General Constrained Dynamic Models in Economics - General Dynamic Theory of Economic Variables - Beyond Walras and Keynes

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General *Dynamic* Theory of Economic Variables

Beyond Walras and Keynes

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Abstract

For more than 100 years economists have tried to describe economics in analogy to physics, more precisely to classical Newtonian mechanics. The development of the Neoclassical General Equilibrium Theory has to be understood as the result of these efforts. But there are many reasons why General Equilibrium Theory is inadequate: 1. No genuine dynamics. 2. The assumption of the existence of utility functions and the possibility to aggregate them to one “master” utility function. 3. The impossibility to describe situations as in “Prisoners Dilemma”, where individual optimization does not lead to a collective optimum. This book aims at overcoming these problems. It illustrates how not only equilibria of economic systems, but also the general dynamics of these systems can be described in close analogy to classical mechanics.

To this end, this book makes the case for an approach based on the concept of constrained dynamics, analyzing the economy from the perspective of “economic forces” and “economic power” based on the concept of physical forces and the reciprocal value of mass. Realizing that accounting identities constitute constraints in the economy, the concept of constrained dynamics, which is part of the standard models of classical mechanics, can be applied to economics. Therefore, it is reasonable to denote such models as General Constraint Dynamic Models (GCD-Models)

Such a framework allows understanding both Keynesian and neoclassical models as special cases of GCD-Models in which the power relationships with respect to certain variables are one-sided. As mixed power relationships occur more frequently in reality than purely one-sided power constellations, GCD-models are better suited to describe the economy than standard Keynesian or Neoclassic models.

A GCD-model can be understood as “Continuous Time”, “Stock Flow Consistent”, “Microfounded”, where the behaviour of the agents is described with a general differential equation for every agent. In the special case where the differential equations can be described with utility functions, the behaviour of every agent can be understood as an individual optimization strategy. He thus seeks to maximize his utility. However, while the core assumption of neoclassical models is that due to the “invisible hand” such egoistic individual behaviour leads to an optimal result for all agents, reality is often defined by “Prisoners Dilemma” situations, in which individual optimization leads to the worst outcome for all. One advantage of GCD-models over standard models is that they are able to describe also such situations, where an individual optimization strategy does not lead to an optimum result for all agents.

In conclusion, the big merit and effort of Newton was, to formalize the right terms (physical force, inertial mass, change of velocity) and to set them into the right relation. Analogously the appropriate terms of economics are economic force, economic power and change of variables. GCD-Models allow formalizing them and setting them into the right relation to each other.

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1. Introduction

For more than 100 years economists have tried to describe the economy in analogy to physics, more precisely to classical mechanics. The neoclassical General Equilibrium Theory has to be understood as the result of these efforts. But the orientation of economics towards physics has been implemented only partially, especially the dynamics of mechanical systems have been omitted completely. So, Field medalist Steve Smale stated in 1998 (Smale 1991; 1997; 1998; Smale Institute 2003) as problem No. 8 of 18 major problems of dynamics to extend the mathematical model of general equilibrium theory to include the dynamics of price adjustments. This book therefore seeks to analyze the dynamics of economic models in perfect analogy to Newtonian mechanics. It shows that not only equilibria, but also the general dynamics of an economic system with all its disequilibria for all variables (including price) can be described using the framework provided by classical mechanics. We refer to the corresponding models as General Constrained Dynamic models (GCD models). They seem to be a contribution to the solution of Steve Smale's problem No. 8.

The formalization of the physical concepts of force and mass by Isaac Newton revolutionized physics and was the basis for the entire following development of the discipline. Similarly, this book aims at developing a formalization of the concepts of economic force and economic power in order to establish a single consistent structure for the description of economic systems. Within these analogies economic power corresponds to the reciprocal value of mass.

The book is divided into 7 sections:

A. Basic Principles,	chap. 2 – 5
B. Microeconomic models	chap. 6
C. Macroeconomic models	chap. 7 – 19
D. Supply, demand and price shock models	chap. 20 – 23
E. GCD with intertemporal utility functions	chap. 24 – 27
F. Summary	chap. 28

In **Section A**, we provide a historical review of attempts to find similarities between economics and physics, and explain why and how GCD models make an essential new genuine contribution to this effort.

Chapter 2 provides a short overview over the historic attempts to find similarities between economics and physics.

In Chapter 3, we illustrate the main ideas of this approach. It describes the formal structure of such "General Constrained Dynamic Models" (GCD models), which is based on the concepts of economic force and economic power, where the concept of economic power is inversely proportional to the concept of inertial mass in physics and closely related to the concept of adjustment speed in economics. The basic idea of GCD models is that the dynamics is determined by the resultant of those forces that market participants exert to optimize their individual interests (individual utility optimization). GCD models are thus an extension of neoclassical models described by the maximization of a single master utility function (overall utility maximization). The term "constrained" refers to the fact that the economy is often subject to constraints. The most important class of such constraints are accounting identities, which lead to economic constraint forces in perfect analogy to the constraint forces in classical mechanics.

We explain the concept of GCD models using the microeconomic model of the Edgeworth Box.

In chapter 4, we give the formal description of GCD models with individual utility functions (individual utility optimization) and the relation to the neoclassical description of economics with a master utility function (overall utility maximization). In this context, the question of the aggregability of individual utility functions to a master utility function plays an essential role.

In chapter 5, we show that GCD models provide the basis for a new and comprehensive understanding of economic models from a mathematical and theoretical perspective. Thus, in a sense, they can be understood as a metatheory for economic models. This basic structure, in which (almost) all economic models can be embedded, can be formally described as a differential-algebraic system of equations. (Almost) all mathematical structures used for economic models can be seen as special cases of a GCD model.

In **Section B** we describe the model of the well-known Edgeworth box, which describes the static general equilibrium in a pure microeconomic exchange economy with only two agents and two goods. A major problem of general equilibrium theory is the fact that no statements can be derived about the so-called tatonnement, i.e. the path of the bargaining process from the initial endowment to the final general equilibrium. We show how GCD models can be defined to model this dynamic bargaining process. This seems to be a contribution to the solution of problem 8 formulated by Steve Smale (Smale 1991; 1997; 1998; Smale Institute 2003).

In **Section C**, we develop step-by-step increasingly complex macroeconomic models. The target is to show how GCD models are built in principle. However, one cannot expect to be able to derive concrete practical economic insights from these models already. To do so, these models and their parameters must first be better adapted to real conditions and tested. This is one of the major tasks that will have to be completed in the future. Only then will it be possible to derive first qualitative and later quantitative economic statements from them.

To facilitate the entry into the practical development of GCD models, the program "GCDconfigurator" was developed. This program is freely accessible via GitHub (Glötzl und Binter 2022) and can be downloaded under

https://github.com/lbinter/gcd

It allows in the 1st step to set up the GCD equation system in a convenient way just from the specification of the utility functions, constraints, power factors and initial conditions. In the 2nd step, the program enables the calculation of the solutions using MATHEMATICA. The results are calculated and plotted graphically as a time evolution of the variables, where the individual parameters can be varied in a convenient way.

All Mathematica program codes used for calculations of the various GCD models can be downloaded under

https://www.dropbox.com/sh/npis47xjqkecggv/AAAMzCVhmhDYIIhoB5MfATFya?dl=0

When developing concrete GCD models, it has proven useful to first describe these models using model graphs that represent the interaction of the different agents and the economic variables. The models start with the simplest model A1, which consists of the two agents (firm and household) and one good that serves as both a consumption good and an investment good. The models are extended step-by-step to include a bank, a central bank and the government. It is also demonstrated, for example, how the monetary policy of the central bank can be modeled in terms of money supply policy or interest rate policy or in terms of the Taylor rule. Finally, Model D2 is a comprehensive model that can still be solved easily with any PC. All the corresponding MATHEMATICA programs can be downloaded in order to be able to analyze and further develop them.

As a final model, we show a simple example of a model that represents the conflicting behaviour of flow and stock variables, as is relevant to many environmental problems: e.g. on the one hand, the burning of fossil fuels (flow variable) fulfills important interests; on the other hand, this leads to the undesirable increase of carbon dioxide concentration in the air.

When analyzing economic dynamics, economic shocks play a very important role. Therefore, we show in **Section D**. how economic shocks can be modeled with GCD models. In the economy, various reasons can lead to a shock, e.g.

- sudden changes in raw material prices
- sudden changes in consumer behaviour due to quarantine regulations
- sudden production restrictions due to a disruption in the supply chain
- etc.

From an economic point of view, there are 2 fundamental issues related to shocks:

(1) Forecasting: How will the economic variables change?
(2) Evaluating countermeasures: What measures can be taken to overcome the shock as quickly as possible or with as little effort as possible?

Since intertemporal utility functions are essential for certain economic models, in **section E** we extend the GCD method to intertemporal utility functions (IGCD models). This allows us to consider GCD models as an alternative to DSGE models. An essential result is, that DSGE models in principle are equivalent to IGCD models with only 1 agent.

The target of this section is to show the principle of defining GCD models with intertemporal utility functions. The actual programming of such GCD models with intertemporal utility functions is much more complex and is therefore still a task for the future.

Section F concludes with an overview over the conceptual and methodological advantages of GCD-models for the understanding of the economy and the dynamics of general economic systems.

A. Basic Principles

2. Historic attempts and literature review

2.1. Economics and Physics

Since the beginnings of modern economics, the endeavor to construct the discipline along the principles of physics has been omnipresent. Already Adam Smith showed his fascination of Newton in 'History of Astronomy' (A. Smith 1795), a fascination that also reveals itself in the methodology of his economic theory as numerous studies show (for an overview over the literature see (Redman 1993)). For instance Smith's theory of value, developed in 'The Wealth of Nations' (A. Smith 1776), is to be regarded as the counterpart to the concept of energy in physics. In its essence the Smithian theory of value was adopted by all following classical economists. In this point of view value is conserved just like energy within the circular flow (Mirowski 1989).

As a result of the impressive scientific advances in the field of physics and chemistry during the 18th and 19th century, the social sciences increasingly tried to imitate the methodology of the natural sciences. Due to the complex and interdependent structure of social phenomena these attempts were of limited success. Only in the field of economics the orientation towards the methodology of physics seemed promising by focusing exclusively on competitive markets, prices and quantities and limiting investigation to rational human behaviour (Rothschild 2002a).

The decisive step in this development was brought by Léon Walras' General Equilibrium Theory (Walras 1874),, and the simultaneously published contributions by Stanley Jevons and the introduction of the 'calculus of pleasure and pain'. This work marked the end of the era of classical economics and was the birth of neoclassical economics. The assumption that the behaviour of all economic agents could be described by utility functions was at the core of this new school of thought. All economic questions involving psychological and social factors were deliberately ignored. Until today these central principles are the foundation of standard economics. The Arrow-Debreu General Equilibrium Model, is seen as the first complete model describing a general equilibrium based on the Walrasian theory (Arrow und Debreu 1954).

The endeavor to identify further similarities between physics and economics, as well as the goal to still increase the orientation of the methodology of economics towards economics was continued by Paul Samuelson. It was his work which was decisive for mathematics to become the standard method in economics. Moreover, Samuelson identified several similarities between physics and economics, arguing that classical thermodynamics and neoclassical economics are related in their common search of a basis for the optimization of observed behaviour. In physics this is achieved by maximizing free energy, in economics by maximizing utility (James B. Cooper 2010); (J. B. Cooper und Russell 2011; James B. Cooper 2010). In a similar vein Smith und Foley (2008) attempt to adopt the model structure of thermodynamics as well as the principle of entropy in economics and show under which circumstances and conditions this is possible (E. Smith und Foley 2008).

In contrast to that, other authors such as Kümmel (2011) have tried to investigate the consequences of the existence of the first and second law of thermodynamics within the economy, rather than trying to find suitable analogies for economics.

To understand the historical developments and why neoclassical economics initially dealt only with equilibrium models, one must always keep the following in mind. The reason was not so much the assumption that real economic systems are in fact mostly in equilibrium, but the fact that without this radical simplification an economic theory was hardly possible. This is because dynamic descriptions of economic systems are so complex that, on the one hand, there is no possibility to find analytical solutions for them and, on the other hand, numerical solutions for differential or difference equation systems were only generally accessible towards the end of the 20th century. The formal foundations of GCD models, on the other hand, consist of differential-algebraic systems of equations. These are usually still much more difficult to solve than differential or difference equation systems. Programs for the numerical solution of algebraic differential equation systems have only become readily generally available in recent years. Therefore, it is understandable that algebraic differential equations have hardly been used systematically for the description of economic systems so far.

2.2. Economics and Power

The goal to imitate physics led to the fact that questions of power were ignored for two distinct reasons. On the one hand there was the idea that while power relations might play a role in the short term, in the long run are irrelevant due to inevitable economic laws. This argument is most prominently made in 'Macht oder ökonomisches Gesetz' by Eugen von Böhm-Bawerk (1914). To some extent the idea can also be found in later discussions, for example in the Lucas-critique. On the other hand, as a result of the self-imposed restriction to follow a strictly mathematical methodology questions of power were left to the disciplines of psychology and the social sciences.

