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del Barrio Castro, Tomás and Osborn, Denise R.

University of the Balearic Islands, University of Manchester

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TOMÁS DEL BARRIO CASTRO*
Department of Applied Economics, University of the Balearic Islands, Palma de Mallorca, 07013, Spain (email: tomas.barrio@uib.es)

DENISE R. OSBORN
Economics, School of Social Sciences, University of Manchester, Oxford Road, Manchester UK, M13 9PL (email: denise.osborn@manchester.ac.uk)

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Summary

Seasonality is pervasive across a wide range of economic time series and it substantially complicates the analysis of unit root non-stationarity in such series. This paper reviews recent contributions to the literature on non-stationary seasonal processes, focussing on periodically integrated (PI) and seasonally integrated (SI) processes. Whereas an SI process captures seasonal non-stationarity essentially through an annual lag, a PI process has (a restricted form of) seasonally-varying autoregressive coefficients. The fundamental properties of both types of process are compared, noting in particular that a simple SI process observed S times a year has S unit roots, in contrast to the single unit root of a PI process. Indeed, for $S > 2$ and even (such as processes observed quarterly or monthly), an SI process has a pair of complex-valued unit roots at each seasonal frequency except the Nyquist frequency, where a single real root applies. Consequently, recent literature concerned with testing the unit roots implied by SI processes employs complex-valued unit root processes, and these are discussed in some detail. A key feature of the discussion is to show how the demodulator operator can be used to convert a unit root process at a seasonal frequency to a conventional zero-frequency unit root process, thereby enabling the well-known properties of the latter to be exploited. Further, circulant matrices are introduced and it is shown how they are employed in theoretical analyses to capture the repetitive nature of seasonal processes. Discriminating between SI and PI processes requires care, since testing for unit roots at seasonal frequencies may lead to a PI process (erroneously) appearing to have an SI form, while an application to monthly US industrial production series illustrates how these types of seasonal non-stationarity can be distinguished in practice. Although univariate processes are discussed, the methods considered in the paper can be used to analyze cointegration, including cointegration across different frequencies.

Keywords: Periodic Integration, Seasonal Integration, Vector of Seasons, Circulant Matrices, Demodulator Operator, Industrial Production.

JEL codes: C32

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1 Introduction

Seasonality complicates the analysis of non-stationary unit root behavior in time series, because unit roots can apply at seasonal frequencies in addition to the zero frequency. This possibility has given rise to a substantial literature concerned with so-called seasonal integration[1]. However, a prominent alternative possibility is that a process evolves with seasonally-dependent dynamics driven by a single unit root, with this known as periodic integration; Ghysels and Osborn (2001) provide an introductory discussion to both types of processes. This paper considers recent research that explores the nature of both types of seasonal unit root processes and the relationships between them. Due to the particular characteristics of seasonal time series, some of the techniques used for analysis are not standard in econometrics and a key purpose here is to provide insights into these techniques and how they are employed in recent research on seasonal processes. In particular, circulant matrices, complex-valued processes and demodulation are discussed. An illustration using monthly US industrial production series provides insights into how seasonal and periodic integration can be distinguished in practice.

To outline the essential issues, consider firstly a purely stochastic time series process observed S times a year ($S = 4$ for quarterly data and $S = 12$ for monthly data) given by

$$y_t = y_{t-S} + \varepsilon_t \tag{1}$$

where the zero mean disturbance process $\varepsilon_t \sim iid(0, \sigma^2)$. The assumption that ε_t follows an *iid* process is made for expositional simplicity, with all key results carrying over to more general stationary and invertible disturbance processes. Similarly, the inclusion of a deterministic component in (1) does not affect the essential features on which we focus; see the discussion in subsection 4.4 below.

The non-stationary process y_t of (1) is referred to as being *seasonally integrated* (SI). It is transformed to stationarity by application of the seasonal difference filter $\Delta_S = 1 - L^S$, with L being the usual lag operator. As discussed in more detail below, this process has autoregressive unit roots at the zero frequency and at each of the so-called seasonal frequencies, $\omega_k = 2\pi k/S,\ k = 1, ..., S/2$ (assuming S even). Indeed, there is a single unit root at $\omega_k = \pi$ and pairs of complex-valued unit roots at each $\omega_k = 2\pi k/S,\ k = 1, ..., (S/2) - 1$. In total, therefore, the SI process has S unit roots, with the pairs of complex-valued roots being a feature of the analysis of such processes.

The second type of process can be illustrated by the first-order periodic autoregressive ($PAR(1)$) process written as[2]

$$y_t = \sum_{s=1}^{S} \phi_s D_{st} y_{t-1} + \varepsilon_t \tag{2}$$

where D_{st} is a dummy variable which is unity when the observation at time t falls in season s ($s = 1, ..., S$) and, again for expositional simplicity, ε_t is assumed to be $iid(0, \sigma^2)$ and no deterministic component is included. The conventional $AR(1)$ is a special case of (2), which is ruled out by requiring that at least one $\phi_s \neq \phi$ for $s = 1, ..., S$. Stationarity of this process requires $\prod_{s=1}^{S} \phi_s < 1$, so that a stationary $PAR(1)$ process can have some individual coefficients that are greater than unity. However, a *periodically integrated (PI)* $PAR(1)$ process satisfies $\prod_{s=1}^{S} \phi_s = 1$. It is key to what follows that, in contrast to the S unit roots of the SI process, a PI process is driven by a single unit root that is transmitted across intra-year observations by the seasonally varying coefficients ϕ_s.

The discussion below focusses on the unit root behaviors of the above processes. In particular, the non-stationary behaviors of seasonal processes that are either PI or have a single unit root (at either the frequency zero or at π) are ruled by a single common stochastic trend that applies across all seasons. However, the behavior of a process whose non-stationarity arises from a (single) pair of complex-valued unit roots at a seasonal frequency $\omega_k = 2\pi k/S,\ k = 1, ..., (S/2) - 1$ is ruled by two common stochastic trends. These stochastic trends give rise to scalar Brownian motions that appear in the asymptotic distributions of the periodic integration and seasonal unit-root test statistics that have been proposed in the literature.

[1] We rule out cases of explosive seasonal non-stationarity, which could arise from, say, roots at seasonal frequencies which lie inside the unit circle. Cases of seasonal fractional integration and multiple unit roots associated with seasonality are also beyond the scope of this paper.

[2] More general periodic autoregressive processes of order p can be considered (see, for example, Boswijk and Franses, 1996), but the first-order process is sufficient to draw out the essential properties we wish to discuss.

Following Osborn (1991), Franses (1994), and others, we sometimes employ the "double subscript" vector of seasons representation in our analysis. This is a multivariate representation of seasonal time series in which each season is treated as an individual time series; this representation facilitates the exploration of cointegrating relationships and/or common stochastic trends shared by the seasons. Assuming that the first time series observation corresponds to the first "season" (usually month or quarter) of a year, $y_{(s\tau)}$ is then the s^{th} observation of year τ; the identity $t = S(\tau - 1) + s$ provides the link between the notation used in (1) and (2) and the vector representation.

Although our focus is seasonality over an annual cycle, the techniques discussed can be applied to non-stationary cycles of longer or shorter durations than a year. In particular, a weekly cycle may be relevant for the analysis of daily observations on (say) retail sales or financial variables.

In the remainder of the paper, Section 2 elaborates the fundamental aspects of seasonal and periodic integration on which formal analyses depend. Sections 3 and 4 then examine the regression used to test for unit roots in a potential SI process, with Section 3 showing how circulant matrices can be used to represent the regressor variables and Section 4 summarizing the asymptotic distributions of the test statistics. The complex pairs of unit roots at seasonal frequencies lead to the discussion in Section 4 dealing with complex processes and demodulation. Section 5 returns PI processes and examines the implications of applying SI unit root test to such processes. Finally, the illustration of Section 6 applies both SI and PI tests to US industrial production series and draws some conclusions about the likely nature of the underlying processes. The concluding remarks in Section 7 point to some on-going research issues.

2 Fundamentals of Seasonal and Periodic Integration

2.1 Seasonal integration

The seminal paper of Hylleberg, Engle, Granger and Yoo (1990) (hereafter referred to as HEGY) has prompted a large literature on seasonal integration and seasonal unit roots. Earlier contributions, including Box and Jenkins (1970) and Dickey, Hasza and Fuller (1984), discussed the removal of seasonal non-stationarity through the use of seasonal differencing, but HEGY were the first authors to focus on the full set of unit roots which are contained in a quarterly SI process. Their test is discussed in the next section for the case of general S.

As already noted, a seasonally integrated process has all S unit roots implied by the use of the seasonal difference operator Δ_S. For S even, $1 - L^S$ can be factorized as

$$\begin{aligned}\left(1-L^S\right) &= (1-L)(1+L+...+L^{S-1}) \\ &= (1-L)(1+L)\prod_{k=1}^{S^*}\left(1-2\cos(\omega_k)L+L^2\right), \end{aligned} \tag{3}$$

where $\omega_k = 2\pi k/S$ for $k = 0, 1, \ldots, S^*$ and $S^* = (S/2) - 1$. The zero frequency ($\omega_k = 0$) unit root is associated with the factor $1 - L$ in (3), the root at frequency π is associated with $1 + L$ and implies oscillations every two periods, while the factor $\left(1 - 2\cos(\omega_k) L + L^2\right) = \left(1 - e^{-i\omega_k} L\right)\left(1 - e^{i\omega_k} L\right)$ at each frequency $\omega_k = 2\pi k/S$, $k = 1, 2, \ldots, S^*$ gives rise to a complex conjugate pair of unit roots with oscillations that complete a full cycle every $2\pi/\omega_k = S/k$ periods. Although odd S can be relevant, such as when examining a weekly cycle in daily observations, we assume that S is even for simplicity[3].

For example, with monthly data ($S = 12$), an SI process has real unit roots at the zero and π frequencies together with five pairs of complex-valued unit roots at the frequencies $\omega_k = 2k\pi/12$ for $k = 1, 2, ..., 5$. For the specific case of $k = 4$ (say), the pair of unit roots at frequency $\omega_4 = 2 \times 4\pi/12 = 2\pi/3$ imply non-stationary cycles that occur every 3 months within the overall annual cycle.

It is also important to note that the application of seasonal differencing to a process that contains only a subset of these S unit roots leads to over-differencing, with the differenced process having a non-invertible moving average component.

Employing the double subscript representation, the SI process of (1) can be written as

$$y_{(s\tau)} = y_{(s,\tau-1)} + \varepsilon_{(s\tau)}, \qquad s = 1, ..., S. \tag{4}$$

[3] When S is odd, $S^* = (S-1)/2$ and the factor $(1+L)$ does not appear in the factorization. The discussion below can easily be amended for S odd.

By treating the observations for each season as distinct series, this representation makes clear that each season follows its own unit root process and, as apparently first noted by Osborn (1993), the observations over the S seasons are not cointegrated. In other words, equivalent to associating the S unit roots of an SI process with the zero and seasonal frequencies, each of the S unit roots can be associated with a season (month or quarter) of the year.

Defining the vectors $Y_\tau = \left[y_{(1\tau)}, y_{(2\tau)}, \ldots, y_{(S\tau)}\right]' = [y_{S(\tau-1)+1}, y_{S(\tau-1)+1}, ..., y_{S\tau}]'$ and $E_\tau = [\varepsilon_{(1\tau)}, \varepsilon_{(2\tau)}, ..., \varepsilon_{(S\tau)}]'$ corresponding to year τ, the vector representation of (4) is

$$Y_\tau = Y_{\tau-1} + E_\tau. \tag{5}$$

The system (5) has characteristic equation

$$|\mathbf{I}_S - \mathbf{I}_S z| = (1 - z^S) = 0 \tag{6}$$

where $\mathbf{I}_S$ is an $S \times S$ identity matrix. This equation has S solutions with $|z| = 1$ and implies the presence of unit roots at the zero and all seasonal frequencies, as discussed in relation to (3).

It is straightforward to see that recursive substitution in (5) yields

$$Y_\tau = \sum_{j=0}^{\tau-1} E_{\tau-j} \tag{7}$$

where, for simplicity of exposition, we assume $Y_0 = 0$. If N the years of observations are available on the process, the asymptotic distribution of the vector Y_τ is then given by

$$\frac{1}{\sqrt{N}} Y_{\lfloor Nr \rfloor} \Rightarrow \sigma W(r) \tag{8}$$

where, here and throughout the paper, $W(r) = [W_1(r), W_2(r), \ldots, W_S(r)]'$ is an $S \times 1$ vector of uncorrelated standard Brownian motion processes[4]. The asymptotic distribution of (8) therefore reflects the separate unit root processes followed by each of the S intra-year observations.

2.2 Periodic integration

Employing the double subscript representation, the $PAR(1)$ process in (2) is

$$y_{(s\tau)} = \phi_s y_{(s-1,\tau)} + \varepsilon_{(s\tau)}, \qquad s = 1, ..., S. \tag{9}$$

It immediately follows that this $PAR(1)$ process has the vector representation

$$\mathbf{A}_0 Y_\tau = \mathbf{A}_1 Y_{\tau-1} + E_\tau \tag{10}$$

where Y_τ and E_τ are as defined in (5), while $\mathbf{A}_0$ and $\mathbf{A}_1$ are $S \times S$ matrices with generic elements

$$\begin{aligned} A_{0(h,j)} &= \begin{cases} 1 & h = j,\ j = 1, ..., S \\ -\phi_h & h = j+1,\ j = 1, ..., S-1 \\ 0 & \text{otherwise} \end{cases} \\ A_{1(h,j)} &= \begin{cases} \phi_1 & h = 1,\ j = S \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{11}$$

in which the subscript (h, j) indicates the $(h, j)^{th}$ element of the respective matrix (see, for example, Osborn, 1991, or Franses, 1994). Due to the form of $\mathbf{A}_0$ and $\mathbf{A}_1$, the characteristic equation for the vector process (10) is

$$|\mathbf{A}_0 - \mathbf{A}_1 z| = 1 - \left(\prod_{s=1}^{S} \phi_s \right) z = 0.$$

[4] The increments in standard Brownian motion have unit standard deviation whereas the increments in (1) have a standard deviation of σ.

Hence the $PAR(1)$ is stationary when $\left|\prod_{s=1}^{S}\phi_s\right| < 1$, but the process integrated when $\prod_{s=1}^{S}\phi_s = 1$. In contrast to the SI process, the PI process has no unit root at any seasonal frequency $\omega_k = 2\pi k/S$, $k = 1, \ldots, S/2$.

Noting that $\mathbf{A}_1$ is non-singular and $\mathbf{A}_0^{-1}\mathbf{A}_1$ is idempotent, recursive substitution in (10) with the simplifying assumption $Y_0 = 0$ yields

$$Y_\tau = \mathbf{A}_0^{-1}E_\tau + \mathbf{A}_0^{-1}\mathbf{A}_1\mathbf{A}_0^{-1}\sum_{j=1}^{\tau-1}E_{\tau-j}. \tag{12}$$

Since $\mathbf{A}_1$ has a single non-zero element, it has rank 1 and so also does $\mathbf{A}_0^{-1}\mathbf{A}_1\mathbf{A}_0^{-1}$. Therefore, as shown by Boswijk and Franses (1996), the $S \times S$ matrix $\mathbf{A}_0^{-1}\mathbf{A}_1\mathbf{A}_0^{-1}$ can be represented as

$$\mathbf{A}_0^{-1}\mathbf{A}_1\mathbf{A}_0^{-1} = \mathbf{a}\mathbf{b}' \tag{13}$$

in which the $S \times 1$ vectors $\mathbf{a}$ and $\mathbf{b}$ are

$$\begin{aligned}
\mathbf{a} &= \begin{bmatrix} 1 & \phi_2 & \phi_2\phi_3 & \cdots & \prod_{s=2}^{S}\phi_s \end{bmatrix}' \\
\mathbf{b} &= \begin{bmatrix} 1 & \phi_1\prod_{s=3}^{S}\phi_s & \phi_1\prod_{s=4}^{S}\phi_s & \cdots & \phi_1 \end{bmatrix}'.
\end{aligned} \tag{14}$$

Substituting (13) into (12) shows that the S elements of Y_τ are driven by the single common trend, given by $\mathbf{b}'\sum_{j=1}^{\tau-1}E_{\tau-j}$.

