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Abstract

We study core allocations in a two-period asymmetric information mixed economy, where the consumption sets are arbitrary subsets of an ordered Banach space, and the feasibility is defined as exact. We prove that (i) the strong blocking by a generalized coalition leads to a weak blocking by some ordinary coalition, implying the equivalence between the Aubin core and the core of an economy with only negligible agents; and (ii) the core can be characterized in terms of the size (and diameter) of the blocking coalitions in an economy with only negligible agents as well as both negligible and non-negligible agents.

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1 Introduction

The core of an economy is a solution concept that acknowledges that coalitions of agents may corporate to improve their welfare. In other words, for any allocation not belonging to the core, there is a coalition whose members achieve better commodity bundles than the non-core allocation by redistributing their initial endowments with themselves. In a classical exchange economy with a continuum of agents, the core coincides with the set of competitive allocations, refer to Aumann [3]. However, the equivalence theorem fails to hold, in general, if there are some non-negligible market participants in addition to the negligible ones (see Shitovitz [28]). Note that the market participants become non-negligible for the following reasons: (i) some agents may be endowed with an *exceptional* initial endowments because their initial ownership of commodities is sufficiently large with respect to the total market endowment. This is typical in monopolistic or, more generally, in oligopolistic markets; and (ii) the other reason is, while the initial endowment is spread over a continuum of negligible agents, some of them may join forces and decide to act as a single agent in the form of cartels, syndicates, or similar institutions. The paper aims to study a few classical problems concerning the core allocations in the exchange economy embodying many agents, some of which are non-negligible.

- *Coincidence of the Aubin core and the core*: It follows from the definitions that the Aubin core is contained in the core of an economy. In a classical economy with an atomless measure space of agents, the validity of the core-equivalence result guarantees that these two core coincides. Therefore, no similar conclusion is plainly guaranteed as soon as there are infinitely many commodities and uncertainty with asymmetric information because there is no core-equivalence theorem in such a setting.

- *Size and diameter of the blocking coalition*: The robustness of the core allocations with respect to the restrictions imposed on the size and diameter of the blocking coalition go back to the seminal contributions of Schmeidler [29], Grodal [18], and Vind [31] in a classical model of an atomless economy with the positive cone of the ℓ-dimensional Euclidean space as the consumption sets of agents and without uncertainty and asymmetric information. More precisely, Schmeidler [29] showed that if a feasible allocation is not in the core of the economy, it can be blocked by coalitions of small measures. Thus, the core (particularly the set of competitive allocations) can be implemented only by forming small coalitions. Such a result is crucial when forming a coalition implies a certain cost proportional to its size. Schmeidler's idea of a blocking mechanism was further extended by Grodal [18], where she showed that for any $\varepsilon > 0$ and any non-core allocation, there is a blocking coalition which can be expressed as the union of atmost $\ell+1$ sub-coalitions, and the measure and diameter of each of these sub-coalitions is less than ε. A coalition whose measure and diameter both are less than ε intuitively means that the coalition consists of relatively *few* agents and that the agents in the coalition resemble one another in chosen characteristics. Finally, Vind [31] established that for any feasible allocation outside the core of an economy and any measure ε less than the measure of the grand coalition, there is a coalition S whose measure is precisely ε such that the non-core allocation is blocked by S. When it combines with the core-equivalence theorem, this theorem shows that the only allocation against which there is no coalition of weight $\alpha \in (0, 1)$ proposing a deviation are the competitive ones. One of the implications of Vind's theorem is normative in the following sense: since arbitrarily large-sized coalitions are entitled to block each non-core allocation, the core can be seen as a solution concept supported by an arbitrarily large majority of agents. The proof of these results relies on the validity of the Lyapunov convexity theorem for the range of a finite-dimensional vector measure. Therefore, no similar simple conclusions hold true as long as there are some atoms in the economy or infinitely many commodities. Furthermore, Vind's [31] theorem depends on the strong monotonicity assumption. Therefore, as the free-disposal condition may not be satisfied for a model involving arbitrary consumption sets, such an assumption cannot be directly applied.

The purpose of the paper is to study the above specified problems in a very general setting. The economic activity is taken into account uncertainty, where agents subscribe to contracts at the time $\tau = 0$ (ex-ante) that are contingent upon the realized state of nature at time $\tau = 1$ (ex-post), in a way so that their expected payoff is maximized. Consumption sets are assumed to be arbitrary subsets of the commodity space, which are primarily motivated by real-life constraints imposed on the consumption of each individual agent. Consumption of certain commodities may be rationed in some scenarios, certain commodities may not feature at all in the consumption of an individual agent, etc, all this corresponds to the consumption set being generally confined to an arbitrary subset of the commodity space. Uncertainty is also a cause for arbitrary consumption sets. In the words of Radner [27], uncertainty lies at the heart of most real-life economic decisions. These uncertainties are reflected in the various states of the world. Thus, each individual agent devising a consumption decision that plans for each possible commodity in each of the possible states of the world posits a serious computational problem. Computational limitations in such scenarios lead consumption to be restricted to a subset of all total possibilities. In this paper, we assume that the consumption set of an individual agent is an arbitrary subset of an ordered (not necessarily separable) Banach space whose positive cone has a non-empty interior. Our extension is primarily motivated by the amendments of Bewley [6] made to the classical Arrow-Debreu-Mckenzie model by permitting the dimensionality of commodity space to be infinite. Infinite dimensional commodity spaces arise quite naturally in general equilibrium analysis. Primarily the following three scenarios lead to studying infinite dimensional commodity spaces: (i) Dealing with an infinite time horizon; (iii) Infinitely many states of nature; and (iii) Infinitely many variations in commodity characteristics. Our economy is also assumed to involve information asymmetry among individuals with the exact feasibility condition. In the above setup, we show that the Aubin core coincides with the core and investigate the core allocations in terms of size and diameter of coalitions, extending those obtained by Schmeidler [29], Grodal [18], and Vind [31]. However, there are three main challenges with our framework.

- The arbitrary consumption set in the case of our economy leads to a violation of the free disposal condition. This, in turn, leads to the failure of standard assumptions like strict monotonicity.

- The asymmetric information setup, along with exact feasibility, possesses certain challenges with the technicalities of the results.

- The Lyapunov convexity theorem fails to hold in its original form due to the infinite dimensionality of the commodity space. It only holds in a weaker form. Moreover, the commodity space is not necessarily separable.

All these challenges together make a huge difference in the contribution of this paper from either the finite-dimensional counterpart or the scenario without asymmetric information and exact feasibilty, or the scenario with the non-negative orthant as the commodity space.[1] We show that all these difficulties can be overcome with a suitable non-satiation condition. In fact, the major findings in this regard are Proposition 3.4 and Proposition 4.1. While Proposition 3.4 claims that the weak Aubin blocking can be replaced by a strong ordinary blocking, Proposition 4.1 is an extension of the Lyapunov convexity theorem in its exact form for Bochner integrable functions in our general setting. As corollaries of these two results, one immediately obtains the equivalence between the Aubin core and the core,[2] and the Schmeider [29] type of result in our framework. We also formulate, as applications of Proposition 4.1, the Grodal [18] theorem and the Vind [31] theorem in their standard formulation (i.e., for ordinary coalitions) for non-atomic agents; the last one of these also requires the validity of an additional result (Proposition 4.4).

Recognized that the atomless model corresponds to an ideal extreme case, as the competition in real economic exchange is far from being perfect. Thus, it is worth considering the above

[1] For comparison with the special cases, we refer to Bhowmik [7], Bhowmik and Cao [8, 9], Bhowmik and Graziano [11, 12], Evren and Hüsseinov [14], Graziano and Romaniello [17], Hervés-Beloso Et al. [19], Hervés-Beloso Et al. [20], Khan [22], Pesce [25, 26], among others.

[2] We also obtain coincidence of the core and the strong core to centain extent, as an application of Proposition 3.4.

problems concerning the size and diameter of blocking coalitions for a mixed-measure space of agents. As a consequence, the blocking procedure in our model involves generalized (or Aubin) coalitions defining the so-called Aubin core (see [4], [11], [24]). Therefore, we can conclude that, with infinitely many commodities, arbitrary consumption sets, information asymmetry and the exact feasibility, and atoms, any given allocation outside the Aubin core can be improved by a coalition of arbitrary small or large size. From a technical point of view, this result, when compared with the one in Vind [31], involves a fourth level of generality: with respect to the measure space, to the commodity space, to the consumption sets, and to the information asymmetry and exact feasibility. Interestingly, the problem related to the presence of atoms for the measure μ does not affect the results in the "Aubin" sense, since, in view of Proposition 3.4 and Proposition 4.1, we construct a correspondence between the (Aubin) core of the mixed economy and the core of an associated atomless economy (see Theorem 4.9 and Theorem 4.10), which is an extension of the relationship between core allocations of the two economies in line with Greenberg and Shitovitz [15]. In virtue of Theorem 4.9 and Theorem 4.10 and the core-equivalence theorem in Angeloni and Martins-da-Rocha [2] for an atomless economy, we can extend the core-equivalence result in Greenberg and Shitovitz [15] and Shitovitz [28] to a mixed with asymmetric information and finitely many commodities. In contrast to them, our result does not depend on the number of atoms, as in the case of Basile et al. [5] for a similar result in a public good setup. As a further result, we formulate and prove a suitable version of Grodal's theorem in [18] for an economy with atoms and generalized coalitions. In our Vind's and Grodal's theorem, we show that small agents in the blocking generalized coalition of a fixed measure behave in the same way as in an ordinary coalition, which means they use their full initial endowments. This significantly extends the scope of the theory, incorporating a much larger class of models as it involves the four aspects together: negligible as well as non-negligible agents, infinite-dimensional commodity spaces, uncertainty with asymmetric information and exact feasibility, and arbitrary consumption sets.

The paper is organized as follows. Section 2 is attributed to describing the economic model. Section 3 introduces various core notions and the relationship between the core notions. Section 4 deals with the Schmeidler-Grodal-Vind theorem in an economy with or without atoms. Section 5 summarizes our findings. Section 6 is the appendix of our paper, which contains all the proof.

2 Description of the model

We consider a standard pure exchange economy with uncertainty and asymmetric information. We assume that the economic activity takes place over two periods $\tau = 0, 1$. The exogenous uncertainty is described by a measurable space $(\Omega, \mathscr{F})$, where Ω is a finite set denoting all possible states of nature at time $\tau = 1$ and the σ-algebra $\mathscr{F}$ denotes all events. At time $\tau = 0$ (ex-ante stage) there is uncertainty about the state of nature that will be realized at time $\tau = 1$ (ex-post stage). At the ex-ante stage, agents arrange contract on redistribution of their initial endowments. At $\tau = 1$, agents carry out previously made agreements, and consumption takes place[3].

Economic agents: The space of **economic agents** is described by a complete probability space $(T, \mathscr{T}, \mu)$, where T represents the set of agents, the σ-algebra $\mathscr{T}$ represents the collections of allowable coalitions whose economic weights on the market are given by μ. Since $\mu(T) < \infty$, the set T of agents can be decomposed in the disjoint union of an atomless sector T_0 of non-influential (small or negligible) agents and the set T_1 of influential (large or non-negligible) agents, which is the union of at most countable family $\{A_1, A_2, \cdots\}$ of atoms of μ. Abusing notation, we also denote by T_1 the collection $\{A_1, A_2, \cdots\}$. Thus, the space of agents not only allow us to investigate in a unified manner the markets that are competitive and the markets that are not, but also deal with the simultaneous action of influential and non-influential agents. This general representation permits to cover simultaneously the case

[3]For simplicity, we assume that there are no endowments and thus no consumption at $\tau = 0$. Hence, agents are only concerned with allocating their second period ($\tau = 1$) endowments.

of an economy with a finite set of agents (when T_0 is empty and T_1 is finite), the case of an atomless economy (when T_1 is empty), the case of mixed markets in which an ocean of negligible agents coexists with few influential agents (when both T_0 and T_1 have positive measure). Moving from this representation, we can also identify two relevant subfamilies from $\mathscr{T}$ by defining

$$\mathscr{T}_0 := \{S \in \mathscr{T} : S \subseteq T_0\} \text{ and } \mathscr{T}_1 := \{S \in \mathscr{T} : T_1 \subseteq S\}.$$

Thus, $\mathscr{T}_0$ is a subfamily of $\mathscr{T}$ containing no atoms whereas $\mathscr{T}_1$ is a subfamily of $\mathscr{T}$ containing all atoms. Finally, we denote by

$$\mathscr{T}_2 := \mathscr{T}_0 \cup \mathscr{T}_1 = \{S \in \mathscr{T} : S \in \mathscr{T}_0 \text{ or } S \in \mathscr{T}_1\}$$

the subfamily of $\mathscr{T}$ formed by coalitions containing either no atoms or all atoms.

Commodity Spaces: The **commodity space** in our model is an ordered Banach space with the interior of the positive cone is non-empty. We denote by $\mathbb{Y}$ the commodity space of our economy whereas the notation $\mathbb{Y}_+$ is employed to denote the positive cone of $\mathbb{Y}$. Let $\mathbb{Y}_{++}$ be the interior of $\mathbb{Y}_+$.

Defining an economy: We introduce a mixed economy with uncertainity and asymmetric information, and an ordered Banach space whose positive cone has non-empty interior as the commodity space.

Definition 2.1. An **economy** is defined as $\mathscr{E} := \{(X_t, \mathscr{F}_t, u_t, e(t,\cdot), \mathbb{P}_t) : t \in T\}$ with the following specifications:

(A) $X_t : \Omega \rightrightarrows \mathbb{Y}$ denotes the (state-contingent) consumption set of agent $t \in T$[4];

(B) $\mathscr{F}_t$ is the σ-algebra generated by a measurable partition $\mathscr{P}_t$ of Ω (i.e. $\mathscr{P}_t \subseteq \mathscr{F}$) denoting the private information of agent t;

[4]Notice that we do not impose non-negative constraints on consumption sets. Thus, short sales are allowed.

(C) $u_t : \Omega \times \mathbb{Y} \to \mathbb{R}$ is the state-dependent utility function of agent t;

(D) $e(t, \cdot) : \Omega \to \mathbb{Y}$ is the random initial endowment of agent t;

(E) $\mathbb{P}_t : \Omega \to [0, 1]$ is the prior of agent t.

Available Information and Expected Utilities: The family of all paritions of Ω is denoted by $\mathfrak{P}$. Since Ω is finite, $\mathfrak{P}$ has only finitely many different elements: $\mathscr{P}_1, \cdots, \mathscr{P}_n$. We assume that $T_i := \{t \in T : \mathscr{P}_t = \mathscr{P}_i\}$ is $\mathscr{T}$-measurable for all $1 \leq i \leq n$. For every $1 \leq i \leq n$, define $\mathscr{G}_i$ to be the set of all functions $\varphi : \Omega \to \mathbb{Y}$ such that φ is $\mathscr{P}_i$-measurable.[5] For any $x : \Omega \to \mathbb{Y}$, define the **ex-ante expected utility** of agent t by the usual formula

$$V_t(x) = \sum_{\omega \in \Omega} u_t(\omega, x(\omega)) \mathbb{P}_t(\omega).$$

We now state our main assumptions to be used throughout the paper.

Assumptions: Consider an economy $\mathscr{E}$ as defined in Definition 2.1.

($\mathbf{A}_1$) For all $(t, \omega) \in T \times \Omega$, $X_t(\omega)$ is a closed convex cone.