Those economic theories which explicitly deal with questions of power, such as Marxian theory where class struggle and distribution put power relations center stage (Foley 1986) or parts of institutional economics, have been marginalized and are a small minority in modern economics. In contrast, neoclassical orthodoxy limits itself to monopoly power of companies and negotiating power of workers on the labor market in its understanding of power, as the AS-AD model which can be found in every standard economics textbook (Blanchard und Illing 2009). This view of power fully neglects the fact that in reality all agents have a more or less pronounced power to assert their interest, be it in the market process or by influencing the political and social framework. Finally, power can not only be a means to economic actions but an end in itself (Rothschild 2002a).

2.3. Closure of economic models

An important body of literature has dealt with the problem of closure of economic models. Closure is the task of making an under- or over-determined equation system, usually including macroeconomic accounting identities, solvable. Therefore, "[…] prescribing closures boils down to stating which variables are endogenous or exogenous[…]"(Taylor 1991, 41), as some behavioural equations need to be omitted to yield a determined system. Already in 1956, Kaldor set out to investigate the model structures of different schools of economic thought and thereby implicitly also discussed diverse closures of Ricardian, Marxist, Keynesian and Neoclassical models (Kaldor 1956). In a similar vein A. Sen (1963) further showed that in fact Neo-classical and Neo-keynesian models of distribution can be derived from the same equation system and differ in their essence the choice of which equations are dropped i.e. in the assumptions about causality. Marglin (1987) on the other hand approaches the problem from the other direction and argues that Neo-classical, Neo-keynesian and Neo-marxist models have a common underdetermined core equational system which is closed using different behavioural rules. More recently, Barbosa-Filho (2004) investigated three alternative closures of Keynesian models with investment, net exports or autonomous consumption as driving force of aggregate demand.

2.4. The invisible hand does not always lead to the optimum

Adam Smith's analysis of the economy and his theory that egoistic behaviour of all agents will lead to the optimal result in the end, often summarized under the metaphor of the 'invisible hand', is a central thought in economics until today. This is the case even though many authors have shown that individual optimization does not necessarily lead to an overall optimum. For instance John Nash, the founder of game theory, showed that individually optimal behaviour can lead to stable equilibria which constitute the worst scenario for all players (Nash 1951). Throughout the second half of the 20th century there has been significant work, not least with experiments, trying to understand to what extent such prisoners dilemmas play a role in reality as (Giza 2013) illustrates.

The method to describe problems of game theory with continuous time and differential equations can be used also for more general problems in game theory (Cvitanic und Zhang 2014). Because of the characterization with differential equations the continuous-time approach is usually easier to solve than the discrete time models (Sannikov 2007; Yuliy Sannikov 2012)

3. The basic principles and easy examples

3.1. From Newton to General Constraint Dynamics (GCD) in economics

What was the great achievement of Newton?

He brought the right terms into the right relationship.

What were the right terms?

The right terms were: time change of velocity v', inertial mass M, physical force f.

What was the right relationship?

Newton's law.

For J forces in dimension 3 Newton's law reads as:

$$v_i' = \frac{1}{M}\sum_{j=1}^{J} f_i^j \qquad i = 1,2,3 \qquad \text{<3.1>}$$

That is, the change in velocity v is equal to the reciprocal of the inertial mass M times the sum of the forces f^j. The sum of the forces is called the resultant (of the forces).

So, what do we need to do if we want to describe the economics?

Put the right terms in the right relationship.

What are the right terms?

The right terms are: time change of economic variable x, economic power μ, economic force f.

What is the right relationship?

General "ex-ante" Dynamics (GD).

For J economic forces and I economic variables this law reads as:

$$x_i' = \sum_{j=1}^{J} \mu_i^j f_i^j \qquad i = 1,2,\ldots,I \qquad j = 1,2,\ldots,J \qquad \text{<3.2>}$$

That is, the temporal change of an economic variable x_i is equal to the sum of the J forces f_i^j each weighted with the power factor μ_i^j.

Constraints lead to additional forces acting on the "ex ante" dynamics. The dynamics under consideration of constraining forces is called "ex post" dynamics. The theory of classical mechanics under constraints was developed about 100 years after Newton by Joseph-Louis

Lagrange. Constraints play a major role in economics, especially in the form of accounting identities. Therefore, the Lagrangian theory of classical mechanics, is the correct basis to develop a theory of the dynamics of economic processes analogous to classical mechanics. We call the corresponding economic models General Constrained Dynamic models (GCD models), see in particular chap. 3.3.

Now about the interpretation and relationship between inertial mass M and power factors μ_i^j:

A physical force f does not directly lead to a change in velocity v. The change in velocity is inversely proportional to the inertial mass M. A large mass causes the velocity to change slowly, a small mass causes the velocity to change quickly. Each force "feels" the same inertial mass, i.e. the inertial mass is independent of the force acting on it.

In the same way, an economic force does not directly lead to a change in the economic variable x. The change in economic variable x is proportional to the power μ_i^j an actor has to change variable x_i when acting with force f_i^j to change x_i. A large power causes the velocity to change quickly, a small power causes the velocity to change slowly. That means that the power factors can be interpreted as the reciprocal of the mass. In contrast to physics, however, each force acts with a different power factor, i.e. the power factors depend on the respective economic force. Because the power factors influence the respective time change of the economic variables, they can also be interpreted in some sense as velocity adjustment factors.

Some may argue that one cannot measure power factors and economic forces in economics in the same way that one measures mass and physical forces in physics. This is only partly correct. In principle, both quantities can only be measured by comparing the real change in velocity or economic variables over time with the respective physical or economic forces. In physics, this is easy because one can make the measurements in simple, reproducible experiments. In economics, this is more difficult but not impossible in principle for two reasons:

1. In contrast to physics, where Newton's law <3.1> has proven to be a generally valid law, the formula <3.2> must first prove to be a useful law for describing economic systems.

2. In economics, as a rule, no simple experiments can be carried out. But the power factors and also the economic forces, if one assumes the validity of <3.2>, can be determined in any case in principle just like the mass and physical forces from the comparison of reality with the respective best fit of the model <3.2> or <3.1>.

We have no doubt that what Kurt Rothschild (2002a) says about the importance of power in economics is true: *„In the end everything in economics is a question of power“*. Therefore, in order to understand economics, it is absolutely necessary to formalize the concept of power.

Of course, there is still much to be done in order to be able to determine economic forces and power factors quantitatively correctly. In any case, however, all GCD models show that qualitative changes in power factors and forces alone also contribute significantly to understanding the economics.

3.2. Utility functions in economics correspond to potential functions in physics.

In physics, if a force f depends only on the spatial coordinates s, i.e. $f = f(s)$ and can be represented as a gradient of a potential U , this force is called a conservative force. In many cases, an economic force f can also be represented as a gradient of a function called a utility function, denoted also by U. Such economic forces are called microfounded forces. Utility functions in economics are thus the analogous terms to potential (functions) in physics.

If in physics all forces in <3.1> are conservative forces, i.e.

$$f^j(s) = grad\, U^j(s) = \frac{\partial U^j(s)}{\partial s} \qquad resp. \quad f_i^j(s) == \frac{\partial U^j(s)}{\partial s_i}$$

equation <3.1> can be written as

$$v_i' = \frac{1}{M}\sum_{j=1}^{J}\frac{\partial U^j(s)}{\partial s_i} \qquad i = 1,2,...,I \qquad j = 1,2,...,J \qquad <3.3>$$

In economics forces in <3.2> are typically microfounded, i.e.

$$f^j(x) = grad\, U^j(x) = \frac{\partial U^j(x)}{\partial x} \qquad resp. \quad f_i^j(x) == \frac{\partial U^j(x)}{\partial x_i}$$

In that case equation <3.2> yields the basic **GCD equation system of ex-ante dynamics for microfounded forces**

$$\boxed{x_i' = \sum_{j=1}^{J}\mu_i^j\frac{\partial U^j(x)}{\partial x_i} \qquad i = 1,2,...,I \qquad <3.4>}$$

Note: in the general case the utility functions not only depend on x but may also depend on the antiderivative X and the derivative x'. For more details see chapter 7.9.

3.3. Constraint dynamics in classical physics and economics

In many cases in physics and economics the solutions of the dynamic system are restricted by a set of constraint conditions Z^k. Thus equations <3.1> resp. <3.2> has to be extended by equations

$$0 = Z^k \qquad k = 1,2,...,K \qquad <3.5>$$

The theory of constraint dynamics was developed by Lagrange and D'Alembert.

Denote the space variables by

$$s_i \qquad i = 1,2,3$$

then

$$v_i = s_i'$$
$$v_i' = s_i''$$

If in physics $Z = Z(s)$ or in economics $Z = Z(X)$, which means that Z only depends on the antiderivatives, the constraint condition Z is called holonomic.

To explain the principle, we will first discuss only conservative forces with one holonomic constraint. (For non-holonomic constraints see chapter 7.9.)

Thus, in physics the Newtonian equation <3.4> has to be extended to

$$v_i' = \frac{1}{M}\sum_{j=1}^{J}\frac{\partial U^j(s)}{\partial x_i} \qquad i = 1,2,3 \qquad j = 1,2,...,J$$
$$0 = Z(s) \qquad \text{<3.6>}$$

But this system of equations consists of 4 equations for 3 variables and is therefore generally not solvable. Typically, $0 = Z(s)$ is an algebraic equation.

A method to make an unsolvable equation system solvable is called closure method. According to Lagrange, to make this system of equations solvable, one must add an additional variable λ , called the Lagrange multiplier, and an additional force f^Z , called the constraint force (We call this method Lagrange closure.)

$$v_i' = \frac{1}{M}\sum_{j=1}^{J}\frac{\partial U^j(s)}{\partial x_i} + \lambda f^Z$$
$$0 = Z(s) \qquad \text{<3.7>}$$

According to D'Alembert in classical mechanics it holds the following principle for the constraint force

$$f^Z = \frac{\partial Z}{\partial s} \qquad \text{<3.8>}$$

This principle is called D'Alembert's principle. It cannot be derived from Newton's axioms, but is an additional axiom which has always proved to be fulfilled in nature like Newton's axioms. D'Alembert's principle yields

$$v_i' = \frac{1}{M}\sum_{j=1}^{J}\frac{\partial U^j(s)}{\partial x_i} + \lambda\frac{\partial Z(s)}{\partial s}$$
$$0 = Z(s) \qquad \text{<3.9>}$$

Typically, $0 = Z(s)$ is an algebraic equation. Therefore, the equation system is called a differential algebraic equation system (DAE).

For the economic system <3.4> with constraint $0 = Z(x)$ we get in complete analogy to classical mechanics

$$x_i' = \sum_{j=1}^{J}\mu_i^j\frac{\partial U^j(x)}{\partial x_i} + \lambda\frac{\partial Z(x)}{\partial x}$$
$$0 = Z(s) \qquad \text{<3.10>}$$

For K constraints Z^k, $k = 1,2,...,K$ <3.10> extends to the **general GCD model equations** for the ex post dynamics

$$
\boxed{
\begin{aligned}
x_i' &= \sum_{j=1}^{J} \mu_i^j \frac{\partial U^j(x)}{\partial x_i} + \sum_{k=1}^{K} \lambda^k \frac{\partial Z^k(x)}{\partial x_i} \qquad && I = 1,2,...,I \\
0 &= Z^k(x) && k = 1,2,...,K
\end{aligned}
}
\qquad \text{<3.11>}
$$

Note: There are good reasons why D'Alembert's principle is probably fulfilled in economics as well (see more in chapter 7.8.2.1).

In economics, a different closure method is sometimes used to make an overdetermined system of equations solvable (A. Sen 1963): one omits some of the equations (we call this method drop closure, see more details in chapter 18). This method is problematic, however, because a lot of information is lost by omitting an equation.

As an illustrative example of constrained dynamics, we describe below the motion on an inclined plane.

Denoting:

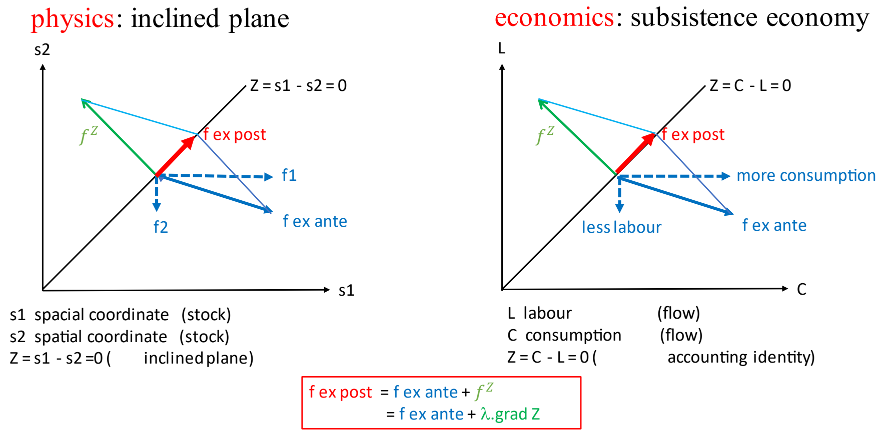
s_1, s_2 the spatial coordinates,
v_1, v_2 the velocity coordinates and
v_1', v_2' their derivatives with respect to time
M the inertial mass
f_1, f_2 the coordinates of the forces exerted on the mass M
$Z(s_1, s_2) = s_1 - s_2 = 0$ the constraint describing the inclined plane with 45°
λ the Lagrange-multiplier

If f_1 denotes a horizontal force and f_2 a vertical force acting on a mass point with mass M the movement of the mass point on the inclined plane is described by the following Newton-Lagrange equations:

$$
\begin{aligned}
v_1' &= \frac{1}{M} f_1 + \lambda \frac{\partial Z}{\partial s_1} = \frac{1}{M} f_1 + \lambda \\
v_2' &= \frac{1}{M} f_2 + \lambda \frac{\partial Z}{\partial s_2} = \frac{1}{M} f_2 - \lambda \\
Z(s_1, s_2) &= s_1 - s_2 = 0
\end{aligned}
\qquad \text{<3.12>}
$$

The respective first terms $\frac{1}{M} f_1$ und $\frac{1}{M} f_2$ describe the coordinates of the “ex-ante” force while the respective second terms $\lambda \frac{\partial Z}{\partial x_1}$ and $\lambda \frac{\partial Z}{\partial x_2}$ describe the coordinates of the ‘constraint force’. The sum of both terms is denoted as “ex-post” force, as it describes the factual resulting movement under the constraint.

