Innovation and inequality from stagnation to growth

Chu, Angus C. and Peretto, Pietro

University of Macau, Macao, Macao, Duke University, Durham , United States

September 2019

Innovation and Inequality from Stagnation to Growth

Angus C. Chu, University of Macau
Pietro F. Peretto, Duke University

July 21, 2023

Abstract

This study explores the evolution of income inequality in an economy featuring an endogenous transition from stagnation to growth. We incorporate heterogeneous households in a Schumpeterian model of endogenous takeoff. In the pre-industrial era, the economy is in stagnation, and income inequality is determined by the unequal distribution of land. When the takeoff occurs, the economy experiences innovation and economic growth, and income inequality gradually rises until the economy reaches the steady state. We calibrate the model for a quantitative analysis and compare the simulation results to historical data in the UK. Extending the analysis to allow for endogenous labor supply, we find that endogenous labor supply introduces a channel through which inequality contributes to shaping the transition path of the economy and that households sort themselves into a leisure class that supplies zero labor and the rest of society that supplies labor.

JEL classification: D30, O30, O40
Keywords: income inequality, innovation, economic growth, endogenous takeoff

Chu: anguscc@gmail.com. Department of Economics, University of Macau, Macau, China. Peretto: peretto@econ.duke.edu. Department of Economics, Duke University, Durham, United States. We are extremely grateful to Jakob Madsen and Ravshonbek Otojanov for sharing their data with us and Xilin Wang for his excellent research assistance. We also would like to thank the helpful comments from seminar participants at Queen Mary University of London, University of Birmingham, University of Liverpool and University of Sheffield.

1 Introduction

What is the historical relationship between growth and inequality and, if any, what drives it? These questions have a long tradition in economics. Kuznets (1955) famously hypothesized that industrialization causes income inequality to rise.[1] Williamson (1980, 1985) provides evidence for this hypothesis, showing that in Britain income inequality increases after the Industrial Revolution and keeps rising until the mid-19th century.[2] A subsequent study by van Zanden (1995) provides evidence that the rise in income inequality in the early modern period was driven by economic development. More broadly, there is abundant evidence that periods characterized by waves of innovation in technology and business organization (e.g., the period straddling the second half of the 19th century and the pre-wars 20th century) display higher and rising inequality. History thus suggests that *innovation-driven growth accelerations* cause rising inequality. The recent study by Madsen *et al.* (2021) makes this point quite forcefully. It carries out "a long-run econometric analysis for 21 OECD countries using annual data over the period 1860–2015" (p. 477) and shows that "intangibles have been a contributing factor in wealth inequality since 1860 and that the marked increase in investment in intangible assets has been a significant driver of the increasing inequality since the 1970s" (p. 477). The takeaway of the paper is that growth accelerations fueled by investment in intangibles cause rising inequality.

To illuminate analytically the mechanism that this evidence points to, we need to understand the origins of the transition from stagnation to growth and how this transition affects the evolution of the wealth and income distributions. In this study we develop a growth-theoretic framework that enables us to characterize analytically the endogenous takeoff of an economy and the evolution of the wealth and income distributions from stagnation to growth.

The framework builds on two branches of growth economics. The first is Unified Growth Theory (Galor and Weil 2000, Galor 2005 and 2011), developed to explain the transition from Malthusian stagnation to modern growth. The second is the theory of endogenous technological change (Romer 1990), developed to formalize the idea that innovation is the key driver of economic growth. Exploiting ideas from both branches, Peretto (2015) extends the Schumpeterian growth model to allow for endogenous takeoff. We incorporate in his model the approach to heterogeneous households in Chu and Cozzi (2018), to obtain a structure that allows us to characterize analytically the endogenous takeoff of the aggregate economy and its transition dynamics from stagnation to growth. The goal is to understand

[1] Kuznets (1955) also hypothesized that income inequality eventually falls as the economy develops, but the evidence in Piketty (2014) shows that inequality has been rising in recent times.

[2] Lindert (2000a, b) also finds a rise in income inequality in Britain in as early as the late 18th century.

the evolution of the personal distributions of wealth and income throughout the process. The source of heterogeneity across households is the unequal initial distribution of assets. Accordingly, our framework builds on the literature, recently revived by Piketty (2014), that considers wealth inequality as the root cause of income inequality that is also determined by the rate of economic growth and the rate of return on assets. One advantage of our analysis is that we do not impose any parametric assumption on the wealth distribution except that it is non-degenerate and has well-defined moments. This property, in turn, allows us to obtain analytical solutions for popular measures of inequality, in particular the Gini coefficient.

Our first main finding is that the economy initially features a pre-industrial era, characterized by stagnation with very slow economic growth, in which income inequality is determined solely by the unequal distribution of land. Although the initial distribution of land is exogenous, the distribution may change over time because our heterogenous households can trade it to achieve intertemporal consumption smoothing in anticipation of the forthcoming industrial era. The industrial era begins when the size of the market becomes sufficiently large due to population growth, and the economy begins to experience innovation. Accordingly, the rate of economic growth begins to rise gradually, until it converges to its steady-state value. This growth acceleration, fueled by a rising rate of return to innovation, causes income inequality to rise gradually, until it reaches a constant steady-state value. Remarkably, this rise in income inequality occurs despite the fact that the wealth distribution becomes stationary in the industrial era. The mechanisms driving wealth and income inequality dynamics in the two eras are thus drastically different. In the preindustrial era, income inequality was mainly caused by the unequal distribution of physical assets, for which the most type was land. The evidence supporting the industrial-era mechanism is also strong: Madsen (2017) shows that the rate of return to financial assets is an important determinant of income inequality and, as stated above, Madsen *et al.* (2021) carries out the most comprehensive study documenting the central role of investment in the intangibles (e.g., R&D) that drive the asset-market valuation of firms.

We calibrate the model to data in the modern UK economy to perform a quantitative analysis. Simulating the transitional paths of the output growth rate and the real interest rate, we find that the increase in the simulated growth rate and the simulated interest rate is also consistent with available historical data in the UK. Then, we simulate the transitional path of income inequality and find that income inequality increases sharply when the takeoff occurs. When the economy reaches the steady state, income inequality becomes almost twice as high as the level prior to the takeoff and is in line with the Gini coefficient of income in the UK in recent time.

We obtain the result discussed above in a baseline model with inelastic labor supply. The model has three main strengths: (i) it is analytically tractable; (ii) it identifies sharply the role of growth accelerations as a main driver of rising income inequality via a higher rate of return on assets as also shown in the evidence by Piketty (2014); (iii) it measures inequality with a well understood and widely used summary statistic of the shape of a non-degenerate distribution of income. Inequality, however, plays no role in shaping the transition from stagnation to growth. When we extend the analysis allowing for labor income inequality due to endogenous labor supply, we find that endogenous labor supply introduces a channel through which inequality contributes to shaping the transition path of the economy, while it preserves the features (i)-(iii) that make the baseline model so useful. Specifically, labor income inequality consists of two margins: an extensive margin along which households sort themselves into a leisure class that supplies zero labor and the rest of society that supplies labor; and an intensive margin along which households supply labor as a decreasing function of their consumption share, which in turn is an increasing function of their wealth share. The leisure class consists of households that are wealthy enough to find optimal to forgo labor income. Our model, therefore, generates endogenously the two-class structure—workers vs. capitalists—that is widely used in the literature on inequality that builds on the classical theory of the distribution of income. Models in this tradition, however, are often silent about the shape of the cross-sectional distribution of income because the imposed within-class homogeneity reduces the cross section of the whole population to two degenerate distributions. Consequently, work that uses this approach tends to measure inequality with grand ratios like the wealth share of GDP (see, e.g., Madsen *et al.* 2021 discussed above). Our structure, in contrast, allows for heterogeneity in wealth, labor supply, and thus overall income, within each class. One of our results is that a simple summary statistic of labor supply heterogeneity captures the channel through which inequality affects aggregate outcomes.

This study relates to the vast literature on innovation and economic growth. The seminal contribution by Romer (1990) features the invention of new products (i.e., horizontal innovation) as the engine of growth. Aghion and Howitt (1992) develop the Schumpeterian creative-destruction model in which economic growth is driven by the development of higher-quality products (i.e., vertical innovation) that displace existing products.[3] Subsequent studies, such as Smulders (1994), Smulders and van de Klundert (1995), Peretto (1994, 1998, 1999) and Dinopoulos and Thompson (1998), combine vertical in-house innovation by incumbent firms and horizontal innovation by entrant firms to develop the class of Schumpeterian creative-accumulation models with endogenous market structure; see Garcia-Macia *et al.* (2019) for evidence that growth is mostly driven by in-house innovation of existing firms (i.e.,

[3] See also Grossman and Helpman (1991) and Segerstrom *et al.* (1990).

creative accumulation).[4] This study contributes to this literature by introducing heterogeneous households to a tractable Schumpeterian creative-accumulation model that features an endogenous takeoff and also fully endogenous growth with the scale effect removed by endogenous market structure. The goal is then to explore the effects of innovation on the evolution of income inequality during the historical transition from stagnation to growth.

This study also relates to the literature on inequality and economic growth. The study most directly related to our work is Madsen *et al.* (2021) discussed above. It uses a first-generation Schumpeterian model to set up an empirical exercise guided by theory, although it restricts attention to the model's steady state. We differ chiefly in that we consider a scale-invariant Schumpeterian model with endogenous market structure and use our model's non-linear dynamics with phase transitions to go after analytical results on the historical relationship between income growth and income inequality.

Early studies in this literature explore how inequality affects economic growth via capital accumulation; see for example, Galor and Zeira (1993) and Aghion and Bolton (1997). Galor and Moav (2004) show that in the early (later) stage of development, in which the accumulation of physical (human) capital is the main engine of growth, inequality stimulates (stifles) economic growth. Subsequent studies consider how inequality affects the demand and supply of resources for innovation in the Romer model; see for example, Chou and Talmain (1996), Zweimuller (2000), Foellmi and Zweimuller (2006) and Garcia-Penalosa and Wen (2008). Recent studies by Jones and Kim (2018) and Aghion *et al.* (2019) focus on the relationship between innovation and top-income inequality in the Schumpeterian model.[5] This study differs from these contributions by considering a Schumpeterian model with endogenous takeoff and analyzing the historical evolution of income inequality from stagnation to growth. The recent study by Madsen and Strulik (2020) also explores the evolution of income inequality, measured as the ratio of land rents to wages, from stagnation to growth arising from land-biased technological change driven by education. We, instead, consider other measures of income inequality, such as the Gini coefficient and the top income share, in a Schumpeterian innovation-driven growth model.

Finally, this study relates to the literature on the Industrial Revolution and the transition to modern economic growth. As mentioned, Unified Growth Theory (Galor and Weil 2000, Galor 2005 and 2011) explores how the quality-quantity trade-off in child-rearing and the associated process of human capital accumulation allow an economy to escape the Malthusian

[4] Laincz and Peretto (2006), Ha and Howitt (2007), Madsen (2008, 2010) and Ang and Madsen (2011) also provide supportive empirical evidence for this broad class of Schumpeterian models.

[5] Other studies, such as Chu (2010), Chu and Cozzi (2018) and Chu *et al.* (2019, 2021), analyze the effects of patent policy and monetary policy on innovation and income inequality.

trap and experience economic growth.[6] Galor, Moav and Vollrath (2009) explore how the inequality of land ownership in the pre-industrial era affects the transition of an economy to the industrial era via the emergence of human-capital promoting institutions. Although the Schumpeterian model in Peretto (2015) features exogenous population growth and does not feature human capital accumulation, the innovation-driven takeoff in the model captures the Industrial Revolution, which is arguably the most important economic takeoff in human history.[7] Furthermore, when we incorporate heterogeneous households into the model, this tractable growth-theoretic framework allows us to study analytically how innovation affects (i) the rate of return on assets as emphasized by Piketty (2014) on its importance on inequality and thereby (ii) the evolution of income inequality.

The rest of this study is organized as follows. Section 2 presents the model. Section 3 analyzes the dynamics and derives the evolution of income inequality. Section 4 performs a quantitative analysis. Section 5 considers labor income inequality due to endogenous labor supply. Section 6 concludes.

2 A Schumpeterian growth model with heterogeneous households and endogenous takeoff

We introduce heterogeneous households as in Chu (2010) and Chu and Cozzi (2018) to the Schumpeterian model of endogenous takeoff in Peretto (2015). Our analysis provides a complete closed-form solution for economic growth and income inequality from stagnation to takeoff and eventually to the steady state.

2.1 Heterogeneous households

There is a continuum of mass one of households indexed by $h \in [0, 1]$. Household h has preferences

$$U(h) = \int_0^\infty e^{-\rho t} \ln c_t(h) dt, \tag{1}$$

[6] See also Galor and Moav (2002), Galor and Mountford (2008) and Ashraf and Galor (2011) for other studies and empirical evidence that supports Unified Growth Theory.

[7] Mokyr (2016) argues that innovations in Europe gave rise to the Industrial Revolution and sustained economic growth that subsequently spread across the world.

where $\rho > 0$ is the subjective discount rate and $c_t(h)$ is household consumption of the final good.[8] The household maximizes (1) subject to

$$\dot{a}_t(h) = r_t a_t(h) + w_t L_t - c_t(h), \tag{2}$$

where $a_t(h)$ is household wealth and r_t is the real interest rate. The household supplies L_t units of labor inelastically to earn wage income $w_t L_t$. The household's labor endowment (the mass of identical household members) grows at rate $\lambda > 0$, i.e., $L_t = L(0)\, e^{\lambda t}$, $L(0) = 1$.

Standard dynamic optimization yields the familiar Euler equation

$$\frac{\dot{c}_t(h)}{c_t(h)} = r_t - \rho.$$

A property of this saving rule that is quite important for our research question is that, due to the homothetic preferences (1), the welfare-maximizing growth rate of consumption is the same across households. Consequently, we can write

$$\frac{\dot{c}_t(h)}{c_t(h)} = \frac{\dot{C}_t}{C_t} = r_t - \rho. \tag{3}$$

where $C_t \equiv \int_0^1 c_t(h)dh$ is aggregate consumption.

2.2 Final good

A competitive representative firm produces a final good G_t that can be consumed, used to produce intermediate goods, invested in the improvement of the quality of existing intermediate goods, or invested in the creation of new intermediate goods. The final good is the numeraire so its price is $P_G \equiv 1$. The production technology is

$$G_t = \int_0^{N_t} X_t^{\theta}(i) \left[Z_t^{\alpha}(i) Z_t^{1-\alpha} L_t^{\gamma}(i) R_t^{1-\gamma} \right]^{1-\theta} di, \tag{4}$$

where $\{\theta, \alpha, \gamma\} \in (0,1)$. N_t is the mass of non-durable intermediate goods, whereas $L_t(i)$ and R_t are, respectively, services of labor and land. The representative firm rents land from households by paying a rental price. The index i reflects the property that both labor $L_t(i)$ and intermediate goods $X_t(i)$ are rival inputs. The index i on labor also implies that the

[8] For simplicity, we assume that flow utility is a function of the household's total consumption, rather than the mass of identical household members multiplied by the utility of consumption per household member. This allows us to abstract from differentiating between household-level consumption and individual-level consumption given that the distinction is not important to our research question.

technology features full dilution of labor across intermediate goods (i.e., $L_t(i) = L_t/N_t$ in equilibrium). Land, instead is non-rival across intermediate goods and labor. Quality is the good's ability to raise the productivity of the other physical factors. The contribution of good i to factor productivity downstream depends on the knowledge stock of firm i, $Z_t(i)$, and on the average knowledge of all firms, $Z_t = \int_0^{N_t} [Z_t(j)/N_t]dj$.

