Problems Book to Accompany Mathematics for Economists

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Chapter 1. Matrix Algebra

Matrix algebra is used with economic models that involve linear-equation systems. For models with n variables systems of n simultaneous linear equations need to be solved. A system of n linear equations in n variables $x_1, x_2, ..., x_n$ can be represented as:

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = d_1 \\
&a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = d_2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = d_n
\end{aligned}$$

This system of linear equations can be written in a matrix format:

$$A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix} \qquad d = \begin{bmatrix} d_1 \\ d_2 \\ ... \\ d_n \end{bmatrix}$$

where the first matrix is the set of coefficients $A = \left[a_{ij} \right]$ or the coefficient matrix. The second set is the set of variables $x_1, x_2, ..., x_n$ also called the variable term vector. The last set is that of the constant terms $d_1, d_2, ..., d_n$ or the constant term vector.

Shortly the system of linear equations can be written as

$$\underset{(n\times n)}{A}\ \underset{(n\times 1)}{x} = \underset{(n\times 1)}{d}$$

We can apply the concept of inverse matrix to the solution of this simultaneous-equation system, where the inverse of matrix A, denoted by A^{-1} is defined only if A is a square matrix such that the following condition is satisfied:

$$AA^{-1} = A^{-1}A = I$$

The result of the multiplication (whether pre- or postmultiplication) is the same. It is the identity matrix. An identity matrix I is a square matrix which has 1 for every element on the principal diagonal from left to right and 0 for all other elements. The identity matrix is similar to the number 1 in scalar algebra since multiplying any matrix by the identity matrix produces the original matrix. It should be noted that not every square matrix has an inverse. Later we will show the conditions for the inverse matrix to exist.

Thus we can solve our system of linear equations through matrix inversion. If the inverse A^{-1} exists, we obtain:

$$A^{-1}Ax = A^{-1}d \quad \text{or}$$

$$\bar{x} = A^{-1}d$$

To find the variable term vector or the variables $x_1, x_2, ..., x_n$ we can first find the inverse A^{-1} for the non-singular matrix A and then multiply it by the constant term vector d .

Conditions for Nonsingularity

The squareness of the coefficient matrix A is a necessary but not a sufficient condition for the nonsingularity of the matrix. What is also necessary is that its rows be linearly independent (or respectively its columns).

An $n \times n$ coefficient matrix A can be considered as an ordered set of row vectors, i.e. a column vector whose elements are themselves row vectors:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} v_1' \\ v_2' \\ \dots \\ v_n' \end{bmatrix}$$

For the row vectors to be linearly independent, none must be a linear combination of the rest. Linear row independence requires that the only set of scalars k_i which can satisfy the vector equation

$$\sum_{i=1}^{n} k_i v_i' = \underset{(1\times n)}{0}$$

be $k_i = 0$ for all i.

Intuitive thinking tells us that in order for a system of equations to possess a determinate solution, the first condition is to have as many equations as there are unknowns. Having fewer equations than variables does not allow finding a unique set of solutions for those variables. The second condition is that equations must be functionally independent of one another, in this case linearly independent. This is to say that no equation should be redundant or should be a repetition of another equation. From looking at the system of equations we may not be immediately able to tell if linear dependence exists. Therefore, we resort to the determinant test of nonsingularity of a matrix.

Determinants and Nonsingularity

The determinant of a square matrix A is a uniquely defined number (scalar) associated with that matrix. For a 2×2 matrix the determinant is defined as:

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Given its dimension, $|A|$ is called a second-order determinant. We can easily illustrate the condition for nonsingularity with the help of the determinant. Consider a determinant where the elements of the second row are a multiple of the elements of the first row respectively. For example,

$$|A| = \begin{vmatrix} a & b \\ ka & kb \end{vmatrix} = kab - kab = 0$$

We find that the determinant to a matrix of linear dependence is zero.

Laplace Expansion

A 3×3 matrix has a third-order determinant the value of which is:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} =$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Each second-order determinant represents a minor $|M_{ij}|$ which is a determinant of the submatrix obtained by deleting the i-th row and the j-th column of the matrix. A cofactor is a minor with a prescribed algebraic sign attached to it such that

$$|C_{ij}| = (-1)^{i+j}|M_{ij}|$$

For odd-number elements for which the sum $i+j$ is an odd number we get that their cofactor will have a different sign from that of their minor. We can use the Laplace expansion process to express a third-order determinant:

$$|A| = a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}| = a_{11}|C_{11}| + a_{12}|C_{12}| + a_{13}|C_{13}| = \sum_{j=1}^{3} a_{1j}|C_{1j}|$$

The value of the determinant of any order n can be found by the Laplace expansion of any row or column:

$$|A| = \sum_{i=1}^{n} a_{ij}|C_{ij}| \text{ (expansion by the } i \text{ th row)}$$

Properties of Determinants

Property 1. The interchange of rows and columns does not change the value of the determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$$

Alternatively, this property implies that the determinants of an original matrix A and its transpose A' are the same. Recall that a transpose to a matrix is a matrix in which the rows of the original matrix are converted into columns and the columns into rows.

Property 2. The interchange of two rows or columns will alter the sign but not the numerical value of the determinant.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad = -(ad - bc)$$

Property 3. The multiplication of any one row or column by a scalar k will change the value of the determinant k times.

$$\begin{vmatrix} a & b \\ kc & kd \end{vmatrix} = kad - kbc = k(ad - bc)$$

As opposed to determinants the factoring of a matrix requires the presence of a common divisor for all elements, not just for one row or one column, for example:

$$\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} = k\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Property 4. The addition (subtraction) of a multiple of any row (or column) to (from) another row (or column) leaves the value of the determinant unchanged.

$$\begin{vmatrix} a & b \\ c + ka & d + kb \end{vmatrix} = ad + kab - bc - kab = ad - bc$$

Property 5. If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero. This property serves to establish whether some rows (columns) are linearly dependent and, hence, whether the matrix is nonsinuglar. Thus, a nonsingular matrix will have a determinant different from 0. When the determinant is 0, then the matrix does not have an inverse.

$$\begin{vmatrix} ka & kb \\ a & b \end{vmatrix} = kab - kab = 0$$

Property 6. The expansion of a determinant by the "wrong" row or column yields a value of zero. If for the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we expand the determinant by using its first-row elements but the cofactors of the second-row elements we get a vanishing determinant.

$$|A| = \sum_{j=1}^{2} a_{1j} |C_{2j}| = a_{11}|C_{21}| + a_{12}|C_{22}| = -ab + ab = 0$$

We use this property in the process of matrix inversion.

Matrix Inversion

Assume that we have an $n \times n$ non-singular matrix

$$\underset{(n\times n)}{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Its cofactor matrix consists of the cofactors to all of its elements:

$$\underset{(n\times n)}{C} = \begin{bmatrix} |C_{11}| & |C_{12}| & \dots & |C_{1n}| \\ |C_{21}| & |C_{22}| & \dots & |C_{2n}| \\ \dots & \dots & \dots & \dots \\ |C_{n1}| & |C_{n2}| & \dots & |C_{nn}| \end{bmatrix}$$

A transpose of the cofactor matrix exists such that:

$$\underset{(n\times n)}{C'} = \begin{bmatrix} |C_{11}| & |C_{21}| & \dots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \dots & |C_{n2}| \\ \dots & \dots & \dots & \dots \\ |C_{1n}| & |C_{2n}| & \dots & |C_{nn}| \end{bmatrix}$$

Another name for the transpose of the cofactor matrix is adjoint of A or an adjoint matrix so

$$AdjA = \underset{(n\times n)}{C'} = \begin{bmatrix} |C_{11}| & |C_{21}| & \dots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \dots & |C_{n2}| \\ \dots & \dots & \dots & \dots \\ |C_{1n}| & |C_{2n}| & \dots & |C_{nn}| \end{bmatrix}$$

where if we multiply the matrices A and C' we obtain

$$\underset{(n\times n)}{AC'} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}|C_{1j}| & \sum_{j=1}^{n} a_{1j}|C_{2j}| & \dots & \sum_{j=1}^{n} a_{1j}|C_{nj}| \\ \sum_{j=1}^{n} a_{2j}|C_{1j}| & \sum_{j=1}^{n} a_{2j}|C_{2j}| & \dots & \sum_{j=1}^{n} a_{2j}|C_{nj}| \\ \dots & \dots & \dots & \dots \\ \sum_{j=1}^{n} a_{nj}|C_{1j}| & \sum_{j=1}^{n} a_{nj}|C_{2j}| & \dots & \sum_{j=1}^{n} a_{nj}|C_{nj}| \end{bmatrix} = \begin{bmatrix} |A| & 0 & \dots & 0 \\ 0 & |A| & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & |A| \end{bmatrix} = |A| I$$

The nonsingularity of matrix A implies that $|A| \neq 0$ so we can divide both sides of the equation and we obtain

$$\frac{AC'}{|A|} = I$$

Pre-multiplying by the inverse matrix A^{-1} we get

$$A^{-1}A\frac{C'}{|A|} = A^{-1}I$$

We get the identity matrix on the left side, therefore, $I\frac{C'}{|A|} = A^{-1}$ or the inverse matrix A^{-1} of A is equal to the transpose of its cofactor matrix divided by its determinant:

$$A^{-1} = \frac{C'}{|A|} \qquad \text{or alternatively} \qquad A^{-1} = \frac{1}{|A|}AdjA$$

We see that in order for any matrix to have an inverse (be non-singular) its determinant must be different from 0 or $|A| \neq 0$.

Cramer's Rule

Sometimes in the variable term vector we may be looking for only one of the unknown variables $x_1, x_2, ..., x_n$. Matrix inversion allows us to use a convenient method to obtain just one of the x variables without having to find all others. This short-cut method of solving is known as the Cramer's rule. Solving a system of linear equations through the method of matrix inversion yields:

$$\bar{x} = A^{-1}d = \frac{C'}{|A|}d$$

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ ... \\ \bar{x}_n \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} |C_{11}| & |C_{21}| & ... & |C_{n1}| \\ |C_{12}| & |C_{22}| & ... & |C_{n2}| \\ ... & ... & ... & ... \\ |C_{1n}| & |C_{2n}| & ... & |C_{nn}| \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \\ ... \\ d_n \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} |C_{11}|d_1 + |C_{21}|d_2 + ... + |C_{n1}|d_n \\ |C_{12}|d_1 + |C_{22}|d_2 + ... + |C_{n2}|d_n \\ ... \\ |C_{1n}|d_1 + |C_{2n}|d_2 + ... + |C_{nn}|d_n \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} \sum_{i=1}^{n}|C_{i1}|d_i \\ \sum_{i=1}^{n}|C_{i2}|d_i \\ ... \\ \sum_{i=1}^{n}|C_{in}|d_i \end{bmatrix}$$

For the first variable x_1 we get

$$\bar{x}_1 = \frac{1}{|A|}\sum_{i=1}^{n}|C_{i1}|d_i = \frac{|A_1|}{|A|}$$

It turns out that the value of first variable is the ratio of two determinants where the determinant $|A_1|$ is the same as $|A|$ but its first column is substituted with the column vector d. The index shows that the first column has been substituted. For the variable x_i we substitute the i th column respectively so

$$\bar{x}_i = \frac{|A_i|}{|A|}$$

Thus, if we want to find the value of the second unknown variable $\bar{x}_2$ we substitute the constant term vector d in the second column of the original determinant. We illustrate the Cramer's rule with a simple numerical example. For the system of two equations with two unknowns

$$5x_1 + 3x_2 = 30$$
$$6x_1 - 2x_2 = 8$$

we must solve for x_1 and x_2. Finding $|A|$,

$$|A| = 5(-2) - 3(6) = -28 \neq 0$$

By Cramer's rule

$$\bar{x}_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 30 & 3 \\ 8 & -2 \end{vmatrix}}{(-28)} = \frac{-60-24}{(-28)} = 3 \qquad \bar{x}_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 5 & 30 \\ 6 & 8 \end{vmatrix}}{(-28)} = \frac{40-180}{(-28)} = 5$$

The Gauss Method

An alternative method for solving systems of linear equations is the Gaussian elimination method. We simply write the system of equations in a matrix form and transform the coefficient matrix into an identity matrix with the help of repeated row operations. We will use the numerical example above to illustrate how this simple method works. Given the system

$$5x_1 + 3x_2 = 30$$
$$6x_1 - 2x_2 = 8$$

we can use the Gauss elimination method to find the values of x_1 and x_2. We write the system in the form of an augmented matrix:

$$A|B = \left[\begin{array}{cc|c} 5 & 3 & 30 \\ 6 & -2 & 8 \end{array}\right]$$

where A is the coefficient matrix. We multiply the first row by $\frac{1}{5}$ to obtain 1 in the a_{11} element.

$$\left[\begin{array}{cc|c} 1 & \frac{3}{5} & 6 \\ 6 & -2 & 8 \end{array}\right]$$

Then we subtract 6 times the first row from the second row to obtain 0 for the first row of the second element and clear the first column.

$$\left[\begin{array}{cc|c} 1 & \frac{3}{5} & 6 \\ 0 & \frac{28}{5} & 28 \end{array}\right]$$

We multiply the second row by $\frac{5}{28}$ to obtain 1 in a_{22}.

$$\left[\begin{array}{cc|c} 1 & \frac{3}{5} & 6 \\ 0 & 1 & 5 \end{array}\right]$$

We multiply the second row by $\frac{3}{5}$ and subtract it from the first row. This gives

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 5 \end{array}\right]$$

Now the solution is on the right or $x_1 = 3$ and $x_2 = 5$.

The Gauss method is further extended to finding an inverse of a matrix. This time from the augmented matrix we try to obtain the identity matrix on the right. After applying a number of row operations we transform the coefficient matrix into the identity matrix and what remains on the right is the inverse matrix. We will again illustrate this using our previous example. For the coefficient matrix

$$A = \begin{bmatrix} 5 & 3 \\ 6 & -2 \end{bmatrix}$$

we set up the augmented matrix

$$\left[\begin{array}{cc|cc} 5 & 3 & 1 & 0 \\ 6 & -2 & 0 & 1 \end{array}\right]$$

and we use similar steps as those for solving linear equations. First, we multiply the first row by $\frac{1}{5}$. This gives us 1 for the first element in the coefficient matrix.

$$\left[\begin{array}{cc|cc} 1 & \frac{3}{5} & \frac{1}{5} & 0 \\ 6 & -2 & 0 & 1 \end{array}\right]$$

Now we multiply the first row by 6 and subtract the second row from it.

$$\left[\begin{array}{cc|cc} 1 & \frac{3}{5} & \frac{1}{5} & 0 \\ 0 & \frac{28}{5} & \frac{6}{5} & -1 \end{array}\right]$$

We multiply the second row by $\frac{5}{28}$ to obtain 1 in the main diagonal of the lead matrix

$$\left[\begin{array}{cc|cc} 1 & \frac{3}{5} & \frac{1}{5} & 0 \\ 0 & 1 & \frac{3}{14} & -\frac{5}{28} \end{array}\right]$$

We only have one more term to clear – to do that we multiply the second row by $\frac{3}{5}$ and subtract it from the first row. Hence,

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{14} & \frac{3}{28} \\ 0 & 1 & \frac{3}{14} & -\frac{5}{28} \end{array}\right]$$

The matrix we obtained on the right is the inverse of the original coefficient matrix. This can easily be checked by multiplying the two matrices such that this should result in the identity matrix:

$$AA^{-1} = A^{-1}A = I$$

$$\begin{bmatrix} 5 & 3 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{14} & \frac{3}{28} \\ \frac{3}{14} & -\frac{5}{28} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Market Equilibrium Analysis

Consider the market for one commodity the demand and supply for which are given by

$$q_d = \alpha - \beta p \qquad \alpha, \beta > 0$$
$$q_s = -\gamma + \delta p \qquad \gamma, \delta > 0$$

where α, β, γ and δ are all positive parameters. Assuming the market is in equilibrium we have quantity demanded equal to quantity supplied at the equilibrium price. We could solve for equilibrium quantity and price either by direct substitution or using the matrix method. Rewriting the example in a matrix form,

$$\beta p + q = \alpha$$
$$\delta p - q = \gamma$$

$$\underset{\substack{coefficient \\ matrix}}{\begin{bmatrix} \beta & 1 \\ \delta & -1 \end{bmatrix}} \underset{\substack{variable \\ matrix}}{\begin{bmatrix} \bar{p} \\ \bar{q} \end{bmatrix}} = \underset{\substack{constant \\ term \\ matrix}}{\begin{bmatrix} \alpha \\ \gamma \end{bmatrix}}$$

$$|A| = -\beta - \delta < 0$$

Solving by the Cramer's rule,

$$\bar{p} = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} \alpha & 1 \\ \gamma & -1 \end{vmatrix}}{-\beta - \delta} = \frac{-\alpha - \gamma}{-\beta - \delta} = \frac{\alpha + \gamma}{\beta + \delta} > 0$$

$$\bar{q} = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} \beta & \alpha \\ \delta & \gamma \end{vmatrix}}{-\beta - \delta} = \frac{\beta\gamma - \alpha\delta}{-\beta - \delta} = \frac{\alpha\delta - \beta\gamma}{\beta + \delta}$$

where we would presume that $\alpha\delta > \beta\gamma$ in order to have positive equilibrium quantity. Graphically this can be shown as

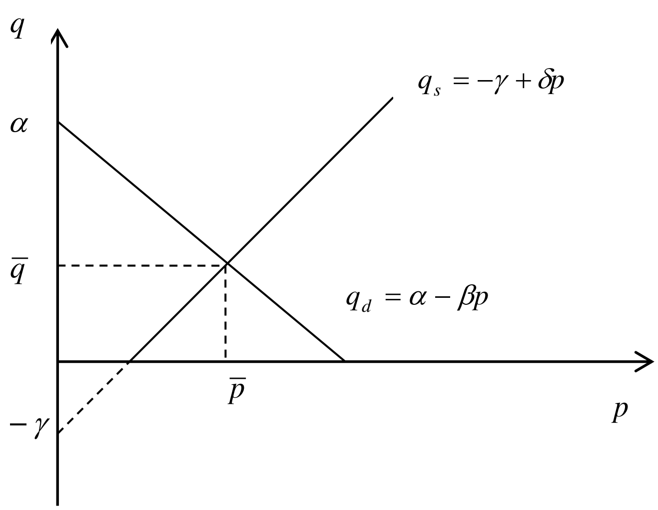

Figure 1

By direct substitution

$$\alpha - \beta\overline{p} = -\gamma + \delta\overline{p}$$

$$\overline{p} = \frac{\alpha+\gamma}{\beta+\delta} \qquad \overline{q} = \alpha - \beta\overline{p} = \alpha - \frac{\beta(\alpha+\gamma)}{\beta+\delta} = \frac{\alpha\delta-\beta\gamma}{\beta+\delta}$$

are the results obtained previously. The example given illustrates the case of partial market equilibrium when the market for only one commodity is in equilibrium. In a general market equilibrium we should have

$$E_i = q_{di} - q_{si} = 0 \qquad i = 1,2,...,n$$

where E_i is the excess demand for the i^{th} commodity and in the conditions of general equilibrium this excess demand is equal to zero for all goods.