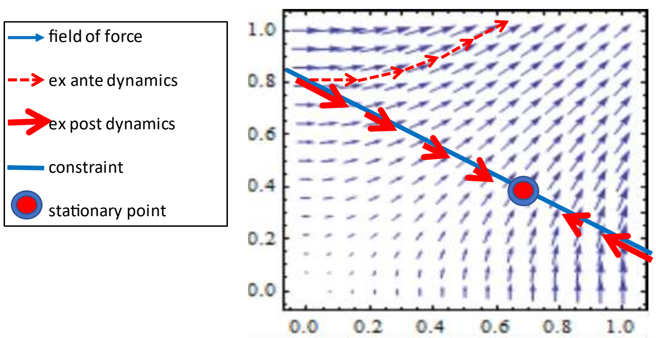
simple examples for constrained dynamics



The analogy between a constrained dynamics in physics and a constrained dynamics in economics can be illustrated by the following simple example from economics. For a subsistence economy, where everything produced by labor L is also consumed, the accounting identity $L = C$ holds. If f_1 describes the interest to eat more (i.e. if on the given state (C,L) the force f_1 is applied) and f_2 describes the interest to work less (i.e. if on the state (C,L) additionally the force f_2 is applied) and both power factors are set equal to $1/M$, formally the same system of equations results as for the dynamics on the inclined plane. We discuss other formally identical models from economics in chap. 18.2.

The interaction of force field, constraint, ex-ante dynamics, ex-post dynamics and steady state, can also be illustrated by the following graph.

constrained dynamics in a vector field of forces



4. Model Equations of General Constrained Dynamic Models (GCD models)

4.1. The general structure of GCD models

For any number of agents (independent from the fact whether these agents are individual economic agents or a representative agent for a certain group or sector) the general concept of GCD models can be described verbally in the following way:

- Starting from an economic state at time t , which is described by I **variables** x_i $(i = 1,2,...,I)$, every one of J agents $(j = 1,2,...,J)$ is interested in changing this state and has an **economic power** μ_i^j to assert his interest.
- Therefore, every agent j employs an economic force f_i^j to change the variable x_i in the direction which is beneficial for him. The effective force is directly proportional to the **economic force** f_i^j he employed and his economic power μ_i^j. The resultant of all forces and power factors determines the 'ex-ante' dynamics.
- K **constraints** Z^k $(k = 1,2,...,K)$ such as accounting identities evoke K additional constraint forces for each variable x_i. The ex-post dynamics is determined by $I.J$ interest-led forces f_i^j times the power factors μ_i^j plus $I.K$ constraint forces. The constraint forces are given analogously to classical mechanics as the K Lagrange multipliers times the gradient of Z^k

The models can be formulated much more easily by using continuous time differential equations instead of difference equations. In general, however, an equivalent formulation in discrete time would always be possible, but using the strong theory of differential equations is much easier and more convenient. Moreover, this reveals the analogies with physics. Adding stochastic terms to the GCD models would not pose any problem. For reasons of simplicity this will not be done in the following.

The **general equation system of GCD models** is

$$\begin{aligned} x_i' &= \sum_{j=1}^{J} \mu_i^j f_i^j(x) + \sum_{k=1}^{K} \lambda^k \frac{\partial Z^k(x)}{\partial x_i} \qquad && I = 1,2,...,I \\ 0 &= Z^k(x) && k = 1,2,...,K \end{aligned}$$

<4.1>

The use of accounting identities as constraints means that GCD models are always SFC (stock-flow consistent). Typically, economic forces can be modeled as gradient of utility functions (see next chapter 4.2). In this case GCD models are also microfounded because the behavioural equations arise from the microeconomic utility optimization of market participants.

Remark 1: infinite power results in algebraic equations

When for a certain i_0 and a certain j_0 it holds that $\mu_{i_0}^{j_0} \to \infty$, then divide in <4.1> the equation $i = i_0$

$$x_{i_0}' = \sum_{j=1}^{J} \mu_{i_0}^{j} f_{i_0}^{j}(x) + \sum_{k=1}^{K} \lambda^k \frac{\partial Z^k(x)}{\partial x_{i_0}} \qquad \text{<4.2>}$$

by $\mu_{i_0}^{j_0}$ and let $\mu_{i_0}^{j_0} \to \infty$. This yields the algebraic equation:

$$0 = f_{i_0}^{j_0}(x)$$

This means that also algebraic behavioural equations can be interpreted as GCD behavioural equations with infinite power factors.

Remark 2: Additional behavioural equations are necessary for parameters in constraint conditions

If an additional variable p occur in the constraints that do not occur in the ex-ante equations, i.e. if the system of equations initially looks like this

$$x_{i_0}' = \sum_{j=1}^{J} \mu_{i_0}^{j} f_{i_0}^{j}(x) + \sum_{k=1}^{K} \lambda^k \frac{\partial Z^k(x)}{\partial x_{i_0}}$$

$$0 = Z^k(x, p)$$

then there is one more variable than equations. Therefore, to create a complete GCD model, an additional behavioral equation is necessary which is linearly independent of the other equations. If there are several additional variables, an additional behavioral equation is necessary for each variable. This behavioural equation can be a differential equation or an algebraic equation (see Remark 1)

This leads e.g., to

$$x_{i_0}' = \sum_{j=1}^{J} \mu_{i_0}^{j} f_{i_0}^{j}(x) + \sum_{k=1}^{K} \lambda^k \frac{\partial Z^k(x)}{\partial x_{i_0}}$$

$$p' = \sum_{j=1}^{J} \mu_{p}^{j} f_{p}^{j}(x, p) + \sum_{k=1}^{K} \lambda^k \frac{\partial Z^k(x, p)}{\partial p}$$

$$0 = Z^k(x, p)$$

or

$$x_{i_0}' = \sum_{j=1}^{J} \mu_{i_0}^{j} f_{i_0}^{j}(x) + \sum_{k=1}^{K} \lambda^k \frac{\partial Z^k(x)}{\partial x_{i_0}}$$
$$0 = f_p(x, p)$$
$$0 = Z^k(x, p)$$

How to proceed with further additional variables $(p_1, p_2, ...)$ in the constraints is obvious.

Remark 3: Interpretation of power factors as adjustment speed factors

If for a given i_1 and a given j_1 , $\mu_{i_1}^{j} = 0$ is true for all $j \neq j_1$ then the differential equation is

$$x_{i_0}' = \mu_{i_1}^{j_1} f_{i_1}^{j_1}(x) + \sum_{k=1}^{K} \lambda^k \frac{\partial Z^k(x)}{\partial x_{i_1}}$$

In this particular case, the power factor $\mu_{i_1}^{j_1}$ can be interpreted as the adjustment speed factor. This interpretation is however only partially adequate because a variable does not adjust on its own, it can only be adjusted by actions of an agent. Thus, the factors μ_i^j are therefore rather characteristics of the agents than of the variables and can however very well be interpreted as the power of agent j to change variable x_i when applying a force f_i^j .

Remark 4: Constraint conditions depend on time derivatives of variables

If a constraint depends not only on $x = (x_1, x_2, ...x_I)$ but also on $x' = (x_1', x_2', ...x_I')$ or higher derivatives $x'' = (x_1'', x_2'', ...x_I'')$, ...

$$0 = Z(x, x', x'', ...)$$

the constraint forces are always to be derived from the highest time derivative of the variables (Flannery 2011), i.e.

$$\begin{aligned} &\frac{\partial Z(x,x')}{\partial x_i'} \; instead\ of \; \frac{\partial Z(x,x')}{\partial x_i} \; resp. \\ &\frac{\partial Z(x,x',x'')}{\partial x_i'} \; instead\ of \; \frac{\partial Z(x,x',x'')}{\partial x_i} \end{aligned} \qquad \text{<4.3>}$$

Remark 5: non vertical constraint forces

In most cases in economics, it is plausible to model the constraint forces analogously to d'Alembert's law in physics as Lagrange multiplier times the gradient of the constraint according to <4.1>. But in economics d'Alembert's law is not to hold as an Axiom like in physics. But in economics, d'Alembert's law does not apply as an axiom as it does in physics. Another type of constraint force that can occur, especially in the case of a constraint force describing a limited resource, is a constraint force that is central to the origin. We therefore refer to this as a "central constraining force" (more on this in chapter 7.8.2.2).

4.2. GCD-models with individual utility functions

For economic models the case in which the economic forces can be described as gradients of individual utility functions U^j of an agent j is of special importance.

$$f^j(x) = grad\, U^j(x) \quad i.e. \quad f_i^j(x) = \frac{\partial U^j(x)}{\partial x_i} \qquad for\ i = 1,2,\ldots,I$$

The path-independent economic force $grad\, U^j(x)$ associated to the utility function $U^j(x)$ describes the "rational" preferences of agent j. For these cases the basic system of GCD equations for ex-ante dynamics <3.2> reads as

$$x_i\,' = \sum_{j=1}^{J} \mu_i^j \frac{\partial U^j(x)}{\partial x_i} \qquad I = 1,2,\ldots,I$$ <4.4>

and the **general system of GCD equations** for ex-post dynamics reads as

$$\boxed{\begin{array}{ll} x_i\,' = \sum_{j=1}^{J} \mu_i^j \frac{\partial U^j(x)}{\partial x_i} + \sum_{k=1}^{K} \lambda^k \frac{\partial Z^k(x)}{\partial x_i} & I = 1,2,\ldots,I \\ 0 = Z^k(x) & k = 1,2,\ldots,K \end{array}}$$ <4.5>

This system of equations can be interpreted in the following way: the more an agent's individual utility will increase, the higher will be the 'rational' preference respectively the economic interest and thereby the economic force an agent will employ in order to change a variable. The factual change arises as a resultant of all these forces and the constraint forces. It is thus the resultant force of the agents' **individual optimization** strategies.

4.3. GCD models with a master utility function

Adam Smith assumed that in a market economy the "invisible hand" leads to an optimal outcome for all market participants, or in other words: If each market participant tries to optimize his individual utility, this leads to the maximization of overall utility. To discuss this problem more formally, we define the following terms:

- aggregability of utility functions,
- master utility function and
- overall utility function

We call utility functions U^j, $j = 1,2,\ldots,J$ to be **"aggregable"** iff there exists a **"master utility function"** MU such that

$$\sum_{j=1}^{J} \mu_i^j \frac{\partial U^j(x)}{\partial x_i} = \frac{\partial MU(x)}{\partial x_i}$$ <4.6>

In this case the basic **GCD equation system of ex-ante dynamics for microfounded forces** are

$$x_i' = \sum_{j=1}^{J} \mu_i^j \frac{\partial U^j(x)}{\partial x_i} = \frac{\partial MU(x)}{\partial x_i} \qquad i = 1,2,...,I \qquad \text{<4.7>}$$

With respect to the master utility function MU to major questions arise:

1. Under what conditions exists such a master utility function
2. Under what conditions does maximizing the master utility function also lead to maximizing the overall utility, if this is defined as the sum of the utilities of all agents? That means, the overall utility is an unweighted utilitarian (or Bentham) social welfare function.

To get a deeper understanding of question 1. one has to go into more theoretical details.

In physics the **Helmholtz decomposition** of forces in 3 dimensions is well known. In general under certain (mild) conditions all forces depending on variables $y = \{y_1, y_2, ..., y_I\}$ not only in dimension 3 but in any dimension can always be decomposed into a gradient force $g = \frac{\partial V(y)}{\partial y}$ and a rotational force r by means of the so-called Helmholtz decomposition for any dimension (Glötzl und Richters 2021b; 2021a). For our purposes in economics, it is not necessary to describe the rotational force r in more detail. Helmholtz decomposition means that a utility function U exists such that

$$f(y) = g(y) + r(y) = \frac{\partial V(y)}{\partial y} + r(y) \qquad \text{<4.8>}$$

If the rotational part r of the force f is equal to $r = 0$, f is called rotation-free. Then

$$f = g = \frac{\partial V}{\partial y}$$

Note: In this case f is path-independent and the so-called integrability conditions are fulfilled.

