Bertrand-Edgeworth game under oligopoly. General results and comparisons with duopoly

De Francesco, Massimo A. and Salvadori, Neri

University of Siena, University of Pisa and Accademia dei Lincei

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Massimo A. De Francesco
University of Siena

Neri Salvadori
University of Pisa

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Abstract

This paper studies price competition among a given number of capacity-constrained producers of a homogeneous commodity under the efficient rationing rule and constant (and identical) marginal cost until full capacity, when demand is a continuous, non-increasing, and non-negative function defined on the set of non-negative prices and is positive, strictly decreasing, twice differentiable and (weakly) concave when positive. The focus is on general properties of equilibria in the region of the capacity space in which no pure strategy equilibria exist. We study how the properties that are known to hold for the duopoly are generalized to the oligopoly and, on the contrary, what properties do not need to hold in oligopoly. Our inquiry reveals, among other properties, the possibility of an atom in the support of a firm smaller than the largest one and the properties that such an atom entails. Although the characterization of equilibria is far from being complete, this paper provides substantial elements in this direction.

Keywords: Bertrand-Edgeworth; Price game; Oligopoly; Duopoly; Mixed strategy equilibrium.

JEL: C72, D43, L13

1 Introduction

Price setting interaction among capacity-constrained sellers has been an active and varied field of research over the last forty years. Part of this

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research has proceeded along the lines of the analysis of duopoly in Levitan and Shubik (1972), where (i) a homogeneous commodity is produced at a constant (and identical across firms) average variable cost up to each firm's capacity, (ii) any rationing of demand at the cheaper firm is made according to the consumers' surplus maximizing rule, and (iii) demand is a non-negative, strictly decreasing, twice differentiable and weakly concave function on the price range from zero to the lowest price where demand is zero. Then for any number of firms the capacity space can be partitioned into three parts: in two of them a pure strategy equilibrium exists; in the third no pure strategy equilibrium exists.[1] For any capacity configuration in this last region, hereafter referred to as "the no-pure strategy equilibrium region", it follows from Das Gupta and Maskin's (1986) existence theorem for discontinuous games that a mixed strategy equilibrium necessarily exists, as was proved constructively by Levitan and Shubik (1972) for the duopoly under linear demand and equal capacity.

The set of equilibria of the price game was subsequently characterized by Kreps and Scheinkman (1983) for the duopoly under constant average variable cost and concave demand, in the context of a two-stage game in which the firms first simultaneously invest in costly capacity and afterwards, in the knowledge of capacity decisions, set prices. That same context was subsequently adopted to allow for convexities in demand (Osborne and Pitchik, 1986) or for differences in unit cost (Deneckere and Kovenock, 1996). These partial departures from the assumptions in Kreps and Scheinkman also led to the possibility that the supports of the equilibrium strategies of the price subgame are disconnected and non-identical for the duopolists. Further generalizations of Kreps and Scheinkman's model consisted in extending the two-stage capacity and price setting to oligopoly (among others, see Madden, 1998, Boccard and Wauthy, 2000, Acemoglu, Bimpikis, and Ozdaglar, 2009). This extension did not require to characterize the equilibria throughout the no-pure strategy region of the capacity space as the determination of the equilibrium profit of the largest firm(s) in any price subgame was enough to determine the subgame perfect equilibrium of the two stage game.

Comparatively less effort has been devoted to the characterization of equilibria throughout the no-pure strategy region of the capacity space, although a few important results have emerged in the recent literature. One major result is that the equilibrium payoff of the largest firm, or any of the

[1] This is a folk theorem. It is obtained along the lines of the analogous Theorem for the duopoly. As far as we know it is not available in the literature and its proof is here provided in Appendix A.

largest firms sharing the same size, is equal to the payoff of the Stackelberg follower when the rivals supply their entire capacity (Boccard and Wauthy 2000 and De Francesco 2003; see also Ubeda, 2007, and Hirata, 2009).[2] Based on this property, Ubeda (2007) proved, among other things, that the maximum and the minimum price at a mixed strategy equilibrium are the maximum and the minimum of the support of the equilibrium strategy of the largest firm (or any of the largest firms sharing the same size).[3] Other results were provided by De Francesco and Salvadori (2010) in a provisional way.

Progress on the characterization of equilibria of the price game under given capacities and downward-sloping demand has been made along several directions. One direction focused on portions of the no-pure strategy region of the capacity space. The equal capacity case for the oligopoly is analized by Vives (1986), where the (symmetric) mixed strategy equilibrium is determined. In a subsequent contribution, besides showing uniqueness of equilibrium in the symmetric case, De Francesco and Salvadori (2011) characterized the unique equilibrium under "almost symmetric" capacity configurations, in which differences in capacities among the firms are sufficiently small. A number of equilibrium features were discovered: the minimum of the support of the equilibrium strategy is the same for all firms so that payoffs are proportional to capacities; all equilibrium distributions are continuously increasing and, finally, for all firms smaller than the second largest one, the maximum of the support is lower than for any larger firm. More recently, De Francesco and Salvadori (2022) have characterized equilibria for capacity configurations in which it is possible to distinguish between two groups of firms according to the following criterion: the total capacity of the "large" firms can meet even the highest level of demand that can arise at an equilibrium whereas the total industry capacity minus the capacity of any of the large firms is lower than (in a specific case, equal to) the smallest level of demand that can arise at an equilibrium. It is proved that, for each large firm, equilibrium features are precisely the same as in the almost symmetric oligopoly previously described. Neat results have also been obtained for small firms. The equilibrium payoff-capacity ratio is the same for each

[2]The proof by Boccard and Wauthy (2000) proceeded along the lines of Kreps and Sheinkman (1983). A mistake in that proof was subsequently pointed out and the proof was correctly finalized in De Francesco (2003).

[3]These results are obtained for a discriminatory auction in which each buyer actually pays the price bid of the seller he purchases from and in which any rationing is such as to maximize consumers' surplus. This context is equivalent to Bertrand-Edgeworth competition under efficient rationing (Ubeda, 2007, p. 10).

small firm and (apart from the aforementioned specific case) it is strictly higher than the analogous ratio for large firms. Furthermore, while there is a continuum of equilibrium distributions for the small firms, the capacity-weighted average of their distributions is unique as well as the minimum and the maximum of the union of the supports of their equilibrium strategies: the minimum is higher than (in the aforementioned specific case, equal to) the minimum of the supports of the large firms while the maximum is always lower than the maximum of the support of each large firm.

A second direction of research focused on restricting the number of competing firms. Hirata (2009) and De Francesco and Salvadori (2010, 2015, 2016) have analyzed the triopoly price game with a decreasing and concave demand function, independently establishing a number of features of equilibria.[4] A third direction of research focused on a different demand function: inelastic market demand was adopted (Acemoglu, Bimpikis, and Ozdaglar, 2009, and Mark Armstrong and John Vickers, 2018).

Having examined how complex it might be to characterize equilibria in portions of the no-pure strategy equilibrium region, the present paper aims at providing some building blocks in order to effectively tackle this task. These are represented by some general results concerning mixed strategy equilibria, which in many cases can be derived by using the properties of the payoff function of any firm in the range between the minimum and the maximum of the union of the supports of the equilibrium strategies. Several properties of a duopolistic mixed strategy equilibrium may be generalized to oligopoly: among them the values of the minimum and maximum of the support of the equilibrium strategy for any firm with the highest capacity and the equilibrium payoff of any firm with the second highest capacity. Unlike what happens in a duopoly, under some circumstances there is a continuum of equilibria as far as "small" firms are concerned and even when a unique equilibrium does exist, the equilibrium distributions do not necessarily increase for all firms, or even for any firm. We identify some circumstances which must necessarily hold when a price lower than the maximum of the union of the supports of equilibrium strategies is charged with strictly positive probability by some firm. Among other things, it is shown that such a firm must necessarily be smaller than the largest one; furthermore, in the event of such an "atom" the union of the supports of the equilibrium strategies is not connected: an open right neighbourhood of the atom is not included in it. Another result relates to the second largest firm. Its equilib-

[4]Basic properties of equilibria under the triopoly were discovered almost at the same time by Hirata (2008) and De Francesco and Salvadori (2008).

rium payoff-capacity ratio equals that of any firm smaller than the largest one whose support includes the minimum of the union of the supports. This implies that the minimum of its support is either equal to the minimum of the union of the supports or larger than a determined threshold (or both, in some specific circumstances). This property is indeed extended to any firm not smaller than a firm whose support includes the minimum of the union of the supports. Finally we investigate a property that concerns the case in which two, or more, firms are equal. If several firms have the largest capacity, then they share the same equilibrium strategies.[5] This symmetry may not hold for equally-sized firms that are smaller than the largest one.

The remainder of the paper is organized as follows. Section 2 describes the assumptions and provides the basic notation; it also identifies the no-pure strategy equilibrium region of the capacity space. Section 3 presents general results for the no-pure strategy equilibrium region. Section 4 discusses four numerical examples of determination of mixed strategy equilibria. Example 1 refers to a quadriopoly in which the whole support of the second largest firm is above the mentioned threshold and clarifies how this may occur; there is also a continuum of equilibrium distributions for the second and the third largest firms and the payoff per unit of capacity is higher for the smallest firm. All the other examples refer to the triopoly; in each of them there is a unique equilibrium and the payoff per unit of capacity is higher for the smallest firm. In Example 2 there is a gap in the support of the intermediate-size firm; in Examples 3 and 4 the smallest firm charges its highest price with positive probability: in either case a thorough discussion is provided of how to determine the minimum of the support of the smallest firm and the gaps in the supports of the two larger firms. Section 5 briefly concludes. Appendix A contains the proof of the Proposition that identifies the regions of the capacity space where pure-strategy equilibria exist, and, consequently, the region where pure-strategy equilibria do not exist. Appendix B begins with Lemma 1, which discusses the properties of the payoff function of any firm in the range between the minimum and the maximum of the union of the supports of the equilibrium strategies; it then proceeds with the proofs of the propositions stated in the main text.

[5]To undertand the relevance of this result consider, for instance, that Vives (1986), in his analysis of symmetric oligopoly, specifies that he is "restricting attention to symmetric equilibria". See also Hirata (2009, p. 5 and n. 7), who leaves open, for the oligopoly, the possibility of asymmetric distributions for the two firms with the same largest capacity.

2 Preliminaries

Assumption 1. There are n firms producing a homogeneous good at the same constant unit cost (normalized to zero), up to capacity.

Let $N = \{1, 2, ..., n\}$ be the set of firms and $N_{-i} = N - \{i\}$. Without loss of generality, we consider the subset of the capacity space $(K_1, K_2, ..., K_n)$ where

$$K_1 \geqslant K_2 \geqslant ... \geqslant K_n > 0 \tag{1}$$

and we define $K = K_1 + K_2 + ... + K_n$.

Assumption 2. The market demand function is given by $D(p)$ (demand as a function of price p) and $P(x)$ (price as a function of quantity x). Function $D(p)$ is strictly positive on some bounded interval $[0, p^\star)$, on which it is continuously differentiable, strictly decreasing and such that $pD(p)$ is strictly concave; it is continuous for $p \geqslant 0$ and equals 0 for $p \geqslant p^\star$; $X = D(0) < \infty$. $P(x) = D^{-1}(x)$ on the bounded interval $[0, X]$; the function $P(x)$ is continuous for $x \geqslant 0$ and equal to 0 for $x \geqslant X$; $p^\star = P(0) < \infty$.