Let us assume a simple two-commodity model where for the first commodity we have

$$q_{d_1} = a_o + a_1 p_1 + a_2 p_2$$
$$q_{s_1} = b_o + b_1 p_1 + b_2 p_2$$

as the quantity demanded and supplied of the first good. The prices of the two goods are respectively p_1 and p_2. Then it must be that the two goods are related. For example, if the coefficient a_2 is positive, it must be that the two goods are substitutes. If it is negative, they must be complements. Naturally we would expect a_1 to be negative as the law of demand would dictate and $b_1 > 0$ in consistency with the law of supply. For the second commodity we have

$$q_{d_2} = \alpha_o + \alpha_1 p_1 + \alpha_2 p_2$$
$$q_{s_2} = \beta_o + \beta_1 p_1 + \beta_2 p_2$$

Similarly for good 2, we expect to have $\alpha_2 < 0$ and $\beta_2 > 0$. As to the value of α_1, again it may be either positive or negative depending on whether the two goods are complements or substitutes. In the conditions of general-market equilibrium we should have

$q_{d_1} = q_{s_1}$ and

$q_{d_2} = q_{s_2}$

Hence, from each set of equations

$$a_o + a_1 p_1 + a_2 p_2 = b_o + b_1 p_1 + b_2 p_2$$
$$\alpha_o + \alpha_1 p_1 + \alpha_2 p_2 = \beta_o + \beta_1 p_1 + \beta_2 p_2$$

Rearranging,

$$(a_1 - b_1) p_1 + (a_2 - b_2) p_2 = b_o - a_o$$
$$(\alpha_1 - \beta_1) p_1 + (\alpha_2 - \beta_2) p_2 = \beta_o - \alpha_o$$

Let us set $a_i - b_i = c_i$ and $\alpha_i - \beta_i = \gamma_i$ for convenience. Thus we get

$$c_1 p_1 + c_2 p_2 = -c_o$$
$$\gamma_1 p_1 + \gamma_2 p_2 = -\gamma_o$$

Solving by the matrix method,

$$\begin{bmatrix} c_1 & c_2 \\ \gamma_1 & \gamma_2 \end{bmatrix} \begin{bmatrix} \overline{p}_1 \\ \overline{p}_2 \end{bmatrix} = \begin{bmatrix} -c_o \\ -\gamma_o \end{bmatrix}$$

$|A| = c_1\gamma_2 - c_2\gamma_1$

Since the determinant $|A|$ should be different from zero we should have $c_1\gamma_2 \neq c_2\gamma_1$. Furthermore,

$$\bar{p}_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} -c_o & c_2 \\ -\gamma_o & \gamma_2 \end{vmatrix}}{c_1\gamma_2 - c_2\gamma_1} = \frac{-c_o\gamma_2 + c_2\gamma_o}{c_1\gamma_2 - c_2\gamma_1} = \frac{c_2\gamma_o - c_o\gamma_2}{c_1\gamma_2 - c_2\gamma_1} > 0$$

$$\bar{p}_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} c_1 & -c_o \\ \gamma_1 & -\gamma_o \end{vmatrix}}{c_1\gamma_2 - c_2\gamma_1} = \frac{-c_1\gamma_o + c_o\gamma_1}{c_1\gamma_2 - c_2\gamma_1} = \frac{c_o\gamma_1 - c_1\gamma_o}{c_1\gamma_2 - c_2\gamma_1} > 0$$

where for both prices we expect to get positive values. For a general-market equilibrium with n commodities we have

$$q_{d_i} = q(p_1, p_2, ..., p_n)$$
$$q_{s_i} = q(p_1, p_2, ..., p_n)$$

where p_i is the price of the i^{th} commodity and $i = 1, 2, ..., n$. For equilibrium $q_{d_i} = q_{s_i}$ or excess demand is exactly zero.

$$E_i = q_{d_i} - q_{s_i} = 0$$

Alternatively, $E_i(p_1, p_2, ..., p_n) = 0$

A Simple National-Income Model

A simple Keynesian national-income model can be introduced where national income is Y, C is the level of consumption, I_o gives the aggregate investment level and government spending is given by G_o. Furthermore, the notation behind the last two variables shows that aggregate investment and government spending are exogenous variables. In the context of economic modelling this means that they depend on factors external to the model and therefore their values cannot be influenced in the model and should be taken for granted. At the same time the other variables are presumed to be endogenous and, hence, they depend on factors internal to the model. In fact, the endogenous variables in the model depend on the exogenous ones as well as on the parameters. Here α is the level of autonomous consumption, the consumption that is unrelated to the level of national income. Whether the nation produces any output or not it would still consume some positive amount. The parameter β gives the share of national income that goes into consumption known as the marginal propensity to consume. Therefore, this is the consumption that depends on the level of output produced in the economy. This consumption is negatively related to the level of savings since the income that is not consumed is saved and vice versa. Therefore, the marginal propensity to consume β and the marginal propensity to save s in a closed economy should equal 1 since the national income would either be spent or saved. We want to solve for the endogenous variables $\bar{Y}$ and $\bar{C}$ in equilibrium.

$$Y = C + I_o + G_o$$
$$C = \alpha + \beta Y \qquad \alpha > 0 \quad \beta \in (0,1) \qquad \text{since } \beta + s = 1$$

Thus formulated, the equations give the so called structural form of the model. When we solve for $\bar{Y}$ or $\bar{C}$, we obtain the reduced form of the model. We have the reduced form solution when the endogenous variable is expressed in terms of the exogenous variables or parameters in the model. Rewriting the equations,

$$Y - C = I_o + G_o$$
$$-\beta Y + C = \alpha$$

$$\underset{\substack{coefficient\\matrix}}{\begin{bmatrix} 1 & -1 \\ -\beta & 1 \end{bmatrix}} \underset{\substack{variable\\matrix}}{\begin{bmatrix} \bar{Y} \\ \bar{C} \end{bmatrix}} = \underset{\substack{constant\\vector\\matrix}}{\begin{bmatrix} I_o + G_o \\ \alpha \end{bmatrix}}$$

$$|A| = 1 - \beta > 0$$

The determinant is clearly positive since the marginal propensity to save is less than 1. Solving for $\bar{Y}$ we get

$$\bar{Y} = \frac{\begin{vmatrix} I_o + G_o & -1 \\ \alpha & 1 \end{vmatrix}}{1-\beta} = \frac{I_o + G_o + \alpha}{1-\beta} > 0$$

From this simple national-income model we obtain that equilibrium national income is positive and is positively related to exogenous aggregate investment, government spending and the autonomous consumption. On the other hand, it is positively related to the marginal propensity to consume which means that it is negatively affected by the marginal propensity to save s. In a closed economy like the one described above, i.e. in the absence of foreign trade we have in equilibrium

$$\bar{Y} = \frac{I_o + G_o + \alpha}{s}$$

For aggregate consumption

$$\bar{C} = \frac{\begin{vmatrix} 1 & I_o + G_o \\ -\beta & \alpha \end{vmatrix}}{1-\beta} = \frac{\alpha + \beta(I_o + G_o)}{1-\beta} = \frac{\alpha + \beta(I_o + G_o)}{s} > 0$$

Equilibrium aggregate consumption is positive, too. As can be expected, it is negatively related to the marginal propensity to save. The more the nation is inclined to save, the lower its consumption would be, the more likely it is to consume, the larger the level of aggregate consumption. We can also see that aggregate consumption is positively related to non-income related consumption and the level of investment and government spending.

Leontief Input-Output Model

One of the most vivid applications of matrix algebra in economic analysis is the Leontief input-output model. If an economy has n industries how much output should each industry supply to satisfy the needs of the economy? It should be noted that the output of each industry is an input for some other industries or even for that industry itself. For example the electrical industry might use some of its own output – electricity. To avoid bottlenecks in the economy the output levels of the n industries must be consistent with the input requirements of all of these n industries. The model makes three assumptions: 1) Each industry produces a homogeneous product (if some industry produces several products, we can hypothetically split that industry into several other industries. 2) All industries use fixed input proportions or factors of production are applied at a particular constant ratio. 3) There are constant returns to scale in each industry which means that doubling inputs doubles output.

Let the input of the i th commodity needed to produce a unit of the j th commodity be a_{ij}. Thus the production of one unit of the j th commodity will require a_{1j} amount of the first commodity, a_{2j} of the second commodity, … a_{nj} of the n th commodity. "A dollar's worth" of each commodity as its unit is assumed where prices are assumed to be given. For an n-industry economy the input coefficients a_{ij} can be arranged as a matrix such that:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

where A is called an input coefficient matrix. To produce a unit (a dollar's worth) of the commodity of the first industry, for example, we need a_{11} units of its own output, a_{21} units of the output of the second industry, … a_{n1} units of the output of the nth industry or the nth commodity.

The open input-output model developed by Leontief has a greater importance to applying matrix algebra to economic analysis as it gives a meaningful solution. The open model contains an "open" or "external" sector, which represents the exogenously determined final demand or the demand of the households for the product of each industry. In return for its demand the household sector provides the primary input or labor. If the household sector is absorbed into the model as just another industry (we add one more row and column to the input coefficient matrix), the model becomes closed. We shall not prove here that the rows of this matrix are linearly dependent but as it has an infinite number of solutions, no unique output mix can be found. What is more interesting is finding the optimal output of the nth industries in the open model.

The sum of the elements in each column of the input coefficient matrix A must be less than 1 that is $\sum_{i=1}^{n} a_{ij} < 1$ where $j = 1, 2, ..., n$. Since the value of output ($1) must be used to pay for other factors of production, the amount by which the column sum falls short of $1 must represent the payment to the primary input of the open sector. Thus the value of the primary input needed to produce a unit of the jth commodity is $1 - \sum_{i=1}^{n} a_{ij}$.

In the open model if the first industry is to satisfy the input requirements of the nth industries and the final demand of the external sector, its output level must be equal to:

$$x_1 = a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n + d_1$$

which transforms into

$$(1 - a_{11})x_1 - a_{12}x_2 - ... - a_{1n}x_n = d_1$$

where d_1 is the final demand of the external or open sector. For all the n industries we obtain the following system of n linear equations:

$$\begin{aligned} &(1 - a_{11})x_1 - a_{12}x_2 - ... - a_{1n}x_n = d_1 \\ &-a_{21}x_1 + (1 - a_{22})x_2 - ... - a_{2n}x_n = d_2 \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &-a_{n1}x_1 - a_{n2}x_2 - ... + (1 - a_{nn})x_n = d_n \end{aligned}$$

Writing the system with the help of matrixes yields:

$$\begin{bmatrix} 1 - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & 1 - a_{22} & \dots & -a_{2n} \\ \dots & \dots & \dots & \dots \\ -a_{n1} & -a_{n2} & \dots & (1 - a_{nn}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \dots \\ d_n \end{bmatrix}$$

or the coefficient matrix is nothing but the identity matrix I_n minus the input coefficient matrix A. The variable term vector is x and the constant term vector is d, respectively:

$(I - A)x = d$

The difference $I - A$ is also called the technology matrix T where

$Tx = d$

and in order to solve for x we have to find the inverse of the technology matrix such that:

$\bar{x} = T^{-1}d$

Problems

1. The demand and supply for two commodities are given by

$q_{d_1} = 53 - 2p_1 - 3p_2$ $\quad$ $q_{d_2} = 64 - 3p_1 - 4p_2$

$q_{s_1} = -4 + 3p_1$ $\quad$ $q_{s_2} = -8 + 2p_2$

where the prices of the two goods are respectively p_1 and p_2, q_{d_i} is the quantity demanded of the respective good and q_{s_i} is the quantity supplied of it. Use the matrix approach to determine the prices and quantities when each of the two markets is in equilibrium. How are the two goods related, if at all?

Solution:

In the conditions of general-market equilibrium we should have

$q_{d_1} = q_{s_1}$ and

$q_{d_2} = q_{s_2}$

From each set of equations

$$53 - 2p_1 - 3p_2 = -4 + 3p_1$$
$$64 - 3p_1 - 4p_2 = -8 + 2p_2$$

$$5p_1 + 3p_2 = 57$$
$$3p_1 + 6p_2 = 72 \ \ (:3)$$

$$5p_1 + 3p_2 = 57$$
$$p_1 + 2p_2 = 24$$

Writing in a matrix form,

$$\begin{bmatrix} 5 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \bar{p}_1 \\ \bar{p}_2 \end{bmatrix} = \begin{bmatrix} 57 \\ 24 \end{bmatrix}$$

$$|A| = 10 - 3 = 7 \neq 0$$

Using Cramer's rule,

$$\bar{p}_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 57 & 3 \\ 24 & 2 \end{vmatrix}}{7} = \frac{114 - 72}{7} = 6$$

$$\bar{p}_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 5 & 57 \\ 1 & 24 \end{vmatrix}}{7} = \frac{120 - 57}{7} = 9$$

From the supply equation for the first good

$\bar{q}_1 = -4 + 3(6) = 14$

For the second good
$\bar{q}_2 = -8 + 2(9) = 10$

From the two demand functions we can easily see the cross-price effect of a change in one price. When the price of good two increases, the quantity demanded for first good decreases which indicates they are complements and are consumed in a bundle. Similarly, when the price of the first good increases the quantity demanded of the second decreases.

2. The demand and supply for two commodities are

$q_{d_1} = 16 - 2p_1 + 2p_2$ $\qquad$ $q_{d_2} = 37 + p_1 - 3p_2$

$q_{s_1} = -10 + 4p_1 + p_2$ $\qquad$ $q_{s_2} = -4 + 2p_1 + 6p_2$

where the prices of the two goods are respectively p_1 and p_2, q_{d_i} is the quantity demanded of the respective good and q_{s_i} is the quantity supplied of it for $i = 1, 2$. Use matrix inversion to find the equilibrium prices and quantities of the two goods. How are the two goods related?

Solution:

In equilibrium on both markets

$q_{d_1} = q_{s_1}$ and

$q_{d_2} = q_{s_2}$

From each set of equations

$16 - 2p_1 + 2p_2 = -10 + 4p_1 + p_2$
$37 + p_1 - 3p_2 = -4 + 2p_1 + 6p_2$

$6p_1 - p_2 = 26$
$p_1 + 9p_2 = 41$

Written in a matrix form,

$$\begin{bmatrix} 6 & -1 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} \bar{p}_1 \\ \bar{p}_2 \end{bmatrix} = \begin{bmatrix} 26 \\ 41 \end{bmatrix}$$

$|A| = 54 + 1 = 55$

To solve by matrix inversion

$$C = \begin{bmatrix} 9 & -1 \\ 1 & 6 \end{bmatrix} \qquad C' = \begin{bmatrix} 9 & 1 \\ -1 & 6 \end{bmatrix}$$

$$\begin{bmatrix} \bar{p}_1 \\ \bar{p}_2 \end{bmatrix} = \frac{1}{|A|} C'd = \frac{1}{55} \begin{bmatrix} 9 & 1 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} 26 \\ 41 \end{bmatrix} = \frac{1}{55} \begin{bmatrix} 275 \\ 220 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

From the supply equation for the first good

$\bar{q}_1 = -10 + 4(5) + 4 = 14$

For the second good

$\bar{q}_2 = -4 + 2(5) + 6(4) = 30$

From the two demand functions we see that when the price of the second good increases, the quantity demanded of first good increases. This shows they are substitutes. Similarly, when the price of the first good increases the quantity demanded of the second increases.

3. Consider the model of general-market equilibrium in an economy with three goods. The demand and supply functions for the three commodities are given by

$q_{d_1} = 3 - p_1 + 2p_2 - p_3$ $\quad$ $q_{d_2} = 28 + p_1 - 2p_2 - 2p_3$ $\quad$ $q_{d_3} = 58 - 2p_1 - p_2 - 4p_3$

$q_{s_1} = -6 + p_1$ $\quad$ $q_{s_2} = -4 + 2p_2$ $\quad$ $q_{s_3} = -5 + 3p_3$

where the price of the respective good is p_i, q_{d_i} is the quantity demanded and q_{s_i} is the quantity supplied for $i = 1,2,3$. Use the matrix approach to determine the equilibrium price and quantity in each market. Can you conclude how the three goods are related to one another?

Solution:

In the conditions of general-market equilibrium we have

$$q_{d_i} = q_{s_i}$$

From each set of equations

$$3 - p_1 + 2p_2 - p_3 = -6 + p_1$$
$$28 + p_1 - 2p_2 - 2p_3 = -4 + 2p_2$$
$$58 - 2p_1 - p_2 - 4p_3 = -5 + 3p_3$$

$$2p_1 - 2p_2 + p_3 = 9$$
$$-p_1 + 4p_2 + 2p_3 = 32$$
$$2p_1 + p_2 + 7p_3 = 63$$

In a matrix form,

$$\begin{bmatrix} 2 & -2 & 1 \\ -1 & 4 & 2 \\ 2 & 1 & 7 \end{bmatrix} \begin{bmatrix} \bar{p}_1 \\ \bar{p}_2 \\ \bar{p}_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 32 \\ 63 \end{bmatrix}$$

Developing the determinant by Laplace expansion,

$$|A| = 2\begin{vmatrix} 4 & 2 \\ 1 & 7 \end{vmatrix} + 2\begin{vmatrix} -1 & 2 \\ 2 & 7 \end{vmatrix} + 1\begin{vmatrix} -1 & 4 \\ 2 & 1 \end{vmatrix} = 2(28-2) + 2(-7-4) - 1 - 8 = 52 - 22 - 9 = 21$$

Using matrix inversion,

$$C = \begin{bmatrix} 26 & 11 & -9 \\ 15 & 12 & -6 \\ -8 & -5 & 6 \end{bmatrix} \qquad C' = \begin{bmatrix} 26 & 15 & -8 \\ 11 & 12 & -5 \\ -9 & -6 & 6 \end{bmatrix}$$

$$\begin{bmatrix} \bar{p}_1 \\ \bar{p}_2 \\ \bar{p}_3 \end{bmatrix} = \frac{1}{21}\begin{bmatrix} 26 & 15 & -8 \\ 11 & 12 & -5 \\ -9 & -6 & 6 \end{bmatrix}\begin{bmatrix} 9 \\ 32 \\ 63 \end{bmatrix} = \frac{1}{21}\begin{bmatrix} 234 + 480 - 504 \\ 99 + 384 - 315 \\ -81 - 192 + 378 \end{bmatrix} = \frac{1}{21}\begin{bmatrix} 210 \\ 168 \\ 105 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 5 \end{bmatrix}$$

From the supply equation for each good

$\bar{q}_1 = -6 + 10 = 4$ $\quad$ $\bar{q}_2 = -4 + 2(8) = 12$ $\quad$ $\bar{q}_3 = -5 + 3(5) = 10$

From the demand function for the first good we see its quantity demanded is positively related to the price of the second good and negatively to that of the third good. This implies goods one and two are substitutes and one and three – complements. The third demand function also shows goods one and three are complements. At the same time the second and the third good seem to be complements, which is logical given that good one and three are complements and good one and two are substitutes.

4. Determine the equilibrium prices of three related goods that satisfy the condition:

$$p_1 + 2p_2 + 2p_3 = 40$$
$$p_1 + 3p_2 + 4p_3 = 67$$
$$p_1 + 4p_2 + 5p_3 = 84$$

Use whichever matrix method you find most suitable.

Solution:

In a matrix form,

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} \overline{p}_1 \\ \overline{p}_2 \\ \overline{p}_3 \end{bmatrix} = \begin{bmatrix} 40 \\ 67 \\ 84 \end{bmatrix}$$

Developing the determinant by the first column,

$$|A| = 15 - 16 - 10 + 8 + 8 - 6 = -1$$

Using matrix inversion,

$$C = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 3 & -2 \\ 2 & -2 & 1 \end{bmatrix} \qquad C' = \begin{bmatrix} -1 & -2 & 2 \\ -1 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \overline{p}_1 \\ \overline{p}_2 \\ \overline{p}_3 \end{bmatrix} = -1 \begin{bmatrix} -1 & -2 & 2 \\ -1 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 40 \\ 67 \\ 84 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ 1 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 40 \\ 67 \\ 84 \end{bmatrix} = \begin{bmatrix} 40 + 134 - 168 \\ 40 - 201 + 168 \\ -40 + 134 - 84 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 10 \end{bmatrix}$$

5. A consumer consumes two goods, the quantities of which are x and y, and whose total amount is 28. The price of x is \$3, that of y is \$5 and his total budget on the two goods is \$120. Use the matrix method to find the quantities of the two goods.

Solution:

$$x + y = 28$$
$$3x + 5y = 120$$

Writing in a matrix form,

$$\begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 28 \\ 120 \end{bmatrix}$$

$$|A| = 5 - 3 = 2 \neq 0$$

Using Cramer's rule,

$$x=\frac{|A_1|}{|A|}=\frac{\begin{vmatrix}28 & 1\\ 120 & 5\end{vmatrix}}{2}=\frac{140-120}{2}=10 \qquad y=\frac{|A_2|}{|A|}=\frac{\begin{vmatrix}1 & 28\\ 3 & 120\end{vmatrix}}{2}=\frac{120-84}{2}=18$$

When the consumer spends his budget fully, he buys 10 units of the first good and 18 units of the second.

6. A firm buys two inputs, labor L and capital K, the total amount of which cannot exceed 50. The wage is \$8 and the rental rate is \$10. The firm can at most spend \$440 on the two inputs. What are the quantities of the two inputs the firm must buy in order to produce a maximum output? Use the matrix approach.

Solution:

$L+K=50$
$8L+10K=440$

Given in a matrix form,

$$\begin{bmatrix}1 & 1\\ 8 & 10\end{bmatrix}\begin{bmatrix}L\\ K\end{bmatrix}=\begin{bmatrix}50\\ 440\end{bmatrix}$$

$|A|=10-8=2\neq 0$

Using Cramer's rule,

$$L=\frac{|A_1|}{|A|}=\frac{\begin{vmatrix}50 & 1\\ 440 & 10\end{vmatrix}}{2}=\frac{500-440}{2}=30 \qquad K=\frac{|A_2|}{|A|}=\frac{\begin{vmatrix}1 & 50\\ 8 & 440\end{vmatrix}}{2}=\frac{440-400}{2}=20$$

The firm can at most buy 20 machines and hire 30 workers.

7. Suppose the demand and supply functions on the market for milk are given by $q_d=90-4p$ and $q_s=6+2p$, respectively.

a) What are the equilibrium quantity and price of milk? Determine the total consumer surplus and producer surplus on the market for milk.
b) If a price floor of 20 is set, what will be the resulting market surplus? Find the deadweight social loss caused by the price floor.
c) Provide a graph that shows the equilibrium point, the consumer and producer surpluses and the effect of the price floor.

Solution:

a) $90-4p=6+2p$
$6p=84$
$p_e=14$

$q_e=6+2(14)=34$ where the intercepts are found as

$q_d(0)=90 \qquad p_d(0)=\frac{90}{4}=22.5 \qquad q_s(0)=6 \qquad p_s(0)=-3$

$CS=\frac{(22.5-14)34}{2}=7.5(17)=\$144.5 \qquad PS=\frac{(6+34)14}{2}=20(14)=\$280$ - area of a trapezoid

b) $p_f=20$

$q_d(20) = 90 - 4(20) = 10$

$q_s(20) = 6 + 2(20) = 46$

The market surplus is the excess supply or $q_s - q_d = 46 - 10 = 36$. Since only 10 units are traded after the trade floor is imposed, there is a deadweight social loss which is net loss in both consumer and producer surplus.

$$DSL = \frac{(20-14)(34-10)}{2} + \frac{(14-2)(34-10)}{2} = 72 + 144 = 216$$

c)

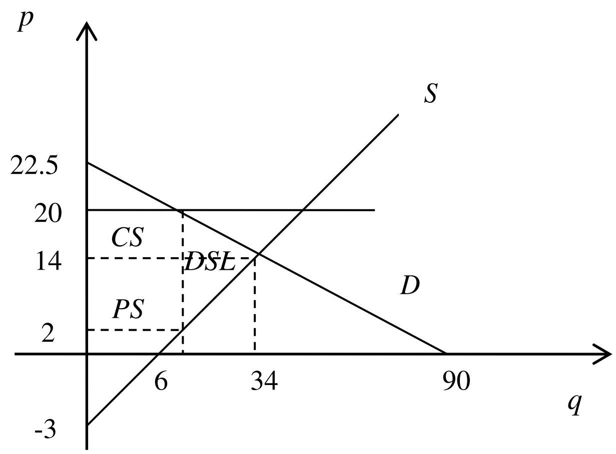

Figure 2

8. Suppose the demand and supply functions on the market for milk are given by $q_d = 70 - p$ and $q_s = \frac{1}{3}p - 10$, respectively.

a) What are the equilibrium quantity and price of milk? Determine the total producer surplus enjoyed by milk producers. What is the consumer surplus?
b) Determine the quantity demanded, the quantity supplied and the shortage resulting from the imposition of a price ceiling of 45 per gallon of milk. Is there a deadweight social loss resulting from the price ceiling?
c) Provide a graph that shows the equilibrium point, the producer surplus and the effect of the price ceiling.