Applying this to the economic force f (the resultant of the individual forces)

$$f = \sum_{j=1}^{J} \mu_i^j \frac{\partial U^j(x)}{\partial x_i} \qquad \text{<4.9>}$$

one gets the fundamental answer to question 1.:

$$U^j,\ j=1,2,...,J\ \textit{are aggregable} \Leftrightarrow$$
$$\Leftrightarrow f_i=\sum_{j=1}^{J}\mu_i^j\frac{\partial U^j(x)}{\partial x_i},\ i=1,2,...,I,\ \textit{is rotation free} \Leftrightarrow$$
$$\Leftrightarrow f=(f_1,f_2,...,f_I)\ \textit{is path independent} \qquad \text{<4.10>}$$
$$\Leftrightarrow \textit{rotation densities}\ \frac{\partial f_i}{\partial x_j}-\frac{\partial f_j}{\partial x_i}=0$$
$$\textit{for all}\ i=1,2,...,I,\ j=1,2,...,J$$

For practical application, the following 3 sufficient conditions for aggregability of individual utility functions U^j are very valuable.

For simplicity we formulate these conditions for two individual utility functions U^A, U^B with two variables (x_1,x_2) and individual power factors $\mu_1^A,\mu_2^A,\mu_1^B,\mu_2^B$.:

A master utility function MU exists such that U^A, U^B are aggregable, i.e.

$$\mu_1^A\frac{\partial U^A(x_1,x_2)}{\partial x_1}+\mu_1^B\frac{\partial U^B(x_1,x_2)}{\partial x_1}=\frac{\partial MU(x_1,x_2)}{\partial x_1}$$
$$\mu_2^A\frac{\partial U^A(x_1,x_2)}{\partial x_2}+\mu_1^B\frac{\partial U^B(x_1,x_2)}{\partial x_2}=\frac{\partial MU(x_1,x_2)}{\partial x_2} \qquad \text{<4.11>}$$

if one of the following 3 sufficient conditions is fulfilled:

1. „ linear ”:

$$\textit{If}\ U^A(x_1,x_2)=a_0+p_1^Ax_1+p_2^Ax_2,$$
$$U^B(x_1,x_2)=b_0+p_1^Bx_1+p_2^Bx_2$$
$$\Rightarrow MU(x_1,x_2)=$$
$$=a_0+(\mu_1^Ap_1^A+\mu_1^Bp_1^B)x_1+b_0+(\mu_2^Ap_2^A+\mu_2^Bp_2^B)x_2$$
$$\textit{is a master utility function}$$

2. „independent “:

$$\textit{If}\ U^A(x_1,x_2)=U^A(x_1)$$
$$U^B(x_1,x_2)=U^B(x_2)$$
$$\Rightarrow MU(x_1,x_2)=\mu_1^AU^A(x_1)+\mu_2^BU^B(x_2)$$
$$\textit{is a master utility function}$$

Note: If $x_1=x^A,\ x_2=x^B$ (which means x_1 is a variable which describes a property of A and x_2 is a variable which describes a property of B respectively U^A depends only on x^A and U^B depends only on x^B), the condition **„independent“** can be called **“self-related”.** In many practical cases this property is assumed to hold for the individual utility functions. Therefore, in these cases a neoclassical approach is reasonable. But especially for prisoner dilemma situations utility functions are not self-related.

3. „uniform power “:

$$\begin{aligned}&\textit{If } \mu^A := \mu_1^A = \mu_2^A \qquad \textit{and} \qquad \mu^B := \mu_1^B = \mu_2^B\\&\Rightarrow MU(x_1,x_2) = \mu^A U^A(x_1,x_2) + \mu^B U^B(x_1,x_2)\\&\quad \textit{is a master utility function}\end{aligned}$$

These 3 sufficient conditions can be easily proved by calculating <4.11>.

Defining overall utility as $GU := U^A + U^B$ it becomes clear from the above examples that in general $GU \neq MU$ and that the maximization of the master utility function MU does not necessarily lead to a maximization of the overall utility function GU. As an answer to the second question, it is immediately apparent from the above conditions that the following applies

$$\begin{aligned}&1. \textit{ If all } \mu_i^j = 1 && \Rightarrow && MU = GU\\&2. \textit{ If all } \mu_i^j = \mu && \Rightarrow && a)\ MU = \mu GU\\&&&&& b)\ MU \textit{ maximal} \Leftrightarrow GU \textit{ maximal}\end{aligned}$$

If the utility functions U^A, U^B are aggregable to a master utility function MU and $MU = GU$ the basic GCD equation system for ex-ante dynamics is

$$\boxed{\begin{aligned}&\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \mu_{x_1}^A \dfrac{\partial U^A}{\partial x_1} \\ \mu_{x_2}^A \dfrac{\partial U^A}{\partial x_2} \end{pmatrix} + \begin{pmatrix} \mu_{x_1}^B \dfrac{\partial U^B}{\partial x_1} \\ \mu_{x_2}^B \dfrac{\partial U^B}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial MU}{\partial x_1} \\ \dfrac{\partial MU}{\partial x_2} \end{pmatrix} = \\&\quad = \begin{pmatrix} \dfrac{\partial GU}{\partial x_1} \\ \dfrac{\partial GU}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial (U^A + U^B)}{\partial x_1} \\ \dfrac{\partial (U^A + U^B)}{\partial x_2} \end{pmatrix} \qquad\qquad <4.12>\end{aligned}}$$

This equation system represents the fact that in the case of aggregable utility functions (with $MU = GU$) the **"individual optimization" strategy** (in the sense of GCD models**)** is equivalent to an **overall utility maximisation" strategy.**

Neoclassical **General Equilibrium Theory** (GE) is another form of an individual optimization strategy. We discuss the relation between the individual optimization strategy in the sense of GCD models, the overall maximization strategy and the individual optimization strategy in the sense of GE Theory in chapter 6.

5. GCD as a metatheory of economic model structures and economic theories

5.1. Basic principles

Model building can basically serve three different targets:

1. the process of insight including a qualitative forecast of the future or
2. the quantitative forecast of the future or
3. the control of the system to achieve an optimal behaviour in the future.

The most important target in the education of economists is to understand economics. In the sense of 1. it is therefore essential to deal with the formation of economic models.

In order to understand the fundamental differences of different model structures, the description of model structures in a unified framework is of great advantage (**metatheory of economic model formation**)

In order to understand the fundamental differences of different economic theories, the description of the different economic theories in a unified framework is of great advantage (**metatheory of economic theories**)

5.2. Metatheory of economic model building

1. The economy is a dynamic system in which time-dependent **flows** "flow" from one node ("agents", "aggregated agents", etc.) to another node and lead to temporal changes in the **stocks** ("balances") of the nodes. Therefore, every economic model is initially characterized by which flows one considers. This automatically determines which nodes and thus which stocks are considered.

2. The **dynamics** of the flows and other variables (prices, other parameters, etc.) is determined by:
   2.1. behavioural equations for the variables, which can be formally represented in the following form:
      2.1.1. differential equations
      2.1.2. algebraic equations
      2.1.3. decisions (especially for ABM and game theoretic models)
   2.2. **accounting identities**, which are formally algebraic equations

3. There are 2 types of behavioural equations
   3.1. determined by the influence of agents **("microfounded")**
   3.2. general without reference to single agents **("not microfounded")**

4. The influence of an agent on a variable (flows, stocks, prices, parameters, etc.) is described by
   4.1 his **interest** in changing this variable (the higher his interest, the higher will be the "economic force" f he will spend to change the variable according to his interest).
   4.2. his **economic power** μ to enforce his interest.

5. The interest or economic force that an agent exerts is described as the sum of rational interests (forces) and irrational interests (forces):
   5.1 Rational interests or **rational economic forces** are described by gradients of utility functions. This means that the forces are **path-independent**.
   5.2 Irrational interests or **irrational economic forces** cannot be described as gradients of utility functions, they are described by the rotational part of the Helmholtz decomposition of the force (see chapter 4.3). This means that the forces are **path-dependent**.

We propose General Constrained Dynamic Models (GCD) as the basis for a novel and encompassing understanding of economic models from a mathematical and theoretical perspective. This basic structure, in which (almost) all economic models can be embedded, can be formally described as a differential-algebraic system of equations. Beside of game theoretical models and ABM models almost all mathematical structures used for economic models can be regarded as special cases of a GCD model. It has to be taken into account that GCD models can be extended in a natural way by a stochastic part or can be transformed into difference equation systems by discretization of time.

A simplified overview of the most important mathematical structures for the description of economic models can be found in the following table.

Unified look at mathematical structure of economic models
because of simplicity without constrained conditions

$0 = \frac{\partial MU(x)}{\partial x_i}$	equilibrium	**GE**
$x_i' = \frac{\partial MU(x)}{\partial x_i}$	shock to equilibrium	**dynamic GE**
$x_i' = \mu_i \frac{\partial MU(x)}{\partial x_i}$	adjustment velocities μ_i (sticky prices etc.)	**Neo-Keynesian GE**
$x_i' = \sum_j \mu_i^j \frac{\partial U^j(x)}{\partial x_i}$	power factors μ_i^j, utility functions U^j	**GCD with utility**
$x_i' = \sum_j \mu_i^j f_i^j$	general economic force f_i^j	**GCD general force**
$y_i' = f_i$	not agent based general economic force f_i	**Post-Keynesian**

In principle, DSGE models can also be seen within this framework. Essentially, it is not an ordinary master utility function but an intertemporal utility function, i.e. the time integral of a discounted master utility function. This variational problem leads to the Euler equations, which are behavioural equations describing the dynamic behaviour of the economic system.

In section E. chapters 24 - 27 we show that GCD models can be modeled not only with ordinary utility functions, but also with intertemporal utility functions such as those used in DSGE models.

5.3. Metatheorie ökonomischer Theorien

The ideas of Amartya Sen in (A. Sen 1963) can be seen as the first meta-theory for economic theories. He shows that economic theories differ precisely in which variables are considered exogenous and which variables are considered endogenous (see also chap. 18.3)

In the structure of GCD models, the difference between different economic theories arises precisely from the different assumptions about the power factors μ_i^j , i.e., from the different assumptions about the economic power of the different agents j with which they can influence the different variables x_i . In particular, this is an extension of the metatheory of Sen. The assumption that one of the agents j has complete power to determine the value of a variable x_i means that the corresponding power factor is $\mu_i^j = \infty$. This just leads to the fact that this variable x_i is to be considered exogenous. If for a given i the power factors μ_i^j are $\mu_i^j < \infty$ for all j , this means that the variable x_i is endogenous.

If for a given i the power factors $\mu_i^j = 0$ for all j, then the corresponding variables x_i are not influenced by agents but only by constraints, which means that they are influenced only by "pure" market forces. Formally, in these cases, the corresponding behavioural equations can be omitted. This is exactly in line with Sen's procedure to characterize the different economic theories. The omission of equations can be called drop closure (see chap. 2.3 and 18). For more details see chap. 18.3 and (Glötzl 2015).

Here, an advantage of GCD models becomes quite apparent. Implicit in many economic models is often an assumption of one-sided power relations as in detail is explained by Richters und Glötzl (2020) along the model SIM of (Godley und Lavoie 2012). In reality, however, power relations are usually not one-sided but mixed. With GCD models, reality can be better described, because they can be used to describe not only one-sided power relationships, but also mixed power relationships.

5.4. Resumé

Models are indispensable for a better understanding of economics.

In order not to be confused by the variety of models, a clear distinction must first be made between model structure and the substantive assumptions with which a model structure is filled (i.e., economic theories).

In order to understand the fundamental differences of various model structures, it is of great advantage to describe the model structures in a unified framework (metatheory of economic modelling)

In order to understand the fundamental differences of various economic theories, the description of the various economic theories in a unified framework is of great advantage (Metatheory of Economic Theories)

General Constrained Dynamic Models (GCD models) can be considered as the basis for both a metatheory of economic modelling and a metatheory of economic theories. GCD models are therefore the basis for being able to represent the most diverse views on economics in a unified framework in the sense of a pluralistic economics.

The economic mainstream, on the other hand, is characterized, among other things, by the fatal "maximization assumption" that forms the basis of general equilibrium theory. This maximization assumption is fatal because it leads to a very simplified model structure that cannot be filled with substantially different economic content, such as different power relations. Moreover, it tempts economists to analyse economic systems almost always from the point of view that they are in equilibrium. However, this is usually not the case for real systems.