Let $p_t(i)$ be the price of good i and q_t be the rental price of land. Profit maximization yields the conditional demand functions:

$$R_t = \frac{(1-\gamma)(1-\theta)}{q_t} G_t; \tag{5}$$

$$L_t(i) = \left\{ \frac{\gamma(1-\theta)}{w_t} X_t^{\theta}(i)[Z_t^{\alpha}(i) Z_t^{1-\alpha} R_t^{1-\gamma}]^{1-\theta} \right\}^{1/[1-\gamma(1-\theta)]}; \tag{6}$$

$$X_t(i) = \left[\frac{\theta}{p_t(i)} \right]^{\frac{1}{1-\theta}} Z_t^{\alpha}(i) Z_t^{1-\alpha} L_t^{\gamma}(i) R_t^{1-\gamma}. \tag{7}$$

Moreover, the final producer pays total compensation to, respectively, suppliers of intermediate goods, labor and land:

$$\int_0^{N_t} p_t(i) X_t(i) \, di = \theta G_t; \tag{8}$$

$$\int_0^{N_t} w_t L_t(i) \, di = \gamma(1-\theta) G_t; \tag{9}$$

$$q_t R_t = (1-\gamma)(1-\theta) G_t. \tag{10}$$

2.3 Intermediate goods and in-house R&D

Monopolistic firm i produces with a technology that requires $X_t(i)$ units of the final good to produce $X_t(i)$ units of good i at quality $Z_t(i)$. The firm also bears a fixed operating cost $\phi Z_t^{\alpha}(i) Z_t^{1-\alpha}$ in units of the final good. The firm can allocate $I_t(i)$ units of the final good to accumulate firm-specific knowledge according to the technology

$$\dot{Z}_t(i) = I_t(i). \tag{11}$$

The firm's gross profit (i.e., profit before-R&D) is

$$\Pi_t(i) = [p_t(i) - 1] X_t(i) - \phi Z_t^{\alpha}(i) Z_t^{1-\alpha}. \tag{12}$$

The value of the firm is

$$V_t(i) = \int_t^{\infty} \exp\left(-\int_t^s r_u du\right) [\Pi_s(i) - I_s(i)]\, ds. \tag{13}$$

The firm maximizes (13) subject to (7) and (11). We solve this problem in Appendix A; here we discuss only the elements needed to address the paper's research question.

The demand curve (7) says that an unconstrained monopolist would charge $p_t(i) = 1/\theta$. However, we assume that competitive fringe firms can produce good i at the same quality $Z_t(i)$ as the monopolist but at the higher marginal cost $\mu \in (1, 1/\theta)$.[9] The value-maximization problem then says that the monopolistic firm sets

$$p_t(i) = \min\{\mu, 1/\theta\} = \mu \tag{14}$$

to price fringe firms out of the market. The problem also delivers the firm's rate of return to quality innovation,

$$r_t^q(i) = \alpha \frac{\Pi_t(i)}{Z_t(i)} = \alpha \left[(\mu - 1)\frac{X_t(i)}{Z_t(i)} - \phi Z_t^{\alpha-1}(i)\, Z_t^{1-\alpha}\right],$$

which is linear in quality-adjusted firm size $x_t(i) \equiv X_t(i)/Z_t(i)$. This property is at the heart of the mechanism that we study: incentives to innovate depend on quality-adjusted firm size, which in turn depends on the size of the market.

In models of this class, the equilibrium of the market for intermediate goods is symmetric: firms start with the same initial knowledge $Z_0(i) = Z_0$ for $i \in [0, N_0]$ and, facing a symmetric environment, make identical decisions. Consequently, they grow at the same rate and symmetry holds at any point in time. Using the limit price (14), we then have

$$x_t = \frac{X_t}{Z_t} = \left(\frac{\theta}{\mu}\right)^{1/(1-\theta)} \left(\frac{L_t}{N_t}\right)^{\gamma} R_t^{1-\gamma}. \tag{15}$$

This variable compresses the three variables L_t (labor input), R_t (land input) and N_t (mass of firms) into a single variable and thus makes the analysis of the model's dynamics very simple. For brevity, henceforth, we refer to x_t as "firm size". With this notation, the rate of return to quality innovation is

$$r_t^q = \alpha \frac{\Pi_t}{Z_t} = \alpha\left[(\mu - 1)\, x_t - \phi\right]. \tag{16}$$

[9] Specifically, we allow for diffusion of knowledge from monopolistic firms to fringe firms that enables the latter to constrain the pricing behavior of the former. This characterization disentangles markups from the technological parameter θ that in this model is a key driver of the functional distribution of income.

2.4 Entrants

A new firm pays βX_t,[10] $\beta > 0$, units of the final good to develop a new intermediate good of average quality, Z_t, set up operations and enter the market.[11] This structure preserves the symmetry of the equilibrium of the intermediate goods market at all times. The asset-pricing equation governing the value of firms (old and new) is

$$r_t = \frac{\Pi_t - I_t}{V_t} + \frac{\dot{V}_t}{V_t}. \tag{17}$$

Entry is positive when the free-entry condition holds, i.e., when

$$V_t = \beta X_t. \tag{18}$$

Substituting (7) and (14) into (12) and then using the resulting expression, (11), (15), (17) and (18) yield the return to entry as

$$r_t^e = \frac{1}{\beta}\left(\mu - 1 - \frac{\phi + z_t}{x_t}\right) + z_t + \frac{\dot{x}_t}{x_t}, \tag{19}$$

where $z_t \equiv \dot{Z}_t / Z_t$ is the growth rate of average quality.

2.5 Value of land

Let v_t denote the value of a unit of land. The asset-pricing equation for v_t is $r_t v_t = q_t + \dot{v}_t$. This equation states that the return on land is determined by the rental price q_t of land and the capital gain $\dot{v}_t$ in land value.

2.6 General equilibrium

The general equilibrium of this economy is a time path of allocations $\{A_t, C_t, G_t, L_t, R_t, X_t(i), I_t(i)\}$ and a time path of prices $\{r_t, w_t, q_t, v_t, p_t(i), V_t(i)\}$ such that:

- households maximize utility taking $\{r_t, w_t, q_t\}$ as given;
- final-good firms maximize profit taking $\{p_t(i), w_t, q_t\}$ as given;

[10] It is useful to note that X_t is the profit-maximizing firm size in the intermediate-good sector at time t.

[11] Peretto and Connolly (2007) discuss alternative specifications of entry costs that yield the same qualitative results. They also show that the cost of entry scaling with market size prevents the cost from vanishing in the presence of population growth. An empirical study by Bollard *et al.* (2016) documents that entry costs do rise with the level of development, providing empirical support for our theoretical specification.

- intermediate-good firms choose $\{p_t(i), I_t(i)\}$ to maximize $V_t(i)$ taking r_t as given;
- entrants make entry decisions anticipating that when in operation they will maximize their value, i.e., they will behave as the incumbents in the previous bullet point;
- aggregate household wealth is the sum of the value of land and of the aggregate value of monopolistic firms, $A_t \equiv \int_0^1 a_t(h)dh = v_t R + V_t N_t$;
- the market for land services clears, $\int_0^1 R_t(h)dh = R$;
- the labor market clears, $\int_0^{N_t} L_t(i)di = N_t L_t(i) = L_t$;
- the market for the final good G_t clears.

2.7 Aggregation

In symmetric equilibrium, (7) and (14) yield the reduced-form representation of final output

$$G_t = (\theta/\mu)^{\theta/(1-\theta)} N_t^{1-\gamma} Z_t L_t^{\gamma} R^{1-\gamma}, \tag{20}$$

where total land endowment R is constant. The associated growth rate of output is

$$g_t \equiv \frac{\dot{G}_t}{G_t} = (1-\gamma)n_t + z_t + \gamma\lambda. \tag{21}$$

This growth rate has three components: (i) the growth rate of the variety of intermediate goods, $n_t \equiv \dot{N}_t/N_t$; (ii) the growth rate of the average quality of intermediate goods, z_t; (iii) the growth rate of the labor force λ multiplied by the labor elasticity of final output γ.

3 Dynamics

This section analyzes the dynamics of the model. See Section 3.1 for the dynamics of the aggregate economy. See Section 3.2 for the dynamics of the wealth distribution. See Section 3.3 for the dynamics of the income distribution. Section 3.4 provides a discussion on income inequality.

3.1 Dynamics of the aggregate economy

The model identifies two eras: the pre-industrial era, where no innovation of any kind takes place, and the industrial era, where variety innovation takes place because the free-entry condition holds with equality. The industrial era consists of two phases: in phase 1, only horizontal innovation occurs; and in phase 2, quality innovation also occurs.[12]

In the pre-industrial era, the demand for each intermediate product is initially so small (i.e., $x_0 < \phi/(\mu-1)$) that a would-be monopolist operating the increasing-returns technology would earn negative profit. Thus, the existing N_0 intermediate goods are produced by competitive firms that do not innovate, make zero profit at the price $p_t(i) = \mu$, and have zero stock-market value. Anticipating this, agents are not willing to pay the sunk entry cost and there is no variety innovation. Initially, therefore, all technologies exhibit constant returns to scale and the demand for each intermediate product grows only because of exogenous population growth. Eventually, the size of the market for intermediate goods is sufficiently large that a would-be monopolist operating the increasing-returns technology could earn a positive profit. We assume, however, that only innovation, in this case a process innovation, allows a new firm to monopolize an existing market. The pre-industrial era, therefore, ends only when the present value of monopolistic firms is sufficiently large that the free-entry condition (18) holds.

We summarize the first important result governing the model's dynamics in the following proposition, which states that two key grand ratios take up era-specific constant values.

Proposition 1 *(Grand Ratios) Define the composite parameter*

$$\Theta \equiv \frac{\rho\beta\theta}{(1-\theta)\,\mu}.$$

The equilibrium consumption-output ratio is

$$\frac{C_t}{G_t} = \left(\frac{C}{G}\right)^* = \begin{cases} 1-\theta & n_t = 0 \\ (1-\theta)(1+\Theta) & n_t > 0 \end{cases},$$

where $n_t = 0$ identifies the pre-industrial era (the free-entry condition does not hold) and $n_t > 0$ identifies the industrial era (the free-entry condition holds). Similarly, the consumption-wealth ratio is

$$\frac{C_t}{A_t} = \left(\frac{C}{A}\right)^* = \begin{cases} \frac{\rho}{1-\gamma} & n_t = 0 \\ \frac{\rho}{1-\gamma}\frac{1+\Theta}{1+\frac{\Theta}{1-\gamma}} & n_t > 0 \end{cases}.$$

Proof. See Appendix A. ■

[12] See Bouscasse *et al.* (2021) for recent evidence that the historical pattern consists of a secular acceleration of economic growth that can be divided in two well-identified phases. There is some debate in the literature about the precise timing of the key events, but there is remarkable agreement on the overall time-profile of the process.

The result that the consumption-output ratio is always constant implies that at all times consumption and output grow at the same rate, i.e.,

$$g_t \equiv \frac{\dot{G}_t}{G_t} = \frac{\dot{C}_t}{C_t}.$$

As the economy progresses through the three phases discussed above, the growth rate is

$$g_t = \begin{cases} \gamma\lambda & n_t = 0 \\ \gamma\lambda + (1-\gamma)n_t & n_t > 0 \text{ and } z_t = 0 \\ \gamma\lambda + (1-\gamma)n_t + z_t & n_t > 0 \text{ and } z_t > 0 \end{cases} .$$

To translate this characterization of the general equilibrium of the model into a state-space representation, in Appendix A we construct the rates of variety growth (entry) n_t and of quality growth z_t as two functions of the state variable x_t that account for the non-negativity constraints $n_t \geq 0$ and $z_t \geq 0$. The end result is

$$g_t = \begin{cases} \gamma\lambda & 0 \leq x \leq x_N \\ \gamma\lambda + (1-\gamma)\left[\frac{1}{\beta}\left(\mu - 1 - \frac{\phi}{x_t}\right) - \rho\right] & x_N < x_t \leq x_Z \\ \alpha\left[(\mu-1)x_t - \phi\right] - \rho & x_t > x_Z \end{cases} , \tag{22}$$

where x_N and x_Z are the firm-size activation thresholds of, respectively, variety and quality innovation; see Appendix A for their derivations.[13] This piecewise function says that in each phase, growth accelerates because one form of Schumpeterian innovation starts occurring. Proposition 1, moreover, says that the consumption-output ratio jumps up when the first phase transition occurs because this event entails the costly creation of a new form of wealth—equity shares in monopolistic firms that accumulate intangible capital—that make households richer. The individual and aggregate effects of this *wealth creation event* depend on how the newly issued shares are distributed across the heterogeneous households (more on this below).

The function $n(x_t)$ constructed in Appendix A yields the equilibrium law of motion of the state variable x_t. We summarize the property as follows.

[13] The inequality $x_N < x_Z$ can be ensured by a parameter restriction that imposes an upper bound on α, which holds in our calibration.

Proposition 2 *(Equilibrium Dynamics) Assume*

$$\rho + \lambda < \min\left\{(1-\alpha)\phi, (1-\alpha)(\mu-1)/\beta\right\}.$$

The key properties of the model's dynamics are as follows. (i) The state variable x_t *obeys the law of motion*

$$\dot{x}_t = \begin{cases} \gamma\lambda x_t & 0 \leq x \leq x_N \\ \gamma\left[\frac{\phi}{\beta} - \left(\frac{\mu-1}{\beta} - \lambda - \rho\right)x_t\right] & x_N < x_t \leq x_Z \\ \gamma\left[\lambda - \frac{[(1-\alpha)(\mu-1)-\rho\beta]x_t-(1-\alpha)\phi+\rho+\gamma\lambda}{\beta x_t-(1-\gamma)}\right]x_t & x_t > x_Z \end{cases}.$$

(ii) There exists a unique, scale-invariant, steady state

$$x^* = \frac{(1-\alpha)\phi - (\rho+\lambda)}{(1-\alpha)(\mu-1) - \beta(\rho+\lambda)} > x_Z. \tag{23}$$

(iii) Given initial condition $x_0 \in (0, x_N)$*, the dynamics are globally stable and* x_t *converges to the steady state* x^**. (iv) The steady state exhibits the scale-invariant growth rate*

$$g^* = \alpha\left[(\mu-1)x^* - \phi\right] - \rho > 0. \tag{24}$$

Proof. See Appendix A. ■

We illustrate the dynamics described by Proposition 2 in two figures. Figure 1 shows that firm size x_t grows throughout the transition, following an S-shaped (i.e., logistic) path, where T_N and T_Z are the activation dates of, respectively, variety and quality innovation.[14] As x_t converges to its steady-state value x^*, we have

$$N_t = \left[\left(\frac{\theta}{\mu}\right)^{1/(1-\theta)} \frac{R^{1-\gamma}}{x^*}\right]^{1/\gamma} L_t,$$

so that the mass of products/firms grows at the rate λ; see Laincz and Peretto (2006), among many others, for empirical evidence that N_t is proportional to L_t in advanced economies.[15]

[14] See Appendix A.

[15] Laincz and Peretto (2006) use the number of establishments as a proxy for the number of products and provide a detailed discussion on this assumption.

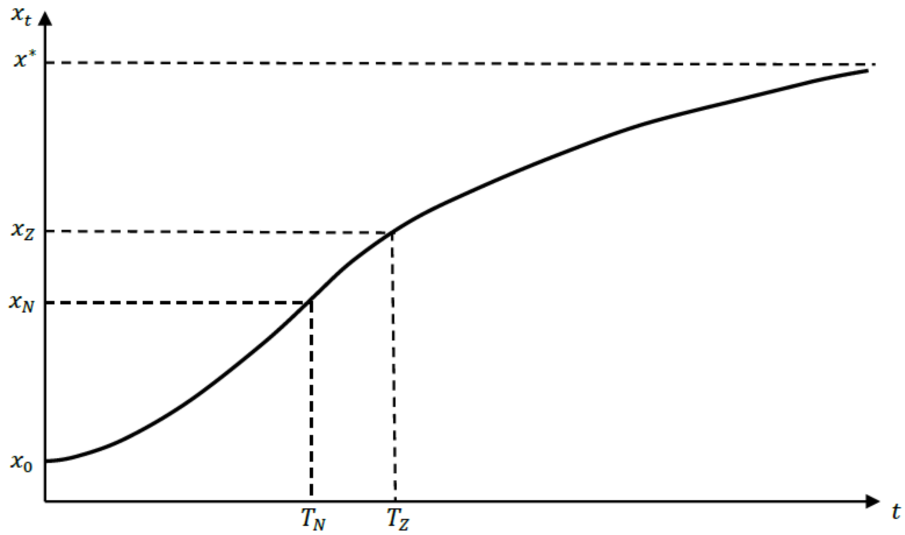

Figure 1: Transition path of the firm size

Figure 2 shows the dynamics of economic growth that we obtain by feeding the path of x_t to the growth rate equation (22). In the pre-industrial era, the growth rate of output is simply $g_t = \gamma\lambda$ due to the absence of innovation. In the industrial era, the growth rate accelerates, initially fueled only by variety innovation and then also by quality innovation. As x_t converges to x^*, the growth rate converges to the steady-state value g^* in (24).