Assumption 3. Any rationing is in accordance with the efficient rule. Consequently, let $\Omega(p)$ be the set of firms charging price p: the residual demand forthcoming to all firms in $\Omega(p)$ is $\max\left\{0, D(p) - \sum_{j:p_j<p} K_j\right\} = Y(p)$. If $\sum_{i\in\Omega(p)} K_i > Y(p)$, the residual demand forthcoming to any firm $i \in \Omega(p)$ is a fraction $\alpha_i(\Omega(p), Y(p))$ of $Y(p)$, namely, $D_i(p_1, p_2, \dots, p_n) = \alpha_i(\Omega(p), Y(p))Y(p)$.[6]

Let p^c be the competitive price, that is

$$p^c = P(K). \tag{2}$$

Necessary and sufficient conditions for the existence of a pure strategy equilibrium for the oligopoly are easily found as straightforward generalizations of similar results for the duopoly; as a consequence no pure-strategy equilibrium actually exists when the competitive price is not an equilibrium. These results are

Proposition 1 *Let Assumptions 1, 2, and 3 hold. (i)* $(p_1, p_2, ..., p_n) = (p^c, p^c, ..., p^c)$ *is an equilibrium if and only if either*

$$K - K_1 \geqslant X, \text{ if } X \leqslant K, \tag{3}$$

[6] Our analysis does not depend on the specific assumption being made on $\alpha_i(\Omega(p), Y(p))$: for example, it is consistent with $\alpha_i(\Omega(p), Y(p)) = K_i / \sum_{r\in\Omega(p)} K_r$ as well as with the assumption that residual demand is shared evenly, apart from capacity constraints, among firms in $\Omega(p)$. In this case, $\alpha_i(\Omega(p), Y(p)) = \min\{K_i/Y(p), \hat{\alpha}(p)\}$ where $\hat{\alpha}(p)$ is the solution in α of equation $\sum_{i\in\Omega(p)} \min\{K_i/Y(p), \alpha\} = 1$.

or

$$K_1 \leqslant -p^c \left[D'(p)\right]_{p=p^c} = -\frac{P(K)}{P'(K)}, \text{ if } X > K. \tag{4}$$

In the former case the set of equilibria includes any strategy profile such that $\sum_{s\in\Omega(0)-\{j\}} K_s \geqslant X$ *for each* $j \in \Omega(0) \neq \emptyset$*. In the latter,* $(p^c, p^c, \dots, p^c)$ *is the unique equilibrium.*

(ii) No pure strategy equilibrium exists if neither (3) nor (4) holds so that

$$K_1 > \max\left\{K - X, -\frac{P(K)}{P'(K)}\right\}. \tag{5}$$

A formal proof is provided in Appendix A. As is well known, condition (3) gives rise to the classic Bertrand equilibrium, whereas condition (4) can be interpreted in terms of the Cournot model of quantity competition among capacity-unconstrained firms. Indeed, condition (4) identifies, in the $(K_1, K_2, ..., K_n)$-space, the region in which each firm's capacity is not higher than its best (capacity-unconstrained) quantity response when the rivals supply their entire capacity (namely, the region that is bounded above by the lower envelope of the Cournot best-response functions).[7] Proposition 1 states that the existence of a pure strategy equilibrium depends upon total capacity and the capacity of the largest firm. This is depicted in Figure 1 which is based on the assumption that $P(x) = a - bx$ when positive and the given number of existing firms is n. According to inequalities (1), the relevant region is that in which $K/n \leqslant K_1 < K$. Note that the relevant region depends on n, but not on $P(x)$. If $P(x) = a - bx$, then $X = a/b$ while $-P(K)/P'(K) = (a/b) - K$ if $K \leqslant a/b$. According to Proposition 1, a pure strategy equilibrium exists in region A (with $p^c > 0$), including the points where $K_1 = -a/b + K$, and in region B (with $p^c = 0$), including the points where $K_1 = -a/b + K$, whereas no pure strategy equilibrium exists in region C.

In order to study equilibria in the region where pure strategy equilibria do not exist, we need to enrich our notation. A strategy by firm i is denoted by $\sigma_i : (0, \infty) \to [0, 1]$, where $\sigma_i(p) = \Pr_{\sigma_i}(p_i < p)$ is the probability of firm i charging less than p under strategy σ_i. Of course, any function $\sigma_i(p)$ is non-decreasing and everywhere continuous except at p' such that $\Pr_{\sigma_i}(p_i =$

[7] It should be noted that Assumption 1 does not guarantee the uniqueness of the Cournot equilibrium. Uniqueness would be ensured if, for instance, one assumed $D'(p) + pD''(p) < 0$ on $(0, p^\star)$. (On this, see Dechenaux and Kovenock, 2007).

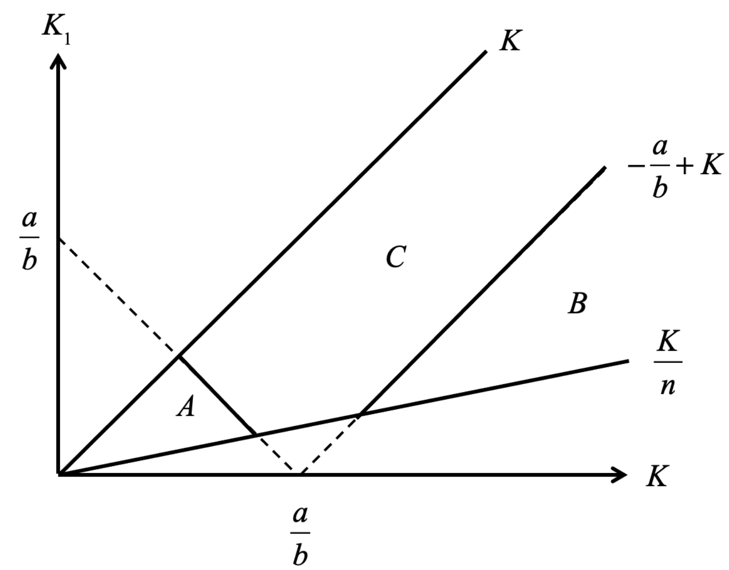

Figure 1: Taxonomy of equilibria

$p') > 0$, where it is left-continuous ($\sigma_i(p'-) = \sigma_i(p')$), but not continuous.[8] An equilibrium is denoted by $\phi = (\phi_1, \phi_2, ..., \phi_n)$, where $\phi_i(p) = \Pr_{\phi_i}(p_i < p)$ and firm i's payoff (expected profit) at strategy profile (σ_i, σ_{-i}) is denoted by $\Pi_i(\sigma_i, \sigma_{-i})$. As a consequence, $\Pi_i(\phi) = \Pi_i(\phi_i, \phi_{-i})$ is firm i's expected profit at the equilibrium strategy profile ϕ; $\Pi_i(p, \phi_{-i})$ is firm i's expected profit when it charges p with certainty and the rivals are playing their equilibrium profile of strategies ϕ_{-i}, and $\Pi_i(\phi_i, \phi_{-i}) \geqslant \Pi_i(\sigma_i, \phi_{-i})$ for each i and each σ_i. When no doubt can arise, and for the sake of brevity, we write $\Pi_i^\star$ rather than $\Pi_i(\phi_i, \phi_{-i})$ and $\Pi_i(p)$ rather than $\Pi_i(p, \phi_{-i})$. Further, we denote by $S_i(\phi_i)$ the support of ϕ_i and by $p_M^{(i)}(\phi_i)$ and $p_m^{(i)}(\phi_i)$ the maximum and minimum of $S_i(\phi_i)$, respectively. More specifically, we say that $p \in S_i(\phi_i)$ when $\phi_i(\cdot)$ is increasing at p, that is, when there is $\delta > 0$ such that $\phi_i(p+h) > \phi_i(p-h)$ for any $0 < h < \delta$. Obviously, $\Pi_i^\star = \Pi_i(p)$ almost everywhere in $S_i(\phi_i)$. Once again, when no doubt can arise and for the sake of brevity, we write S_i, $p_M^{(i)}$, and $p_m^{(i)}$ rather than $S_i(\phi_i)$, $p_M^{(i)}(\phi_i)$, and $p_m^{(i)}(\phi_i)$, respectively. If S_i is not connected, i.e. if $\phi_i(p)$ is constant in an open interval $(\widetilde{p}, \widetilde{\widetilde{p}})$ and is

[8] In order to shorten notation, we denote $\lim_{p\to h+} f(p)$ and $\lim_{p\to h-} f(p)$ as $f(h+)$ and $f(h-)$, respectively, for any function $f(p)$.

increasing in $\widetilde{p}$ and in $\widetilde{\widetilde{p}}$, then the interval $(\widetilde{p}, \widetilde{\widetilde{p}})$ will be referred to as a *gap* in S_i.

So long as firm i's rivals' equilibrium strategies $\phi_{-i}(p)$ are continuous in p, $\Pi_i(p) = Z_i(p; \phi_{-i}(p))$, where

$$Z_i(p; \varphi_{-i}) := p \sum_{\psi \in \mathcal{P}(N_{-i})} q_{i,\psi}(p) \prod_{r \in \psi} \varphi_r \prod_{s \in N_{-i} - \psi} (1 - \varphi_s), \tag{6}$$

$\varphi_j \in [0, 1]$ (each $j \neq i$) is an independent variable, $\mathcal{P}(N_{-i})$ is the power set of N_{-i}, and $q_{i,\psi}(p) = \max\{0, \min\{D(p) - \sum_{r \in \psi} K_r, K_i\}\}$ is firm i's output when it charges p, any firm $r \in \psi$ charges less than p and any firm $s \in N_{-i} - \psi$ charges more than p.[9] If instead $\Pr_{\phi_j}(p_j = p') > 0$ for some $j \neq i$, then $Z_i(p'; \phi_{-i}(p')) \geqslant \Pi_i(p') \geqslant Z_i(p'+; \phi_{-i}(p'+))$.[10] The RHS of (6) is a weighted arithmetic mean of the functions $pq_{i,\psi}(p)$'s since $\sum_{\psi \in \mathcal{P}(N_{-i})} \prod_{r \in \psi} \varphi_r \prod_{s \in N_{-i} - \psi} (1 - \varphi_s) = 1$. As a consequence, $pq_{i,N-i}(p) \leqslant Z_i(p; \phi_{-i}) \leqslant pq_{i,\emptyset}(p)$. Note that the coefficient of $pq_{i,\psi}(p)$ is positive if and only if $\varpi_i \supseteq \psi \supseteq \Xi$, where $\varpi_i = \{h \in N_{-i} : \varphi_h > 0\}$ and $\Xi = \{h \in N_{-i} : \varphi_h = 1\}$. Indeed $\varphi_h = 0$ for some $h \in \psi$ if and only if $\varpi_i \not\supseteq \psi$ and $\varphi_h = 1$ for some $h \in N_{-i} - \psi$ if and only if $\psi \not\supseteq \Xi$ (obviously ϖ_i and Ξ depend on ϕ). Our analysis below will rely on a number of properties of the functions $Z_i(p; \varphi_{-i})$ that are established in Lemma 1 in Appendix B.

Finally, let us define[11]

$$p_M = \min \arg \max_{p \geqslant 0} pq_{1,N_{-1}}(p), \tag{7}$$

$$p_m = \min \left\{ p : pq_{1,\emptyset}(p) = \max_{p \geqslant 0} pq_{1,N_{-1}}(p) \right\}. \tag{8}$$

Note that $p_M = p_m = p^c$ if either inequalities (3) or inequalities (4) hold; otherwise $p_m < p_M$. If firm 1 charges with certainty p_M, then it will get a profit higher than or equal to $\max_p pq_{1,N_{-1}}(p)$. Hence $\Pi_1^\star \geqslant \max_p pq_{1,N_{-1}}(p)$ (obviously the same is true for any i such that $k_i = k_1$). On the other hand, if firm 1 charges with certainty p_m, it will get a profit lower than or equal to $\max_p pq_{1,N_{-1}}(p)$. Hence, if firm 1 charges a price lower than p_m, it will get a profit lower than $\Pi_1^\star$ and therefore $p_m^{(i)} \geqslant p_m$ for each i such that $K_i = K_1$.

[9]Note that $\prod_{r \in \psi} \varphi_r$ is the empty product, hence equal to 1, when $\psi = \emptyset$; and it is similarly $\prod_{s \in N_{-i} - \psi} (1 - \varphi_s) = 1$ when $\psi = N_{-i}$.

[10]The exact value of $\Pi_i(p')$ when $\Pr_{\phi_j}(p_j = p') > 0$ for some $j \neq i$ depends on function $\alpha_i(\Omega(p), Y(p))$.

[11]Equation (7) reads $p_M = \arg\max_{p \geqslant 0} pq_{1,N_{-1}}(p)$ unless inequalities (3) hold.

3 Equilibria under oligopoly when no pure strategy equilibrium exists: some general results

The analysis developed in this section refers to the region of the capacity space where no pure strategy equilibrium exists, i.e. the region where inequalities (1) and (5) hold. The definitions of p_M and p_m make it possible to characterize the region where inequalities (1) and (5) hold by substituting inequality (5) with inequality

$$P(K) < p_m. \tag{9}$$

Indeed, if $K_1 \leqslant K - X$, then $p_m = P(K) = 0$ whereas if $K_1 \leqslant -\frac{P(K)}{P'(K)}$, then $p_m = p_M = P(K) \geqslant 0$. Conversely, if inequality (5) holds, then inequality (9) holds too. Finally, note that in the region where inequalities (5) and (1) hold we have:

$$p_M = \arg\max_p p\left[D(p) - \sum_{j\neq 1} K_j\right] \tag{10}$$

$$p_m = \max\{\widehat{p}, \widehat{\widehat{p}}\}, \tag{11}$$

where

$$\widehat{p} = \frac{\max_p p[D(p) - \sum_{j\neq 1} K_j]}{K_1} \tag{12}$$

$$\widehat{\widehat{p}} = \min\left\{p : pD(p) = \max_p p\left[D(p) - \sum_{j\neq 1} K_j\right]\right\}. \tag{13}$$

Note that $\widehat{\widehat{p}} \geqslant \widehat{p}$ if and only if $D(\widehat{\widehat{p}}) \leqslant D(\widehat{p}) \leqslant K_1$. This is so since $\widehat{\widehat{p}}D(\widehat{\widehat{p}}) = \widehat{p}K_1$, the demand function is decreasing, and the function $pD(p)$ is increasing throughout the range $[0, p_M]$.