B. Microeconomic models

6. Microeconomic example: Edgeworth-Box

6.1. General description

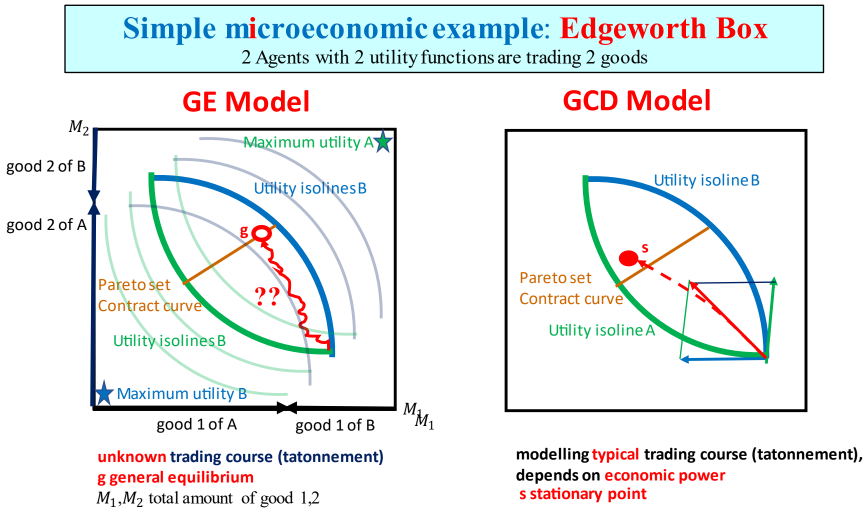
An important and instructive example of a GCD model can be derived from the Edgeworth-Box. An Edgeworth-Box is a graphic tool in microeconomics designed to describe the equilibrium in a pure exchange economy with only two agents A, B and two goods, good 1 and good 2. $x_1^A, x_2^A, x_1^B, x_2^B$ denote the amounts of goods 1, 2 of the agents A, B. The utility function U^A depends only on (x_1^A, x_2^A) and U^B depends only on (x_1^B, x_2^B),

$$\begin{aligned} &U^A(x_1^A, x_2^A) \\ &U^B(x_1^B, x_2^B) \end{aligned} \qquad \text{<6.1>}$$

The allocation of goods 1, 2 of agents A, B before the exchange is called endowment and denoted by $x_1^A0,\ x_2^A0,\ x_1^B0,\ x_2^B0$. The agents reach a Pareto optimum by trading along an unknown unspecified process called tatonnement. All Pareto optima lie on a curve called the contract curve. In these Pareto optima, no agent's utility can be increased without simultaneously decreasing that of another agent. In the standard general equilibrium (GE) model an auctioneer is assumed to set the prices such that there is no excess supply and no excess demand. This special Pareto-optimum is called general equilibrium.

General Equilibrium Theory makes no assertions about the tatonnement, i.e. the way how the general equilibrium is reached. The nature of an GCD model lies exactly in describing the dynamics of the tatonnement. Obviously, in the general case, it is not possible to predict the tatonnement on which the agents negotiate, whether they reach a stationary point or whether they reach a mutually beneficial outcome (Pareto optimum) or the maximum of an overall utility. However, it is reasonable to model the typical negotiation path of two agents in terms of a GCD model as follows:

The negotiation strategy of both agents is based on optimizing their individual utility function under constraints. Each agent will therefore employ an economic force in the direction which corresponds to the highest increase of his utility function. The more his gain in utility, the higher will be the force he employs. The direction and magnitude can be described by the gradient of the utility function, which is perpendicular to the lines of constant utility. The extent to which an agent can achieve his goal does not only depend on the force he and the other agent employed, but also on their respective economic power. The ex-ante change in the allocation of goods will therefore be directed towards the resulting force of the economic forces employed by the agents, weighted by their respective power factors. To get the ex-post change one has to add the constraint forces, which arise from the so-called budget constraints and the constraints which guarantees that excess demand and excess supply are zero. Since budget constraints depend on the prices of goods, a complete model requires behavioral equations for prices that correspond to the bargaining process or the behavior of an auctioneer.



6.2. Aggregability, maximum of overall utility, contract curve

Note that U^A depends only on (x_1^A, x_2^A) and U^B depends only on (x_1^B, x_2^B). Thus, the utilities are "independent" resp. "self-related" (see chapter 4.3). If

$$\begin{aligned} \mu^A_{x_1^A} &= \mu^A_{x_2^A} =: \mu^A \\ \mu^B_{x_1^B} &= \mu^B_{x_2^B} =: \mu^B \end{aligned} \qquad \text{<6.2>}$$

then the utility functions U^A, U^B are **aggregable** to the master utility function

$$MU = MU(x_1^A, x_2^A, x_1^B, x_2^B) = \mu^A U^A(x_1^A, x_2^A) + \mu^B U^B(x_1^B, x_2^B)$$

because

$$\mu^A_{x_1^A} \frac{\partial U^A(x_1^A, x_2^A)}{\partial x_1^A} = \mu^A \frac{\partial U^A(x_1^A, x_2^A)}{\partial x_1^A} = \frac{\partial MU}{\partial x_1^A}$$

$$\mu^B_{x_1^B} \frac{\partial U^B(x_1^B, x_2^B)}{\partial x_1^B} = \mu^A \frac{\partial U^A(x_1^A, x_2^A)}{\partial x_2^A} = \frac{\partial MU}{\partial x_2^A}$$

$$\mu^A_{x_2^A} \frac{\partial U^A(x_1^A, x_2^A)}{\partial x_2^A} = \mu^B \frac{\partial U^B(x_1^B, x_2^B)}{\partial x_1^B} = \frac{\partial MU}{\partial x_1^B}$$

$$\mu^B_{x_2^B} \frac{\partial U^B(x_1^B, x_2^B)}{\partial x_2^B} = \mu^B \frac{\partial U^B(x_1^B, x_2^B)}{\partial x_2^B} = \frac{\partial MU}{\partial x_2^B}$$

If $\mu^A = \mu^B = 1$ the master utility MU equals the overall utility

$$GU = U^A + U^B .$$

Typically, the utility functions are assumed to be of the Cobb-Douglas-type

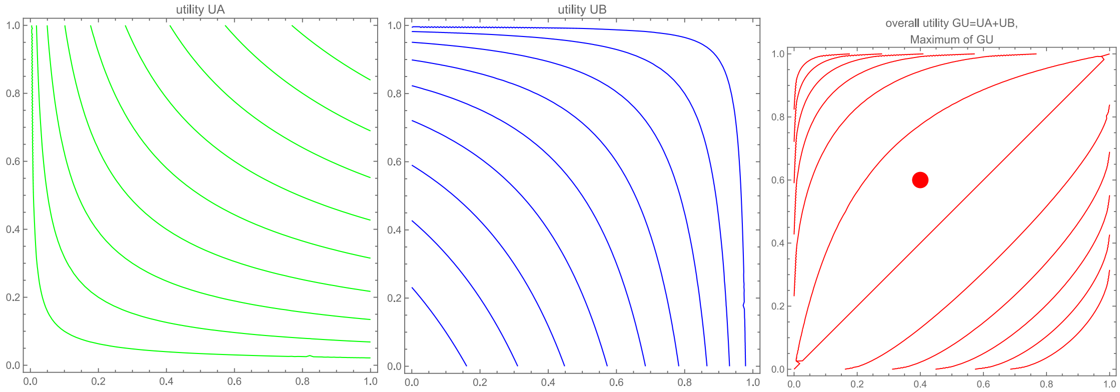
$$U^A(x_1^A, x_2^A) = \left(x_1^A\right)^{\alpha^A} \left(x_2^A\right)^{\left(1-\alpha^A\right)} \qquad 0 < \alpha^A < 1$$
$$U^B(x_1^B, x_2^B) = \left(x_1^B\right)^{\alpha^B} \left(x_2^B\right)^{\left(1-\alpha^B\right)} \qquad 0 < \alpha^B < 1 \qquad \text{<6.3>}$$

In this case GU is convex and has a unique maximum $(max_1^A, max_2^A, max_1^B, max_2^B)$ in the region $x_1^A > 0, x_2^A > 0, x_1^B > 0, x_2^B > 0$

The following graphics show the isolines of

$$U^A \ with \ \alpha^A = 0.4, U^B \ with \ \alpha^B = 0.6, GU = U^A + U^B \quad ,$$

the red point is indicating the unique maximum of GU .



Let M_1, M_2 denote the total amount of good 1,2 then for the overall utility GU written in amounts of good 1 results

$$GU(x_1^A, x_2^A) = U^A(x_1^A, x_2^A) + U^B(M_1 - x_1^A, M_2 - x_2^A) =$$
$$= \left(x_1^A\right)^{\alpha^A} \left(x_2^A\right)^{\left(1-\alpha^A\right)} + \left(M_1 - x_1^A\right)^{\alpha^B} \left(M_2 - x_2^A\right)^{\left(1-\alpha^B\right)}$$

and for the **maximum of** GU holds:

$$0 = \frac{\partial GU(max_1^A, max_2^A)}{\partial max_1^A} =$$

$$= \frac{\partial U^A(max_1^A, max_2^A)}{\partial max_1^A} + \frac{\partial U^B(max_1^A, max_2^A)}{\partial max_1^A} =$$

$$= \alpha^A \left(max_1^A\right)^{\left(\alpha^A - 1\right)} \left(max_2^A\right)^{\left(1-\alpha^A\right)} -$$

$$-\alpha^B \left(M_1 - max_1^A\right)^{\left(\alpha^B - 1\right)} \left(M_2 - max_2^A\right)^{\left(1-\alpha^B\right)}$$

$$0 = \frac{\partial GU(max_1^A, max_2^A)}{\partial max_2^A} =$$

$$= \frac{\partial U^A(max_1^A, max_2^A)}{\partial max_2^A} + \frac{\partial U^B(max_1^A, max_2^A)}{\partial max_2^A} =$$

$$= \left(1-\alpha^A\right)\left(max_1^A\right)^{\left(\alpha^A\right)} \left(max_2^A\right)^{\left(-\alpha^A\right)} -$$

$$-\left(1-\alpha^B\right)\left(M_1 - max_1^A\right)^{\left(\alpha^B\right)} \left(M_2 - max_2^A\right)^{\left(-\alpha^B\right)} \qquad \text{<6.4>}$$

The Pareto optima and thus the **contract curve** is defined by the fact that the gradient of the utility function of A is directed opposite to the gradient of the utility function of B, i.e.

$$\begin{pmatrix} \dfrac{\partial U^A}{\partial x_1^A} \\ \dfrac{\partial U^A}{\partial x_2^A} \end{pmatrix} = -\delta \begin{pmatrix} \dfrac{\partial U^B}{\partial x_1^A} \\ \dfrac{\partial U^B}{\partial x_2^A} \end{pmatrix} \qquad \textit{for some } \delta > 0 \qquad \text{<6.5>}$$

This results in

$$0 = \frac{\partial U^A}{\partial x_1^A} + \delta \frac{\partial U^B}{\partial x_1^A} =$$

$$= \alpha^A \left(max_1^A\right)^{\left(\alpha^A - 1\right)} \left(max_2^A\right)^{\left(1-\alpha^A\right)} -$$

$$-\delta\,\alpha^B \left(M_1 - max_1^A\right)^{\left(\alpha^B - 1\right)} \left(M_2 - max_2^A\right)^{\left(1-\alpha^B\right)}$$

<6.6>

$$0 = \frac{\partial U^A}{\partial x_2^A} + \delta \frac{\partial U^B}{\partial x_2^A} =$$

$$= \left(1-\alpha^A\right)\left(max_1^A\right)^{\left(\alpha^A\right)} \left(max_2^A\right)^{\left(-\alpha^A\right)} -$$

$$-\delta\left(1-\alpha^B\right)\left(M_1 - max_1^A\right)^{\left(\alpha^B\right)} \left(M_2 - max_2^A\right)^{\left(-\alpha^B\right)}$$

Obviously with $\delta = 1$ the maximum of GU lies on the contract curve and is identical with the maximum on the contract curve.

6.3. (Static) GE model

Denote $x_1^A 0,\ x_2^A 0,\ x_1^B 0,\ x_2^B 0$ the endowment (the amounts of goods 1, 2 of the agents A, B before exchange),

$x_1^A, x_2^A, x_1^B, x_2^B$ the amounts of goods 1, 2 of the agents A, B

$U^A(x_1^A, x_2^A)$ and $U^B(x_1^B, x_2^B)$ the utility functions of A, B

$p = (p_1, p_2)$ prices of good 1 and good 2.

There are 2 equivalent formal descriptions of the GE model:

- the **standard description**: individual optimization of both agents under budget constraint + no excess demand
- the **alternative description**: individual optimization of one agent under budget constraint + Pareto-optimum + no excess demand

6.3.1. Standard description

For any given price p Agent A maximizes his utility U^A under the “budget constraint”

$$0 = Z^A(x_1^A, x_2^A, p_1, p_2) = p_1 x_1^A + p_2 x_2^A - (p_1 x_1^A 0 + p_2 x_2^A 0) \qquad \text{<6.7>}$$

For any given price p Agent B maximizes his utility U^B under the “budget constraint”

$$0 = Z^B(x_1^B, x_2^B, p_1, p_2) = p_1 x_1^B + p_2 x_2^B - (p_1 x_1^B 0 + p_2 x_2^B 0) \qquad \text{<6.8>}$$

This yields the first order conditions

$$\begin{aligned}
B1: &\quad 0 = \frac{\partial U^A}{\partial x_1^A} - \lambda^A p_1 \\
B2: &\quad 0 = \frac{\partial U^A}{\partial x_2^A} - \lambda^A p_2 \\
B3: &\quad 0 = \frac{\partial U^B}{\partial x_1^B} - \lambda^B p_1 \qquad \text{<6.9>} \\
B4: &\quad 0 = \frac{\partial U^B}{\partial x_2^A} - \lambda^A p_2 \\
Z^A: &\quad 0 = Z^A(x_1^A, x_2^A, p_1, p_2) = p_1 x_1^A + p_2 x_2^A - (p_1 x_1^A 0 + p_2 x_2^A 0) \\
Z^B: &\quad 0 = Z^B(x_1^B, x_2^B, p_1, p_2) = p_1 x_1^B + p_2 x_2^B - (p_1 x_1^B 0 + p_2 x_2^B 0)
\end{aligned}$$

Since the total amount of good 1 and the total amount of good 2 do not change as a result of the exchange, the following constraints must be met:

$$\begin{aligned}
Z_1: &\quad 0 = z_1(x_1^A, x_1^B) = x_1^A + x_1^B - (x_1^A 0 + x_1^B 0) \\
Z_2: &\quad 0 = z_2(x_2^A, x_2^B) = x_2^A + x_2^B - (x_2^A 0 + x_2^B 0)
\end{aligned} \qquad \text{<6.10>}$$

z_1, z_2 describe the excess demand ($z_i > 0$) resp. excess supply ($z_i < 0$) of good 1,2. The conditions Z_1, Z_2 therefore describe the assumption that in equilibrium the excess demand resp. excess supply for both goods are zero.