A further consequence of (12) and (13) is that the asymptotic distribution of the each element of the vector Y_τ is a function of a single scalar Brownian motion $w_p(r)$. In particular, as shown by del Barrio Castro and Osborn (2008, Lemma 1) or in the more general Lemma of Boswijk and Franses (1996),

$$\begin{aligned}
\frac{1}{\sqrt{N}}Y_{\lfloor Nr\rfloor} &\Rightarrow \sigma\mathbf{A}_0^{-1}\mathbf{A}_1\mathbf{A}_0^{-1}W(r) = \sigma\mathbf{a}\mathbf{b}'W(r) \\
&= \sigma\varpi\mathbf{a}w_p(r)
\end{aligned} \tag{15}$$

where $w_p(r) = \varpi^{-1}\mathbf{b}'W(r)$ and $\varpi = (\mathbf{b}'\mathbf{b})^{1/2}$ is a scaling term. Therefore, in contrast to the S Brownian motions underlying the asymptotic behaviour of the elements of Y_τ in (8) for the SI process, (15) shows the PI process is driven by the single stochastic trend $w_p(r)$ and hence there are $S-1$ linearly independent cointegrating relationships between the elements of Y_τ.

All statistics proposed in the literature to test the PI null hypothesis have distributions that follow functionals of the scalar Brownian motion $w_p(r)$ associated with the common stochastic trend shared by the seasons of Y_τ. In particular, Boswijk and Franses (1996) propose a likelihood ratio test statistic for the PI null hypothesis in the general $PAR(p)$ model, which can be written as

$$LR = T\,\ln\left(\frac{RSS_0}{RSS_1}\right) \tag{16}$$

where RSS_0 denotes the residual sum of squares computed under the null hypothesis, namely imposing $\prod_{s=1}^{S}\phi_s = 1$ for the $PAR(1)$ case of (2), and RSS_1 is the residual sum of squares from the unrestricted $PAR(1)$ model. Boswijk and Franses (1996) show that under the PI null hypothesis

$$LR \Rightarrow \left[\int w_p(r)\,dw(r)\right]^2 \Big/ \left[\int w_p(r)^2\,dr\right] \tag{17}$$

where $w_p(r)$ is defined in (15). A key feature of (17) is that it is identical to the asymptotic distribution of the squared Dickey-Fuller t-statistic in a conventional (nonperiodic) $I(1)$ processes, namely $LR \Rightarrow \left[\int w(r)\,dw(r)\right]^2 / \left[\int w(r)^2\,dr\right]$, where $w(r)$ is again scalar standard Brownian motion. Hence testing for a PI process is analogous to testing for a single unit root in an conventional AR process.

2.3 Constant parameter representation of a PI process

Having set out the basics of $SI(1)$ and $PI(1)$ processes, we now turn an important aspect of the relationship between them, by examining the constant parameter representation of a $PI(1)$ process. This is representation can be obtained analytically as the conventional (constant parameter) process that has identical autocovariance properties to the $PI(1)$ process when the latter is analyzed as if there is no parameter variation over the S seasons; it is considered in Osborn (1991), Ghysels and Osborn (2001) and del Barrio Castro and Osborn (2008).

Recursively substituting for the $S-1$ observations $y_{(s-1,\tau)},...,y_{(s+1,\tau-1)}$ on the right-hand side of (9), we obtain

$$\begin{aligned} y_{(s\tau)} &= \left(\prod_{j=1}^{S}\phi_j\right)y_{(s,\tau-1)}+\varepsilon_{(s\tau)}+\phi_s\varepsilon_{(s-1,\tau)}+\phi_s\phi_{s-1}\varepsilon_{(s-1,\tau)}+\cdots+\left(\prod_{j=0}^{S-1}\phi_{s-j}\right)\varepsilon_{(s-(S-1),\tau)}, \\ s &= 1,...,S. \end{aligned} \quad (18)$$

In this expression, it is understood that $y_{(s-i,\tau)}=y_{(S-(s-i),\tau-1)}$ for $s-i\leq 0$ and also that $\phi_{s-j}=\phi_{S-(s-i)}$ for $s-j\leq 0$. Note that in this representation the AR coefficient $\prod_{j=1}^{S}\phi_j$ applies for all $s=1,...,S$ whereas the coefficients of the $MA\,(S-1)$ are seasonally varying. Using the properties of moving averages, the representation (18) gives rise to the constant parameter representation of a $PAR(1)$ process.

For the case of periodic integration, $\prod_{j=1}^{S}\phi_j=1$ and the autoregressive part of (18) is a seasonal first difference, so that

$$\begin{aligned} y_{(s\tau)}-y_{(s,\tau-1)} &= \varepsilon_{(s\tau)}+\phi_s\varepsilon_{(s-1,\tau)}+\phi_s\phi_{s-1}\varepsilon_{(s-1,\tau)}+\cdots+\left(\prod_{j=0}^{S-1}\phi_{s-j}\right)\varepsilon_{(s-(S-1),\tau)}, \\ s &= 1,...,S. \end{aligned} \quad (19)$$

Note that (19) gives the expression for a generic season s of $y_{(s\tau)}$ in the final equation representation of the vector process (10)-(11)[5] under periodic integration (del Barrio Castro and Osborn, 2008) and is also equivalent to (12) above.

If the process (19) is considered as one with constant parameters, then this $PI(1)$ process appears to be an $MA(S-1)$ in the annual difference $\Delta_s y_t=y_t-y_{t-S}$, since analyzing it as a constant parameter one effectively averages across the seasonally varying MA processes of (19) and the sum (or average) of $MA(q)$ processes is also an $MA(q)$. Hence, using the conventional single subscript notation, the constant parameter representation of the $PI(1)$ process (2) has the form

$$\Delta_S y_t=\left(1+\theta_1 L+\cdots+\theta_{S-1}L^{S-1}\right)\eta_t, \quad (20)$$

which is (except for special cases) an invertible $MA(S-1)$ in the annual difference series; see del Barrio Castro and Osborn (2008, section 2.2).

An important feature of the constant parameter representation of the $PI(1)$ process in (20) that is it appears to be $SI(1)$, with $\Delta_S y_t$ being an invertible MA. However, such an SI conclusion is a misspecification, since the true data generating process has only a single unit root rather than the S unit roots of an $SI(1)$ process.

The constant parameter representation sheds light on the seasonal patterns that can arise from a $PI(1)$ process. Figure 1 illustrates this in the frequency domain, by showing the average periodogram based on 10,000 replications of a simulated quarterly PI process (2) with $\phi_1=0.8$, $\phi_2=1$, $\phi_3=0.5$, $\phi_4=1/\left(\phi_1\phi_2\phi_3\right)$, and $\varepsilon_t\sim Niid\,(0,1)$. Clearly, Figure 1 shows spectral power at the zero, $\pi/2$ and π frequencies[6], so that the process exhibits zero frequency and seasonal behavior. As in del Barrio

[5] That is, writing (10)-(11) as $|A_0-A_1B|\,Y_\tau=adj\,(A_0-A_1B)\,E_\tau$, where $|\cdot|$ and $adj\,(\cdot)$ denote the determinat and the adjoint matrix respectivelly and B is the annual backshift (lag) operator such that $BY_\tau=Y_{\tau-1}$. In del Barrio Castro and Osborn (2008) it is also shown that for $adj\,(A_0-A_1B)=(\Theta_0+\Theta_1B)=D\,(B)$, it is possible to write $D\,(B)=\mathbf{ab}'$ with $\mathbf{a}$ and $\mathbf{b}$ defined in (14).

[6] It well know that a non-periodic $AR(1)$ $y_t=\phi y_{t-1}+\varepsilon_t$ has spectral power only around the zero and π frequencies with positive or negative values of ϕ, respectively (see for example Wei (2006, Figure 12.5). In order to obtain expectral power at an harmonic frequency, it will be necesary to move to an $AR(2)$ with complex conjugate roots. See, also the folowing sections.

Castro and Osborn (2008, section 2.2), the invertible constant-parameter representation associated with this process has the form of (20) with[7] $\theta_1 = 0.8497$, $\theta_2 = 0.5976$, $\theta_3 = 0.4133$, and $\sigma_\eta^2 = 2.8139$. The $MA(3)$ polynomial can be factorized as: $\left(1 + 0.8497L + 0.5976L^2 + 0.4133L^3\right) = (1 + 0.7703L)$ $(1 + [0.0397 + 0.7314i]\, L)\,(1 + [0.0397 - 0.7314i]\, L)$ and hence has no factor in common with the seasonal difference operator $\left(1 - L^4\right) = (1 - L)\,(1 + L)\,(1 - iL)\,(1 + iL)$. The spectral power in Figure 1 at the zero frequency is higher than at frequencies $\pi/2$ and π, due to the spectral power at the Nyquist frequency being lowered by the factor $(1 + 0.7703L)$ and that at frequency $\pi/2$ by the complex conjugate factors $(1 + [0.0397 \mp 0.7314i]\, L)$. Nevertheless, power is evident at the seasonal frequencies.

Although in principle it is always possible to compute the constant parameter representation associated with a PI process, for a quarterly process this implies solving a nonlinear system of four equations and evaluating which one of the eight solutions implies an invertible MA, as in section 2.2 of del Barrio Castro and Osborn (2008). The situation becomes more involved with periodicities higher than quarterly ($S = 4$), with monthly data ($S = 12$) requiring solving a nonlinear system of 12 equations. To avoid this, Appendix 1 provides a simpler method for examining the contributions of each zero and seasonal frequency to the overall spectral power of a PI process.

3 Circulant Matrices and the SI Test Regression

For the case of a seasonally integrated process, this section examines the properties of the standard regression used to test for seasonal unit roots. Following much of the recent literature, we employ circulant matrices to represent relevant quantities in the analysis.

3.1 Circulant matrices

A *circulant matrix* is a square matrix where the elements in each row are shifted by one place to the right compared with the row above. Hence, all elements of a circulant matrix can be defined in terms of the elements of its first row. A 4×4 circulant matrix $\mathbf{C}$ has the form

$$\mathbf{C} = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ c_4 & c_1 & c_2 & c_3 \\ c_3 & c_4 & c_1 & c_2 \\ c_2 & c_3 & c_4 & c_1 \end{bmatrix}.$$

If $\mathbf{C}$ is an $S \times S$ circulant matrix with first row $(c_1, c_2, c_3, \cdots, c_S)$, writing $\mathbf{C} = \mathrm{Ci}\,rc\,[c_1, c_2, c_3, \cdots, c_S]$ defines all elements of the matrix. Note that all diagonal elements of $\mathbf{C}$ take the value c_1. The identity matrix $\mathbf{I}_S$ and a matrix in which all elements are equal to unity are trivial examples of circulant matrices. The properties of circulant matrices are considered in detail by Davis (1979) and Gray (2006).

It is a property of circulant matrices that $\mathbf{C} = \mathbf{F}^* \Lambda \mathbf{F}$, where $\mathbf{F}$[8] is an $S \times S$ complex-valued matrix associated with the eigenvectors of $\mathbf{C}$, $\mathbf{F}^*$ is the conjugate transpose of $\mathbf{F}$ and $\Lambda = \mathbf{diag}\,[\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_S]$, where λ_j for $j = 1, 2, 3, \cdots, S$, are the eigenvalues of $\mathbf{C}$. Indeed $\mathbf{F}$ has the same (known) elements across all circulant matrices, so that such matrices differ only in their eigenvalues.

Circulant matrices have useful properties that can be exploited to establish the properties of SI processes. In particular, if $\mathbf{D}$ is another circulant matrix with $\mathbf{D} = \mathrm{Ci}\,rc\,[d_1, d_2, d_3, \cdots, , d_S]$ and hence $\mathbf{D} = \mathbf{F}^* \tilde{\Lambda} \mathbf{F}$ with $\tilde{\Lambda} = \mathbf{diag}\left[\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3, \cdots, \tilde{\lambda}_S\right]$, and k is a scalar constant, then (Davis, 1979, Theorem

[7] Rounding to the fourth decimal place.

[8] $\mathbf{F}$ is defined for all the circulant matrices of dimesion $S \times S$ as:

$$\mathbf{F} = \frac{1}{\sqrt{S}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-i\frac{2\pi}{S}} & e^{-i\frac{4\pi}{S}} & \cdots & e^{-i\frac{2(S-1)\pi}{S}} \\ 1 & e^{-i\frac{4\pi}{S}} & e^{-i\frac{8\pi}{S}} & \cdots & e^{-i\frac{4(S-1)\pi}{S}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-i\frac{2(S-1)\pi}{S}} & e^{-i\frac{2(S-1)\pi}{S}} & \cdots & e^{-i\frac{2(S-1)^2\pi}{S}} \end{bmatrix}.$$

Note that the matrices associated with the eigenvector of the circulant matrix, that is, $\mathbf{F}$ and $\mathbf{F}^*$, each form an orthogonal basis in the Fourier analysis.

3.2.4, and Gray, 2005, Theorem 3.1)

$$\begin{aligned} \mathbf{CD} &= \mathbf{F}^{*}\mathbf{diag}\left[\lambda_1\tilde{\lambda}_1, \lambda_2\tilde{\lambda}_2, \lambda_3\tilde{\lambda}_3, \cdots, \lambda_S\tilde{\lambda}_S\right]\mathbf{F} & (21) \\ \mathbf{C}+\mathbf{D} &= \mathbf{F}^{*}\mathbf{diag}\left[\lambda_1+\tilde{\lambda}_1, \lambda_2+\tilde{\lambda}_2, \lambda_3+\tilde{\lambda}_3, \cdots, \lambda_S+\tilde{\lambda}_S\right]\mathbf{F} & (22) \\ k\mathbf{C} &= \mathbf{F}^{*}\mathbf{diag}\left[k\lambda_1, k\lambda_2, k\lambda_3, \cdots, k\lambda_S\right]\mathbf{F}. & (23) \end{aligned}$$

3.2 SI Test regression

As already noted, the analysis of HEGY has stimulated a large econometrics literature relating to the analysis of seasonal unit root processes. For S seasons per year and S even, the seasonal unit root test procedure developed from HEGY is based on the following regression (see, for example, Smith, Taylor and del Barrio Castro, 2009, or del Barrio Castro, Osborn and Taylor, 2012):

$$\Delta_S y_t = \pi_0 y_{0,t-1} + \pi_{S/2} y_{S/2,t-1} + \sum_{k=1}^{S^*}\left(\pi_k^{\alpha} y_{k,t-1}^{\alpha} + \pi_k^{\beta} y_{k,t-1}^{\beta}\right) + \varepsilon_t \tag{24}$$

where $S^* = (S/2) - 1$ as in (3) and throughout the paper. For simplicity, the expression in (24) ignores any deterministic component. Further, augmentation $\sum_{j=1}^{p}\gamma_j\Delta_s y_{t-j}$ will be generally be included in this test regression in order to render ε_t white noise. Although this is important in practice, appropriate augmentation does not affect key features of the asymptotic distributions of the test statistics and is also omitted for simplicity of exposition.

The variables $y_{0,t}$, $y_{S/2,t}$ and $y_{k,t}^{\alpha}$, $y_{k,t}^{\beta}$ $(k = 1, ..., S^*)$ are defined as

$$\begin{aligned} y_{0,t} &= \sum_{j=0}^{S-1} y_{t-j}, \qquad y_{S/2,t} = \sum_{j=0}^{S-1}\cos[(j+1)\pi]y_{t-j} \\ y_{k,t}^{\alpha} &= \sum_{j=0}^{S-1}\cos[(j+1)\omega_k]y_{t-j}, \quad k = 1,2,..,S^* & (25) \\ y_{k,t}^{\beta} &= -\sum_{j=0}^{S-1}\sin[(j+1)\omega_k]y_{t-j}, \quad k = 1,2,\ldots,S^*. \end{aligned}$$

These are associated with the frequencies for which the characteristic equation (6) has unit roots: in particular, $y_{0,t}$ and $y_{S/2,t}$ are associated with the real roots at frequencies zero and π, respectively, while $y_{k,t}^{\alpha}$, $y_{k,t}^{\beta}$ are associated with the pairs of complex conjugate roots at frequencies $\omega_k = 2\pi k/S$ for $k = 1, 2, \ldots, S^*$. Note also that the variables $y_{0,t}$ and $y_{S/2,t}$ impose all unit roots in (3) except for the roots 1 and -1, respectively, while the pair $y_{k,t}^{\alpha}$ and $y_{k,t}^{\beta}$ impose all unit roots except for the complex pair of unit roots at frequency ω_k arising from $\left(1 - 2\cos(\omega_k)L + L^2\right)$. It is straightforward to see that for quarterly data ($S = 4$), (24) is the test regression proposed by HEGY (see expression (3.8) in HEGY, with no augmentation).

The values of the regressor variables in (24) can be collected into annual vectors, defined as: $Y_{0,\tau} = \left[y_{0,S(\tau-1)}, y_{0,S(\tau-1)+1}, \ldots, y_{0,S\tau-1}\right]'$, $Y_{S/2,\tau} = \left[y_{S/2,S(\tau-1)}, y_{S/2,S(\tau-1)+1}, \ldots, y_{S/2,S\tau-1}\right]'$, $Y_{k,\tau}^{\alpha} = \left[y_{k,S(\tau-1)}^{\alpha}, y_{k,S(\tau-1)+1}^{\alpha}, \ldots, y_{k,S\tau-1}^{\alpha}\right]'$ and $Y_{k,\tau}^{\beta} = \left[y_{k,S(\tau-1)}^{\beta}, y_{k,S(\tau-1)+1}^{\beta}, \ldots, y_{k,S\tau-1}^{\beta}\right]'$ for $k = 1, 2, \ldots, S^*$.