Since Kreps and Sheinkman (1983) it is known that, in a duopoly,

D.1 $\Pi_1^\star = \max_p pq_{1,N_{-1}}(p)$;

D.2 $p_M^{(1)} = p_M^{(2)} = p_M$;

D.3 $p_m^{(1)} = p_m^{(2)} = p_m$;

D.4 $\Pi_2^\star = p_m K_2$;

D.5 if $K_1 = K_2$, then $\phi_1(p) = \phi_2(p)$ throughout $[p_m, p_M]$ and $\phi_1(p_M) = \phi_2(p_M) = 1$, whereas if $K_1 > K_2$, $\phi_1(p_M) < \phi_2(p_M) = 1$.

Some of these results also hold in the oligopoly, as will be shown in this section. More precisely in an oligopoly

O.1 $\Pi_1^\star = \max_p pq_{1,N_{-1}}(p)$;

O.2 there is $h \in N_{-1}$ such that $p_M^{(1)} = p_M^{(h)} = p_M$;

O.3 there is $h \in N_{-1}$ such that $p_m^{(1)} = p_m^{(h)} = p_m$; moreover either $p_m^{(2)} = p_m$ or $p_m^{(2)} \geqslant P\left(K_1 + \sum_{h:p_M^{(h)} \leqslant p_m^{(2)}} K_h\right)$, or both;

O.4 $\Pi_2^\star = p_m K_2$;

O.5 if $K_1 = K_2$, then $\phi_1(p) = \phi_2(p)$ throughout $[p_m, p_M]$ and $\phi_1(p_M) = \phi_2(p_M) = 1$, whereas if $K_1 > K_2$, $\phi_1(p_M) < \phi_2(p_M) = 1$.

The following proposition states in our formalism a proposition available in the literature. It generalizes to oligopoly statement **D.1** and part of statement **D.2**. For a complete proof see Boccard and Wauthy (2000) and De Francesco (2003). See also Ubeda, 2007, Loertscher, 2008, and Hirata (2009).

Proposition 2 *Let Assumptions 1, 2, and 3 and inequality (9) hold. In any equilibrium $\phi_j(p_M) = 1$ for any j such that $K_j < K_1$; $p_M^{(i)} = p_M = \max_{j \in N} p_M^{(j)}$ and $p_m^{(i)} = p_m = \min_{j \in N} p_m^{(j)}$ for some i such that $K_i = K_1$, and*

$$\Pi_i^\star = \max_p p \left[D(p) - \sum_{j \neq 1} K_j \right] \tag{14}$$

for any i such that $K_i = K_1$.

Corollary 1 If $K_1 = K_2$, then $\Pr_{\phi_i}(p_i = p_M) = 0$ for any $i \in N$.

Corollary 2 If $\widehat{p} > \widehat{\widehat{p}}$, then for any i such that $K_i = K_1$, the equilibrium payoff can also be written $\Pi_i^\star = p_m K_1$.

Corollary 3 If $\widehat{\widehat{p}} \geqslant \widehat{p}$, then the equilibrium payoff of firm 1 can also be written $\Pi_1^\star = p_m D(p_m)$ and $K_1 \geqslant D(p_m) > D(p_M) > \sum_{j \neq 1} K_j$.

Let $M = \{i \in N : p_M^{(i)} = p_M\}$ and $L = \{i \in N : p_m^{(i)} = p_m\}$. The following proposition establishes basic properties of equilibria in the region defined by inequalities (5) and (1). Proofs of parts (i)-(vi) can be found in De Francesco and Salvadori (2022, Proposition 1(i)-(ii)). The proof of parts (vii)-(ix) is in Appendix B. Parts (i) and (vi) generalize statement **D.3** to the oligopoly, by establishing the first part of statement **O.3**, whereas part (v) completes the generalization of statement **D.2**. A similar generalization was also provided by Ubeda (2007) in a different context.

Proposition 3 *Let Assumptions 1, 2, and 3 and inequality ((9) hold. In any equilibrium ϕ:*

(i) $\#L \geqslant 2$;

(ii) $D(p_m) < \sum_{i \in L} K_i$;

(iii) $\Pr(p_j = p_m) = 0$ (each j);

(iv) for any $i \in L - \{1\}$, $\Pi_i^\star = p_m K_i$;

(v) $\#M \geqslant 2$;

(vi) $p_m^{(i)} = p_m$ for any firm i such that $K_i = K_1$;

(vii) for any $i \in N$, $\Pi_i^\star = \Pi_i(p)$ for p in the interior of S_i and for $p = p_m^{(i)}$;

(viii) if $\alpha \subset S_i$, where α is a non degenerate interval, then $\alpha \subset \cup_{j \neq i} S_j$;

(ix) for any $i \neq 1$ such that

$$p_M^{(i)} \geqslant P\left(K_1 + \sum_{h: p_M^{(h)} < p_M^{(i)}} K_h\right), \tag{15}$$

$\Pi_i^\star = p_m K_i$.

Example 1 in the following section is useful to understand the role of inequality (15).

Corollary 4 $\Pi_j^\star \geqslant p_m K_j$ (each $j \neq 1$).

In the duopoly, equilibrium distributions are continuously increasing and uniquely determined for both firms throughout the range $[p_m, p_M)$: $\Pr_{\phi_i}(p_i = p) = 0$ and $\phi_i'(p) > 0$ (each $i = 1, 2$, each $p \in [p_m, p_M)$).[12] As mentioned in the introduction, this need not be so with more than two firms. The following proposition, whose proof is in Appendix B, provides

[12]Osborne and Pitchik(1986) showed that this need not to be so if function $pD(p)$ is not concave.

general results on any atom in the support of any equilibrium strategy in the range $[p_m, p_M)$. In Example 2 in the next section a support is not connected, but there is no such an atom. In Examples 3 and 4 an atom exists in S_3 at $p_M^{(3)}$ in a triopoly. $\bigcap_{i=N} S_i = \{p_m^{(3)}\}$ in Example 3, whereas $\bigcap_{i=N} S_i$ is a non-degenerate interval $[p_m^{(3)}, \widetilde{\widetilde{p}}]$, with $\widetilde{\widetilde{p}} < p_M^{(3)}$, in Example 4.

Proposition 4 *Let Assumptions 1, 2, and 3 and inequality (9) hold. Assume that ϕ is an equilibrium in which $\phi_j(\widetilde{p}) < \phi_j(\widetilde{p}+)$ for some j and some $\widetilde{p} \in [p_m, p_M)$. Then*

(i) $\Pi_j^\star = \Pi_j(\widetilde{p}) = Z_j(\widetilde{p}; \phi_{-j}(\widetilde{p}))$;

(ii.a) there is no $\chi \subseteq \varpi$ such that $\sum_{h\in\chi\cup\Xi} K_h \geqslant D(\widetilde{p}) \geqslant K_j + \sum_{h\in\varpi\cup\Xi} K_h - K_m$, where $\varpi = \{h \in N_{-j}, p_m^{(h)} \leqslant \widetilde{p} \leqslant p_M^{(h)}\}$, $\Xi = \{h \in N_{-j}, p_M^{(h)} \leqslant \widetilde{p}\}$, and $m = \max \chi$;

(ii.b) if $\phi_h(\widetilde{p}) = 0$ (some $h \in N$), then $p_m^{(h)} > \widetilde{p}$ and $\Pi_h^\star > \Pi_h(\widetilde{p})$;

(iii) $p_m < \widetilde{p} < P(K_1)$;

(iv) $\phi_i(\widetilde{p}+) = \phi_i(\widetilde{p})$ (each $i \neq j$);

(v) there is $p^\circ > \widetilde{p}$ such that $(\cup_{i\in N} S_i) \cap (\widetilde{p}, p^\circ) = \emptyset$ and $p^\circ \in \cup_{i\in N} S_i$;

(vi) there is a right neighborhood of $\widetilde{p}$ in which the function $Z_j(p, \phi_{-j}(\widetilde{p}))$ is decreasing in p;

(vii) $K_j < K_1$.

The following Proposition 5 (proof in Appendix B) generalizes to oligopoly statement **D.5**; furthermore, it shows that if several firms have the largest capacity, then their equilibrium strategies are necessarily the same. This result depends on the fact that $Z_1(p, \phi_{-1}(p))$ depends on $\phi_i(p)$ (any $i \neq 1$) throughout $[p_m, p_M]$: as a consequence, if $\phi_1(p) \neq \phi_2(p)$ and $K_1 = K_2$, then $\Pi_2(p) \neq \Pi_1(p)$, which in turn inevitably leads to a contradiction.

Proposition 5 *Let Assumptions 1, 2, and 3 and inequality (9) hold. In any equilibrium ϕ:*

(i) if $K_1 > K_2$, then $\phi_1(p_M) < 1$;

(ii) if $K_1 = K_j$ (some $j \neq 1$), then $\phi_j(p) = \phi_1(p)$ throughout $[p_m, p_M]$;

(iii) if $K_1 = K_2 = ... = K_s > K_{s+1}$, then $p_M^{(s+1)}, p_M^{(s+2)}, ..., p_M^{(n)} < p_M$.

The following Proposition (proof in Appendix B) demonstrates that no firm can have, in equilibrium, a profit-to-capacity ratio lower than that any larger firm has.[13] An immediate consequence is that if the minimum of the

[13]See also Proposition 10 in Ubeda (2007) where the result of Proposition 6(i) was obtained in the context of a discriminatory auction.

support of a firm is larger than p_m whereas the minimum of the support of a smaller firm equals p_m, then the two firms have, in equilibrium, the same profit-to-capacity ratio p_m. Finally, although equally-sized firms smaller than the largest one have the same equilibrium profit, this does not imply that the equilibrium distributions are the same, as is the case with firms with the same largest capacity (see Proposition 5(ii)). The reason is that, if $K_i = K_j < K_1$, then $Z_i(p, \phi_{-i}(p))$ may not depend on $\phi_j(p)$ on a left neighborhood of p_M. As a consequence, there might be a subset of that neighbourhood in which, for instance, $p \in S_i \cap S_j$ even if $\phi_i(p) > \phi_j(p)$.

Proposition 6 *Let Assumptions 1, 2, and 3 and inequality (9) hold and $K_1 > K_i \geqslant K_j$. Then:*

(i) in any equilibrium $\frac{\Pi_j^\star}{K_j} \geqslant \frac{\Pi_i^\star}{K_i}$;

(ii.a) in any equilibrium in which $j \in L$, $\Pi_i^\star = p_m K_i$ and (ii.b) there is $\chi \subseteq \{h : p_m^{(h)} \leqslant p_m^{(i)}\} - \{h \in L : K_h \leqslant K_i\}$ such that $\Pi_i^\star = (1 - \Pi_{h\in\chi}\phi_h(p_m^{(i)}))p_m^{(i)} K_i$ and $\Pi_l(p_m^{(i)}) = (1 - \Pi_{h\in\chi}\phi_h(p_m^{(i)}))p_m^{(i)} K_l = \Pi_l^\star$, each $l \in \{h \in L : K_h \leqslant K_i\}$; (ii.c) as a consequence either $p_m^{(i)} = p_m$ or $p_m^{(i)} \geqslant P\left(\sum_{h\in\chi\cup\{h:p_M^{(h)}\leqslant p_m^{(i)}\}} K_h\right)$,[14] or both.[15]

(iii.a) If $K_1 > K_i = K_j$, then in any equilibrium $\Pi_j^\star = \Pi_i^\star$; further, in an equilibrium in which $j \in L$ and $\phi_j(p') < \phi_i(p') < 1$ for some $p' \in S_i \cup S_j$ (iii.b) either $p' \in S_i - S_j$, or there is $\chi \subseteq \{h : p_m^{(h)} \leqslant p'\} - \{h \in L : K_h \leqslant K_i\}$ such that $\Pi_i(p') = (1 - \Pi_{h\in\chi}\phi_h(p'))p' K_i$. More precisely, (iii.c) if $p' \in S_i \cap S_j$, then there is $\chi \subseteq \{h : p_m^{(h)} < p'\} - \{h \in L : K_h \leqslant K_i\}$ such that $\Pi_i(p') = (1 - \Pi_{h\in\chi}\phi_h(p'))p' K_i$; whereas (iii.d) if there is no $\chi \subseteq \{h : p_m^{(h)} \leqslant p'\} - \{h \in L : K_h \leqslant K_i\}$ such that $\Pi_i(p') = (1 - \Pi_{h\in\chi}\phi_h(p'))p' K_i$, then there is $p'' > p'$ such that $S_j \cap [p', p'') = \emptyset$ and there is $\chi \subseteq \{h : p_m^{(h)} \leqslant p''\} - \{h \in L : K_h \leqslant K_i\}$ such that $\Pi_i(p'') = (1 - \Pi_{h\in\chi}\phi_h(p''))p'' K_i$.