($\mathbf{A}_2$) The correspondence $\mathbf{\Theta} : T \times \Omega \rightrightarrows \mathbb{Y}$, defined by $\mathbf{\Theta}(t, \omega) := X_t(\omega)$, is such that $\mathbf{\Theta}(\cdot, \omega)$ is $\mathscr{T}$-measurable for all $\omega \in \Omega$.

($\mathbf{A}_3$) The mapping $e(\cdot, \omega) : T \to \mathbb{Y}$ is $\mathscr{T}$-measurable for all $\omega \in \Omega$ and $e(t, \omega)$ is an interior point of $X_t(\omega)$ for all $\omega \in \Omega$.

($\mathbf{A}_4$) The mapping $\varphi : T \to [0, 1]^\Omega$, defined by $\varphi(t) = \mathbb{P}_t$, is $\mathscr{T}$-measurable.

($\mathbf{A}_5$) For all $(t, \omega) \in T_1 \times \Omega$, $u_t(\omega, \cdot)$ is quasi-concave.

($\mathbf{A}_6$) For all $(t, \omega) \in T \times \Omega$, $u_t(\omega, \cdot)$ is continuous and for all $x \in \mathbb{Y}$, $t \mapsto u_t(\omega, x)$ is $\mathscr{T}$-measurable.

($\mathbf{A}_7$) For all $(t, \omega) \in T \times \Omega$, $u_t(\omega, y) > u_t(\omega, x)$ for all $x, y \in X_t(\omega)$ with $y \geq x$ and $x \neq y$.

[5]By $\mathscr{P}_i$-measurability, we mean the measurability with respect to the σ-algebra generated by $\mathscr{P}_i$.

($\mathbf{A}_8$) For all $(t,\omega) \in T \times \Omega$, $x \in X_t$, separable closed linear subspace $\mathbb{Z}$ of $\mathbb{Y}^\Omega$ satisfying $e(T,\cdot) \subseteq \mathbb{Z}$ and $\varepsilon > 0$, there is an $y \in \mathbb{Z} \cap \mathscr{G}_t \cap \mathbb{B}(0,\varepsilon)^\Omega$ such that $x+y \in X_t$ and $u_t(\omega, x(\omega)+y(\omega)) > u_t(\omega, x(\omega))$[6].

($\mathbf{A}_8'$) For all $(t,\omega) \in T \times \Omega$, $x \in X_t$, separable closed linear subspace $\mathbb{Z}$ of $\mathbb{Y}^\Omega$ satisfying $e(T,\cdot) \subseteq \mathbb{Z}$ and $\varepsilon > 0$, there is an $y \in \mathbb{Z} \cap \bigcap\{\varepsilon\mathscr{G}_i : i \in \mathbb{K}\} \cap \mathbb{B}(0,\varepsilon)^\Omega$ such that $x+y \in \mathrm{int}X_t$ and $u_t(\omega, x(\omega)+y(\omega)) > u_t(\omega, x(\omega))$.

Remark 2.2. The assumptions in ($\mathbf{A}_1$)-($\mathbf{A}_7$) are standard in the literature of general equilibrium in economies with asymmetric information and/ or restricted consumption sets. Assumptions ($\mathbf{A}_8$) and ($\mathbf{A}_8'$) are satisfied under the monotonity assumption whenever $X_t(\omega) = \mathbb{Y}_+$ for all $(t,\omega) \in T \times \Omega$.

3 The Ex-ante Core

Our aim in this section is to introduce the ex-ante (Aubin) core allocations in an economy with a mixed-measure space of agents by considering ordinary (generalized) coalitions and providing the equivalence between the ex-ante Aubin core allocations and ex-ante core allocations. To this end, we introduce the notion of strong and weak blocking for a two-period economy with uncertainty. We assume implicitly that the trade takes place at time $\tau = 0$ and that contracts are binding: they are carried out after the resolution of uncertainty, and there is no possibility of their renegotiation. Moreover, the consumption of each agent is compatible with her private information. We now introduce the concept of an allocation, which is a specification of the amount of commodities assigned to each agent.

Definition 3.1. An **allocation** in $\mathscr{E}$ is a Bochner integrable function $f : T \times \Omega \to \mathbb{Y}$ such that

(i) $f(t,\omega) \in X_t(\omega)$ for all $(t,\omega) \in T \times \Omega$; and

[6] $\mathbb{B}(0,\varepsilon)$ denotes the closed ball centered at 0 and radius ε in $\mathbb{Y}$.

(ii) $f(t,\cdot) \in \mathscr{G}_i$ for all $t \in T_i$ and all $1 \leq i \leq n$.

It is said to be **feasible** if $\int_T f(\cdot,\omega)d\mu = \int_T e(\cdot,\omega)d\mu$ for all $\omega \in \Omega$. We assume that e is an allocation.

An element of $\mathscr{T}$ with positive measure is interpreted as an **ordinary coalition** or simply, a **coalition** of agents. Each $S \in \mathscr{T}$ can be regarded as a function $\chi_S : T \to \{0,1\}$, defined by

$$\chi_S(t) := \begin{cases} 1, & \text{if } t \in S; \\ 0, & \text{otherwise.} \end{cases}$$

Here, $\chi_S(t)$ means the degree of membership of agent $t \in T$ to the coalition S. Following this interpretation for an ordinary coalition, it is natural to introduce a family of **generalized coalitions** as follows (see [24]). To this end, for any function $\gamma : T \to \mathbb{R}$, define the **support** of the function γ as

$$S_\gamma := \{t \in T : \gamma(t) \neq 0\}.$$

An **Aubin** or a **generalized coalition** of $\mathscr{E}$ is a simple, measurable function $\gamma : T \to \mathbb{R}$ whose support has a positive measure. It is worthwhile to point out that $\gamma(t)$ represents the share of resources employed by agent t. By identifying $S \in \mathscr{T}$ with χ_S, we can treat S as a generalized coalition. The **weight of a generalized coalition** γ, denoted by $\mu^A(\gamma)$, is given by $\mu^A(\gamma) := \int_T \gamma \, d\mu$. For any ordinary coalition, this weight simply coincides with the measure of the coalition itself.

Our first notion of (Aubin) core aims to study the blocking mechanism under the assumptions that a coalition deviates from a proposed allocation if its members guarantee a *strictly better* commodity bundles for themselves by the redistribution.

Definition 3.2. An allocation f is **ex-ante blocked by a generalized coalition** γ if there is an allocation g such that $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on S_γ, and

$$\int_T \gamma g(\cdot,\omega)d\mu = \int_T \gamma e(\cdot,\omega)d\mu$$

for all $\omega \in \Omega$. The **ex-ante Aubin core** of $\mathscr{E}$, denoted by $\mathscr{C}^A(\mathscr{E})$, is the set of feasible allocations that are not ex-ante blocked by any generalized coalition. If the generalized coalitions are replaced with ordinary coalitions, the corresponding set of allocations is called the **core** of $\mathscr{E}$, denoted by $\mathscr{C}(\mathscr{E})$.

The next formalization of core differs from the earlier one in the sense that agents within a blocking generalized coalition are not *worse-off* by the re-distribution whereas some are *strictly better-off*. To formally define, we consider a **sub-coalition** of a generalized coalition γ is a generalized coalition ρ such that $S_\rho \subseteq S_\gamma$.

Definition 3.3. An allocation f is **ex-ante weakly blocked by a generalized coalition** γ if there is a sub-coalition ρ of γ and an allocation g such that

(i) $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on S_ρ;

(ii) $V_t(g(t,\cdot)) \geq V_t(f(t,\cdot))$ μ-a.e. on S_γ; and

(iii) $\int_T \gamma g(\cdot,\omega)d\mu = \int_T \gamma e(\cdot,\omega)d\mu$ for all $\omega \in \Omega$.

The **ex-ante Aubin strong core** of $\mathscr{E}$, denoted by $\mathscr{C}^{AS}(\mathscr{E})$, is the set of feasible allocations that are not ex-ante weakly blocked by any generalized coalition. If the generalized coalitions are replaced with ordinary coalitions, the corresponding set of allocations is called the **ex-ante strong core** of $\mathscr{E}$, denoted by $\mathscr{C}^S(\mathscr{E})$.

Recognized that if an allocation f is ex-ante blocked by a generalized coalition γ, it is also weakly blocked by the same coalition. For the converse, we additionally assume in our next result that if an allocation f is ex-ante weakly blocked by a generalized coalition γ via some allocation g and if ρ is a sub-coalition of γ in which members of S_ρ strictly prefer g to f, then the information available to both coalitions are the same, i.e, $\mathbb{I}_{S_\gamma} = \mathbb{I}_{S_\rho}$, where

$$\mathbb{I}_{S_\gamma} := \{i : \mu(S_\gamma \cap T_i) > 0\}.$$

The basic intuition is that members belonging to $R_i := R \cap T_i$, where $R := S_\gamma \setminus S_\rho$ and $i \in \mathbb{I}_R$, can be allocated $\mathscr{P}_i$-measurable consumption bundles that give higher utilities by reducing the utility level of the members of $S_\rho \cap T_i$ due to continuity and strong monotonicity. However, such an argument cannot be applied readily in arbitrary consumption sets. In what follows, we establish this result in a continuum economy by showing that if an allocation is ex-ante weakly blocked by a generalized coalition, it can also be blocked by an ordinary coalition. In this regard, Lemma 6.1 and Lemma 6.2 in Appendix play vital roles.

Theorem 3.4. *Let $\mathscr{E}$ be a continuum economy satisfying $(\mathbf{A}_1)$-$(\mathbf{A}_8)$. Suppose that γ is a generalized coalition and g is an allocation such that $V_t(g(t,\cdot)) \geq V_t(f(t,\cdot))$ μ-a.e. on S_γ. Assume further that the measurable set B, defined by*

$$B := \{t \in S_\gamma : V_t(g(t,\cdot)) > V_t(f(t,\cdot))\},$$

has positive measure and $\mathbb{I}_{S_\gamma} = \mathbb{I}_B$. Let $\mathbb{Z}$ be a separable closed linear subspace of $\mathbb{Y}^\Omega$ such that

$$f(T,\cdot) \cup g(T,\cdot) \cup e(T,\cdot) \subseteq \mathbb{Z}.$$

Then there are coalitions E, R, an element $\lambda_0 \in (0,1)$, an element $\eta > 0$, and an allocation y such that

(i) $R \subseteq E \subseteq S_\gamma$ *and* $\mathbb{I}_R = \mathbb{I}_E = \mathbb{I}_{S_\gamma}$;

(ii) $\int_E (y - e)\, d\mu = \lambda_0 \int_T \gamma(g - e)\, d\mu$;

(iii) $y(t,\cdot) + z \in \mathbb{Z} \cap X_t$ *for all* $z \in \mathbb{Z} \cap \mathbb{B}(0,\eta)^\Omega$ *and μ-a.e. on R; and*

(iv) $V_t(y(t,\cdot) + z) > V_t(f(t,\cdot))$ *for all* $z \in \mathbb{Z} \cap \mathbb{B}(0,\eta)^\Omega$ *and μ-a.e. on R.*

Proof. The proof of the proposition is relegated to Appendix. □

Corollary 3.5. *For a continuum economy $\mathscr{E}$ satisfying $(\mathbf{A}_1)$-$(\mathbf{A}_8)$ and an allocation f, assume that f is ex-ante weakly blocked by a generalized coalition γ via some allocation*

g satisfying $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on S_ρ for some sub-coalition ρ of γ satisfying $\mathbb{I}_{S_\gamma} = \mathbb{I}_{S_\rho}$. Then there is a coalition E such that f is blocked by E.

Remark 3.6. Notice that, in the proof of Proposition 3.4, the number λ_0 can be chosen sufficiently close to 1. Moreover, it also follows that, if γ is replaced with some ordinary coalition S, then the coalition E can be choosen so that $\mu(E) \geq \lambda_0 \mu(S)$.

In view of the above proposition, we have the following theorem whose proof is immediate.

Theorem 3.7. *Suppose that $\mathscr{E}$ is a continuum economy satisfying $(\mathbf{A}_1)$-$(\mathbf{A}_8)$. Then $\mathscr{C}^A(\mathscr{E}) = \mathscr{C}(\mathscr{E})$*

4 The Schmeidler-Grodal-Vind theorems

In this section, we study the characterizations of ex-ante core allocations by means of the size of coalitions in an economy either containing a continuum of negligible agents or comprised of both negligible and non-negligible agents. To establish the results of a mixed economy, we introduce an atomless economy associated with the mixed economy and investigate the connection between the core allocations of these two economies.

4.1 The size of blocking coalitions in a continuum economy

In this subsection, we address the issues related to the size of a blocking coalition, extending the corresponding results of Schmeidler [29], Grodal [18] and Vind [31] to the case of an atomless economy with arbitrary consumption sets and private information.

Extending the Schmeidler theorem: The insight of Schmeidler theorem was that, in an atomless economy, if a feasible non-core allocation is blocked by some coalition S, then it can also be blocked by a coalition of any given measure less than that of S. The immediate implication of this theorem includes the fact that the core (and thus, the set of competitive allocations) can be implemented by the formation of small coalitions only. In what follows, we extend this result to our framework. This extends the corresponding result of Bhowmik and Graziano [12] to a certain extent. It is worthwhile to point out that the techniques adopted in the proof of Bhowmik and Graziano [12] are inappropriate in our setup of infinitely many commodities. Thus, we first establish the following proposition to obtain the Schmeidler theorem in our framework. This proposition can be considered an extension of the Lyapunov convexity theorem.

Proposition 4.1. *Let $\mathscr{E}$ be a mixed economy and let the assumptions $(\mathbf{A}_1)$-$(\mathbf{A}_8)$ be satisfied. Suppose that ψ, f and g are allocations such that $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on some coalition $S \in \mathscr{T}_0$. Let $\mathbb{Z}$ be a separable closed linear subspace of $\mathbb{Y}^\Omega$ such that*

$$f(T,\cdot) \cup g(T,\cdot) \cup \psi(T,\cdot) \subseteq \mathbb{Z}.$$

Assume further that there is a sub-coalition R of S such that $\mathbb{I}_R = \mathbb{I}_S$ and for each $t \in R$ there is some $\eta_t > 0$ such that $g(t,\cdot) + z \in X_t$ for all $z \in \mathbb{Z} \cap \mathbb{B}(0,\eta_t)^\Omega$. Assume further that $\mu(S \cap H) \geq \alpha$ for some coalition H of $\mathscr{E}$ and some $\alpha > 0$. Then there are an $\eta_0 > 0$, two coalitions B and C, and an allocation φ such that

(i) $C \subseteq B \subseteq S$, $\mathbb{I}_C = \mathbb{I}_B = \mathbb{I}_S$ *and* $\mu(B \cap H) = \alpha$;

(ii) $\varphi(t,\cdot) + z \in \mathbb{Z} \cap X_t$ *for all* $z \in \mathbb{Z} \cap \mathbb{B}(0,\eta_0)^\Omega$ *and μ-a.e. on* C;

(iii) $V_t(\varphi(t,\cdot) + z) > V_t(f(t,\cdot))$ *for all* $z \in \mathbb{B}(0,\eta_0)^\Omega$ *and μ-a.e. on* C;

(iv) $V_t(\varphi(t,\cdot)) > V_t(f(t,\cdot))$ *μ-a.e. on* $B \setminus C$; *and*

(v) $\int_B(\varphi(\cdot,\omega) - \psi(\cdot,\omega))d\mu = \frac{\alpha}{\mu(S\cap H)} \int_S(g(\cdot,\omega) - \psi(\cdot,\omega))d\mu$ *for all* $\omega \in \Omega$.