Since the conditions Z^A, Z^B, Z_1, Z_2 are linearly dependent one of these conditions can be omitted. Therefore only the relative price $\frac{p_2}{p_1}$ is determined by the equations of the model. Usually one assume that good 1 is a numeraire, which means $p_1 = 1$.

With $GU = U^A + U^B$ the equation system <6.9> + <6.10> yields the equation system for the general equilibrium $g_1^A, g_2^A, g_1^B, g_2^B$

$$
\begin{aligned}
&\textit{maximization of } GU \textit{ under budget constraints } Z^A, Z^B \\
&B1: \quad 0 = \frac{\partial U^A}{\partial g_1^A} - \lambda^A p_1 = \frac{\partial GU}{\partial g_1^A} - \lambda^A p_1 \\
&B2: \quad 0 = \frac{\partial U^A}{\partial g_2^A} - \lambda^A p_2 = \frac{\partial GU}{\partial g_2^A} - \lambda^A p_2 \\
&B3: \quad 0 = \frac{\partial U^B}{\partial g_1^B} - \lambda^B p_1 = \frac{\partial GU}{\partial g_1^B} - \lambda^B p_1 \\
&B4: \quad 0 = \frac{\partial U^B}{\partial g_2^A} - \lambda^B p_2 = \frac{\partial GU}{\partial g_2^A} - \lambda^B p_2 \\
&\textit{budget constraints} \\
&Z^A: \quad 0 = Z^A(g_1^A, g_2^A, p_1, p_2) = p_1 g_1^A + p_2 g_2^A - (p_1 g_1^A 0 + p_2 g_2^A 0) \\
&Z^B: \quad 0 = Z^B(g_1^B, g_2^B, p_1, p_2) = p_1 g_1^B + p_2 g_2^B - (p_1 g_1^B 0 + p_2 g_2^B 0) \\
&\textit{excess supply conditions} \\
&Z_1: \quad 0 = z_1(g_1^A, g_1^B) = g_1^A + g_1^B - (g_1^A 0 + g_1^B 0) \\
&Z_2: \quad 0 = z_2(g_2^A, g_2^B) = g_2^A + g_2^B - (g_2^A 0 + g_2^B 0)
\end{aligned}
\tag{6.11}
$$

The equation system <6.11> for the general equilibrium consists of 8 equations for 8 variables $(x_1^A, x_2^A, x_1^B, x_2^B, p_1, p_2, \lambda^A, \lambda^B)$. If the utility functions fulfill the **SMD-conditions** (Sonnenschein, Mantel, Debreux conditions) there exists a unique solution. Since the conditions Z^A, Z^B, Z_1, Z_2 are linearly dependent one get a solution for

$$(x_1^A, x_2^A, x_1^B, x_2^B, \frac{p_2}{p_1}, \lambda^A, \lambda^B)$$

Assuming good 1 to be a numeraire, i.e $p_1 = 1$, the solution for the Cobb-Douglas-type utility functions <6.3> is given by

https://www.dropbox.com/s/09xshhos19uz7lr/L%C3%B6sungen%20GE%20Version%203.nb?dl=0

$$
\begin{aligned}
&\mathrm{p1} \to 1, \\
&\mathrm{p2} \to \frac{\mathrm{g1a0}+\mathrm{g1b0}-\mathrm{g1a0}\alpha\mathrm{a}-\mathrm{g1b0}\alpha\mathrm{b}}{\mathrm{g2a0}\alpha\mathrm{a}+\mathrm{g2b0}\alpha\mathrm{b}}, \\
&\mathrm{g1a} \to \frac{\alpha\mathrm{a}(-\mathrm{g1b0g2a0}(-1+\alpha\mathrm{b})+\mathrm{g1a0}(\mathrm{g2a0}+\mathrm{g2b0}\alpha\mathrm{b}))}{\mathrm{g2a0}\alpha\mathrm{a}+\mathrm{g2b0}\alpha\mathrm{b}}, \\
&\mathrm{g2a} \to \frac{(-1+\alpha\mathrm{a})(-\mathrm{g1b0g2a0}(-1+\alpha\mathrm{b})+\mathrm{g1a0}(\mathrm{g2a0}+\mathrm{g2b0}\alpha\mathrm{b}))}{\mathrm{g1a0}(-1+\alpha\mathrm{a})+\mathrm{g1b0}(-1+\alpha\mathrm{b})},
\end{aligned}
\tag{6.12}
$$

$$\text{g1b} \to \frac{(\text{g1b0}(\text{g2b0}+\text{g2a0}\alpha\text{a})+\text{g1a0}(\text{g2b0}-\text{g2b0}\alpha\text{a}))\alpha\text{b}}{\text{g2a0}\alpha\text{a}+\text{g2b0}\alpha\text{b}},$$

$$\text{g2b} \to \frac{(\text{g1b0}(\text{g2b0}+\text{g2a0}\alpha\text{a})+\text{g1a0}(\text{g2b0}-\text{g2b0}\alpha\text{a}))(-1+\alpha\text{b})}{\text{g1a0}(-1+\alpha\text{a})+\text{g1b0}(-1+\alpha\text{b})}$$

6.3.2. Alternative description

Conditions B3, B4 can be substituted by the condition that the general equilibrium must lie on the contract curve. If the utilities U^A, U^B are given in coordinates of good 1, the contract curve is defined by the condition that the gradient of U^A is opposite to the gradient of U^B:

$$\textit{contract curve condition: it exists } \delta > 0 \textit{ such that}$$

$$\begin{aligned} C1 \qquad 0 &= \frac{\partial U^A(g_1^A, g_2^A)}{\partial g_1^A} + \\ &+\delta \frac{\partial U^B(g_1^A 0 + g_1^B 0 - g_1^A, g_2^A 0 + g_2^B 0 - g_2^A)}{\partial g_1^A} \\ C2 \qquad 0 &= \frac{\partial U^A(g_1^A, g_2^A)}{\partial g_2^A} + \\ &+\delta \frac{\partial U^B(g_1^A 0 + g_1^B 0 - g_1^A, g_2^A 0 + g_2^B 0 - g_2^A)}{\partial g_2^A} \end{aligned} \qquad \text{<6.13>}$$

6.4. (Dynamic) GCD models

6.4.1. Basic equations

If one uses the same utility functions as in the GE model, one obtains, according to <3.4>, as GCD ex ante equation system

$$\begin{aligned} B1: \quad & x_1^{A\prime} = \mu_{x_1^A}^A \frac{\partial U^A(x_1^A, x_2^A)}{\partial x_1^A} \\ B1: \quad & x_1^{B\prime} = \mu_{x_1^B}^B \frac{\partial U^B(x_1^B, x_2^B)}{\partial x_1^B} \\ B1: \quad & x_2^{A\prime} = \mu_{x_2^A}^A \frac{\partial U^A(x_1^A, x_2^A)}{\partial x_2^A} \\ B1: \quad & x_2^{B\prime} = \mu_{x_2^B}^B \frac{\partial U^B(x_1^B, x_2^B)}{\partial x_2^B} \end{aligned} \qquad \text{<6.14>}$$

For the GE model the budget conditions are formulated for the stock variables $x_1^A, x_2^A, x_1^B, x_2^B$:

$$Z^A: \quad 0 = Z^A(x_1^A, x_2^A, p_1, p_2) = $$
$$= p_1 x_1^A + p_2 x_2^A - (p_1 x_1^A 0 + p_2 x_2^A 0)$$

<6.15>

$$Z^B: \quad 0 = Z^B(x_1^B, x_2^B, p_1, p_2) = $$
$$= p_1 x_1^B + p_2 x_2^B - (p_1 x_1^B 0 + p_2 x_2^B 0)$$

Since in the GCD model the prices changes over time, one must formulate the budget conditions for a differentially small exchange. This leads to the budget condition for the flow variables $x_1^A{}', x_2^A{}', x_1^B{}', x_2^B{}'$

$$Z^A*: \quad 0 = Z^A * (x_1^A{}', x_2^A{}', p_1, p_2) = p_1 x_1^A{}' + p_2 x_2^A{}'$$
$$Z^B*: \quad 0 = Z^B * (x_1^B{}', x_2^B{}', p_1, p_2) = p_1 x_1^B{}' + p_2 x_2^B{}'$$

<6.16>

The conditions for the excess supply can be formulated equivalently for the stock variables

$$Z_1: \quad 0 = z_1(x_1^A, x_1^B) = x_1^A + x_1^B - (x_1^A 0 + x_1^B 0)$$
$$Z_2: \quad 0 = z_2(x_2^A, x_2^B) = x_2^A + x_2^B - (x_2^A 0 + x_2^B 0)$$

<6.17>

or for the flow variables

$$Z_1*: \quad 0 = z_1 * (x_1^A{}', x_1^B{}') = x_1^A{}' + x_1^B{}'$$
$$x_1^A(0) = x_1^A 0$$
$$x_2^A(0) = x_2^A 0$$
$$Z_2*: \quad 0 = z_2 * (x_2^A{}', x_2^B{}') = x_2^A{}' + x_2^B{}'$$
$$x_1^B(0) = x_1^B 0$$
$$x_2^B(0) = x_2^B 0$$

Note that the budget constraints <6.17> depend on p_1, p_2. Therefore, according to remark 2 in chapter 4.1, 2 further behavioral equations are required to obtain a complete model. Any 2 behavioural equations for p_1, p_2 can be used. This results in the following GCD ex-post equations:

$$B1: \quad x_1^{A\prime} = \mu_{x_1^A}^A \frac{\partial U^A(x_1^A, x_2^A)}{\partial x_1^A} + \lambda^A p_1 + \lambda_1$$

$$B2: \quad x_2^{A\prime} = \mu_{x_2^A}^A \frac{\partial U^A(x_1^A, x_2^A)}{\partial x_2^A} + \lambda^A p_2 + \lambda_2$$

$$B3: \quad x_1^{B\prime} = \mu_{x_1^B}^B \frac{\partial U^B(x_1^B, x_2^B)}{\partial x_1^B} + \lambda^B p_1 + \lambda_1$$

$$B4: \quad x_2^{B\prime} = \mu_{x_2^B}^B \frac{\partial U^B(x_1^B, x_2^B)}{\partial x_2^B} + +\lambda^B p_2 + \lambda_2$$

$$B5: \quad behavioural\ equation\ for\ p_1$$

$$B6: \quad behavioural\ equation\ for\ p_2$$

$$Z^A*: \quad 0 = Z^A*(x_1^A{}', x_2^A{}', p_1, p_2) = p_1 x_1^A{}' + p_2 x_2^A{}'$$

$$Z^B*: \quad 0 = Z^B*(x_1^B{}', x_2^B{}', p_1, p_2) = p_1 x_1^B{}' + p_2 x_2^B{}'$$

$$Z_1: \quad 0 = z_1(x_1^A, x_1^B) = x_1^A + x_1^B - (x_1^A 0 + x_1^B 0)$$

$$Z_2: \quad 0 = z_2(x_2^A, x_2^B) = x_2^A + x_2^B - (x_2^A 0 + x_2^B 0) \qquad \text{<6.18>}$$

We now discuss some possible behavioural equations B5, B6.

6.4.2. Model 1

The auctioneer P knows the general equilibrium price $\bar{p}_1, \bar{p}_2$ and tries to change the actual price p_1, p_2 with a force which is proportional to the difference of the actual price and the equilibrium price. This is equivalent that he acts with a force which is proportional to the gradient of the utility function

$$U^P(p_1, p_2) = -\frac{1}{2}(\bar{p}_1 - p_1)^2 - \frac{1}{2}(\bar{p}_2 - p_2)^2 \qquad \text{<6.19>}$$

and a power $\mu_{p_1}^P, \mu_{p_2}^P$ to influence the price.