Finally the following proposition, whose proof is in Appendix B, generalizes to oligopoly statement **D.4** and completes the generalization of statement **D.3** by establishing the second part of statement **O.3**.

[14] Note that necessarily $1 \in \chi$ and therefore $P\left(\sum_{h\in\chi\cup\{h:p_M^{(h)}\leqslant p_m^{(i)}\}} K_h\right) \leqslant P\left(K_1 + \sum_{h:p_M^{(h)}\leqslant p_m^{(i)}} K_h\right)$. The equality holds if and only if $\Pi_{h\in\chi}\phi_h(p_m^{(i)}) = \phi_1(p_m^{(i)})$; that is $\phi_h(p_m^{(i)}) = 1$ for each $h \in \chi - \{1\}$.

[15] It is easily recognized that both conditions hold when $p_m \geqslant P\left(\sum_{h\in\chi} K_h\right)$ and $\chi \subseteq \{h \in L : K_h > K_i\}$, which is certainly the case when $p_m \geqslant P(K_1)$.

Proposition 7 *Let Assumptions 1, 2, and 3 and inequality (9) hold. In any equilibrium ϕ:*

(i) $\Pi_i^\star = p_m K_i$ for any i such that $K_i = K_2$;

(ii) if $K_1 > K_2$, then for any i such that $K_i = K_2$, $\Pi_i^\star = (1-\phi_1(p_m^{(i)}))p_m^{(i)} K_2$ and $\Pi_j(p_m^{(i)}) = (1-\phi_1(p_m^{(i)})p_m^{(i)} K_j$, each $j \in L - \{1\}$; as a consequence either $p_m^{(i)} = p_m$ or $p_m^{(i)} \geqslant P\left(K_1 + \sum_{h:p_M^{(h)} \leqslant p_m^{(i)}} K_h\right)$, or both;

(iii) if $K_1 > K_2 = K_j$ and $\phi_j(p') < \phi_2(p') < 1$ for some $p' \in S_2 \cup S_j$, then either $p' \in S_2$, or $\Pi_2(p') = (1-\phi_1(p'))p'K_2$, or both; further, if $p' \in S_2 \cap S_j$, then $\Pi_2(p') = (1-\phi_1(p'))p'K_2$; whereas if $\Pi_2(p') \neq (1-\phi_1(p'))p'K_2$, then there is $p'' > p'$ such that $\Pi_2(p'') = (1-\phi_1(p''))p''K_2$ and $S_j \cap [p', p'') = \emptyset$.

Example 1 in the following section discusses a case in which $p_m^{(2)} > p_m$, highlighting the role of the inequalities appearing in Propositions 6(i.c) and 7(ii).

4 Examples

In this section we present four numerical examples which are useful in understanding the Propositions stated in the previous section.

4.1 Example 1

Here is an example in which $\Pi_2(p_m^{(2)}) = \left[1 - \phi_1(p_m^{(2)})\right] p_m^{(2)} K_2$, $\Pi_j(p_m^{(2)}) = \left[1 - \phi_1(p_m^{(2)})\right] p_m^{(2)} K_j$, each j such that $p_m^{(j)} = p_m$, and $p_m^{(2)} > p_m$. Let $D(p) = 14.4 - p$; $(K_1, K_2, K_3, K_4) = (13.2; 4; 3; 0.4)$. $P(K_1 + K_2) = P(K_1 + K_3) = 0 < P(K_1 + K_4) = 0.8 < p_m = 0.928\overline{030} < P(K_1) = 1.2 < p_M = 3.5$. Therefore, $\Pi_1^\star = 12.25$ and $\Pi_2^\star = 3.7\overline{12}$, by Propositions 2 and 6(ii).

We first prove that a continuum of equilibria in which $L = \{1, 3\}$ exists. Any such equilibria has the following features: $S_1 = [p_m, p_M]$, $S_4 = [p_m^{(4)}, p_M^{(4)}] = [1.001838002, 1.086817875]$, and $p_m^{(2)} \geqslant p_M^{(4)}$. Therefore, $\phi_1(p) = \phi_{13}^\star(p) := \frac{(p-p_m)K_3}{p[K_1+K_3-D(p)]}$ and $\phi_3(p) = \phi_3^\star(p) := \frac{(p-p_m)K_1}{p[K_1+K_3-D(p)]}$ over the range $[p_m, p_m^{(4)}]$, $\phi_{13}^\star(p)$ and $\phi_3^\star(p)$ being the solutions to equations $p_m K_3 = Z_3(p; \varphi_1, 0, 0)$ and $p_m K_1 = Z_1(p; 0, \varphi_3, 0)$, respectively. Note that $\Pi_4^\star > p_m K_4$ since $Z_4(p, \phi_{13}^\star(p), 0, \phi_3^\star(p))$ is increasing on a right neighbourhood of p_m. More specifically, $p_m^{(4)} = \arg\max_{p \in (p_m, P(K_1))} Z_4(p, \phi_{13}^\star(p), 0, \phi_3^\star(p))$ and $\Pi_4^\star = \max_{p \in (p_m, P(K_1))} Z_4(p, \phi_{13}^\star(p), 0, \phi_3^\star(p)) = 0.3793489392$. Over the range $[p_m^{(4)}, p_M^{(4)}]$, $(\phi_1(p), \phi_2(p), \phi_3(p), \phi_4(p)) = (\phi_1^\circ(p), 0, \phi_3^\circ(p), \phi_4^\circ(p))$, where $(\phi_1^\circ(p), \phi_3^\circ(p), \phi_4^\circ(p))$ is the solution in $(\varphi_1, \varphi_3, \varphi_4)$ of the equation system

$$\begin{aligned} p_m K_i &= Z_i(p, \varphi_{-i-2}, 0) \quad (i = 1, 3), \\ \Pi_4^\star &= Z_4(p, \varphi_{-4-2}, 0). \end{aligned}$$

Therefore, in any such equilibrium, $[p_m, p_M^{(4)}) \subseteq S_1 \cap S_3 - S_2$. Note that inequality $p_m^{(2)} \geqslant P\left(K_1 + \sum_{h:p_M^{(h)} \leqslant p_m^{(2)}} K_h\right)$ (see Propositions 3(ix), 6(ii.b), and 7(ii)) is equivalent to $p_m^{(2)} \geqslant p_M^{(4)}$. Indeed, if $p_m^{(2)} < p_M^{(4)}$, then $P\left(K_1 + \sum_{h:p_M^{(h)} \leqslant p_m^{(i)}} K_h\right) = P(K_1)$ and the inequality cannot hold. On the contrary, if $p_m^{(2)} \geqslant p_M^{(4)}$, then $P\left(K_1 + \sum_{h:p_M^{(h)} \leqslant p_m^{(i)}} K_h\right) = P(K_1 + K_4) < p_m$ and the inequality holds.

Let us complete the analysis: over the range $[p_M^{(4)}, p_M]$ $\phi_1(p) = \frac{p-p_m}{p}$ whereas $\phi_2(p)$ and $\phi_3(p)$ are such that $\phi_2(p)K_2 + \phi_3(p)K_3 = D(p) - K_4 - \frac{p_m K_1}{p}$. Notice that, over that range, $p_m^{(2)}$ can be any $p \in [p_M^{(4)}, \overline{p}_m^{(2)}]$, where $\overline{p}_m^{(2)} = 1.257359313$ is the solution in p to the equation $p_m K_1 = Z_1(p, 0, 1, 1)$ (namely, $p_m K_1 = p[D(p) - K_3 - K_4]$), and $p_M^{(2)}$ can be any $p \in [\overline{p}_M^{(2)}, p_M]$ where $\overline{p}_M^{(2)} = 1.895466445$ is the solution in p to the equation $p_m K_1 = Z_1(p, 1, \phi_3(p_M^{(4)}), 1)$.

Finally, we will prove that in no equilibrium $L \neq \{1, 3\}$. If $L = \{1, 2\}$, then $\phi_1(p) = \phi_{12}^\star(p) := \frac{(p-p_m)K_2}{p[K_1+K_2-D(p)]}$ and $\phi_2(p) = \phi_2^\star(p) := \frac{(p-p_m)K_1}{p[K_1+K_2-D(p)]}$ over the range $[p_m, \min\{p_m^{(3)}, p_m^{(4)}\}]$, $\phi_{12}^\star(p)$ and $\phi_2^\star(p)$ being the solutions to equations $p_m K_2 = Z_2(p; \varphi_1, 0, 0)$ and $p_m K_1 = Z_1(p; \varphi_2, 0)$, respectively. Then $\Pi_i^\star > p_m K_i$ (each $i \in \{3, 4\}$) since, as one can check, $Z_i(p, \phi_{12}^\star(p), \phi_2^\star(p), 0) > p_m K_i$ $(i \in \{3, 4\})$ for p larger than and close enough to p_m. Then $\Pi_i(p_M^{(i)})^- \geqslant (1 - \phi_1(p_M^{(3)}))p_M^{(i)} K_i > p_m K_i$ (each $i \in \{3, 4\}$) and therefore $\Pi_2(p_M^{(i)})^- = (1 - \phi_1(p_M^{(i)}))p_M^{(i)} K_2 > p_m K_2$, which contradicts Proposition 3(ix). If $L \supset \{1, 2\}$, the two-equation system $Z_2(p, \phi_{-2}) = p_m K_2$ and $Z_i(p, \phi_{-i}) = p_m K_i$ $(i \in L - \{1, 2\})$ implies that $K_2(1 - \phi_2(p)) = K_3(1 - \phi_3(p))$ on a right neighbourhood of p_m, which contradicts the requirement that $\lim_{p \to p_m} \phi_2(p) = \lim_{p \to p_m} \phi_i(p) = 0$. If $L = \{1, 4\}$, then $\phi_1(p) = \phi_{14}^\star(p) := \frac{(p-p_m)K_4}{p[K_1+K_4-D(p)]}$ and $\phi_4(p) = \phi_4^\star(p) := \frac{(p-p_m)K_1}{p[K_1+K_4-D(p)]}$ on a right neighbourhood of p_m where, similarly to the above, $\phi_{14}^\star(p)$ and $\phi_4^\star(p)$ are the solutions to equations $p_m K_4 = Z_4(p; \varphi_1, 0, 0)$ and $p_m K_1 = Z_1(p; 0, 0, \varphi_4)$, respectively. Then, it follows that $Z_2(p, \phi_{14}^\star(p), 0, \phi_4^\star(p)) < p_m K_2$ and $Z_3(p, \phi_{14}^\star(p), 0, \phi_4^\star) < p_m K_3$ throughout $(p_m, P(K_1))$, whereas $\phi_4^\star(p) = 1$ at $p = 0.9378221735 < P(K_1)$. This contradicts Proposition 3(viii).