Notice these definitions incorporate the lagged values, so that the first element in the vector for year τ relates to the final period (S) of year $\tau - 1$.

Relating to these vectors, consider the circulant matrices

$$\begin{aligned} \mathbf{C}_0 &= \mathrm{Ci}\,rc\,[1,1,1,\cdots,1], \quad \mathbf{C}_{S/2} = \mathrm{Ci}\,rc\,[1,-1,1,\cdots,-1] \\ \mathbf{C}_k^{\alpha} &= \mathrm{Ci}\,\mathbf{rc}\left[\cos(0\omega_k), \cos(\omega_k), \cos(2\omega_k), \ldots, \cos([S-1]\omega_k)\right] & (26) \\ \mathbf{C}_k^{\beta} &= \mathrm{Ci}\,\mathbf{rc}\left[\sin(0\omega_k), \sin([S-1]\omega_k), \sin([S-2]\omega_k), \ldots, \sin(\omega_k)\right]. \end{aligned}$$

where $\mathbf{C}_0$ and $\mathbf{C}_{S/2}$ both have rank 1, while $\mathbf{C}_k^\alpha$ and $\mathbf{C}_k^\beta$ have rank 2. In effect, the elements of these circulant matrices associate the coefficients of lagged values y_{t-j} in the definitions of the variables of (25) with the appropriate season for the relevant regressor in (24).

For example, consider the regressor $y^\alpha_{k,t-1}$ in (24). The period $t = S(\tau - 1) + 1$, which corresponds to season 1 of year τ, has regressor $y^\alpha_{k,t-1} = y^\alpha_{k,S(\tau-1)} = y^\alpha_{k,(S,\tau-1)}$ where the last expression uses the double sunscript notation. From (25),

$$\begin{aligned} y^\alpha_{k,(S,\tau-1)} &= \cos(\omega_k) y_{(S,\tau-1)} + \cos(2\omega_k) y_{(S-1,\tau-1)} + ... + \cos(S\omega_k) y_{(1,\tau-1)} \\ &= \cos(0\omega_k) y_{(1,\tau-1)} + \cos(\omega_k) y_{(2,\tau-1)} + ... + \cos([S-1]\omega_k) y_{(S,\tau-1)}. \end{aligned}$$

Notice that the order of the lagged variables is reversed in the second line, which also uses the properties of the *cosine* function for $\omega_k = 2\pi k/S$: for example, $\cos(S\omega_k) = \cos(2\pi k) = 1 = \cos(0\omega_k)$ and $\cos([S-1]\omega_k) = \cos(2\pi k - [2\pi k/S]) = \cos(2\pi k/S) = \cos(\omega_k)$ as the *cosine* function is symmetric around $2\pi k$ for any integer k. Hence, except for "end effects" the transformation $\mathbf{C}_k^\alpha Y_\tau$ yields the elements of the vector $Y^\alpha_{k,\tau}$.

The relationship between the circulant matrices and the regressor variables of (24) is formalized by del Barrio Castro, Osborn, and Taylor (2012, Lemma 2) as

$$\begin{array}{ll} Y_{0,\tau} = \mathbf{C}_0 Y_\tau + 0_p(1) & Y_{S/2,\tau} = \mathbf{C}_{S/2} Y_\tau + 0_p(1) \\ \left.\begin{array}{l} Y^\alpha_{k,\tau} = \mathbf{C}_k^\alpha Y_\tau + 0_p(1) \\ Y^\beta_{k,\tau} = \mathbf{C}_k^\beta Y_\tau + 0_p(1) \end{array}\right\}, & k = 1, 2, \ldots, S^*. \end{array} \tag{27}$$

Since the circulant coefficient matrices in (27) are non-stochastic these relationships, together with the asymptotic distribution of the vector SI process in (8), can be used to obtain the asymptotic distributions of the HEGY auxiliary variables. The properties of circulant matrices, specifically (21), imply that the circulant matrices corresponding to different frequencies are orthogonal; that is,

$$\begin{aligned} \mathbf{C}_0 \mathbf{C}_{S/2} &= 0 \\ \mathbf{C}_0 \mathbf{C}_k^\alpha &= \mathbf{C}_0 \mathbf{C}_k^\beta = \mathbf{C}_{S/2} \mathbf{C}_k^\alpha = \mathbf{C}_{S/2} \mathbf{C}_k^\beta = 0 \\ \mathbf{C}_k^\alpha \mathbf{C}_j^\alpha &= \mathbf{C}_k^\beta \mathbf{C}_j^\beta = \mathbf{C}_k^\alpha \mathbf{C}_j^\beta = 0, \;\; k, j = 1, ..., S^*, \;\; k \neq j. \end{aligned} \tag{28}$$

Therefore, the HEGY variables at distinct frequencies are asymptotically uncorrelated. However, the matrices $\mathbf{C}_k^\alpha$ and $\mathbf{C}_k^\beta$ for a given seasonal frequency ω_k ($k = 1, ..., S^*$) are not orthogonal[9], which has implications for the distribution of t-ratio tests associated with $y^\alpha_{k,t-1}$ and $y^\beta_{k,t-1}$ in (24) when serial correlation is present or when the process does not contain all unit roots roots in Δ_S; see Burridge and Taylor (2001), del Barrio Castro (2007), Smith, Taylor and del Barrio Castro (2009), del Barrio Castro and Osborn (2011) and del Barrio Castro, Osborn and Taylor (2012). However, despite $\mathbf{C}_k^\alpha$ and $\mathbf{C}_k^\beta$ not being orthogonal, $Y^\alpha_{k,\tau}$ and $Y^\beta_{k,\tau}$ are asymptotically uncorrelated, as shown in Appendix 2 below[10]. From a theoretical perspective, $\mathbf{C}_k^\alpha$ and $\mathbf{C}_k^\beta$ not being orthogonal is inconvenient and obscures the roles of the S individual unit roots that apply in an SI process.

Since the unit roots at the seasonal frequencies ω_k for $k = 1, ..., S^*$ exist as complex conjugate pairs, examination of the individual roots at these frequencies involves the use of complex algebra. In order to consider these, del Barrio Castro, Rodrigues and Taylor (2018, 2019), del Barrio Castro and Rachinger (2021) and del Barrio Castro, Cubadda and Osborn (2022) employ transformations of $\mathbf{C}_k^\alpha$ and $\mathbf{C}_k^\beta$, specifically the complex-valued circulant matrices

$$\begin{aligned} \mathbf{C}_k^- &= \mathrm{Ci}\,\mathbf{rc}\left[1, e^{-i(S-1)\omega_k}, e^{-i(S-2)\omega_k}, \cdots, e^{-i\omega_k}\right] \\ \mathbf{C}_k^+ &= \mathrm{Ci}\,\mathbf{rc}\left[1, e^{i(S-1)\omega_k}, e^{i(S-2)\omega_k}, \cdots, e^{i\omega_k}\right]. \end{aligned} \tag{29}$$

[9] In fact we have $\mathbf{C}_k^\alpha \mathbf{C}_k^\beta = S/2\mathbf{C}_k^\beta$, note also that $\mathbf{C}_k^\alpha$ is symmetric, but for $\mathbf{C}_k^\beta$ we have $\left(\mathbf{C}_k^\beta\right)' = -\mathbf{C}_k^\beta$.

[10] We show that $Y^{\alpha\prime}_{k,\tau}$ and $Y^\beta_{k,\tau}$ are asymptotically orthogonal, as their scaled cross-product $\frac{1}{T^2}\sum_{s=1}^S \sum_{\tau=1}^N Y^{\alpha\prime}_{k,\tau-1} Y^\beta_{k,\tau-1} \Rightarrow 0$.

Using the properties of the sin and cos functions for $\omega_k = 2\pi k/S$ together with the identities $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and $e^{-i\theta} = \cos(\theta) - i\sin(\theta)$, these matrices are related to $\mathbf{C}_k^{\alpha}$ and $\mathbf{C}_k^{\beta}$ through the one-to-one identities

$$\begin{aligned}\mathbf{C}_k^{-} &= \mathbf{C}_k^{\alpha} - i\mathbf{C}_k^{\beta} \\ \mathbf{C}_k^{+} &= \mathbf{C}_k^{\alpha} + i\mathbf{C}_k^{\beta}.\end{aligned} \tag{30}$$

The matrices $\mathbf{C}_k^{-}$ and $\mathbf{C}_k^{+}$ are attractive in that

$$\mathbf{C}_k^{-}\mathbf{C}_j^{-} = \mathbf{C}_k^{+}\mathbf{C}_j^{+} = \mathbf{C}_k^{-}\mathbf{C}_j^{+} = 0,\ k, j = 1, ..., S^*.$$

As seen in subsection 4.3, $\mathbf{C}_k^{-}$ and $\mathbf{C}_k^{+}$ are useful when analyzing the properties associated with the individual complex roots that apply at seasonal frequencies for an SI process. Finally, matrices $\mathbf{C}_k^{-}$ and $\mathbf{C}_k^{+}$ also play an important role in our analysis in section 5.

4 SI Test Asymptotic Distributions

It is convenient to discuss the cases of tests for the single unit roots at the zero and Nyquist frequencies, before turning to the pairs of complex unit roots at the harmonic seasonal frequencies $\omega_k = 2\pi k/S$ for $S = 1, ..., S^*$.

4.1 Zero frequency test

Consider first the zero frequency unit root in the SI test regression (24), for which the corresponding regressor $y_{0,t-1}$ is defined in (25). Using the vector representation $Y_{0,\tau}$, the result in (27) together with the asymptotic distribution of the SI process Y_τ implies that

$$\begin{aligned}\frac{1}{\sqrt{N}}Y_{0,\lfloor Tr \rfloor} &\Rightarrow \mathbf{C}_0 W(r) = \sigma \mathbf{d}_0 \mathbf{d}_0' W(r) \\ &= \sigma S^{1/2}\mathbf{d}_0 w_0(r)\end{aligned} \tag{31}$$

where the $S \times 1$ vector $\mathbf{d}_0 = [1, 1, \cdots, 1]'$ and the scalar standard Brownian motion $w_0(r) = (S)^{-1/2}$ $\mathbf{d}_0' W(r) = S^{-1/2}\sum_{j=1}^{S} W_j(r)$[11]. Therefore, the scalar Brownian motion driving all elements of $Y_{0,\tau}$ is the (scaled) sum of the S independent Brownian motions driving the elements of the vector Y_τ. Since the innovations to each element of $W(r)$ have unit variance, those for the sum $\mathbf{d}_0' W(r)$ have variance S; hence $(S)^{-1/2}\,\mathbf{d}_0' W(r)$ is standard Brownian motion.

As will be seen, the general form of (31) carries over to the distributions of the variables associated with the individual unit root processes at each of seasonal frequency: that is, the process is driven by a single Brownian motion process which is formed as a scaled linear combination of the elements of $W(r)$. Also, the form of the linear combination, specifically the vector $\mathbf{d}_0'$ in (31), is determined by the form of the corresponding circulant matrix.

A zero frequency unit root test applied in the SI regression (24) tests $H_0 : \pi_0 = 0$ against the one-sided alternative $H_1 : \pi_0 < 0$. Due to the asymptotic orthogonality of the auxiliary variables, given by (28), the regressors related to the unit roots at seasonal frequencies are asymptotically irrelevant when the process is SI. Hence the asymptotic distributions relating to $\widehat{\pi}_0$ can be obtained by considering

$$\Delta_S y_t = \pi_0 y_{0,t-1} + \varepsilon_t.$$

Since $\Delta_S = (1-L)(1+L+...+L^{S-1})$, then $\Delta_S y_t = (1+L+...+L^{S-1})y_t - (1+L+...+L^{S-1})y_{t-S} = \Delta y_{0,t}$, this model can be written as

$$\Delta y_{0,t} = \pi_0 y_{0,t-1} + \varepsilon. \tag{32}$$

[11] This result, together with those below for the other HEGY variables, is established using circulant matrices for more general SI processes than the seasonal random walk of (1) by Smith, Taylor and del Barrio Castro (2009, Theorem 1) and del Barrio Castro, Osborn and Taylor (2012, Lemma 1).

The regression (32) is a Dickey-Fuller regression and the conventionally calculated t-ratio for $\widehat{\pi}_0$ under the null hypothesis asymptotically follows the Dickey-Fuller distribution (Dickey and Fuller, 1979), namely

$$t_{\pi_0} \Rightarrow \frac{\int w_0(r)dw_0(r)}{\sqrt{\int w_0^2(r)dr}}. \tag{33}$$

Hence the usual Dickey-Fuller t-test can be applied to test for a zero frequency unit root in the SI test regression (24). As already noted, this distribution also underlies the test for a unit root in a PI process.

The result in (33), namely that the usual Dickey-Fuller distribution applies for a test on π_0 in the SI test regression was shown formally by HEGY for the quarterly case, and by Rodrigues and Taylor (2004) and del Barrio Castro, Osborn and Taylor (2012), among others, for the general case of S seasons.

4.2 Nyquist frequency test

The relevant regressor variable in (24) for a test at the Nyquist frequency π is $y_{S/2,t-1}$, with vector equivalent $Y_{S/2,\tau}$. In particular, the relationship of (27) and the asymptotic distribution of (8) together imply that

$$\begin{aligned} \frac{1}{\sqrt{N}} Y^{S/2}_{\lfloor Tr \rfloor} &\Rightarrow \mathbf{C}_{S/2} W(r) = \sigma \mathbf{d}_{S/2} \mathbf{d}'_{S/2} W(r) \\ &= \sigma S^{1/2} \mathbf{d}_{S/2} w_{S/2}(r) \end{aligned} \tag{34}$$

where the $S \times 1$ vector $\mathbf{d}_{S/2} = [-1, 1, -1, \cdots, 1]'$ and $w_{S/2}(r) = S^{-1/2}\mathbf{d}'_{S/2}W(r)$ is scalar standard Brownian motion. It is also useful to note that, due to the properties of the cosine function, we can write $w_{S/2}(r) = (S)^{-1/2}\sum_{j=1}^{S}\cos(j\pi)W_j(r)$. Once again, the scaling $S^{-1/2}$ used in defining $w_{S/2}(r)$ is required to ensure the implied innovations have unit variance.

Now consider a process $x_{s/2,t}$ such that $x_{s/2,t} + x_{s/2,t-1} = \varepsilon_t$, which has a unit root only at the Nyquist frequency. A unit root test regression corresponding to this process would have the form

$$(1+L)x_{S/2,t} = \alpha_{S/2} x_{S/2,t-1} + \varepsilon_t \tag{35}$$

with the test of $H_0 : \alpha_{S/2} = 0$ considered against the one-sided alternative of stationarity, namely $H_1 : \alpha_{S/2} > 0$. The t-ratio for $\widehat{\alpha}_{S/2}$ in (37) under the null hypothesis has the asymptotic distribution of the form

$$t_{\alpha_{S/2}} \Rightarrow -\frac{\displaystyle\int w(r)\,dw(r)}{\sqrt{\displaystyle\int w(r)^2\,dr}} \tag{36}$$

where $w(r)$ is standard Brownian motion. The distribution in (35) is again the Dickey-Fuller distribution except for the minus sign, and hence it has a mirror image property in relation to the distribution of the zero frequency test statistic in (17). This property was pointed out in the relatively early literature concerned with seasonal unit roots, including Fuller (1996) and Chan and Wei (1988).

To relate this to the SI case, define

$$\Lambda_{S/2}(L) = \frac{\Delta_S}{(1+L)} = (1-L)\prod_{j=1}^{S^*}\left(1-e^{-i\omega_k}L\right)\left(1-e^{i\omega_k}L\right)$$

and note that, from (3), $\Delta_S = (1+L)\Lambda_{S/2}(L)$. Then $\Lambda_{S/2}(L)y_t = -y_{S/2,t}$, where $y_{S/2,t}$ is defined in (25). Due to the orthogonality properties of (28), the unit root test applied to $\alpha_{S/2}$ in (35) is asymptotically equivalent to a test of $\pi_{S/2} = 0$ in

$$(1+L)y_{S/2,t} = \pi_{S/2} y_{S/2,t-1} + \varepsilon_t. \tag{37}$$

Note that a minus sign is incorporated in the definition of $y_{S/2,t}$ compared with $x_{S/2,t}$ above, and hence the alternative hypothesis in (37) is $\pi_{S/2} < 0$ and the asymptotic distribution of $t_{\pi_{S/2}}$ for an SI process is the Dickey-Fuller distribution, or more precisely it is

$$t_{\pi_{S/2}} \Rightarrow \frac{\displaystyle\int w_{S/2}(r)\, dw_{S/2}(r)}{\sqrt{\displaystyle\int w_{S/2}(r)^2\, dr}} \tag{38}$$

where $w_{S/2}(r)$ is the standard Brownian motion of (34). The use of circulant matrices simplifies the derivation of (34) and hence the result in (38) compared with that of early studies of seasonal unit roots.