Proof. The proof of the proposition is relegated to Appendix. □

The following theorem is an immediate implication of the above proposition, which extends Schmeidler's [29] theorem to our framework.

Theorem 4.2. *Consider an atomless economy $\mathscr{E}$ and assume that the assumptions* $(\mathbf{A}_1)$-$(\mathbf{A}_8)$ *are satisfied. Let f be an allocation of $\mathscr{E}$ blocked by some coalition S. Then, for any $\varepsilon \in (0, \mu(S))$, there is a coalition R such that $\mu(R) = \varepsilon$ and f is ex-ante blocked by R.*

Proof. The proof of the theorem is relegated to Appendix. □

Extending the Grodal Theorem: Given an $\varepsilon > 0$, it was shown in [18] for an atomless economy that the blocking coalition S can be chosen as a union of finitely many disjoint sub-coalitions, each of which having measure and diameter less than ε. A coalition whose measure and diameter both are less than ε intuitively means that the coalition consists of relatively *few* agents and that the agents in the coalition resemble one another in chosen characteristics. Next, we extend this result to an atomless economy, where the consumption sets of agents are arbitrary subsets of an ordered Banach space having the non-empty interior of the positive cone.

Theorem 4.3. *Let $\mathscr{E}$ be an atomless economy such that $\mathscr{E}$ satisfies* $(\mathbf{A}_1)$-$(\mathbf{A}_8)$*. Suppose that T is endowed with a pseudometric, which makes T a separable topological space such that $\mathscr{B}(T) \subseteq \mathscr{T}$. For any feasible allocation $f \notin \mathscr{C}(\mathscr{E})$ and any $\varepsilon, \delta > 0$, there exists a coalition S with $\mu(S) \leq \varepsilon$ ex-ante blocking f and satisfying the following:*

(i) *There exists some $\alpha > 0$ such that any coalition $F \subseteq S$ satisfying $\mu(S \setminus F) < \alpha$ ex-ante blocking f; and*

(ii) *$S = \bigcup\{S_i : 1 \leq i \leq n\}$ for a finite collection of coalitions $\{S_1, \cdots, S_n\}$ with diameter of S_i smaller than δ for all $i = 1, \cdots, n$.*

Extending the Vind theorem: Vind's theorem (refer to [31]) states that, in an atomless economy, if a feasible allocation is not in the core of the economy, then there is a blocking coalition of any given measure less than the measure of the grand coalition. Thus, the core allocations (and hence, the competitive allocations) can also be characterized by means of coalitions of arbitrarily large sizes. We now intend to show a similar result in our framework.

To this end, we first establish the following result, which claims that if an allocation is blocked by a coalition S via some allocation g then there is another allocation h in which everyone achieves better utility than she gets under f. This Proposition extends the corresponding results in Bhowmik and Cao [10] and Hervés-Beloso and Moreno-García [21].

Proposition 4.4. *Let $\mathscr{E}$ be an atomless economy such that the assumptions $(\mathbf{A}_1)$-$(\mathbf{A}_8)$ are satisfied. Suppose that f and g are two allocations such that $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on some coalition S. Let $\mathbb{Z}$ be a separable closed linear subspace of $\mathbb{Y}^\Omega$ such that $f(T,\cdot) \cup g(T,\cdot) \subseteq \mathbb{Z}$. Assume further that there is a sub-coalition R of S such that $\mathbb{I}_R = \mathbb{I}_S$ and for each $t \in R$ there is some $\eta_t > 0$ such that $g(t,\cdot) + z \in X_t$ for all $z \in \mathbb{Z} \cap \mathbb{B}(0,\eta_t)^\Omega$. Then, for any $0 < \delta < 1$, there exist some allocation h and some $\eta > 0$ such that $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on S; $h(t,\cdot) + z \in \mathbb{Z} \cap X_t$ for all $z \in \mathbb{Z} \cap \mathbb{B}(0,\eta)^\Omega$ for some $\eta > 0$ and μ-a.e. on G for some sub-coalition G of S with $\mathbb{I}_G = \mathbb{I}_S$; and*

$$\int_S h(\cdot,\omega)d\mu = \int_S (\delta g(\cdot,\omega) + (1-\delta) f(\cdot,\omega))d\mu$$

for all $\omega \in \Omega$.

Proof. The proof of the proposition is relegated to Appendix. □

Corollary 4.5. Consider now a mixed economy where all large agents have continuous and quasi-concave utility functions. For any large agent A and $x, y \in X_A$, if $V_A(y) > V_A(x)$ and $0 < \delta < 1$ then, by Lemma 5.26 of Aliprantis and Border [1], we have $V_A(\delta y + (1-\delta)x) > V_A(x)$. In view of this, the conclusion of Proposition 4.4 can be obtained in a mixed model.

Next, we formulate a version of Vind's (1972) theorem on blocking by an arbitrary coalition.

Theorem 4.6. *Consider a continuum economy $\mathscr{E}$ in which the assumptions $(\mathbf{A}_1)$-$(\mathbf{A}'_8)$ are satisfied. Let f be a feasible allocation such that $f \notin \mathscr{C}(\mathscr{E})$. Then for any $\varepsilon \in (0,1)$, there is some coalition R such that $\mu(R) = \varepsilon$ and f is blocked by R.*

Proof. The proof of the theorem is relegated to Appendix. □

4.2 Interpretation via a mixed economy

In this subsection, we associate $\mathscr{E}$ with an atomless economy $\widetilde{\mathscr{E}}$ and study the connection between the ex-ante (Aubin) core allocations of these two economies. This extends the result of Greenberg and Shitovitz [15] and some of its follow-up papers as mentioned in Section 1.

Given the economy $\mathscr{E}$, the economy $\widetilde{\mathscr{E}}$ is obtained by *splitting* each large agent into a continuum of small agents whose characteristics are the same as that of large agent. Therefore, the space of agents of $\widetilde{\mathscr{E}}$, denoted by $(\widetilde{T}, \widetilde{\mathscr{T}}, \widetilde{\mu})$, satisfies the following: (i) $\widetilde{T}_0 = T_0$ and $\widetilde{\mu}(\widetilde{T}_1) = \mu(T_1)$, where $\widetilde{T}_1 := T \setminus T_0$; (ii) $\widetilde{\mathscr{T}}$ and $\widetilde{\mu}$ are obtained by the direct sum of $\mathscr{T}$ and μ restricted to T_0 and the Lebesgue atomless measure space over $\widetilde{T}_1$; and (iii) each atom A_i one-to-one corresponds to a Lebesgue measurable subset $\widetilde{A_i}$ of $\widetilde{T}_1$ such that $\mu(A_i) = \widetilde{\mu}(\widetilde{A_i})$, where $\{\widetilde{A_i} : i \geq 1\}$ can be expressed as the disjoint union of the intervals $\{\widetilde{A_i} : i \geq 1\}$ given by $\widetilde{A_1} := [\mu(T_0), \mu(T_0) + \mu(A_1))$, and

$$\widetilde{A_i} := \left[\mu(T_0) + \mu\left(\bigcup_{j=1}^{i-1} A_j\right),\ \mu(T_0) + \mu\left(\bigcup_{j=1}^{i} A_j\right)\right),$$

for all $i \geq 2$. Furthermore, the space of states of nature and the commodity space of $\widetilde{\mathscr{E}}$ are the same as those of $\mathscr{E}$. Finally, the characteristics $(\widetilde{X}_t \widetilde{\mathscr{F}}_t, \widetilde{u}_t, \widetilde{e}(t, \cdot), \widetilde{\mathbb{P}}_t)$ of each agent $t \in \widetilde{T}$ in $\widetilde{\mathscr{E}}$ is defined as follows:

$$\widetilde{X}_t := \begin{cases} X_t, & \text{if } t \in T_0; \\ X_{A_i}, & \text{if } t \in \widetilde{A_i}, \end{cases}$$

$$\widetilde{\mathscr{F}_t} := \begin{cases} \mathscr{F}_t, & \text{if } t \in T_0; \\ \mathscr{F}_{A_i}, & \text{if } t \in \widetilde{A_i}, \end{cases}$$

$$\widetilde{u}_t := \begin{cases} u_t, & \text{if } t \in T_0; \\ u_{A_i}, & \text{if } t \in \widetilde{A_i}, \end{cases}$$

$$\widetilde{e}(t,\cdot) := \begin{cases} e(t,\cdot), & \text{if } t \in T_0; \\ e(A_i,\cdot), & \text{if } t \in \widetilde{A_i}, \end{cases}$$

and

$$\widetilde{\mathbb{P}}_t := \begin{cases} \mathbb{P}_t, & \text{if } t \in T_0; \\ \mathbb{P}_{A_i}, & \text{if } t \in \widetilde{A_i}. \end{cases}$$

We now introduce some notations for the rest of the section. To an allocation f in $\mathscr{E}$, we associate an allocation $\widetilde{f} := \Xi[f]$ in $\widetilde{\mathscr{E}}$, defined by

$$\widetilde{f}(t,\omega) := \begin{cases} f(t,\omega), & \text{if } (t,\omega) \in T_0 \times \Omega; \\ f(A_i,\omega), & \text{if } (t,\omega) \in \widetilde{A_i} \times \Omega. \end{cases}$$

Reciprocally, for each allocation $\widetilde{f}$ in $\widetilde{\mathscr{E}}$, we define an allocation $f := \Phi[\widetilde{f}]$ in $\mathscr{E}$ such that

$$f(t,\omega) := \begin{cases} \widetilde{f}(t,\omega), & \text{if } (t,\omega) \in T_0 \times \Omega; \\ \frac{1}{\widetilde{\mu}(\widetilde{A_i})} \int_{\widetilde{A_i}} \widetilde{f}(\cdot,\omega)\, d\widetilde{\mu}, & \text{if } t = A_i \text{ and } \omega \in \Omega. \end{cases}$$

Recognized that if f is a feasible allocation in $\mathscr{E}$ then $\Xi[f]$ is a feasible allocation in $\widetilde{\mathscr{E}}$. Similarly, for each feasible allocation $\widetilde{f}$ in $\widetilde{\mathscr{E}}$, the allocation $\Phi[\widetilde{f}]$ is feasible in $\mathscr{E}$.

We show that an allocation is in the ex ante core of a mixed economy assigns indifferent consumption plans to all large agents. This is due to the fact that all agents have the same characteristics.

Proposition 4.7. *Let the assumptions* $(\mathbf{A}_1)$-$(\mathbf{A}'_8)$ *be satisfied for a mixed economy* $\mathscr{E}$. *Let* R *be a coalition in* $\mathscr{T}_1$[7] *having the same characteristics. If* f *is in the ex ante core of* $\mathscr{E}$ *then* $V_t(f(t,\cdot)) = V_t(\mathbf{x}_f)$ μ*-a.e. on* R*, where*

$$\mathbf{x}_f(\omega) := \frac{1}{\mu(R)} \int_R f(\cdot,\omega) d\mu$$

[7] If T_1 is empty then R contains only negligible agents.

for all $\omega \in \Omega$.

Proof. The proof of the proposition is relegated to Appendix. $\square$

Remark 4.8. If $\mathbb{Y}$ is finite-dimensional, then one can dispense with the assumption $(\mathbf{A}_8')$. In fact, the assumption $(\mathbf{A}_8')$ helps us to apply Proposition 4.1 in the proof of Proposition 4.7. In the case of finite dimension, we can use $(\mathbf{A}_8)$ and apply the Lyapunov convexity theorem instead of Proposition 4.1.

Theorem 4.9. *Let* $\mathscr{E}$ *be a mixed economy satisfying the assumptions* $(\mathbf{A}_1)$*-*$(\mathbf{A}_8')$*. If* $\widetilde{f} \in \mathscr{C}(\widetilde{\mathscr{E}})$ *then* $f := \Phi[\widetilde{f}] \in \mathscr{C}^A(\mathscr{E})$.

Proof. The proof of the theorem is relegated to Appendix. $\square$

Theorem 4.10. *Let* $\mathscr{E}$ *be a mixed economy satisfying the assumptions* $(\mathbf{A}_1)$*-*$(\mathbf{A}_8')$*. Suppose also that* $R \in \mathscr{T}_1$ *is a coalition having the same characteristics. Then* $f \in \mathscr{C}(\mathscr{E}) \Rightarrow \widetilde{f} := \Xi[f] \in \mathscr{C}(\widetilde{\mathscr{E}})$ *if either of the following two conditions are true:*

(i) $R = T_1$ *has at least two elements.*

(ii) T_1 *has exactly one element and* $\mu(R \setminus T_1) > 0$.

Proof. The proof of the theorem is relegated to Appendix. $\square$

4.3 The size of blocking coalitions in a mixed economy

In this subsection, we generalize the main results of Subsection 4.1 to a mixed economy by applying the results of

Extending the Grodal Theorem: Next, an extension of Theorem 4.3 to an economy with a mixed measure space of agents is presented. Basically, we show that for any given $\varepsilon, \delta > 0$, there is a generalized coalition γ whose measure is less than ε and γ can be expressed as some of the pairwise disjoint generalized coalitions each of its diameters is less than δ. To this end, we say that two generalized coalitions γ_1 and γ_2 are **disjoint**

if $(\gamma_1 \wedge \gamma_2)(t) := \min\{\gamma_1(t), \gamma_2(t)\} = 0$ for all $t \in T$. As a consequence of this, we have $S_{\gamma_1} \cap S_{\gamma_2} = \emptyset$. Following Gerla and Volpe [16] (see also Bhowmik and Graziano [11]), the **diameter** of a generalized coalition γ is defined by

$$\operatorname{diam}(\gamma) := \sup\left\{\min\{\alpha, \beta\}\|a - b\| : \lambda_a^\alpha, \lambda_b^\beta \text{ are fuzzy points of } \gamma\right\},$$

where a **fuzzy point** λ_a^ξ is a function $\lambda_a^\xi : T \to (0,1]$ for each $a \in T$ and $\xi \in (0,1]$, such that $\lambda_a^\xi(t) = 0$ if $t \neq a$ and $\lambda_a^\xi(t) = \xi$ if $t = a$.

Theorem 4.11. *Let $\mathscr{E}$ be a mixed economy such that $\mathscr{E}$ satisfies $(\mathbf{A}_1)$-$(\mathbf{A}_8)$. Suppose further that T is endowed with a pseudometric which makes T a separable topological space such that $\mathscr{B}(T) \subseteq \mathscr{T}$ and $f \notin \mathscr{C}^A(\mathscr{E})$. For any $\varepsilon, \delta > 0$, there exist a generalized coalition γ with $\mu^A(\gamma) \leq \varepsilon$ and a finite collection $\{\gamma_1, \cdots, \gamma_n\}$ of pairwise disjoint characteristics functions*[8] *of ordinary coalitions such that the diameter of γ_i smaller than δ and $S_{\gamma_i} \subseteq T_0$ for all $i \in \{1, \cdots, n\}$, f is blocked by γ and*

$$\gamma = \begin{cases} \sum_{i=1}^n \gamma_i + \sum_{k \in \mathbb{K}} \alpha_k \chi_{A_k}, & \text{if } \mathbb{K} \neq \emptyset; \\ \sum_{i=1}^n \gamma_i, & \text{if } \mathbb{K} = \emptyset, \end{cases}$$

where $\mathbb{K} := \{k : A_k \in S_\gamma\}$ and $\alpha_k \in (0,1]$ if $k \in \mathbb{K}$.