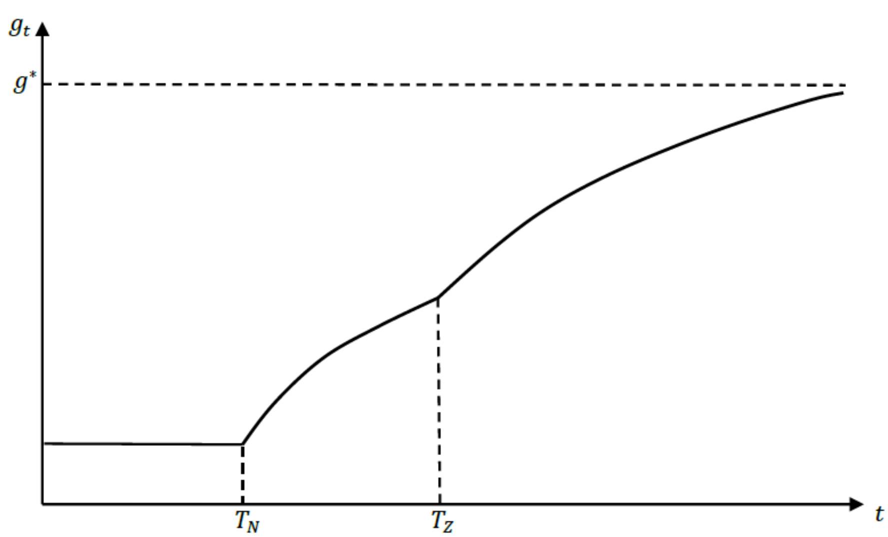

Figure 2: Transition path of the growth rate

This gradual acceleration of economic growth is consistent with historical data for the UK. Corresponding to the growth rate of output in Figure 2, we plot the log of the UK real GDP from 1700 to 2016 in Figure 3.[16] The slope of the plot is the growth rate. According to the data, the average growth rate of real GDP in the UK was 0.71% in the first half of the 18th century, 1.24% in the second half of the 18th century, 1.86% in the first half of the 19th century, 2.23% in the second half of the 19th century, 1.50% in the first half of the 20th century and 2.55% from the second half of the 20th century onwards. Except for the wartime periods in the first half of the 20th century, the UK has experienced a gradually rising growth rate as in our model. This acceleration in economic growth is due to the activation of innovation in our Schumpeterian model of endogenous takeoff. Using historical data in the UK from 1661 to 1851, Sullivan (1989) shows that there was an acceleration in the number of patents starting in 1757 that was caused by an acceleration in patentable invention and followed by an increase of the growth rate of total factor productivity. We are interested in the implications of these dynamics for inequality.

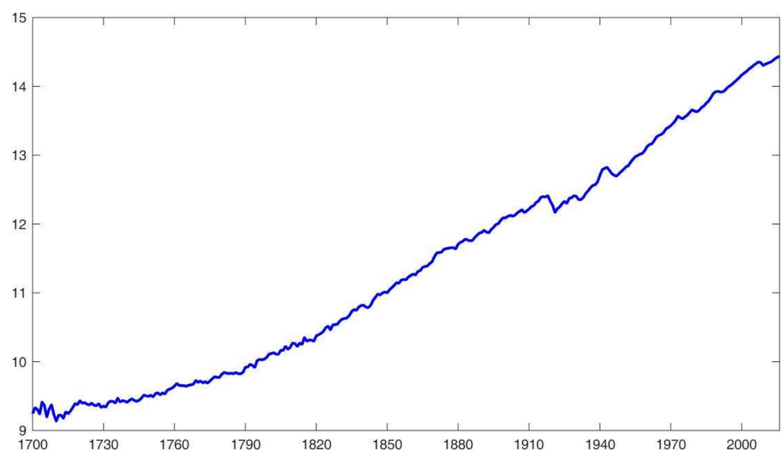

Figure 3: Log of real GDP in the UK from 1700 to 2016

3.2 Dynamics of the wealth distribution

Let $s_{c,t}(h) \equiv c_t(h)/C_t$ and $s_{a,t}(h) \equiv a_t(h)/A_t$ be, respectively, the share of consumption and of wealth of household h at time t. Equation (3) says that households want to achieve the same consumption growth rate, since they face the common interest rate r_t and they have

[16] Data source: Federal Reserve Bank of St. Louis.

the same discount rate ρ. It follows that the household consumption share $s_{c,t}(h)$ is always constant. In particular, it jumps at time $t = 0$ and remains constant during the pre-industrial era. It then jumps again at time $t = T_N$ and remains constant thereafter. We denote these constant values $s^*_{c,0}(h)$ and $s^*_{c,T_N}(h)$, respectively. This behavior of household consumption incorporates all information about the future, including the arrival of the industrial era, and drives the dynamics of the wealth share.

The following proposition summarizes our main formal result on the dynamics of the household consumption and wealth shares.

Proposition 3 *(Household Shares) Consider household h for $h \in [0,1]$. Let $s_{R,0}(h) \geq 0$ be the household's land share at time $t = 0$ and let $s_{N,T_N}(h)$ be the household's industrial share at $t = T_N$. In equilibrium, the household's consumption and wealth shares are, respectively:*

$$s^*_{c,0}(h) - 1 = \frac{1-\gamma}{1-\gamma+\Theta e^{-\rho T_N}} \left\{ (1-\gamma)\left[s_{R,0}(h) - 1\right] + \Theta e^{-\rho T_N}\left[s_{N,T_N}(h) - 1\right] \right\};$$

$$s^*_{c,T_N}(h) - 1 = \frac{1-\gamma+\Theta}{(1+\Theta)\left(1-\gamma+\Theta e^{-\rho T_N}\right)} \left\{ (1-\gamma)\left[s_{R,0}(h) - 1\right] + \Theta e^{-\rho T_N}\left[s_{N,T_N}(h) - 1\right] \right\}.$$

$$s_{a,t}(h) - 1 = \begin{cases} \frac{1-\gamma+\Theta e^{\rho(t-T_N)}}{1-\gamma+\Theta e^{-\rho T_N}}\left[s_{R,0}(h) - 1\right] + \frac{e^{-\rho T_N} - e^{\rho(t-T_N)}}{1-\gamma+\Theta e^{-\rho T_N}}\Theta\left[s_{N,T_N}(h) - 1\right] & 0 \leq t \leq T_N \\ \frac{1-\gamma}{1-\gamma+\Theta e^{-\rho T_N}}\left[s_{R,0}(h) - 1\right] + \frac{\Theta e^{-\rho T_N}}{1-\gamma+\Theta e^{-\rho T_N}}\left[s_{N,T_N}(h) - 1\right] & t > T_N \end{cases}.$$

Proof. See Appendix A. ■

The proposition says that agents incorporate in their decisions all available information at $t = 0$. The implication for the dynamics of the wealth shares is twofold. In the pre-industrial era, our heterogenous households trade land to achieve intertemporal consumption smoothing in anticipation of the forthcoming industrial era. This process creates the *endogenous* distribution of land with which the economy begins the industrial era. Second, Proposition 1 says that the grand ratios that households face in the industrial era are constant and no further changes are expected to occur. Therefore, households need not trade wealth to achieve intertemporal consumption smoothing in anticipation of a future change in the economic landscape and, consequently, the wealth distribution stabilizes. To complete the model's solution we need to take a stand on the distribution of industrial wealth at $t = T_N$. Since the model does not contain forces that relate it endogenously to fundamentals, we treat it as an exogenous parameter that allows us to construct different scenarios.[17]

[17] In future work, we plan to make the initial distribution of industrial wealth endogenous by developing a component of the model that determines how heterogenous households participate in the formation of new firms.

According to this characterization, in the pre-industrial era the distribution of wealth is determined by the initial exogenous distribution of land and by the households trading land to achieve intertemporal consumption smoothing in anticipation of the arrival of the industrial era. When the economy enters the industrial era, a new form of wealth appears—equity shares in industrial firms that develop and apply new technology. We find that the distribution of wealth, now consisting of both land and industrial shares, becomes stationary and jointly determined by the endogenous distribution of land inherited from the pre-industrial era and by the initial distribution of industrial shares at the time of the takeoff. Throughout this process, the economy features transition dynamics determined by the evolution of firm size. Note that in the pre-industrial era

$$\dot{s}_{R,t}\left(h\right)=\frac{\rho\Theta e^{\rho(t-T_N)}}{1-\gamma+\Theta e^{-\rho T_N}}\left[s_{R,0}\left(h\right)-s_{N,T_N}\left(h\right)\right].$$

There is thus a precise relationship between the households shares of land at $t=0$ and of industrial wealth at $t=T_N$ that determines whether the household relatively dissaves or saves in the run-up to the industrial revolution. The derivative is positive for $s_{R,0}\left(h\right)>s_{N,T_N}\left(h\right)$. This suggests a form of *wealth smoothing*: relatively land rich households who expect to be relatively industry poor acquire land to compensate and maintain their overall wealth share.

We now use our explicit solution to look at measures of wealth inequality. The model produces any other summary statistics that we might wish to consider. The variance of the wealth share is

$$\sigma^2_{a,t}\equiv\int_0^1[s_{a,t}(h)-1]^2dh.$$

With the expressions in Proposition 3, we calculate

$$\sigma^2_{a,t}=\begin{cases}\int_0^1\left[\frac{1-\gamma+\Theta e^{\rho(t-T_N)}}{1-\gamma+\Theta e^{-\rho T_N}}\left[s_{R,0}\left(h\right)-1\right]+\frac{e^{-\rho T_N}-e^{\rho(t-T_N)}}{1-\gamma+\Theta e^{-\rho T_N}}\Theta\left[s_{N,T_N}\left(h\right)-1\right]\right]^2dh & 0\leq t\leq T_N\\ \int_0^1\left[\frac{1-\gamma}{1-\gamma+\Theta e^{-\rho T_N}}\left[s_{R,0}\left(h\right)-1\right]+\frac{\Theta e^{-\rho T_N}}{1-\gamma+\Theta e^{-\rho T_N}}\left[s_{N,T_N}\left(h\right)-1\right]\right]^2dh & t>T_N\end{cases}.$$

The first branch is time varying due to the time-varying weights attached to the two exogenous shares $s_{R,0}\left(h\right)$ and $s_{N,T_N}\left(h\right)$. The second branch features a constant value, which we denote σ^2_{a,T_N}. We focus on the first branch to assess what happens to this indicator of wealth inequality in the run up to industrialization. A standard result in statistics gives

$$\begin{aligned}\sigma^2_{R,t} &= \left[\frac{1-\gamma+\Theta e^{\rho(t-T_N)}}{1-\gamma+\Theta e^{-\rho T_N}}\right]^2 Var\left(s_{R,0}\left(h\right)\right)+\left[\frac{e^{-\rho T_N}-e^{\rho(t-T_N)}}{1-\gamma+\Theta e^{-\rho T_N}}\Theta\right]^2 Var\left(s_{N,T_N}\left(h\right)\right)\\ &\quad +2\frac{1-\gamma+\Theta e^{\rho(t-T_N)}}{1-\gamma+\Theta e^{-\rho T_N}}\frac{e^{-\rho T_N}-e^{\rho(t-T_N)}}{1-\gamma+\Theta e^{-\rho T_N}}\Theta Cov\left(s_{R,0}\left(h\right),s_{N,T_N}\left(h\right)\right).\end{aligned}$$

The coefficients of the two variance terms say that land inequality rises over time with initial land inequality and falls over time with the anticipated industrial wealth inequality. The coefficient of the covariance term is non-monotonic in t. Over time, therefore, land inequality is subject to competing forces that depend on several factors: the exogenous distributions of land at time 0 and of industrial wealth at time T_N (the time of the takeoff), the duration of the pre-industrial era T_N, the aggregate land share $1-\gamma$, and the fundamentals determining industrial wealth, the factor Θ. This result follows from the fact that households with expected industrial wealth share lower than the initial land share *acquire* land to engage in a form of wealth smoothing that, to our knowledge, is not discussed in the existing literature.

To close this subsection, we sort households in ascending order of wealth and define the Gini coefficient of wealth at time t,

$$\sigma_{a,t}^G \equiv 1 - 2\int_0^1 \mathcal{L}_{a,t}(h)dh,$$

where the Lorenz curve of wealth inside the integral is

$$\mathcal{L}_{a,t}(h) \equiv \frac{\int_0^h a_t(\chi)d\chi}{\int_0^1 a_t(\chi)d\chi} = \frac{\int_0^h a_t(\chi)d\chi}{A_t} = \int_0^h s_{a,t}(\chi)d\chi.$$

Similarly, we can define the share of wealth of the top ε households at time t,

$$S_{a,t}^{\varepsilon} \equiv \int_{1-\varepsilon}^1 s_{a,t}(h)dh.$$

We focus on the Gini coefficient because of its prominent role in the literature.

3.3 Dynamics of the income distribution

Household h earns income $y_t(h) \equiv r_t a_t(h) + w_t L_t$. Aggregating across households yields

$$Y_t \equiv \int_0^1 y_t(h)dh = r_t A_t + w_t L_t.$$

Let $s_{y,t}(h) \equiv y_t(h)/Y_t$ denote the income share of household h. Proposition 3 highlights that the economics of our model dictates that we look at variables in deviation from the mean. We follow that insight and write the following equilibrium relation (see Appendix A for the derivation):

$$s_{y,t}(h) - 1 = \frac{r_t}{r_t + \frac{w_t L_t}{A_t}}\left[s_{a,t}(h) - 1\right].$$

Filling in the era-specific values for the aggregate variables, we obtain:

$$s_{y,t}(h) - 1 = \begin{cases} \left(1 + \frac{\rho}{\rho+\gamma\lambda}\frac{\gamma}{1-\gamma}\right)^{-1} [s_{a,t}(h) - 1] & 0 \leq x_t \leq x_N \\ \left(1 + \frac{\rho}{\rho+g_t}\frac{\gamma}{1-\gamma+\Theta}\right)^{-1} [s_{a,t}(h) - 1] & x_t > x_N \end{cases}. \tag{25}$$

This equation says that in the pre-industrial era the household income share relative to the mean is a constant multiple of the household wealth share relative to the mean. In the industrial era, in contrast, the multiplier is an increasing function of growth rate g_t. This property constitutes the main transmission channel of macro events to household income and thus to the cross-sectional distribution of income.

Equation (25) allows us to derive any summary statistic of the income distribution. The variance of the income share is

$$\sigma^2_{y,t} \equiv \int_0^1 [s_{y,t}(h) - 1]^2 dh = \begin{cases} \left(1 + \frac{\rho}{\rho+\gamma\lambda}\frac{\gamma}{1-\gamma}\right)^{-2} \sigma^2_{a,t} & 0 \leq x_t \leq x_N \\ \left(1 + \frac{\rho}{\rho+g_t}\frac{\gamma}{1-\gamma+\Theta}\right)^{-2} \sigma^2_{a,T_N} & x_t > x_N \end{cases}.$$

The Gini coefficient of income is

$$\sigma^G_{y,t} = \begin{cases} \left(1 + \frac{\rho}{\rho+\gamma\lambda}\frac{\gamma}{1-\gamma}\right)^{-1} \sigma^G_{a,t} & 0 \leq x_t \leq x_N \\ \left(1 + \frac{\rho}{\rho+g_t}\frac{\gamma}{1-\gamma+\Theta}\right)^{-1} \sigma^G_{a,T_N} & x_t > x_N \end{cases}. \tag{26}$$

The income share of the top ε households is

$$S^\varepsilon_{y,t} \equiv \int_{1-\varepsilon}^1 s_{y,t}(h) dh = \frac{r_t A_t S^\varepsilon_{a,t} + w_t L_t \varepsilon}{r_t A_t + w_t L_t}.$$

We can use (26) to rewrite $S^\varepsilon_{y,t}$ as

$$S^\varepsilon_{y,t} = \frac{\sigma^G_{y,t}}{\sigma^G_{a,t}} \left(S^\varepsilon_{a,t} - \varepsilon\right) + \varepsilon. \tag{27}$$

This expression says that the top ε income share is increasing in the ratio of the Gini indices $\sigma^G_{y,t}/\sigma^G_{a,t}$ if and only if $S^\varepsilon_{a,t} > \varepsilon$. In other words, a rising Gini coefficient of income does not necessarily yield a rising top ε income share. The condition for this to happen is that the share of wealth of the top ε household be larger than ε. This condition clearly holds in the data since wealth is highly concentrated.