Both the zero and Nyquist frequency tests retain their asymptotic Dickey-Fuller distributions in the presence of serial correlation in the SI process, provided that the regression (24) is appropriately augmented.

4.3 Tests at other seasonal frequencies

In order to facilitate the discussion of unit root tests at the seasonal frequencies $\omega_k = 2k\pi/S$ for $S = 1, ..., S^*$, this subsection first considers some properties of complex-valued unit root processes.

4.3.1 Complex-valued unit root processes

Complex-valued unit root processes play a key role in the analysis of SI processes and the properties of such processes are considered in detail in a series of papers by Gregoir (1999, 2006, 2010).

In order to concentrate on the principal issues, consider the simplest case which gives rise to a pair of complex-valued unit roots, namely where a real-valued process x_t^k is given by

$$[1 - 2\cos(\omega_k)L + L^2]x_t^k = (1 - e^{-i\omega_k}L)(1 - e^{i\omega_k}L)x_t^k = \varepsilon_t \tag{39}$$

in which $\varepsilon_t \sim iid(0, \sigma^2)$. In order to focus on one of the pair of unit roots, define $x_t^{k-} = (1 - e^{i\omega_k}L)x_t^k$, so that (39) implies

$$x_t^{k-} = (1 - e^{i\omega_k}L)x_t^k = e^{-i\omega_k}x_{t-1}^{k-} + \varepsilon_t. \tag{40}$$

Clearly x_t^{k-} in (40) is complex-valued. Further, it has a single unit root at frequency ω_k, and hence $x_t^{k-} \sim I_{\omega_k}(1)$.

Recursive substitution in (40) shows that

$$\begin{aligned} x_t^{k-} &= e^{-i\omega_k} \sum_{j=1}^{t} e^{i\omega_k j}\varepsilon_j \\ &= e^{-i\omega_k t}x_t^{(0)-} \end{aligned} \tag{41}$$

where, for simplicity, we assume $x_0^{k-} = 0$ and we also define

$$x_t^{(0)-} = e^{i\omega_k t}x_t^{k-} = \sum_{j=1}^{t} e^{i\omega_k j}\varepsilon_j. \tag{42}$$

Examining the form of $x_t^{(0)-}$, note that (42) implies that it satisfies

$$x_t^{(0)-} = x_{t-1}^{(0)-} + e^{i\omega_k t}\varepsilon_t. \tag{43}$$

Because $e^{i\omega_k t}\varepsilon_t$ is stationary, (43) shows that $x_t^{(0)-}$ is integrated of order one at the zero frequency, $x_t^{(0)-} \sim I_0(1)$.

The relationship (41) is a key one, since it shows that multiplication by $e^{-i\omega_k t}$ converts the (complex-valued) process $x_t^{(0)-}$ integrated at the zero frequency, $x_t^{(0)-} \sim I_0(1)$, to $x_t^{k-} \sim I_{\omega_k}(1)$. The operator $e^{-i\omega_k t}$ is known as the *demodulation operator* and it is valuable because it means that the properties of (conventional) zero-frequency unit root processes can be exploited when considering unit roots at seasonal frequencies. Clearly, demodulation can also be used to convert from the zero frequency to the ω_k frequency, through $x_t^{(0)} = e^{i\omega_k t}x_t^{k-}$.

Example

Say data is observed monthly, but is generated by the process

$$[1-2\cos(\omega_4)L+L^2]x_t^4=(1-e^{-i\omega_4}L)(1-e^{i\omega_4}L)x_t^4=\varepsilon_t$$

where $\omega_4=2\times 4\pi/12=2\pi/3$. As noted in Section 2.1, the pair of unit roots at frequency ω_4 correspond to non-stationary cycles of three months duration within the twelve months of annual data. The real-valued process x_t^4 has two complex-valued unit roots at frequency ω_4.

To focus on one unit root, x_t^4 is transformed to the complex-valued series $x_t^4\sim I_{2\pi/3}(1)$ using

$$\begin{aligned} x_t^{4-} &= (1-e^{i\omega_4}L)x_t^4=(1-e^{(2\pi/3)i}L)x_t^4 \\ &= x_t^4-[\cos(2\pi/3)+i\sin(2\pi/3)]x_{t-1}^4. \end{aligned}$$

Applying the demodulator operator of (42) to x_t^{4-} yields the complex-valued series $x_t^{(0)-}=e^{i\omega_4 t}x_t^{4-}=[\cos(2\pi/3)+i\sin(2\pi/3)]x_t^{4-}$, which is integrated at the zero frequency.

Returning to general x_t^{k-}, the innovation $e^{i\omega_k t}\varepsilon_t$ to the process (43) is complex-valued and, as the sample size T increases, the asymptotic distribution of $x_t^{(0)-}$ has the form

$$\frac{1}{\sqrt{T}}x_t^{(0)-}\Rightarrow\left(\sigma^2/2\right)^{1/2}\left[w_R^k(r)+iw_I^k(r)\right] \tag{44}$$

in which $w_R^k(r)$ and $w_I^k(r)$ are independent standard Brownian motion processes. In (44), $w_R^k(r)$ and $w_I^k(r)$ are distinct (real) Brownian motion processes and $w_R^k(r)+iw_I^k(r)$ is a single complex-valued Brownian motion process. The implication of (44) is that the complex-valued series $x_t^{(0)-}$ asymptotically behaves as a scalar complex-valued Brownian motion process.

Considering again the real-valued process x_t^k of (39) and multiplying by $(1-e^{i\omega_k}L)$ yields the complex conjugate process to (40), namely

$$x_t^{k+}=(1-e^{-i\omega_k}L)x_t^k=e^{i\omega_k t}x_{t-1}^{k+}+\varepsilon_t \tag{45}$$

where $x_t^{k+}\sim I_{\omega_k}(1)$. Following the same line of reasoning as above, this can be transformed to $x_t^{(0)+}\sim I_0(1)$ with $x_t^{(0)+}=x_{t-1}^{(0)+}+e^{-i\omega_k t}\varepsilon_t$, with $x_t^{(0)-}$ and $x_t^{(0)+}$ also forming a complex conjugate pair.

For the situation where x_t^k is observed S times a year, define the vectors $X_\tau^{k-}=[x_{S(\tau-1)+1}^{k-},....,x_{S\tau}^{k-}]'$ and $X_\tau^{k+}=[x_{S(\tau-1)+1}^{k+},....,x_{S\tau}^{k+}]'$. Then (44) implies that the asymptotic distributions of X_τ^{k-} and X_τ^{k+} are given by

$$\begin{aligned} \frac{1}{\sqrt{N}}X_{\lfloor Nr\rfloor}^{k-} &\Rightarrow\sigma\left(S/2\right)^{1/2}\mathbf{C}_k^{-}W(r)=\sigma\left(S/2\right)^{1/2}\mathbf{d}_{\omega_k}^{-}\mathbf{d}_{\omega_k}^{+\prime}W(r) \\ &=\sigma\left(S/2\right)^{1/2}\mathbf{d}_{\omega_k}^{-}\left[w_R^k(r)+iw_I^k(r)\right] \end{aligned}\tag{46}$$

and

$$\begin{aligned} \frac{1}{\sqrt{N}}X_{\lfloor Tr\rfloor}^{k+} &\Rightarrow \sigma\left(S/2\right)^{1/2}\mathbf{C}_k^{+}W(r)=\sigma\left(S/2\right)^{1/2}\mathbf{d}_{\omega_k}^{+}\mathbf{d}_{\omega_k}^{-\prime}W(r) \\ &= \sigma\left(S/2\right)^{1/2}\mathbf{d}_{\omega_k}^{+}\left[w_R^k(r)-iw_I^k(r)\right] \end{aligned}\tag{47}$$

where $\mathbf{C}_k^-$ and $\mathbf{C}_k^+$ are defined in (29), $\mathbf{d}_{\omega_k}^-$ and $\mathbf{d}_{\omega_k}^+$ are $S\times 1$ vectors defined as $\mathbf{d}_{\omega_k}^-=\begin{bmatrix} e^{-i\omega_k} & e^{-i2\omega_k} & e^{-i3\omega_k} & \cdots & e^{-iS\omega_k}\end{bmatrix}'$and $\mathbf{d}_{\omega_k}^+=\begin{bmatrix} e^{i\omega_k} & e^{i2\omega_k} & e^{i3\omega_k} & \cdots & e^{iS\omega_k}\end{bmatrix}'$ respectively, while $[w_R(r)+iw_I(r)]$ and $\left[w_R^k(r)-iw_I^k(r)\right]$ are scalar complex-valued standard Brownian motions with $\left[w_R^k(r)\pm iw_I^k(r)\right]=(S/2)^{-1/2}\mathbf{d}_{\omega_k}^{\pm\prime}W(r)$. Note that $w_R^k(r)=(S/2)^{-1/2}\sum_{j=1}^{S}\cos[j\omega_k]W_j(r)$ and $w_I^k(r)=(S/2)^{-1/2}\sum_{j=1}^{S}\sin[j\omega_k]W_j(r)$, where $W_j(r)$ is the j^{th} element of $W(r)$.

The key features of (46) and (47) are that one of the complex conjugate pair of scalar Brownian motions separately drives each of these two vector processes X_τ^{k-} and X_τ^{k+}, with the form of these relationships being analogous to (31) and (34) related to the zero and Nyquist frequency, respectively. The zero frequency unit roots $\left[w_R^k(r)\pm iw_I^k(r)\right]$ are shifted in (46) and (47) to unit roots at frequency ω_k through demodulation, achieved using the vectors $\mathbf{d}_{\omega_k}^-$ and $\mathbf{d}_{\omega_k}^+$, with the cyclical property of $e^{\mp i\omega_k t}$ (with a cycle every S/k periods) implying that $e^{\mp i\omega_k t}=e^{\mp i\omega_k[t\pm(S/k)]}$. The scaling $S^{1/2}$ enters the right-hand side of (46) and (47) due to the distribution being considered in relation to the number of years N, rather than the number of observations T in (44).

4.3.2 Real-valued process

The real-valued process x_t^k of (39) involves both x_t^{k-} and x_t^{k+}, with the S elements of the real-valued annual vector $X_\tau^k = [x_{S(\tau-1)+1}, ...x_{S\tau}]'$ being driven by two complex Brownian motion processes, namely $w_R^k(r) \pm iw_I^k(r)$. In particular, using the partial fraction decomposition (see Gregoir, 1999, and Tanaka, 2008),

$$\begin{aligned}\frac{1}{(1-2\cos(\omega_k)z+z^2)} &= -\frac{e^{-i\omega_k}}{2i\sin(\omega_k)}\frac{1}{(1-e^{-i\omega_k}z)} \\ &\quad +\frac{e^{i\omega_k}}{2i\sin(\omega_k)}\frac{1}{(1-e^{-i\omega_k}z)}\end{aligned}$$

and hence, from (39),

$$x_t^k = -\frac{e^{-i\omega_k}}{2i\sin(\omega_k)}x_t^{k-} + \frac{e^{i\omega_k}}{2i\sin(\omega_k)}x_t^{k+}.$$

Therefore, from (46) and (47), the asymptotic distribution of X_τ^k is given by

$$\begin{aligned}\frac{1}{\sqrt{N}}X_{\lfloor Nr\rfloor}^k \Rightarrow & -\frac{e^{-i\omega_k}}{2i\sin(\omega_k)}\sigma\left(\frac{S}{2}\right)^{1/2}\mathbf{d}_{\omega_k}^{-}\left[w_R^k(r)+iw_I^k(r)\right] \\ & +\frac{e^{i\omega_k}}{2i\sin(\omega_k)}\sigma\left(\frac{S}{2}\right)^{1/2}\mathbf{d}_{\omega_k}^{+}\left[w_R^k(r)-iw_I^k(r)\right].\end{aligned} \tag{48}$$

Although the asymptotic distributions of the least squares estimators in a seasonal unit root process like (39) are analyzed by Ahtola and Tiao (1987), Chan and Wei (1988), and Tanaka (2008), use of (48) permits a more straightforward derivation of these distributions than used by earlier authors. A test regression corresponding to (39) is

$$[1-2\cos(\omega_k)L+L^2]x_t^k = [\phi_1 - 2\cos(\omega_k)]x_{t-1}^k + (\phi_2+1)x_{t-2}^k + \varepsilon_t. \tag{49}$$

The hypothesis $H_0 : \phi_2 + 1 = 0$ is considered against the alternative $H_a : \phi_2 + 1 > 0$ to test the null of a pair of complex conjugate unit roots, while $H_0 : \phi_1 - 2\cos(\omega_k) = 0$ versus $H_a : \phi_1 - 2\cos(\omega_k) \neq 0$ is a test of the frequency allocation of the unit roots. The t-ratio statistics have asymptotic distributions

$$\begin{aligned} t_{\hat{\phi}_1-2\cos(\omega_k)} &\Rightarrow \left[\frac{\sin(\omega_k)\left(\int w_R^k(r)\,dw_I^k(r) - \int w_I^k(r)\,dw_R^k(r)\right)}{\sqrt{\int\left[w_R^k(r)\right]^2dr + \int\left[w_I^k(r)\right]^2dr}}\right. \\ &\quad \left. +\frac{\cos(\omega_k)\left(\int w_R^k(r)\,dw_R^k(r) + \int w_I^k(r)\,dw_I^k(r)\right)}{\sqrt{\int\left[w_R^k(r)\right]^2dr + \int\left[w_I^k(r)\right]^2dr}}\right] \\ t_{\hat{\phi}_2+1} &\Rightarrow -\frac{\int w_R^k(r)\,dw_R^k(r) + \int w_I^k(r)\,dw_I^k(r)}{\sqrt{\int\left[w_R^k(r)\right]^2dr + \int\left[w_I^k(r)\right]^2dr}}.\end{aligned} \tag{50}$$

Clearly, the real and complex parts of the two underlying complex-valued Brownian motion processes enter both asymptotic distributions. Dickey, Haza and Fuller (1984) obtain the distribution of $-t_{\hat{\phi}_2+1}$ in (50) for a seasonal unit root test with two seasons per year. This distribution is also reported by del Barrio Castro and Sanso (2015)[12] and they show that the distribution of the joint F-type test for the null of $\phi_1 - 2\cos(\omega_k) = 0$ and $\phi_2 + 1 = 0$ is given by (see expression (3.4) in their proposition 3.1)

$$F_k = \frac{\left(\int w_R^k(r)\,dw_R^k(r) - \int w_I^k(r)\,dw_I^k(r)\right)^2 + \left(\int w_R^k(r)\,dw_R^k(r) + \int w_I^k(r)\,dw_I^k(r)\right)^2}{2\left(\int\left[w_R^k(r)\right]^2dr + \int\left[w_I^k(r)\right]^2dr\right)}. \tag{51}$$

[12]See expressions (3.2) and (3.3) in proposition 3.1 of del Barrio Castro and Sansó Rossello (2015). Note that these expressions are more general than those reported in (50) as they allow for serial correlation and appropriate augmentation in the test regression.

Early discussions of unit root tests at the harmonic seasonal frequencies ω_k for $k = 1, ..., S^*$, including HEGY, were based on (49) and often involved using a sequence of t-type tests relating to, firstly, ϕ_2 and then ϕ_1. However, as these test statistics are correlated, more recent analyses have focussed on a representation transformation which yields uncorrelated test statistics. More specifically, Smith and Taylor (1999) employ the regression

$$[1 - 2\cos(\omega_k) L + L^2]x_t^k = \pi_k^\alpha[-\cos(\omega_k)x_{t-1}^k + x_{t-2}^k] + \pi_k^\beta \sin(\omega_k)x_{t-2}^k + \varepsilon_t \tag{52}$$

for which the unit root null hypothesis is $\pi_k^\alpha = \pi_k^\beta = 0$. Unlike those in (49), the regressors in (52) are asymptotically orthogonal.