Remark 4.12. Notice that each sub-coalition γ_i of $\gamma|_{T_0}$ is chosen as the set of agents sharing their full initial endowments and the diameter of γ_j is exactly the same as that of S_{γ_j}. Therefore, agents in γ_i have δ-similar characteristics in the ordinary sense, implying the second part of Theorem 4.3 as a simple corollary. For a mixed economy, the blocking coalition γ contains atmost finitely many atoms, which means that each sub-coalition of diameter δ is either δ-similar non-atomic agents or a single atom, not a neighborhood of points contained in an atom. Since an atom can be treated as δ-similar to itself for any $\delta > 0$, our approach for taking a neighborhood containing a single atom does not violate Grodal's requirements.

[8]Thus, γ_i can be treated as an ordinary coalition.

Therefore, similar to Grodal [18]. Therefore, it can be considered as an extension Grodal's theorem to a mixed economy.

Extending the Vind Theorem: In view of above results and Theorem 4.6, one can readily derive the following result as in Bhowmik and Graziano [11].

Theorem 4.13. *Consider a mixed economy $\mathscr{E}$ in which the assumptions* $(\mathbf{A}_1)$-$(\mathbf{A}'_8)$ *are satisfied. Let f be a feasible allocation such that $f \notin \mathscr{C}^A(\mathscr{E})$. Then for any $\varepsilon \in (0,1)$, there is some Aubin coalition γ such that $\mu^{\mathrm{A}}(\gamma) = \varepsilon$ and f is Aubin blocked by γ.*

Proof. The proof of the theorem is relegated to Appendix. □

Remark 4.14. It is clear from the proof of Theorem 4.13 that for an $\varepsilon > 0$, there exists a generalized coalition γ such that f is blocked by γ with $\tilde{\mu}(\gamma) = \varepsilon$ and $\gamma(t) = 1$ if $t \in S_\gamma \cap T_0$. Thus, as in the case of atomless economies, non-atomic agents in S_γ use their full initial endowments. However, the atomic agents in γ only use parts of their initial endowments and the share α_i for an atomic agent A_i depends on the size of γ. So, Theorem 4.13 can be treated as an extension of that in an atomless economy.

5 Concluding remarks

We have investigated the blocking of any allocation not belonging to the (Aubin, strong) core of a mixed economy with asymmetric information and exact feasibility, infinitely many commodities, and arbitrary consumption sets. We showed that the Aubin core coincides with the core under mild assumptions in an atomless economy. This conclusion trivially follows from Proposition 3.4. It has also been shown, as an application of Proposition 3.4, that the ex-ante core can be characterized by means of coalitions of a given size less than that of the grand coalition in an atomless economy, extending the results in Schmeidler [29] and Vind [31]. Thus, it is enough to consider coalitions of either arbitrarily small sizes or arbitrarily large sizes to find the core of an atomless economy. It is further shown that

Grodal's theorem (refer to Grodal [18]) holds true in an atomless economy. All of these results have also been carried out to a mixed economy by considering generalized coalitions instead of standard coalitions. To do this, we associated an atomless economy with a mixed economy and showed that the Aubin core of the mixed economy is equivalent to the core of the atomless economy. All of these results have been obtained without assuming the separability condition on the commodity space. The difficulties that arise in all of these results are due to the following facts: (i) the standard Lyapunov convexity theorem does not hold, it holds only in a weaker form in an infinite dimensional commodity space; (ii) the consumption sets are arbitrary subsets of the commodity space which may not satisfy the free-disposal condition $X_t + \mathbb{Y}_+^{\Omega} \subseteq X_t$ for $t \in T$, thus the strong monotonicity condition may not be applied whenever required; and (iii) information asymmetry, which further restricts the consumption of each agent and prevents to maintain the exact feasibility condition smoothly in the blocking mechanism. All of these difficulties are taken care of by establishing several key propositions, out of which Proposition 3.4 has its own interest. In fact, as a consequence of this proposition, we conclude that the ex-ante strong core is equivalent to the ex-ante core in an atomless economy under some restriction on information structure.

We close this section with further remarks dealing with possible extensions and applications of our results.

Remark 5.1. Hervés-Beloso and Moreno-García [21] provides a characterization of Walrasian allocations in terms of robustly efficient allocations in an atomless economy. Later, it was extended by Bhowmik and Cao [10] to a mixed economy with asymmetric information and an ordered separable Banach space whose positive cone has an interior point as the commodity space by applying Vind's theorem and a result similar to Proposition 4.4. Thus, in view of Proposition 4.4 and Theorem 4.6, it would be interesting to know whether the main result of Bhowmik and Cao [10] can be extended to our framework.

Remark 5.2. Our paper is confined to infinite dimensional commodity spaces with a non-empty positive interior. However, modeling many different economic scenarios require infinite dimensional spaces without having interior points in their positive cones. For instance, commodity differentiation is modeled through the space $\mathcal{M}(K)$ of signed Borel measure on a compact topological space. It would be interesting to investigate our results in those settings.

6 Appendix

Lemma 6.1. *Suppose that f and g are two allocations such that $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on some coalition S, and $\mathbb{Z}$ is a separable, closed, linear subspace of $\mathbb{Y}^\Omega$ such that $f(S,\cdot) \cup g(S,\cdot) \subseteq \mathbb{Z}$. Assume further that, for each $t \in S$ there is some $\eta_t > 0$ such that $g(t,\cdot) + z \in X_t$ for all $z \in \mathbb{Z} \cap \mathbb{B}(0,\eta_t)^\Omega$. Then for any $0 < \varepsilon < \mu(S)$, there are some $\eta > 0$ and a sub-coalition R of S such that*

(i) $\mu(R) > \mu(S) - \varepsilon$;

(ii) $g(t,\cdot) + z \in \mathbb{Z} \cap X_t$ *for all* $z \in \mathbb{Z} \cap \mathbb{B}(0,\eta)^\Omega$ *and* $(t,\omega) \in R \times \Omega$; *and*

(iii) $V_t(g(t,\cdot) + z) > V_t(f(t,\cdot))$ *for all* $z \in \mathbb{Z} \cap \mathbb{B}(0,\eta)^\Omega$ *and μ-a.e. on R.*

Proof. Define a correspondence $\boldsymbol{\Upsilon} : S \rightrightarrows \mathbb{R}_+$ by letting

$$\boldsymbol{\Upsilon}(t) := \left\{\eta \in (0,\infty) : g(t,\cdot) + z \in X_t \text{ and } V_t(g(t,\cdot) + z) > V_t(f(t,\cdot)) \text{ for all } z \in \mathbb{Z} \cap \mathbb{B}(0,\eta)^\Omega\right\}.$$

By the continuity of preferences and the fact that $g(t,\cdot) + z \in X_t$ for all $z \in \mathbb{Z} \cap \mathbb{B}(0,\eta_t)^\Omega$ for some $\eta_t > 0$ and μ-a.e. on S, we have $\boldsymbol{\Upsilon}(t) \neq \emptyset$ μ-a.e. on S. As $\boldsymbol{\Upsilon}(t)$ is bounded from above, the function $\varphi : S \to \mathbb{R}_+$, defined by $\varphi(t) := \sup \boldsymbol{\Upsilon}(t)$, is well-defined. We show that φ is $\mathscr{T}_S$-measurable. To this end, note that the function $\psi : S \times \mathbb{Z} \to \mathbb{R}$, defined by $\psi(t,z) := V_t(g(t,\cdot) + z) - V_t(f(t,\cdot))$, is a Carathéodory function, and thus, it is $\mathscr{T}_S \otimes \mathscr{B}(\mathbb{Z})$-measurable. Define a correspondence $\mathbf{G} : S \rightrightarrows \mathbb{Z}$ by letting $\mathbf{G}(t) := \{z \in \mathbb{Z} : \psi(t,z) > 0\}$. It follows that $\mathbf{G}$ is non-empty valued and has $\mathscr{T}_S \otimes \mathscr{B}(\mathbb{Z})$-measurable graph, as $\mathrm{Gr}_{\mathbf{G}} = \psi^{-1}(0,\infty)$.

Consider a correspondence $\mathbf{H} : S \rightrightarrows \mathbb{Z}$ defined by

$$\mathbf{H}(t) := \{z \in \mathbb{Z} : g(t,\omega) + z(\omega) \in X_t(\omega) \text{ for all } \omega \in \Omega\}.$$

Due to the closeness of $X_t(\omega)$, $\mathbf{H}(t)$ can be equivalently expressed as[9]

$$\mathbf{H}(t) = \{z \in \mathbb{Z} : \mathrm{dist}(g(t,\omega) + z(\omega), X_t(\omega)) = 0 \text{ for all } \omega \in \Omega\}.$$

In view of the fact that $0 \in \mathbf{H}(t)$, we have $\mathbf{H}(t) \neq \emptyset$ μ-a.e. on S. Moreover, $\mathrm{Gr}_{\mathbf{H}}$ is $\mathscr{T}_S \otimes \mathscr{B}(\mathbb{Z})$-measurable as $\mathrm{Gr}_{\mathbf{H}} = y^{-1}(\{0\})$, where $y : S \times \mathbb{Z} \to \mathbb{R}$ is defined by $y(t,\omega) := \mathrm{dist}(g(t,\omega) + z(\omega), X_t(\omega))$, is $\mathscr{T}_S \otimes \mathscr{B}(\mathbb{Z})$-measurable. Finally, define a correspondence $\mathbf{\Phi} : S \rightrightarrows \mathbb{Z}$ such that $\mathbf{\Phi}(t) := \mathbf{G}(t) \cap \mathbf{H}(t)$ for all $t \in S$. As $0 \in \mathbf{\Phi}(t)$, we have $\mathbf{\Phi}(t) \neq \emptyset$ μ-a.e. on S. Moreover, $\mathrm{Gr}_{\mathbf{\Phi}}$ is $\mathscr{T}_S \otimes \mathscr{B}(\mathbb{Z})$-measurable. Analogously, the correspondence $\mathbf{\Theta}_\eta : S \rightrightarrows \mathbb{Z}$, defined by $\mathbf{\Theta}_\eta(t) := \mathbb{Z} \cap \mathbb{B}(0,\eta)^\Omega$, has $\mathscr{T}_S \otimes \mathscr{B}(\mathbb{Z})$-measurable graph, for all $\eta > 0$. Thus,

$$\mathbf{\Upsilon}(t) = \{\eta \in (0,\infty) : \mathbf{\Theta}_\eta(t) \subseteq \mathbf{\Phi}(t)\} = \{\eta \in (0,\infty) : \mathbf{\Lambda}_\eta(t) = \emptyset\},$$

where $\mathbf{\Lambda}_\eta : S \rightrightarrows \mathbb{Z}$, defined as $\mathbf{\Lambda}_\eta(t) := \mathbf{\Theta}_\eta(t) \cap (\mathbb{Z} \setminus \mathbf{\Phi}(t))$, has $\mathscr{T}_S$-measurable graph. Finally, the $\mathscr{T}_S$-measurability of φ follows from the fact that for each $\alpha > 0$, we have

$$\{t \in S : \varphi(t) < \alpha\} = \bigcup_{\eta \in \mathbb{Q} \cap (0,\alpha)} \mathrm{Proj}_S \mathbf{\Lambda}_\eta.$$

For each $\eta \in \mathbb{Q} \cap (0,1)$, define $B_\eta := \{t \in S : \varphi(t) \geq \eta\}$. Thus, $\{B_\eta : \eta \in \mathbb{Q} \cap (0,1)\}$ is a family of $\mathscr{T}_S$-measurable sets such that $B_\eta \subseteq B_{\eta'}$ if and only if $\eta \geq \eta'$ and $S \sim \bigcup\{B_\eta : \eta \in \mathbb{Q} \cap (0,1)\}$[10]. Let $\varepsilon \in (0, \mu(S))$. Then there is some $\eta_0 \in \mathbb{Q} \cap (0,1)$ such that $\mu(B_{\eta_0}) > \mu(S) - \varepsilon$.

[9]For any $x \in \mathbb{Y}$ and $A \subseteq \mathbb{Y}$, the distance between x and A, denoted by $\mathrm{dist}(x,A)$. defined as

$$\mathrm{dist}(x,A) := \inf\{\|x - y\| : y \in A\}.$$

[10]$C \sim D$ means $\mu(C \Delta D) = 0$, where $C \Delta D = (C \setminus D) \cup (D \setminus C)$.

Set $R := B_{\eta_0}$ and note that, for $t \in R$, as $\varphi(t) \geq \eta_0$, we have $\mathbb{Z} \cap \mathbb{B}(0, \eta_0)^{\Omega} \subseteq \mathbf{\Phi}(t)$. This completes the proof. $\square$

The following lemma on the convexity of vector measure is an application of the infinite-dimensional version of the Lyapunov convexity theorem (refer to Uhl [30]), whose proof can be found in Bhowmik and Cao [10] and Evren and Hüsenov [14].

Lemma 6.2. *Consider a continuum economy and assume that $f \in L_1\left(\mu, \mathbb{Y}^{\Omega}\right)$. Suppose also that S, R are two coalitions of $\mathscr{E}$ such that $\mu(S \cap R) > 0$. Then,*

$$H := \operatorname{cl}\left\{\left(\mu(B \cap R), \int_B f d\mu\right) : B \in \mathscr{T}_S\right\}$$

is a convex subset of $\mathbb{R} \times \mathbb{Y}^{\Omega}$. Moreover, for any $0 < \delta < 1$, there is a sequence $\{G_n : n \geq 1\} \subseteq \mathscr{T}_S$ such that $\mu(G_n \cap R) = \delta\mu(S \cap R)$ for all $n \geq 1$ and

$$\lim_{n \to \infty} \int_{G_n} f(\cdot, \omega) d\mu = \delta \int_S f(\cdot, \omega) d\mu$$

for all $\omega \in \Omega$.

Proof of Proposition 3.4: It is given that

(i) $V_t(g(t, \cdot)) \geq V_t(f(t, \cdot))$ μ-a.e. on S_γ; and

(ii) $V_t(g(t, \cdot)) > V_t(f(t, \cdot))$ for all $t \in B$ and $\mathbb{I}_B = \mathbb{I}_{S_\gamma}$.