Solution:

a) $70 - p = \frac{p}{3} - 10$

$\frac{4p}{3} = 80$

$p_e = 60$

From the demand function

$q_e = 70 - 60 = 10$

$CS = \frac{(70-60)10}{2} = 50$ $\qquad$ $PS = \frac{(60-30)10}{2} = 150$

b) $p_c = 45$

$$q_d(45) = 70 - 45 = 25$$

$$q_s(45) = \frac{45}{3} - 10 = 5$$

The price ceiling causes a market shortage of 20 units. Only 5 units are traded after the trade ceiling is imposed instead of the equilibrium 10. This causes a deadweight social loss to the amount

$$DSL = \frac{(65-60)(10-5)}{2} + \frac{(60-45)(10-5)}{2} = \frac{25}{2} + \frac{75}{2} = 50$$

c)

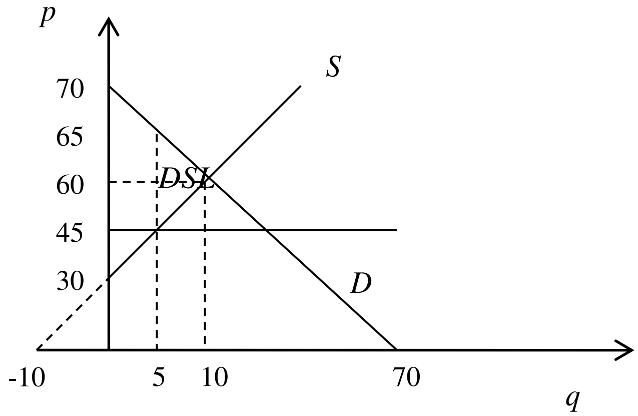

Figure 3

9. Suppose the demand and supply functions on the market for milk are given by $q_d = 50 - p$ and $q_s = \frac{1}{3}p - 6$, respectively.

a) What are the equilibrium quantity and price of milk? Determine the producer and the consumer surplus.
b) Determine the quantity demanded, the quantity supplied and the shortage resulting from the imposition of a price ceiling of 30 per gallon of milk. What is the deadweight social loss caused by the price ceiling?
c) Provide a graph that shows the equilibrium point, the producer and consumer surplus and the effect of the price ceiling.

Solution:

a) $50 - p = \frac{p}{3} - 6$

$$\frac{4p}{3} = 56$$

$$p_e = 42$$

From the demand function

$$q_e = 50 - 42 = 8$$

$$CS = \frac{(50-42)8}{2} = 32 \qquad PS = \frac{(42-18)8}{2} = 96$$

b) $p_c = 30$

$$q_d(30) = 50 - 30 = 20$$

$$q_s(30) = \frac{30}{3} - 6 = 4$$

The market shortage caused by the price ceiling is 16 units. Only 4 units are traded after its imposition instead of the equilibrium 8. Thus the deadweight social loss is

$$DSL = \frac{(46-42)(8-4)}{2} + \frac{(42-30)(8-4)}{2} = 8 + 24 = 32$$

c)

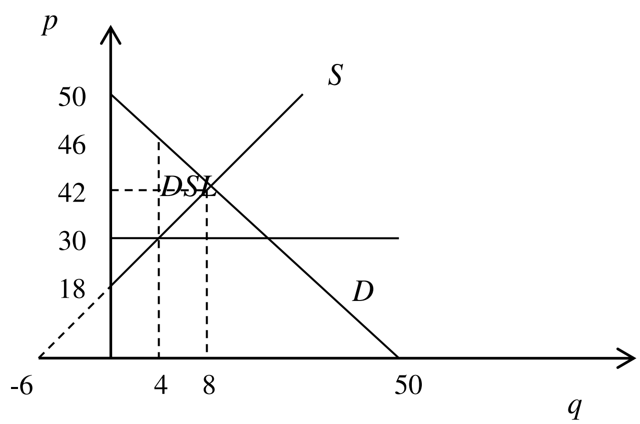

Figure 4

10. Suppose the demand and supply functions on the market for milk are given by $q_d = 60 - p$ and $q_s = \frac{p}{3} - 4$, respectively.

a) What are the equilibrium quantity and price of milk? Determine the total producer surplus enjoyed by milk producers. What is the consumer surplus?
b) Determine the quantity demanded, the quantity supplied and the shortage resulting from the imposition of a price ceiling of 45 per gallon of milk. What is the deadweight social loss from this price control?
c) Provide a graph that shows the equilibrium point, the producer surplus and the effect of the price ceiling.

Solution:

a) $60 - p = \frac{p}{3} - 4$

$$\frac{4p}{3} = 64$$

$$p_e = 48$$

From the demand function

$$q_e = 60 - 48 = 12$$

$$CS = \frac{(60-48)12}{2} = 72 \qquad PS = \frac{(48-12)12}{2} = 216$$

b) $p_c = 45$

$$q_d(45) = 60 - 45 = 15$$

$$q_s(45) = \frac{45}{3} - 4 = 11$$

The market shortage caused by the price ceiling is 4 units. After it is imposed the quantity traded falls from 12 to 11. Thus the deadweight social loss is

$$DSL = \frac{(49-48)(12-11)}{2} + \frac{(48-45)(12-11)}{2} = \frac{1}{2} + \frac{3}{2} = 2$$

c)

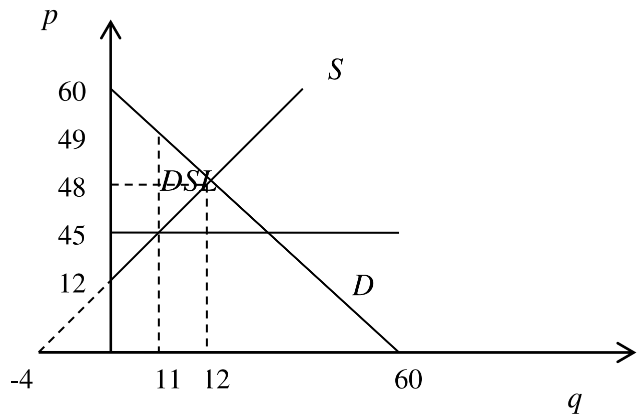

Figure 5

11. Suppose the demand and supply functions on the market for milk are given by $q_d = 100 - 2p$ and $q_s = 5 + 3p$, respectively.

a) What are the equilibrium quantity and price of milk? Find the total consumer surplus and producer surplus on the market for milk.
b) If a price floor of 30 is set, what will be the resulting market surplus? What is the amount of the deadweight social loss caused by the ceiling?
c) Provide a graph that shows the equilibrium point, the consumer and producer surpluses and the effect of the price floor.

Solution:

a) $100 - 2p = 5 + 3p$

$5p = 95$

$p_e = 19$

From the supply function

$q_e = 5 + 3(19) = 62$

$CS = \frac{(50-19)62}{2} = 961$ $\qquad$ $PS = \frac{(5+62)19}{2} = \frac{1273}{2}$ - area of a trapezoid

b) $p_f = 30$

$q_d(30) = 100 - 2(30) = 40$

$q_s(30) = 5 + 3(30) = 95$

The market surplus is the excess supply or 55. Since 40 units are traded instead of 62, there is a net loss in consumer and producer surplus.

$$DSL = \frac{(30-19)(62-40)}{2} + \frac{(19-35/3)(62-40)}{2} = 121 + \frac{22(11)}{3} = \frac{605}{3}$$

c)

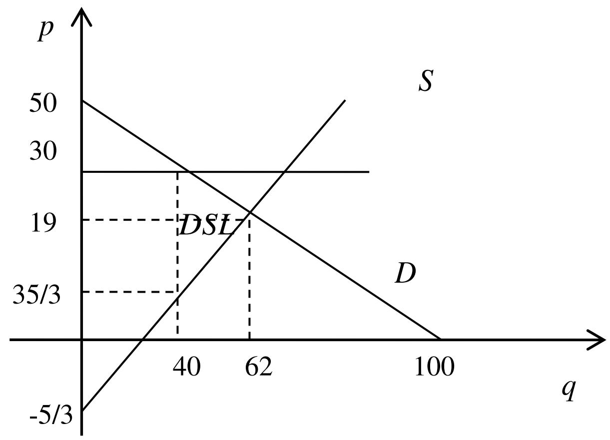

Figure 6

12. The demand and supply functions are given by

$$p = 120 - 5q_d$$

$$p = 24 + \frac{q_s}{3}$$

where p, q_d and q_s are, respectively, the price, quantity demanded and quantity supplied of the good. Find the equilibrium quantity and price. If the government imposes a per unit sales tax of \$32, what is the new quantity traded, the price that consumers pay and the price that producers receive? Who pays more of the tax?

Solution:

Prior to the tax

$$120 - 5q = 24 + \frac{q}{3}$$

$$\frac{16q}{3} = 96$$

$$\bar{q} = 18$$

From the supply equation

$$\bar{p} = 24 + \frac{18}{3} = 30$$

After the imposition of the tax we have

$$p_b - p_s = t$$

Substituting,

$$120 - 5q_t - 24 - \frac{q_t}{3} = 32$$

where q_t is the quantity traded after tax. Solving,

$$\frac{16q_t}{3} = 64$$

$$q_t = 12$$

As can be expected, after the tax is imposed quantity traded goes down to 12 units. The new price that consumers now pay is

$$p_b = 120 - 5(12) = 60$$

For consumers

$$p_s = 24 + \frac{12}{3} = 28$$

is the new price they receive. It can be easily seen that the difference is exactly the value of the tax, \$32. Thus the tax is shared between the two groups, but while producers only lose \$2 per unit, consumers seem much more hurt by the tax. Instead of paying a price of \$30, they end up paying 60. Thus the price doubles for them. The reason is that consumers are much more inelastic than producers in this market – the slope of the demand function is 5, while that of supply only $1/3$. Expressed in the form of total tax burden this is

Tax paid by buyers $(p_b - \bar{p})q_t = (60 - 30)12 = \$360$

Tax paid by suppliers $(\bar{p} - p_s)q_t = (30 - 28)12 = \$24$

The total tax revenue to the government is the sum of the two or \$384.

13. On the market for bananas demand is given by $p = 25 - 0.05q_d$ and supply by $p = 4 + 0.02q_s$. Find the equilibrium price and quantity. When a sales tax of \$5 per unit is imposed quantity traded decreases by one third.

a) Find the new quantity traded, the price paid by consumers and that received by banana suppliers. Which market group bears more of the tax incidence? Explain why. What are the total tax revenues to the government?

b) Find the loss in the consumer and producer surplus resulting from the imposition of the sales tax. Express the deadweight social loss caused by the sales tax. Provide a graph showing the pre-tax and the after-tax situation.

Solution:

a) In equilibrium

$$25 - 0.05q = 4 + 0.02q$$
$$0.07q = 21$$
$$\bar{q} = 300$$
$$\bar{p} = 25 - 0.05(300) = 10$$

After the imposition of the tax the quantity traded becomes $q_t = 200$ which is one third of the equilibrium quantity. Along the demand curve

$p_b = 25 - 0.05(200) = 15$ is the new price the consumers end up paying. Similarly for banana suppliers

$$p_s = 4 + 0.02(200) = 8$$

They now receive a price of 8 compared to the previous price of 10. At the same time consumers end up paying 15 instead of the equilibrium level 10. Expressed in the form of total tax burden this is

Tax paid by buyers $(p_b - \bar{p})q_t = (15 - 10)200 = \$1000$

Tax paid by suppliers $(\bar{p} - p_s)q_t = (10 - 8)200 = \$400$

The total tax revenue to the government is the sum of the two or \$1,400. As the numbers show, most of the tax incidence is on consumers. This is because they are the less elastic market force. From the demand function we can deduce that its slope is 0.05, while that of the supply function is 0.02 which shows supply is more elastic than demand. Therefore, banana suppliers bear less of the tax incidence.

b) To find the loss in the consumer surplus, we simply subtract the new surplus from the old one, i.e. the area of an upper triangle from the one before the imposition of the tax. The consumer surplus at the Pareto-optimal point is

$$CS = \frac{(25-10)300}{2} = \$2250$$

and after the tax is imposed

$$CS_t = \frac{(25-15)200}{2} = \$1000$$

Hence, the loss in the consumer surplus is

$$CS - CS_t = 2250 - 1000 = \$1250$$

Alternatively, we can find the area of the trapezoid that represents the loss of that surplus.

Loss of CS $\frac{(200+300)(15-10)}{2} = \$1250$ which gives the same result. Similarly for banana suppliers the loss in producer surplus is

$$PS - PS_t = \frac{(200+300)(10-8)}{2} = \$500$$

To find the deadweight social loss from the tax which is the uncompensated loss of consumer or producer surplus, we should subtract the total tax revenue to the government from the total loss of social surplus. The loss of total social surplus thus is

$$1250 + 500 = \$1750$$

Out of this total loss $1400 is compensated for in the form of government revenue form the tax. Therefore, the net loss or the deadweight social loss is

$$DSL = 1750 - 1400 = 350$$

We have seen that since the tax levied on bananas moves trade away from the Pareto-optimal equilibrium quantity traded, this inevitably leads to deadweight social loss. Graphically, this can be represented as

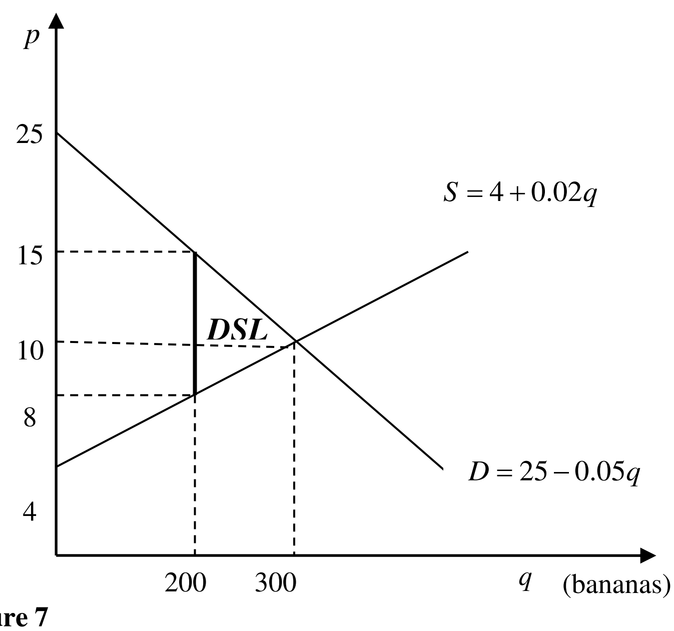

Figure 7

14. On the market for apples demand is given by $p = 88 - 0.8q_d$ and supply by $p = 10 + 0.5q_s$. Find the equilibrium price and quantity. When a sales tax of $39 per unit is levied quantity traded on the market for apples decreases by one half.

a) Find the new quantity traded, the price paid by consumers and that received by producers. Which market group bears more of the tax incidence? Explain why. What are the total tax revenues to the government?

b) Find the loss in the consumer and producer surplus resulting from the imposition of the sales tax. Express the deadweight social loss caused by the sales tax. Provide a graph showing the pre-tax and the after-tax situation on the market for apples.

Solution:

a) In equilibrium

$$88 - 0.8q = 10 + 0.5q$$
$$1.3q = 78$$
$$\bar{q} = 60$$
$$\bar{p} = 10 + 0.5(60) = 40$$

After the tax the quantity traded goes down by half which means it is $q_t = 30$. Along the demand curve

$p_b = 88 - 0.8(30) = 64$ is the new price the consumers pay. For apple producers

$p_s = 10 + 0.5(30) = 25$ is the new price producers get. This is less than the original equilibrium price of 40 by 15. Consumers, on the other hand, end up paying 64 compared to the previous price of 40 which is an increase of 24. The total tax burden thus is

Tax paid by buyers $(p_b - \bar{p})q_t = (64 - 40)30 = \$720$

Tax paid by suppliers $(\bar{p} - p_s)q_t = (40 - 25)30 = \$450$

The total tax revenue to the government therefore is $1,170. Most of the burden of the tax is on consumers. As can be seen from the demand function, they are the less elastic market force – the slope of the demand function is 0.8, while that of the supply function is 0.5. However, since the difference in the elasticity is rather small, the two market forces end up paying nearly the same amount of the tax. In other words, the burden of the tax is somewhat shared between the two market groups.

b) To find the loss in the consumer surplus, we find the old and the new consumer surplus. The consumer surplus at the Pareto-optimal point is

$$CS = \frac{(88 - 40)60}{2} = \$1440$$

and after the tax is imposed

$$CS_t = \frac{(88 - 64)30}{2} = \$360$$

Therefore, the loss in consumer surplus due to the imposition of the tax is

$$CS - CS_t = 1440 - 360 = \$1080$$

Alternatively, from the area of the trapezoid lost

Loss of CS $\dfrac{(30 + 60)(64 - 40)}{2} = \$1080$

is the result obtained previously. For apple producers the loss in producer surplus is

$$PS - PS_t = \frac{(30 + 60)(40 - 25)}{2} = \$675$$

To find the deadweight social loss from the tax which is the net loss in consumer or producer surplus, we should subtract the total tax revenue to the government from the total loss of social surplus. The loss of total social surplus thus is

$$1080 + 675 = \$1755$$

Out of this total loss \$1170 is compensated for in the form of government revenue form the tax. Therefore, the net loss or the deadweight social loss is

$$DSL = 1755 - 1170 = \$585$$

Since the tax levied on apples reduces the equilibrium quantity traded, this inevitably leads to deadweight social loss. Graphically, this can be represented as

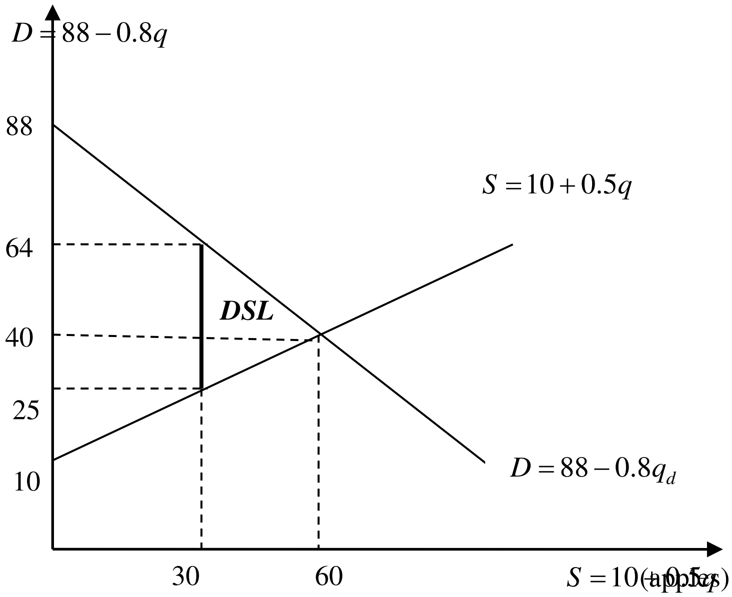

Figure 8

15. The demand and supply functions for a good are $q_d = 60 - \frac{p}{2}$ and $q_s = 10 + 2p$. Find the equilibrium price and quantity. Suppose a tax of \$2 per unit is imposed on the good. What is the new quantity traded? What is the price that consumers pay and producers get? What are the total tax revenues to the government? What is the loss to the producers?

Solution:

In equilibrium

$$60 - \frac{p}{2} = 10 + 2p$$

$$\frac{5p}{2} = 50$$

$\bar{p} = 20$ $\qquad\qquad$ $\bar{q} = 10 + 2(20) = 50$

After the tax is imposed

$60 - \frac{p+2}{2} = 10 + 2p$ $\qquad$ and $\qquad$ $60 - \frac{p}{2} = 10 + 2(p - 2)$

$\frac{5p}{2} = 49$ $\qquad\qquad$ $\frac{5p}{2} = 54$

$p_s = 19.6$ $\qquad\qquad$ $p_b = 21.6$

Thus after the tax the price that buyers pay is 21.6 and what producers receive is 19.6, the difference being the per unit tax. For the equilibrium quantity we have

$q_d = 60 - \frac{21.6}{2} = 49.2$ or alternatively $q_s = 10 + 2(19.6) = 49.2$

Therefore, the quantity traded goes down. The total tax revenue to the government is

$$T = tq_t = 2(49.2) = 98.4$$

Before tax

$$TR_1 = 20(50) = 1000$$

$$TR_2 = 19.6(49.2) = 964.32$$

$$TR_2 - TR_1 = 964.32 - 1000 = -35.68$$

16. The demand and supply functions for a good are $q_d = a - bp$ and $q_s = \alpha + \beta p$. Find the equilibrium price and quantity. Suppose a tax of t per unit is imposed on the good. What is the new quantity traded and by how much does quantity fall? What is the price that consumers pay and producers get? What is the total tax revenue the government collects with the tax?