As shown by del Barrio Castro and Sansó Rossello (2015), the connection between the distributions of the t-ratio tests t_{α_k} and t_{β_k} of the transformed variables and the t-ratio tests $t_{\hat{\phi}_1 - 2\cos(\omega_k)}$ and $t_{\hat{\phi}_2+1}$ in (50) is

$$\begin{bmatrix} t_{\pi_k^\alpha} \\ t_{\pi_k^\beta} \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ \frac{1}{\sin(\omega_k)} & \cos(\omega_k) \end{bmatrix} \times \begin{bmatrix} t_{\hat{\phi}_1 - 2\cos(\omega_k)} \\ t_{\hat{\phi}_2+1} \end{bmatrix}.$$

Hence, for the simple $SI(1)$ process of (1) with no serial correlation, the asymptotic distributions for the t-ratios associated with the coefficients in (54) are

$$\begin{aligned} t_{\pi_k^\alpha} &\Rightarrow \frac{\left[\int w_R^k(r)\, dw_R^k(r) + \int w_I^k(r)\, dw_I^k(r)\right]}{\sqrt{\int \left[w_R^k(r)\right]^2 dr + \int \left[w_I^k(r)\right]^2 dr}} \\ t_{\pi_k^\beta} &\Rightarrow \frac{\left[\int w_I^k(r)\, dw_R^k(r) - \int w_R^k(r)\, dw_I^k(r)\right]}{\sqrt{\int \left[w_R^k(r)\right]^2 dr + \int \left[w_I^k(r)\right]^2 dr}}. \end{aligned} \tag{53}$$

The roles of the asymptotic distributions for the partial sum variables associated with each of the complex pair of roots, (46) and (47) are clear in these expressions. Indeed, the distributions of (50) can be viewed as analogous to those of (33) and (38) for the zero and Nyquist frequencies, respectively, except that the distributions in (50) relate to a scalar complex standard Brownian motion rather than the real standard Brownian motions at 0 and π frequencies. Further, as the distributions of $t_{\pi_k^\alpha}$ and $t_{\pi_k^\beta}$ are asymptotically uncorrelated, a joint F-type test F_k of the null $\alpha_k = \beta_k = 0$ has the same asymptotic distribution as given in (51).

4.3.3 SI test regression

To relate the above discussion to the SI test regression (24), the pair of unit roots at the seasonal frequency ω_k can be isolated by noting that

$$\Lambda_k(L) = \frac{\Delta_S}{(1 - 2\cos(\omega_k) L + L^2)} = (1-L)(1+L) \prod_{\substack{j=1 \\ j \neq k}}^{S^*} \left(1 - e^{-i\omega_j} L\right)\left(1 - e^{i\omega_j} L\right)$$

and hence, from (3),

$$\Lambda_k(L)[1 - 2\cos(\omega_k) L + L^2] = (1 - L^S).$$

Therefore, defining[13] $y_{k,t} = \Lambda_k(L)y_t$, the SI null hypothesis implies that the process $y_{k,t}$ of (25) is asymptotically equivalent to x_t^k in (39).

Also, due to the asymptotic orthogonality of the regressors and under the SI null hypothesis, testing for a pair of complex unit roots at the seasonal frequency ω_k in the HEGY test regression (24) is asymptotically equivalent to testing $\pi_k^\alpha = 0$ and $\pi_k^\beta = 0$ in (52). From (52) and due to the asymptotic equivalence of x_t^k and $y_{k,t}$, we can therefore consider

$$\begin{aligned} [1 - 2\cos(\omega_k) L + L^2]y_{k,t} &= \pi_k^\alpha[-\cos(\omega_k)y_{k,t-1} + y_{k,t-2}] + \pi_k^\beta \sin(\omega_k)y_{k,t-2} + \varepsilon_t \\ &= \pi_k^\alpha y_{k,t-1}^\alpha + \pi_k^\beta y_{k.t-1}^\beta + \varepsilon_t \end{aligned} \tag{54}$$

[13] Note that we define $\Lambda_{S/2}(L) = \Delta_S/(1+L)$ and $\Lambda_k(L) = \Delta_S/\left(1 - 2\cos(\omega k) L + L^2\right)$ for $k = 1, 2, \ldots, S^*$, which are the operators denoted $\Delta_{S/2}(L)$ and $\Delta_k(L)$, respectively, by Smith and Taylor (1999) and del Barrio Castro and Sansó (2015); that is by $\Delta_{S/2}(L) = -\Lambda_{S/2}(L)$ and $\Delta_k(L) = -\Lambda_k(L)$.

with regressors $y^{\alpha}_{k,t-1} = -\cos(\omega_k)y_{k,t-1} + y_{k,t-2}$ and $y^{\beta}_{k,t-1} = \sin(\omega_k)y_{k,t-1}$. Smith and Taylor (1999, equation 2.17) show that $y^{\alpha}_{k,t}$ and $y^{\beta}_{k,t}$ are equal to $y^{\alpha}_{k,t}$ and $y^{\beta}_{k,t}$ defined in (25).

As already noted, the SI test regression (24) requires augmentation to account for serial correlation in a general SI process. However, as shown by Burridge and Taylor (2001), del Barrio Castro and Osborn (2011), and del Barrio Castro, Osborn, and Taylor (2012), the t-ratio tests $t_{\pi^{\alpha}_k}$ and $t_{\pi^{\beta}_k}$ for $k = 1, 2, \ldots, S^*$ in the presence of serial correlation have asymptotic distributions which depend on nuisance parameters. Hence the distributions (53) apply only in very special circumstances. Nevertheless, the F-type tests F_k for $k = 1, 2, \ldots, S^*$ remain pivotal (that is, their distribution is free from nuisance parameters) when correctly augmented with asymptotic distribution given by (51). Hence, it is better in practice to use this joint F_k test associated with the presence of a pair of complex conjugates at frequency $\omega_k = 2\pi k/S$ for $k = 1, 2, \ldots, S^*$ than the two t-ratio tests $t_{\pi^{\alpha}_k}$ and $t_{\pi^{\beta}_k}$.

4.4 General comments

Note that, despite all the scalar Brownian motions appearing in (31), (34), (46) and (47) being defined as linear combinations of the elements of the same Brownian motion vector $W(r)$, they are independent from each other, as the coefficients of the linear combinations are based on the elements of vectors of dimension $S\times 1$ that are orthogonal. Also, the circulant matrices C_0, $C_{S/2}$, C^{α}_k, and C^{β}_k for $k = 1, 2, \ldots, S^*$ represent the transformations used to construct the regressor variables of the SI test regression (24) and, as already noted, the circulant matrices corresponding to different frequencies are orthogonal to one another.

Although tests for unit roots are often considered separately at each of the zero and seasonal frequencies, the implied level of significance needs to be carefully controlled when joint inference is being made about about the presence of unit roots across frequencies. A straightforward way to control the level of significance is to use of F-type tests when testing the overall null hypothesis of SI or the null hypothesis of unit roots at all seasonal frequencies, the latter allowing a possible zero frequency unit root. An implication of the orthogonality relationships noted in the previous paragraph is that the distributions of these test statistics can be immediately obtained. Under the null hypothesis, a joint test of the overall SI null hypothesis, namely $H_0 : \pi_0 = \pi_{S/2} = \pi^{\alpha}_1 = \pi^{\beta}_1 = ... = \pi^{\alpha}_{S^*} = \pi^{\beta}_{S^*} = 0$ in (24), has the asymptotic distribution

$$F_{01\ldots S} = \frac{1}{S}\left\{(t_{\pi_0})^2 + (t_{\pi_{S/2}})^2 + \sum_{k=1}^{S^*}[(t_{\pi^{\alpha}_k})^2 + (t_{\pi^{\beta}_k})^2]\right\} \tag{55}$$

where t_{π_0} and $t_{\pi_{S/2}}$ are as in (33) and (38), respectively, and $t_{\pi^{\alpha}_k}$, $t_{\pi^{\beta}_k}$ $(k = 1, ..., S^*)$ in (53). Similarly, a joint F-type test of the null hypothesis that the process has unit roots at all seasonal frequencies $\omega_k = 2\pi k/S$ for $S = 1, ..., S^*$, corresponding to $H_0 : \pi_{S/2} = \pi^{\alpha}_1 = \pi^{\beta}_1 = ... = \pi^{\alpha}_{S^*} = \pi^{\beta}_{S^*} = 0$ in (24), has the asymptotic distribution

$$F_{seas} = \frac{1}{(S-1)}\left\{(t_{\pi_{S/2}})^2 + \sum_{k=1}^{S^*}[(t_{\pi^{\alpha}_k})^2 + (t_{\pi^{\beta}_k})^2]\right\}. \tag{56}$$

Another practical consideration when testing for unit roots in a seasonal context is whether the distributions discussed above relating to individual (zero and seasonal) frequencies continue to apply when not all unit roots are present, in other words when the underlying process does not have the SI form. Reassuringly, Smith, Taylor and del Barrio Castro (2009) show that, with appropriate augmentation for serial correlation, the asymptotic distributions for t_{π_0} and $t_{\pi_{S/2}}$ in (33) and (38) continue to apply in this situation, as does the asymptotic distribution of F_k in (51) for testing a pair of complex unit roots at the seasonal frequency ω_k. However, the distributions of (53) for the individual regressors $y^{\alpha}_{k,t-1}$ and $y^{\beta}_{k,t-1}$ do not hold in this case, so that the joint F-type test should always be applied to test for a unit root at a seasonal frequency ω_k, $k = 1, ..., S^*$.

If a deterministic part is included in (24), such as a constant, seasonal dummies, a constant and a linear trend, seasonal dummies and a linear trend, or seasonal dummies and trends, the scalar Brownian motions $w_0(r)$, $w_{S/2}(r)$, $w^k_R(r)$, and $w^k_I(r)$ for $k = 1, 2, \ldots, S^*$ are replaced by de-meaned or de-meaned and de-trended Brownian motion, as explained in Smith, Taylor, and del Barrio Castro (2009), for example. The use of GLS de-trending, mean recursive de-trending, etc., in the HEGY approach is analyzed by Rodrigues and Taylor (2004, 2007) and Taylor (2002), while del Barrio Castro, Osborn and Taylor (2016) compare the performances of various approaches to detrending and determining the appropriate lag length for testing SI in the presence of autocorrelation. Finally, the use of bootstrap techniques to deal with volatility in the SI regression is analyzed by Burridge and Taylor (2004) and Cavaliere, Skrobotov, and Taylor (2019).

5 Testing for Seasonal Unit Roots in PI Processes

In the light of the key differences in the unit root properties of SI and PI processes, it is relevant to ask what the consequences are of a potential misspecification of the form of the process under investigation. Except for deterministic seasonal effects, the analysis of seasonal processes is typically undertaken assuming the parameters are constant over the seasons. Hence it is important to ask what the consequences are of applying seasonal unit root tests to a PI process.

This situation is analyzed by Boswijk and Franses (1996) for the Dickey, Hasza and Fuller (1984) test and extended by del Barrio Castro and Osborn (2008) to the HEGY procedure in the quarterly case ($S = 4$) for an integrated $PAR(1)$ process. Essentially, del Barrio Castro and Osborn (2008) use the PI process asymptotic distribution (15) and the circulant matrices associated with the SI test auxiliary variables (26) to obtain the distributions of the auxiliary variables when applied to a PI process. For the PI process of (2), which has S seasons and vector representation given by (10), these asymptotic distributions (after appropriate scaling) are:

$$\begin{aligned}\frac{1}{\sqrt{N}}\mathbf{C}_0 Y_{\lfloor Nr\rfloor} &\Rightarrow \sigma\varpi\mathbf{C}_0\mathbf{a}w_p(r)=\sigma\varpi\mathbf{d}_0\mathbf{d}_0'\mathbf{a}w_p(r)\\ &= \sigma\varpi\left(\sum_{j=1}^{S}\mathbf{a}_j\right)\mathbf{d}_0 w_p(r)\end{aligned} \tag{57}$$

$$\begin{aligned}\frac{1}{\sqrt{N}}\mathbf{C}_{S/2} Y_{\lfloor Nr\rfloor} &\Rightarrow \sigma\varpi\mathbf{C}_{S/2}\mathbf{a}w_p(r)=\sigma\varpi\mathbf{d}_{S/2}\mathbf{d}_{S/2}'\mathbf{a}w_p(r)\\ &= \sigma\varpi\left(\sum_{j=1}^{S}\cos(\pi j)\,\mathbf{a}_j\right)\mathbf{d}_{S/2} w_p(r)\end{aligned} \tag{58}$$

$$\begin{aligned}\frac{1}{\sqrt{N}}\mathbf{C}_k^{\alpha} Y_{\lfloor Nr\rfloor} &\Rightarrow \sigma\varpi\mathbf{C}_k^{\alpha}\mathbf{a}w_p(r)=\sigma\varpi\frac{1}{2}\left(\mathbf{C}_k^{+}+\mathbf{C}_k^{-}\right)\mathbf{a}w_p(r)\\ &= \sigma\varpi\frac{1}{2}\mathbf{d}_k^{+}\mathbf{d}_k^{-\prime}\mathbf{a}w_p(r)+\sigma\varpi\frac{1}{2}\mathbf{d}_k^{-}\mathbf{d}_k^{+\prime}\mathbf{a}w_p(r)\\ &= \sigma\varpi\left(\sum_{j=1}^{S}\cos(\omega_k j)\,\mathbf{a}_j\right)\mathbf{R}_k^{\alpha} w_p(r)\\ &\quad +\sigma\varpi\left(\sum_{j=1}^{S}\sin(\omega_k j)\,\mathbf{a}_j\right)\mathbf{R}_k^{\beta} w_p(r)\end{aligned} \tag{59}$$

$$\begin{aligned}\frac{1}{\sqrt{N}}\mathbf{C}_k^{\beta} Y_{\lfloor Nr\rfloor} &\Rightarrow \sigma\varpi\mathbf{C}_k^{\beta}\mathbf{a}w_p(r)\\ &= \sigma\varpi\left(\sum_{j=1}^{S}\cos(\omega_k j)\,\mathbf{a}_j\right)\mathbf{R}_k^{\beta} w_p(r)\\ &\quad -\sigma\varpi\left(\sum_{j=1}^{S}\sin(\omega_k j)\,\mathbf{a}_j\right)\mathbf{R}_k^{\alpha} w_p(r)\end{aligned} \tag{60}$$

where ϖ is a scaling term, $\mathbf{a}$ and $w_p(r)$ relate to the PI process, see (14) and (15), while all the remaining quantities in (57)-(60) effectively carry over from the analysis of SI processes in the preceding section. In particular, the circulant matrices are defined in (26) and (29), while $\mathbf{R}_k^{\alpha}$ and $\mathbf{R}_k^{\beta}$ are defined as $\mathbf{R}_k^{\alpha} = 1/2\left(\mathbf{d}_k^{+} + \mathbf{d}_k^{-}\right)$ $\mathbf{R}_k^{\alpha} = 1/(2i)\left(\mathbf{d}_k^{+} + \mathbf{d}_k^{-}\right)$, and finally note that they are also defined in Appendix 1 (see (63)).

The key conclusion from (57)-(60) us that the distribution of all the auxiliary variables used in the SI test regressions are functions of the single scalar Brownian motion $w_p(r)$. This scalar Brownian motion is associated with the common stochastic trend that is shared by all the seasons of the PI process. Hence, unlike the case for an SI process discussed above, the distributions of the test statistics are not asymptotically uncorrelated when the SI tests are applied to a PI process.

Based on distributional results of the form of (57)-(60) for quarterly data (see Lemma 3 in del Barrio Castro and Osborn, 2008), del Barrio Castro and Osborn (2008, Theorems 2 and 3) obtain the asymptotic distributions of the t-ratios associated with $y_{0,t}$ and $y_{S/2,t}$ and the F-type test associated with the joint exclusion of $y_{k,t}^{\alpha}$ and $y_{k,t}^{\beta}$ in (24) when the true process is PI. Not surprisingly, the distributions for t_{π_0} and $t_{\pi_{S/2}}$ have the form of (33), which is the usual t^{DF} distribution for a zero frequency unit root test, while the F-type test associated with the joint exclusion of $y_{k,t}^{\alpha}$ and $y_{k,t}^{\beta}$ has an asymptotic distribution of the form $\left(t^{DF}\right)^2$. However, because the single Brownian motion process $w_p(r)$ underlies the asymptotic distributions for the tests at the zero and all seasonal frequencies, effectively the same unit root is being examined at all these frequencies. Therefore, when the true process is PI, the asymptotic distributions for joint F-type tests discussed in subsections 4.3 for 4.4 for SI processes will not apply.

The analysis of del Barrio Castro and Osborn (2008) employs the constant parameter representation of a PI process, which is discussed above in subsection 2.3. Due to the $MA(S-1)$ component present in this representation, as in (20), the theoretical analysis assumes appropriate augmentation in the SI test regression in order to capture the resulting autocorrelation in the (misspecified) seasonally differenced process. In practice, however, augmentation is an empirical matter and the roots of the invertible MA in the constant parameter representation may be relatively close to the corresponding unit roots of the AR seasonal difference, leading to less spectral power at the seasonal frequencies than at the zero frequency, as in the example of subsection 2.3.

