the objective function is the constrained maximization

problem (as would be the case for the consumer's constrained utility-maximization problem or the producer's constrained output-maximization problem). On the other, the arbitrary functional form exhibits concavity when the objective function is the unconstrained maximization problem (as would be the case for the consumer's unconstrained utility-maximization problem or the producer's unconstrained output-maximization problem), whereas the Wold-Juréen (1953) function is non-concave when the objective function is the unconstrained maximization problem.

References

Baldani, J., J. Bradfield, and R. Turner (2005), Mathematical Economics, 2nd edition (Chicago: Harcourt Brace College Publishers, Dryden Press).

Chiang, A., and K. Wainwright (2005), Fundamental Methods of Mathematical Economics, 4th edition (New York: McGraw-Hill).

Chipman, J.S., and J-S. Lenfant (2002), "Slutsky's 1915 article: How it came to be found and interpreted," History of Political Economy 34 (3), 553–597.

Cornes, R. (1992), Duality and Modern Economics (Cambridge: Cambridge University Press).

Hands, D.W. (1991), Introductory Mathematical Economics (Lexington, MA: Heath)

Heijman, W., and P.G. van Mouche (2011), "On simple concrete Giffen utility functions: Old and new results," in W. Heijman and P.G. van Mouche, editors, New Insights into the Theory of Giffen Behaviour (Berlin: Springer).

Jensen, R.T., and N.H. Miller (2008), "Giffen behavior and subsistence consumption," American Economic Review 98 (4), 1553-77.

Moffatt, P.G. (2002), "Is Giffen behaviour compatible with the axioms of consumer theory?," Journal of Mathematical Economics 37 (4), 259-267.

Moffatt, P.G. (2011), "A class of indirect utility functions predicting Giffen behaviour," in W. Heijman and P.G. van Mouche, editors, New Insights into the Theory of Giffen Behaviour (Berlin: Springer), 127-141.

Roberts, B. and D.L. Schulze (1976), Modern Mathematics and Economic Analysis (New York: Norton).

Sproule, R. (2013), "A systematic analysis of the links amongst the Marshallian, Hicksian, and Frischian demand functions: A note," Economics Letters 121, 555–557.

Sproule, R. (2015), “An argument for the use of the square-root functional form in teaching undergraduate microeconomics,” International Review of Economics Education 20, 37-45.

Sproule, R. (2020), “The delimitation of Giffenity for The Wold-Juréen (1953) utility function using relative prices: A note,” Economics: The Open-Access, Open-Assessment E-Journal, Kiel Institute for the World Economy, Kiel, Germany.

Sproule, R. (2022), “In search of a Giffen input: A comprehensive analysis of the Wold-Juréen (1953) production function,” Journal of Mathematical Economics and Finance 8 (2), 87-100.

Sproule, R., and M. Karras (2022), “Two conditions which induce Giffen behavior in any numerical analysis if applied to the Wold-Juréen (1953) utility function,” Theoretical and Applied Economics 4, Winter, 197-204

Stigler, G. (1947), “Notes on the history of the Giffen paradox ,” Journal of Political Economy 55, 152-156.

Vives, X. (1987), “Small income effects: A Marshallian theory of consumer surplus and downward sloping demand,” Review of Economic Studies 54 (1), 87–103.

Weber, C.E. (1997), “The case of a Giffen good: Comment,” Journal of Economic Education 28, 36-44.

Wold, H. (1948), “On Giffen’s paradox,” Nordisk tidsskrift for teknisk økonomi 12, 283–90.

Wold, H., and L. Juréen (1953), Demand Analysis: A Study in Econometrics (New York: Wiley).