Solution:

In equilibrium

$$a - bp = \alpha + \beta p$$

$$\bar{p} = \frac{a-\alpha}{b+\beta} \qquad \bar{q} = a - \frac{b(a-\alpha)}{b+\beta} = \frac{a\beta + \alpha b}{b+\beta}$$

After the tax

$$a - b(p+t) = \alpha + \beta p \qquad \text{and} \qquad a - bp = \alpha + \beta(p-t)$$

$$a - bp - bt = \alpha + \beta p \qquad\qquad a - bp = \alpha + \beta p - \beta t$$

$$p_s = \frac{a - \alpha - bt}{b+\beta} \qquad\qquad p_b = \frac{a - \alpha + \beta t}{b+\beta}$$

It could easily be checked that the price producers get is lower than the price consumers pay exactly by the amount of the tax. To find the new quantity traded we could take either function

$$q_t = a - \frac{b(a - \alpha + \beta t)}{b+\beta} = \frac{a\beta + \alpha b - b\beta t}{b+\beta}$$

We can easily see the quantity falls down by $\frac{b\beta t}{b+\beta}$. The total tax revenue to the government is

$$T = tq_t = \frac{t(a\beta + \alpha b - b\beta t)}{b+\beta} = \frac{(a\beta + \alpha b)t - b\beta t^2}{b+\beta}$$

17. Consider the nonlinear market equilibrium model where the functions of demand and supply are given by

$$q_d = 5 - p^2 \qquad\qquad q_s = 12p - 8$$

Find the equilibrium price and quantity. Provide a graph.

Solution:

For equilibrium

$q_d = q_s$
$5 - p^2 = 12p - 8$
$p^2 + 12p - 13 = 0$

$D = 36 + 13 = 49$

$p_1 = -6 - 7 = -13 < 0 \qquad p_2 = \bar{p} = -6 + 7 = 1$

From the supply function

$\bar{q} = 12(1) - 8 = 4$

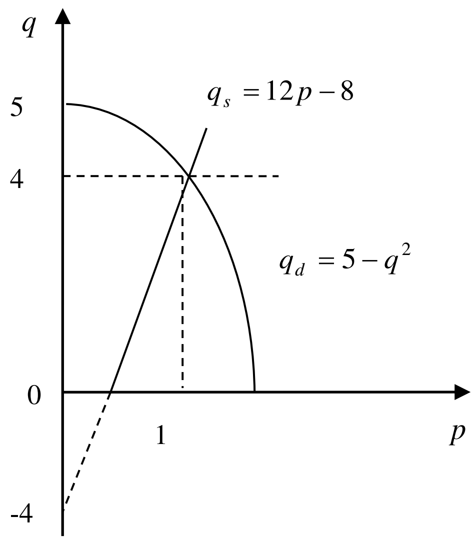

Figure 9

18. Given the supply and demand functions

$p = q_s^2 + 2q_s + 20$
$p = -q_d^2 - 4q_d + 76$

find the equilibrium price and quantity. Provide a graph.

Solution:

Assuming equilibrium

$q^2 + 2q + 20 = -q^2 - 4q + 76$
$2q^2 + 6q - 56 = 0$

$D = 9 + 112 = 121$

$q_1 = \frac{-3-11}{2} = -7 < 0 \qquad q_2 = \frac{-3+11}{2} = 4$

Therefore, $\bar{q} = 4$. From the supply equation

$\bar{p} = 4^2 + 2(4) + 20 = 44$

Graphing the functions,

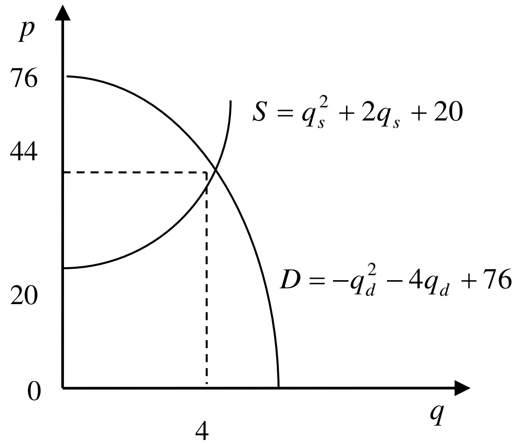

Figure 10

19. The demand for bananas is given by the function $q_d = \alpha - \beta p + \gamma I$ where p is the price of bananas and I is consumer income. The supply of bananas is $q_s = \delta + \varepsilon p$. It is known that $\alpha, \beta, \gamma, \delta, \varepsilon$ are all positive parameters. Express equilibrium price and quantity on the market for bananas. Are bananas a normal or an inferior good? How are equilibrium price and quantity of bananas related to consumer income?

Solution:

$$q_d = q_s$$

$$\alpha - \beta \overline{p} + \gamma I = \delta + \varepsilon \overline{p}$$

$$(\beta + \varepsilon)\overline{p} = \alpha + \gamma I - \delta$$

$$\overline{p} = \frac{\alpha + \gamma I - \delta}{\beta + \varepsilon} \qquad \overline{q} = \delta + \frac{\varepsilon(\alpha + \gamma I - \delta)}{\beta + \varepsilon} = \frac{\beta\delta + \varepsilon(\alpha + \gamma I)}{\beta + \varepsilon}$$

Since demand increases with consumer income ($\gamma > 0$), bananas are a normal good. Equilibrium price and quantity are also positively related to income.

20. A two-commodity market model is given where Q_{d_1} and Q_{s_1} are demand and supply for the first commodity, Q_{d_2} and Q_{s_2} are, respectively, demand and supply for the second commodity and the prices of the two commodities are P_1 and P_2 :

$$Q_{d_1} = m_o + m_1 P_1 + m_2 P_2 \qquad Q_{d_2} = p_o + p_1 P_1 + p_2 P_2$$

$$Q_{s_1} = n_o + n_1 P_1 + n_2 P_2 \qquad Q_{s_2} = q_o + q_1 P_1 + q_2 P_2$$

Equating $Q_d = Q_s = Q$ and using the demand and supply functions for the second commodity construct the coefficient matrix, the variable-term vector and the constant-term vector. Solve for $\overline{P}_1$ and $\overline{P}_2$ with the help of Crammer's rule.

Solution:

$$\left| \begin{array}{l} p_1 P_1 + p_2 P_2 = Q - p_o \\ q_1 P_1 + q_2 P_2 = Q - q_o \end{array} \right.$$

$$\begin{bmatrix} p_1 & p_2 \\ q_1 & q_2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} Q - p_o \\ Q - q_o \end{bmatrix}$$

coefficient matrix — variable-term matrix — constant-term matrix

$$|A| = \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} = p_1 q_2 - p_2 q_1 \neq 0$$

$$\overline{P}_1 = \frac{\begin{vmatrix} Q - p_o & p_2 \\ Q - q_o & q_2 \end{vmatrix}}{p_1 q_2 - p_2 q_1} = \frac{Q q_2 - p_o q_2 - Q p_2 + p_2 q_o}{p_1 q_2 - p_2 q_1} = \frac{Q(q_2 - p_2) + p_2 q_o - p_o q_2}{p_1 q_2 - p_2 q_1} > 0$$

$$\overline{P}_2 = \frac{\begin{vmatrix} p_1 & Q - p_o \\ q_1 & Q - q_o \end{vmatrix}}{p_1 q_2 - p_2 q_1} = \frac{Q p_1 - p_1 q_o - Q q_1 + p_o q_1}{p_1 q_2 - p_2 q_1} = \frac{Q(p_1 - q_1) + p_o q_1 - p_1 q_o}{p_1 q_2 - p_2 q_1} > 0$$

21. For the previous problem assuming a general market equilibrium find the two equilibrium prices $\overline{P_1}$ and $\overline{P_2}$ through matrix inversion.

Solution:

$$Q_{d_1} = m_o + m_1 P_1 + m_2 P_2$$
$$Q_{s_1} = n_o + n_1 P_1 + n_2 P_2$$
$$Q_{d_2} = p_o + p_1 P_1 + p_2 P_2$$
$$Q_{s_2} = q_o + q_1 P_1 + q_2 P_2$$

$$\left| \begin{array}{l} m_o - n_o + (m_1 - n_1) P_1 + (m_2 - n_2) P_2 = 0 \\ p_o - q_o + (p_1 - q_1) P_1 + (p_2 - q_2) P_2 = 0 \end{array} \right.$$

$$\begin{bmatrix} m_1 - n_1 & m_2 - n_2 \\ p_1 - q_1 & p_2 - q_2 \end{bmatrix} \begin{bmatrix} \overline{P}_1 \\ \overline{P}_2 \end{bmatrix} = \begin{bmatrix} -m_o + n_o \\ -p_o + q_o \end{bmatrix}$$

$$|A| = (m_1 - n_1)(p_2 - q_2) - (m_2 - n_2)(p_1 - q_1) \neq 0$$

$$C = \begin{bmatrix} p_2 - q_2 & -p_1 + q_1 \\ -m_2 + n_2 & m_1 - n_1 \end{bmatrix} \qquad C' = \begin{bmatrix} p_2 - q_2 & -m_2 + n_2 \\ -p_1 + q_1 & m_1 - n_1 \end{bmatrix}$$

$$\begin{bmatrix} \overline{P}_1 \\ \overline{P}_2 \end{bmatrix} = \frac{1}{(m_1 - n_1)(p_2 - q_2) - (m_2 - n_2)(p_1 - q_1)} \begin{bmatrix} p_2 - q_2 & -m_2 + n_2 \\ -p_1 + q_1 & m_1 - n_1 \end{bmatrix} \begin{bmatrix} -m_o + n_o \\ -p_o + q_o \end{bmatrix}$$

22. In the two-commodity general equilibrium model solve for the two equilibrium prices $\overline{p}_1, \overline{p}_2$ using the matrix inversion method. Use that demand and supply for the first commodity are

$$q_{d_1} = a_o + a_1 p_1 + a_2 p_2$$
$$q_{s_1} = b_o + b_1 p_1 + b_2 p_2$$

and for the second

$$q_{d_2} = \alpha_o + \alpha_1 p_1 + \alpha_2 p_2$$
$$q_{s_2} = \beta_o + \beta_1 p_1 + \beta_2 p_2$$

Solution:

In the conditions of general-market equilibrium we should have

$q_{d_1} = q_{s_1}$ and

$q_{d_2} = q_{s_2}$

Therefore,

$$a_o + a_1 p_1 + a_2 p_2 = b_o + b_1 p_1 + b_2 p_2$$
$$\alpha_o + \alpha_1 p_1 + \alpha_2 p_2 = \beta_o + \beta_1 p_1 + \beta_2 p_2$$

Rearranging,

$$(a_1 - b_1) p_1 + (a_2 - b_2) p_2 = b_o - a_o$$
$$(\alpha_1 - \beta_1) p_1 + (\alpha_2 - \beta_2) p_2 = \beta_o - \alpha_o$$

Setting $a_i - b_i = c_i$ and $\alpha_i - \beta_i = \gamma_i$ we get

$$c_1 p_1 + c_2 p_2 = -c_o$$
$$\gamma_1 p_1 + \gamma_2 p_2 = -\gamma_o$$

Writing in a matrix form,

$$\begin{bmatrix} c_1 & c_2 \\ \gamma_1 & \gamma_2 \end{bmatrix} \begin{bmatrix} \bar{p}_1 \\ \bar{p}_2 \end{bmatrix} = \begin{bmatrix} -c_o \\ -\gamma_o \end{bmatrix}$$

$$|A| = c_1\gamma_2 - c_2\gamma_1 \neq 0$$

Since the determinant $|A|$ should be different from zero we should have $c_1\gamma_2 \neq c_2\gamma_1$. Furthermore,

$$C = \begin{bmatrix} \gamma_2 & -\gamma_1 \\ -c_2 & c_1 \end{bmatrix} \qquad C' = \begin{bmatrix} \gamma_2 & -c_2 \\ -\gamma_1 & c_1 \end{bmatrix}$$

$$\begin{bmatrix} \bar{p}_1 \\ \bar{p}_2 \end{bmatrix} = \frac{1}{c_1\gamma_2 - c_2\gamma_1} \begin{bmatrix} \gamma_2 & -c_2 \\ -\gamma_1 & c_1 \end{bmatrix} \begin{bmatrix} -c_o \\ -\gamma_o \end{bmatrix} = \frac{1}{c_1\gamma_2 - c_2\gamma_1} \begin{bmatrix} -c_o\gamma_2 + c_2\gamma_o \\ c_o\gamma_1 - c_1\gamma_o \end{bmatrix}$$

which is the same as the result obtained previously,

$$\bar{p}_1 = \frac{c_2\gamma_o - c_o\gamma_2}{c_1\gamma_2 - c_2\gamma_1} > 0 \qquad \text{and} \qquad \bar{p}_2 = \frac{c_o\gamma_1 - c_1\gamma_o}{c_1\gamma_2 - c_2\gamma_1} > 0$$

where we expect both prices to be positive.

23. From the national-income model find the equilibrium values of $\bar{Y}$ and $\bar{C}$.

$$Y = C + I_o + G_o$$
$$C = 18 + 0.8Y - 0.002Y^2$$

for $I_o = 12$ and $G_o = 10$.

Solution:

Substituting the second into the first equation,

$$Y = 18 + 0.8Y - 0.002Y^2 + 12 + 10$$

$$0.002Y^2 + 0.2Y - 40 = 0$$

$0.001Y^2 + 0.1Y - 20 = 0$

$D = 0.01 + 0.08 = 0.09$

$\bar{Y} = \dfrac{-0.1 - 0.3}{2(0.001)} = -200 < 0$ $\quad\quad$ $\bar{Y} = \dfrac{-0.1 + 0.3}{2(0.001)} = 100 > 0$

For equilibrium aggregate consumption

$\bar{C} = 18 + 0.8(100) - 0.002(100)^2 = 18 + 80 - 20 = 78$

As we can see, most of the national income, 78 percent is in the form of consumption, 12 percent is taken by aggregate investment and government spending is 10 percent.

24. Consider the national-income model

$Y = C + I_o + G_o$
$C = 16 + 4\sqrt{Y}$

Find the equilibrium values of $\bar{Y}$ and $\bar{C}$ for $I_o = 10$ and $G_o = 6$.

Solution:

Substituting the second into the first equation,

$Y = 16 + 4\sqrt{Y} + 16$
$Y - 4\sqrt{Y} - 32 = 0$

$D = 4 + 32 = 36$

$\sqrt{Y} = 2 - 6 = -4 < 0$ $\quad\quad$ $\sqrt{Y} = 2 + 6 = 8$
$Y_1 = 16$ $\quad\quad$ $Y_2 = 64$

Clearly the first root is implausible. For equilibrium aggregate consumption we have

$\bar{C} = 16 + 4\sqrt{64} = 48$

25. National income is the sum of aggregate consumption and aggregate savings. Income that is not consumed is being saved and vice versa. Therefore,

$Y = C + S$

If the specific aggregate consumption function is given by

$C = 16 + 0.7Y$

express the aggregate savings function S and the two marginal propensities – to consume and to spend. Graph the two aggregate functions in relation to national income in the general case of $C = \alpha + \beta Y$. How are consumption and savings interrelated?

Solution:

From the main identity

$S = Y - C$

Substituting for consumption

$S = Y - 16 - 0.7Y = 0.3Y - 16$

From the consumption function we can easily see that

$MPC = 0.7$

and form the savings function

$MPS = 0.3$

Evidently

$MPC + MPS = 1$

where 70 percent of national income is consumed and 30 percent is saved. Both functions are positively related to income which could be graphed by

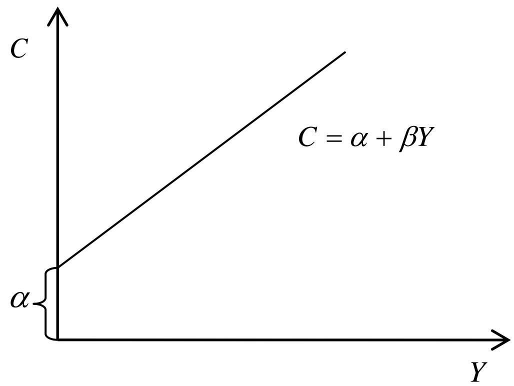

Figure 11

where in this specific case we have for the autonomous consumption $\alpha = 16$ and for the marginal propensity to consume $\beta = 0.7$ which gives the slope of the consumption function. Thus the higher the MPC , the steeper the line of the consumption function. This means that the higher the MPC , the more rapidly consumption increases with the increase in income. For the savings function

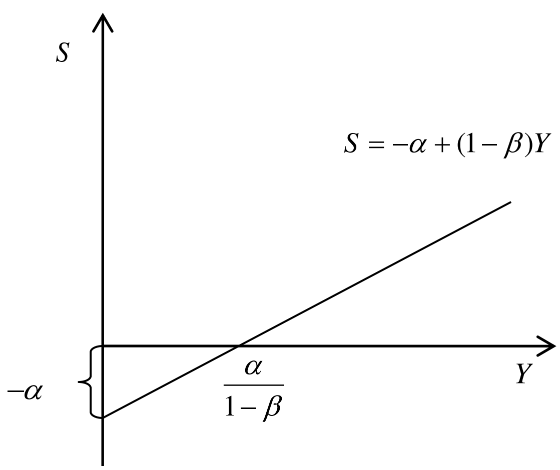

Figure 12

We see that savings also increase with income but initially, when no income is generated in the economy, their value is negative – it is exactly the amount of autonomous consumption. This implies that when the nation is not producing anything, it still has to consume some positive amount and this amount comes from negative savings, that is, the net borrowing the nation has to resort to, if it is not producing anything. In the general case the aggregate savings function is

$$S = -\alpha + (1 - \beta)Y = -\alpha + sY$$

where the marginal propensity to save s in our specific example is exactly 0.3. The higher the value of MPS is, the faster savings grow, as national income increases.

26. The National-Income Model is:

$$Y = C + I_o + G_o \qquad a>0 \qquad b\in(0,1)$$
$$C = a + b(Y - T) \qquad d>0 \qquad t\in(0,1)$$
$$T = d + tY$$

List the variables in the order $\overline{C}, \overline{Y}, \overline{T}$ and solve by matrix inversion.

Solution:

$$-C + Y = I_o + G_o$$
$$C - bY + bT = a$$
$$-tY + T = d$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -b & b \\ 0 & -t & 1 \end{bmatrix}\begin{bmatrix} \overline{C} \\ \overline{Y} \\ \overline{T} \end{bmatrix} = \begin{bmatrix} I_o + G_o \\ a \\ d \end{bmatrix}$$

$$|A| = -1\begin{vmatrix} -b & b \\ -t & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & b \\ 0 & 1 \end{vmatrix} = -(-b+bt) - 1 = b - bt - 1 \neq 0$$

The determinant is definitely different from 0 since b is less than 1. Thus the determinant is negative.

$C_{11} = -b + bt \qquad C_{12} = -1 \qquad C_{13} = -t \qquad C_{21} = -1 \qquad C_{22} = -1$

$C_{23} = -t \qquad C_{31} = b \quad C_{32} = -(-b) = b \qquad C_{33} = b - 1$

$$C = \begin{bmatrix} bt-b & -1 & -t \\ -1 & -1 & -t \\ b & b & b-1 \end{bmatrix} \qquad C' = \begin{bmatrix} bt-b & -1 & b \\ -1 & -1 & b \\ -t & -t & b-1 \end{bmatrix}$$

$$\begin{bmatrix} \overline{C} \\ \overline{Y} \\ \overline{T} \end{bmatrix} = \frac{1}{b-bt-1}\begin{bmatrix} bt-b & -1 & b \\ -1 & -1 & b \\ -t & -t & b-1 \end{bmatrix}\begin{bmatrix} I_o + G_o \\ a \\ d \end{bmatrix}$$

$$\begin{bmatrix} \overline{C} \\ \overline{Y} \\ \overline{T} \end{bmatrix} = \frac{1}{bt-b+1}\begin{bmatrix} b-bt & 1 & -b \\ 1 & 1 & -b \\ t & t & 1-b \end{bmatrix}\begin{bmatrix} I_o + G_o \\ a \\ d \end{bmatrix}$$

$$\begin{bmatrix} \overline{C} \\ \overline{Y} \\ \overline{T} \end{bmatrix} = \frac{1}{bt-b+1}\begin{bmatrix} b(1-t)(I_o+G_o) + a - bd \\ I_o + G_o + a - bd \\ (I_o+G_o)t + at + d(1-b) \end{bmatrix}$$

27. The National-Income Model is:

$$Y = C + I_o + G$$
$$C = a + b(Y - T_o) \qquad a>0\,, \qquad 0<b<1$$
$$G = gY \qquad 0<g<1$$

List the variables in the order $\overline{C}, \overline{Y}, \overline{G}$ and solve using matrix inversion. Set the conditions for a unique set of endogenous variables to exist in your equilibrium model.

Solution:

$$-C + Y - G = I_o$$
$$C - bY = a - bT_o$$
$$-gY + G = 0$$

$$\begin{bmatrix} -1 & 1 & -1 \\ 1 & -b & 0 \\ 0 & -g & 1 \end{bmatrix}\begin{bmatrix} \overline{C} \\ \overline{Y} \\ \overline{G} \end{bmatrix} = \begin{bmatrix} I_o \\ a-bT_o \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} -b & -1 & -g \\ g-1 & -1 & -g \\ -b & -1 & b-1 \end{bmatrix} \qquad C' = \begin{bmatrix} -b & g-1 & -b \\ -1 & -1 & -1 \\ -g & -g & b-1 \end{bmatrix}$$

$$\begin{bmatrix} \overline{C} \\ \overline{Y} \\ \overline{G} \end{bmatrix} = \frac{1}{g+b-1}\begin{bmatrix} -b & g-1 & -b \\ -1 & -1 & -1 \\ -g & -g & b-1 \end{bmatrix}\begin{bmatrix} I_o \\ a-bT_o \\ 0 \end{bmatrix} = \frac{1}{1-b-g}\begin{bmatrix} b & 1-g & b \\ 1 & 1 & 1 \\ g & g & 1-b \end{bmatrix}\begin{bmatrix} I_o \\ a-bT_o \\ 0 \end{bmatrix} =$$

$$= \frac{1}{1-b-g}\begin{bmatrix} bI_o + (1-g)(a-bT_o) \\ I_o + a - bT_o \\ gI_o + g(a-bT_o) \end{bmatrix}$$

We need $b + g \neq 1$ in order to have a unique set of solutions.