Hence, we can conclude that a unit root at the zero frequency is very likely to be detected by the HEGY approach when applied to a PI process, while detection of apparent unit roots at the Nyquist and harmonic seasonal frequencies will depend on the order of augmentation used in the test for seasonal integration and the values of the coefficients that give rise to the PI unit root, $\prod_{s=1}^{S} \phi_s = 1$. In general, however, spurious unit roots could be detected. The simulation analysis of del Barrio Castro and Osborn (2008) confirms this conclusion, which is also illustrated in the empirical example of the next section.

Finnaly expressions (57), (58), (59) and (60) also link with expressions (73), (74) and (75) in Appendix 1, note that in (57) we have $\left(\sum_{j=1}^{S} \mathbf{a}_j\right) = \mathbf{R}_0'\mathbf{a}$, note that $\Phi = \phi_1 \mathbf{a}$. Hence in expression (57), (58), (59) and (60) we have terms that are quivalent to the expression collected in (73), (74) and (75), that we propose to use to the following section to provide information about the relative importance of each (zero and seasonal) frequency of the nonstationary behaviour of a PI process.

6 Empirical Illustration

In this section we illustrate the use of the seasonal unit root and periodic integration tests through their application to seasonally unadjusted monthly US industrial production series. In particular, we examine six important components of the industrial production index (IPI), namely Business Equipment, Business Supplies, Construction Supplies, Durable Consumer Goods, Nondurable Consumer Goods and Nondurable Goods Materials. Seasonally unadjusted data are available on each of these components at the monthly frequency starting in January 1947. The sample ends in December 2020, yielding $N = 74$ years of data[14]. As usual, the data are analyzed after the logarithmic transformation, with the series (in log form) shown in Figure 2.

All six series exhibit clear upward trends over time, with the extent of seasonal variation varying

[14] Our sample ends in 2020 to avoid dealing with the disruptions to production caused by the COVID-19 epidemic. The data was downloaded from the FRED Economic Data of the St. Louis Fed.

over series. In particular, Figure 2 indicates that Business Equipment exhibits relatively small seasonal variations, with these much more marked in the case of Construction Supplies and Durable Consumer Goods. It is also notable that the extent of seasonality sometimes varies over time, with (for example) the extent of seasonality apparently declining for Durable Consumer Goods. To obtain further insights into the seasonality in the six IPI time series, Figure 3 reports the raw periodograms of the series. Clearly, all six show a high peak associated with the zero frequency, but peaks are also seen at all the seasonal frequencies, namely at $\pi/6$, $\pi/3$, $\pi/2$, $2\pi/3$, $5\pi/6$ and π, corresponding to 1, 2, 3, 4, 5, 6 cycles per year, respectively. The peaks at the seasonal frequencies are generally lower than at the zero frequency, with little peak evident for Business Equipment and Nondurable Goods Materials at frequency $\pi/6$ (corresponding to one cycle per year).

Tables 1.a, 1.b, 2.a and 2.b present the results of unit root tests applied at the zero and each seasonal frequency for the SI test regression (24). The regression is augmented to deal with possible presence of serial correlation using the $MAIC$ information criteria. Ordinary least squares (OLS) detrending is applied in Table 1.a and 2.a, with generalized least squares (GLS) detrending in Tables 1.b and 2.b. Tables 1.a and 1.b allow for both seasonal dummies and seasonal trends in the test regression, while seasonal dummies are used in conjunction with a single trend in the corresponding parts of Table 2; see del Barrio Castro, Osborn and Taylor (2016) for an evaluation of methods that are used for lag selection and detrending in the SI test regression. The critical values used for the tests are provided in Tables 1.c and 2.c, where these are asymptotic critical values obtained by simulation.

Tables 1 and 2 point to the presence of a unit root at the zero frequency for all six series, irrespective of form of trend (seasonal or not) and detrending method. Results for the Nyquist frequency (ω_π) are also relatively straightforward, with the presence of a unit root rejected in all cases at a level of significance of 10% or (generally) below. The situation in respect of F-type tests for pairs of unit roots at other seasonal frequencies is a little less clear, with unit roots not rejected in some cases at frequencies $\pi/6$, $\pi/3$ and $\pi/2$, which correspond to one, two and three cycles per year in data observed monthly. However, the evidence against unit roots is strong at frequencies $2\pi/3$ and $5\pi/6$, with the overall F_{seas} statistic for the null of unit roots at all seasonal frequencies also being decisively rejected. Therefore, none of these series appear to be of the SI form and the conclusion might be drawn that the seasonality observed in Figure 3 is either stationary in form or associated with deterministic seasonality. However, these SI tests do not consider the possibility of seasonality being of PI form.

Table 3 considers the possibility that the IPI series are periodically integrated, and includes tests for both periodically varying coefficients and a unit root. The model considered is a generalization of (2) for monthly data, with

$$\begin{aligned} y_t &= \sum_{s=1}^{12} D_{st}\alpha_s + \sum_{s=1}^{12} D_{st}\beta_s t \\ &\quad + \sum_{s=1}^{12} \phi_{1s} D_{st} y_{t-1} + \cdots + \sum_{s=1}^{12} \phi_{ps} D_{st} y_{t-p} + \varepsilon_t. \end{aligned} \tag{61}$$

This general specification includes both constant and trends that are allowed to vary over the months of the year, while the lag order p is selected for each series using AIC; the maximum lag considered is 4 due to the highly parameterized nature of (61).

The first statistic reported, labelled F_{per}, provides a test that the AR coefficients are periodically varying, which employs the joint null hypothesis $\phi_{js} = \phi_j$ for $j = 1, \ldots, p$ (see Boswijk and Franses (1996) for details); since no unit root is under test, the conventional F distribution is applied, with all six series showing strong evidence of coefficient variation over the year. The possibility that the series are periodically integrated, PI, is then considered through the LR test of Boswijk and Franses (1996), which is discussed above; see (16) and (17). Using conventional significance levels, none of the series provides evidence against the PI unit root null hypothesis. In order to check that the unit root uncovered by the LR test has a periodic form, an F-test is undertaken of the null that all coefficients associated with the periodic integration restriction are equal to one. Representing the relevant coefficients as φ_s with $\prod_{s=1}^{S} \varphi_s = 1$ (where $\varphi_s = \phi_s$ for a $PAR(1)$ process such as (2)), this last test has $H_0 : \varphi_s = 1$ for $s = 1, 2, \ldots, S$. Hence the null hypothesis is that the conventional zero frequency unit root applies and that differencing of the form $(1 - L)$ is appropriate, with the process being PI under the alternative. Five of the six series reject the null hypothesis at a significance level of 10% or lower, with only Business Supplies supporting the conventional zero frequency unit root. However, the test is not significant for

Nondurable Goods Materials at 5%.

Overall, the results in Tables 1 to 3 indicate that the IPI series examined are predominantly PI in form, with none apparently having the full set of unit roots at seasonal frequencies required for SI processes. Business Supplies is a partial exception, since unit roots are not rejected at some seasonal frequencies for this series and the first differencing null hypothesis is not rejected at conventional levels against the periodic alternative.

Finally Figures 4.a, 4.b and 4.c use the estimated coefficients from (61) to provide information about the relative importance of each (zero and seasonal) frequency to the nonstationary behaviour of the six IPI series. In particular, Figures 4.a and 4.b employ the seasonal dummy and seasonal trend coefficients, employing the trignometric transformation to convert these to coefficients associated with the zero and seasonal frequencies. Figure 4.c, on the other hand, applies the trignometric transformation to the cumulation of the estimated φ_s coefficients associated with a possible PI unit root; Appendix 1 provides further information on these transformations. It might be noted here, however, that the cumulation associated with each month can be represented as the vector $\Phi^c = \left[\varphi_1,\ \varphi_1\varphi_2,\ \varphi_1\varphi_2\varphi_3,\ \ldots,\ \prod_{j=1}^{12}\varphi_j\right]'$, which is a scaling (multiplication by φ_1) of the common trend coefficients for a PI process in (14). Hence Figure 4.c gives information of how the PI coefficients contribute to the spectral power at each frequency. To put Figure 4.c in context, note if the seasonal unit root process of (39) is analyzed in this way, it will have a non-zero contribution only at the frequency ω_k.

Figures 4.a and 4.b show that both the seasonal dummy variables and the seasonally varying trends generally contribute strongly to the seasonal effects seen in the IPI series. For example, the seasonal dummy variable coefficients imply particularly strong cycles at frequencies $\pi/6$ and $\pi/2$ for Durable Consumption Goods; these are further enhanced by the seasonal trend effects for this series, with trend seasonality also strong at frequencies $\pi/3$ and $2\pi/3$.

In contrast to the strong contributions in Tables 3.a and 3.b of the dummy variables and trends to seasonal effects, the PI coefficients in Table 3.c add very little. The dominant contribution of these coefficients is at the zero frequency, which suggests that the estimated PI coefficients are (in some sense) all close to unity. Nevertheless, the conventional unit root is rejected for most series in Table 3, and the periodic coefficients consequently do make some contributions in Figure 3 at the seasonal frequencies.

7 Concluding Remarks

This paper compares the properties of seasonally integrated (SI) and periodically integrated (PI) processe, with the aim of providing insights into the methods and results used in the analysis of non-stationary seasonal processes. A substantial theoretical literature has built on the analysis of Hylleberg, Engle, Granger and Yoo (1990), which was the first to consider the separate unit roots at the zero and seasonal frequencies implied by seasonal integration. Since the unit roots at seasonal frequencies (except at the frequency $\omega_k = \pi$) imply the existence of pairs of complex-valued unit roots, the methods of analysis can appear somewhat different to the usual zero frequency unit root case. However, as the discussion of this paper aims to draw out (particularly in Section 4), the theoretical analysis dealing with complex unit root processes is, in essence, analogous to that associated with conventional $I(1)$ processes. The theoretical analysis dealing with PI processes is relatively more straightforward than the seasonal integrated case, because (as discussed in subsection 2.2) periodic integration quite naturally leads to a vector representation to which conventional cointegration analysis can be applied. Nevertheless, the crucial difference between the two types of seasonal non-stationarity is that an SI process has a unit root associated with each of S "seasons" of the year (typically a month or quarter), and hence S unit roots in total, in contrast to the single unit root underlying the movements of a PI process.

Although they are more straightforward than SI processes to analyze from a theoretical perspective, PI processes have attracted less study and relatively few empirical applications appear to have been made to macroeconomic data. Nevertheless, when the approaches are compared in practice, and as reflected in the empirical analysis of Section 6, Franses and Romijn (1993) find more support for periodic than seasonal integration for UK series. It should also be noted that an empirical PI model will often involve more parameters than an SI model, especially for higher frequency data, such as monthly or weekly, due to the inherent use of seasonally-varying coefficients in the former. Useful directions for further theoretical and empirical analyses are, firstly, the discrimination between SI and PI processes and, secondly, effective methods to reduce the number of coefficients to be estimated in a PI model for higher frequency data.

Although the present paper has discussed only univariate techniques, the analytical methods considered here can be used to analyze cointegration issues. In particular, del Barrio Castro, Cubadda and Osborn (2022) apply demodulation in conjunction with seasonally (or, more generally, cyclically varying) coefficients to show that cointegration can exist across different frequencies. This generalizes the previously considered cointegration possibilities of seasonal cointegration (Engle, Granger, Hylleberg and Lee, 1994, Johansen and Schaumburg, 1999) and periodic cointegration (Boswijk and Franses 1995).

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APPENDIX 1
Trigonometric Representation of a PI Process

At this stage, it is useful to introduce the one to one connection between the dummy variable and trigonometric representations of seasonal dummy variables; see Ghysels and Osborn (2001, section 2.2) for more details in the context of deterministic seasonality.

As shown in Ghysels and Osborn (2001, section 2.2), there is a one-to-one correspondence between the coefficients of the full set of S seasonal dummy variables and the coefficients of a trigonometric representation of the seasonal pattern. To be specific, the two representations of the deterministic seasonally-varying mean are

$$\sum_{s=1}^{S} \gamma_s D_{st} = \sum_{k=0}^{S^*} \left[\alpha_k \cos\left(\frac{2\pi kt}{S}\right) + \beta_k \sin\left(\frac{2\pi kt}{S}\right) \right] \tag{62}$$

where D_{st} is as defined for (2) while $S^* = (S/2) - 1$, assuming S is even, as in (3) above. The left-hand side of (62) is the dummy variable representation of deterministic seasonality, where γ_s is the coefficient (or intercept) associated with season $s = 1, 2, \ldots, S$. The trigonometric representation of deterministic seasonality is given by the right-hand side of (62). Here the coefficients α_0 and $\alpha_{S/2}$ are associated with the zero and Nyquist (π) frequencies, respectively, as $\sin(2\pi 0t/S) = \sin(0) = \sin(\pi t) = 0$, while the coefficients α_k and β_k are associated with seasonal waves at the harmonic frequencies $\omega_k = 2\pi k/S$ ($k = 1, \ldots, S^*$) that complete a full cycle every $2\pi/\omega_k = S/k$ periods.

Defining the $S \times 1$ coefficient vectors $\Gamma = \left[\begin{array}{cccc} \gamma_1 & \gamma_2 & \cdots & \gamma_S \end{array}\right]'$ and $\mathbf{B} = \left[\begin{array}{ccccccc} \alpha_0 & \alpha_1 & \beta_1 & \cdots & \alpha_{(S-1)/2} & \beta_{(S-1)/2} & \alpha_{S/2} \end{array}\right]'$, the equivalence in (62) implies that

$$\Gamma = \mathbf{RB}$$

where the $S \times S$ matrix $\mathbf{R}$ is

$$\mathbf{R} = \left[\begin{array}{ccccccc} \mathbf{R}_0 & \mathbf{R}_1^{\alpha} & \mathbf{R}_1^{\beta} & \cdots & \mathbf{R}_{(S-1)/2}^{\alpha} & \mathbf{R}_{(S-1)/2}^{\beta} & \mathbf{R}_{S/2} \end{array}\right] \tag{63}$$

with j^{th} row

$$\mathbf{R}_{\cdot j} = \left[\begin{array}{cccccccc} 1 & \cos\left(\frac{2\pi j}{S}\right) & \sin\left(\frac{2\pi j}{S}\right) & \cdots & \cos\left(\frac{2\pi[(S-1)/2]j}{S}\right) & \sin\left(\frac{2\pi[(S-1)/2]j}{S}\right) & \cos(\pi j) \end{array}\right].$$

Note that the columns of matrix $\mathbf{R}$ are orthogonal to each other and that $\mathbf{R}_i'\mathbf{R}_i = S$ for $i = 0$ and $i = S/2$, $\mathbf{R}_i^{\ell\prime}\mathbf{R}_i^{\ell} = S/2$ for $i = 1, 2, \ldots, (S-1)/2$ and $\ell = \alpha$, β; see, for example, Fuller (1996, Theorem 3.1.1) or Wei (2006, expressions 11.2.13-11.2.15). Finally, it is also possible to obtain the trigonometric coefficients $\mathbf{B}$ from the dummy coefficients $\mathbf{\Gamma}$ as

$$\mathbf{B} = \mathbf{R}^{-1}\mathbf{\Gamma}. \tag{64}$$

As noted in Ghysels and Osborn (2001), since the columns of matrix $\mathbf{R}$ for an orthogonal basis, it is straightforward to see that $\mathbf{R}'\mathbf{R}$ is a diagonal matrix and its inverse is

$$\mathbf{R}^{-1} = diag\left[\begin{array}{ccccccc} 1/S & 2/S & 2/S & \cdots & 2/S & 2/S & 1/S \end{array}\right] \times \mathbf{R}'. \tag{65}$$

The relationships just discussed apply whenever the full set of seasonal dummy variables is used. Hence, for a periodically integrated PAR, of the form

$$y_{(s\tau)} = \mu_s + \beta_s \tau + \phi_s y_{(s-1,\tau)} + \sum_{j=1}^{p-1} \gamma_{s,j} \left(y_{(s-j,\tau)} - \phi_{s-j} y_{(s-j-1,\tau)} \right) + \varepsilon_{(s\tau)},$$

we can obtain the contributions of the seasonal intercepts to the zero frequency movement by defining the $S \times 1$ vector $\mathbf{\Gamma}_\mu = \left[\begin{array}{cccc} \mu_1 & \mu_2 & \cdots & \mu_S \end{array}\right]'$, with the zero frequency contribution being

$$\frac{1}{S}\mathbf{R}_0'\mathbf{\Gamma}_\mu = \frac{1}{S}\sum_{s=1}^{S} \mu_s = \mu_0. \tag{66}$$

The contribution at the Nyquist frequency is then:

$$\frac{1}{S}\mathbf{R}_{S/2}'\mathbf{\Gamma}_{\mu}=\frac{1}{S}\sum_{s=1}^{S}\cos\left(\pi s\right)\mu_{s}=\mu_{S/2}. \tag{67}$$

and the contributions at each harmonic frequencies $\omega_k = 2\pi k/S$ for $k = 1, 2, \ldots, (S-1)/2$ are:

$$\begin{aligned}\frac{2}{S}\mathbf{R}_{j}^{\alpha\prime}\mathbf{\Gamma}_{\mu} &= \frac{2}{S}\sum_{s=1}^{S}\cos\left(\frac{2\pi s}{S}j\right)\mu_{s}=\mu_{k}^{\cos} \\ \frac{2}{S}\mathbf{R}_{j}^{\beta\prime}\mathbf{\Gamma}_{\mu} &= \frac{2}{S}\sum_{s=1}^{S}\sin\left(\frac{2\pi s}{S}j\right)\mu_{s}=\mu_{k}^{\sin}.\end{aligned} \tag{68}$$

To summarize the total effect at frequency $\omega_k = 2\pi k/S$ for $k = 1, 2, \ldots, (S-1)/2$, we can then use

$$\left[\left(\mu_{k}^{\cos}\right)^{2}+\left(\mu_{k}^{\sin}\right)^{2}\right]. \tag{69}$$

And in the case of the zero and Nyquist frequencies in order to make comparable the information we also summarize the effect and the zero and Nyquist frequencies with $(\mu_0)^2$ and $\left(\mu_{S/2}\right)^2$, see for example Wei (2006).