Define $\varphi : T \times \Omega \times (0, 1) \to \mathbb{Y}$ by letting $\varphi(t, \omega, \lambda) := \lambda g(t, \omega) + (1 - \lambda)e(t, \omega)$. By Lemma 5.28 in Aliprantis and Border [1], we conclude that $\varphi(t, \omega, \lambda)$ is an interior point of $X_t(\omega)$ for all $(t, \omega, \lambda) \in T \times \Omega \times (0, 1)$. Furthermore, $\varphi(T, \cdot, \cdot) \subseteq \mathbb{Z}$. For each $t \in B$, we define

$$\lambda_t := \inf\left\{\lambda \in (0, 1) : V_t(\varphi(t, \cdot, \lambda)) > V_t(f(t, \cdot))\right\}$$

By the continuity of preference, λ_t exists for each $t \in B$. Furthermore, the mapping $t \mapsto \lambda_t$ is measurable. This follows from the following equality and $\mathscr{T}$-measurability of the function

$\theta(\cdot,\lambda): B \to \mathbb{R}$, defined by $\theta(t,\lambda) := V_t(\varphi(t,\cdot,\lambda)) - V_t(f(t,\cdot))$ for all $(t,\lambda) \in B \times (0,1)$.

$$\{t \in B : \lambda_t > \alpha\} = \bigcup_{r \in \mathbb{Q} \cap (\alpha,1)} \{t \in B : V_t(\varphi(t,\cdot,r)) \leq V_t(f(t,\cdot))\}.$$

For each $\lambda \in (0,1) \cap \mathbb{Q}$, define $B_\lambda := \{t \in B : \lambda \geq \lambda_t\}$. Thus, $\{B_\lambda : \lambda \in \mathbb{Q} \cap [0,1)\}$ is a family of $\mathscr{T}_B$-measurable sets such that $B_\lambda \subseteq B_{\lambda'}$ if and only if $\lambda \leq \lambda'$. Furthermore,

$$B = \bigcup \{B_\lambda : \lambda \in \mathbb{Q} \cap (0,1)\}.$$

Let $\varepsilon > 0$ be such that $\varepsilon < \min\{\mu(B_i) : i \in \mathbb{I}_B\}$. Choose an $\lambda_0 \in (0,1) \cap \mathbb{Q}$ such that $\mu(B_{\lambda_0}) > \mu(B) - \varepsilon$, which implies $\mathbb{I}_{B_{\lambda_0}} = \mathbb{I}_B = \mathbb{I}_{S_\gamma}$. Applying Lemma 6.1, there are some $\eta > 0$ and a sub-coalition $\widehat{B}$ of B_{λ_0} such that

(a) $\mathbb{I}_{\widehat{B}} = \mathbb{I}_{B_{\lambda_0}}$;

(b) $\varphi(t,\cdot,\lambda_0) + z \in X_t$ for all $z \in \mathbb{Z} \cap \mathbb{B}(0,\eta)^\Omega$ and $t \in \widehat{B}$; and

(c) $V_t(\varphi(t,\cdot,\lambda_0) + z) > V_t(f(t,\cdot))$ for all $z \in \mathbb{Z} \cap \mathbb{B}(0,\eta)^\Omega$ and $t \in \widehat{B}$.

Since γ is simple and measurable, there is a collection $\{Q_1, \cdots, Q_m\}$ of pairwise disjoint measurable sets such that $\gamma(t) := \gamma_j$ for some $\gamma_j \in [0,1]$ and all $t \in Q_j$. We define $\mathbb{J} := \{j : \gamma_j \neq 0\}$. So, the support of γ is given by

$$S_\gamma = \bigcup \{Q_j : j \in \mathbb{J}\}.$$

Let $\mathbb{K} := \{(i,j) \in \mathbb{I}_{\widehat{B}} \times \mathbb{J} : \mu(\widehat{B}_i \cap Q_j) > 0\}$. Pick an element $(i,j) \in \mathbb{K}$. By Lemma 6.2, there exists a sequence $\{G_n : n \geq 1\} \subseteq \mathscr{T}_{\widehat{B}_i \cap Q_j}$ of coalitions such that $\mu(G_n) = \gamma_j \mu(\widehat{B}_i \cap Q_j)$ and for all $\omega \in \Omega$,

$$\lim_{n\to\infty} \int_{G_n} (\varphi(\cdot,\cdot,\lambda_0) - e(\cdot,\omega)) d\mu = \gamma_j \int_{\widehat{B}_i \cap Q_j} (\varphi(\cdot,\cdot,\lambda_0) - e(\cdot,\omega))\, d\mu.$$

The function $\xi_n : \Omega \to \mathbb{Y}$, defined by

$$\xi_n(\omega) = \gamma_j \int_{\widehat{B}_i \cap Q_j} (\varphi(\cdot, \cdot, \lambda_0) - e(\cdot, \omega))\, d\mu - \int_{G_n} (\varphi(\cdot, \cdot, \lambda_0) - e(\cdot, \omega)) d\mu,$$

satisfies $\xi_n \in \mathbb{Z} \cap \mathscr{G}_i$ for all $n \geq 1$ and $\{\|\xi_n(\omega)\| : n \geq 1\}$ converges to 0 for all $\omega \in \Omega$. Define

$$\kappa := \min \left\{ \gamma_j \mu(\widehat{B}_i \cap Q_j) : (i, j) \in \mathbb{K} \right\}.$$

Choose an integer $n_{ij} \geq 1$ such that $\xi_{n_{ij}}(\omega) \in \mathbb{B}\left(0, \frac{\eta\kappa}{3m}\right)$ for all $\omega \in \Omega$. It follows that

$$\sum_{\{j : (i,j) \in \mathbb{K}\}} \xi_{n_{ij}}(\omega) \in \mathbb{B}\left(0, \frac{\eta\kappa}{3}\right).$$

We define $R := \bigcup \left\{ G_{n_{ij}} : (i, j) \in \mathbb{K} \right\}$. Letting $F := S_\gamma \setminus \widehat{B}$, we note that $\mathbb{I}_F \subseteq \mathbb{I}_{S_\gamma}$. Define

$$\mathbb{M} := \{(i, j) \in \mathbb{I}_F \times \mathbb{J} : \mu(F_i \cap Q_j) > 0\}.$$

For any $(i, j) \in \mathbb{M}$, similar to above, there is a subcoalition H_{ij} of $F_i \cap Q_j$ such that $\mu(H_{ij}) = \lambda_0 \gamma_j \mu(F_i \cap Q_j)$ and $b_{ij} \in \mathbb{Z} \cap \mathscr{G}_i$ with

$$b_{ij}(\omega) := \lambda_0 \gamma_j \int_{F_i \cap Q_j} (g(\cdot, \omega) - e(\cdot, \omega))\, d\mu - \int_{H_{ij}} (g(\cdot, \omega) - e(\cdot, \omega))\, d\mu \in \mathbb{B}\left(0, \frac{\eta\kappa}{3m}\right)$$

for all $\omega \in \Omega$. As a consequence, we have

$$\sum_{\{j : (i,j) \in \mathbb{M}\}} b_{ij}(\omega) \in \mathbb{B}\left(0, \frac{\eta\kappa}{3}\right).$$

Pick an $(i, j) \in \mathbb{M}$, and define

$$\mathbb{D}_i := \mathbb{Z} \cap \mathscr{G}_i \cap \mathbb{B}\left(0, \frac{\eta\kappa}{3m}\right)^\Omega.$$

As in Lemma 6.1, the correspondence $\mathbf{F}_{ij} : H_{ij} \rightrightarrows \mathbb{D}_i$, defined by

$$\mathbf{F}_{ij}(t) := \{z \in \mathbb{D}_i : g(t, \cdot) + z \in X_t \text{ and } V_t(g(t, \cdot) + z) > V_t(f(t, \cdot))\},$$

is non-empty valued and has $\mathscr{T}_{H_{ij}} \otimes \mathscr{B}(\mathbb{D}_i)$-measurable graph, which further implies the existence of a $\mathscr{T}_{H_{ij}}$-measurable selection h_{ij} of $\mathbf{F}_{ij}$. Define

$$\zeta_{ij} := \frac{1}{\mu(H_{ij})} \int_{H_{ij}} h_{ij} d\mu.$$

By properties of the Bochner integral (see Diestel and Uhl [13], Corollary 8, p. 48), one has $\zeta_{ij} \in \overline{\text{co}}\{h_{ij}(t) : t \in H_{ij}\}$[11], which, in view of the fact that $\mathbb{D}_i$ is closed and convex, immediately implies that $\zeta_{ij} \in \mathbb{D}_i$. Therefore, $\beta_{ij} := \zeta_{ij}\mu(H_{ij}) \in \mathbb{D}_i$. Consequently,

$$\sum_{\{j:(i,j)\in\mathbb{M}\}} \beta_{ij}(\omega) \in \mathbb{B}\left(0, \frac{\eta\kappa}{3}\right).$$

Let $C := \bigcup\{H_{ij} : (i, j) \in \mathbb{M}\}$. For each $i \in \mathbb{I}_{S_\gamma}$, let $x_i : \Omega \to \mathbb{Y}$ be a function defined by

$$x_i(\omega) := \begin{cases} \sum_{\{j:(i,j)\in\mathbb{K}\}} \xi_{n_{ij}}(\omega) + \sum_{\{j:(i,j)\in\mathbb{M}\}}[b_{ij}(\omega) - \beta_{ij}(\omega)], & \text{if } \omega \in \Omega \text{ and } i \in \mathbb{I}_F; \\ \sum_{\{j:(i,j)\in\mathbb{K}\}} \xi_{n_{ij}}(\omega), & \text{if } \omega \in \Omega \text{ and } i \notin \mathbb{I}_F. \end{cases}$$

It follows that $x_i(\omega) \in \mathbb{B}(0, \eta\kappa)$. Finally, we define a function $y : T \times \Omega \to \mathbb{Y}$ defined by[12]

$$y(t, \omega) := \begin{cases} \varphi(t, \omega, \lambda_0) + \frac{x_i(\omega)}{\|\{j:(i,j)\in\mathbb{K}\}\|\mu(G_{n_{ij}})}, & \text{if } (t, \omega) \in G_{n_{ij}} \times \Omega \text{ and } (i, j) \in \mathbb{K}; \\ g(t, \omega) + h_{ij}(t, \omega), & \text{if } (t, \omega) \in H_{ij} \times \Omega \text{ and } (i, j) \in \mathbb{M}; \\ g(t, \omega), & \text{otherwise.} \end{cases}$$

Recognized that y is an allocation with $V_t(y(t, \cdot)) > V_t(f(t, \cdot))$ μ-a.e. on $E := C \cup R$.

[11]Here, $\overline{\text{co}}$ stands for the closed convex hull.

[12]$\xi(t, \omega)$ denotes the ω^{th}-coordinate of $\xi(t)$.

Recognized that

$$\begin{aligned}
\int_E (y(\cdot,\omega) - e(\cdot,\omega))d\mu &= \int_R (y(\cdot,\omega) - e(\cdot,\omega)d\mu + \int_C (y(\cdot,\omega) - e(\cdot,\omega)d\mu \\
&= \sum_{(i,j)\in\mathbb{K}} \left[\int_{G_{n_{ij}}} (\varphi(\cdot,\omega,\lambda_0) - e(\cdot,\omega))d\mu + \frac{x_i(\omega)}{\| \{j : (i,j) \in \mathbb{K}\} \|} \right] \\
&+ \sum_{(i,j)\in\mathbb{M}} \left[\int_{H_{ij}} (g(\cdot,\omega) - e(\cdot,\omega))d\mu + \int_{H_{ij}} h_{ij}(\cdot,\omega)d\mu \right] \\
&= \sum_{(i,j)\in\mathbb{K}} \int_{G_{n_{ij}}} (\varphi(\cdot,\omega,\lambda_0) - e(\cdot,\omega))d\mu + \sum_{i\in\mathbb{I}_B} x_i(\omega) \\
&+ \sum_{(i,j)\in\mathbb{M}} \int_{H_{ij}} (g(\cdot,\omega) - e(\cdot,\omega))d\mu + \sum_{(i,j)\in\mathbb{M}} \beta_{ij}(\omega) \\
&= \sum_{(i,j)\in\mathbb{K}} \gamma_j \int_{\hat{B}_i\cap Q_j} (\varphi(\cdot,\omega,\lambda_0) - e(\cdot,\omega))d\mu - \sum_{(i,j)\in\mathbb{K}} \zeta_{n_{ij}}(\omega) + \sum_{(i,j)\in\mathbb{K}} \zeta_{n_{ij}}(\omega) \\
&+ \sum_{(i,j)\in\mathbb{M}} b_{ij}(\omega) - \sum_{(i,j)\in\mathbb{M}} \beta_{ij}(\omega) + \sum_{(i,j)\in\mathbb{M}} \lambda_0\gamma_j \int_{F_i\cap Q_j} (g(\cdot,\omega) - e(\cdot,\omega))d\mu \\
&- \sum_{(i,j)\in\mathbb{M}} b_{ij}(\omega) + \sum_{(i,j)\in\mathbb{M}} \beta_{ij}(\omega) \\
&= \sum_{(i,j)\in\mathbb{K}} \lambda_0\gamma_j \int_{\hat{B}_i\cap Q_j} (g(\cdot,\omega) - e(\cdot,\omega))d\mu + \sum_{(i,j)\in\mathbb{M}} \lambda_0\gamma_j \int_{F_i\cap Q_j} (g(\cdot,\omega) - e(\cdot,\omega))d\mu \\
&= \lambda_0 \int_{\hat{B}} (\gamma g(\cdot,\omega) - \gamma e(\cdot,\omega))d\mu + \lambda_0 \int_F (\gamma g(\cdot,\omega) - \gamma e(\cdot,\omega))d\mu \\
&= \lambda_0 \int_{S_\gamma} \gamma(g(\cdot,\omega) - e(\cdot,\omega))d\mu.
\end{aligned}$$

For each $t \in R_i$, define

$$\eta_i := \min \left\{ \eta - \operatorname{dist}\left(0, \frac{x_i(\omega)}{\| \{j : (i,j) \in \mathbb{K}\} \| \mu(G_{n_{ij}})}\right) : \omega \in \Omega \text{ and } (i,j) \in \mathbb{K} \right\}.$$

Let $\eta_0 := \min\{\eta_i : i \in \mathbb{I}_R\}$. As a consequence, we have $y(t,\cdot) + z \in X_t$ and $V_t(y(t,\cdot) + z) > V_t(f(t,\cdot))$ for all $z \in \mathbb{Z} \cap \mathbb{B}(0,\eta_0)^\Omega$ and μ-a.e. on R. This completes the proof. □

Proof of Proposition 4.1: Let $\varepsilon > 0$ be such that $\varepsilon < \min\{\mu(R_i) : i \in \mathbb{I}_S\}$. By Lemma 6.1, one can find an $\eta > 0$ and a sub-coalition C of R such that

(i) $\mu(C) > \mu(R) - \varepsilon$;

(ii) $g(t,\cdot) + z \in \mathbb{Z} \cap X_t$ for all $z \in \mathbb{Z} \cap \mathbb{B}(0,\eta)^\Omega$ and μ-a.e. on C; and

(iii) $V_t(g(t,\cdot) + z) > V_t(f(t,\cdot))$ for all $z \in \mathbb{Z} \cap \mathbb{B}(0,\eta)^\Omega$ and μ-a.e. on C.