28. For the specific national-income model given:

$$Y = C + I_o + G_o$$
$$C = 60 + 0.9(Y - T)$$
$$T = 30 + 0.2Y$$

where $I_o = 31$ and $G_o = 20$ find the values of the endogenous variables solving by matrix inversion.

Solution:

$$Y - C = 31 + 20$$
$$-0.9Y + C + 0.9T = 60$$
$$-0.2Y + T = 30$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -0.9 & 1 & 0.9 \\ -0.2 & 0 & 1 \end{bmatrix}\begin{bmatrix} \overline{Y} \\ \overline{C} \\ \overline{T} \end{bmatrix} = \begin{bmatrix} 51 \\ 60 \\ 30 \end{bmatrix}$$

Expanding the determinant along the first row,

$$|A| = 1 + 1\begin{vmatrix} -0.9 & 0.9 \\ -0.2 & 1 \end{vmatrix} = 1 - 0.9 + 0.18 = 0.28 \neq 0$$

To find the inverse matrix

$$C = \begin{bmatrix} 1 & 0.72 & 0.2 \\ 1 & 1 & 0.2 \\ -0.9 & -0.9 & 0.1 \end{bmatrix} \qquad C' = \begin{bmatrix} 1 & 1 & -0.9 \\ 0.72 & 1 & -0.9 \\ 0.2 & 0.2 & 0.1 \end{bmatrix}$$

$$\begin{bmatrix} \overline{Y} \\ \overline{C} \\ \overline{T} \end{bmatrix} = \frac{1}{0.28}\begin{bmatrix} 1 & 1 & -0.9 \\ 0.72 & 1 & -0.9 \\ 0.2 & 0.2 & 0.1 \end{bmatrix}\begin{bmatrix} 51 \\ 60 \\ 30 \end{bmatrix} = \frac{1}{0.28}\begin{bmatrix} 51+60-27 \\ 36.72+60-27 \\ 10.2+12+3 \end{bmatrix} = \begin{bmatrix} 300 \\ 249 \\ 90 \end{bmatrix}$$

More than 80 percent of the national income is consumption and the rest is taken by investment and government spending. Of all the consumption nearly one fourth is autonomous, while the nonautonomous is 90 percent of the disposable income.

29. In the following national-income model:

$$Y = C + I_o + G_o$$
$$C = 22 + 0.8(Y - T)$$
$$T = 15 + 0.2Y$$

where $I_o = 15$ find the values of the endogenous variables, if you know that the nation maintains a balanced budget, that is, the government spending is exactly equal to the total tax revenues. Use the matrix approach.

Solution:

With the assumption of balanced budget we have $G_o = T$ and we can write

$$Y = C + I_o + G_o$$
$$C = 22 + 0.8(Y - T)$$
$$T = 15 + 0.2Y$$

$$Y - C - T = 15$$
$$-0.8Y + C + 0.8T = 22$$
$$-0.2Y + T = 15$$

$$\begin{bmatrix} 1 & -1 & -1 \\ -0.8 & 1 & 0.8 \\ -0.2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{Y} \\ \bar{C} \\ \bar{T} \end{bmatrix} = \begin{bmatrix} 15 \\ 22 \\ 15 \end{bmatrix}$$

Expanding the determinant along the last row,

$$|A| = -0.2(-0.8 + 1) + 1(1 - 0.8) = -0.04 + 0.2 = 0.16$$

To find the inverse matrix

$$C = \begin{bmatrix} 1 & 0.64 & 0.2 \\ 1 & 0.8 & 0.2 \\ 0.2 & 0 & 0.2 \end{bmatrix} \qquad C' = \begin{bmatrix} 1 & 1 & 0.2 \\ 0.64 & 0.8 & 0 \\ 0.2 & 0.2 & 0.2 \end{bmatrix}$$

$$\begin{bmatrix} \bar{Y} \\ \bar{C} \\ \bar{T} \end{bmatrix} = \frac{1}{0.16} \begin{bmatrix} 1 & 1 & 0.2 \\ 0.64 & 0.8 & 0 \\ 0.2 & 0.2 & 0.2 \end{bmatrix} \begin{bmatrix} 15 \\ 22 \\ 15 \end{bmatrix} = \frac{1}{0.16} \begin{bmatrix} 15 + 22 + 3 \\ 9.6 + 17.6 \\ 3 + 4.4 + 3 \end{bmatrix} = \begin{bmatrix} 250 \\ 170 \\ 65 \end{bmatrix}$$

With a balanced budget the government could spend no more than the total tax collected, which is 65.

30. Solve the national-income model:

$$Y = C + I_o + G$$
$$C = 26 + 0.8(Y - T_o)$$
$$G = 0.25Y$$

where $I_o = 50$ and $T_o = 120$. Does the government incur a budget surplus or a budget deficit according to this model? Choose whichever matrix method you deem more suitable.

Solution:

$$Y - C - G = 50$$
$$-0.8Y + C = 26 - 0.8(120)$$
$$-0.25Y + G = 0$$

$$Y - C - G = 50$$
$$-0.8Y + C = -70$$
$$-0.25Y + G = 0$$

Written in a matrix form

$$\begin{bmatrix} 1 & -1 & -1 \\ -0.8 & 1 & 0 \\ -0.25 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{Y} \\ \bar{C} \\ \bar{G} \end{bmatrix} = \begin{bmatrix} 50 \\ -70 \\ 0 \end{bmatrix}$$

$$|A| = -0.25 + 1 - 0.8 = -0.05$$

$$\bar{Y} = \frac{|A_1|}{|A|} = -\frac{\begin{vmatrix} 50 & -1 & -1 \\ -70 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}}{0.05} = -\frac{50-70}{0.05} = 400$$

$$\bar{C} = -\frac{\begin{vmatrix} 1 & 50 & -1 \\ -0.8 & -70 & 0 \\ -0.25 & 0 & 1 \end{vmatrix}}{0.05} = -\frac{-0.25(-70) - 70 + 0.8(50)}{0.05} = -\frac{17.5-30}{0.05} = 250$$

$$\bar{G} = -\frac{\begin{vmatrix} 1 & -1 & 50 \\ -0.8 & 1 & -70 \\ -0.25 & 0 & 0 \end{vmatrix}}{0.05} = \frac{0.25(70-50)}{0.05} = 100$$

Comparing equilibrium government spending $\bar{G} = 100$ with taxes $T_o = 120$, we see that taxes exceed government spending by 20. Thus the country has a budget surplus.

31. In the Keynesian model

$$Y = C + I + G_o$$
$$C = a + b(Y - T_o)$$
$$I = \alpha - \beta i$$
$$M_{so} = \gamma Y - \delta i$$

Y is national income, C is aggregate consumption, I is investment, G is government spending, T_o is tax collection, M_{so} is money supply and i is the interest rate. The parameters $a, b, \alpha, \beta, \gamma, \delta$ are all positive where for the marginal propensity to consume $b \in (0,1)$. Find the equilibrium values of Y and i.

Solution:

Substituting in the first equation and rewriting the last one we get

$$Y = a + b(Y - T_o) + \alpha - \beta i + G_o$$

$M_{so} = \gamma Y - \delta i$

$(1-b)Y + \beta i = a - bT_o + \alpha + G_o$
$\gamma Y - \delta i = M_{so}$

Writing in a matrix form,

$$\begin{bmatrix} 1-b & \beta \\ \gamma & -\delta \end{bmatrix}\begin{bmatrix} Y \\ i \end{bmatrix} = \begin{bmatrix} a - bT_o + \alpha + G_o \\ M_{so} \end{bmatrix}$$

The determinant is

$|A| = -\delta(1-b) - \beta\gamma < 0$

$$C = \begin{bmatrix} -\delta & -\gamma \\ -\beta & 1-b \end{bmatrix} \qquad C' = \begin{bmatrix} -\delta & -\beta \\ -\gamma & 1-b \end{bmatrix}$$

$$\begin{bmatrix} \bar{Y} \\ \bar{i} \end{bmatrix} = \frac{1}{-\delta(1-b) - \beta\gamma}\begin{bmatrix} -\delta & -\beta \\ -\gamma & 1-b \end{bmatrix}\begin{bmatrix} a - bT_o + \alpha + G_o \\ M_{so} \end{bmatrix} =$$
$$= \frac{1}{\delta(1-b) + \beta\gamma}\begin{bmatrix} \delta & \beta \\ \gamma & -(1-b) \end{bmatrix}\begin{bmatrix} a - bT_o + \alpha + G_o \\ M_{so} \end{bmatrix} =$$
$$= \frac{1}{\delta(1-b) + \beta\gamma}\begin{bmatrix} \delta(a - bT_o + \alpha + G_o) + \beta M_{so} \\ \gamma(a - bT_o + \alpha + G_o) - (1-b)M_{so} \end{bmatrix}$$

32. For the Keynesian *IS-LM* model given in the previous problem try to solve using four equations instead of two:

$Y = C + I + G_o$
$C = a + b(Y - T_o)$
$I = \alpha - \beta i$
$M_{so} = \gamma Y - \delta i$

Solution:

Rewriting the equations

$Y - C - I = G_o$
$-bY + C = a - bT_o$
$I + \beta i = \alpha$
$\gamma Y - \delta i = M_{so}$

Writing in a matrix form,

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ -b & 1 & 0 & 0 \\ 0 & 0 & 1 & \beta \\ \gamma & 0 & 0 & -\delta \end{bmatrix}\begin{bmatrix} \bar{Y} \\ \bar{C} \\ \bar{I} \\ \bar{i} \end{bmatrix} = \begin{bmatrix} G_o \\ a - bT_o \\ \alpha \\ M_{so} \end{bmatrix}$$

The determinant is most easily found by the last row

$|A| = -\beta\gamma - \delta(1-b) < 0$

Since the determinant is nonzero, the system has a viable set of solutions.

$$C=\begin{bmatrix} -\delta & -b\delta & \beta\gamma & -\gamma \\ -\delta & -\beta\gamma-\delta & \beta\gamma & -\gamma \\ -\delta & -b\delta & -\delta(1-b) & -\gamma \\ -\beta & -b\beta & -\beta(1-b) & 1-b \end{bmatrix}$$

$$C'=\begin{bmatrix} -\delta & -\delta & -\delta & -\beta \\ -b\delta & -\beta\gamma-\delta & -b\delta & -b\beta \\ \beta\gamma & \beta\gamma & -\delta(1-b) & -\beta(1-b) \\ -\gamma & -\gamma & -\gamma & 1-b \end{bmatrix}$$

$$\begin{bmatrix} \bar{Y} \\ \bar{C} \\ \bar{I} \\ \bar{i} \end{bmatrix}=\frac{1}{|A|}\begin{bmatrix} -\delta & -\delta & -\delta & -\beta \\ -b\delta & -\beta\gamma-\delta & -b\delta & -b\beta \\ \beta\gamma & \beta\gamma & -\delta(1-b) & -\beta(1-b) \\ -\gamma & -\gamma & -\gamma & 1-b \end{bmatrix}\begin{bmatrix} G_o \\ a-bT_o \\ \alpha \\ M_{so} \end{bmatrix}=$$

$$=\frac{1}{\delta(1-b)+\beta\gamma}\begin{bmatrix} \delta & \delta & \delta & \beta \\ b\delta & \beta\gamma+\delta & b\delta & b\beta \\ -\beta\gamma & -\beta\gamma & \delta(1-b) & \beta(1-b) \\ \gamma & \gamma & \gamma & -(1-b) \end{bmatrix}\begin{bmatrix} G_o \\ a-bT_o \\ \alpha \\ M_{so} \end{bmatrix}=$$

$$=\frac{1}{\delta(1-b)+\beta\gamma}\begin{bmatrix} \delta(G_o+a-bT_o+\alpha)+\beta M_{so} \\ b\delta(G_o+\alpha)+(\beta\gamma+\delta)(a-bT_o)+b\beta M_{so} \\ -\beta\gamma(G_o+a-bT_o)+(1-b)(\alpha\delta+\beta M_{so}) \\ \gamma(G_o+a-bT_o+\alpha)-(1-b)M_{so} \end{bmatrix}$$

Comparing the results from a previous solution for $\bar{Y}$ and $\bar{i}$ we see that they are identical

$$\begin{bmatrix} \bar{Y} \\ \bar{i} \end{bmatrix}=\frac{1}{\delta(1-b)+\beta\gamma}\begin{bmatrix} \delta(a-bT_o+\alpha+G_o)+\beta M_{so} \\ \gamma(a-bT_o+\alpha+G_o)-(1-b)M_{so} \end{bmatrix}$$

33. In the Keynesian *IS-LM* model

$$Y=C+I$$
$$C=100+0.8Y$$
$$I=1000-20i$$
$$M_{so}=0.5Y-30i$$

where $M_{so}=2350$ find the equilibrium values of national income Y and interest rate i.

Solution:

Substituting in the first equation and rewriting the last one yields

$$Y=100+0.8Y+1000-20i$$
$$0.5Y-30i=2350$$

$$0.2Y+20i=1100$$
$$0.5Y-30i=2350$$

Written in a matrix form,

$$\begin{bmatrix} 0.2 & 20 \\ 0.5 & -30 \end{bmatrix}\begin{bmatrix} \bar{Y} \\ \bar{i} \end{bmatrix}=\begin{bmatrix} 1100 \\ 2350 \end{bmatrix}$$

$|A| = -6 - 10 = -16$

$$\bar{Y} = -\frac{\begin{vmatrix} 1100 & 20 \\ 2350 & -30 \end{vmatrix}}{16} = \frac{33,000 + 47,000}{16} = 5,000 \qquad \bar{i} = -\frac{\begin{vmatrix} 0.2 & 1100 \\ 0.5 & 2350 \end{vmatrix}}{16} = -\frac{470 - 550}{16} = 5\%$$

34. In the Keynesian *IS-LM* model

$Y = C + I$
$C = 800 + 0.8Y$
$I = 4000 - 300i$
$L = 0.85Y - 400i$
$M_{so} = L$

in equilibrium money supply equals money demand L on the money market and it is known that $M_{so} = 10,350$. Find the equilibrium values of national income Y and interest rate i . How does money demand depend on national income?

Solution:

Substituting in the first equation and rewriting the last one yields

$Y = 800 + 0.8Y + 4000 - 300i$
$0.85Y - 400i = 10,350$

$0.2Y + 300i = 4800$
$0.85Y - 400i = 10,350$

Written in a matrix form,

$$\begin{bmatrix} 0.2 & 300 \\ 0.85 & -400 \end{bmatrix} \begin{bmatrix} \bar{Y} \\ \bar{i} \end{bmatrix} = \begin{bmatrix} 4800 \\ 10,350 \end{bmatrix}$$

$|A| = -80 - 255 = -335$

$$\bar{Y} = -\frac{\begin{vmatrix} 4800 & 300 \\ 10,350 & -400 \end{vmatrix}}{335} = \frac{(19,200 + 31,050)100}{335} = 15,000$$

$$\bar{i} = -\frac{\begin{vmatrix} 0.2 & 4800 \\ 0.85 & 10,350 \end{vmatrix}}{335} = -\frac{2070 - 4080}{335} = 6\%$$

Money demand is positively related to income and negatively to interest rate. As the nation's income grows, its transaction demand for money increases. If the interest rate grows, the opportunity cost of spending money increases and therefore, people would be willing to put their money in the bank. Demand for money thus falls.

35. Consider the Keynesian *IS-LM* model

$Y = C + I$
$C = 400 + 0.9Y$
$I = 1100 - 40i$
$L_o = 0.6Y - 60i$
$p = 0.03Y - 8i$

where $L_o = 6750$. The average price level p is given that depends positively on national income and negatively on the interest rate. Find the equilibrium values of Y, i and p using the matrix method.

Solution:

Substituting in the first equation and rewriting

$$Y = 400 + 0.9Y + 1100 - 40i$$
$$0.6Y - 60i = 6750$$
$$0.03Y - 8i - p = 0$$

$$0.1Y + 40i = 1500$$
$$0.6Y - 60i = 6750$$
$$0.03Y - 8i - p = 0$$

Writing in a matrix form,

$$\begin{bmatrix} 0.1 & 40 & 0 \\ 0.6 & -60 & 0 \\ 0.03 & -8 & -1 \end{bmatrix} \begin{bmatrix} \bar{Y} \\ \bar{i} \\ \bar{p} \end{bmatrix} = \begin{bmatrix} 1500 \\ 6750 \\ 0 \end{bmatrix}$$

The determinant is

$$|A| = 6 + 24 = 30$$

$$\bar{Y} = \frac{\begin{vmatrix} 1500 & 40 & 0 \\ 6750 & -60 & 0 \\ 0 & -8 & -1 \end{vmatrix}}{30} = \frac{90,000 + 270,000}{30} = 12,000$$

$$\bar{i} = \frac{\begin{vmatrix} 0.1 & 1500 & 0 \\ 0.6 & 6750 & 0 \\ 0.03 & 0 & -1 \end{vmatrix}}{30} = \frac{-675 + 900}{30} = 7.5\%$$

$$\bar{p} = \frac{\begin{vmatrix} 0.1 & 40 & 1500 \\ 0.6 & -60 & 6750 \\ 0.03 & -8 & 0 \end{vmatrix}}{30} = \frac{0.03(270,000 + 90,000) + 8(675 - 900)}{30} = \frac{10,800 - 1800}{30} = 300$$

36. Consider the following open economy

$$Y = C + I + G_o + X - M$$
$$C = 80 + 0.8Y$$
$$I = 50 - 20i$$
$$X = 0.3Y - 14i$$
$$M = 50 + 0.2Y$$

in which $G_o = 190$ and the trade deficit is known to be 20. Find $\bar{Y}$ and $\bar{i}$. What is the value of exports and imports? How much is the marginal propensity to import?

Solution:

Since the country faces a trade deficit we have

$$X - M = -20$$

Substituting in the national-income equation

$$Y = 80 + 0.8Y + 50 - 20i + 190 - 20$$

and from the trade equation we have

$$0.3Y - 14i - 50 - 0.2Y = -20$$

Transforming these two equations

$$0.2Y + 20i = 300$$
$$0.1Y - 14i = 30$$

Written in a matrix form,

$$\begin{bmatrix} 0.2 & 20 \\ 0.1 & -14 \end{bmatrix} \begin{bmatrix} \bar{Y} \\ \bar{i} \end{bmatrix} = \begin{bmatrix} 300 \\ 30 \end{bmatrix}$$

$$|A| = -2.8 - 2 = -4.8$$

$$\bar{Y} = -\frac{\begin{vmatrix} 300 & 20 \\ 30 & -14 \end{vmatrix}}{4.8} = \frac{4,200 + 600}{4.8} = 1,000 \qquad \bar{i} = -\frac{\begin{vmatrix} 0.2 & 300 \\ 0.1 & 30 \end{vmatrix}}{4.8} = -\frac{6 - 30}{4.8} = 5\%$$

$$X = 0.3Y - 14i = 0.3(1000) - 14(5) = 300 - 70 = 230$$
$$M = 50 + 0.2Y = 50 + 0.2(1000) = 50 + 200 = 250$$

As the model assumes, the trade balance of the country is negative at -20. This implies the nation is importing more than it is exporting. The marginal propensity to import shows the share of national income that is dedicated to imports, that is, $MPI = 0.2$.

37. Consider an ideal global economy consisting of two countries, Home and Foreign, trading with each other.

$$Y_1 = C_1 + I_{1o} + X_1 - M_1 \qquad Y_2 = C_2 + I_{2o} + X_2 - M_2$$
$$C_1 = 320 + 0.8Y_1 \qquad C_2 = 220 + 0.7Y_2$$
$$M_1 = 0.2Y_1 \qquad M_2 = 0.1Y_2$$

The first model describes Home, in which exogenous investment is $I_{1o} = 100$, and the second shows the Foreign economy in which $I_{2o} = 200$. With mutual trade the exports of Home are imports for Foreign and vice versa. Find the GDP levels of the two countries. How much is the trade between them and what is the trade balance?

Solution:

Since the exports of Home are imports for Foreign and vice versa, we have

$$X_1 = M_2 \qquad \text{and} \qquad X_2 = M_1$$

Substituting in the two national-income identities,

$$Y_1 = 320 + 0.8Y_1 + 100 + 0.1Y_2 - 0.2Y_1$$

and for Foreign

$$Y_2 = 220 + 0.7Y_2 + 200 + 0.2Y_1 - 0.1Y_2$$

Rearranging,

$$0.4Y_1 - 0.1Y_2 = 420 \quad (:0.1)$$
$$-0.2Y_1 + 0.4Y_2 = 420 \quad (:0.2)$$

$$4Y_1 - Y_2 = 4200$$
$$-Y_1 + 2Y_2 = 2100$$

Written in a matrix form,

$$\begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} \bar{Y}_1 \\ \bar{Y}_2 \end{bmatrix} = \begin{bmatrix} 4200 \\ 2100 \end{bmatrix}$$

$$|A| = 8 - 1 = 7$$

$$\bar{Y}_1 = \frac{\begin{vmatrix} 4200 & -1 \\ 2100 & 2 \end{vmatrix}}{7} = \frac{8,400 + 2100}{7} = 1500$$

$$\bar{Y}_2 = \frac{\begin{vmatrix} 4 & 4200 \\ -1 & 2100 \end{vmatrix}}{7} = \frac{8,400 + 4200}{7} = 1800$$

For the volume of trade

$$X_1 = M_2 = 0.1Y_2 = 0.1(1800) = 180$$
$$X_2 = M_1 = 0.2Y_1 = 0.2(1500) = 300$$

The value of Home's exports is 180 which is exactly the value of imports for Foreign. Home imports from Foreign goods and services worth 300. Hence, Foreign exports more than it imports and realizes a positive trade balance with Home.