The case of seasonal deterministic trends β_s is analogous to that just discussed. By defining the $S \times 1$ vector $\mathbf{\Gamma}_{\beta}=\left[\begin{array}{cccc}\beta_1 & \beta_2 & \cdots & \beta_S\end{array}\right]'$, the zero frequency contribution is:

$$\frac{1}{S}\mathbf{R}_{0}'\mathbf{\Gamma}_{\beta}=\frac{1}{S}\sum_{s=1}^{S}\beta_{s}=\beta_{0}, \tag{70}$$

and that for the Nyquist frequency:

$$\frac{1}{S}\mathbf{R}_{S/2}'\mathbf{\Gamma}_{\beta}=\frac{1}{S}\sum_{s=1}^{S}\cos\left(\pi s\right)\beta_{s}=\beta_{S/2}. \tag{71}$$

At the harmonic frequencies $\omega_k = 2\pi k/S$ for $k = 1, 2, \ldots, (S-1)/2$:

$$\begin{aligned}\frac{2}{S}\mathbf{R}_{j}^{\alpha\prime}\mathbf{\Gamma}_{\beta} &= \frac{2}{S}\sum_{s=1}^{S}\cos\left(\frac{2\pi s}{S}j\right)\beta_{s}=\beta_{k}^{\cos} \\ \frac{2}{S}\mathbf{R}_{j}^{\beta\prime}\mathbf{\Gamma}_{\beta} &= \frac{2}{S}\sum_{s=1}^{S}\sin\left(\frac{2\pi s}{S}j\right)\beta_{s}=\beta_{k}^{\sin},\end{aligned} \tag{72}$$

with the summary total effect at frequency ω_k:

$$\left[\left(\beta_{k}^{\cos}\right)^{2}+\left(\beta_{k}^{\sin}\right)^{2}\right].$$

And in the case of the zero and Nyquist frequencies we also use $(\beta_0)^2$ and $\left(\beta_{S/2}\right)^2$

Finally in the case of the coefficients ϕ_s, $s = 1, 2, \ldots, S$, of a PI process associated to the periodic integration restriction $\prod_{s=1}^{S}\phi_s = 1$, we use the $S \times 1$ vector $\Phi = \left[\begin{array}{ccccc}\phi_1 & \phi_1\phi_2 & \phi_1\phi_2\phi_3 & \cdots & \prod_{s=1}^{S}\phi_s\end{array}\right]'$. The cumulation of the coefficients in defining Φ is important, as it reflects the interactions between adjacent observations. The contribution of the zero frequency will then be measured by:

$$\frac{1}{S}\mathbf{R}_{0}'\Phi=\phi_{0}, \tag{73}$$

that associated with the Nyquist (π) frequency by:

$$\frac{1}{S}\mathbf{R}_{S/2}'\Phi = \phi_{S/2} \tag{74}$$

and, finally, with the harmonic frequencies $\omega_k = 2\pi k/S$ for $k = 1, 2, \ldots, (S-1)/2$ by:

$$\begin{aligned} \frac{2}{S}\mathbf{R}_j^{\alpha\prime}\Phi &= \phi_k^{\cos} \\ \frac{2}{S}\mathbf{R}_j^{\beta\prime}\Phi &= \phi_k^{\sin}. \end{aligned} \tag{75}$$

and the summary total effect at frequency $\omega_k = 2\pi k/S$:

$$\left[\left(\phi_k^{\cos}\right)^2 + \left(\phi_k^{\sin}\right)^2\right].$$

And as in the previous two cases in order to compare the results with the zero and Nyquist frequencies we summarize the information for these frequencies with $(\phi_0)^2$ and $\left(\phi_{S/2}\right)^2$ respectively.

Note that it is also possible to compute (73), (74) and (75) using the coefficients that capture the common trend in the PI process, namely $\mathbf{a} = \left[\begin{array}{ccccc} 1 & \phi_2 & \phi_2\phi_3 & \cdots & \prod_{s=2}^{S}\phi_s \end{array}\right]'$. However, for our empirical illustration, the results using $\mathbf{a}$ are very similar to those using the form of Φ of the previous paragraph and hence only results using this Φ are included in the paper.

It should also be pointed out that it does not make sense to define the $S \times 1$ vector Φ as $\Phi = \left[\begin{array}{ccccc} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_S \end{array}\right]'$ because if we have, for example, the process $(1+L)\, x_t^{\pi} = \varepsilon_t$ which is a particular case of a PI process with $\Phi^{\pi} = \left[\begin{array}{cccc} -1 & -1 & \cdots & -1 \end{array}\right]'$ then $\frac{1}{S}\mathbf{R}_{S/2}'\Phi^{\pi} = -\frac{1}{S}\sum_{s=1}^{S}\cos(\pi s) = 0$. Or, for a process $\left(1 - e^{-i\omega_k}L\right) x_t^{k-} = \varepsilon_t$, we would have $\Phi^{\omega_k} = \left[\begin{array}{cccc} e^{-i\omega_k} & e^{-i\omega_k} & \cdots & e^{-i\omega_k} \end{array}\right]'$ and $\frac{2}{S}\mathbf{R}_k^{\alpha\prime}\Phi^{\omega_k} = e^{-i\omega_k}\frac{2}{S}\sum_{s=1}^{S}\cos(\omega_k s) = 0$ or $\frac{2}{S}\mathbf{R}_k^{\beta\prime}\Phi^{\omega_k} = e^{-i\omega_k}\frac{2}{S}\sum_{s=1}^{S}\sin(\omega_k s) = 0$. However, by cumulating the coefficients as in Φ of the previous paragraph, the contribution at the frequency of the unit root is no longer zero while the contributions at other frequencies are zero.

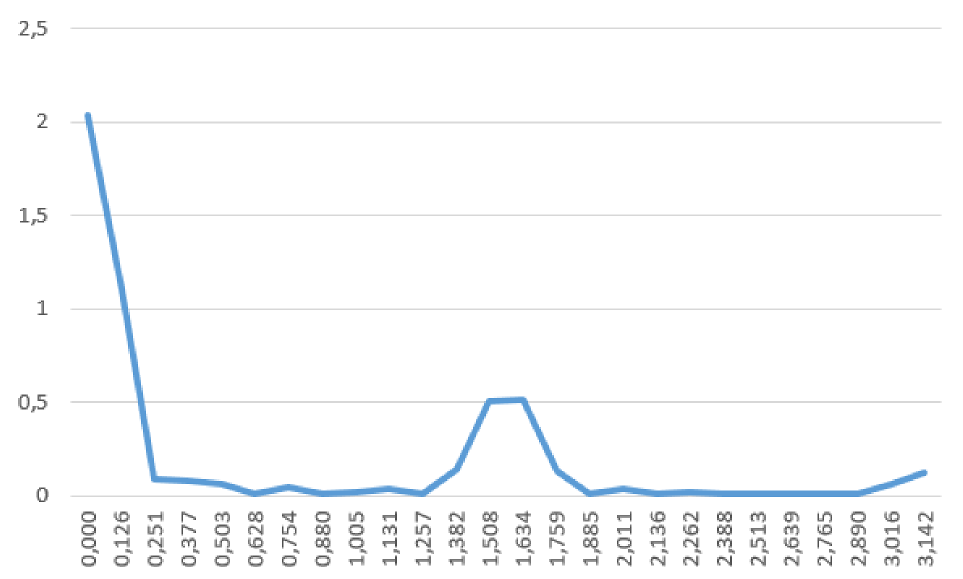
APPENDIX 2
Asymptotic Independence of HEGY Regressors

Although the circulant matrices $\mathbf{C}_k^{\alpha}$ and $\mathbf{C}_k^{\beta}$ are not orthogonal, it is noted in subsection 3.2 that the regressor variables $y_{k,t-1}^{\alpha}$ and $y_{k,t-1}^{\beta}$ in (24) are asymptotically independent when y_t is generated by the SI process (1). This can be established using (27), since

$$\begin{aligned} \frac{1}{T^2}\sum_{t=1}^{T} y_{k,t-1}^{\alpha} y_{k,t-1}^{\beta} &= \frac{1}{S^2N^2}\sum_{\tau=1}^{N}\left(Y_{k,\tau}^{\alpha}\right)' Y_{k,\tau}^{\beta} + o_p(1) \\ &= \frac{1}{S^2N^2}\sum_{\tau=1}^{N}\left(\mathbf{C}_k^{\alpha}Y_{\tau}\right)'\mathbf{C}_k^{\beta}Y_{\tau} + o_p(1) \\ &= \frac{1}{S^2N^2}\sum_{\tau=1}^{N} Y_{\tau}'\mathbf{C}_k^{\beta}Y_{\tau} + o_p(1) \\ &= \frac{1}{S2N^2}\sum_{\tau=1}^{N} Y_{\tau}'\mathbf{v}_k^{\alpha}\left(\mathbf{v}_k^{\beta}\right)' Y_{\tau} + o_p(1) \\ &\Longrightarrow \frac{1}{S2}\int W(r)'\,\mathbf{v}_k^{\alpha}\left(\mathbf{v}_k^{\beta}\right)' W(r)\,dr \\ &= -\int w_R^k(r)\, w_I^k(r)\,dr + \int w_I^k(r)\, w_R^k(r)\,dr = 0. \end{aligned}$$

To obtain this result, we use the fact that $\mathbf{C}_k^\alpha$ is symmetric and that $\mathbf{C}_k^\alpha \mathbf{C}_k^\beta = S/2\mathbf{C}_k^\beta \mathbf{C}_k^\beta$ while $\mathbf{C}_k^\beta = \mathbf{v}_k^\alpha \left(\mathbf{v}_k^\beta\right)'$; see Smith, Taylor and del Barrio Castro (2009, p 560) for more details.

Figure 1: Average Periodogram for Example Periodically Integrated Process



Notes: The example process has $\phi_1 = 0.8$, $\phi_2 = 1$, $\phi_3 = 0.5$, $\phi_4 = 1/(\phi_1\phi_2\phi_3)$; see text (subsection 2.3).

Figure 2: Series used in the Empirical Illustration

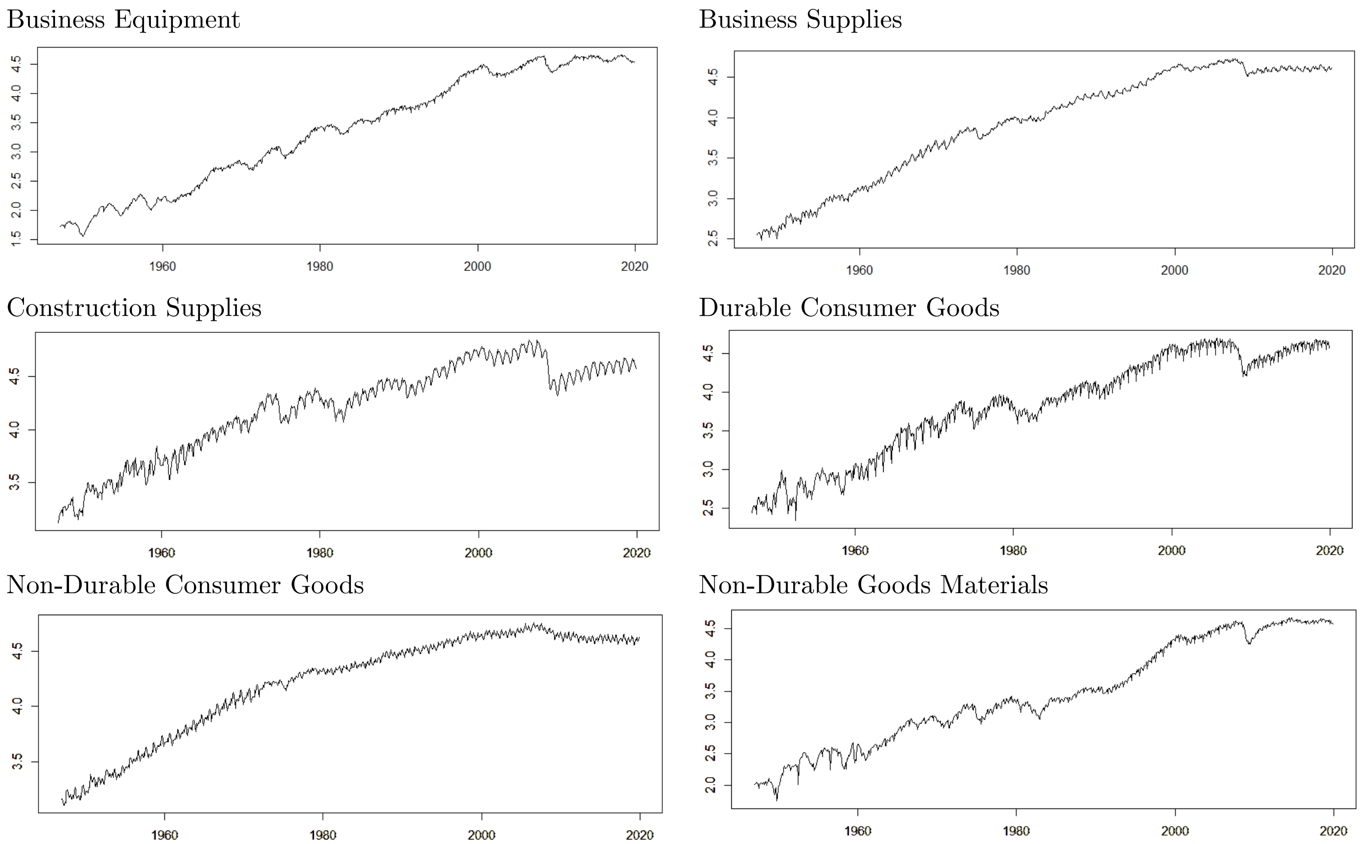
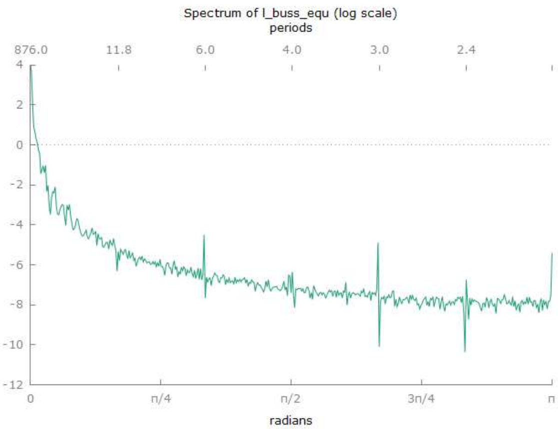
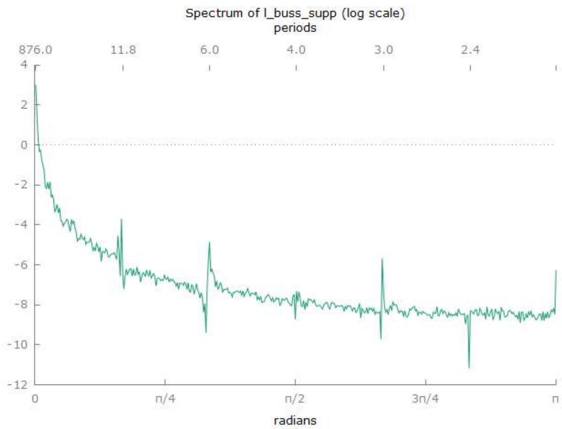