We summarize our result on the evolution of income inequality as follows. In the pre-industrial era, the dynamics of income inequality is driven by the dynamics of land ownership inequality. At the beginning of the industrial era, income inequality becomes a multiple of the constant wealth inequality, with a multiplier that is an increasing function of the growth rate g_t, whose dynamics are described by equation (22). In phase 1, the growth rate is fueled only by variety innovation and rises gradually. Eventually, phase 2 starts and quality innovation adds its contribution, providing a new acceleration of the growth rate with final convergence to the steady state g^* described in (24).

3.4 Discussion

Our model says that income inequality is initially driven only by land inequality. The wealth distribution in the pre-industrial era evolves endogenously due to the fact that households trade land among each other to achieve intertemporal consumption smoothing in anticipation of the arrival of the industrial era. This component of our mechanism is entirely expectations driven and, to the best of our knowledge, previously unreported in the literature.

After the takeoff, in contrast, the wealth distribution stabilizes and income inequality rises during the transition because the growth acceleration that takes place throughout the industrial era is a manifestation of the rising rate of return to innovation, which in equilibrium manifests itself as a rising rate of return to corporate equity. The rising rate of return propagates through the households, continuously spreading out the distribution of their incomes because of the heterogeneity in their assets holdings. This dynamic mechanism emphasizes the property mentioned above that in this model wealth inequality is the root cause of income inequality. This insight is consistent with the thrust of the recent literature based on Piketty (2014), which sees fundamental differences across households in their sources of income—capital vs. labor—as the root cause of income inequality. Our main mechanism differs in that the dynamics of income inequality are driven by growth accelerations, not by the mere fact that the interest rate is higher than the growth rate.

To summarize, in this model the income distribution is non-degenerate, endogenous and analytically tractable. The dynamics produces a clear insight: the secular acceleration of the growth rate in the aftermath of the Industrial Revolution produced a secular rise of income inequality. The underlying wealth distribution is stationary because our households have identical homothetic preferences over consumption and thus want parallel log-consumption paths with intercepts that differ because of the unequal distribution of wealth that they inherit from the pre-industrial era and of the exogenous distribution of industrial wealth at the onset of the industrial era. In particular, the stationarity result is due to the fact that in the industrial era, households expect no further future changes in the economic landscape.

3.5 A useful special case

We now consider the special case in which the industrial wealth share is identical to the initial land share, $s_{N,T_N}(h) = s_{R,0}(h)$. Our solution above says that the land share in the pre-industrial era remains stationary (i.e., $\dot{s}_{R,t}(h) = 0$) so that $s_{a,t}(h) = s_{R,0}(h)$ for all $t \geq 0$. This means that households make initial consumption decisions that put them of paths that preserve the initial wealth share forever. Accordingly, all our measures of wealth inequality remain constant. The variance of the wealth share is

$$\sigma_{a,t}^2 = \sigma_{a,0}^2 = \int_0^1 [s_{R,0}(h) - 1]^2 \, dh$$

The Gini coefficient of wealth is

$$\sigma_{a,t}^G = \sigma_{a,0}^G = 1 - 2\int_0^1 \mathcal{L}_{a,0}(h)dh,$$

where the Lorenz curve of wealth inside the integral becomes

$$\mathcal{L}_{a,t}(h) = \int_0^h s_{a,t}(\chi)d\chi = \int_0^h s_{a,0}(\chi)d\chi = \int_0^h s_R(\chi)d\chi \equiv \mathcal{L}_{a,0}(h),$$

which is exogenously determined at time 0. The share of wealth of the top ε households is

$$S_{a,t}^{\varepsilon} \equiv \int_{1-\varepsilon}^1 s_{a,t}(h)dh = \int_{1-\varepsilon}^1 s_R(h)dh \equiv S_{a,0}^{\varepsilon}.$$

This result is the strongest manifestation of our expectation driven mechanism driving wealth inequality. Households want to preserve the consumption share and when they expect their initial share of industrial wealth at time T_N to equal their initial land share, they chose consumption paths that freeze the wealth distribution at $t = 0$.

With the household wealth share pinned down at $t = 0$, the variance of the household income share becomes

$$\sigma_{y,t}^2 \equiv \int_0^1 [s_{y,t}(h) - 1]^2 dh = \begin{cases} \left(1 + \frac{\rho}{\rho+\gamma\lambda}\frac{\gamma}{1-\gamma}\right)^{-1} \sigma_{a,0}^2 & 0 \leq x_t \leq x_N \\ \left(1 + \frac{\rho}{\rho+g_t}\frac{\gamma}{1-\gamma+\Theta}\right)^{-1} \sigma_{a,0}^2 & x_t > x_N \end{cases}.$$

The Gini coefficient of income simplifies to

$$\sigma_{y,t}^G = \begin{cases} \left(1 + \frac{\rho}{\rho+\gamma\lambda}\frac{\gamma}{1-\gamma}\right)^{-1} \sigma_{a,0}^G & 0 \leq x_t \leq x_N \\ \left(1 + \frac{\rho}{\rho+g_t}\frac{\gamma}{1-\gamma+\Theta}\right)^{-1} \sigma_{a,0}^G & x_t > x_N \end{cases}.$$

In other words, because wealth inequality stays constant at all time, income inequality remains constant in the pre-industrial era. Then it jumps up at the time of the takeoff and gradually rises in the industrial era. Figure 4 summarizes the dynamics of the Gini index $\sigma_{y,t}$ from stagnation to takeoff and eventually to the steady state.

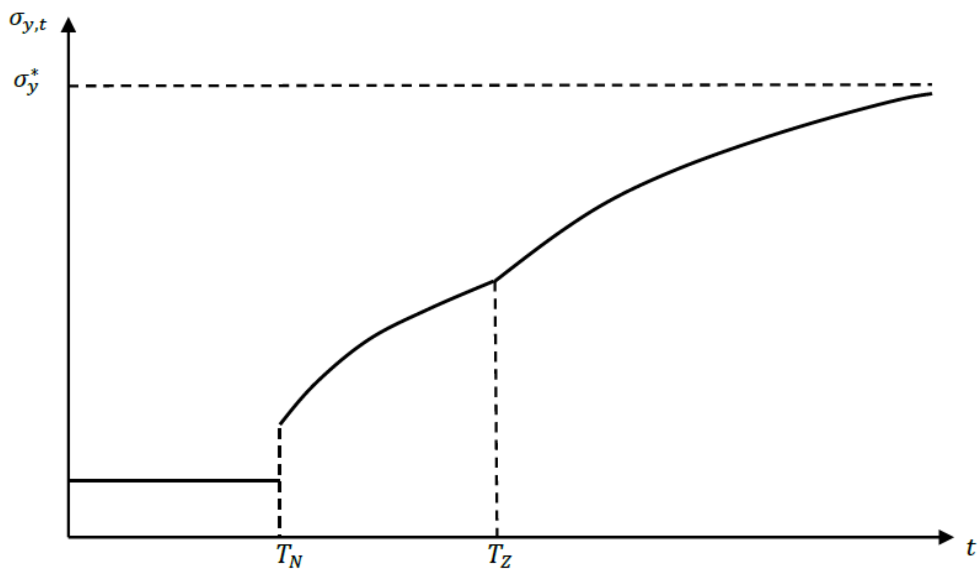

Figure 4: Transition path of income inequality

This special case is interesting also because it replicates the major qualitative property of our model when we impose one of two restrictions that suppress our expectation driven mechanism driving wealth dynamics. The first is the restriction that we used in our previous working paper (Chu and Peretto 2021), namely, that households cannot trade land. As we showed there, under this assumption households must preserve their initial wealth distribution, which can change only when the industrial era begins. As a consequence, the model predicts that the wealth distribution is a step function of time, changing discretely at $t = T_N$. The second restriction is that households do not foresee the arrival of the industrial era and thus take no action to smooth consumption and wealth intertemporally in the pre-industrial era. Strictly speaking, such a restriction is inconsistent with the solution procedure for the rest of the model, which uses rational expectations and perfect foresight, but it might appeal to some readers. We stress that the special restrictions discussed in this subsection do not change the model's implications for the industrial era; they only affect the characterization of wealth dynamics in the pre-industrial era.

4 Quantitative analysis

In this section, we calibrate the model to UK data in order to perform a quantitative analysis. To keep things simple and firmly focused on the growth acceleration of the industrial era, we work with the model's solution for the special case just discussed. One reason for doing so is that there exists very little information on the dynamics of the wealth distribution in the pre-industrial era. It is thus reasonable to work with the version of the model that keeps it constant.

The model features the following parameters: $\{\rho, \alpha, \lambda, \theta, \beta, \gamma, \mu, \phi\}$. We set the discount rate ρ to 0.04. We follow Iacopetta *et al.* (2019) to set the degree of technology spillovers $1-\alpha$ to 0.833. In the UK, the long-run population growth rate λ is 0.6%.[18] Then, we calibrate the remaining parameters $\{\theta, \beta, \gamma, \mu, \phi\}$ by matching the following moments for the UK economy: 52.6% for labor income as a share of output,[19] 74.4% for consumption as a share of output,[20] 12.3% for housing rents as a share of output,[21] 2.5% for the growth rate of output,[22] and 18.4% for investment as a share of output.[23] Table 1 summarizes the calibrated parameter values.[24] These parameter values imply a rate of asset returns of 6.5% and R&D as a share of output of 2.0%, which are in line with UK data.

Table 1: Calibrated parameter values

ρ	α	λ	θ	β	γ	μ	ϕ
0.040	0.167	0.006	0.351	14.468	0.810	2.138	0.245

Figure 5 presents the simulated paths of the output growth rate g_t and the real interest rate r_t along with the HP-filter trends of the GDP growth rate and the rate of return on non-residential fixed capital in the UK.[25] We choose an initial value x_0 such that the takeoff occurs in the late 18th century.[26] This figure shows that the output growth rate increases from about 0.5% in the late 18th century to 2.5% in recent time. This gradual increase in the growth rate and the magnitude of the increase are in line with historical data in the UK. Figure 5 also shows that the real interest rate increases from 4.5% in the late 18th century to an average of 5.9% in the 19th century and reaches an average of 6.4% in the 20th century.

[18] Data source: Maddison Project Database.

[19] Data source: Office for National Statistics.

[20] Data source: Office for National Statistics.

[21] Data source: New Economics Foundation.

[22] Data source: Federal Reserve Bank of St. Louis.

[23] Data source: Office for National Statistics. To compute this moment from the model, we add up expenses on intermediate goods and horizontal/vertical R&D. One can think of the intermediate goods in our model as investment in capital that depreciates rapidly.

[24] The calibrated value of μ seems high but implies a reasonable profit share of output of 11.5%.

[25] Here we use a smoothing parameter of 1000 on the annual data in order to extract a smoother trend.

[26] According to Ashton (1998), the Industrial Revolution started in as early as 1760.

The average rates of return on non-residential fixed capital in the UK were 5.1% in the 18th century, 6.0% in the 19th century, and 7.0% from the 20th century onwards.[27] Therefore, the increase in the rate of return on assets and the magnitude of the increase in asset returns predicted by our model are also in line with historical data; see Table 2 for a summary.

Table 2: Real interest rates			
century	18th	19th	20th
data	5.1%	6.0%	7.0%
model	4.5%	5.9%	6.4%

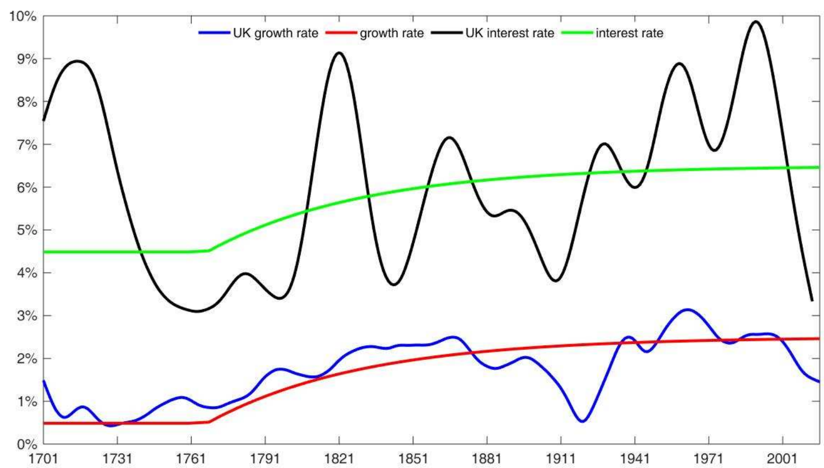

Figure 5: Simulated paths of the growth rate and the interest rate

The increase in the real interest rate in Figure 5 implies an increase in income inequality in our model. Here we focus on the special case $s_{N,T_N}(h) = s_{R,0}(h)$ due to the lack of data on the industrial wealth distribution at the time of Industrial Revolution. Figure 6 presents the simulated path of income inequality in terms of percent changes from its initial value prior to the takeoff. This figure shows that income inequality increases sharply by about 50% when the takeoff occurs. When the economy reaches the balanced growth path, income inequality would have almost doubled. Our model takes the degree of wealth inequality as given. If we consider a Gini coefficient of wealth of 0.732 in recent time,[28] then we can also simulate the Gini coefficient of income. Figure 7 reports the simulated path of income

[27] It is useful to note that the rate of return on non-residential fixed capital is not simply the bond rate which has decreased and moved in the opposite direction as all other assets since the early 20th century; see Madsen (2017). The authors are grateful to Jakob Madsen for sharing this data series.

[28] Data source: Credit Suisse Global Wealth Databook.

inequality along with the Gini coefficient of income in the UK from 1961 to 2017.[29] It shows that the simulated Gini coefficient of income increases from 0.15 before the takeoff to 0.29 in the steady state. Given the lack of historical data on the Gini coefficient of income, we will examine the historical data series on another measure of income inequality.

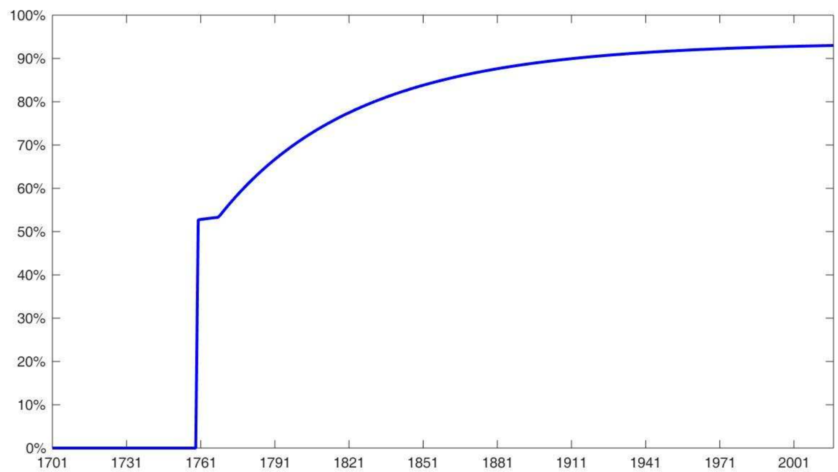

Figure 6: Simulated path of income inequality (percent change)

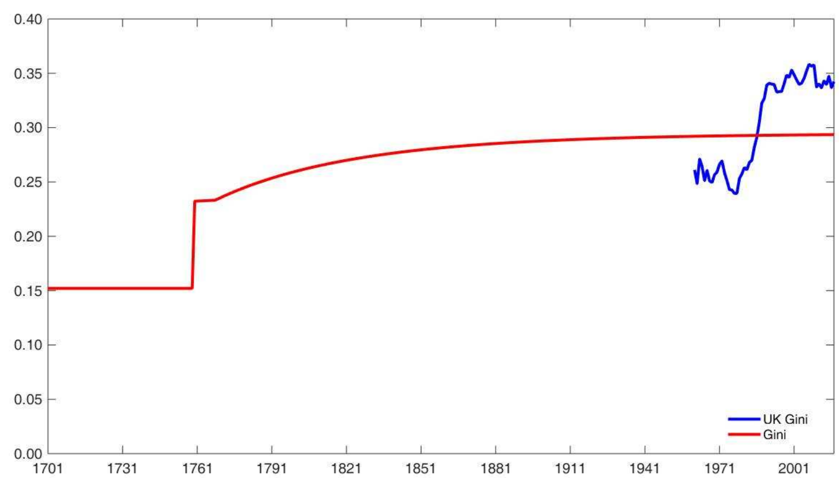

Figure 7: Simulated path of income inequality (Gini coefficient)

[29]Data source: Institute for Fiscal Studies. Data available from 1961.