To simplify we assume good 1 to be a numeraire, i.e. assuming $p_1 = \bar{p}_1 = 1$From <6.12> results

$$\bar{p}_2 = \frac{(1-\alpha^A)x_1^A 0 + (1-\alpha^b)x_1^B 0}{\alpha^A x_2^A 0 + \alpha^b x_2^B 0}$$

This yields the behavioural equations for p_1, p_2 :

$$B5: \quad p_1 = 1$$

$$B6: \quad p_2' = \mu_{p_2}^P \frac{\partial U^P}{\partial p_2} = \mu_{p_2}^P (\bar{p}_2 - p_2) =$$

$$= \mu_{p_2}^P \left(\frac{(1-\alpha^A)x_1^A 0 + (1-\alpha^b)x_1^B 0}{\alpha^A x_2^A 0 + \alpha^b x_2^B 0} - p_2\right)$$

6.4.3. Model 2

Again, we assume the good 1 to be a numeraire, i.e. $p_1 = 1$.

The auctioneer tries to change the actual price p_1, p_2 with a force that reduces the excess supply. The model is described in detail in (Glötzl, Glötzl, und Richters 2019). It results in

$$B5 \qquad p_1 = 1$$

$$B6 \qquad p_2' = \mu_{p_2}^P \left(\frac{1}{1+(p_2)^2} \right) \mu^A \left(\frac{\partial U^A}{\partial x_2^A} - p_2 \frac{\partial U^A}{\partial x_1^A} \right) + \\ + \mu_{p_2}^P \left(\frac{1}{1+(p_2)^2} \right) \mu^B \left(\frac{\partial U^B}{\partial x_2^B} - p_2 \frac{\partial U^B}{\partial x_1^B} \right)$$

6.4.4. Model 3

In reality, price negotiations usually follow the following pattern: If A sells a product to B then A proposes a selling price p_2^A that is advantageous for him and B also proposes a purchase price p_2^B that is advantageous for him. Depending on the negotiating power (negotiating skill) $\mu_{p_2}^A$ and $\mu_{p_2}^B$ of A and B, respectively, a weighted average value for the price p_2 is agreed

$$p_2 = \frac{\mu_{p_2}^A}{\mu_{p_2}^A + \mu_{p_2}^B} p_2^A + \frac{\mu_{p_2}^B}{\mu_{p_2}^A + \mu_{p_2}^B} p_2^B$$

If $\frac{\partial U^A}{\partial x_2^A}$ is high A will offer a high selling price p_2^A and if $\frac{\partial U^B}{\partial x_2^B}$ is low B will offer a low purchasing price p_2^B, For the sake of simplicity, we can therefore assume that

$$p_2^A = \rho^A \frac{\partial U^A}{\partial x_2^A} \qquad p_1^A = \rho^A \frac{\partial U^A}{\partial x_1^A} = 1 \qquad \Rightarrow p_2^A = \frac{\frac{\partial U^A}{\partial x_2^A}}{\frac{\partial U^A}{\partial x_1^A}}$$

$$p_2^B = \rho^B \frac{\partial U^B}{\partial x_2^B} \qquad p_1^B = \rho^B \frac{\partial U^B}{\partial x_1^B} = 1 \qquad \Rightarrow p_2^B = \frac{\frac{\partial U^B}{\partial x_2^B}}{\frac{\partial U^B}{\partial x_1^B}}$$

This results in **model 3a**

$$B5 \qquad p_1 = 1$$

$$B6 \qquad p_2 = \frac{\mu_{p_2}^A}{\mu_{p_2}^A + \mu_{p_2}^B} p_2^A + \frac{\mu_{p_2}^B}{\mu_{p_2}^A + \mu_{p_2}^B} p_2^B =$$

$$= \frac{\mu_{p_2}^A}{\mu_{p_2}^A + \mu_{p_2}^B} \frac{\frac{\partial U^A}{\partial x_2^A}}{\frac{\partial U^A}{\partial x_1^A}} + \frac{\mu_{p_2}^B}{\mu_{p_2}^A + \mu_{p_2}^B} \frac{\frac{\partial U^B}{\partial x_2^B}}{\frac{\partial U^B}{\partial x_1^B}}$$

However, it is also possible that this price does not arise immediately, but that the price negotiation process causes the current price to move in the direction of this desired price. This results in **model 3b** and can be modelled by the behavioural equation

$$B6 \qquad p_2{}' = \mu \left(\frac{\mu_{p_2}^A}{\mu_{p_2}^A + \mu_{p_2}^B} \frac{\frac{\partial U^A}{\partial x_2^A}}{\frac{\partial U^A}{\partial x_1^A}} + \frac{\mu_{p_2}^B}{\mu_{p_2}^A + \mu_{p_2}^B} \frac{\frac{\partial U^B}{\partial x_2^B}}{\frac{\partial U^B}{\partial x_1^B}} - p_2 \right)$$

Here μ is a parameter such that $\frac{1}{\mu}$ expresses the rigidity of prices. $\mu \to \infty$ results in $B6 \to B6$

6.5. Numerical calculations

https://www.dropbox.com/s/5ja8lrbkwb9iqb0/Edgeworth%20%20Buch%20model%201%2B2%2B3%20Version%203.nb?dl=0

Note:

In model 1 and 2 μ^P is a measure for the power of the auctioneer. For $\mu^P = 0$ model 1 and model 2 are equivalent

In model 3b μ is a measure such that $\frac{1}{\mu}$ expresses the rigidity of prices

Model 3a corresponds to model 3b with $\mu \to \infty$

The following graph shows the tatonnement of model 1,2,3a,3b for

$power\ factors$	$\mu^P = \mu = 0$	
$endowment$	$x_1^A = 30$	$x_1^B = 5$
	$x_2^A = 10$	$x_2^B = 20$
$start\ price$	$p_0 = 1$	
$power\ factors$	$\mu^A = 1.5$	$\mu^B = 1$
$Cobb-Douglas\ parameter$	$\alpha^A = 0.3$	$\alpha^B = 0.6$

The x-axis is x_1^A , the y-axis is x_1^B .

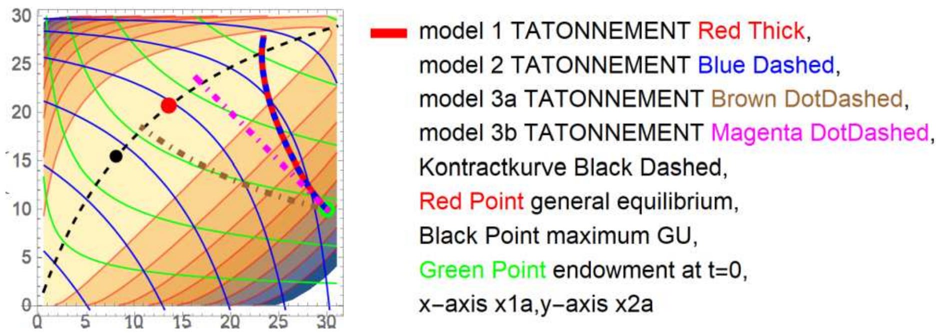
The contract curve is black dashed.

Isolines of U^A are blue, isolines of U^B are green, Isolines of the overall utility $GU = U^A + U^B$ are brown

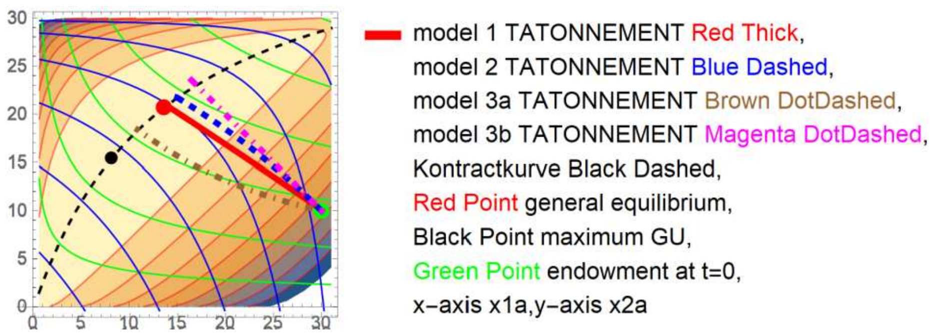
The green point is the endowment and thereby the starting point of the tatonnement.

The red point is the general equilibrium point.

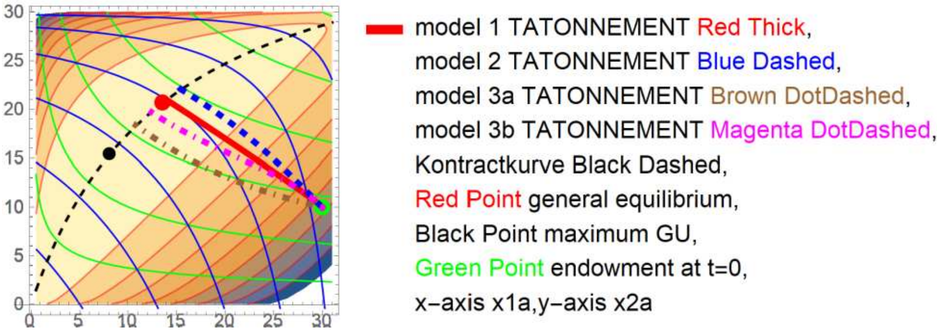
The black point is the maximum of the overall utility $GU = U^A + U^B$



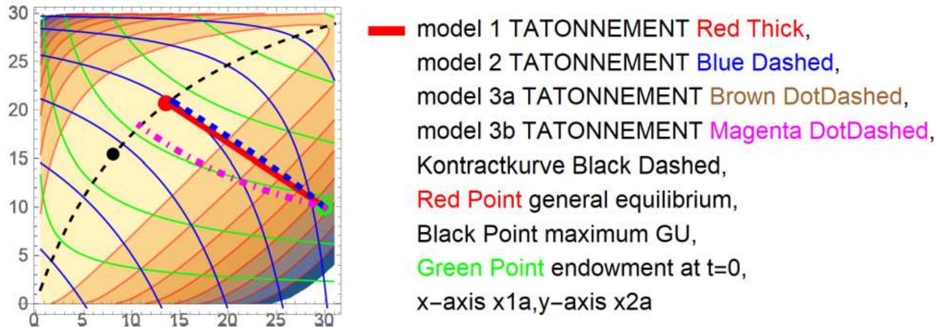
The higher the power μ^P of the auctioneer, the more likely Model 1 and Model 2 converge to general equlibrium. Model 1 converges faster than model 2. The tatonnements of model 1 and model 2 are shown for $\mu^P = 1$ in the following graph:



The higher μ the more likely the tatonnement of model 3b converge to tatonnement of model 3a. This shown in the next graph with $\mu = 0.05$ ($\mu^P = 1$ remains unchanged)



For high μ^P (e.g. $\mu^P = 5$) tatonnement of model 1 and model 2 converge to the general equilibrium point. For high μ (e.g. $\mu = 2$) the tatonnement of model 3a und model 3b are identical, but neither converge to the general equilibrium point nor to the maximum of GU.



If one changes the parameters (endowment, power factors, initial price and Cobb-Douglas parameters), the behaviour of the models differs to a greater or lesser extent in each case. The best summary is that model 1 typically converges best to the general equilibrium.

C. Macroeconomic models

7. The principle set up of GCD models

7.1. The model graph

It has proved to be extremely helpful to present each model in the form of a model graph. This provides an immediate overview of the agents, stock variables and flow variables. Using model A2 we also show how the constraints can be systematically determined from the model graph (see chapter 8.2.). Another possibility for the systematic representation of a model results from specifying the corresponding transaction matrices. This method is often used to describe SFC models (stock flow consistent models). Constraints can also be derived from this in a systematic way (see Chap. 8.3). However, we prefer the description of a model with model graphs, as long as the models are not so complex that the graphs become unclear.

In detail a GCD model consists of the following elements:

7.2. Agents

In principle, any number of any agents is possible, e.g:

- One or more households
- One or more firms
- One or more banks
- A central bank
- One State
- Any other agents

7.3. Goods

Agents exchange goods (flows) and/or store them (stocks) or create or destroy them. In GCD models it is useful to consider not only money but also all other goods that are usually exchanged for money at a certain price.

In principle any number of any goods is possible, e.g:

- Money
- Goods
- Services
- Labour
- debt notes (promissory notes)

  (receivables = positive stock of debt notes, liabilities = negative stock of debt notes). The immediate price of a debt note is usually 1 (e.g.: for lending 100 € you get 100 debt notes). However, debt notes usually trigger corresponding interest payments.

- Energy

- Raw materials

- etc.

7.4. Variables

All stocks, all flows and all creation and destruction processes are represented by time-dependent variables.

It is important to distinguish between 2 types of variables: Differentially defined variables and algebraically defined variables.

We first assume that only differentially defined variables occur. This means that the behavioural equations of all variables that appear in the utility functions are given by the differential equations of the general GCD model equations in the form <3.11>. We therefore refer to these variables as differentially defined variables. However, in the models variables are also possible for which the behavioural equations are not given by a differential equation but by an algebraic equation, e.g. by assuming a certain production function

$$Y(t) = \beta L^{\alpha} K^{(1-\alpha)}$$

or a specific rule for determining the amount of household income tax

$$T^{H}(t) = 0.3wL$$

In chapter 7.11 the algebraically defined variables are explained in more detail.