Let $\delta \in (0,1]$ be such that $\alpha = \delta\mu(S \cap H)$. Pick an $i \in \mathbb{I}_S$. By Lemma 6.2, there exists a sequence $\{G_n : n \geq 1\} \subseteq \mathscr{T}_{C_i}$ such that $\mu(G_n) = \delta\mu(C_i \cap H)$ and for all $\omega \in \Omega$,

$$\lim_{n\to\infty} \int_{G_n} (g(\cdot,\omega) - \psi(\cdot,\omega)) d\mu = \delta \int_{C_i} (g(\cdot,\omega) - \psi(\cdot,\omega))\, d\mu.$$

The function $\xi_n : \Omega \to \mathbb{Y}$, defined by

$$\xi_n(\omega) = \delta \int_{C_i} (g(\cdot,\omega) - \psi(\cdot,\omega))\, d\mu - \int_{G_n} (g(\cdot,\omega) - \psi(\cdot,\omega))\, d\mu,$$

satisfies $\xi_n \in \mathbb{Z} \cap \mathscr{G}_i$ for all $n \geq 1$ and $\{\|\xi_n(\omega)\| : n \geq 1\}$ converges to 0 for all $\omega \in \Omega$. Choose an $n_i \geq 1$ such that

$$\xi_{n_i}(\omega) \in \mathbb{B}\left(0, \frac{\eta\delta\mu(C_i \cap H)}{2}\right)$$

for all $\omega \in \Omega$. Define $D := S \setminus C$. Similar to above, for each $i \in \mathbb{I}_D$, there exist some $F_i \in \mathscr{T}_{D_i}$ and $b_i \in \mathbb{Z} \cap \mathscr{G}_i$ such that $\mu(F_i) = \delta\mu(D_i \cap H)$ and

$$b_i(\omega) := \delta \int_{D_i} (g(\cdot,\omega) - \psi(\cdot,\omega))\, d\mu - \int_{F_i} (g(\cdot,\omega) - \psi(\cdot,\omega))\, d\mu \in \mathbb{B}\left(0, \frac{\eta\delta\mu(C_i \cap H)}{2}\right).$$

For each $\omega \in \Omega$, define $z_i(\omega) := b_i(\omega)$ if $i \in \mathbb{I}_D$; and $z_i(\omega) := 0$, if $i \in \mathbb{I}_S \setminus \mathbb{I}_D$. Analogously, define

$$B_i := \begin{cases} G_{n_i} \cup F_i, & \text{if } i \in \mathbb{I}_D; \\ G_{n_i}, & \text{if } i \in \mathbb{I}_S \setminus \mathbb{I}_D. \end{cases}$$

Recognized that, for each $i \in \mathbb{I}_S$, we have

$$S_i := \begin{cases} C_i \cup D_i, & \text{if } i \in \mathbb{I}_D; \\ C_i, & \text{if } i \in \mathbb{I}_S \setminus \mathbb{I}_D. \end{cases}$$

Further, note that $\mathbb{I}_C = \mathbb{I}_B = \mathbb{I}_S$ and $\mu(B) = \delta\mu(S \cap H) = \alpha$. For each $i \in \mathbb{I}_S$, define a function $\varphi^i : T_i \times \Omega \to \mathbb{Y}$ such that

$$\varphi^i(t,\omega) := \begin{cases} g(t,\omega) + \frac{1}{\delta\mu(C_i \cap H)}(\xi_{n_i}(\omega) + z_i(\omega)), & \text{if } (t,\omega) \in G_{n_i} \times \Omega; \\ g(t,\omega), & \text{otherwise.} \end{cases}$$

It follows that $\varphi^i(t,\cdot) \in \mathbb{Z}$ for all $t \in T_i$. Furthermore, in light of (ii) and (iii), we have $\varphi^i(t,\cdot) \in X_t$ and $V_t(\varphi^i(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on B_i. Lastly, note that

$$\int_{B_i} (\varphi^i(\cdot,\omega) - \psi(\cdot,\omega))d\mu = \delta \int_{S_i} (g(\cdot,\omega) - \psi(\cdot,\omega))d\mu$$

for all $\omega \in \Omega$. Let $B := \bigcup\{B_i : i \in \mathbb{I}_S\}$ and

$$\eta_0 := \min\left\{\eta - \operatorname{dist}\left(0, \frac{1}{\delta\mu(C_i \cap H)}(\xi_{n_i}(\omega) + z_i(\omega))\right) : i \in \mathbb{I}_S \text{ and } \omega \in \Omega\right\}.$$

Thus, the function $\varphi : T \times \Omega \to \mathbb{Y}$, defined by $\varphi(t,\omega) := \varphi^i(t,\omega)$ for all $(t,\omega) \in T_i \times \Omega$, satisfies the requied properties for the above choices of B, C and η_0. □

Proof of Theorem 4.2: Let h be an allocation such that f is blocked by S via h. We choose a separable closed linear subspace $\mathbb{Z}$ of $\mathbb{Y}^\Omega$ such that $f(T,\cdot) \cup h(T,\cdot) \cup e(T,\cdot) \subseteq \mathbb{Z}$. Pick an $\varepsilon \in (0, \mu(S))$. In view of Theorem 3.4 and Remark 3.6, we can choose an $\eta > 0$, two coalitions E, R and an allocation g such that (i) $\mu(E) > \varepsilon$; (ii) $R \subseteq E \subseteq S$ and $\mathbb{I}_R = \mathbb{I}_E = \mathbb{I}_S$; (ii) f is blocked by E via g; and (iii) $g(t,\cdot) + z \in \mathbb{Z} \cap X_t$ for all $z \in \mathbb{Z} \cap \mathbb{B}(0,\eta)^\Omega$ and μ-a.e. on R. Let $\delta \in (0,1)$ be such that $\varepsilon = \delta\mu(E)$. By Proposition 4.1, there are a coalition B

and an allocation φ such that $\mu(B) = \varepsilon$; $V_t(\varphi(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on B; and

$$\int_B (\varphi(\cdot,\omega) - e(\cdot,\omega))d\mu = \delta \int_E (g(\cdot,\omega) - e(\cdot,\omega))d\mu$$

for all $\omega \in \Omega$. Consequently, $\int_B(\varphi(\cdot,\omega) - e(\cdot,\omega))d\mu = 0$ for all $\omega \in \Omega$. This means that B blocks f. $\square$

Proof of Theorem 4.3: Let f be an allocation and R be a coalition blocking f via some allocation g. We choose a separable closed linear subspace $\mathbb{Z}$ of $\mathbb{Y}^\Omega$ such that $f(T,\cdot) \cup g(T,\cdot) \cup e(T,\cdot) \subseteq \mathbb{Z}$. Let $\varepsilon > 0$. Invoking the proof of Theorem 4.2, we derive, in view of Proposition 4.1, that there are an $\eta > 0$, two sub-coalitions B, C of R and an allocation φ such that

(i) $\mathbb{I}_C = \mathbb{I}_B = \mathbb{I}_R$ and $\mu(B) \leq \varepsilon$;

(ii) $\varphi(t,\cdot) + z \in \mathbb{Z} \cap X_t$ for all $z \in \mathbb{Z} \cap \mathbb{B}(0,\eta)^\Omega$ and μ-a.e. on C;

(iii) $V_t(\varphi(t,\cdot) + z) > V_t(f(t,\cdot))$ for all $z \in \mathbb{Z} \cap \mathbb{B}(0,\eta)^\Omega$ and μ-a.e. on C; and

(iv) f is blocked by B via φ.

Let $i \in \mathbb{I}_B$. By absolute continuity of the Bochner integral, there exists some $\zeta_i > 0$ such that

$$\frac{2}{\mu(C_i)} \int_{G_i} (\varphi - e)\ d\mu \in \mathbb{Z} \cap \mathscr{G}_i \cap \mathbb{B}(0,\eta)^\Omega$$

for all $G_i \in \mathscr{T}_{B_i}$ satisfying $\mu(G_i) < \zeta_i$. Define

$$\alpha := \min\left\{\zeta_i,\ \frac{\mu(C_i)}{2} : i \in \mathbb{I}_B\right\}.$$

For all $i \in \mathbb{I}_B$, pick any $D_i \in \mathscr{T}_{B_i}$ such that $\mu(B_i \setminus D_i) < \alpha$. Therefore, $\mu(C_i \cap D_i) > \frac{\mu(C_i)}{2}$. It follows that

$$x_i := \frac{1}{\mu(C_i \cap D_i)} \int_{B_i \setminus D_i} (\varphi - e)\, d\mu \in \mathbb{Z} \cap \mathscr{G}_i \cap \mathbb{B}(0,\eta)^\Omega.$$

Let $h_i : T \times \Omega \to \mathbb{Y}$ such that

$$h_i(t,\omega) := \begin{cases} \varphi(t,\omega) + x_i(\omega), & \text{if } (t,\omega) \in (C_i \cap D_i) \times \Omega; \\ \varphi(t,\omega), & \text{otherwise.} \end{cases}$$

By (ii) and (iii), it follows that $h_i(t,\cdot) \in \mathbb{Z} \cap \mathscr{G}_i \cap X_t$ and $V_t(h_i(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on $C_i \cap D_i$. Furthermore,

$$\int_{D_i} (h_i(\cdot,\omega) - e(\cdot,\omega))d\mu = \int_{B_i} (\varphi(\cdot,\omega) - e(\cdot,\omega))\, d\mu$$

for all $\omega \in \Omega$. Let $S \in \mathscr{T}_B$ be a coalition such that $\mu(B \backslash S) < \alpha$. It follows that $\mu(B_i \backslash S_i) < \alpha$ for all $i \in \mathbb{I}_B$. Consequently, for each $i \in \mathbb{I}_B$, there is an allocation h_i such that $V_t(h_i(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on S_i, and

$$\int_{S_i} (h_i(\cdot,\omega) - e(\cdot,\omega))d\mu = \int_{B_i} (\varphi(\cdot,\omega) - e(\cdot,\omega))\, d\mu$$

for all $\omega \in \Omega$. We consider an allocation $h : T \times \Omega \to \mathbb{Y}_+$, defined by $h(t,\omega) = h_i(t,\omega)$ if $(t,\omega) \in S_i \times \Omega$, $i \in \mathbb{I}_B$ and $h(t,\omega) = g(t,\omega)$, otherwise. Recognized that S blocks the allocation f via h.

For the second part, choose $\delta > 0$. Let $\{t_n : n \geq 1\}$ be a countable dense subset of R. For all $n \geq 1$, define

$$G_n := B \cap \mathbb{B}\left(t_n, \frac{\delta}{2}\right)$$

Letting $F_n := \bigcup\{G_k : 1 \leq k \leq n\}$ for all $n \geq 1$, we see that $\{F_n : n \geq 1\}$ is an ascending sequence and $B = \bigcup\{F_n : n \geq 1\}$. Thus, there is an $n_0 \geq 1$ such that $\mu(B \setminus F_{n_0}) < \frac{\alpha}{2}$. Thus, F_{n_0} is blocking f. Let H be a sub-coalition of F_{n_0} such that $\mu(F_{n_0} \setminus H) < \frac{\alpha}{2}$. It follows that $\mu(B \setminus H) < \alpha$. Thus, H is also blocking f. $\square$

Proof of Proposition 4.4: Let $0 < \delta < 1$. In view of Proposition 4.1, there are an $\eta_0 > 0$, two non-null coalitions B and C, and an allocation φ such that

(i) $C \subseteq B \subseteq S$, $\mathbb{I}_C = \mathbb{I}_B = \mathbb{I}_S$ and $\mu(B) = \delta\mu(S)$;

(ii) $\varphi(t,\cdot) + z \in \mathbb{Z} \cap X_t$ for all $z \in \mathbb{Z} \cap \mathbb{B}(0,\eta_0)^\Omega$ and μ-a.e. on C;

(iii) $V_t(\varphi(t,\cdot) + z) > V_t(f(t,\cdot))$ for all $z \in \mathbb{Z} \cap \mathbb{B}(0,\eta_0)^\Omega$ and μ-a.e. on C;

(iii) $V_t(\varphi(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on $B \setminus C$; and

(iv) $\int_B (\varphi(\cdot,\omega) - f(\cdot,\omega)) d\mu = \delta \int_S (g(\cdot,\omega) - f(\cdot,\omega)) d\mu$ for all $\omega \in \Omega$.

Let $E := S \setminus B$ and define, for each $i \in \mathbb{I}_E$, the set $\mathbb{D}_i := \mathbb{Z} \cap \mathscr{G}_i \cap \mathbb{B}\left(0, \eta_0\mu(C_i)\right)^\Omega$, where $C_i := C \cap T_i$. As in Lemma 6.1, the correspondence $\mathbf{F}_i : E_i \rightrightarrows \mathbb{D}_i$, defined by

$$\mathbf{F}_i(t) := \left\{ z \in \mathbb{D}_i : f(t,\cdot) + z \in X_t \text{ and } V_t(f(t,\cdot) + z) > V_t(f(t,\cdot)) \right\},$$

has a $\mathscr{T}_{E_i} \otimes \mathscr{B}(\mathbb{D}_i)$-measurable graph. By our stated assumptions, $\mathbf{F}_i(t) \neq \emptyset$ for all $t \in E_i$. By the Aumann-Saint-Beuve measurable selection theorem, there is a $\mathscr{T}_{E_i}$-measurable selection ξ_i of $\mathbf{F}_i$. Define

$$\zeta_i := \frac{1}{\mu(E_i)} \int_{E_i} \xi_i d\mu.$$

As in the proof of Theorem 3.4, one can show that $\zeta_i \in \mathbb{D}_i$. So, $\varepsilon_i := \zeta_i \mu(E_i) \in \mathbb{D}_i$ and

$$b_i := \frac{\varepsilon_i}{\mu(C_i)} \in \mathbb{Z} \cap \mathscr{G}_i \cap \mathbb{B}\left(0, \eta_0\right)^\Omega.$$

Let $h : S \times \Omega \to \mathbb{Y}$ be a function such that

$$h(t,\omega) := \begin{cases} \varphi(t,\omega) - b_i(\omega), & \text{if } (t,\omega) \in C_i \times \Omega \text{ and } i \in \mathbb{I}_E; \\ f(t,\omega) + \xi_i(t,\omega), & \text{if } (t,\omega) \in E_i \times \Omega \text{ and } i \in \mathbb{I}_E; \\ \varphi(t,\omega), & \text{otherwise.} \end{cases}$$

It is evident that h is an allocation and $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on S. It can be readily verified that

$$\int_S h(\cdot,\omega) d\mu = \int_S (\delta g(\cdot,\omega) + (1-\delta) f(\cdot,\omega)) d\mu$$

for all $\omega \in \Omega$. Define

$$\eta := \min \{\eta_0 - \operatorname{dist}(0, b_i(\omega))) : i \in \mathbb{I}_E \text{ and } \omega \in \Omega\}.$$

This completes the proof. $\square$

Proof of Theorem 4.6: Since f is a non-core allocation, there exist some $\eta > 0$, coalitions R, S and an allocation g such that

(i) f is blocked by S via g;

(ii) $g(t, \cdot) + z \in \mathbb{Z} \cap X_t$ for all $z \in \mathbb{Z} \cap \mathbb{B}(0, \eta)^{\Omega}$ and μ-a.e. on R; and

(iii) $V_t(g(t, \cdot) + z) > V_t(f(t, \cdot))$ for all $z \in \mathbb{Z} \cap \mathbb{B}(0, \eta)^{\Omega}$ and μ-a.e. on R.

Then, for each $\varepsilon \in (0, \mu(S))$, by Theorem 4.2, there is a coalition G such that $\mu(G) = \varepsilon$ and f is blocked by G. If $\mu(S) = \mu(T)$, there is nothing more to verify. Thus, we assume that $\mu(S) < \mu(T)$ and choose an $\varepsilon \in (\mu(S), \mu(T))$. Define

$$\delta := 1 - \frac{\varepsilon - \mu(S)}{\mu(T \setminus S)}.$$

Let $\mathbb{Z}$ be a separable closed linear subspace of $\mathbb{Y}^{\Omega}$ such that $f(T, \cdot) \cup g(T, \cdot) \cup e(T, \cdot) \subseteq \mathbb{Z}$. By Proposition 4.1, there are an $\eta_0 > 0$, two non-null coalitions B and C, and an allocation φ such that

(A) $C \subseteq B \subseteq S$, $\mathbb{I}_C = \mathbb{I}_B = \mathbb{I}_S$ and f is blocked by B via φ;

(B) $\varphi(t, \cdot) + z \in \mathbb{Z} \cap X_t$ for all $z \in \mathbb{Z} \cap \mathbb{B}(0, \eta_0)^{\Omega}$ and μ-a.e. on C; and

(C) $V_t(\varphi(t, \cdot) + z) > V_t(f(t, \cdot))$ for all $z \in \mathbb{Z} \cap \mathbb{B}(0, \eta_0)^{\Omega}$ and μ-a.e. on C.