Williamson (1980, 1985) and Lindert (2000a, 2000b) examine historical data in Britain and document that income inequality, based on different measures, increases in the late 18th century/early 19th century and levels off after the mid-19th century. Then, income inequality, measured by the top 1% income share, decreases from the early 20th century to the late 1970's.[30] As for the Gini coefficient of income, it decreases from 0.27 in the early 1960's to 0.24 in the late 1970's before rising again to as high as 0.36 in recent time with an average value of 0.30 from 1961 to 2017 in the UK. Therefore, the long-run level of income inequality predicted by our model is in line with recent data in the UK. Furthermore, our model is able to deliver the pattern of rising income inequality in the late 18th century/early 19th century and its leveling off in the late 19th century. However, our model is unable to explain the decrease in income inequality from the early 20th century to the late 1970's. The reason is that this decrease in income equality is driven by a decrease in wealth inequality,[31] whereas our model takes wealth inequality as given.

To address this issue, we consider historical data on the income and wealth shares owned by the top households, which have longer time series than the Gini coefficient. Therefore, we now use historical data on the top 10% wealth share in the UK along with the rate of return to assets computed from our model to simulate the top 10% income share. Figure 8 presents the simulated path of the top 10% income share along with data in the UK from 1900 to 2010.[32] Given the data on wealth inequality, our model now predicts that income inequality rises in the 19th century and falls from the early 20th century to the 1970's. After that, income inequality becomes rising again. This pattern matches the data. Furthermore, the average value of the top 10% income share in the UK from 1900 to 2010 is 0.37, whereas our model predicts an average value of 0.36 in this period.

[30] World Inequality Database documents a decrease in the top 1% income share from 20% in the early 20th century to 5% in the late 1970's.

[31] World Inequality Database documents a decrease in the top 1% wealth share from 70% in the early 20th century to less than 20% in the early 1980's.

[32] Data source: Piketty (2014). Data on the top 10% wealth (income) share is available from 1810 (1900).

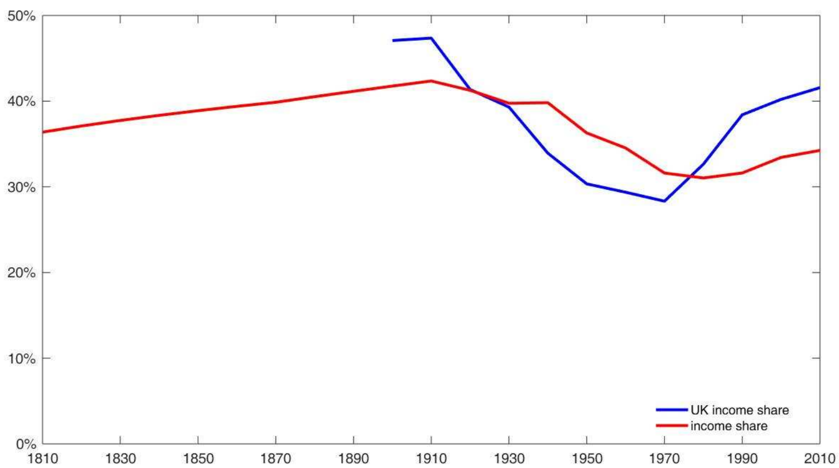

Figure 8: Simulated path of the top 10% income share

5 Labor income inequality

Our baseline model has three main strengths: (i) it is eminently tractable; (ii) it identifies sharply the role of growth accelerations as a main driver of rising income inequality; (iii) it measures inequality with a well understood and widely used summary statistic of the shape of a non-degenerate distribution of income. The baseline model, on the other hand, has one main weakness: inequality plays no role in shaping the transition from stagnation to growth.[33] In this section, we extend the analysis allowing for labor income inequality due to endogenous labor supply. The advantage of doing this is twofold. First, endogenous labor supply is interesting per se, and including it makes the analysis more empirically relevant. Second, endogenous labor supply introduces a channel through which inequality contributes to shaping the transition path of the economy.

5.1 The model with endogenous labor supply

We generalize the utility function of household $h \in [0,1]$ to

$$U(h) = \int_0^\infty e^{-\rho t} \left\{ \ln c_t(h) + \frac{\eta}{1-1/\omega} [1 - l_t(h)/L_t]^{1-1/\omega} \right\} dt, \tag{28}$$

[33] For example, Galor, Moav and Vollrath (2009) provide evidence that inequality in landownership affected the transition of the US economy from agriculture to industry.

where $l_t(h)$ is the household's labor supply and $1 - l_t(h)/L_t$ is leisure per member of the household. The parameter $\eta > 0$ determines the importance of leisure, whereas the parameter $\omega > 0$ determines the elasticity of intertemporal substitution for leisure. The budget constraint is

$$\dot{a}_t(h) = r_t a_t(h) + w_t l_t(h) - c_t(h). \tag{29}$$

The novel element is the household's endogenous supply of labor

$$\frac{l_t(h)}{L_t} = 1 - \left[\frac{\eta c_t(h)}{w_t L_t}\right]^{\omega}. \tag{30}$$

The rest of the model is the same as before.

5.2 Special case: $\omega = 1$

We begin our analysis with the special case $\omega = 1$, which gives log-log utility. Aggregating the labor supply (30) yields

$$\frac{l_t}{L_t} = \int_0^1 \frac{l_t(h)}{L_t} dh = 1 - \int_0^1 \frac{\eta c_t(h)}{w_t L_t} dh = 1 - \frac{\eta C_t}{w_t L_t}.$$

We stress that because labor supply is linear in consumption, the heterogeneity across households washes out. Next, we use the labor demand (9), note that Proposition 1 holds in this extension as well, and rearrange terms to write

$$\frac{\eta C_t}{w_t L_t} = \frac{\eta C_t}{w_t l_t} \frac{l_t}{L_t} = \frac{\eta C_t}{\gamma (1-\theta) G_t} \frac{l_t}{L_t} = \frac{\eta}{\gamma (1-\theta)} \left(\frac{C}{G}\right)^* \frac{l_t}{L_t}.$$

The resulting equilibrium employment ratio is

$$\frac{l_t}{L_t} = \left(\frac{l}{L}\right)^* \equiv \begin{cases} \left(1 + \frac{\eta}{\gamma}\right)^{-1} & 0 \leq x_t \leq x_N \\ \left[1 + \frac{\eta}{\gamma}(1+\Theta)\right]^{-1} & x_t > x_N \end{cases}. \tag{31}$$

Since it depends on the consumption-output ratio, the employment ratio jumps when the consumption ratio jumps.

The simplicity of this special case allows us to show immediately the implications of endogenous labor supply for the model's dynamics. To express the dynamics in terms of a pre-determined state variable that does not jump, we note that in this extension

$$\frac{X_t}{Z_t} = \left(\frac{\theta}{\mu}\right)^{1/(1-\theta)} \left(\frac{l_t}{L_t}\right)^{\gamma} \left(\frac{L_t}{N_t}\right)^{\gamma} R^{1-\gamma}.$$

We thus modify the definition of x_t to

$$x_t \equiv \left(\frac{L_t}{l_t}\right)^{\gamma} \frac{X_t}{Z_t} = \left(\frac{\theta}{\mu}\right)^{1/(1-\theta)} \left(\frac{L_t}{N_t}\right)^{\gamma} R^{1-\gamma}. \tag{32}$$

The interpretation of x_t is no longer firm size, but the mapping to that concept is transparent.

Employment growth is $\dot{l}_t/l_t = \lambda$ and the growth rate of output is

$$g_t = \begin{cases} \gamma\lambda & 0 \le x_t \le x_N \\ \gamma\lambda + (1-\gamma)\left\{\frac{1}{\beta}\left[\mu - 1 - \left[1 + \frac{\eta}{\gamma}(1+\Theta)\right]^{\gamma} \frac{\phi}{x_t}\right] - \rho\right\} & x_N < x_t \le x_Z \\ \alpha\left\{(\mu-1)\left[1+\frac{\eta}{\gamma}(1+\Theta)\right]^{-\gamma} x_t - \phi\right\} - \rho & x_t > x_Z \end{cases}. \tag{33}$$

In this extension as well, the state variable x_t grows from an initial value x_0 and gradually converges to the steady-state value x^* following an S-shaped path. The value $[(l/L)^*]^{\gamma}x^*$ is the same as x^* in (23) in the baseline case due to the model's scale-invariance property. The growth rate g_t is constant in the pre-industrial era and then gradually increases throughout the industrial era, converging to the same value g^* in (24) as in the baseline case due to scale-invariance. Moreover, we show in Appendix A that the dynamics of the economy are qualitatively the same as those discussed in Propositions 2-3.

We now derive the implications of this structure for the income distribution. Since the wealth share is constant, we rewrite the budget constraint (29) as $c_t(h) = (r_t - g_t)a_t(h) + w_t l_t(h)$ since $\dot{a}_t(h)/a_t(h) = \dot{A}_t/A_t = g_t$. Using this result and (30), we write labor income as

$$w_t l_t(h) = \frac{1}{1+\eta}\left[w_t L_t - \eta(r_t - g_t)a_t(h)\right]. \tag{34}$$

This result says that, since $r_t > g_t$, wealthier households supply less labor and earn lower labor income. Accounting for this labor income inequality, the household's income share obeys the relation

$$s_{y,t}(h) - 1 = \frac{(r_t + \eta g_t)A_t}{(r_t + \eta g_t)A_t + w_t L_t}\left[s_{a,t}(h) - 1\right]. \tag{35}$$

Our extended result therefore is that the evolution of the household income share is determined by the evolution of $r_t + \eta g_t$ rather than r_t. The additional term ηg_t captures the effect on labor income of standard labor supply behavior.

The Gini coefficient of income is

$$\sigma^G_{y,t} = \frac{(r_t + \eta g_t)A_t}{(r_t + \eta g_t)A_t + w_t L_t}\sigma^G_{a,t}. \tag{36}$$

In the pre-industrial era, we have

$$\frac{(r_t+\eta g_t)A_t}{w_tL_t}=\frac{\rho+(1+\eta)\gamma\lambda}{\rho}\left(\frac{1-\gamma}{\gamma}\right)\frac{l_t}{L_t}=\frac{\rho+(1+\eta)\gamma\lambda}{\rho}\left(\frac{1-\gamma}{\gamma+\eta}\right).$$

Substituting this expression in (36) yields an expression similar to (26), except for the addition of the parameter η. Finally, in the industrial era

$$\frac{(r_t+\eta g_t)A_t}{w_tL_t}=\frac{\rho+(1+\eta)g_t}{\rho}\left(\frac{1-\gamma+\Theta}{\gamma}\right)\frac{l_t}{L_t}=\frac{\rho+(1+\eta)g_t}{\rho}\left[\frac{1-\gamma+\Theta}{\gamma+\eta(1+\Theta)}\right].$$

Substituting this expression in (36) yields an expression similar to (26). In the industrial era, income inequality, $\sigma_{y,t}$, gradually increases until growth, g_t, converges to the same steady-state value g^* as in our baseline case.

We stress that in this structure the employment ratio l_t/L_t is the only channel through which the equilibrium of the labor market affects the economy's dynamics. This property is important to understand the transmission mechanism of the heterogeneity in labor income that we discuss next.

5.3 General case: $\omega \neq 1$

We now consider the general case $\omega \neq 1$. With the preferences in (28) that yield labor supply (30), we have two new properties: (i) the curvature of $l_t(h)/L_t$ with respect to the consumption per capita-to-wage ratio, $\left[c_t\left(h\right)/L_t\right]/w_t$, yields that household heterogeneity in labor supply, and therefore in labor income, matters for aggregate outcomes; (ii) household behavior allows for $l_t(h)/L_t=0$ for some h. To make the exposition as clear as possible, we focus first on property (i) and then discuss property (ii).

5.3.1 The role of labor income inequality

In this subsection we shut down property (ii), that is, we work with a parameters configuration such that $l_t(h)/L_t>0$ for all $h\in[0,1]$. Aggregating the labor supply (30) yields

$$\frac{l_t}{L_t}=\int_0^1\frac{l_t\left(h\right)}{L_t}dh=\int_0^1\left(1-\left[\frac{s_c^*(h)}{\frac{w_tL_t}{\eta C_t}}\right]^\omega\right)dh=1-\int_0^1\left[\frac{s_c^*(h)}{\frac{w_tL_t}{\eta C_t}}\right]^\omega dh=1-\left(\frac{\eta C_t}{w_tL_t}\Delta_c^*\right)^\omega,$$

where the operator

$$\Delta_c^*\equiv\left(\int_0^1\left[s_c^*(h)\right]^\omega dh\right)^{\frac{1}{\omega}}$$

accounts for the heterogeneity in labor supply behavior, which in this specification does not wash out. Proceeding as in the previous case, we obtain the labor market clearing condition

$$\frac{l_t}{L_t} = 1 - \left[\left(\frac{C}{G} \right)^* \frac{\eta \Delta_c^*}{\gamma (1-\theta)} \right]^\omega \left(\frac{l_t}{L_t} \right)^\omega.$$

This is an equation in the endogenous variable l_t/L_t and two objects, Δ_c^* and $(C/G)^*$, that are functions of the model's parameters. The left-hand side is increasing; the right hand side is decreasing. Therefore, we have the unique solution

$$\left(\frac{l}{L} \right)^* \equiv \arg\text{solve} \left\{ \frac{l_t}{L_t} = 1 - \left[\left(\frac{C}{G} \right)^* \frac{\eta \Delta_c^*}{\gamma (1-\theta)} \right]^\omega \left(\frac{l_t}{L_t} \right)^\omega \right\}, \tag{37}$$

with the important comparative statics property that

$$\frac{\partial \left(\frac{l}{L} \right)^*}{\partial \Delta_c^*} < 0.$$

Substituting this result in the household labor supply yields

$$\frac{l_t(h)}{L_t} = 1 - \left[\left(\frac{C}{G} \right)^* \frac{\eta s_c^*(h)}{\gamma (1-\theta)} \left(\frac{l}{L} \right)^* \right]^\omega > 0 \quad \text{with } s_c^*(h) < \frac{\gamma(1-\theta)}{\eta \left(\frac{C}{G} \right)^* \left(\frac{l}{L} \right)^*}.$$

As stated, we first rule out the corner solution $l_t(h)/L_t = 0$ so that all households work.

Since the employment ratio is always constant, in this case as well, and the case that we study in the next subsection, the differential equation governing the dynamics of the household wealth share yields that the wealth share is constant at all times. The formal proof is identical to that developed in the previous subsection.

The presence of the operator Δ_c^* in the solution (37) is our property (i), namely, equilibrium aggregate labor supply, and therefore equilibrium employment, depends on the heterogeneity across household in their individual labor supply. The question then is, what is Δ_c^*, the term accounting for such heterogeneity? The answer is that Δ_c^* is the power mean of the consumption shares, where, because of the unit continuum of households, the consumption share is also consumption relative to mean consumption. The parameter ω drives how the operator "penalizes" or "rewards" the dispersion of consumption relative to the mean. Specifically, for $\omega = 1$ we have $\Delta_c^* = 1$ regardless of the dispersion of consumption relative to the mean. For $\omega \neq 1$, instead, Δ_c^* deviates from unity unless $s_c^*(h) = 1$ for all $h \in [0, 1]$ (i.e., a completely equal society). In particular, an unequal society has $\Delta_c^* > 1$ for $\omega > 1$ and $\Delta_c^* < 1$ for $\omega < 1$. Moreover, Δ_c^* is increasing in consumption inequality for $\omega > 1$ and decreasing in it for $\omega < 1$. Finally, given that the employment ratio l_t/L_t is decreasing in the dispersion index Δ_c^*, an unequal society features lower employment than an equal one under $\omega > 1$ and higher employment under $\omega < 1$. Similarly, the employment ratio l_t/L_t is decreasing in consumption inequality under $\omega > 1$ and increasing under $\omega < 1$.

To understand why the value of ω determines how inequality affects employment, we plot (31) in Figure 9. This figure shows that when ω is greater (less) than 1, the decrease in labor supply by rich households, which have above average consumption, is greater (less) than the increase in labor supply by poor households, which have below average consumption; as a result, inequality that gives rise to rich and poor households reduces (raises) aggregate employment.