4.2 Example 2

The example here is a case in which there is a unique equilibrium, all distributions are continuous over the range (p_m, p_M) and there is a gap in the support of the strategy of one of the firms. Let $D(p) = 20 - p$, $n = 3$, and $(K_1, K_2, K_3) = (16, 5, 0.8)$. Then $p_M = 7.1$ and hence $\Pi_1^\star = 50.41$, $p_m = 3.150625$ and $\Pi_2^\star = 15.753125$. Furthermore, $L = \{1, 2\}$ and $[p_m, p_m^{(3)}] \subset S_1 \cap S_2$, so that, over the range $[p_m, p_m^{(3)}]$, $\phi_1(p) = \phi_{12}^\star(p)$ and $\phi_2(p) = \phi_2^\star(p)$, where $\phi_{12}^\star(p)$ and $\phi_2^\star(p)$ are defined as in Example 1. Therefore, $\Pi_3(p) = Z_3(p, \phi_{12}^\star(p), \phi_2^\star(p))$ over the range $[p_m, p_m^{(3)}]$. Next, let $\beta := \arg\max_{p \in (p_m, p_M)} Z_3(p, \phi_{12}^\star(p), \phi_2^\star(p)) = 3.426978457$ and $\Pi_3^\star(\beta) := Z_3(\beta, \phi_{12}^\star(\beta), \phi_2^\star(\beta)) = 2.618610069$, and denote by $(\phi_1^\circ(p, \Pi_3^\star(\beta)), \phi_2^\circ(p, \Pi_3^\star(\beta)), \phi_3^\circ(p, \Pi_3^\star(\beta)))$ the solution in $(\varphi_1, \varphi_2, \varphi_3)$ of the equation system

$$\Pi_i^\star = Z_i(p, \varphi_{-i}) \quad (i = 1, 2),$$
$$\Pi_3^\star(\beta) = Z_3(p, \varphi_{-3}),$$

and by $p_M^{(3)}(\beta) = 3.719278412$ the solution to the equation $\phi_3^\circ(p, \Pi_3^\star(\beta))) = 1$. It is found that $\phi_1^\circ(p, \Pi_3^\star(\beta))$ and $\phi_3^\circ(p, \Pi_3^\star(\beta))$ are both increasing over the range $(\beta, p_M^{(3)}(\beta))$ whereas $\phi_2^\circ(p, \Pi_3^\star(\beta))$ is first increasing and then decreasing over that range, with $\phi_2^\circ(p_M^{(3)}(\beta), \Pi_3^\star(\beta)) > \phi_2^\circ(\beta, \Pi_3^\star(\beta))$. It follows that there is an equilibrium in which $\Pi_3^\star = \Pi_3^\star(\beta)$. In this equilibrium, $S_3 = [p_m^{(3)}, p_M^{(3)}] = [\beta, p_M^{(3)}(\beta)]$, $S_1 = [p_m, p_M]$, and $S_2 = [p_m; \gamma] \cup [p_M^{(3)}; p_M]$, where $\gamma = 3.600955035$ is such that $\phi_2(\gamma) = \phi_2^\circ(\gamma, \Pi_3^\star(\beta)) = \phi_2(p_M^{(3)}) = \phi_2^\circ(p_M^{(3)}, \Pi_3^\star(\beta))$. Clearly, over the range $(\gamma, p_M^{(3)})$, $\phi_1(p)$ and $\phi_3(p)$ are the solutions in φ_1 and φ_3 to the equations $Z_3(p, \varphi_1, \phi_2^\circ(\gamma, \Pi_3^\star(\beta))) = \Pi_3^\star$ and $Z_1(p, \phi_2^\circ(\gamma, \Pi_3^\star(\beta)), \varphi_3)) = \Pi_1^\star$, respectively. Finally, over the range $[p_M^{(3)}, p_M]$, $\phi_1(p) = \phi_1^{\star\star}(p)$ and $\phi_2(p) = \phi_2^{\star\star}(p)$, where $\phi_1^{\star\star}(p)$ and $\phi_2^{\star\star}(p)$ are the solutions to equation $p_m K_2 = Z_2(p, \varphi_1, 1)$ and $p_m K_1 = Z_1(p, 1, \varphi_2)$, respectively. We omit the proof of uniqueness of the equilibrium, which is tedious although straightforward.

4.3 Example 3

Here is an example of a triopoly in which there is an atom in S_3 at $p_M^{(3)}$ and $\bigcap_{i=N} S_i = \{p_m^{(3)}\}$. Let $D(p) = 20 - p$, $(K_1, K_2, K_3) = (16; 5; 0.2)$. Then $p_M = 7.40$, $\Pi_1^\star = 54.76$, $p_m = 3.4225$, and $\Pi_2^\star = 17.1125$. Note that $P(K_1 + K_2) = 0 < p_m < P(K_1 + K_3) = 3.8$. In order to determine $p_m^{(3)}$, $p_M^{(3)}$, and $\Pi_3^\star$, note that, at any $p \in S_1 \cap S_2 \cap S_3$, $(\phi_1(p), \phi_2(p), \phi_3(p)) =$

$(\phi_1^\circ(p,\Pi_3^\star), \phi_2^\circ(p,\Pi_3^\star), \phi_3^\circ(p,\Pi_3^\star))$, the solution to the three-equation system

$$\begin{aligned} p_m K_i &= Z_i(p, \varphi_{-i}) \quad (i = 1, 2), \\ \Pi_3^\star &= Z_3(p, \varphi_{-3}). \end{aligned} \tag{16}$$

Such a solution is $\left(\sqrt{\frac{K_2}{K_1}\frac{pK_3-\Pi_3^\star}{pK_3}}, \sqrt{\frac{K_1}{K_2}\frac{pK_3-\Pi_3^\star}{pK_3}}, (p-p_m)\sqrt{\frac{K_1K_2}{(pK_3-\Pi_3^\star)K_3 p}} + \frac{D(p)-K_1-K_2}{K_3}\right)$ for $p \in [p_m, P(K_1 + K_3)]$. Therefore, it cannot be $L = \{1, 2, 3\}$. Otherwise $\Pi_3^\star = p_m K_3$ because of Proposition 3(iv) and, as a consequence, $\phi_3(p_m)$ would be equal to $\frac{D(p_m)-K_1-K_2}{K_3} < 0$. Furthermore, it cannot be $L = \{1, 3\}$, because of Proposition 3(ii). Therefore $L = \{1, 2\}$ and $\phi_1(p) = \phi_{12}^\star(p)$ and $\phi_2(p) = \phi_2^\star(p)$ over the range $[p_m, p_m^{(3)}]$, where $\phi_{12}^\star(p)$ and $\phi_2^\star(p)$ are defined as in Example 1. As a consequence, $\Pi_3^* > p_m K_3$ since $Z_3(p, \phi_{12}^\star(p), \phi_2^\star(p)) = p(1 - \phi_{12}^\star(p)\phi_2^\star(p))K_3$ is increasing on a right neighbourhood of p_m. This in turn implies that $p_M^{(3)} < \min\{P(K_1), p_M\} = P(K_1)$ since otherwise Proposition 3(ix) would be violated; therefore, $\phi_1(p) = \phi_{12}^{\star\star}(p)$ and $\phi_2(p) = \phi_2^{\star\star}(p)$ over the range $[p_M^{(3)}, p_M] \cap (S_1 \cap S_2)$, where $\phi_{12}^{\star\star}(p)$ and $\phi_2^{\star\star}(p)$ are defined as in Example 2.

Actually, $Z_3(p, \phi_{12}^\star(p), \phi_2^\star(p))$ is increasing over the range $(p_m, P(K_1 + K_3))$ and decreasing over the range $(P(K_1 + K_3), P(K_1))$: hence $p^{\circ\circ} := \arg\max_{p\in(p_m,P(K_1))} Z_3(p, \phi_{12}^\star(p), \phi_2^\star(p)) = P(K_1 + K_3) = 3.8$ and $\pi_m := \max_{p\in(p_m,P(K_1))} Z_3 = \frac{3212971}{4377600} \approx 0.7339571913$. A distinguishing feature of this example, though, is that even $Z_3(p; \phi_{12}^\star(P(K_1+K_3)), \phi_2^\star(P(K_1+K_3)) = Z_3(p; \phi_1^\circ(P(K_1 + K_3), \pi_m), \phi_2^\circ(P(K_1 + K_3), \pi_m))$ is decreasing on the right of $p^{\circ\circ}$. Moreover, one can easily check that there is no equilibrium in which $p_m^{(3)} = p_M^{(3)} = P(K_1 + K_3)$ since $\phi_{12}^\star(p)$ and $\phi_2^\star(p)$ are proportional to capacities of firms 1 and 2, whereas $\phi_1^{\star\star}(p)$ and $\phi_2^{\star\star}(p)$ are not.

In order to determine $p_m^{(3)}$ when $p_m^{(3)} < p^{\circ\circ}$, it must preliminarily be understood that $p \notin \cap S_1 \cap S_2 \cap S_3$ for p larger than and close enough to $p_m^{(3)}$. More specifically, there is $p' > p^{\circ\circ}$ such that $[p_m^{(3)}, p'] \cap S_2 = \{p_m^{(3)}, p'\}$.[16] Hence, on the right of $p_m^{(3)}$, whatever this may be, $\phi_1(p)$ and $\phi_3(p)$ are the solutions of the equations in φ_1 and φ_3 $\Pi_3^\star = Z_3(p, \varphi_1, \phi_2^\star(p_m^{(3)})]$ and $\Pi_1^\star = Z_1(p, \phi_2^\star(p_m^{(3)}), \varphi_3]$, respectively. But then again $Z_3(p, \phi_1(P(K_1 +$

[16]Here is a sketch of proof. If $p_m^{(3)} < p^{\circ\circ}$ then $\Pi_3^\star = Z_3(p_m^{(3)}, \phi_1^\star(p_m^{(3)}), \phi_2^\star(p_m^{(3)})) = Z_3(p, \phi_1^\circ(p,\Pi_3^\star), \phi_1^\circ(p,\Pi_3^\star)) < Z_3(p, \phi_1^\star(p), \phi_2^\star(p))$ for p larger than and close enough to $p_m^{(3)}$. Then, because of Lemma 1(v)&(v.b)&(v.f) in Appendix B, $\phi_1^\circ(p,\Pi_3^\star) > \phi_{12}^\star(p)$, $\phi_2^\circ(p,\Pi_3^\star) > \phi_2^\star(p)$, and $\phi_3^\circ(p,\Pi_3^\star) < 0$, an obvious contradiction. Thus there is p' such that either $[p_m^{(3)}, p'] \cap S_1 = \{p_m^{(3)}, p'\}$ or $[p_m^{(3)}, p'] \cap S_2 = \{p_m^{(3)}, p'\}$. It is easy to check that the former alternative leads to a contradiction.

$K_3), \phi_2(P(K_1+K_3))$ is decreasing in p on the right of $p^{\circ\circ} = P(K_1+K_3) = 3.8$, implying that $p_M^{(3)} = P(K_1+K_3)$. Finally, it is checked that an equilibrium exists in which $p_m^{(3)} < p_M^{(3)} = P(K_1+K_3)$, $\Pr(p_3 = P(K_1+K_3)) = 0.90793742 > 0$, $S_1 = [p_m, p^{\circ\circ}] \cup [p^\circ, p_M]$, $S_2 = [p_m, p_m^{(3)}] \cup [p^\circ, p_M]$, $S_3 = [p_m^{(3)}, p^{\circ\circ}]$, and $p^\circ \in (p^{\circ\circ}, P(K_1))$:

$$\phi_1(p) = \begin{cases} \phi_{12}^\star(p) = \frac{5(p-p_m)}{p(1+p)} & p \in [p_m, p_m^{(3)}] \\ \frac{pK_3 - \Pi_3^\star}{pK_3\phi_2^\star(p_m^{(3)})} & p \in [p_m^{(3)}, P(K_1+K_3)] \\ \frac{P(K_1+K_3)K_3 - \Pi_3^\star}{P(K_1+K_3)K_3\phi_2^\star(p_m^{(3)})} & p \in [P(K_1+K_3), p^\circ] \\ \phi_{12}^{\star\star}(p) = \frac{p-p_m}{p} & p \in [p^\circ; p_M] \end{cases}$$

$$\phi_2(p) = \begin{cases} \phi_2^\star(p) = \frac{16(p-p_m)}{p(1+p)} & p \in [p_m, p_m^{(3)}] \\ \phi_2^\star(p_m^{(3)}) & p \in [p_m^{(3)}, p^\circ] \\ \phi_2^{\star\star}(p) = \frac{p(19.8-p)-54.76}{5p} & p \in [p^\circ, p_M] \end{cases}$$

$$\phi_3(p) = \begin{cases} 0 & p \in [p_m, p_m^{(3)}] \\ \frac{K_1}{K_3}\frac{p-p_m}{p\phi_2^\star(p_m^{(3)})} + \frac{D(p)-K_1-K_2}{K_3} & p \in [p_m^{(3)}, P(K_1+K_3)) \\ 1 & p \in [P(K_1+K_3), p_M] \end{cases}$$

where $(p_m^{(3)}, \Pi_3^\star, p^\circ) = (3.7982466455, 0.7338170986, 3.821618795)$ is the solution to system

$$\Pi_3^\star = p_m^{(3)}\left[1 - \phi_{12}^*(p_m^{(3)})\phi_2^*(p_m^{(3)})\right]K_3 \tag{17}$$

$$\frac{P(K_1+K_3)K_3 - \Pi_3^\star}{P(K_1+K_3)K_3\phi_2^\star(p_m^{(3)})} = \phi_{12}^{\star\star}(p^\circ) \tag{18}$$

$$\phi_2^\star(p_m^{(3)}) = \phi_2^{\star\star}(p^\circ). \tag{19}$$

For the sake of brevity, we omit the proof of the uniqueness of equilibrium.