38. Two countries, Home and Foreign, trade with each other. Their economies are described by the following two models.

$$Y_1 = C_1 + I_{1o} + X_1 - M_1 \qquad Y_2 = C_2 + I_{2o} + X_2 - M_2$$
$$C_1 = \alpha + \beta Y_1 \qquad C_2 = \gamma + \delta Y_2 \qquad \alpha, \gamma > 0$$
$$M_1 = \mu Y_1 \qquad M_2 = \lambda Y_2 \qquad 0 < \beta, \delta, \lambda, \mu < 1$$

The first model describes Home, in which exogenous investment is I_{1o}, and the second shows Foreign in which investment is I_{2o}. With mutual trade the exports of Home are imports for Foreign and vice versa. Find the equilibrium GDP levels of the two countries.

Solution:

Since the exports of Home are imports for Foreign and vice versa, we have

$X_1 = M_2$ and $X_2 = M_1$

Substituting in the two national-income identities,

$$Y_1 = \alpha + \beta Y_1 + I_{1o} + \lambda Y_2 - \mu Y_1 \quad \text{and}$$
$$Y_2 = \gamma + \delta Y_2 + I_{2o} + \mu Y_1 - \lambda Y_2$$

Rearranging,

$$(1 - \beta + \mu)Y_1 - \lambda Y_2 = \alpha + I_{1o}$$
$$-\mu Y_1 + (1 - \delta + \lambda)Y_2 = \gamma + I_{2o}$$

Written in a matrix form,

$$\begin{bmatrix} 1 - \beta + \mu & -\lambda \\ -\mu & 1 - \delta + \lambda \end{bmatrix} \begin{bmatrix} \bar{Y}_1 \\ \bar{Y}_2 \end{bmatrix} = \begin{bmatrix} \alpha + I_{1o} \\ \gamma + I_{2o} \end{bmatrix}$$

$$|A| = (1-\beta+\mu)(1-\delta+\lambda) - \lambda\mu = (1-\beta)(1-\delta+\lambda) + \mu(1-\delta) + \lambda\mu - \lambda\mu =$$
$$= (1-\beta)(1-\delta+\lambda) + \mu(1-\delta)$$

$$\bar{Y}_1 = \frac{\begin{vmatrix} \alpha + I_{1o} & -\lambda \\ \gamma + I_{2o} & 1-\delta+\lambda \end{vmatrix}}{|A|} = \frac{(\alpha+I_{1o})(1-\delta+\lambda) + \lambda(\gamma+I_{2o})}{(1-\beta)(1-\delta+\lambda)+\mu(1-\delta)}$$

$$\bar{Y}_2 = \frac{\begin{vmatrix} 1-\beta+\mu & \alpha + I_{1o} \\ -\mu & \gamma + I_{2o} \end{vmatrix}}{|A|} = \frac{(1-\beta+\mu)(\gamma+I_{2o}) + \mu(\alpha+I_{1o})}{(1-\beta)(1-\delta+\lambda)+\mu(1-\delta)}$$

39. Consider the expanded national-income model:

$Y = C + I_o + G_o + X_o - M$

$C = \alpha + \beta Y \qquad\qquad \alpha > 0$

$M = M_o + \mu Y \qquad\qquad 0 < \beta, \mu < 0$

where M_o is the level of autonomous (non-income related) imports and μ is the marginal propensity to import. Solve for $\bar{Y}, \bar{C}$ and $\bar{M}$ using the matrix approach.

Solution:

$Y - C + M = I_o + G_o + X_o$

$-\beta Y + C = \alpha$

$-\mu Y + M = M_o$

$$\begin{bmatrix} 1 & -1 & 1 \\ -\beta & 1 & 0 \\ -\mu & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{Y} \\ \bar{C} \\ \bar{M} \end{bmatrix} = \begin{bmatrix} I_o + G_o + X_o \\ \alpha \\ M_o \end{bmatrix}$$

$|A| = \mu + 1 - \beta = 1 - \beta + \mu > 0$

Since $1-\beta > 0$, we get a positive determinant.

$$C = \begin{bmatrix} 1 & \beta & \mu \\ 1 & 1+\mu & -\mu \\ -1 & \beta & 1-\beta \end{bmatrix} \qquad C' = \begin{bmatrix} 1 & 1 & -1 \\ \beta & 1+\mu & \beta \\ \mu & -\mu & 1-\beta \end{bmatrix}$$

$$\begin{bmatrix} \bar{Y} \\ \bar{C} \\ \bar{M} \end{bmatrix} = \frac{1}{1-\beta+\mu} \begin{bmatrix} 1 & 1 & -1 \\ \beta & 1+\mu & \beta \\ \mu & -\mu & 1-\beta \end{bmatrix} \begin{bmatrix} I_o + G_o + X_o \\ \alpha \\ M_o \end{bmatrix} = \frac{1}{1-\beta+\mu} \begin{bmatrix} I_o + G_o + X_o + \alpha - M_o \\ \beta(I_o + G_o + X_o + M_o) + (1+\mu)\alpha \\ \mu(I_o + G_o + X_o - \alpha) + (1-\beta)M_o \end{bmatrix}$$

40. In the *IS-LM* model savings are given as a share of national income and equal private and public investment.

$sY = I + G_o \qquad\qquad 0 < s < 1$

$I = I_o - \alpha i \qquad\qquad \alpha, \beta, \gamma > 0$

$M_{so} = \beta Y - \gamma i$

Y is national income, I is investment, I_o is some initial level of investment, G_o is government spending, M_{so} is money supply and i is the interest rate. Use the matrix approach to find Y and i.

Solution:

Substituting the second in the first equation

$$sY = I_o - \alpha i + G_o$$
$$M_{so} = \beta Y - \gamma i$$

Rearranging to solve in a matrix form

$$sY + \alpha i = I_o + G_o$$
$$\beta Y - \gamma i = M_{so}$$

Writing in a matrix format

$$\begin{bmatrix} s & \alpha \\ \beta & -\gamma \end{bmatrix} \begin{bmatrix} \bar{Y} \\ \bar{i} \end{bmatrix} = \begin{bmatrix} I_o + G_o \\ M_{so} \end{bmatrix}$$

The determinant is

$$|A| = -s\gamma - \alpha\beta < 0$$

Using Cramer's rule

$$\bar{Y} = \frac{\begin{vmatrix} I_o + G_o & \alpha \\ M_{so} & -\gamma \end{vmatrix}}{|A|} = -\frac{(I_o + G_o)\gamma + \alpha M_{so}}{|A|} = \frac{(I_o + G_o)\gamma + \alpha M_{so}}{s\gamma + \alpha\beta}$$

$$\bar{i} = \frac{\begin{vmatrix} s & I_o + G_o \\ \beta & M_{so} \end{vmatrix}}{|A|} = \frac{sM_{so} - \beta(I_o + G_o)}{|A|} = \frac{\beta(I_o + G_o) - sM_{so}}{s\gamma + \alpha\beta}$$

Using matrix inversion,

$$C = \begin{bmatrix} -\gamma & -\beta \\ -\alpha & s \end{bmatrix} \qquad C' = \begin{bmatrix} -\gamma & -\alpha \\ -\beta & s \end{bmatrix}$$

$$\begin{bmatrix} \bar{Y} \\ \bar{i} \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} -\gamma & -\alpha \\ -\beta & s \end{bmatrix} \begin{bmatrix} I_o + G_o \\ M_{so} \end{bmatrix} = -\frac{1}{|A|} \begin{bmatrix} \gamma(I_o + G_o) + \alpha M_{so} \\ \beta(I_o + G_o) - sM_{so} \end{bmatrix}$$

which proves the same result.

41. If we introduce a tax rate t in the *IS-LM* model, savings are a share of a smaller disposable income such that

$$s(1-t)Y = I + G_o \qquad 0 < s, t < 1$$
$$I = I_o - \alpha i \qquad \alpha, \beta, \gamma > 0$$
$$M_{so} = \beta Y - \gamma i$$

where Y is national income, I is investment, I_o is some initial level of investment, G_o is government spending, M_{so} is money supply and i is the interest rate. Use Cramer's rule to find equilibrium Y and i.

Solution:

Substituting the second into the first equation

$$s(1-t)Y = I_o - \alpha i + G_o$$
$$M_{so} = \beta Y - \gamma i$$

Rearranging to solve in a matrix form

$$s(1-t)Y + \alpha i = I_o + G_o$$
$$\beta Y - \gamma i = M_{so}$$

Writing in a matrix format

$$\begin{bmatrix} s(1-t) & \alpha \\ \beta & -\gamma \end{bmatrix} \begin{bmatrix} \bar{Y} \\ \bar{i} \end{bmatrix} = \begin{bmatrix} I_o + G_o \\ M_{so} \end{bmatrix}$$

The determinant is

$$|A| = -s\gamma(1-t) - \alpha\beta < 0$$

Using Cramer's rule

$$\bar{Y} = \frac{\begin{vmatrix} I_o + G_o & \alpha \\ M_{so} & -\gamma \end{vmatrix}}{|A|} = -\frac{(I_o + G_o)\gamma + \alpha M_{so}}{|A|} = \frac{(I_o + G_o)\gamma + \alpha M_{so}}{s\gamma(1-t) + \alpha\beta}$$

$$\bar{i} = \frac{\begin{vmatrix} s(1-t) & I_o + G_o \\ \beta & M_{so} \end{vmatrix}}{|A|} = \frac{s(1-t)M_{so} - \beta(I_o + G_o)}{|A|} = \frac{\beta(I_o + G_o) - s(1-t)M_{so}}{s\gamma(1-t) + \alpha\beta}$$

42. Consider the expanded linear *IS-LM* model

$$Y = C + I + G_o$$
$$C = a + b(Y - T)$$
$$T = t_o + t_1 Y$$
$$I = \alpha - \beta i$$
$$M_{so} = \gamma Y - \delta i$$

where Y is national income, C is aggregate consumption, I is investment, G_o is government spending, T is total tax collection, M_{so} is money supply and i is the interest rate. The parameters $a, b, \alpha, \beta, \gamma, \delta$ are all positive. Furthermore, t_o is non-income tax rate, t_1 is income tax rate and $0 < b, t < 1$. Find equilibrium Y and i.

Solution:

Substituting in the first equation and rewriting the last one we get

$$Y = a + b(Y - t_o - t_1 Y) + \alpha - \beta i + G_o$$
$$M_{so} = \gamma Y - \delta i$$

Rearranging in a matrix form

$$(1 - b + bt_1)Y + \beta i = a - bt_o + \alpha + G_o$$
$$\gamma Y - \delta i = M_{so}$$

$$\begin{bmatrix} 1-b+bt_1 & \beta \\ \gamma & -\delta \end{bmatrix}\begin{bmatrix} \bar{Y} \\ \bar{i} \end{bmatrix} = \begin{bmatrix} a-bt_o+\alpha+G_o \\ M_{so} \end{bmatrix}$$

For the determinant

$$|A| = -\delta(1-b+bt_1) - \beta\gamma < 0$$

By Cramer's rule

$$\bar{Y} = \frac{\begin{vmatrix} a-bt_o+\alpha+G_o & \beta \\ M_{so} & -\delta \end{vmatrix}}{|A|} = \frac{-\delta(a-bt_o+\alpha+G_o)-\beta M_{so}}{-\delta(1-b+bt_1)-\beta\gamma} = \frac{\delta(a-bt_o+\alpha+G_o)+\beta M_{so}}{\delta(1-b+bt_1)+\beta\gamma} > 0$$

$$\bar{i} = \frac{\begin{vmatrix} 1-b+bt_1 & a-bt_o+\alpha+G_o \\ \gamma & M_{so} \end{vmatrix}}{|A|} = \frac{(1-b+bt_1)M_{so}-\gamma(a-bt_o+\alpha+G_o)}{|A|} =$$

$$= -\frac{(1-b+bt_1)M_{so}-\gamma(a-bt_o+\alpha+G_o)}{\delta(1-b+bt_1)+\beta\gamma} < 0$$

43. The input-output matrix for a two-sector economy is given by

$$A = \begin{bmatrix} 0.25 & 0.40 \\ 0.10 & 0.10 \end{bmatrix}$$

Following Leontief's input-output open model find the technology matrix T . If the external demand for the output of the two sectors is said to be $D = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$, find their production levels.

Solution:

$$T = I - A = \begin{bmatrix} 1-0.25 & 0-0.40 \\ 0-0.10 & 1-0.10 \end{bmatrix} = \begin{bmatrix} 0.75 & -0.4 \\ -0.1 & 0.9 \end{bmatrix}$$

$$|\bar{T}| = 0.75(0.9) - 0.1(0.4) = 0.675 - 0.04 = 0.635 \neq 0$$

By Cramer's rule

$$\bar{x}_1 = \frac{|T_1|}{|T|} = \frac{\begin{vmatrix} 10 & -0.4 \\ 20 & 0.9 \end{vmatrix}}{0.635} = \frac{9+8}{0.635} \approx 26.77 \qquad \bar{x}_2 = \frac{|T_2|}{|T|} = \frac{\begin{vmatrix} 0.75 & 10 \\ -0.1 & 20 \end{vmatrix}}{0.635} = \frac{15+1}{0.635} \approx 25.20$$

44. The input-output matrix for a 3-sector economy is

$$A = \begin{bmatrix} 0.3 & 0 & 0 \\ 0.3 & 0.3 & 0.4 \\ 0.2 & 0.5 & 0.4 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$$

Find the outputs of the three industries needed to satisfy total demand.

Solution:

$$T = I - A = \begin{bmatrix} 0.7 & 0 & 0 \\ -0.3 & 0.7 & -0.4 \\ -0.2 & -0.5 & 0.6 \end{bmatrix} \qquad |T| = 0.7(0.42-0.2) = 0.154$$

By Cramer's rule

$$\bar{x}_1 = \frac{\begin{vmatrix} 10 & 0 & 0 \\ 20 & 0.7 & -0.4 \\ 30 & -0.5 & 0.6 \end{vmatrix}}{0.154} = \frac{10(0.42-0.2)}{0.154} = \frac{2.2}{0.154} \approx 14.29$$

$$\bar{x}_2 = \frac{\begin{vmatrix} 0.7 & 10 & 0 \\ -0.3 & 20 & -0.4 \\ -0.2 & 30 & 0.6 \end{vmatrix}}{0.154} = \frac{0.7(12+12)-10(-0.18-0.08)}{0.154} = \frac{16.8+2.6}{0.154} \approx 125.97$$

$$\bar{x}_3 = \frac{\begin{vmatrix} 0.7 & 0 & 10 \\ -0.3 & 0.7 & 20 \\ -0.2 & -0.5 & 30 \end{vmatrix}}{0.154} = \frac{0.7(21+10)+10(0.15+0.14)}{0.154} = \frac{21.7+2.9}{0.154} = \frac{24.6}{0.154} \approx 159.74$$

45. The input-output matrix for a 3-sector economy and the external-demand matrix are:

$$A = \begin{bmatrix} 0.3 & 0 & 0 \\ 0.3 & 0.3 & 0.4 \\ 0 & 0.5 & 0.4 \end{bmatrix} \qquad D = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$$

Find the technology matrix T and the outputs of the three industries as specified by the Leontief input-output model using Cramer's rule.

Solution:

$$T = \begin{bmatrix} 0.7 & 0 & 0 \\ -0.3 & 0.7 & -0.4 \\ 0 & 0.5 & 0.6 \end{bmatrix}$$

$$|T| = 0.7(0.42-0.2) = 0.154$$

$$\bar{x}_1 = \frac{\begin{vmatrix} 10 & 0 & 0 \\ 20 & 0.7 & -0.4 \\ 30 & -0.5 & 0.6 \end{vmatrix}}{0.154} = \frac{10(0.42-0.2)}{0.154} = 14.29$$

$$\bar{x}_2 = \frac{\begin{vmatrix} 0.7 & 10 & 0 \\ -0.3 & 20 & -0.4 \\ 0 & 30 & 0.6 \end{vmatrix}}{0.154} = \frac{0.7.24+1.8}{0.154} = 120.78$$

$$\bar{x}_3 = \frac{\begin{vmatrix} 0.7 & 0 & 10 \\ -0.3 & 0.7 & 20 \\ 0 & -0.5 & 30 \end{vmatrix}}{0.154} = \frac{21.7+1.5}{0.154} = 150.65$$

46. Consider a simple model of an economy in which only three goods are produced: electricity, coal, and steel. Outside or external demand from businesses, other industries, households and the government amount to $1,156,500 worth of electricity, $685,000 worth of coal, and $1,668,000 worth of steel. Given the input-output table for this three-sector economy presented below, determine the output of the second sector, i.e. the coal industry, that will meet both the external and the internal demand of the economy. Once the external demand for coal is satisfied what will be the amount of coal left for the three industries in the economy to consume?

Items Consumed (input)	**Items Produced (output)**		
	Electricity	*Coal*	*Steel*
Electricity	0.15	0.15	0.09
Coal	0.50	0	0.10
Steel	0.10	0.03	0.20

Solution:

$$T = \begin{bmatrix} 0.85 & -0.15 & -0.09 \\ -0.5 & 1 & -0.1 \\ -0.1 & -0.03 & 0.8 \end{bmatrix}$$

$$|T| = 0.85(0.8-0.003)+0.15(-0.4-0.01)-0.09(0.015+0.1) = 0.67745-0.0615-0.01035 =$$
$$= 0.6056$$

$$\overline{x_2} = \frac{\begin{vmatrix} 0.85 & 1156,5 & -0.09 \\ -0.5 & 685 & -0.1 \\ -0.1 & 1668 & 0.8 \end{vmatrix}}{0.6056} = \frac{-1156,5(-0.4-0.01)+685(0.68-0.009)-1668(-0.085-0.045)}{0.6056} =$$
$$= \frac{1,150,640}{0.6056} = \$1,900,000 \text{ worth of coal}$$

Leftover coal: $1,900,000 - 685,000 = \$1,215,000$

47. For the input-output matrix in the previous problem find the amount of steel that will meet total demand and the amount that will be left for the three industries to consume.

Solution:

$$T = \begin{bmatrix} 0.85 & -0.15 & -0.09 \\ -0.5 & 1 & -0.1 \\ -0.1 & -0.03 & 0.8 \end{bmatrix}$$

$$|T| = 0.85(0.8-0.003)+0.15(-0.4-0.01)-0.09(0.015+0.1) = 0.67745-0.0615-0.01035 =$$
$$= 0.6056$$

$$\overline{x_3} = \frac{\begin{vmatrix} 0.85 & -0.15 & 1156.5 \\ -0.5 & 1 & 685 \\ -0.1 & -0.03 & 1668 \end{vmatrix}}{0.6056} = \frac{1156,5(0.015+0.1)-685(-0.0255-0.015)+1668(0.85-0.075)}{0.6056} =$$
$$= \frac{1,453,440}{0.6056} = \$2,400,000 \text{ worth of steel}$$

Leftover steel: $2,400,000 - 1,668,000 = \$732,000$

48. Consider a simple model of an economy in which only three goods are produced: electricity, coal, and steel. Outside or external demand from businesses, other industries, households and the government amount to $2,440,000 worth of electricity, $2,000,000 worth of coal, and $1,680,000 worth of steel. Given the input-output table for this three-sector economy presented below, determine the output of the third sector, i.e., the steel industry that will meet both the external and the internal demand of the economy. Once the external demand for steel is satisfied what will be the amount of steel left for the three industries in the economy to consume? How much worth of labor is used in the production of steel?

Items Consumed (input)	**Items Produced (output)** *Electricity*	*Coal*	*Steel*
Electricity	0.05	0.15	0.08
Coal	0.6	0	0.5
Steel	0.03	0.2	0.2

Solution:

$$T = \begin{bmatrix} 0.95 & -0.15 & -0.08 \\ -0.6 & 1 & -0.5 \\ -0.03 & -0.2 & 0.8 \end{bmatrix}$$

$$|T| = 0.95(0.8 - 0.1) + 0.15(-0.48 - 0.015) - 0.08(0.12 + 0.03) = 0.57875$$

$$\overline{x_3} = \frac{\begin{vmatrix} 0.95 & -0.15 & 2440 \\ -0.6 & 1 & 2000 \\ -0.03 & -0.2 & 1680 \end{vmatrix}}{0.57875} = \frac{2440(0.12 + 0.03) - 2000(-0.19 - 0.0045) + 1680(0.95 - 0.09)}{0.57875} =$$

$$= \frac{2199.8}{0.57875} \approx \$3,800,950 \text{ worth of steel}$$

Leftover steel: $3,800,950 - 1,680,000 = \$2,120,950$

Labor share: $(1 - 0.08 - 0.5 - 0.2)3,800,950 = \$836,209$

49. Consider a simple model of an economy in which only three goods are produced: electricity, coal, and steel. Outside or external demand from businesses, other industries, households and the government amount to $2,500,000 worth of electricity, $1,850,000 worth of coal, and $1,600,000 worth of steel. Given the input-output table for this three-sector economy presented below, determine the output of the second sector, i.e. the coal industry, that will meet both the external and the internal demand of the economy. Once the external demand for coal is satisfied what will be the amount of coal left for the three industries in the economy to consume? How much labor is used in the production of coal?