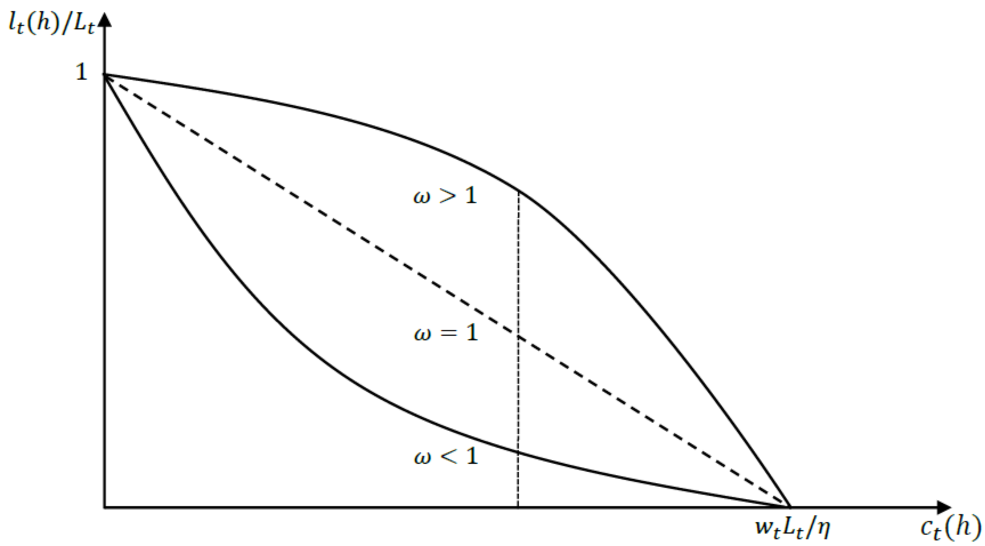

Figure 9: Equation (31)

A society with more unequal land ownership has a more unequal distribution of consumption and thus a more unequal distribution of labor supply. Such labor supply behavior yields lower employment under $\omega > 1$ and higher employment under $\omega < 1$. This intratemporal causal chain traces how wealth inequality propagates throughout the economy, affecting its scale of operation and thereby its growth path. It also stresses the importance of the parameter ω that regulates the responsiveness of labor supply to consumption. The intertemporal part of the causal chain is that in our scale-invariant model the employment ratio determines the threshold x_N for our state variable, x_t, and thus determines the overall shape of the transition path via its effect on the timing of key events, even though it does not affect steady-state growth. Specifically, the takeoff date is $T_N = \ln(x_N/x_0)/\lambda$, with $x_N = \phi/[(\mu - 1 - \beta\rho)(l/L)^\gamma]$ being decreasing in the employment ratio l/L. Consequently, a more unequal society takes off later under $\omega > 1$ and earlier under $\omega < 1$. These differences in the timing of takeoff never wash out, holding constant everything else. Thus, initial wealth inequality, which in our scheme is the root of all inequality, has effects that echo for centuries and are amplified by the growth acceleration that occurs with the takeoff.

5.3.2 Endogenous formation of the leisure class

Property (ii) occurs when the inequality

$$\left[\frac{\eta c_t(h)}{w_t L_t}\right]^{\omega} \geq 1 \Rightarrow \frac{c_t(h)}{C_t}\frac{\eta C_t}{w_t L_t} \geq 1 \Rightarrow s_c^*(h) \geq \frac{w_t L_t}{\eta C_t}$$

holds for some h. We thus write household labor supply as

$$\frac{l_t(h)}{L_t} = \begin{cases} 1 - \left[\frac{s_c^*(h)}{\frac{w_t L_t}{\eta C_t}}\right]^{\omega} & s_c^*(h) < \frac{w_t L_t}{\eta C_t} \\ 0 & s_c^*(h) \geq \frac{w_t L_t}{\eta C_t} \end{cases}.$$

Next, we note that if we sort households over the unit interval in ascending order of consumption share, the condition $s_c^*(h) \geq \frac{w_t L_t}{\eta C_t}$ defines the cutoff value

$$\bar{h}_t = \arg\text{solve}\left\{s_c^*(h) = \frac{w_t L_t}{\eta C_t}\right\}$$

such that households in the set $[0, \bar{h}_t)$ supply labor and households in the set $\left[\bar{h}_t, 1\right]$ do not. The model, therefore, generates endogenously a *leisure class*, i.e., households who do not work and live off asset income, which consists of industrial dividends, land rents and capital gains on the prices of industrial shares and land.

To construct the equilibrium, we need to check how the individual household decision depends on the aggregate state of the labor market. That is, we need to derive the equilibrium expression for the wage by aggregating across households and then check how the individual household responds to that wage, allowing for the possibility of the corner solution.

Aggregation gives us

$$\frac{l_t}{L_t} = \int_0^1 \frac{l_t(h)}{L_t} dh = \int_0^{\bar{h}_t} \left(1 - \left[\frac{s_c^*(h)}{\frac{w_t L_t}{\eta C_t}}\right]^{\omega}\right) dh + \int_{\bar{h}_t}^1 0 dh = \bar{h}_t \left[1 - \left(\frac{\eta C_t}{w_t L_t}\bar{\Delta}_c^*\right)^{\omega}\right],$$

where

$$\bar{\Delta}_c^*\left(\bar{h}_t\right) \equiv \left(\frac{1}{\bar{h}_t}\int_0^{\bar{h}_t} \left[s_c^*(h)\right]^{\omega} dh\right)^{\frac{1}{\omega}}.$$

We note that $\bar{\Delta}_c^*$ is increasing in $\bar{h}_t$ because the function $s_c^*(h)$ representing the sorted households is convex. Proceeding as in the previous case, we obtain the labor market clearing condition

$$\frac{l_t}{L_t} = \bar{h}_t \left\{ 1 - \left[\left(\frac{C}{G} \right)^* \frac{\eta \bar{\Delta}_c^* \left(\bar{h}_t \right)}{\gamma \left(1 - \theta \right)} \frac{l_t}{L_t} \right]^\omega \right\}.$$

Next, we define the new object

$$\bar{h}_t = \bar{h} \left(\frac{l_t}{L_t} \right) \equiv \arg\text{solve} \left\{ s_c^*(h) = \frac{\gamma \left(1 - \theta \right)}{\eta \left(\frac{C}{G} \right)^* \frac{l_t}{L_t}} \right\},$$

with $\bar{h}' \left(\cdot \right) < 0$, and write

$$\frac{l_t}{L_t} = \bar{h} \left(\frac{l_t}{L_t} \right) \left\{ 1 - \left[\frac{\eta \left(\frac{C}{G} \right)^*}{\gamma \left(1 - \theta \right)} \right]^\omega \left[\frac{l_t}{L_t} \bar{\Delta}_c^* \left(\bar{h}_t \left(\frac{l_t}{L_t} \right) \right) \right]^\omega \right\}.$$

Since $(C/G)^*$ is a function of the model's parameters, this is an implicit equation in the endogenous variable l_t/L_t. The left-hand side is increasing; the right hand side is decreasing if the term

$$\frac{l_t}{L_t} \bar{\Delta}_c^* \left(\bar{h}_t \left(\frac{l_t}{L_t} \right) \right)$$

is increasing because the direct effect via l_t/L_t dominates the indirect effect via $\bar{\Delta}_c^* \left(\bar{h}_t \left(l_t/L_t \right) \right)$. Therefore, we have the unique solution

$$\left(\frac{l}{L} \right)^* \equiv \arg\text{solve} \left\{ \frac{l_t}{L_t} = \bar{h} \left(\frac{l_t}{L_t} \right) \left[1 - \left[\frac{\eta \left(\frac{C}{G} \right)^*}{\gamma \left(1 - \theta \right)} \right]^\omega \left[\frac{l_t}{L_t} \bar{\Delta}_c^* \left(\bar{h}_t \left(\frac{l_t}{L_t} \right) \right) \right]^\omega \right] \right\}. \tag{38}$$

Substituting this result in the household labor supply yields

$$\frac{l_t \left(h \right)}{L_t} = \begin{cases} 1 - \left[\left(\frac{C}{G} \right)^* \frac{\eta s_c^*(h)}{\gamma(1-\theta)} \left(\frac{l}{L} \right)^* \right]^\omega & s_c^*(h) < \frac{\gamma(1-\theta)}{\eta \left(\frac{C}{G} \right)^* \left(\frac{l}{L} \right)^*} \\ 0 & s_c^*(h) \geq \frac{\gamma(1-\theta)}{\eta \left(\frac{C}{G} \right)^* \left(\frac{l}{L} \right)^*} \end{cases}.$$

This expression identifies which households want to go to the corner solution.

In this characterization, the equilibrium of the labor market is the standard intersection of labor demand and labor supply. What differs from the standard approach is that here labor supply is the joint solution of two equations. The first says that labor supply is the integral over the set of households that supply labor $\left[0, \bar{h} \right]$, the second determines the set of such households. Consequently, the model allows for heterogeneity in labor supply over two margins: the extensive margin, where the household determines whether to supply labor or not; the intensive margin, where the household determines the fraction of time spent working, conditional on having determined that the fraction is positive.

The operator $\bar{\Delta}_c^*$ plays the same role as the operator Δ_c^* discussed in the previous case. However, we must note that in the variable $s_{c,t}(h) = c_t(h)/C_t$, the C_t in the denominator is aggregate consumption. To remove the consumption of the leisure class, we define

$$C_t^l \equiv \int_0^{\bar{h}} c_t(h)\, dh$$

and for $h \in [0, \bar{h}]$ we write

$$s_{c,t}(h) = \frac{c_t(h)}{C_t^l/\bar{h}_t} \frac{C_t^l/\bar{h}_t}{C_t} \Rightarrow s_c^*(h) = \left(\frac{c(h)}{C^l/\bar{h}}\right)^* \left(\frac{C^l}{C}\right)^* \frac{1}{\bar{h}}$$

Then, we write

$$\bar{\Delta}_c^* = \left(\frac{1}{\bar{h}} \int_0^{\bar{h}} [s_c^*(h)]^\omega\, dh\right)^{\frac{1}{\omega}} = \frac{1}{\bar{h}} \left(\frac{C^l}{C}\right)^* \left(\frac{1}{\bar{h}} \int_0^{\bar{h}} \left[\left(\frac{c(h)}{C^l/\bar{h}}\right)^*\right]^\omega dh\right)^{\frac{1}{\omega}},$$

and note that the variable in the integral is household consumption relative to mean consumption for the set of households that supply labor. We thus have the same interpretation as before for the operator $\bar{\Delta}_c^*$, with the refinement that it is the power mean of the consumption relative to the mean of the households that supply labor, adjusted for the endogenous two-classes structure of society, the term $1/\bar{h}$ that pushes it up due to the extensive margin of heterogeneity, and for the consumption share of the households that supply labor, the term $\left(C^l/C\right)^*$.

6 Conclusion

This study explored the historical origins of income inequality from stagnation to growth in a tractable Schumpeterian model with endogenous takeoff and heterogeneous households. Our first result can be summarized as follows. In the pre-industrial era, the economy is in stagnation and income inequality is determined solely by the unequal distribution of land ownership. In the industrial era, the gradually rising growth rate causes income inequality to increase over time until the economy reaches the balanced growth path. We calibrate the model to perform a quantitative analysis and find that the simulation results are roughly in line with historical data for the UK.

The result above obtains in a baseline model with inelastic labor supply that has three main strengths: (i) it is analytically tractable; (ii) it identifies sharply the role of growth accelerations as a main driver of rising income inequality; (iii) it measures inequality with a well understood and widely used summary statistic of the shape of a non-degenerate distribution of income. The baseline model, however, says that inequality plays no role in shaping the transition from stagnation to growth. To address this weakness, we extended the analysis allowing for labor income inequality due to endogenous labor supply. The advantage of doing this is twofold. First, the aggregate dynamics of our economy remain eminently tractable and consist of the two-phase secular transition documented for the simple baseline model with inelastic labor supply. Second, allowing for endogenous labor supply extends considerably the scope of our analysis: inequality affects the employment ratio and thus, through that standard role that scale plays in Schumpeterian models, contributes to shaping the transition path of the aggregate economy. Specifically, inequality that results in higher employment produces an earlier takeoff and a higher growth rate during the transition path. However, as the equilibrium growth rate converges to the steady state, this effect disappears due to the scale-invariance of our Schumpeterian growth model.

In this scheme, labor income inequality consists of two margins: an extensive margin along which households sort themselves into a leisure class that supplies zero labor and the rest of society that supplies labor; an intensive margin along which households supply labor as a decreasing function of their consumption share, which in turn is an increasing function of their wealth share. The leisure class consists of households that are wealthy enough to find optimal to forgo labor income. Because of this property our model generates endogenously the two-class structure—workers vs. capitalists—that is widely used in the literature on inequality that builds on the classical theory of the distribution of income. The practice there is to postulate the two classes with fixed size and with homogeneity within each class. The resulting models are then silent about the shape of the cross-sectional distribution of income since the imposed within-class homogeneity reduces the cross section of the whole population to two degenerate distributions. Consequently, work that uses this approach tends to measure inequality with grand ratios like the wealth share of GDP (see, e.g., Madsen *et al.* 2021, which we discussed in detail). Our structure, in contrast, allows for heterogeneity in wealth, labor supply, and thus overall income, within each class.

References

[1] Aghion, P., Akcigit, U., Bergeaud, A., Blundell, R., and Hemous, D., 2019. Innovation and top income inequality. *Review of Economic Studies*, 86, 1-45.

[2] Aghion, P., and Bolton, P., 1997. A theory of trickle-down growth and development. *Review of Economic Studies*, 64, 151-172.

[3] Aghion, P., and Howitt, P., 1992. A model of growth through creative destruction. *Econometrica*, 60, 323-351.

[4] Ang, J., and Madsen, J., 2011. Can second-generation endogenous growth models explain the productivity trends and knowledge production in the Asian miracle economies?. *Review of Economics and Statistics*, 93, 1360-1373.

[5] Ashraf, Q., and Galor, O., 2011. Dynamics and stagnation in the Malthusian epoch. *American Economic Review*, 101, 2003-2041.

[6] Ashton, T. S., 1998. *The Industrial Revolution, 1760-1830*. Oxford University Press.

[7] Bollard, A., Klenow, P., and Li, H., 2016. Entry costs rise with development. Discussion paper.

[8] Bouscasse, P., Nakamura, E., and Steinsson, J., 2021. When did growth begin? New estimates of productivity growth in England from 1250 to 1870. NBER Working Papers 28623.

[9] Chou, C.-F., and Talmain, G., 1996. Redistribution and growth: Pareto improvements. *Journal of Economic Growth*, 1, 505-523.

[10] Chu, A., 2010. Effects of patent policy on income and consumption inequality in an R&D-based growth model. *Southern Economic Journal*, 77, 336-350.

[11] Chu, A., and Cozzi, G., 2018. Effects of patents versus R&D subsidies on income inequality. *Review of Economic Dynamics*, 29, 68-84.

[12] Chu, A., Cozzi, G., Fan, H., Furukawa, Y., and Liao, C., 2019. Innovation and inequality in a monetary Schumpeterian model with heterogeneous households and firms. *Review of Economic Dynamics*, 34, 141-164.

[13] Chu, A., Furukawa, Y., Mallick, S., Peretto, P., and Wang, X., 2021. Dynamic effects of patent policy on innovation and inequality in a Schumpeterian economy. *Economic Theory*, 71, 1429-1465.

[14] Clark, G., 2008. *A Farewell to Alms: A Brief Economic History of the World*. Princeton University Press.

[15] Dinopoulos, E., and Thompson, P., 1998. Schumpeterian growth without scale effects. *Journal of Economic Growth*, 3, 313-335.

[16] Foellmi, R., and Zweimuller, J., 2006. Income distribution and demand-induced innovations. *Review of Economic Studies*, 73, 941-960.

[17] Galor, O., 2011. *Unified Growth Theory*. Princeton University Press.

[18] Galor, O., 2005. From stagnation to growth: Unified growth theory. *Handbook of Economic Growth*, 1, 171-293.

[19] Galor, O., and Moav, O., 2002. Natural selection and the origin of economic growth. *Quarterly Journal of Economics*, 117, 1133-1192.

[20] Galor, O., and Moav, O., 2004. From physical to human capital accumulation: inequality and the process of development. *Review of Economic Studies*, 71, 1001-1026.