Define

$$\mathbb{D} := \bigcap \{\mathscr{G}_i : 1 \leq i \leq n\} \cap \mathbb{Z} \cap \mathbb{B}\left(0, \eta_0 \delta \mu(C)\right)^{\Omega}.$$

As in Lemma 6.1, the correspondence $\mathbf{F} : T \setminus B \rightrightarrows \mathbb{D}$, defined by[13]

$$\mathbf{F}(t) := \{z \in \mathbb{D} : f(t,\cdot) + z \in \mathrm{int}X_t \text{ and } V_t(f(t,\cdot)+z) > V_t(f(t,\cdot))\},$$

is non-empty valued and has $\mathscr{T}_{T\setminus B} \otimes \mathscr{B}(\mathbb{D})$-measurable graph, which further implies the existence of a $\mathscr{T}_{T\setminus B}$-measurable selection ξ of $\mathbf{F}$. Define

$$\zeta := \frac{1}{\mu(T\setminus B)} \int_{T\setminus B} \xi d\mu.$$

As in the proof of Theorem 3.4, one can show that $\zeta \in \mathbb{D}$. So, $\varepsilon := \zeta\mu(T\setminus B) \in \mathbb{D}$ and

$$\gamma := \frac{\varepsilon}{\delta\mu(C)} \in \bigcap\{\mathscr{G}_i : 1 \le i \le n\} \cap \mathbb{Z} \cap \mathbb{B}(0,\eta_0)^{\Omega}.$$

In view of Proposition 4.1, there exist a coalition F and an allocation ψ such that

(a) $\mu(F) = (1-\delta)\mu(T\setminus S)$;

(b) $V_t(\psi(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on F; and

(c) $\int_F(\psi(\cdot,\omega) - e(\cdot,\omega))d\mu = (1-\delta)\int_{T\setminus S}(f(\cdot,\omega) + \xi(\cdot,\omega) - e(\cdot,\omega))d\mu$ for all $\omega \in \Omega$.

Let $\widetilde{g} : T \times \Omega \to \mathbb{Y}$ be an allocation such that

$$\widetilde{g}(t,\omega) := \begin{cases} \varphi(t,\omega) - \gamma(\omega), & \text{if } (t,\omega) \in C \times \Omega; \\ \varphi(t,\omega), & \text{otherwise.} \end{cases}$$

By Proposition 4.4, there exist some allocation h such that $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on S, and

$$\int_S h(\cdot,\omega)d\mu = \int_S (\delta\widetilde{g}(\cdot,\omega) + (1-\delta)f(\cdot,\omega))d\mu$$

[13]Here, $\mathrm{int}X_t$ is the interior of X_t under the relative topology of the norm-totology of Y^{Ω} on X_t.

for all $\omega \in \Omega$. We define a function $y : T \times \Omega \to \mathbb{Y}$ by setting

$$y(t,\omega) := \begin{cases} \psi(t,\omega), & \text{if } (t,\omega) \in F \times \Omega; \\ h(t,\omega), & \text{otherwise.} \end{cases}$$

Recognized that y is an allocation with $V_t(y(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on $E := F \cup S$. It can be readily verified that $\mu(E) = \varepsilon$ and

$$\int_E (y(\cdot,\omega) - e(\cdot,\omega)) d\mu = (1-\delta) \int_T (f(\cdot,\omega) - e(\cdot,\omega)) d\mu = 0.$$

This completes the proof. □

Proof of Proposition 4.7: Denoting by X_R, $\mathscr{F}_R$, V_R, and $e_R(\cdot)$ the common values of X_t, $\mathscr{F}_t$, V_t, and $e(t,\cdot)$, respectively. Suppose, on contrary, that $V_R(\mathbf{x}_f) > V_R(f(t,\cdot))$ for all $t \in B$ for some sub-coalition B of R. Without loss of generality, we may assume that $\mu(R) < \mu(T)$. Otherwise, f will be be blocked by B via $\mathbf{x}_f$. This can be seen as follows:

$$\int_B \mathbf{x}_f d\mu = \frac{\mu(B)}{\mu(R)} \int_R \mathbf{x}_f d\mu = \frac{\mu(B)}{\mu(R)} \int_R e d\mu = \int_B e d\mu.$$

Therefore, we assume that $\mu(R) < \mu(T)$. Then there are an $\lambda \in (0,1)$ and a sub-coalition D of B such that $V_R(\lambda \mathbf{x}_f + (1-\lambda)e_R) > V_R(f(t,\cdot))$ for all $t \in D$. By Lemma 5.28 of Aliprantis and Border [1], we have[14] $\lambda \mathbf{x}_f + (1-\lambda)e_R$ is an interior point of X_R. It follows that there are an $\eta > 0$ and a sub-coalition E of D such that

$$V_R(\lambda \mathbf{x}_f + (1-\lambda)e_R - z) > V_R(f(t,\cdot))$$

for all $z \in \mathbb{B}(0,\eta)^\Omega$ and $t \in E$. Let $\mathbb{Z}$ be a separable closed linear subspace of $\mathbb{Y}^\Omega$ such that

[14]Since X_t is closed and convex, as in the proof of Theorem 3.4 we can readily show that $\mathbf{x}_f \in X_R$.

$f(T,\cdot)\cup e(T,\cdot)\subseteq\mathbb{Z}$. Let $\delta\in(0,1]$ be such that $\mu(E)=\delta\mu(R)$. Define

$$\mathbb{D}:=\bigcap\{\mathscr{G}_i:1\leq i\leq n\}\cap\mathbb{Z}\cap\mathbb{B}\left(0,\eta\mu(E)\right)^{\Omega}.$$

As before, one can find an allocation $\xi:T_0\times\Omega\to\mathbb{Y}$ such that

(i) $\xi(t,\cdot)\in\mathbb{D}$ μ-a.e. on T_0;

(ii) $f(t,\cdot)+\xi(t,\cdot)\in\mathrm{int}X_t$ μ-a.e. on T_0; and

(iii) $V_t(f(t,\cdot)+\xi(t,\cdot))>V_t(f(t,\cdot))$ μ-a.e. on T_0.

By Proposition 4.1, there exists a coalition $C\in\mathscr{T}_{T\setminus R}$ and an allocation φ such that

(A) $\mu(C)=\delta\mu(T\setminus R)$;

(B) $V_t(\varphi(t,\cdot))>V_t(f(t,\cdot))$ μ-a.e. on C; and

(C) $\displaystyle\int_C(\varphi-e)d\mu=\lambda\delta\int_{T\setminus R}(f+\xi-e)d\mu.$

Define

$$\zeta:=\frac{1}{\mu(T\setminus R)}\int_{T\setminus R}\xi d\mu.$$

As in the proof of Theorem 3.4, one can show that $\zeta\in\mathbb{D}$, which further implies $\alpha:=\lambda\delta\zeta\mu(T\setminus R)\in\mathbb{D}$. Consequently,

$$\gamma:=\frac{\alpha}{\mu(E)}\in\bigcap\{\mathscr{G}_i:1\leq i\leq n\}\cap\mathbb{Z}\cap\mathbb{B}\left(0,\eta\right)^{\Omega}.$$

Finally, we define an assignment $y:T\times\Omega\to\mathbb{Y}$ defined by

$$y(t,\omega):=\begin{cases}\lambda\mathbf{x}_f(\omega)+(1-\lambda)e_R(\omega)-\gamma(\omega), & \text{if } (t,\omega)\in E\times\Omega;\\[2ex] \varphi(t,\omega), & \text{otherwise.}\end{cases}$$

It can be readily verified that $S:=C\cup E$ blocks f via y. This is a contradiction. Hence, $V_R(f(t,\cdot))\geq V_R(\mathbf{x}_f)$ μ-a.e. on R. Let G be a sub-coalition of R such that$V_R(f(t,\cdot))>V_R(\mathbf{x}_f)$

for all $t \in G$. By applying Jensen's inequality, one obtains

$$V_R\left(\frac{1}{\mu(G)}\int_G f d\mu\right) > V_R(\mathbf{x}_f)$$

and

$$V_R\left(\frac{1}{\mu(R \setminus G)}\int_{R\setminus G} f d\mu\right) \geq V_R(\mathbf{x}_f).$$

Let $\kappa := \frac{\mu(G)}{\mu(R)}$. By Lemma 5.26 in Aliprantis and Border (2005), one has

$$\begin{aligned} V_R(\mathbf{x}_f) &= V_R\left(\frac{\kappa}{\mu(G)}\int_G f d\mu + \frac{1-\kappa}{\mu(R \setminus G)}\int_{R\setminus G} f d\mu\right) \\ &> V_R(\mathbf{x}_f), \end{aligned}$$

which is a contradiction. Therefore, $V_R(f(t,\cdot)) = V_R(\mathbf{x}_f)$ μ-a.e. on R. $\square$

Proof of Theorem 4.9: Let $\widetilde{f} \in \mathscr{C}(\widetilde{\mathscr{E}})$. Suppose by the way of contradiction that $f := \Phi[\widetilde{f}] \notin \mathscr{C}^A(\mathscr{E})$. Consequently, there exists a generalized coalition γ and an allocation g such that $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on S_γ and

$$\int_{S_\gamma} \gamma g(\cdot,\omega)\, d\mu = \int_{S_\gamma} \gamma e(\cdot,\omega)\, d\mu$$

for all $\omega \in \Omega$. Define $\mathbb{J} := \{j : A_j \subseteq S_\gamma\}$. By Theorem 3.7, we may assume that $\mathbb{J} \neq \emptyset$. Therefore,

$$\int_{S_\gamma \cap T_0} \gamma(g-e)\, d\mu + \sum_{j\in\mathbb{J}} \gamma(A_j)\mu(A_j)(g(A_j) - e(A_j)) = 0.$$

Let $\mathbb{Z}$ be a separable closed linear subspace of $\mathbb{Y}^\Omega$ such that $f(T,\cdot) \cup g(T,\cdot) \cup e(T,\cdot) \subseteq \mathbb{Z}$. In view of Theorem 3.4, there exist an $r_0 \in (0,1)$, a sub-coalition E of $S_\gamma \cap T_0$, and an allocation y such that $V_t(y(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on E and

$$\int_E (y-e)\, d\mu = r_0 \int_{S_\gamma \cap T_0} \gamma(g-e)\, d\mu.$$

By the Lyapunov convexity theorem, there is a sub-coalition $\widetilde{B_j}$ of $\widetilde{A_j}$ such that

$$\widetilde{\mu}(\widetilde{B_j}) = r_0\gamma(A_j)\widetilde{\mu}(\widetilde{A_j}).$$

Define an allocation $\varphi : T \to \mathbb{Y}$ by letting

$$\varphi(t,\omega) := \begin{cases} \widetilde{g}(t,\omega), & \text{if } (t,\omega) \in \widetilde{B_j} \times \Omega \text{ and } j \in \mathbb{J}; \\ y(t,\omega), & \text{otherwise,} \end{cases}$$

where $\widetilde{g} := \Xi[g]$. Define

$$\widetilde{S} := E \cup \bigcup \{\widetilde{B_j} : j \in \mathbb{J}\}.$$

It follows that $\int_{\widetilde{S}} (\varphi - e)\, d\mu = 0$. Pick an $j \in \mathbb{J}$. Since

$$f(A_j, \cdot) = \frac{1}{\widetilde{\mu}(\widetilde{A_j})} \int_{\widetilde{A_j}} \widetilde{f} d\widetilde{\mu},$$

applying Proposition 4.7 to a continuum economy with $R = \widetilde{A_j}$, we have $V_t(f(A_j, \cdot)) = V_t(\widetilde{f}(t, \cdot))$ μ-a.e. on $\widetilde{A_j}$. Therefore, μ-a.e. on $\widetilde{B_j}$, we have

$$V_t(\varphi(t,\cdot)) = V_{A_i}(g(A_i,\cdot)) > V_{A_i}(f(A_i,\cdot)) = V_{A_j}(\widetilde{f}(t,\cdot)).$$

Hence, $\widetilde{f}$ is blocked by $\widetilde{S}$ via φ, which leads to a contradiction. □

Proof of Theorem 4.10: First, we define $\mathbf{x}_{\widetilde{f}} : \Omega \to \mathbb{Y}$ by letting

$$\mathbf{x}_{\widetilde{f}}(\omega) := \frac{1}{\widetilde{\mu}(R)} \int_R \widetilde{f}(\cdot,\omega) d\widetilde{\mu}$$

for all $\omega \in \Omega$. Thus, consider a feasible allocation $\widetilde{f}^A : \widetilde{T} \times \Omega \to \mathbb{Y}$ such that

$$\widetilde{f}^A(t,\omega) := \begin{cases} \widetilde{f}(t,\omega), & \text{if } (t,\omega) \in (T \setminus R) \times \Omega; \\ \mathbf{x}_{\widetilde{f}}(\omega), & \text{otherwise.} \end{cases}$$

In view of Proposition 4.7, we have $V_t(\widetilde{f}(t,\cdot)) = V_t(\widetilde{f}^A(t,\cdot))$ μ-a.e. on R. Suppose, by the way of contradiction, that $\widetilde{f} \notin \mathscr{C}(\widetilde{\mathscr{E}})$. Thus, $\widetilde{f}^A$ is not in the core of $\widetilde{\mathscr{E}}$, which means that it is blocked by some coalition $\widehat{S}$ via some allocation $\widehat{g}$. Let $\mathbb{Z}$ be a separable closed linear subspace of $\mathbb{Y}^\Omega$ such that $f(\widetilde{T},\cdot) \cup \widehat{g}(\widetilde{T},\cdot) \cup e(\widetilde{T},\cdot) \subseteq \mathbb{Z}$.

Case 1. $R = T_1$ and $|T_1| \geq 2$. Choose an element $A_0 \in T_1$ and let $\mu(A_0) = \varepsilon > 0$. By Theorem 4.6, $\widetilde{f}^A$ is blocked by a coalition $\widetilde{B}$ of $\widetilde{\mathscr{E}}$ via the allocation $\widetilde{g}$ with $\widetilde{\mu}(\widetilde{B}) = \widetilde{\mu}(T_0) + \varepsilon$, which gives $\widetilde{\mu}(\widetilde{B} \cap \widetilde{T}_1) \geq \varepsilon$. Moreover, it can be checked that there is a sub-coalition G of $\widetilde{B}$ such that $\mathbb{I}_G = \mathbb{I}_{\widetilde{B}}$ and for each $t \in G$ there is some $\eta > 0$ such that $g(t,\cdot) + z \in \mathbb{Z} \cap X_t$ for all $z \in \mathbb{Z} \cap \mathbb{B}(0,\eta)^\Omega$. Therefore, in the light of Proposition 4.1, there exist a coalition $\widetilde{E}$ and an allocation $\widetilde{y}$ such that $\widetilde{f}^A$ will be blocked by $\widetilde{E}$ via $\widetilde{y}$ and $\widetilde{\mu}(\widetilde{E} \cap \widetilde{T}_1) = \varepsilon$. Define a coalition S of $\mathscr{E}$ such that $S := (\widetilde{E} \cap T_0) \cup A_0$, and define a function $y : T \times \Omega \to \mathbb{Y}$ by

$$y(t,\omega) = \begin{cases} \widetilde{y}(t,\omega), & \text{if } (t,\omega) \in (T_0 \setminus A_0) \times \Omega; \\ \frac{1}{\varepsilon} \int_{\widetilde{E} \cap \widetilde{T}_1} \widetilde{y}(\cdot,\omega) d\widetilde{\mu}, & \text{otherwise.} \end{cases}$$

Recognized that y is an allocation of $\mathscr{E}$ such that

$$\int_S y(\cdot,\omega) d\mu = \int_S e(\cdot,\omega) d\mu$$

for all $\omega \in \Omega$. Furthermore, by the quasi-concavity of V_{T_1}, we have $V_t(y(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on S, which leads to a contradiction.