Items Consumed (input)	**Items Produced (output)** *Electricity*	*Coal*	*Steel*
Electricity	0.05	0.15	0.08
Coal	0.6	0	0.5
Steel	0.03	0.2	0.2

Solution:

$$T = \begin{bmatrix} 0.95 & -0.15 & -0.08 \\ -0.6 & 1 & -0.5 \\ -0.03 & -0.2 & 0.8 \end{bmatrix}$$

$$|T| = 0.95(0.8 - 0.1) + 0.15(-0.48 - 0.015) - 0.08(0.12 + 0.03) = 0.57875$$

$$\bar{x}_2 = \frac{\begin{vmatrix} 0.95 & 2500 & -0.08 \\ -0.6 & 1850 & -0.5 \\ -0.03 & 1600 & 0.8 \end{vmatrix}}{0.57875} = \frac{-2500(-0.48-0.015)+1850(0.76-0.0024)-1600(-0.475-0.048)}{0.57875} =$$

$$= \frac{3475.86}{0.57875} \approx \$6,005,806 \text{ worth of coal}$$

Leftover coal: $6,005,806 - 1,850,000 = \$4,155,806$

Labor share: $(1-0.15-0.2)6,005,806 = \$3,903,744$

50. Consider a simple economy in which three goods are produced: electricity, coal, and steel. Monthly external demand from businesses, households and the government amounts to $200,000 worth of electricity, $220,000 worth of coal, and $180,000 worth of steel. Using the input-output table below, determine the output of the second sector, i.e. the coal industry, which will meet both the external and the internal demand of the economy. Once the external demand for coal is satisfied what will be the amount of coal left for the three industries in the economy to consume?

Items Consumed (input)	**Items Produced (output)**		
	Electricity	*Coal*	*Steel*
Electricity	0.2	0.2	0.1
Coal	0.4	0	0
Steel	0.2	0.1	0.8

Solution:

$$T = \begin{bmatrix} 0.8 & -0.2 & -0.1 \\ -0.4 & 1 & 0 \\ -0.2 & -0.1 & 0.2 \end{bmatrix}$$

$$|T| = -0.1(0.04+0.2)+0.2(0.8-0.08) = -0.024+0.144 = 0.12$$

$$|T_2| = \begin{vmatrix} 0.8 & 200 & -0.1 \\ -0.4 & 220 & 0 \\ -0.2 & 180 & 0.2 \end{vmatrix} = -0.1(-72+44)+0.2(176+80) = 2.8+51.2 = 54$$

$$\bar{x}_2 = \frac{54,000}{0.12} = \$450,000 \text{ worth of coal}$$

Coal for the three industries: $450,000 - 220,000 = \$230,000$

51. Consider a simple model of an economy in which only three goods are produced: electricity, coal, and steel. Outside or external demand from businesses, other industries, households and the government amount to $250 million worth of electricity, $180 million worth of coal, and $220 worth of steel. Given the input-output table for this three-sector economy presented below, determine the output of the first sector, i.e., the electrical industry that will meet both the external and the internal demand of the economy. How much worth of labor is used in the production of electricity?

Items Consumed (input)	**Items Produced (output)**		
	Electricity	*Coal*	*Steel*
Electricity	0.05	0.15	0.08
Coal	0.8	0.4	0.2
Steel	0	0.2	0.4

Solution:

$$T = \begin{bmatrix} 0.95 & -0.15 & -0.08 \\ -0.8 & 0.6 & -0.2 \\ 0 & -0.2 & 0.6 \end{bmatrix}$$

$$|T| = 0.2(-0.19 - 0.064) + 0.6(0.57 - 0.12) = -0.0508 + 0.27 = 0.2192$$

$$\bar{x}_1 = \frac{\begin{vmatrix} 250 & -0.15 & -0.08 \\ 180 & 0.6 & -0.2 \\ 220 & -0.2 & 0.6 \end{vmatrix}}{0.2192} = \frac{250(0.36 - 0.04) - 180(-0.09 - 0.016) + 220(0.03 + 0.048)}{0.2192} =$$

$$= \frac{116.24}{0.2192} \approx \$530.3 \text{ million worth of electricity}$$

Share of labor: $(1 - 0.05 - 0.8)530.3 = (0.15)530.3 \approx \$80$ million

52. Consider a simple model of an economy in which only three goods are produced: electricity, coal, and steel. Outside or external demand from businesses, other industries, households and the government amount to \$2,700,000 worth of electricity, \$2,380,000 worth of coal, and \$1,750,000 worth of steel. Given the input-output table for this three-sector economy presented below, determine the output of the third sector, i.e., the steel industry that will meet both the external and the internal demand of the economy. Once the external demand for steel is satisfied what will be the amount of steel left for the three industries in the economy to consume? How much worth of labor is used in the production of steel?

Items Consumed (input)	**Items Produced (output)**		
	Electricity	*Coal*	*Steel*
Electricity	0.05	0.15	0.08
Coal	0.6	0	0.5
Steel	0.03	0.2	0.2

Solution:

$$T = \begin{bmatrix} 0.95 & -0.15 & -0.08 \\ -0.6 & 1 & -0.5 \\ -0.03 & -0.2 & 0.8 \end{bmatrix}$$

$$|T| = 0.95(0.8 - 0.1) + 0.15(-0.48 - 0.015) - 0.08(0.12 + 0.03) = 0.57875$$

$$\bar{x}_3 = \frac{\begin{vmatrix} 0.95 & -0.15 & 2700 \\ -0.6 & 1 & 2380 \\ -0.03 & -0.2 & 1750 \end{vmatrix}}{0.57875} = \frac{2700(0.12 + 0.03) - 2380(-0.19 - 0.0045) + 1750(0.95 - 0.09)}{0.57875} =$$

$$= \frac{2372.91}{0.57875} \approx \$4,100,060 \text{ worth of steel}$$

Leftover steel: $4,100,060 - 1,750,000 = \$2,350,060$

Labor share: $(1 - 0.08 - 0.5 - 0.2)4,100,060 = \$902,013.2$

53. For the previous problem find the output of the coal industry that will meet the external and internal demand of the economy. What will be the amount of coal left for the three industries in the economy to consume? How much labor is used in the production of coal?

Solution:

$$T = \begin{bmatrix} 0.95 & -0.15 & -0.08 \\ -0.6 & 1 & -0.5 \\ -0.03 & -0.2 & 0.8 \end{bmatrix}$$

$$|T| = 0.95(0.8-0.1) + 0.15(-0.48-0.015) - 0.08(0.12+0.03) = 0.57875$$

$$\bar{x}_2 = \frac{\begin{vmatrix} 0.95 & 2700 & -0.08 \\ -0.6 & 2380 & -0.5 \\ -0.03 & 1750 & 0.8 \end{vmatrix}}{0.57875} = \frac{2700(-0.48-0.015) + 2380(0.76-0.0024) - 1750(-0.475-0.048)}{0.57875} =$$

$$= \frac{4054.838}{0.57875} \approx \$7,006,200 \text{ worth of coal}$$

Leftover coal: $7,006,200 - 2,380,000 = \$4,626,200$

Labor share: $(1-0.15-0.2)7,006,200 = \$4,554,030$

54. Consider a simple model of an economy in which only three goods are produced: electricity, coal, and steel. Outside or external demand from businesses, other industries, households and the government amount to $240 million worth of electricity, $180 million worth of coal, and $220 million worth of steel. Given the input-output table for this three-sector economy presented below, determine the output of the second sector, i.e., the coal industry that will meet both the external and the internal demand of the economy. How much worth of labor is used in the production of coal?

Items Consumed (input)	**Items Produced (output)**		
	Electricity	*Coal*	*Steel*
Electricity	0.04	0.15	0.05
Coal	0.7	0.4	0.5
Steel	0.02	0	0.1

Solution:

$$T = \begin{bmatrix} 0.96 & -0.15 & -0.05 \\ -0.7 & 0.6 & -0.5 \\ -0.02 & 0 & 0.9 \end{bmatrix}$$

$$|T| = -0.02(0.075+0.03) + 0.9(0.576-0.105) = 0.4218$$

$$\bar{x}_2 = \frac{\begin{vmatrix} 0.96 & 240 & -0.05 \\ -0.7 & 180 & -0.5 \\ -0.02 & 220 & 0.9 \end{vmatrix}}{0.4218} = \frac{-240(-0.63-0.01) + 180(0.864-0.001) - 220(-0.48-0.035)}{0.4218} =$$

$$= \frac{422.24}{0.4218} \approx \$1,000 \text{ million worth of coal}$$

Labor share: $(1-0.15-0.4)1000 \approx \$450$ million

55. Consider a simple economy in which three goods are produced: electricity, coal, and steel. Monthly external demand from businesses, households and the government amounts to $130,000 worth of electricity, $250,000 worth of coal, and $240,000 worth of steel. Using the input-output table below, determine the output of the first sector, i.e. the electricity industry, which will meet both the external and the internal demand of the economy. Once the external demand for electricity is satisfied what will be the amount of electricity left for the three industries in the economy to consume?

Items Consumed (input)	**Items Produced (output)** *Electricity*	*Coal*	*Steel*
Electricity	0.1	0.2	0.2
Coal	0.5	0	0
Steel	0.2	0.3	0.6

Solution:

$$T = \begin{bmatrix} 0.9 & -0.2 & -0.2 \\ -0.5 & 1 & 0 \\ -0.2 & -0.3 & 0.4 \end{bmatrix}$$

$$|T| = -0.2(0.15 + 0.2) + 0.4(0.9 - 0.1) = -0.07 + 0.32 = 0.25$$

$$|T_1| = \begin{vmatrix} 130 & -0.2 & -.2 \\ 250 & 1 & 0 \\ 240 & -0.3 & 0.4 \end{vmatrix} = 130.0.4 - 250(-0.08 - 0.06) + 240(0.2) = 52 + 35 + 48 = 135$$

$$\overline{x}_1 = \frac{135{,}000}{0.25} = \$540{,}000 \text{ worth of electricity}$$

$$540{,}000 - 130{,}000 = \$410{,}000 \text{ worth of electricity}$$

56. For the previous problem find the outputs of the second and the third sector, that is, coal and steel production as well as the amounts of both left for the three industry in the economy to consume once the external demand is satisfied.

Solution:

$$T = \begin{bmatrix} 0.9 & -0.2 & -0.2 \\ -0.5 & 1 & 0 \\ -0.2 & -0.3 & 0.4 \end{bmatrix}$$

$$|T| = -0.2(0.15 + 0.2) + 0.4(0.9 - 0.1) = -0.07 + 0.32 = 0.25$$

$$|T_2| = \begin{vmatrix} 0.9 & 130 & -0.2 \\ -0.5 & 250 & 0 \\ -0.2 & 240 & 0.4 \end{vmatrix} = 130.0.2 + 250(0.36 - 0.04) - 240(0 - 0.1) = 26 + 80 + 24 = 130$$

$$\overline{x}_2 = \frac{|T_2|}{|T|} = \frac{130}{0.25} = \$520{,}000 \text{ worth of coal}$$

Leftover coal: $520{,}000 - 250{,}000 = \$270{,}000$

$$|T_3| = \begin{vmatrix} 0.9 & -0.2 & 130 \\ -0.5 & 1 & 250 \\ -0.2 & -0.3 & 240 \end{vmatrix} = 130(0.15 + 0.2) - 250(-0.27 - 0.04) + 240(0.9 - 0.1) = 45.5 + 77.5 + 192 = 315$$

$$\overline{x}_3 = \frac{|T_3|}{|T|} = \frac{315}{0.25} = \$1{,}260{,}000 \text{ worth of steel}$$

Leftover steel: $1,260,000 - 240,000 = \$1,020,000$

57. Consider a simple model of an economy in which only three goods are produced: electricity, coal, and steel. Outside or external demand from businesses, other industries, households and the government amount to \$280 million worth of electricity, \$300 million worth of coal, and \$260 million worth of steel. Given the input-output table for this three-sector economy presented below, determine the output of the first sector, i.e., the electricity industry that will meet both the external and the internal demand of the economy. How much worth of labor is used in the production of electricity?

Items Consumed (input)	**Items Produced (output)** *Electricity*	 *Coal*	 *Steel*
Electricity	0.04	0.1	0.1
Coal	0.8	0.5	0.5
Steel	0.02	0	0.1

Solution:

$$T = \begin{bmatrix} 0.96 & -0.1 & -0.1 \\ -0.8 & 0.5 & -0.5 \\ -0.02 & 0 & 0.9 \end{bmatrix}$$

$$|T| = -0.02(0.05 + 0.05) + 0.9(0.48 - 0.08) = -0.002 + 0.36 = 0.358 > 0$$

$$|T_1| = \begin{vmatrix} 280 & -0.1 & -0.1 \\ 300 & 0.5 & -0.5 \\ 260 & 0 & 0.9 \end{vmatrix} = 280(0.45) - 300(-0.09) + 260(0.05 + 0.05) = 126 + 27 + 26 = 179$$

$$\bar{x}_1 = \frac{|T_1|}{|T|} = \frac{179}{0.358} = \$500 \text{ million}$$

Labor share: $(1 - 0.04 - 0.8 - 0.02)500 = \$70 \text{ million}$

58. Given the input-output table for the three-sector economy in the previous problem, determine the output of the third sector, i.e., the steel industry that will meet both the external and the internal demand of the economy. How much worth of labor is used in the production of steel?

Solution:

$$T = \begin{bmatrix} 0.96 & -0.1 & -0.1 \\ -0.8 & 0.5 & -0.5 \\ -0.02 & 0 & 0.9 \end{bmatrix}$$

$$|T| = -0.02(0.05 + 0.05) + 0.9(0.48 - 0.08) = -0.002 + 0.36 = 0.358 > 0$$

$$|T_3| = \begin{vmatrix} 0.96 & -0.1 & 280 \\ -0.8 & 0.5 & 300 \\ -0.02 & 0 & 260 \end{vmatrix} = 280(0.01) - 300(-0.002) + 260(0.48 - 0.08) = 2.8 + 0.6 + 104 = 107.4$$

$$\bar{x}_3 = \frac{|T_3|}{|T|} = \frac{107.4}{0.358} = \$300 \text{ million}$$

Labor share: $(1 - 0.1 - 0.5 - 0.1)300 = \$90 \text{ million}$

59. The input-output matrix for a 2-sector economy is the following:

$$A = \begin{bmatrix} 0.25 & 0.40 \\ 0.14 & 0.12 \end{bmatrix}$$

Find the production levels of each sector for each of the external demand vectors given below:

	External Demands				
Sector 1	10	15	20	20	40
Sector 2	20	20	20	40	30

Solution:

$$T = I - A = \begin{bmatrix} 0.75 & -0.4 \\ -0.14 & 0.88 \end{bmatrix} \qquad |T| = 0.66 - 0.056 = 0.604$$

To find $\bar{x}_1$ and $\bar{x}_2$ for all external demand vectors, it is better to use the matrix inversion method.

$$C = \begin{bmatrix} 0.88 & 0.14 \\ 0.4 & 0.75 \end{bmatrix} \qquad C' = \begin{bmatrix} 0.88 & 0.4 \\ 0.14 & 0.75 \end{bmatrix}$$

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \frac{1}{0.604}\begin{bmatrix} 0.88 & 0.4 \\ 0.14 & 0.75 \end{bmatrix}\begin{bmatrix} 10 \\ 20 \end{bmatrix} = \frac{1}{0.604}\begin{bmatrix} 16.8 \\ 16.4 \end{bmatrix} = \begin{bmatrix} 27.81 \\ 27.15 \end{bmatrix}$$

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \frac{1}{0.604}\begin{bmatrix} 0.88 & 0.4 \\ 0.14 & 0.75 \end{bmatrix}\begin{bmatrix} 15 \\ 20 \end{bmatrix} = \frac{1}{0.604}\begin{bmatrix} 21.20 \\ 17.1 \end{bmatrix} = \begin{bmatrix} 35.09 \\ 28.31 \end{bmatrix}$$

60. The input-output matrix for a 3-sector economy is $A = \begin{bmatrix} 0.3 & 0.4 & 0 \\ 0 & 0.3 & 0.2 \\ 0.5 & 0 & 0.3 \end{bmatrix}$. Find the three output vectors $\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix}$ denoting the production levels of the three sectors for each set of the external demands:

	External Demands		
Sector 1	100	100	200
Sector 2	100	200	200
Sector 3	200	300	400

Solution:

$$T = \begin{bmatrix} 0.7 & -0.4 & 0 \\ 0 & 0.7 & -0.2 \\ -0.5 & 0 & 0.7 \end{bmatrix}$$

$$|T| = 0.7(0.49) + 0.4(-0.1) = 0.303$$

$$C = \begin{bmatrix} 0.49 & 0.1 & 0.35 \\ 0.28 & 0.49 & 0.2 \\ 0.08 & 0.14 & 0.49 \end{bmatrix} \qquad C' = \begin{bmatrix} 0.49 & 0.28 & 0.08 \\ 0.1 & 0.49 & 0.14 \\ 0.35 & 0.2 & 0.49 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.303}\begin{bmatrix} 0.49 & 0.28 & 0.08 \\ 0.1 & 0.49 & 0.14 \\ 0.35 & 0.2 & 0.49 \end{bmatrix}\begin{bmatrix} 100 \\ 100 \\ 200 \end{bmatrix} = \begin{bmatrix} 306.9 \\ 287.12 \\ 504.95 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.303}\begin{bmatrix} 0.49 & 0.28 & 0.08 \\ 0.1 & 0.49 & 0.14 \\ 0.35 & 0.2 & 0.49 \end{bmatrix}\begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix} = \begin{bmatrix} 425.7 \\ 495 \\ 723.67 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.303}\begin{bmatrix} 0.49 & 0.28 & 0.08 \\ 0.1 & 0.49 & 0.14 \\ 0.35 & 0.2 & 0.49 \end{bmatrix}\begin{bmatrix} 200 \\ 200 \\ 400 \end{bmatrix} = \begin{bmatrix} 613.86 \\ 574.25 \\ 1009.9 \end{bmatrix}$$

61. The input-output matrix for a 3-sector economy is $A = \begin{bmatrix} 0.4 & 0.4 & 0 \\ 0 & 0.3 & 0.4 \\ 0.1 & 0 & 0.2 \end{bmatrix}$. Find the three output vectors $\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix}$ denoting the production levels of the three sectors for each set of the external demands:

	External Demands		
Sector 1	200	200	400
Sector 2	200	400	400
Sector 3	500	500	200

Solution:

$$T = \begin{bmatrix} 0.6 & -0.4 & 0 \\ 0 & 0.7 & -0.4 \\ -0.1 & 0 & 0.8 \end{bmatrix}$$

$$|T| = 0.6(0.56) + 0.4(-0.04) = 0.32$$

$$C = \begin{bmatrix} 0.56 & 0.04 & 0.07 \\ 0.32 & 0.48 & 0.04 \\ 0.16 & 0.24 & 0.42 \end{bmatrix} \qquad C' = \begin{bmatrix} 0.56 & 0.32 & 0.16 \\ 0.04 & 0.48 & 0.24 \\ 0.07 & 0.04 & 0.42 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.32}\begin{bmatrix} 0.56 & 0.32 & 0.16 \\ 0.04 & 0.48 & 0.24 \\ 0.07 & 0.04 & 0.42 \end{bmatrix}\begin{bmatrix} 200 \\ 200 \\ 500 \end{bmatrix} = \begin{bmatrix} 800 \\ 700 \\ 500 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.32}\begin{bmatrix} 0.56 & 0.32 & 0.16 \\ 0.04 & 0.48 & 0.24 \\ 0.07 & 0.04 & 0.42 \end{bmatrix}\begin{bmatrix} 200 \\ 400 \\ 500 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1000 \\ 750 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.32}\begin{bmatrix} 0.56 & 0.32 & 0.16 \\ 0.04 & 0.48 & 0.24 \\ 0.07 & 0.04 & 0.42 \end{bmatrix}\begin{bmatrix} 400 \\ 400 \\ 200 \end{bmatrix} = \begin{bmatrix} 1200 \\ 800 \\ 400 \end{bmatrix}$$

62. For the previous problem find the three output vectors for each set of the external demands below:

	External Demands		
Sector 1	400	400	600
Sector 2	200	300	300
Sector 3	200	400	500

Solution:

$$T = \begin{bmatrix} 0.6 & -0.4 & 0 \\ 0 & 0.7 & -0.4 \\ -0.1 & 0 & 0.8 \end{bmatrix}$$

$$|T| = 0.6(0.56) + 0.4(-0.04) = 0.32$$

$$C = \begin{bmatrix} 0.56 & 0.04 & 0.07 \\ 0.32 & 0.48 & 0.04 \\ 0.16 & 0.24 & 0.42 \end{bmatrix} \qquad C' = \begin{bmatrix} 0.56 & 0.32 & 0.16 \\ 0.04 & 0.48 & 0.24 \\ 0.07 & 0.04 & 0.42 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.32}\begin{bmatrix} 0.56 & 0.32 & 0.16 \\ 0.04 & 0.48 & 0.24 \\ 0.07 & 0.04 & 0.42 \end{bmatrix}\begin{bmatrix} 400 \\ 200 \\ 200 \end{bmatrix} = \begin{bmatrix} 1000 \\ 500 \\ 375 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.32}\begin{bmatrix} 0.56 & 0.32 & 0.16 \\ 0.04 & 0.48 & 0.24 \\ 0.07 & 0.04 & 0.42 \end{bmatrix}\begin{bmatrix} 400 \\ 300 \\ 400 \end{bmatrix} = \begin{bmatrix} 1200 \\ 800 \\ 650 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.32}\begin{bmatrix} 0.56 & 0.32 & 0.16 \\ 0.04 & 0.48 & 0.24 \\ 0.07 & 0.04 & 0.42 \end{bmatrix}\begin{bmatrix} 600 \\ 300 \\ 500 \end{bmatrix} = \begin{bmatrix} 1600 \\ 900 \\ 825 \end{bmatrix}$$

63. The input-output matrix for a 3-sector economy is $A = \begin{bmatrix} 0.2 & 0.4 & 0.2 \\ 0 & 0.2 & 0.4 \\ 0 & 0 & 0.4 \end{bmatrix}$. Find the three output vectors $\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix}$ denoting the production levels of the three sectors for each set of the external demands:

	External Demands		
Sector 1	60	20	40
Sector 2	40	40	40
Sector 3	120	60	60

Solution:

$$T = \begin{bmatrix} 0.8 & -0.4 & -0.2 \\ 0 & 0.8 & -0.4 \\ 0 & 0 & 0.6 \end{bmatrix}$$

$$|T| = 0.6(0.8)0.8 = 0.384$$

$$C = \begin{bmatrix} 0.48 & 0 & 0 \\ 0.24 & 0.48 & 0 \\ 0.32 & 0.32 & 0.64 \end{bmatrix} \qquad C' = \begin{bmatrix} 0.48 & 0.24 & 0.32 \\ 0 & 0.48 & 0.32 \\ 0 & 0 & 0.64 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.384} \begin{bmatrix} 0.48 & 0.24 & 0.32 \\ 0 & 0.48 & 0.32 \\ 0 & 0 & 0.64 \end{bmatrix} \begin{bmatrix} 60 \\ 40 \\ 120 \end{bmatrix} = \begin{bmatrix} 200 \\ 150 \\ 200 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.384} \begin{bmatrix} 0.48 & 0.24 & 0.32 \\ 0 & 0.48 & 0.32 \\ 0 & 0 & 0.64 \end{bmatrix} \begin{bmatrix} 20 \\ 40 \\ 60 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.384} \begin{bmatrix} 0.48 & 0.24 & 0.32 \\ 0 & 0.48 & 0.32 \\ 0 & 0 & 0.64 \end{bmatrix} \begin{bmatrix} 40 \\ 40 \\ 60 \end{bmatrix} = \begin{bmatrix} 125 \\ 100 \\ 100 \end{bmatrix}$$

64. For the previous problem find the output of the three industries given the three sets of external demand below:

	External Demands		
Sector 1	30	30	90
Sector 2	20	20	20
Sector 3	60	30	30

Solution:

$$T = \begin{bmatrix} 0.8 & -0.4 & -0.2 \\ 0 & 0.8 & -0.4 \\ 0 & 0 & 0.6 \end{bmatrix}$$

$$|T| = 0.6(0.8)0.8 = 0.384$$

$$C = \begin{bmatrix} 0.48 & 0 & 0 \\ 0.24 & 0.48 & 0 \\ 0.32 & 0.32 & 0.64 \end{bmatrix} \qquad C' = \begin{bmatrix} 0.48 & 0.24 & 0.32 \\ 0 & 0.48 & 0.32 \\ 0 & 0 & 0.64 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.384}\begin{bmatrix} 0.48 & 0.24 & 0.32 \\ 0 & 0.48 & 0.32 \\ 0 & 0 & 0.64 \end{bmatrix}\begin{bmatrix} 30 \\ 20 \\ 60 \end{bmatrix} = \frac{1}{0.384}\begin{bmatrix} 38.4 \\ 28.8 \\ 38.4 \end{bmatrix} = \begin{bmatrix} 100 \\ 75 \\ 100 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.384}\begin{bmatrix} 0.48 & 0.24 & 0.32 \\ 0 & 0.48 & 0.32 \\ 0 & 0 & 0.64 \end{bmatrix}\begin{bmatrix} 30 \\ 20 \\ 30 \end{bmatrix} = \begin{bmatrix} 75 \\ 50 \\ 50 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.384}\begin{bmatrix} 0.48 & 0.24 & 0.32 \\ 0 & 0.48 & 0.32 \\ 0 & 0 & 0.64 \end{bmatrix}\begin{bmatrix} 90 \\ 20 \\ 30 \end{bmatrix} = \begin{bmatrix} 150 \\ 50 \\ 50 \end{bmatrix}$$

65. The input-output matrix for a 3-sector economy is $A = \begin{bmatrix} 0.2 & 0.4 & 0 \\ 0 & 0.3 & 0.4 \\ 0.3 & 0 & 0.2 \end{bmatrix}$. Find the three output vectors $\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix}$ denoting the production levels of the three sectors for each set of the external demands:

	External Demands		
Sector 1	200	200	200
Sector 2	100	200	100
Sector 3	400	400	100

Solution:

$$T = \begin{bmatrix} 0.8 & -0.4 & 0 \\ 0 & 0.7 & -0.4 \\ -0.3 & 0 & 0.8 \end{bmatrix}$$

$$|T| = 0.8(0.56) + 0.4(-0.12) = 0.4$$

$$C = \begin{bmatrix} 0.56 & 0.12 & 0.21 \\ 0.32 & 0.64 & 0.12 \\ 0.16 & 0.32 & 0.56 \end{bmatrix} \qquad C' = \begin{bmatrix} 0.56 & 0.32 & 0.16 \\ 0.12 & 0.64 & 0.32 \\ 0.21 & 0.12 & 0.56 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.4}\begin{bmatrix} 0.56 & 0.32 & 0.16 \\ 0.12 & 0.64 & 0.32 \\ 0.21 & 0.12 & 0.56 \end{bmatrix}\begin{bmatrix} 200 \\ 100 \\ 400 \end{bmatrix} = \frac{1}{0.4}\begin{bmatrix} 208 \\ 216 \\ 278 \end{bmatrix} = \begin{bmatrix} 520 \\ 540 \\ 695 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.4}\begin{bmatrix} 0.56 & 0.32 & 0.16 \\ 0.12 & 0.64 & 0.32 \\ 0.21 & 0.12 & 0.56 \end{bmatrix}\begin{bmatrix} 200 \\ 200 \\ 400 \end{bmatrix} = \begin{bmatrix} 600 \\ 700 \\ 725 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.4}\begin{bmatrix} 0.56 & 0.32 & 0.16 \\ 0.12 & 0.64 & 0.32 \\ 0.21 & 0.12 & 0.56 \end{bmatrix}\begin{bmatrix} 200 \\ 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 400 \\ 300 \\ 275 \end{bmatrix}$$

66. For the previous problem solve for the outputs of the three industries where three sets of external demand are given:

	External Demands		
Sector 1	200	200	200
Sector 2	100	100	200
Sector 3	300	500	300

Solution:

$$T = \begin{bmatrix} 0.8 & -0.4 & 0 \\ 0 & 0.7 & -0.4 \\ -0.3 & 0 & 0.8 \end{bmatrix}$$

$$|T| = 0.8(0.56) + 0.4(-0.12) = 0.4$$

$$C = \begin{bmatrix} 0.56 & 0.12 & 0.21 \\ 0.32 & 0.64 & 0.12 \\ 0.16 & 0.32 & 0.56 \end{bmatrix} \qquad C' = \begin{bmatrix} 0.56 & 0.32 & 0.16 \\ 0.12 & 0.64 & 0.32 \\ 0.21 & 0.12 & 0.56 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.4}\begin{bmatrix} 0.56 & 0.32 & 0.16 \\ 0.12 & 0.64 & 0.32 \\ 0.21 & 0.12 & 0.56 \end{bmatrix}\begin{bmatrix} 200 \\ 100 \\ 300 \end{bmatrix} = \begin{bmatrix} 480 \\ 460 \\ 555 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.4}\begin{bmatrix} 0.56 & 0.32 & 0.16 \\ 0.12 & 0.64 & 0.32 \\ 0.21 & 0.12 & 0.56 \end{bmatrix}\begin{bmatrix} 200 \\ 100 \\ 500 \end{bmatrix} = \begin{bmatrix} 560 \\ 620 \\ 835 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.4}\begin{bmatrix} 0.56 & 0.32 & 0.16 \\ 0.12 & 0.64 & 0.32 \\ 0.21 & 0.12 & 0.56 \end{bmatrix}\begin{bmatrix} 200 \\ 200 \\ 300 \end{bmatrix} = \begin{bmatrix} 560 \\ 620 \\ 585 \end{bmatrix}$$

67. The input-output matrix for a 3-sector economy is $A = \begin{bmatrix} 0.5 & 0.4 & 0 \\ 0 & 0.2 & 0.2 \\ 0.5 & 0 & 0.5 \end{bmatrix}$. Find the three output vectors $\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix}$ denoting the production levels of the three sectors for each set of the external demands:

	External Demands		
Sector 1	100	200	100
Sector 2	100	200	200
Sector 3	400	300	400

Solution:

$$T = \begin{bmatrix} 0.5 & -0.4 & 0 \\ 0 & 0.8 & -0.2 \\ -0.5 & 0 & 0.5 \end{bmatrix}$$

$$|T| = 0.5(0.4) + 0.4(-0.1) = 0.16$$

$$C = \begin{bmatrix} 0.4 & 0.1 & 0.4 \\ 0.2 & 0.25 & 0.2 \\ 0.08 & 0.1 & 0.4 \end{bmatrix} \qquad C' = \begin{bmatrix} 0.4 & 0.2 & 0.08 \\ 0.1 & 0.25 & 0.1 \\ 0.4 & 0.2 & 0.4 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.16}\begin{bmatrix} 0.4 & 0.2 & 0.08 \\ 0.1 & 0.25 & 0.1 \\ 0.4 & 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 100 \\ 100 \\ 400 \end{bmatrix} = \begin{bmatrix} 575 \\ 468 \\ 1375 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.16}\begin{bmatrix} 0.4 & 0.2 & 0.08 \\ 0.1 & 0.25 & 0.1 \\ 0.4 & 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 200 \\ 200 \\ 300 \end{bmatrix} = \begin{bmatrix} 900 \\ 625 \\ 1500 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.16}\begin{bmatrix} 0.4 & 0.2 & 0.08 \\ 0.1 & 0.25 & 0.1 \\ 0.4 & 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 100 \\ 200 \\ 400 \end{bmatrix} = \begin{bmatrix} 700 \\ 625 \\ 1500 \end{bmatrix}$$

68. The input-output matrix for a 3-sector economy is $A = \begin{bmatrix} 0.2 & 0 & 0.2 \\ 0 & 0.2 & 0.2 \\ 0.4 & 0 & 0.4 \end{bmatrix}$. Use matrix inversion to find the two output vectors $\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix}$ denoting the production levels of the three sectors for each set of the external demands given below:

	External Demands	
Sector 1	40	40
Sector 2	40	60
Sector 3	60	80

Solution:

$$T = \begin{bmatrix} 0.8 & 0 & -0.2 \\ 0 & 0.8 & -0.2 \\ -0.4 & 0 & 0.6 \end{bmatrix}$$

$$|T| = 0.8(0.48 - 0.08) = 0.32$$

$$C = \begin{bmatrix} 0.48 & 0.08 & 0.32 \\ 0 & 0.4 & 0 \\ 0.16 & 0.16 & 0.64 \end{bmatrix} \qquad C' = \begin{bmatrix} 0.48 & 0 & 0.16 \\ 0.08 & 0.4 & 0.16 \\ 0.32 & 0 & 0.64 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{|T|} C'd = \frac{1}{0.32} \begin{bmatrix} 0.48 & 0 & 0.16 \\ 0.08 & 0.4 & 0.16 \\ 0.32 & 0 & 0.64 \end{bmatrix} \begin{bmatrix} 40 \\ 40 \\ 60 \end{bmatrix} = \frac{1}{0.32} \begin{bmatrix} 19.2+9.6 \\ 3.2+16+9.6 \\ 12.8+38.4 \end{bmatrix} = \frac{1}{0.32} \begin{bmatrix} 28.8 \\ 28.8 \\ 51.2 \end{bmatrix} = \begin{bmatrix} 90 \\ 90 \\ 160 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.32} \begin{bmatrix} 0.48 & 0 & 0.16 \\ 0.08 & 0.4 & 0.16 \\ 0.32 & 0 & 0.64 \end{bmatrix} \begin{bmatrix} 40 \\ 60 \\ 80 \end{bmatrix} = \frac{1}{0.32} \begin{bmatrix} 19.2+12.8 \\ 3.2+24+12.8 \\ 12.8+51.2 \end{bmatrix} = \begin{bmatrix} 100 \\ 125 \\ 200 \end{bmatrix}$$

69. The input-output matrix for a 3-sector economy is $A = \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0 & 0.5 & 0 \\ 0.4 & 0 & 0.4 \end{bmatrix}$. Use matrix inversion to find the two output vectors $\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix}$ denoting the production levels of the three sectors for each set of the external demands given below:

	External Demands	
Sector 1	100	150
Sector 2	50	100
Sector 3	60	80

Solution:

$$T = \begin{bmatrix} 0.8 & -0.2 & -0.2 \\ 0 & 0.5 & 0 \\ -0.4 & 0 & 0.6 \end{bmatrix}$$

$$|T| = 0.5(0.48 - 0.08) = 0.2$$

$$C = \begin{bmatrix} 0.3 & 0 & 0.2 \\ 0.12 & 0.4 & 0.08 \\ 0.1 & 0 & 0.4 \end{bmatrix} \qquad C' = \begin{bmatrix} 0.3 & 0.12 & 0.1 \\ 0 & 0.4 & 0 \\ 0.2 & 0.08 & 0.4 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.2} \begin{bmatrix} 0.3 & 0.12 & 0.1 \\ 0 & 0.4 & 0 \\ 0.2 & 0.08 & 0.4 \end{bmatrix} \begin{bmatrix} 100 \\ 50 \\ 60 \end{bmatrix} = 5 \begin{bmatrix} 42 \\ 20 \\ 48 \end{bmatrix} = \begin{bmatrix} 210 \\ 100 \\ 240 \end{bmatrix}$$

$$\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix} = \frac{1}{0.2} \begin{bmatrix} 0.3 & 0.12 & 0.1 \\ 0 & 0.4 & 0 \\ 0.2 & 0.08 & 0.4 \end{bmatrix} \begin{bmatrix} 150 \\ 100 \\ 80 \end{bmatrix} = 5 \begin{bmatrix} 65 \\ 40 \\ 70 \end{bmatrix} = \begin{bmatrix} 325 \\ 200 \\ 350 \end{bmatrix}$$

70. The input-output matrix for a 3-sector economy is $A = \begin{bmatrix} 0.2 & 0.4 & 0 \\ 0.6 & 0.2 & 0.5 \\ 0 & 0.1 & 0.4 \end{bmatrix}$. Use matrix inversion to find the two output vectors $\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix}$ denoting the production levels of the three sectors for each set of the external demands given below:

	External Demands	
Sector 1	400	600
Sector 2	200	300
Sector 3	200	200

Solution:

$$T = \begin{bmatrix} 0.8 & -0.4 & 0 \\ -0.6 & 0.8 & -0.5 \\ 0 & -0.1 & 0.6 \end{bmatrix}$$

$$|T| = 0.8(0.48 - 0.05) + 0.4(-0.36) = 0.344 - 0.144 = 0.2$$

$$C = \begin{bmatrix} 0.43 & 0.36 & 0.06 \\ 0.24 & 0.48 & 0.08 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} \qquad C' = \begin{bmatrix} 0.43 & 0.24 & 0.2 \\ 0.36 & 0.48 & 0.4 \\ 0.06 & 0.08 & 0.4 \end{bmatrix}$$

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \frac{1}{0.2}\begin{bmatrix} 0.43 & 0.24 & 0.2 \\ 0.36 & 0.48 & 0.4 \\ 0.06 & 0.08 & 0.4 \end{bmatrix}\begin{bmatrix} 400 \\ 200 \\ 200 \end{bmatrix} = 5\begin{bmatrix} 172+48+40 \\ 144+96+80 \\ 24+16+80 \end{bmatrix} = \begin{bmatrix} 1300 \\ 1600 \\ 600 \end{bmatrix}$$

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \frac{1}{0.2}\begin{bmatrix} 0.43 & 0.24 & 0.2 \\ 0.36 & 0.48 & 0.4 \\ 0.06 & 0.08 & 0.4 \end{bmatrix}\begin{bmatrix} 600 \\ 300 \\ 200 \end{bmatrix} = 5\begin{bmatrix} 258+72+40 \\ 216+144+80 \\ 36+24+80 \end{bmatrix} = \begin{bmatrix} 1850 \\ 2200 \\ 700 \end{bmatrix}$$

71. For the previous problem solve the model using the following two sets of external demand:

	External Demands	
Sector 1	200	400
Sector 2	100	200
Sector 3	100	150

Solution:

$$T = \begin{bmatrix} 0.8 & -0.4 & 0 \\ -0.6 & 0.8 & -0.5 \\ 0 & -0.1 & 0.6 \end{bmatrix}$$

$$|T| = 0.8(0.48 - 0.05) + 0.4(-0.36) = 0.344 - 0.144 = 0.2$$

$$C = \begin{bmatrix} 0.43 & 0.36 & 0.06 \\ 0.24 & 0.48 & 0.08 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} \qquad C' = \begin{bmatrix} 0.43 & 0.24 & 0.2 \\ 0.36 & 0.48 & 0.4 \\ 0.06 & 0.08 & 0.4 \end{bmatrix}$$

$$\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\\ \bar{x}_3\end{bmatrix}=\frac{1}{0.2}\begin{bmatrix}0.43 & 0.24 & 0.2\\ 0.36 & 0.48 & 0.4\\ 0.06 & 0.08 & 0.4\end{bmatrix}\begin{bmatrix}200\\ 100\\ 100\end{bmatrix}=5\begin{bmatrix}86+24+20\\ 72+48+40\\ 12+8+40\end{bmatrix}=\begin{bmatrix}650\\ 800\\ 300\end{bmatrix}$$

$$\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\\ \bar{x}_3\end{bmatrix}=\frac{1}{0.2}\begin{bmatrix}0.43 & 0.24 & 0.2\\ 0.36 & 0.48 & 0.4\\ 0.06 & 0.08 & 0.4\end{bmatrix}\begin{bmatrix}400\\ 200\\ 150\end{bmatrix}=5\begin{bmatrix}172+48+30\\ 144+96+60\\ 24+16+60\end{bmatrix}=\begin{bmatrix}1250\\ 1500\\ 500\end{bmatrix}$$

72. The input-output matrix for a 3-sector economy is $A=\begin{bmatrix}0.2 & 0.4 & 0.2\\ 0 & 0.2 & 0.2\\ 0 & 0 & 0.1\end{bmatrix}$. Use matrix inversion to find the two output vectors $\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\\ \bar{x}_3\end{bmatrix}$ denoting the production levels of the three sectors for each set of the external demands given below:

	External Demands	
Sector 1	60	50
Sector 2	60	40
Sector 3	90	90

Solution:

$$T=\begin{bmatrix}0.8 & -0.4 & -0.2\\ 0 & 0.8 & -0.2\\ 0 & 0 & 0.9\end{bmatrix}$$

$$|T|=0.8(0.72)=0.576$$

$$C=\begin{bmatrix}0.72 & 0 & 0\\ 0.36 & 0.72 & 0\\ 0.24 & 0.16 & 0.64\end{bmatrix}\qquad C'=\begin{bmatrix}0.72 & 0.36 & 0.24\\ 0 & 0.72 & 0.16\\ 0 & 0 & 0.64\end{bmatrix}$$

$$\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\\ \bar{x}_3\end{bmatrix}=\frac{1}{0.576}\begin{bmatrix}0.72 & 0.36 & 0.24\\ 0 & 0.72 & 0.16\\ 0 & 0 & 0.64\end{bmatrix}\begin{bmatrix}60\\ 60\\ 90\end{bmatrix}=\frac{1}{0.576}\begin{bmatrix}43.2+21.6+21.6\\ 43.2+14.4\\ 57.6\end{bmatrix}=\begin{bmatrix}150\\ 100\\ 100\end{bmatrix}$$

$$\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\\ \bar{x}_3\end{bmatrix}=\frac{1}{0.576}\begin{bmatrix}0.72 & 0.36 & 0.24\\ 0 & 0.72 & 0.16\\ 0 & 0 & 0.64\end{bmatrix}\begin{bmatrix}50\\ 40\\ 90\end{bmatrix}=\frac{1}{0.576}\begin{bmatrix}36+14.4+21.6\\ 28.8+14.4\\ 57.6\end{bmatrix}=\begin{bmatrix}125\\ 75\\ 100\end{bmatrix}$$

73. The input-output matrix for a 3-sector economy is $A=\begin{bmatrix}0.2 & 0.3 & 0.2\\ 0 & 0.2 & 0.2\\ 0 & 0 & 0.1\end{bmatrix}$. Use matrix inversion to find the two output vectors $\begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix}$ denoting the production levels of the three sectors for each set of the external demands given below:

	External Demands	
Sector 1	110	70
Sector 2	60	60
Sector 3	90	90

Solution:

$$T = \begin{bmatrix} 0.8 & -0.3 & -0.2 \\ 0 & 0.8 & -0.2 \\ 0 & 0 & 0.9 \end{bmatrix}$$

$$|T| = 0.8(0.72) = 0.576$$

$$C = \begin{bmatrix} 0.72 & 0 & 0 \\ 0.27 & 0.72 & 0 \\ 0.22 & 0.16 & 0.64 \end{bmatrix} \qquad C' = \begin{bmatrix} 0.72 & 0.27 & 0.22 \\ 0 & 0.72 & 0.16 \\ 0 & 0 & 0.64 \end{bmatrix}$$

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \frac{1}{0.576}\begin{bmatrix} 0.72 & 0.27 & 0.22 \\ 0 & 0.72 & 0.16 \\ 0 & 0 & 0.64 \end{bmatrix}\begin{bmatrix} 110 \\ 60 \\ 90 \end{bmatrix} = \frac{1}{0.576}\begin{bmatrix} 79.2+16.2+19.8 \\ 43.2+14.4 \\ 57.6 \end{bmatrix} = \begin{bmatrix} 200 \\ 100 \\ 100 \end{bmatrix}$$

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \frac{1}{0.576}\begin{bmatrix} 0.72 & 0.27 & 0.22 \\ 0 & 0.72 & 0.16 \\ 0 & 0 & 0.64 \end{bmatrix}\begin{bmatrix} 70 \\ 60 \\ 90 \end{bmatrix} = \frac{1}{0.576}\begin{bmatrix} 50.4+16.2+19.8 \\ 43.2+14.4 \\ 57.6 \end{bmatrix} = \begin{bmatrix} 150 \\ 100 \\ 100 \end{bmatrix}$$