[21] Galor, O., Moav, O., and Vollrath, D., 2009. Inequality in landownership, the emergence of human-capital promoting institutions, and the great divergence. *Review of Economic Studies*, 76, 143-179.

[22] Galor, O., and Mountford, A., 2008. Trading population for productivity: Theory and evidence. *Review of Economic Studies*, 75, 1143-1179.

[23] Galor, O., and Weil, D., 2000. Population, technology and growth: From the Malthusian regime to the demographic transition. *American Economic Review*, 110, 806-828.

[24] Galor, O., and Zeira, J., 1993. Income distribution and macroeconomics. *Review of Economic Studies*, 60, 35-52.

[25] Garcia-Macia, D., Hsieh, C., and Klenow, P., 2019. How destructive is innovation?. *Econometrica*, 87, 1507-1541.

[26] Garcia-Penalosa, C., and Wen, J.-F., 2008. Redistribution and entrepreneurship with Schumpeterian growth. *Journal of Economic Growth*, 13, 57-80.

[27] Grossman, G., and Helpman, E., 1991. Quality ladders in the theory of growth. *Review of Economic Studies*, 58, 43-61.

[28] Ha, J., and Howitt, P., 2007. Accounting for trends in productivity and R&D: A Schumpeterian critique of semi-endogenous growth theory. *Journal of Money, Credit, and Banking*, 33, 733-774.

[29] Iacopetta, M., Minetti, R., and Peretto, P., 2019. Financial markets, industry dynamics and growth. *Economic Journal*, 129, 2192-2215.

[30] Jones, C., and Kim, J., 2018. A Schumpeterian model of top income inequality. *Journal of Political Economy*, 126, 1785-1826.

[31] Kuznets, S., 1955. Economic growth and income inequality. *American Economic Review*, 45, 1-28.

[32] Laincz, C., and Peretto, P., 2006. Scale effects in endogenous growth theory: An error of aggregation not specification. *Journal of Economic Growth*, 11, 263-288.

[33] Lindert, P., 2000a. Three centuries of inequality in Britain and America. *Handbook of Income Distribution*, 1, 167-216.

[34] Lindert, P., 2000b. When did inequality rise in Britain and America?. *Journal of Income Distribution*, 9, 11-25.

[35] Madsen, J., 2008. Semi-endogenous versus Schumpeterian growth models: Testing the knowledge production function using international data. *Journal of Economic Growth*, 13, 1-26.

[36] Madsen, J., 2010. The anatomy of growth in the OECD since 1870. *Journal of Monetary Economics*, 57, 753-767.

[37] Madsen, J., 2017. Is inequality increasing in $r - g$? Piketty's principle of capitalist economics and the dynamics of inequality in Britain, 1210-2013. CAMA Working Paper No. 63/2017.

[38] Madsen, J., 2019. Wealth and inequality over eight centuries of British capitalism. *Journal of Development Economics*, 138, 246-260.

[39] Madsen, J., Minniti, A., and Venturini, F., 2021. Wealth inequality in the long run: A Schumpeterian growth perspective. *Economic Journal*, 131, 476-497.

[40] Madsen, J., and Strulik, H., 2020. Technological change and inequality in the very long run. *European Economic Review*, 129, 103532.

[41] Mokyr, J., 2016. *A Culture of Growth: The Origins of the Modern Economy*. Princeton University Press.

[42] Peretto, P., 1994. *Essays on Market Structure and Economic Growth*. Ph.D. dissertation, Yale University.

[43] Peretto, P., 1998. Technological change and population growth. *Journal of Economic Growth*, 3, 283-311.

[44] Peretto, P., 1999. Cost reduction, entry, and the interdependence of market structure and economic growth. *Journal of Monetary Economics*, 43, 173-195.

[45] Peretto, P., 2015. From Smith to Schumpeter: A theory of take-off and convergence to sustained growth. *European Economic Review*, 78, 1-26.

[46] Peretto, P., and Connolly, M., 2007. The Manhattan metaphor. *Journal Economic Growth*, 12, 329-350.

[47] Piketty, T., 2014. *Capital in the Twenty-First Century*. Harvard University Press.

[48] Romer, P., 1990. Endogenous technological change. *Journal of Political Economy*, 98, S71-S102.

[49] Segerstrom, P., Anant, T., and Dinopoulos, E., 1990. A Schumpeterian model of the product life cycle. *American Economic Review*, 80, 1077-91.

[50] Smulders, S., 1994. *Growth, Market Structure and the Environment: Essays on the Theory of Endogenous Economic Growth*. Ph.D. dissertation, Tilburg University.

[51] Smulders, S. and van de Klundert T., 1995. Imperfect competition, concentration and growth with firm-specific R&D. *European Economic Review*, 39, 139-160.

[52] Sullivan, R. J., 1989. England's "Age of invention": The acceleration of patents and patentable invention during the industrial revolution. *Explorations in Economic History*, 26, 424-452.

[53] van Zanden, J. L., 1995. Tracing the beginning of the Kuznets curve: Western Europe during the early modern period. *Economic History Review*, 48, 643-664.

[54] Williamson, J., 1980. Earnings inequality in nineteenth-century Britain. *Journal of Economic History*, 40, 457-475.

[55] Williamson, J., 1985. *Did British Capitalism Breed Inequality*?. Allen & Unwin.

[56] Zweimuller, J., 2000. Schumpeterian entrepreneurs meet Engel's law: The impact of inequality on innovation-driven growth. *Journal of Economic Growth*, 5, 185-206.

Appendix A (online publication only)

Rate of return to quality innovation. The current-value Hamiltonian of firm i is

$$H_t(i) = \Pi_t(i) - I_t(i) + \eta_t(i)\dot{Z}_t(i) + \xi_t(i)[\mu - p_t(i)],$$

where $\eta_t(i)$ is the co-state variable on (11) and $\xi_t(i)$ is the multiplier on $p_t(i) \leq \mu$. Substituting (7), (11) and (12) into $H_t(i)$, we have:

$$\frac{\partial H_t(i)}{\partial p_t(i)} = 0 \Rightarrow \frac{\partial \Pi_t(i)}{\partial p_t(i)} = \xi_t(i); \tag{39}$$

$$\frac{\partial H_t(i)}{\partial I_t(i)} = 0 \Rightarrow \eta_t(i) = 1; \tag{40}$$

$$\frac{\partial H_t(i)}{\partial Z_t(i)} = \alpha \left\{ [p_t(i) - 1] \left[\frac{\theta}{p_t(i)}\right]^{1/(1-\theta)} L_t^{\gamma}(i) R^{1-\gamma} - \phi \right\} \frac{Z_t^{1-\alpha}}{Z_t^{1-\alpha}(i)} = r_t \eta_t(i) - \dot{\eta}_t(i). \tag{41}$$

If $p_t(i) < \mu$, then $\xi_t(i) = 0$; in this case, $\partial \Pi_t(i) / \partial p_t(i) = 0$ yields $p_t(i) = 1/\theta$. If the constraint on $p_t(i)$ is binding, then $\xi_t(i) > 0$; in this case, $p_t(i) = \mu$. Given $\mu < 1/\theta$, we have $p_t(i) = \mu$. We use (40), (15) and $p_t(i) = \mu$ in (41) and impose symmetry for (16). ■

Proof of Proposition 1. Aggregation of the budget constraints of the heterogeneous households yields

$$\dot{A}_t = r_t A_t + w_t L_t - C_t. \tag{42}$$

As a result of the market structure described above, wealth in the pre-industrial era consists only of land, i.e., $A_t = R v_t$, and (42) reduces to

$$R\dot{v}_t = (q_t + \dot{v}_t) R + w_t L_t - C_t \Rightarrow C_t = q_t R + w_t L_t,$$

which says that in this era consumption equals income, the sum of land income and labor income. Using the factor payments (9)-(10), the expression yields

$$\frac{C_t}{G_t} = \left(\frac{C}{G}\right)^* = 1 - \theta.$$

Using the Euler equation (3), the factor payment (9) and this result we write (42) as

$$\frac{\dot{C}_t}{C_t} - \frac{\dot{A}_t}{A_t} = \frac{C_t}{A_t} - \rho - \frac{\gamma(1-\theta) G_t}{A_t} = \frac{C_t}{A_t}\left[1 - \gamma(1-\theta)\left(\frac{G}{C}\right)^*\right] - \rho.$$

This unstable differential equation says that to satisfy the households' transversality condition the consumption-wealth ratio, C_t/A_t, jumps to the steady-state value

$$\frac{C_t}{A_t} = \left(\frac{C}{A}\right)^* = \frac{\rho}{1-\gamma\left(1-\theta\right)\left(\frac{G}{C}\right)^*} = \frac{\rho}{1-\gamma}.$$

In contrast, in the industrial era the free-entry condition (18) holds, the value of monopolistic firms is $N_t V_t = \beta\theta G_t/\mu$ and wealth is $A_t = Rv_t + \beta\theta G_t/\mu$. We then write (42) as

$$R\dot{v}_t + \frac{\beta\theta}{\mu}\dot{G}_t = \left(q_t + \dot{v}_t\right)R + r_t\frac{\beta\theta}{\mu}G_t + w_t L_t - C_t.$$

Using the factor payments (9)-(10) and the saving rule (3), we reduce this expression to

$$\frac{\dot{C}_t}{C_t} - \frac{\dot{G}_t}{G_t} = \left(\frac{C_t}{G_t} - 1 + \theta\right)\frac{\mu}{\beta\theta} - \rho.$$

This unstable differential equation says that to satisfy the households' transversality condition, the consumption-output ratio, C_t/G_t, jumps to the steady-state value

$$\frac{C_t}{G_t} = \left(\frac{C}{G}\right)^* = 1 - \theta + \rho\beta\theta/\mu = \left(1-\theta\right)\left(1+\Theta\right).$$

Proceeding as in the previous case, we obtain that the consumption-wealth ratio, C_t/A_t, jumps to the steady-state value

$$\frac{C_t}{A_t} = \left(\frac{C}{A}\right)^* = \frac{\rho}{1-\gamma\left(1-\theta\right)\left(\frac{G}{C}\right)^*} = \frac{\rho\left(1-\theta+\rho\beta\theta/\mu\right)}{\left(1-\gamma\right)\left(1-\theta\right)+\rho\beta\theta/\mu} = \frac{\rho\left(1+\Theta\right)}{1-\gamma+\Theta}.$$

■

Proof of Proposition 2. Given $x_t = \left(\theta/\mu\right)^{1/(1-\theta)}\left(L_t/N_t\right)^{\gamma} R^{1-\gamma}$, the growth rate of x_t is given by

$$\frac{\dot{x}_t}{x_t} = \gamma(\lambda - n_t). \tag{43}$$

In the pre-industrial era, the variety growth rate n_t is zero; in this case, the dynamics of x_t is simply given by $\dot{x}_t = \gamma\lambda x_t$.

In the first industrial era ($x_t > x_N$), the growth rate of variety is given by

$$n_t = \frac{1}{\beta}\left(\mu - 1 - \frac{\phi + z_t}{x_t}\right) - \rho, \tag{44}$$

which is obtained by substituting (21), (43) and $r_t = \rho + g_t$ into (19). In this era, the quality growth rate z_t is zero, so the variety growth rate n_t is positive if and only if $x_t >$

$\phi/(\mu - 1 - \beta\rho) \equiv x_N$, where x_N is a threshold for the firm size x_t above which variety innovation starts to occur at time $T_N = \ln(x_N/x_0)/\lambda$. Equation (44) shows that when $x_t > x_N$, variety innovation occurs (i.e., $n_t > 0$); in this case, we substitute (44) into (43) to derive the dynamics of x_t in the first industrial era.

In the second industrial era ($x_t > x_Z > x_N$), quality innovation also occurs (i.e., $z_t > 0$). Substituting (16) and (21) into $r_t = \rho + g_t$ yields

$$g_t = (1-\gamma)n_t + z_t + \gamma\lambda = \alpha\left[(\mu-1)\,x_t - \phi\right] - \rho. \tag{45}$$

Then, we combine (44) and (45) to solve for $n(x_t)$:

$$n_t = \frac{\left[(1-\alpha)(\mu-1) - \rho\beta\right] x_t - (1-\alpha)\phi + \rho + \gamma\lambda}{\beta x_t - (1-\gamma)}. \tag{46}$$

Substituting (46) into (43) yields

$$\dot{x}_t = \frac{\gamma}{\beta - (1-\gamma)/x_t}\left(d_1 - d_2 x_t\right), \tag{47}$$

where we define

$$\begin{aligned} d_1 &\equiv (1-\alpha)\phi - \lambda - \rho, \\ d_2 &\equiv \beta\left[\frac{(1-\alpha)(\mu-1)}{\beta} - \lambda - \rho\right]. \end{aligned}$$

We approximate $(1-\gamma)/x_t \approx 0$ in (47). The resulting linearized dynamics of x_t has a unique steady state that is stable if $d_1 > 0$ and $d_2 > 0$ from which we obtain $\rho + \lambda < \min\{(1-\alpha)\phi, (1-\alpha)(\mu-1)/\beta\}$. Then, $\dot{x}_t = 0$ yields $x^* = d_1/d_2$ in (24). We impose parameter restrictions to ensure $x^* > x_Z$, where

$$x_Z \equiv \underset{x}{\arg\text{solve}}\left\{\left[(\mu-1)\,x - \phi\right]\left(\alpha - \frac{1-\gamma}{\beta x}\right) = \gamma(\rho+\lambda)\right\},$$

which is obtained by combining (44) and (45) to solve for $z(x_t)$ and then setting $z_t = 0$. ■

Proof of Proposition 3. We know that the household consumption share is constant but possibly different in the two eras. We denote its era-specific values $s^*_{c,0}(h)$ and $s^*_{c,T_N}(h)$. We split wealth in its two components, land and industrial shares, and write $A_t = v_t R + V_t N_t$. Accordingly, the household's wealth share is

$$s_{a,t}(h) = s_{R,t}(h)\frac{v_t R}{A_t} + s_{N,t}(h)\frac{V_t N_t}{A_t},$$

where the composition of wealth in the industrial era is

$$\frac{v_t R}{A_t} = \frac{q_t R}{\rho A_t} = \frac{\left(\frac{qR}{G}\right)^*}{\rho\left(\frac{A}{G}\right)^*} = \frac{1-\gamma}{1-\gamma+\Theta} \equiv \Omega \quad \text{and} \quad \frac{V_t N_t}{A_t} = 1-\Omega.$$

Recall that the grand ratios are all determined at the beginning of the respective periods. Specifically, we have

$$\left(\frac{C}{A}\right)_0^* = \frac{\rho}{1-\gamma} \quad \text{and} \quad \left(\frac{C}{A}\right)_{T_N}^* = \frac{\rho(1+\Theta)}{1-\gamma+\Theta}.$$

We write the differential equations driving the household's wealth share in the pre-industrial era and the industrial era, respectively:

$$\dot{s}_{a,t}(h) = \left[1 - s_{c,0}^*(h)\right]\left(\frac{C}{A}\right)_0^* - \rho + \rho s_{a,t}(h); \tag{48}$$

$$\dot{s}_{a,t}(h) = \left[1 - s_{c,T_N}^*(h)\right]\left(\frac{C}{A}\right)_{T_N}^* - \rho + \rho s_{a,t}(h); \tag{49}$$

We solve these equations for:

$$s_{a,t}(h) = e^{\rho t} s_{a,0}(h) + \frac{e^{\rho t}-1}{\rho}\left[\left[1 - s_{c,0}^*(h)\right]\left(\frac{C}{A}\right)_0^* - \rho\right]; \tag{50}$$

$$s_{a,t-T_N}(h) = e^{\rho(t-T_N)} s_{a,T_N}(h) + \frac{e^{\rho(t-T_N)}-1}{\rho}\left[\left[1 - s_{c,T_N}^*(h)\right]\left(\frac{C}{A}\right)_{T_N}^* - \rho\right]. \tag{51}$$

The first solution holds over the period $t \in [0, T_N]$, the second over the period $t \in [T_N, \infty)$.