4.4 Example 4

Here is an example in which there is again an atom in S_3 at $p_M^{(3)}$, but, unlike in Example 3, $\bigcap_{i=N} S_i = [p_m^{(3)}, \widetilde{\widetilde{p}}]$ is a non-degenerate interval, $p_m^{(3)} = p^{\circ\circ}$, and $\Pi_3^\star = \pi_m$.

Let $D(p) = 30 - p$, $(K_1, K_2, K_3) = (24; 12; 0.4)$. Then $p_M = 8.8$, $\Pi_1^\star = 77.44$, $p_m = 3.22\overline{6}$, and $\Pi_2^\star = 38,72$. Note that $P(K_1 + K_2) = 0 < p_m = 3.22\overline{6} < P(K_1 + K_3) = 5.6 < P(K_1) = 6 < p_M$. Just as in Example 3, $L = \{1, 2\}$, $\Pi_3^\star > p_m K_3$ and $p_M^{(3)} < P(K_1)$. Now $p^{\circ\circ} \approx 4.168813866 < P(K_1 + K_3) = 5.6$ and $\pi_m \approx 1.430313412$. We will see that there is an equilibrium in which $p_m^{(3)} = p^{\circ\circ}$ and $\Pi_3^\star = \pi_m$. In order to characterize it, it must be noted that $\phi_3^\circ(P(K_1 + K_3), \pi_m) < 1$ and that $Z_3(p; \phi_1^\circ(P(K_1 + K_3), \pi_m), \phi_2^\circ(P(K_1 + K_3), \pi_m))$ is decreasing on the right of $P(K_1 + K_3)$. Taking into account the discussion of Example 3, there exists an equilibrium in which $p_M^{(3)} = P(K_1 + K_3)$ is an atom in S_3, $\Pr_{\phi_3}(p_3 = p_M^{(3)}) = 0.693830492$, $S_2 = [p_m, \widetilde{\widetilde{p}}] \cup [p^\circ, p_M]$ and $S_1 = [p_m, P(K_1 + K_3)] \cup [p^\circ, p_M]$, where $\widetilde{\widetilde{p}}$ and p° are to be determined. Prices $\widetilde{\widetilde{p}}$ and p° must be such that

$$\phi_2^\circ(\widetilde{\widetilde{p}}, \pi_m) = \phi_2^{\star\star}(p^\circ),$$

$$\phi_1(P(K_1 + K_3)) = \phi_1^{\star\star}(p^\circ),$$

that is

$$\sqrt{\frac{K_1}{K_2} \frac{\widetilde{\widetilde{p}} K_3 - \pi_m}{\widetilde{\widetilde{p}} K_3}} = \frac{p^\circ(D(p^\circ) - K_3) - p_m K_1}{p^\circ K_2}, \tag{20}$$

$$\frac{P(K_1 + K_3)K_3 - \pi_m}{P(K_1 + K_3)K_3 \sqrt{\frac{K_1}{K_2} \frac{\widetilde{\widetilde{p}} K_3 - \pi_m}{\widetilde{\widetilde{p}} K_3}}} = \frac{p^\circ - p_m}{p^\circ}. \tag{21}$$

The relevant solution for p° is the unique solution to the third-degree equation

$$\frac{pK_2[P(K_1 + K_3)K_3 - \pi_m]}{P(K_1 + K_3)K_3[p(D(p) - K_3) - p_m K_1]} = \frac{p - p_m}{p} \tag{22}$$

over the range $(P(K_1 + K_3), P(K_1))$, that is $p^\circ \approx 5.6161244$. Then $\widetilde{\widetilde{p}}$ is easily and uniquely found: $\widetilde{\widetilde{p}} \approx 5.594998554$. Then it is checked that the equilibrium we are looking for is given by the following distributions:

$$\phi_1(p) = \begin{cases} \phi^{\star}_{12}(p) = \frac{12[p-3.22\overline{6}]}{p[6+p]} & p \in [p_m, p_m^{(3)}] \\ 1.118033989\sqrt{\frac{0.4p-1.430313412}{p}} & p \in [p_m^{(3)}, \widetilde{\widetilde{p}}] \\ \frac{2.942616835(0.4p-1.430313412)}{p} & p \in [\widetilde{\widetilde{p}}, P(K_1+K_3)] \\ 0.4254638187 & p \in [P(K_1+K_3), p^{\circ}] \\ \phi^{\star\star}_{12}(p) = \frac{p-3.22\overline{6}}{p} & p \in [p^{\circ}; p_M] \end{cases}$$

$$\phi_2(p) = \begin{cases} \phi^{\star}_{2}(p) = \frac{24[p-3.22\overline{6}]}{p[6+p]} & p \in [p_m, p_m^{(3)}] \\ 2.236067978\sqrt{\frac{0.4p-1.430313412}{p}} & p \in [p_m^{(3)}, \widetilde{\widetilde{p}}] \\ 0.8495839383 & p \in [\widetilde{\widetilde{p}}, p^{\circ}] \\ \phi^{\star\star}_{2}(p) = \frac{p(29.6-p)-77.44000001}{12p} & p \in [p^{\circ}, p_M] \end{cases}$$

$$\phi_3(p) = \begin{cases} 0 & p \in [p_m, p_m^{(3)}] \\ \frac{26.83281574(p-3.22\overline{6})}{0.4p--1.4303134127}\sqrt{\frac{0.4p-1.430313412}{p}} - 15 - 2.5p & p \in [p_m^{(3)}, \widetilde{\widetilde{p}}) \\ 2.942616835 \times \frac{24p-77.44000001-0.8495839383p(6+p)}{p} & p \in [\widetilde{\widetilde{p}}, P(K_1+K_3)) \\ 1 & p \in [P(K_1+K_3), p_M] \end{cases}$$

For the sake of brevity, we omit the proof of the uniqueness of equilibrium.

5 Concluding remarks

This paper has first shown what equilibrium features which need to hold in the no-pure strategy equilibrium region of a duopoly need to hold in the analogous region of an oligopoly. Secondly, it has shown how varied equilibrium features can be, in that region, under oligopoly: unlike in the duopoly, the supports of the equilibrium strategies need not coincide, the supports are not always connected for all the firms, and even the union of the supports may not be connected. Indeed, it is not connected when there is an atom in the support of a firm with a capacity lower than the largest one, and we have analized the logic behind the event of such an atom. Furthermore, Examples 3 and 4 have constructively proved for the triopoly that such an atom can arise in the support of the smallest firm's equilibrium strategy at its maximum price. We have also proved that the minimum of the support of the second largest firm is either p_m, the minimum of the union of the supports, or a price not below a threshold - that we have identified - in such a way that its equilibrium profit is in any case equal to $p_m K_2$. This property has been extended to other firms with an equilibrium payoff-capacity ratio equal to p_m. Finally we have proved that, while equilibrium distributions are necessarily symmetric for firms with the same largest capacity, this need not be the case for equally-sized firms smaller than the largest one and we have made explicit a number of properties concerning this issue.

Further research would clearly be needed in order to fully characterize equilibria throughout the entire no-pure strategy equilibrium region. In particular, a task still to be fully accomplished is that of finding the equilibrium payoffs of firms that are smaller than the two largest ones. Other research questions that also deserve to be addressed are whether uniqueness of each firm's equilibrium payoffs generally holds in the event of multiple equilibria and whether multiple equilibria necessarily mean a continuum of equilibria, as suggested by the results obtained so far. We are confident that the results of this paper will provide helpful insights for substantial progress on these and other relevant issues concerning mixed strategy equilibria.

6 Appendix A

Proof of Proposition 1

(i) Let $K \geqslant X$. A best response for firm i to all rivals charging $p^c = 0$ is p^c if and only if $\sum_{j \neq i} K_j \geqslant X$. This holds for each i if and only if $\sum_{j \neq 1} K_j \geqslant X$. Any strategy profile such that $\sum_{s \in \Omega(0)-\{j\}} K_s \geqslant X$ for each $j \in \Omega(0) \neq \emptyset$ is an equilibrium: all firms get a profit equal to 0, but no firm may get more by deviating. Let $X > K$. A best response for firm i to all rivals charging $p^c > 0$ is p^c if and only if $\left[d[p(D(p) - \sum_{j \neq i} K_j)]/dp\right]_{p=p^c} \leqslant 0$. Also because of Assumption 2, this holds for each i if and only if $K_1 \leqslant -p^c \left[D'(p)\right]_{p=p^c}$. There are no further equilibria. First of all $\max \cup_{j \in N} S_j = \overline{p} \geqslant p^c$ since no $p < p^c$ can be in S_i, any $i \in N$, otherwise firm i can obtain a profit of $p^c K_i > p K_i$. Then it is enough to prove that $\overline{p} = p^c$. By way of contradiction, let $p^c < \overline{p} \in S_i$ for some i and assume first that $\overline{p}$ is charged with zero probability by any firm $j \neq i$ - whether firm i charges $\overline{p}$ with zero or positive probability. Then firm i earns a profit lower than $p^c K_i$ by charging $\overline{p}$ or a price very close to $\overline{p}$. If, instead, $\overline{p}$ is charged with positive probability by more than one firm, then for at least one of them it would pay to charge a price slightly less than $\overline{p}$: indeed, expected output when charging $\overline{p}$ is lower than $\min\{K_i, D(\overline{p}) - \sum_{j:p_j<\overline{p}} K_j\}$ whereas slightly undercutting the other most expensive firms would make residual demand jump to $\min\{K_i, D(\overline{p}) - \sum_{j:p_j<\overline{p}} K_j\}$.

(ii) In the assumed circumstances $(p^c, p^c, \dots, p^c)$ is not an equilibrium. Hence we just have to rule out pure strategy profiles such that $\overline{p} := \max_{j\in N} p_j > p^c$. Assume first that $D(\overline{p}) - \sum_{j:p_j<\overline{p}} K_j > 0$. If $\#\Omega(\overline{p}) < n$, then any firm $h \notin \Omega(\overline{p})$ is selling its entire capacity, but it would still do so if it raised the price to any level less than $\overline{p}$. If $\#\Omega(\overline{p}) = n$, then residual demand is less than capacity for at least some firm $j \in \Omega(\overline{p})$, whereas its output would jump up by undercutting rivals. Next assume $D(\overline{p}) - \sum_{j:p_j<\overline{p}} K_j \leqslant 0$. Firm i, any $i \in \#\Omega(\overline{p})$ has failed to make a best response unless $p^c = 0$ and $\sum_{j\in\Omega(0)} K_j \geqslant X$. The first condition implies that $K \geqslant X$. The second implies a contradiction: if $1 \notin \Omega(0)$, inequality (5) cannot hold; if $1 \in \Omega(0)$, firm 1 has not made a best response because of inequality (5). ■

7 Appendix B

This appendix is devoted to providing proofs to Propositions 3(vii)-(ix) and 4-7. Some of these proofs are obtained by exploiting the properties of functions $Z_i(p;\varphi_{-i})$ (see equations (6)) when $p \in (p_m, p_M)$ and therefore the following inequalities hold

A.1 $0 < \varphi_1 < 1$, because of Proposition 2;

A.2 $\varphi_h > 0$ for some $h \neq 1$, because of Proposition 3(i);

A.3 the set $\varpi_i = \{h \in N_{-i} : \varphi_h > 0\}$ is such that $D(p) < K_i + \sum_{h\in\varpi_i} K_h$, because of Proposition 3(ii) since for $p > p_m$, $D(p) < D(p_m) < \sum_{j\in L} K_j \leqslant \sum_{i:p_m^{(i)}\leqslant p} K_i$;

A.4 $D(p) > K - K_1$, since $D(p) > D(p_M)$.

These properties will be explored by the following Lemma 1. Sometimes we factorize φ_j and $(1-\varphi_j)$ (some $j \neq i$) in equation (6) to obtain

$$Z_i(p;\varphi_{-i-j},\varphi_j) = \varphi_j Z_i(p;\varphi_{-i-j},1) + (1-\varphi_j) Z_i(p;\varphi_{-i-j},0). \tag{23}$$

In equation (23) $Z_i(p;\varphi_{-i-j},1)$ and $Z_i(p;\varphi_{-i-j},0))$ have a clear interpretation: they are the payoffs of firm i when it charges p with certainty conditional on $p_j < p$ and $p_j > p$, respectively, in the assumption that $\varphi_r = \Pr(p_r < p)$ and $1-\varphi_r = \Pr(p_r > p)$, any $r \neq i, j$.

Lemma 1 Function $Z_i(p;\varphi_{-i})$, defined by equation (6), has the following properties when $p \in (p_m, p_M)$ and therefore inequalities **A.1**, **A.2**, **A.3**, and **A.4** hold.