Figure 3: Periodograms of the Series used in the Empirical Illustration

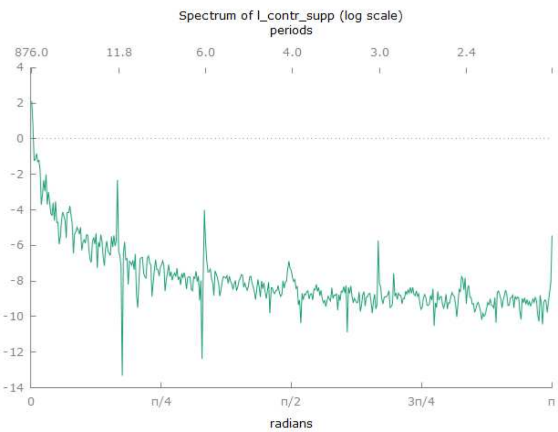
Business Equipment



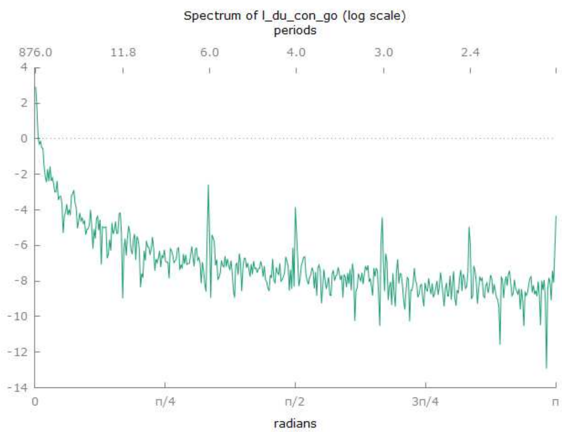
Business Supplies



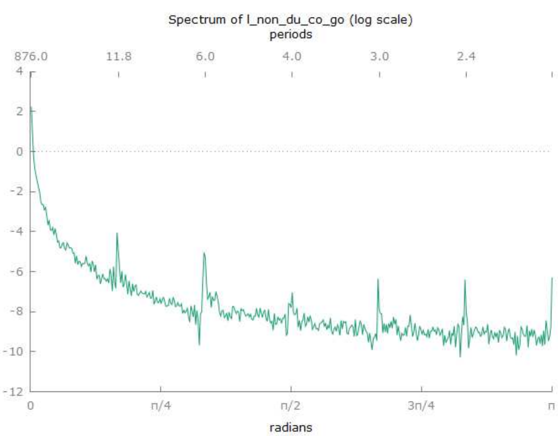
Construction Supplies



Durable Consumer Goods



Non-Durable Consumer Goods



Non-Durable Good Materials

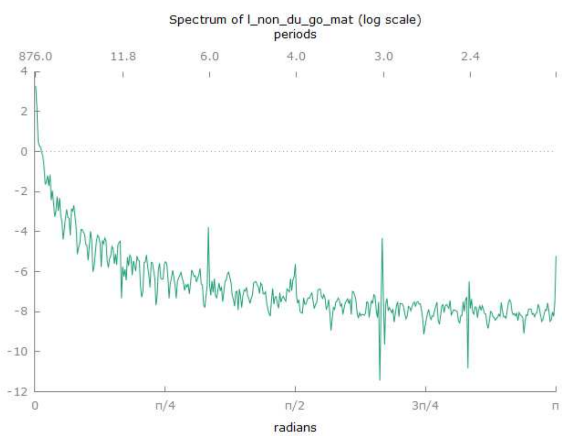


Table 1.a. Seasonal Integration Test Results: Dummies and Trends, OLS

	Bus. Equip.	Bus. Supp.	Const. Supp.	Durable Cons. G.	Nondur. Cons. G.	Nondur. Goods Mat.
t_0	-1.0199	-0.3721	-2.2007	-1.6419	-0.4419	-2.2810
t_π	-3.8454*	-4.9540***	-5.1180***	-3.9023**	-5.3235***	-4.1153***
$F_{\pi/6}$	30.6731***	9.5620**	20.1071***	28.7250***	11.2715**	40.1376***
$F_{\pi/3}$	13.6166***	6.3918	21.7623***	11.1394**	8.0808	26.5298***
$F_{\pi/2}$	10.1656**	24.5148***	14.5043***	14.2796***	16.0250***	27.1045***
$F_{2\pi/3}$	17.3084***	19.5288***	29.8845***	15.0662***	29.7554***	15.4465***
$F_{5\pi/6}$	14.8094***	17.6557***	17.7820***	12.7341***	14.6024***	16.0091***
F_{seas}	16.5453***	14.5460***	19.5753***	15.4758***	14.3971***	24.0702***
Aug	17	14	14	13	13	15

Table 1.b. Seasonal Integration Test Results: Dummies and Trends, GLS

	Bus. Equip.	Bus. Supp.	Const. Supp.	Durable Cons. G.	Nondur. Cons. G.	Nondur. Goods Mat.
t_0	-1.7169	-0.3888	-1.4548	-1.4582	0.0956	-2.3119
t_π	-3.0662**	-3.1018**	-4.0988***	-3.0316**	-4.4457***	-4.0456***
$F_{\pi/6}$	12.1288***	2.6282	12.8731***	17.4216***	2.9525	35.3733***
$F_{\pi/3}$	7.5390**	2.8271	9.6179***	10.7939***	6.2010*	22.3992***
$F_{\pi/2}$	7.3705**	7.6471**	3.9172	7.1728**	7.5806**	20.7182***
$F_{2\pi/3}$	11.1962***	9.2270***	12.0667***	11.8378***	22.8782***	11.0395***
$F_{5\pi/6}$	9.4426***	8.4188**	7.3779**	11.5378***	7.0826**	14.1270***
F_{seas}	8.8746***	5.5785**	8.4681***	11.0124***	8.4028***	19.7138***
Aug	17	14	14	13	13	15

Notes: *, **, *** indicate significance at 10%, 5%, 1% respectively.
Aug. is the level of augmentation selected according to MAIC. The maxim lag is determined with $l_{12} = \left[12\,(T/100)^{1/4}\right]$.

Table 1.c: Critical Values for Tables 1.a and 1.b

OLS	1%	5%	10%
t	-3.9359	-3.3883	-3.1011
F_k	12.0285	9.5506	8.3986
F_{seas}	8.0830	7.1697	6.7214

GLS	1%	5%	10%
t	-3.5677	-3.0157	-2.7385
F_k	9.0715	6.9961	6.0407
F_{seas}	5.7873	5.0916	4.7418

Table 2.a. Seasonal Integration Test Results: Dummies and Constant Trend, OLS

	Bus. Equip.	Bus. Supp.	Const. Supp.	Durable Cons. G.	Nondur. Cons. G.	Nondur. Goods Mat.
t_0	-1.0787	-0.4687	-2.4504	-1.6737	-0.4501	-2.2647
t_π	-3.6959***	-3.7582***	-3.1263**	-3.1401**	-3.0506**	-4.4611***
$F_{\pi/6}$	23.9860***	10.1851***	16.0720***	20.9269***	6.4100*	24.6796***
$F_{\pi/3}$	12.1480***	1.5896	4.8845	7.4819**	2.2600	22.1213***
$F_{\pi/2}$	9.9454***	6.7779**	6.4869*	12.7344***	5.0386	15.4746***
$F_{2\pi/3}$	10.7473***	11.5759***	19.4741***	10.2388***	15.5256***	14.3397***
$F_{5\pi/6}$	8.8004***	10.1580***	11.5983***	8.6555**	7.5868**	13.7899***
F_{seas}	12.3765***	7.3775***	10.6444***	11.3031***	6.9249***	18.5360***
Aug	17	14	13	13	18	19

Table 2.b. Seasonal Integration Test Results: Dummies and Constant Trend, GLS

	Bus. Equip.	Bus. Supp.	Const. Supp.	Durable Cons. G.	Nondur. Cons. G.	Nondur. Goods Mat.
t_0	-1.6452	-0.3761	-1.5971	-1.386	-0.0620	-2.1881
t_π	-3.4177***	-2.1293*	-1.5940	-2.7610***	-3.0717***	-4.5881***
$F_{\pi/6}$	7.7239***	0.2928	4.9614***	6.8328***	0.7618	12.1027***
$F_{\pi/3}$	7.2495***	0.3309	1.8323	4.9575**	0.4025	22.4505***
$F_{\pi/2}$	8.5721***	4.7960**	3.4230**	3.6822**	3.7156**	15.3410***
$F_{2\pi/3}$	4.7374**	3.9178***	1.7992	6.5218***	4.6616**	5.3308***
$F_{5\pi/6}$	4.2740**	8.8063***	7.1080***	3.8781**	4.3232**	13.1237***
F_{seas}	6.0127***	3.2640***	3.4753***	4.7578**	2.5533***	13.6979***
Aug	17	14	13	13	18	19

Notes: *, **, *** indicate significance at 10%, 5%, 1% respectively.
Aug. is the level of augmentation selected according to MAIC. The maxim lag is determined with

$$l_{12} = \left[12\,(T/100)^{1/4}\right].$$

Table 2.c: Critical Values for Tables 2.a and 2.b

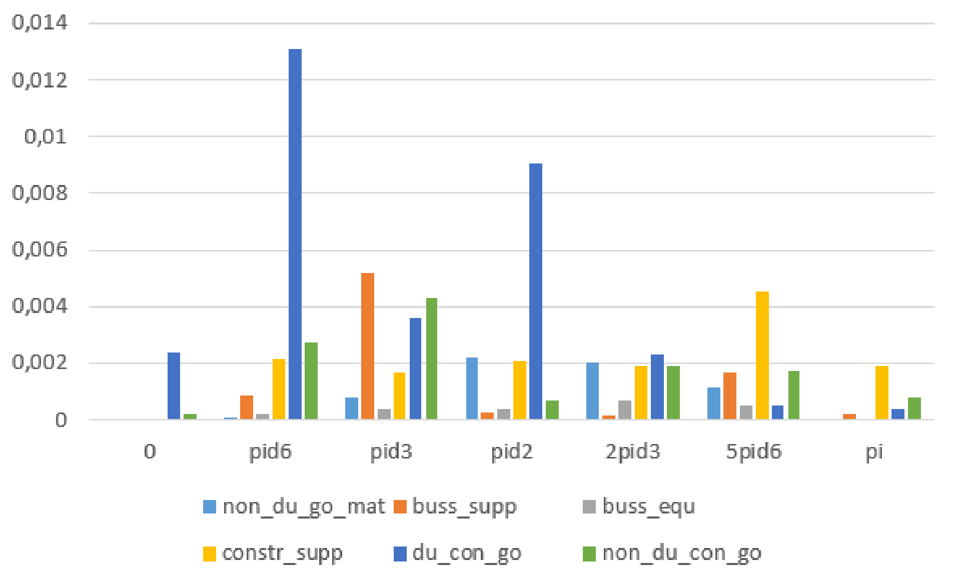
OLS	1%	5%	10%
t_0	-3,9359	-3,3883	-3,1011
t_π	-3,3993	-2,8326	-2,5452
F_k	8,6594	6,5190	5,5180
F_{seas}	5,2075	4,4694	4,0963
GLS	1%	5%	10%
t_0	-3,5677	-3,0157	-2,7385
t_π	-2,7455	-2,1526	-1,8552
F_k	4,9418	3,3184	2,6024
F_{seas}	2,5516	2,0649	1,8302

Table 3. Periodic Tests

	F_{per}	LR	$F_{(1-L)}$	order
Business Equip.	3.411***	0.227	3.367***	2
Business Supplies	7.238***	0.011	0.919	2
Contruc. Supplies	5.196***	2.960	2.118**	2
Durable Cons. Goods	4.468***	4.586	9.678***	3
Nondur. Cons. Goods	5.850***	0.472	2.621***	2
Nondur. Goods Mat.	4.930***	8.470	1.657*	2

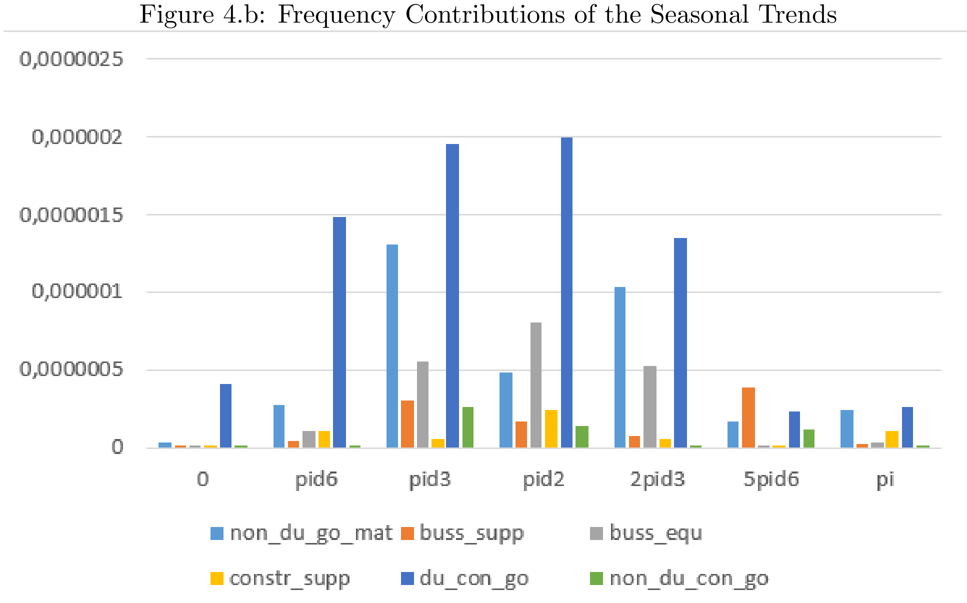
Notes: *, **, *** indicate significance at 10%, 5%, 1% respectively.
Order is the value of p selected according to AIC in (61.
See text for discussion of the tests. Maximun lag in PAR models 4.

Figure 4.a: Frequency Contributions of the Seasonal Intercepts



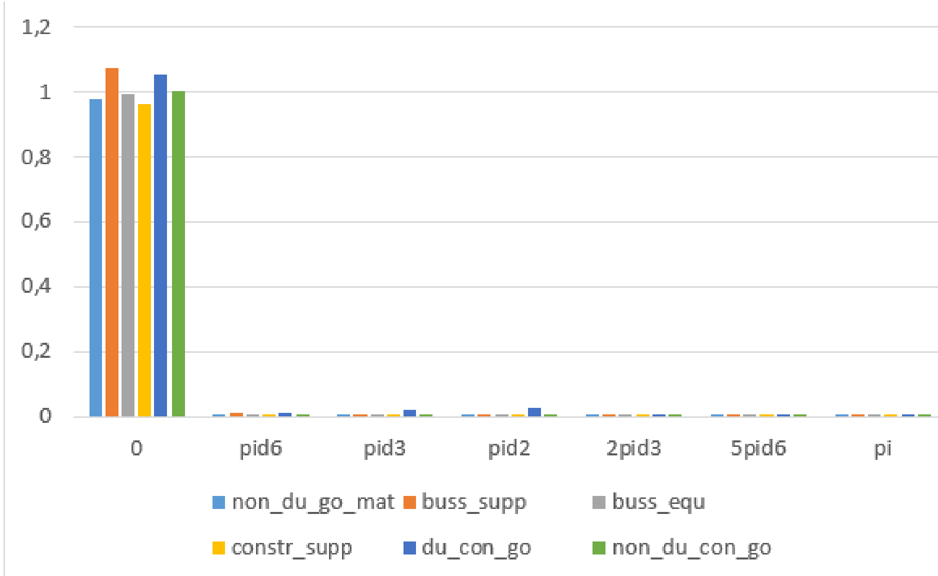
Notes: Values shown are obtained from the Appendix expressions (66) $(\mu_0)^2$, (71) $\left(\mu_{S/2}\right)^2$ and (72) for the relevant $\omega_k = 2\pi k/12$, $k = 0, 1, ..., 6$.

Figure 4.b: Frequency Contributions of the Seasonal Trends



Notes: Values shown are obtained from the Appendix expressions (70) $(\beta_0)^2$, (71) $\left(\beta_{S/2}\right)^2$ and (72) for the relevant $\omega_k = 2\pi k/12$, $k = 0, 1, ..., 6$.

Figure 4.c: Frequency Contributions of Periodic Integration Coefficients



Notes: Values shown are obtained from the Appendix expressions (73) $(\phi_0)^2$, (74) $\left(\phi_{S/2}\right)^2$ and (75) for the relevant $\omega_k = 2\pi k/12$, $k = 0, 1, ..., 6$ using φ_s with $\Phi^c = \left[\varphi_1, \varphi_1\varphi_2, \varphi_1\varphi_2\varphi_3, \ldots, \prod_{j=1}^{S} \varphi_j\right]'$.