Case 2. $\mu(R \setminus T_1) > 0$. Define $C := R \cap T_0$. By Theorem 3.4, we conclude that there are some $\eta > 0$, coalitions $\widetilde{B}$, $\widetilde{G}$ and an allocation $\widetilde{y}$ such that

(A) $\widetilde{G} \subseteq \widetilde{B} \subseteq \widehat{S}$ and $\mathbb{I}_{\widetilde{G}} = \mathbb{I}_{\widetilde{B}} = \mathbb{I}_{\widehat{S}}$;

(B) $\widetilde{y}(t,\cdot) + z \in \mathbb{Z} \cap X_t$ for all $z \in \mathbb{Z} \cap \mathbb{B}(0,\eta)^\Omega$ and μ-a.e. on $\widetilde{G}$; and

(C) $\widetilde{f}^A$ will be blocked by $\widetilde{B}$ via $\widetilde{y}$.

If $\widetilde{B} \subseteq T_0$, there is noting more to verify. Thereofore, we assume that $\widetilde{\mu}(\widetilde{B} \cap \widetilde{T}_1) > 0$. Let $\varepsilon := \widetilde{\mu}(\widetilde{B} \cap \widetilde{T}_1)$. Define a function $\widetilde{y}^A : \widetilde{T} \times \Omega \to \mathbb{Y}$ by

$$\widetilde{y}^A(t,\omega) := \begin{cases} \frac{1}{\varepsilon} \int_{\widetilde{B} \cap \widetilde{T}_1} \widetilde{y}(\cdot,\omega) d\widetilde{\mu}, & \text{if } (t,\omega) \in (\widetilde{B} \cap \widetilde{T}_1) \times \Omega; \\ \widetilde{y}(t,\omega), & \text{otherwise.} \end{cases}$$

It follows that $V_{T_1}(\widetilde{y}^A(t,\cdot)) > V_{T_1}(\widetilde{f}^A(t,\cdot))$ μ-a.e. on $\widetilde{B}$ and

$$\int_{\widetilde{B} \cap T_0} (\widetilde{y}^A - e) d\mu + \varepsilon(\widetilde{y}^A - e_{T_1}) = 0. \tag{6.1}$$

If $\mu(C) \geq \varepsilon$ then we choose a coalition $\widehat{R} \subseteq C$ such that $\mu(\widehat{R}) = \varepsilon$. Consequently, by Equation (6.1), we have

$$\int_{\widetilde{B} \cap T_0} (\widetilde{y}^A - e) d\mu + \mu(\widehat{R})(\widetilde{y}^A - e_{T_1}) = 0.$$

If $\mu(C) < \varepsilon$ then first choose an $\alpha \in (0,1)$ such that $\mu(C) = \alpha\varepsilon$. By Proposition 4.1, there are some $\eta_0 > 0$, two coalitions $\widehat{K}$ and $\widehat{D}$ and an allocation φ such that

(a) $\widehat{K} \subseteq \widehat{D} \subseteq \widetilde{B} \cap T_0$ with $\mathbb{I}_{\widehat{K}} = \mathbb{I}_{\widehat{D}} = \mathbb{I}_{\widetilde{B} \cap T_0}$;

(b) $\varphi(t,\cdot) + z \in \mathbb{Z} \cap X_t$ for all $z \in \mathbb{Z} \cap \mathbb{B}(0,\eta_0)^\Omega$ and μ-a.e. on $\widehat{K}$;

(c) $V_t(\varphi(t,\cdot)) > V_t(f(t\cdot))$ for all $t \in \widehat{D}$; and

(d) $\displaystyle\int_{\widehat{D}} (\varphi - e) d\mu = \alpha \int_{\widetilde{B} \cap T_0} (\widetilde{y}^A - e) d\mu.$

In view of Equation (6.1), we have $\mathbb{I}_{\widehat{K} \cup C} = \mathbb{I}_{\widehat{D} \cup C} = \mathbb{I}_{\widetilde{B}}$ and

$$\int_{\widehat{D}} (\varphi - e) d\mu + \mu(C)(\widetilde{y}^A - e_{T_1}) = 0.$$

Hence, in either of these cases, there are coalitions D, K, N and allocation ξ such that

(i) $K \subseteq D \subseteq \widetilde{B} \cap T_0$ and $N \subseteq C$ such that $\mathbb{I}_{K \cup N} = \mathbb{I}_{D \cup N} = \mathbb{I}_{\widetilde{B}}$;

(ii) $\xi(t,\cdot)+z \in \mathbb{Z}\cap X_t$ for all $z \in \mathbb{Z}\cap \mathbb{B}(0,\widetilde{\eta})^{\Omega}$ and μ-a.e. on K;

(iii) $V_t(\xi(t,\cdot)) > V_t(f(t,\cdot))$ for all $t \in K$; and

(iv) $\int_D (\xi - e)d\mu + \mu(N)(\widetilde{y}^A - e_{T_1}) = 0$.

If $\mu(D\cap N) = 0$ then $D \cup N$ blocks the allocation $\widetilde{f}^A$ via ζ, where the allocation ζ is defined by

$$\zeta(t,\omega) = \begin{cases} \widetilde{y}^A(t,\omega), & \text{if } (t,\omega) \in N \times \Omega; \\ \xi(t,\omega), & \text{otherwise.} \end{cases}$$

If $\mu(D\cap N) > 0$ then we define $E := (D \setminus N) \cup (N \setminus D)$ and $G := D \cap N$. Recognized that there is an $\widetilde{\eta} > 0$ such that $\zeta(t,\omega) + z \in \mathbb{Z}\cap X_t$ for all $z \in \mathbb{Z}\cap\mathbb{B}(0,\widetilde{\eta})^{\Omega}$ and μ-a.e. on H for some sub-coalition H of $K \cup N$ satisfying $\mathbb{I}_H = \mathbb{I}_E$. By Proposition 4.1, there is some coalition $F \subseteq E$ and an allocation h such that

$$\int_F (h-e)d\widetilde{\mu} = \frac{1}{2}\int_E (\zeta - e)d\widetilde{\mu}.$$

By Proposition 4.4, there exist an allocation ι and a sub-coalition V of G such that $\mathbb{I}_V = \mathbb{I}_G$, $V_t(\iota(t,\cdot)) > V_t(f(t,\cdot))$; and

$$\int_G (\iota - e)d\mu = \frac{1}{2}\int_G (\xi - e)d\mu + \frac{1}{2}\int_G (\widetilde{y}^A - e)d\mu.$$

Then $S := F \cup G$ blocks the allocation $\widetilde{f}^A$ via ψ, where the allocation ψ is defined by

$$\psi(t,\omega) = \begin{cases} h(t,\omega), & \text{if } (t,\omega) \in F \times \Omega; \\ \iota(t,\omega), & \text{otherwise.} \end{cases}$$

This contradicts with the fact that f is in the ex-ante core of $\mathscr{E}$. $\square$

Proof of Theorem 4.11: Let us choose $\varepsilon, \delta > 0$. Let $f \notin \mathscr{C}^A(\mathscr{E})$. Defining $\widetilde{f} := \Xi[f]$, we note that $f = \Phi[\widetilde{f}]$. Thus, by Theorem 4.9, we have $\widetilde{f} \notin \mathscr{C}(\widetilde{\mathscr{E}})$. In view of Theorem 4.3, we have a coalition S with $\widetilde{\mu}(S) \le \varepsilon$ ex-ante blocking f and satisfying the following:

(i) There exists an $\tau > 0$ such that such that for any coalition F of S satisfying $\mu(S \backslash F) < \tau$ ex-ate blocking f; and

(ii) $S = \bigcup_{i=1}^{n} S_i$ for a finite collection of coalitions $\{S^1, \cdots, S^n\}$ with diameter of S_i smaller than δ for all $i = 1, \cdots, n$.

Let

$$B^1 := S^1 \text{ and } B^i = S^i \setminus \bigcup \{S^j : 1 \leq j < i\}$$

for all $i \geq 2$. Define $G^i := B^i \cap T_0$ for each $i \in \{1, \cdots, n\}$ and note that

$$S = \bigcup \{G^i : 1 \leq i \leq n\} \cup (S \cap \widetilde{T}_1).$$

Put, $\mathbb{I} := \left\{ k : \widetilde{\mu}(\widetilde{A_k} \cap S) > 0 \right\}$. For $\mathbb{I} \neq \emptyset$, choose a finite subset $\mathbb{K}$ of $\mathbb{I}$ such that $\mathbb{K} = \mathbb{I}$ if $\mathbb{I}$ is finite; and $\sum_{k \in \mathbb{I} \setminus \mathbb{K}} \mu(A_k) < \tau$, otherwise. We define $R := S$ if $\mathbb{I} = \emptyset$; and

$$R := \bigcup \{G^i : 1 \leq i \leq n\} \cup \bigcup \{\widetilde{A_k} \cap S : k \in \mathbb{K}\},$$

otherwise. In view of the fact that $\mu(S \setminus R) < \tau$, we conclude that R ex-ante blocks f. Moreover, it contains either no atom or finitely many atoms. For $\mathbb{K} \neq \emptyset$, let $\gamma : T \to [0, 1]$ be an Aubin coalition such that

$$\gamma(t) := \begin{cases} 1, & \text{if } t \in R \cap T_0; \\ \alpha_k, & \text{if } t = A_k,\ k \in \mathbb{K}; \\ 0, & \text{otherwise,} \end{cases}$$

and for $\mathbb{K} = \emptyset$, define an Aubin coalition $\gamma : T \to [0, 1]$ such that

$$\gamma(t) := \begin{cases} 1, & \text{if } t \in R \cap T_0; \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\alpha_k := \frac{\widetilde{\mu}(\widetilde{A_k} \cap S)}{\widetilde{\mu}(\widetilde{A_k})}.$$

For all $1 \leq i \leq n$, let $\gamma_i : T \to [0,1]$ be an Aubin coalition such that $\gamma_i := \chi_{G^i}$. Thus, $\{\gamma_1, \cdots, \gamma_n\}$ is a finite collection of pairwise disjoint generalized coalitions and $S_{\gamma_i} \subseteq T_0$ for all $1 \leq i \leq n$. It follows from the definition of the diameter of a generalized coalition by taking $\alpha = \beta = 1$ that

$$\operatorname{diam}(\gamma_i) = \sup\{\|a - b\| : a, b \in S_{\gamma_i}\} = \operatorname{diam}(S_{\gamma_i}) < \delta.$$

Furthermore, it can be readily verified that

$$\gamma = \begin{cases} \sum_{i=1}^n \gamma_i + \sum_{k\in\mathbb{K}} \alpha_k \chi_{A_k}, & \text{if } \mathbb{K} \neq \emptyset; \\ \sum_{i=1}^n \gamma_i, & \text{if } \mathbb{K} = \emptyset, \end{cases}$$

This completes the proof. $\square$

Proof of Theorem 4.13: Let f be a feasible allocation of $\mathscr{E}$ such that $f \notin \mathscr{C}^A(\mathscr{E})$ and let $\varepsilon \in (0,1)$. Letting $\widetilde{f} := \Xi[f]$, we note that $f = \Phi[\widetilde{f}]$. Thus, applying Theorem 4.9, one has $\widetilde{f} \notin \mathscr{C}(\widetilde{\mathscr{E}})$. Therefore, in view of Theorem 4.6, one can find a coalition S and an allocation $\widetilde{g}$ in $\widetilde{\mathscr{E}}$ such that $\widetilde{\mu}(S) = \varepsilon$ and $\widetilde{f}$ is ex-ante blocked by the coalition S via some allocation $\widetilde{g}$. Put $\mathbb{J} = \{j : \widetilde{\mu}(S \cap \widetilde{A_i}) > 0\}$. The rest of the proof is decomposed into two cases:

Case 1. $\mathbb{J} \neq \emptyset$. In this case, we have

$$\int_{S\cap T_0} \widetilde{g}\, d\widetilde{\mu} + \sum_{j\in\mathbb{J}} \int_{S\cap\widetilde{A_j}} \widetilde{g}\, d\widetilde{\mu} = \int_{S\cap T_0} \widetilde{e}\, d\widetilde{\mu} + \sum_{j\in\mathbb{J}} \int_{S\cap\widetilde{A_j}} \widetilde{e}\, d\widetilde{\mu}.$$

For each $j \in \mathbb{J}$, choose some $\gamma_j \in (0,1]$ such that $\widetilde{\mu}(S \cap \widetilde{A_j}) = \gamma_j \mu(A_j)$ and define

$$g_j := \frac{1}{\widetilde{\mu}(S \cap \widetilde{A_j})} \int_{S\cap\widetilde{A_j}} \widetilde{g}\, d\widetilde{\mu}.$$

By Jensen's inequality, we have $V_{A_j}(g_j) > V_{A_j}(f(A_j,\cdot))$ for all $j \in \mathbb{J}$ and

$$\int_{S\cap T_0} \widetilde{g}\, d\widetilde{\mu} + \sum_{j\in\mathbb{J}} \gamma_j g_j \mu(A_j) = \int_{S\cap T_0} \widetilde{e}\, d\widetilde{\mu} + \sum_{j\in\mathbb{J}} \gamma_j e_j \mu(A_j).$$

Define an allocation $g : T \times \Omega \to \mathbb{Y}_+$ by

$$g(t,\omega) := \begin{cases} \widetilde{g}(t,\omega), & \text{if } (t,\omega) \in (S \cap T_0) \times \Omega; \\ g_j(\omega), & \text{if } t = A_j,\ \omega \in \Omega \text{ and } j \in \mathbb{J}; \\ f(t,\omega), & \text{otherwise,} \end{cases}$$

and an Aubin coalition $\gamma : T \to [0,1]$ by

$$\gamma(t) := \begin{cases} 1, & \text{if } (t,\omega) \in (S \cap T_0) \times \Omega; \\ \gamma_j, & \text{if } t = A_j,\ \omega \in \Omega \text{ and } j \in \mathbb{J}; \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, we have $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on S_γ and

$$\int_T \gamma g(\cdot,\omega)\, d\mu = \int_T \gamma e(\cdot,\omega)\, d\mu$$

for all $\omega \in \Omega$. Furthermore, note that

$$\int_T \gamma\, d\mu = \mu(S \cap T_0) + \sum_{j\in\mathbb{J}} \int_{A_j} \gamma_j\, d\mu = \mu(S) = \varepsilon.$$

Case 2. $\mathbb{J} = \emptyset$. Analogous to **Case 1**, one can show that f is blocked by an Aubin coalition γ via g, where the function $g : T \times \Omega \to \mathbb{Y}_+$ is defined by

$$g(t,\omega) := \begin{cases} \widetilde{g}(t,\omega), & \text{if } (t,\omega) \in (S \cap T_0) \times \Omega; \\ f(t,\omega), & \text{otherwise,} \end{cases}$$

and the Aubin coalition $\gamma : T \times \Omega \to [0,1]$ is defined by

$$\gamma(t,\omega) := \begin{cases} 1, & \text{if } (t,\omega) \in (S \cap T_0) \times \Omega; \\ 0, & \text{otherwise.} \end{cases}$$

Recognized that $\int_T \gamma \, d\mu = \widetilde{\mu}(S \cap T_0) = \mu(S) = \varepsilon$. $\square$

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