(i) $0 < Z_i(p;\varphi_{-i}) < pK_i$ (each i), and (i.a) $Z_i(p;\varphi_{-i}) \geqslant (1-\varphi_1)pK_i$ (each $i \neq 1$).

(ii) $Z_i(p;\varphi_{-i})$ (each i) is continuous, and almost everywhere twice differentiable in p throughout (p_m, p_M). Singular points may only exist at $p = P(K_1 + \sum_{j\in\omega} K_j) \in (p_m, p_M)$, where $\omega \in \mathcal{P}(N_{-1})$. More specifically,

- if either $i = 1$ or $i \neq 1$ and $i \in \omega$, then at $p = P(K_1 + \sum_{j\in\omega} K_j)$ the right derivative jumps down by $-\left[pD'(p)\right]_{p=P(K_1+\sum_{j\in\omega} K_j)} \prod_{r\in\psi} \varphi_r \prod_{s\in N_{-i}-\psi}(1-\varphi_s)$, where $\psi = \omega \cup \{1\} - \{i\}$;
- if $i \neq 1$ and $i \notin \omega$, then at $p = P(K_1 + \sum_{j\in\omega} K_j)$ the right derivate jumps up by $-\left[pD'(p)\right]_{p=P(K_1+\sum_{j\in\omega} K_j)} \prod_{r\in\psi} \varphi_r \prod_{s\in N_{-i}-\psi}(1-\varphi_s)$, where $\psi = \omega \cup \{1\}$.

Jumps up and down may happen simultaneously when there is χ such that $\omega \cap \chi = \emptyset$, $\sum_{j\in\chi} K_j = K_i$, and either $i = 1$ or $i \neq 1$ and $i \in \omega$.

(iii) $Z_i(p;\varphi_{-i})$ is concave in p over any range in which it is differentiable and if it is not strictly concave, then it is increasing in p.

(iv) $Z_1(p;\varphi_{-1})$ is strictly concave and increasing in p.

(v) $Z_i(p;\varphi_{-i})$ is continuous and differentiable in φ_j and $\partial Z_i/\partial\varphi_j \leqslant 0$ (each i and $j \neq i$). Moreover, (v.a) $\partial Z_i/\partial\varphi_j = 0$ if and only if there is no $\psi \in \mathcal{P}(N_{-i})$ such that $\varpi_i - \{j\} \supseteq \psi \supseteq \Xi$, where $\Xi = \{h \in \varpi_i - \{j\} : \varphi_h = 1\}$, and $K_i + K_j + \sum_{h\in\psi} K_h > D(p) > \sum_{h\in\psi} K_h$, so that

$$Z_i(p;\varphi_{-i}) = \sum_{\psi\in\Psi} \prod_{h\in\psi} \varphi_h \prod_{h\in\varpi_i-\{j\}-\psi} (1-\varphi_h)pK_i, \tag{24}$$

where $\Psi = \{\psi \in \mathcal{P}(N_{-i}) : \varpi_i - \{j\} \supseteq \psi \supseteq \Xi, D(p) \geqslant K_i + K_j + \sum_{h\in\psi} K_h\}$. More precisely, (v.b) $\partial Z_i/\partial\varphi_j = 0$ if and only if there is $\chi \subseteq \varpi_i - \{j\} - \Xi$ such that $\sum_{h\in\chi\cup\Xi} K_h \geqslant D(p) \geqslant K_i + \sum_{h\in\varpi_i\cup\{j\}} K_h - K_m$, where $m = \max\chi$, so that

$$Z_i(p;\varphi_{-i}) = (1 - \Pi_{h\in\chi}\varphi_h)pK_i. \tag{25}$$

Note that (v.c) $K_m \geqslant K_i + K_j$; (v.d) $\partial Z_i/\partial\varphi_j = 0$ if and only if $\partial Z_j/\partial\varphi_i = 0$; (v.e) if $l \in \varpi_i - \{j\} - \Xi - \chi$, then $\partial Z_i/\partial\varphi_l = 0$; (v.f) $\partial Z_1/\partial\varphi_h < 0$ and $\partial Z_h/\partial\varphi_1 < 0$, each $h \neq 1$; (v.g) $\partial Z_2/\partial\varphi_h = 0$ and $\partial Z_h/\partial\varphi_2 = 0$ (each $h \neq 1, 2$) if and only if $p \geqslant P(K_1 + \sum_{r\in\Xi} K_r)$.

(vi) If $K_1 > K_2 = K_j$, $j > 2$, and $\varphi_2 \geqslant \varphi_j$, then $Z_2(p;\varphi_{-2}) \geqslant Z_j(p;\varphi_{-j})$, with $Z_2(p;\varphi_{-2}) > Z_j(p;\varphi_{-j})$ if and only if $\varphi_2 > \varphi_j$ and $\partial Z_2/\partial\varphi_j < 0$.

(vii) If $K_1 > K_i \geqslant K_j$, $j > i$, and $\varphi_j \geqslant \varphi_i = 0$, then $(K_j/K_i)Z_i(p;\varphi_{-i}) \leqslant Z_j(p;\varphi_{-j})$; strict inequality holds if $\partial Z_i/\partial\varphi_j < 0$ and $\varphi_j > 0$.

Proof

(i) $Z_i(p;\varphi_{-i})$ is a sum of nonnegative functions; hence to prove that $Z_i(p;\varphi_{-i}) > 0$ it is enough to prove that at least one of the addends is positive. Let $\chi = \varpi_i - \{1\}$. Then $q_{i,\chi}(p)\prod_{r\in\chi}\varphi_r\prod_{s\in N_{-i}-\chi}(1-\varphi_s) > 0$. Indeed, if $i \neq 1$, then $q_{i,\chi}(p) = K_i$ and $\prod_{r\in\chi}\varphi_r\prod_{s\in N_{-i}-\chi}(1-\varphi_s) = (1-\varphi_1)\prod_{r\in\varpi_i-\{1\}}\varphi_r > 0$, whereas, if $i = 1$, then $0 < q_{1,\chi}(p) < K_1$ and $\prod_{r\in\chi}\varphi_r\prod_{s\in N_{-i}-\chi}(1-\varphi_s) = \prod_{r\in\chi}\varphi_r > 0$.[17]

$Z_i(p;\varphi_{-i})$ is an average of functions lower than or equal to pK_i; hence to prove that $Z_i(p;\varphi_{-i}) < pK_i$ it is enough to prove that at least one of them is lower than pK_i and its weight is positive. Indeed, $q_{i,\varpi_i}(p) = \max\{0, D(p) - \sum_{j\in\varpi_i}K_j\} < K_i$ and $\prod_{r\in\varpi_i}\varphi_r\prod_{s\in N_{-i}-\varpi_i}(1-\varphi_s) = \prod_{r\in\varpi_i}\varphi_r > 0$.

(i.a) $Z_i(p;\varphi_{-1-i},\varphi_1) \geqslant (1-\varphi_1)Z_i(p;\varphi_{-1-i},0)$ because of equation (23) and part (i). $Z_i(p;\varphi_{-1-i},0)$ is an average of functions $pq_{i,\psi}(p)$; but if $\psi \in N_{-1-i}$, then $pq_{i,\psi}(p) = pK_i$.

(ii) For given φ_{-i}, $Z_i(p;\varphi_{-i})$ is a weighted arithmetic mean of functions $pq_{i,\psi}(p)$, each of which is almost everywhere twice differentiable. Singular points and their properties are easily determined by taking into account the fact that $pq_{i,\psi}(p)$ has two singular points, one at $p = P(K_i + \sum_{j\in\psi}K_j)$, where $pq_{i,\psi}(p)$ is locally concave with the right derivative jumping down by $-\left[pD'(p)\right]_{p=P(K_i+\sum_{j\in\psi}K_j)}$, and one at $p = P(\sum_{j\in\psi}K_j)$, where $pq_{i,\psi}(p)$ is locally convex with the right derivative jumping up by $-\left[pD'(p)\right]_{p=P(\sum_{j\in\psi}K_j)}$. Note that if there is χ such that $\omega\cap\chi = \emptyset$, $\sum_{j\in\chi}K_j = K_i$, and either $i = 1$ or $i \neq 1$ and $i \in \omega$, then $q_{i,\omega\cup\{1\}}(p) = K_i$ and $q_{i,\omega\cup\chi\cup\{1\}}(p) = 0$ at $p = P(K_1 + \sum_{j\in\omega}K_j)$.

(iii) Functions $pq_{i,\psi}(p)$ are either strictly concave or linearly increasing whenever they are differentiable and positive.

(iv) Since $\sum_{j\in\psi}K_j < D(p_M)$ each $\psi \subseteq N_{-1}$, $pq_{1,\psi}(p)$ is increasing, concave, and positive. To prove strict concavity it is enough to remark that $pq_{1,\varpi}(p)$ is strictly concave and its weight is positive.

(v) For any given p and φ_{-i-j}, $Z_i(p;\varphi_j,\varphi_{-i-j})$ is a polynomial of degree 1 (or lower) in φ_j. Hence $Z_i(p;\varphi_j,\varphi_{-i-j})$ is everywhere continuously differentiable with respect to φ_j. Partial differentiation of (23) yields $\frac{\partial Z_i}{\partial\varphi_j} = Z_i(p;\varphi_{-i-j},1) - Z_i(p;\varphi_{-i-j},0) \leqslant 0$.

(v.a) $\frac{\partial Z_i}{\partial\varphi_j} = 0$ if and only if $Z_i(p;\varphi_{-i})$ is independent of φ_j, that is $Z_i(p;\varphi_{-i}) = Z_i(p;\varphi_{-i-j},1) = Z_i(p;\varphi_{-i-j},0)$. This requires that $q_{i,\psi}(p) =$

[17] Note that the argument holds also when $\varpi_i - \{1\} = \emptyset$.

$q_{i,\psi\cup\{j\}}(p)$ for each $\psi \subseteq N-\{i,j\}$ such that $\prod_{h\in\psi}\varphi_h \prod_{s\in N_{-i-j-\psi}}(1-\varphi_s) > 0$, that is $\varpi_i - \{j\} \supseteq \psi \supseteq \Xi$. Note that $q_{i,\psi}(p) \geqslant q_{i,\psi\cup\{j\}}(p)$ and $q_{i,\psi}(p) = q_{i,\psi\cup\{j\}}(p)$ when either $q_{i,\psi\cup\{j\}}(p) = K_i$ or $q_{i,\psi}(p) = 0$. In other words $Z_i(p;\varphi_{-i})$ is independent of φ_j if and only if for each $\psi \subseteq N-\{i,j\}$ such that $\varpi_i - \{j\} \supseteq \psi \supseteq \Xi$ either

1. $D(p) \leqslant \sum_{h\in\psi} K_h$, or
2. $D(p) \geqslant K_i + K_j + \sum_{h\in\psi} K_h$.

Hence if $\varpi_i - \{j\} \supseteq \psi \supseteq \Xi$, the two inequalities $K_i + K_j + \sum_{h\in\psi} K_h > D(p) > \sum_{h\in\psi} K_h$ cannot both hold.

(v.b) Let us first prove the if part. Let $\varpi_i - \{j\} \supseteq \psi \supseteq \Xi$. If $\psi \supseteq \chi$, the first alternative mentioned in the proof of part (v.a) holds. If $\psi \not\supseteq \chi$, then there is $l \in \chi$ such that $\psi \subseteq \varpi_i - \{l,j\}$ and therefore $\sum_{h\in\psi} K_h \leqslant \sum_{h\in\varpi_i-\{l,j\}} K_h \leqslant \sum_{h\in\varpi_i-\{m,j\}} K_h$. Hence the second alternative holds. Let us prove now the only if part. Let $\chi_l = \chi_{l-1} - \{m_{l-1}\}$, where $m_{l-1} = \max \chi_{l-1}$ and $\chi_0 = \varpi_i - \{j\} - \Xi$. Since $K_i + \sum_{h\in\varpi_i} K_h > D(p)$, we obtain from part (v.a) that $\sum_{h\in\chi_0\cup\Xi} K_h \geqslant D(p)$. If $D(p) \geqslant K_i + \sum_{h\in\varpi_i\cup\{j\}} K_h - K_{m_0}$, then $\chi_0 = \chi$. If $D(p) < K_i + \sum_{h\in\varpi_i\cup\{j\}} K_h - K_{m_0}$, then, because of part (v.a), $\sum_{h\in\chi_1\cup\Xi} K_h \geqslant D(p)$. Then, by iterating the same procedure we obtain χ since $D(p) > K_i + \sum_{h\in\varpi_i\cup\{j\}} K_h - K_1 \geqslant K - K_1$.