Next, we let $\nu_{a,t}(h)$ denote the shadow value of $a_t(h)$. The TVC associated to (51) is

$$\lim_{t\to\infty} e^{-\rho(t-T_N)} \nu_{a,t}(h)\, a_t(h) = \lim_{t\to\infty} e^{-\rho(t-T_N)} \nu_{a,t}(h)\, s_{a,t}(h)\, A_t = 0.$$

Using the model's dynamics of wealth, we obtain

$$\begin{aligned} \lim_{t\to\infty} e^{-\rho(t-T_N)} \nu_{a,T_N}(h)\, e^{-(\bar{r}-\rho)(t-T_N)} s_{a,t}(h)\, A_{T_N} e^{\bar{g}(t-T_N)} &= 0 \\ \nu_{a,T_N}(h)\, A_{T_N} \cdot \lim_{t\to\infty} e^{-\rho(t-T_N)} e^{-(\bar{r}-\rho)(t-T_N)} e^{\bar{g}(t-T_N)} s_{a,t}(h) &= 0. \end{aligned}$$

Noting that

$$e^{-(\bar{r}-\rho)(t-T_N)} e^{\bar{g}(t-T_N)} = 1,$$

we obtain

$$\begin{aligned} s_{a,T_N}(h) + \lim_{t\to\infty} \frac{1-e^{-\rho(t-T_N)}}{\rho} \left[\left[1 - s^*_{c,T_N}(h)\right] \left(\frac{C}{A}\right)^*_{T_N} - \rho \right] &= 0 \\ s_{a,T_N}(h) + \frac{1}{\rho} \left[\left[1 - s^*_{c,T_N}(h)\right] \left(\frac{C}{A}\right)^*_{T_N} - \rho \right] &= 0. \end{aligned}$$

The TVC, therefore, requires

$$s^*_{c,T_N}(h) = 1 + \frac{\rho}{\left(\frac{C}{A}\right)^*_{T_N}} \left[s_{a,T_N}(h) - 1\right], \tag{52}$$

where

$$s_{a,T_N}(h) = s_{R,T_N}(h) \frac{v_{T_N} R}{A_{T_N}} + s_{N,T_N}(h) \frac{V_{T_N} N_{T_N}}{A_{T_N}} = \Omega s_{R,T_N}(h) + (1-\Omega) s_{N,T_N}(h).$$

This relation describes the combinations of consumption share and wealth share that satisfy the TVC in the industrial era. Note that $s_{a,T_N}(h)$ is *endogenous* because $s_{R,T_N}(h)$ is endogenous. Formally, recalling that land is the only available form of wealth in the pre-industrial era, we use (50) to obtain

$$s_{R,T_N}(h) = 1 + e^{\rho T_N} \left[s_{R,0}(h) - 1\right] - \frac{e^{\rho T_N} - 1}{\rho} \left(\frac{C}{A}\right)^*_0 \left[s^*_{c,0}(h) - 1\right]. \tag{53}$$

The household knows that at $t = T_N$ it starts the industrial era with wealth share

$$\begin{aligned} s_{a,T_N}(h) &= s_{R,T_N}(h)\,\Omega + s_{N,T_N}(h)(1-\Omega) \\ &= \Omega + e^{\rho T_N} \Omega \left[s_{R,0}(h) - 1\right] - \frac{e^{\rho T_N} - 1}{\rho} \Omega \left(\frac{C}{A}\right)^*_0 \left[s^*_{c,0}(h) - 1\right] + s_{N,T_N}(h)(1-\Omega) \\ &= 1 + e^{\rho T_N} \Omega \left[s_{R,0}(h) - 1\right] - \frac{e^{\rho T_N} - 1}{\rho} \Omega \left(\frac{C}{A}\right)^*_0 \left[s^*_{c,0}(h) - 1\right] + (1-\Omega)\left[s_{N,T_N}(h) - 1\right], \end{aligned}$$

and must satisfy the TVC (52). Hence, we obtain the relation

$$s^*_{c,T_N}(h) - 1 = \frac{\rho \left[e^{\rho T_N} \Omega \left[s_{R,0}(h) - 1\right] - \frac{e^{\rho T_N} - 1}{\rho} \Omega \left(\frac{C}{A}\right)^*_0 \left[s^*_{c,0}(h) - 1\right] + (1-\Omega)\left[s_{N,T_N}(h) - 1\right] \right]}{\left(\frac{C}{A}\right)^*_{T_N}}. \tag{54}$$

This is a relation between the two era-specific, constant consumption shares. To obtain a solution we need another such relation.

To construct our solution, we compute the aggregate consumption jump at $t = T_N$ and recall that output is continuous, obtaining

$$C_{T_N^+} - C_{T_N^-} = (1-\theta)\left[(1+\Theta)\, G_{T_N^+} - G_{T_N^-}\right] = (1-\theta)\, G_{T_N}\Theta.$$

We follow a guess and verify approach. Reasoning by analogy we postulate for the individual household

$$c_{T_N^+}(h) - c_{T_N^-}(h) = (1-\theta)\left[(1+\Theta k(h))\, G_{T_N^+} - G_{T_N^-}\right] = (1-\theta)\, G_{T_N}\Theta k(h),$$

where

$$\int_0^1 k(h)\, dh = 1.$$

Our guess thus says that the individual household consumption jumps in proportion to the aggregate consumption jump. Without this property, our solution would fail the aggregation test. We now translate this expression into a jump of the consumption share

$$\begin{aligned}
\frac{c_{T_N^+}(h)}{C_{T_N^+}} C_{T_N^+} - \frac{c_{T_N^-}(h)}{C_{T_N^-}} C_{T_N^-} &= (1-\theta)\, G_{T_N}\Theta k(h) \\
\frac{c_{T_N^+}(h)}{C_{T_N^+}} (1-\theta)(1+\Theta)\, G_{T_N^+} - \frac{c_{T_N^-}(h)}{C_{T_N^-}} (1-\theta)\, G_{T_N^-} &= (1-\theta)\, G_{T_N}\Theta k(h) \\
\frac{c_{T_N^+}(h)}{C_{T_N^+}} (1+\Theta) - \frac{c_{T_N^-}(h)}{C_{T_N^-}} &= \Theta k(h),
\end{aligned}$$

which gives us

$$s^*_{c,T_N}(h)(1+\Theta) - s^*_{c,0}(h) = \Theta k(h).$$

We manipulate this expression to write

$$s^*_{c,T_N}(h) - 1 = \frac{\Theta\left[k(h) - 1\right] + \left[s^*_{c,0}(h) - 1\right]}{1+\Theta}.$$

We then substitute this result in the TVC (54) (repeated here for convenience),

$$s^*_{c,T_N}(h) - 1 = \frac{e^{\rho T_N}(1-\gamma)\left[s_{R,0}(h) - 1\right] + \Theta\left[s_{N,T_N}(h) - 1\right] - \left(e^{\rho T_N} - 1\right)\left[s^*_{c,0}(h) - 1\right]}{1+\Theta},$$

to obtain:

$$s^*_{c,0}(h) - 1 = (1-\gamma)\left[s_{R,0}(h) - 1\right] + e^{-\rho T_N}\Theta\left[s_{N,T_N}(h) - k(h)\right];$$

$$s^*_{c,T_N}(h) - 1 = \frac{\Theta\left[k(h) - 1\right] + (1-\gamma)\left[s_{R,0}(h) - 1\right] + e^{-\rho T_N}\Theta\left[s_{N,T_N}(h) - k(h)\right]}{1+\Theta}.$$

Substituting this solution in the expression for the wealth share dynamics, we obtain

$$\begin{aligned} s_{R,t}(h) - 1 &= e^{\rho t}\left[s_{R,0}(h) - 1\right] - \left(e^{\rho t} - 1\right)\left[s^*_{c,0}(h) - 1\right]\frac{1}{1-\gamma} \\ &= s_{R,0}(h) - 1 - \frac{e^{\rho(t-T_N)} - e^{-\rho T_N}}{1-\gamma}\Theta\left[s_{N,T_N}(h) - k(h)\right]. \end{aligned}$$

Then

$$s_{R,T_N}(h) - 1 = s_{R,0}(h) - 1 - \frac{1 - e^{-\rho T_N}}{1-\gamma}\Theta\left[s_{N,T_N}(h) - k(h)\right].$$

The next question is: what is $k(h)$ and what determines it?

On reflection, the factor by which we scale household consumption ought to reflect the household's wealth share at the time of the jump. Accordingly, we use

$$k(h) = s_{R,T_N}(h)\,\Omega + s_{N,T_N}(h)(1-\Omega).$$

The idea is that the change in household consumption is due to the arrival of the new industrial wealth *and* to the endogenous accumulation/decumulation of land wealth, which is the household's optimal response to the arrival of the new industrial wealth .

With this specification we obtain:

$$s^*_{c,0}(h) - 1 = (1-\gamma)\left[s_{R,0}(h) - 1\right] + e^{-\rho T_N}\Theta\Omega\left[s_{N,T_N}(h) - s_{R,T_N}(h)\right];$$

$$s^*_{c,T_N}(h) - 1 = \frac{\Theta\left[s_{N,T_N}(h) - 1\right] + (1-\gamma)\left[s_{R,0}(h) - 1\right] - \left(1 - e^{-\rho T_N}\right)\Theta\Omega\left[s_{N,T_N}(h) - s_{R,T_N}(h)\right]}{1+\Theta}.$$

Substituting this solution in the expression for the wealth share dynamics, we obtain

$$\begin{aligned} s_{R,t}(h) - 1 &= e^{\rho t}\left[s_{R,0}(h) - 1\right] - \left(e^{\rho t} - 1\right)\left[s^*_{c,0}(h) - 1\right]\frac{1}{1-\gamma} \\ &= s_{R,0}(h) - 1 - \frac{e^{\rho(t-T_N)} - e^{-\rho T_N}}{1-\gamma}\Theta\Omega\left[s_{N,T_N}(h) - s_{R,T_N}(h)\right]. \end{aligned}$$

This equation contains the endogenous term $s_{R,T_N}(h) - 1$. To solve for it we evaluate the equation at $t = T_N$, obtaining

$$s_{R,T_N}(h) - 1 = \frac{(1-\gamma)\left[s_{R,0}(h) - 1\right] - \left(1 - e^{-\rho T_N}\right)\Theta\Omega\left[s_{N,T_N}(h) - 1\right]}{1 - \gamma - \left(1 - e^{-\rho T_N}\right)\Theta\Omega}.$$

Substituting this expression in the land share dynamics equation, we get

$$s_{R,t}(h) - 1 = s_{R,0}(h) - 1 + \frac{\left(e^{\rho(t-T_N)} - e^{-\rho T_N}\right)\Theta\Omega}{1-\gamma-\left(1-e^{-\rho T_N}\right)\Theta\Omega}\left[s_{R,0}(h) - s_{N,T_N}(h)\right].$$

To test that this solution makes sense we look at two limiting cases. The first is $T_N \to 0 \Rightarrow t - T_N \to 0$ (immediate takeoff) and gives

$$s_{R,t}(h) - 1 = s_{R,0}(h) - 1 + \frac{1-1}{1-\gamma}\left[s_{R,0}(h) - s_{N,T_N}(h)\right] = s_{R,0}(h) - 1.$$

The second is $T_N \to \infty$ (no takeoff) and gives

$$s_{R,T_N}(h) - 1 = s_{R,0}(h) - 1 + \frac{0-0}{1-\gamma-\Theta\Omega}\left[s_{R,0}(h) - s_{N,T_N}(h)\right] = s_{R,0}(h) - 1.$$

Finally, we calculate:

$$s^*_{c,0}(h) - 1 = \frac{1-\gamma}{1-\gamma-\left(1-e^{-\rho T_N}\right)\Theta\Omega}\left[(1-\gamma-\Theta\Omega)\left[s_{R,0}(h)-1\right] + e^{-\rho T_N}\Theta\Omega\left[s_{N,T_N}(h)-1\right]\right];$$

$$s^*_{c,T_N}(h) - 1 = \frac{(1-\gamma)^2\left[s_{R,0}(h)-1\right] + \left[(1-\gamma)\Theta - (1-\gamma+\Theta)\Theta\Omega\left(1-e^{-\rho T_N}\right)\right]\left[s_{N,T_N}(h)-1\right]}{(1+\Theta)\left[1-\gamma-\left(1-e^{-\rho T_N}\right)\Theta\Omega\right]}.$$

After the takeoff the household wealth share is

$$\begin{aligned} s_{R,T_N}(h)\Omega + s_{N,T_N}(h)(1-\Omega) - 1 \quad = \quad & \frac{(1-\gamma)\Omega}{1-\gamma-\left(1-e^{-\rho T_N}\right)\Theta\Omega}\left[s_{R,0}(h)-1\right] \\ & + \frac{(1-\gamma)(1-\Omega) - \left(1-e^{-\rho T_N}\right)\Theta\Omega}{1-\gamma-\left(1-e^{-\rho T_N}\right)\Theta\Omega}\left[s_{N,T_N}(h)-1\right]. \end{aligned}$$

Recall that this expression accounts for the arrival of the newly created industrial wealth and thus exhibits a discontinuity with respect to the expression $s_{R,T_N}(h)$ derived above. Filling in the relation $\Omega = \frac{1-\gamma}{1-\gamma+\Theta}$ and rearranging terms yields the expressions shown in the main text.

We note that our solution for the pre-industrial era is consistent with: (1) aggregation constraint due to the era-specific aggregate consumption/wealth ratio; (2) aggregation constraint that all of the expressions for household shares give 1 when integrated across households; (3) the special case that for $s_{R,0}(h) = s_{N,T_N}(h)$ households maintain the wealth share $s_{R,0}(h)$ throughout the process; (4) the solution produces the expected results in the special cases $T_N = 0$ (immediate takeoff) and $T_N \to \infty$ (no takeoff). ∎

Gini coefficient of income. The income received by household h is given by

$$y_t(h) = r_t a_t(h) + w_t L_t = r_t A_t s_{a,t}(h) + w_t L_t.$$

We order households in ascending order of wealth and of income. The Lorenz curves of, respectively, wealth and income are:

$$\mathcal{L}_{a,t}(h) = \int_0^h s_{a,t}(\chi) d\chi;$$

$$\mathcal{L}_{y,t}(h) \equiv \frac{\int_0^h y_t(\chi) d\chi}{\int_0^1 y_t(\chi) d\chi} = \frac{r_t A_t \int_0^h s_{a,t}(\chi) d\chi + w_t L_t \int_0^h 1 d\chi}{Y_t}.$$

The Gini coefficients of, respectively, wealth and income are:

$$\sigma_{a,t}^G \equiv 1 - 2 \int_0^1 \mathcal{L}_{a,t}(h) dh; \tag{55}$$

$$\sigma_{y,t}^G \equiv 1 - 2 \int_0^1 \mathcal{L}_{y,t}(h) dh, \tag{56}$$

Substituting (55) into (56), and noting that $\int_0^h 1 d\chi = h$, yields

$$\sigma_{y,t}^G = 1 - \frac{2 r_t A_t}{Y_t} \left[\int_0^1 \mathcal{L}_{a,t}(h) dh + \frac{w_t L_t}{r_t A_t} \int_0^1 h dh \right],$$

where $\int_0^1 h dh = 0.5$. Substituting the Gini coefficient of wealth into this expression yields the Gini coefficient of income in the main text. ■

Household wealth share with endogenous labor supply. The growth rate of the household wealth share is

$$\frac{\dot{s}_{a,t}(h)}{s_{a,t}(h)} = \frac{\dot{a}_t(h)}{a_t(h)} - \frac{\dot{A}_t}{A_t} = \frac{w_t l_t(h) - s_c^*(h) C_t}{a_t(h)} - \frac{w_t l_t - C_t}{A_t}.$$

Collecting the consumption-wealth ratio C_t/A_t, and using the factor payments and Proposition 1, we obtain

$$\begin{aligned} \frac{\dot{s}_{a,t}(h)}{s_{a,t}(h)} &= \left(\frac{C}{A}\right)^* \frac{\frac{w_t l_t(h)}{G_t} \left(\frac{G}{C}\right)^* - s_c^*(h)}{a_t(h)/A_t} - \frac{w_t l_t - C_t}{A_t} \\ &= \left(\frac{C}{A}\right)^* \left[\frac{\gamma(1-\theta)\left(\frac{G}{C}\right)^* \frac{l_t(h)}{l_t} - s_c^*(h)}{s_{a,t}(h)} - \gamma(1-\theta)\left(\frac{G}{C}\right)^* + 1 \right]. \end{aligned}$$

The new term here is the household's relative labor supply, $l_t(h)/l_t$, which is constant. We thus have that the characterization of the dynamics of the wealth shares is mathematically the same as in the baseline model with inelastic labor supply. ■