(v.c) Inequality $\sum_{h\in\chi} K_h \geqslant K_i + \sum_{h\in\varpi_i\cup\{j\}} K_h - K_m$ implies that $K_m - K_i \geqslant \sum_{h\in\varpi_i\cup\{j\}-\chi} K_h \geqslant K_j$.

(v.d) A straightforward consequence of the fact that $K_i + \sum_{h\in\varpi_i\cup\{j\}} K_h = K_j + \sum_{h\in\varpi_j\cup\{i\}} K_h$.

(v.e) Clearly $\chi \subseteq \varpi_i - \{j\} - \{l\} - \Xi \subseteq \varpi_i - \{l\} - \Xi$ and $D(p) \geqslant K_i + \sum_{h\in\varpi_i\cup\{j\}} K_h - K_m \geqslant K_i + \sum_{h\in\varpi_i\cup\{l\}} K_h - K_m$ since $l \in \varpi_i$.

(v.f) Follows straightforwardly from parts (v.c) and (v.d), according to which neither i nor j can be 1.

(v.g) Follows straightforwardly from parts (v.c) and (v.d), according to which if either $i = 2$ or $j = 2$, then $m = 1$, and therefore $\chi = \{1\}$.

(vi) Since $K_2 = K_j$, $Z_2(p;\varphi_{-2-j},\beta) = Z_j(p;\varphi_{-2-j},\beta)$. Hence, taking into account equation (23), $Z_2(p;\varphi_{-2}) - Z_j(p;\varphi_{-j}) = \varphi_j Z_2(p;\varphi_{-2-j},1) + (1-\varphi_j)Z_2(p;\varphi_{-2-j},0) - \varphi_2 Z_2(p;\varphi_{-2-j},1) - (1-\varphi_2)Z_2(p;\varphi_{-2-j},0) = (\varphi_j - \varphi_2)\left[Z_2(p;\varphi_{-2-j},1) - Z_2(p;\varphi_{-2-j},0)\right] = (\varphi_j - \varphi_2)\partial Z_2/\partial\varphi_j$. Part (v) completes the proof.

(vii) From equation (23), we obtain that $(K_j/K_i)Z_i(p;\varphi_{-i}) - Z_j(p;\varphi_{-j}) = \varphi_j(K_j/K_i)[Z_i(p;\varphi_{-i-j},1) - Z_i(p;\varphi_{-i-j},0)] + [(K_j/K_i)Z_i(p;\varphi_{-i-j},0) - Z_j(p;\varphi_{-i-j},0)]$. The first bracket is strictly negative if $\partial Z_i/\partial\varphi_j < 0$, otherwise it is non-positive; the second bracket is non-positive since $(K_j/K_i)q_{i,\psi}(p) - q_{j,\psi}(p) \leqslant 0$, where $\psi \in N_{-i-j}$. ■

Proof of parts (vii)-(ix) of Proposition 3

(vii) Suppose contrariwise that $\Pi_i(p') < \Pi_i^\star$ for some p' in the interior of S_i. The inequality can only arise if $\Pr_{\phi_j}(p_j = p') > 0$ (some $j \neq i$); but then *a fortiori* $\Pi_i(p) < \Pi_i^\star$ on a right neighborhood of p', contrary to p' being internal to S_i. The event of $\Pi_i(p_m^{(i)}) < \Pi_i^\star$ is similarly ruled out.

(viii) By way of contradiction, let $\phi_{-i}(p)$ be constant over a subset of α, say $\phi_{-i}(p) = \phi_{-i}(p')$. Then $d\Pi_i(p)/dp = \partial Z_i(p;\phi_{-i}(p'))/\partial p \neq 0$ over that subset of S_i; the inequality derives from Lemma 1(iii)-(iv).

(ix) If $i \in L$, the claim holds because of part (iv). Let $i \notin L$; because of parts (i) and (iv) $\Pi_j^\star = p_m K_j$ for $j \in L - \{1\} \neq \emptyset$. If $p \geqslant P(K_1 + \sum_{h:p_M^{(h)}<p_M^{(i)}} K_h)$, then $D(p) \leqslant K_1 + \sum_{h:p_M^{(h)}<p_M^{(i)}} K_h$ and therefore $\Pi_i(p) = (1-\phi_1(p))pK_i$; hence if $\Pi_i(p) > p_m K_i$, then $\Pi_j(p) \geqslant (1-\phi_1(p))pK_j > p_m K_j$ and firm j has not made a best response; the first inequality is a consequence of Lemma 1(i.a). ■

Proof of Proposition 4

(i) Otherwise firm j has not made a best response by charging $\widetilde{p}$ with positive probability.

(ii.a) Otherwise, in a right neighborhood of $\widetilde{p}$, $Z_j(p;\phi_{-j}(p)) = (1 - \Pi_{h\in\chi}\phi_h(p))pK_j$, because of Lemma 1(v.b), while $Z_h(p;\phi_{-h}(p)) < \Pi_h^\star$ (each $h \in \chi$). Then $\phi_{-j}(p) = \phi_{-j}(\widetilde{p})$ on that neighbourhood and hence, as a consequence of part (i), $Z_j(p;\phi_{-j}(p)) > \Pi_j^\star$.

(ii.b) If $p_m^{(h)} = \widetilde{p}$, then $\widetilde{p} \in S_h$, part (ii.a) applies and $\left[\frac{\partial Z_j}{\partial \varphi_h}\right]_{\varphi_{-j}=\phi_{-j}(\widetilde{p})} < 0$ because of Lemma 1(v)&(v.b). Hence $\Pi_h(\widetilde{p}) < \Pi_h(\widetilde{p}-) \leqslant \Pi_h^\star$ and $p_m^{(h)} > \widetilde{p}$.

(iii) Proposition 3(iii) implies that $p_m < \widetilde{p}$. Let $\widetilde{p} \geqslant P(K_1)$: then $\Pi_j(p) = p(1-\phi_1(\widetilde{p}))K_j > \Pi_j^\star$ for p larger than and close enough to $\widetilde{p}$, which is an obvious contradiction. The inequality is a consequence of part (i).

(iv) It is an obvious consequence of part (ii).

(v) Because of part (ii), $\Pi_h(p) < Z_h(\widetilde{p},\phi_{-h}(\widetilde{p})) \leqslant \Pi_h^\star$ (each $h \neq j$) on a right neighbourhood of $\widetilde{p}$ and hence $(\widetilde{p},p^\circ) \cap (\cup_{i\neq j} S_j) = \emptyset$ (some $p^\circ \in (\widetilde{p}, p_M)$); therefore, by Proposition 3(vii), $(\widetilde{p},p^\circ) \cap S_j = \emptyset$ too.

(vi) Otherwise $\Pi_j(p) > \Pi_j(\widetilde{p}) = \Pi_j^\star$ for p larger than and close enough to $\widetilde{p}$ because of parts (i) and (v) and Lemma 1(iii)-(iv).

(vii) Otherwise, because of Lemma 1(iv), part (vi) cannot hold. ■

Proof of Proposition 5

(i) See the proof of Proposition 1(iv) in De Francesco and Salvadori (2022).

(ii) By way of contradiction, let $\phi_j(p') < \phi_1(p')$ at some $p' \in (p_m, p_M)$. Then, because of Lemma 1(v.f), $\Pi_j(p') < Z_1(p', \phi_{-1}(p')) \leqslant \Pi_1^\star = \Pi_j^\star$ and hence $p' \notin S_j$. Then there should be $p'' := \min\{p \in S_j : p > p'\}$ and $\phi_j(p) = \phi_j(p')$ in the whole range (p', p''); but this is not possible because of Lemma 1(v.f).

(iii) If $p_M^{(j)} = p_M$, then $0 < \Pi_j^\star = Z_j(p_M; \phi_{-j}(p_M)) = \sum_{i \neq j} p_M q_i(p_M)$.Then, $\frac{\partial Z_j(p;1,1,\ldots,1)}{\partial p} \mid_{p=p_M} = D(p_M) + p_M D'(p_M) - K + K_j < D(p_M) + p_M D'(p_M) - K + K_1 = 0$ whenever $K_j < K_1$, and therefore $\Pi_j(p)$ is decreasing in a left neighborhood of $p_M^{(j)} = p_M$ whenever $K_j < K_1$. ■

Proof of Proposition 6

(i) By way of contradiction, let $\frac{\Pi_i^\star}{K_i} > \frac{\Pi_j^\star}{K_j}$. Then $p_m^{(i)} > p_m$ because of Proposition 3(iv) and Corollary 4. But then $\Pi_i^\star = [Z_i(p, \varphi_{-i-j}, \varphi_j)]_{p=p_m^{(i)}, \varphi=\phi(p_m^{(i)})} \leqslant \frac{K_i}{K_j} [Z_j(p, \varphi_{-i-j}, 0)]_{p=p_m^{(i)}, \varphi=\phi(p_m^{(i)})} \leqslant \frac{K_i}{K_j} \Pi_j^\star$; the first weak inequality is a consequence of Lemma 1(vii).

(ii.a) It is an obvious consequence of Proposition 3(iv), Corollary 4, and part (i).

(ii.b) The set $\{h \in L : K_h \leqslant K_i\}$ is not empty since $j \in \{h \in L : K_h \leqslant K_i\}$. Because of part (ii.a) the claim holds if $p_m^{(i)} = p_m$. Let $p_m^{(i)} > p_m$. Then, either $[\partial Z_i / \partial \varphi_l]_{\varphi=\phi(p_m^{(i)})} = 0$ or $[\partial Z_i / \partial \varphi_l]_{\varphi=\phi(p_m^{(i)})} < 0$, each $l \in \{h \in L : K_h \leqslant K_i\}$, because of Lemma 1(v). The latter case leads to a contradiction: $p_m K_i = (K_i/K_l)\Pi_l^\star \geqslant (K_i/K_l) Z_l(p_m^{(i)}; \phi_{-l}(p_m^{(i)})) > Z_i(p_m^{(i)}; \phi_{-i}(p_m^{(i)})) = \Pi_i^\star = p_m K_i$. The equalities and the first inequality are obvious. The second inequality is a consequence of Lemma 1(vii) since $[\partial Z_i / \partial \varphi_l]_{\varphi=\phi(p_m^{(i)})} < 0$ and $\phi_l(p_m^{(i)}) > \phi_i(p_m^{(i)}) = 0$. Hence $[\partial Z_i / \partial \varphi_l]_{\varphi=\phi(p_m^{(i)})} = 0$ each $l \in \{h \in L : K_h \leqslant K_i\}$ and the claim is obtained by Lemma 1(v.b)&(v.d).

(ii.c) A straightforward consequence of part (ii.b) and Lemma 1(v.a)&(v.b).

(iii.a) An obvious consequence of part (i).

(iii.b) $Z_j(p'; \phi_{-j}(p')) \leqslant Z_i(p'; \phi_{-i}(p'))$ because of Lemma 1(v). If $Z_j(p'; \phi_{-j}(p')) < Z_i(p'; \phi_{-i}(p'))$, then $p' \in S_i - S_j$. If $Z_j(p'; \phi_{-j}(p')) = Z_i(p'; \phi_{-i}(p'))$, then $[\partial Z_i / \partial \varphi_j]_{\varphi=\phi(p')} = 0$ and the claim is obtained by Lemma 1(v.b)&(v.d).

(iii.c) Otherwise, by Lemma 1(v.b)&(v.d), $\Pi_j^\star = Z_j(p'; \phi_{-j}(p')) < Z_i(p'; \phi_{-i}(p')) = \Pi_i^\star$, contrary to part (iii.a).

(iii.d) Since parts (iii.b)-(iii.c) hold, $p' \in S_i - S_j$. Let $p'' = \min\,[p', p_M] \cap S_j$. Clearly, $S_j \cap [p', p'') = \emptyset$, $p'' \in S_j$, and $\phi_j(p'') = \phi_j(p') < \phi_i(p') \leqslant \phi_i(p'')$;

then there is $\chi \subseteq \{h : p_m^{(h)} \leqslant p''\} - \{h \in L : K_h \leqslant K_i\}$ such that $\Pi_i(p'') = (1 - \Pi_{h\in\chi}\phi_h(p''))p''K_i$ because of parts (iii.b) and (iii.c), whether $p'' \in S_i$ or not. ■

Proof of Proposition 7

(i) If $K_1 = K_2$, the claim holds because of Corollaries 2 and 3. If $K_1 > K_2$, the claim holds because of Propositions 6(i) and 3(i)&(iv).

(ii)-(iii) Are obvious consequences of Proposition 6(ii)&(iii), respectively, and Lemma 1(v.g).

■

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