7.5. Constraint conditions

For every agent and every good, the following conservation equation, which is called a constraint, must necessarily apply:

Incoming goods - outgoing goods + production of goods – *- destruction of goods - change in stock of goods = 0*

E.g. for a company that produces a number $Y(t)$ of machines, designate

$C(t)$ the part of the machines which are sold,
$S(t)$ the stock in the warehouse,
$K(t)$ the number of machines used for production, i.e. the real capital stock and
$I(t)$ the investment, i.e the part of production used for its own further production, the following constraint holds

$$Y(t) - C(t) - S'(t) - I(t) = Y(t) - C(t) - S'(t) - K'(t) = 0$$

We avoid the formulation of this constraint by valuation at market prices p

$$pY(t) - pC(t) - pS'(t) - pI(t) = 0$$

because only the term $pC(t)$ corresponds to a real flow, namely the flow of money when machines are sold, whereas the other terms correspond to a flow of values. However, since valuations can change very easily, the conservation equation for values generally applies only to a very limited extent and must be applied with great caution.

In addition to the above-mentioned constraints, which are derived from the conservation equations for each good for each agent, there are also other constraints imposed by model assumptions, such as the assumption that all consumer goods are consumed immediately and not stored.

Model graphs in the form of flow charts and/or transaction matrices for all goods are very helpful in establishing the constraints. We show model graphs in the form of flow charts for each model. We explain the use of the corresponding transaction matrices with an example in chapter 8.3.

Note: The conservation equations for GCD models are closely related to the conservation equations of physics and chemistry, e.g:

1st law of thermodynamics (conservation of energy)
1st law of chemistry (conservation of mass)

Since debts (liabilities) and accounts (receivables) always arise simultaneously and in the same amount, it applies that in a closed system the sum of debts (liabilities) must always be the same as the sum of accounts (receivables). This analogy to the conservation laws of physics makes it reasonable to call this fundamental relationship for a monetary economy "1st law of economics" (Glötzl 1999; 2009)

7.6. Utility functions for each agent

The behaviour of an agent is described by its utility function. These utility functions are not subject to any restrictions and can basically depend on all variables (stocks and flows) and any parameters.

In equilibrium models, as for example also in DSGE models, the utility functions must always be required to be able to be aggregated, because otherwise a description via a maximisation is basically not possible. (For the definition of aggregability see Chapter 4.3). GCD models are not subject to this restriction.

7.7. Power factors for each agent for each variable

An agent's interest in changing variables does not per se lead to actual change, because the agent must also have the power or opportunity to actually implement its desire for change. This is described by the so-called power factor
μ_x^A, which can assume values between 0 and ∞. A high-power factor leads to a rapid temporal adjustment of the variables. The power factors in some sense can therefore also be interpreted as speed adjustment factors.

7.8. GCD model equations for the simple case (utility functions and constraints depend only on (x_1, x_2))

7.8.1. Ex-ante equations of motion

We explain the principle for 2 agents A, B and 2 variables x_1, x_2.

The utility functions of A, B are $U^A(x_1, x_2), U^B(x_1, x_2)$. The interest of A is to change x_1, x_2 so that the increase of his utility function is maximal. This is given, if the change of x_1, x_2 is done in the direction of the gradient of $U^A(x_1, x_2)$ $U^A(x_1, x_2)$, i.e.

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \quad proportional \quad \begin{pmatrix} \dfrac{\partial U^A}{\partial x_1} \\ \dfrac{\partial U^B}{\partial x_1} \end{pmatrix}$$

The interest of A in a change of the variables does not lead alone to an actual change, because the household must have also the power and/or possibility of actually implementing its change desire. For example, a household cannot or can only partially enforce its additional consumption desire, e.g., to go to the cinema or go on vacation, because it is possibly quarantined or the borders are closed. This limitation of the possibility to enforce his consumption change requests is described by a (possibly time-dependent and endogenously determined) "power factor" μ_C^H. In general, the change request for each of the variables is described by "power factors" $\mu_{x_1}^A, \mu_{x_2}^A, \mu_{x_1}^B, \mu_{x_2}^B$. Considering the power factors, the following applies to the change of x_1, x_2 (due to the interest of A and the power of A to enforce this interest)

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \quad proportional \quad \begin{pmatrix} \mu_{x_1}^A \dfrac{\partial U^A}{\partial x_1} \\ \mu_{x_2}^A \dfrac{\partial U^A}{\partial x_2} \end{pmatrix}$$

Just as A has an interest, to change x_1, x_2, also B has an interest to change these two variables. The actual change is therefore the result of the two individual efforts to change, weighted with the power factors. We therefore refer to this behaviour as **"individual utility optimisation"**.

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \mu_{x_1}^A \dfrac{\partial U^A}{\partial x_1} \\ \mu_{x_2}^A \dfrac{\partial U^A}{\partial x_2} \end{pmatrix} + \begin{pmatrix} \mu_{x_1}^B \dfrac{\partial U^B}{\partial x_1} \\ \mu_{x_2}^B \dfrac{\partial U^B}{\partial x_2} \end{pmatrix} \qquad \text{<7.1>}$$

In case there is a "master utility function" MU such that

$$\begin{pmatrix} \mu_{x_1}^A \dfrac{\partial U^A}{\partial x_1} \\ \mu_{x_2}^A \dfrac{\partial U^A}{\partial x_2} \end{pmatrix} + \begin{pmatrix} \mu_{x_1}^B \dfrac{\partial U^B}{\partial x_1} \\ \mu_{x_2}^B \dfrac{\partial U^B}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial MU}{\partial x_1} \\ \dfrac{\partial MU}{\partial x_2} \end{pmatrix} \tag{<7.2>}$$

the two utility functions can be **aggregated**. Then

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \dfrac{\partial MU}{\partial x_1} \\ \dfrac{\partial MU}{\partial x_2} \end{pmatrix} \tag{<7.3>}$$

Equation <7.3> describes the temporal change of the variable along the gradient of MU. If MU is concav, (x_1,x_2) converges to the maximum value of MU, i.e.

$$\lim_{t\to\infty}(x_1(t),x_2(t)) = (x_1^{\max},x_2^{\max}) \quad with\ MU(x_1^{\max},x_2^{\max}) = maximal$$

Define the overall utility function $GU = U^A + U^B$.(see also chapter 4.3). If the overall utility function equals the master utility function, i.e. $GU = MU$, we therefore refer to

$$\begin{aligned} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} &= \begin{pmatrix} \mu_{x_1}^A \dfrac{\partial U^A}{\partial x_1} \\ \mu_{x_2}^A \dfrac{\partial U^A}{\partial x_2} \end{pmatrix} + \begin{pmatrix} \mu_{x_1}^B \dfrac{\partial U^B}{\partial x_1} \\ \mu_{x_2}^B \dfrac{\partial U^B}{\partial x_2} \end{pmatrix} = \\ &= \begin{pmatrix} \dfrac{\partial MU}{\partial x_1} \\ \dfrac{\partial MU}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial GU}{\partial x_1} \\ \dfrac{\partial GU}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial (U^A+U^B)}{\partial x_1} \\ \dfrac{\partial (U^A+U^B)}{\partial x_2} \end{pmatrix} \end{aligned} \tag{<7.4>}$$

as **"overall utility maximisation"**.

These equations of motion <7.1> resp. <7.4> describe the dynamics of (x_1,x_2) under the condition that there are no constraints that restrict the dynamics. It is therefore referred to as the **ex-ante equation of motion**.

7.8.2. Ex-post equations of motion

7.8.2.1. Vertical constraint forces

If a constraint

$$Z(x_1,x_2) = 0$$

has to be fulfilled, an additional constraint force f^Z has to be added to the ex-ante force

$$x_i' = \sum_{j=1}^{J} \mu_i^j f_i^j + f^Z \qquad i = 1,2,\ldots,I \tag{<7.5>}$$

to ensure the constraint Z to be fulfilled at all times. In physics, this constraint force f^Z is perpendicular to the constraint at all times due to the so-called d'Alembert principle, i.e.

$$f^Z(x_1,x_2) = \begin{pmatrix} f_1^Z(x_1,x_2) \\ f_2^Z(x_1,x_2) \end{pmatrix} = \lambda \begin{pmatrix} \dfrac{\partial Z(x_1,x_2)}{\partial x_1} \\ \dfrac{\partial Z(x_1,x_2)}{\partial x_2} \end{pmatrix} \qquad \text{<7.6>}$$

We therefore refer to this type of constraint forces as **"vertical constraint forces"**. The time-dependent factor $\lambda = \lambda(t)$ is called Lagrange multiplier, as in the case of optimisation under constraints.

Vertical constraint forces can also be characterised by the following equivalent principles. This is because the theorem (Glötzl 2018) holds that the following principles are equivalent:

Theorem:

(1) **d'Alembert's principle** (constraint forces do no work)

(2) **vertical constraint forces** (constraint forces are perpendicular to the manifold of constraint conditions)

(3) **Gaussian principle of least constraint** (those constraint forces f^{Z_i} occur for which $\left\| f^{Z_i} \right\| \to minimal$)

(4) **unnamed principle**

If x is a solution of

$$\begin{aligned} x' &= f(x) + f^Z(x) \\ 0 &= Z(x) \end{aligned}$$

then f^Z satisfies the unnamed principle $:\Leftrightarrow \dfrac{d\|x'\|}{dt} = \dfrac{\langle x', f \rangle}{\|x'\|}$

Note: If one of the equivalent principles is satisfied, then the constraint force has no effect on $\|x'\|$ but only on the direction of x'. Note, however, that the inverse does not hold.

It is therefore plausible in many cases to model constraint forces in economics in an analogous way to physics in terms of d'Alembert's principle respectively as vertical constraint forces.

From <7.1> and <7.6> results the "equation of motion considering the constraint condition", called **ex-post equation of motion**:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \mu_{x_1}^A \dfrac{\partial U^A(x_1,x_2)}{\partial x_1} \\ \mu_{x_2}^A \dfrac{\partial U^A(x_1,x_2)}{\partial x_2} \end{pmatrix} + \begin{pmatrix} \mu_{x_1}^B \dfrac{\partial U^B x_1,x_2}{\partial x_1} \\ \mu_{x_2}^B \dfrac{\partial U^B(x_1,x_2)}{\partial x_2} \end{pmatrix} +$$

$$+\lambda \begin{pmatrix} \dfrac{\partial Z(x_1,x_2)}{\partial x_1} \\ \dfrac{\partial Z(x_1,x_2)}{\partial x_2} \end{pmatrix} \qquad \text{<7.7>}$$

$$0 = Z(x_1,x_2)$$

If U^A, U^B can be aggregated to a master utility function MU , the equation of motion is as follows

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \dfrac{\partial MU(x_1,x_2)}{\partial x_1} \\ \dfrac{\partial MU(x_1,x_2)}{\partial x_2} \end{pmatrix} + \lambda \begin{pmatrix} \dfrac{\partial Z(x_1,x_2)}{\partial x_1} \\ \dfrac{\partial Z(x_1,x_2)}{\partial x_2} \end{pmatrix} \qquad \text{<7.8>}$$

$$0 = Z(x_1,x_2)$$

and if the master utility function MU is concave, (x_1,x_2) converge to a local maximum value of MU under the constraint Z , i.e.

$$\lim_{t\to\infty}(x_1(t),x_2(t)) = (x_1^{\max,Z}, x_2^{\max,Z})$$
$$with\ MU(x_1^{\max,Z}, x_2^{\max,Z}) = maximal\ under\ constraint\ Z$$

and it holds that the dynamics at $(x_1^{\max,Z}, x_2^{\max,Z})$ is stationary, i.e.

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \dfrac{\partial MU(x_1^{\max,Z}, x_2^{\max,Z})}{\partial x_1} \\ \dfrac{\partial MU(x_1^{\max,Z}, x_2^{\max,Z})}{\partial x_2} \end{pmatrix} + \lambda \begin{pmatrix} \dfrac{\partial Z(x_1^{\max,Z}, x_2^{\max,Z})}{\partial x_1} \\ \dfrac{\partial Z(x_1^{\max,Z}, x_2^{\max,Z})}{\partial x_2} \end{pmatrix} = 0 \qquad \text{<7.9>}$$

or equivalently

$$\begin{pmatrix} \dfrac{\partial MU(x_1^{\max,Z}, x_2^{\max,Z})}{\partial x_1} \\ \dfrac{\partial MU(x_1^{\max,Z}, x_2^{\max,Z})}{\partial x_2} \end{pmatrix} = -\lambda \begin{pmatrix} \dfrac{\partial Z(x_1^{\max,Z}, x_2^{\max,Z})}{\partial x_1} \\ \dfrac{\partial Z(x_1^{\max,Z}, x_2^{\max,Z})}{\partial x_2} \end{pmatrix} \qquad \text{<7.10>}$$

In general, for

J agents with the designations $\quad j \qquad j=1,2,...,J$

I Variables with the designations $x_i \qquad i=1,2,...,I$

$$x=(x_1,x_2,...,x_I)$$

K Constraints with the designations $Z^k \qquad k=1,2,...,K$

the I **general GCD model equations for vertical constraint forces** are obtained analogously

$$\boxed{x_i' = \sum_{j=0}^{J} \mu_{x_i}^j \frac{\partial U^j}{\partial x_i} + \sum_{k=1}^{K} \lambda^k \frac{\partial Z^k}{\partial x_i} \qquad i=1,2,...I \qquad \text{<7.11>}}$$

If there is a "master utility function" MU such that

$$\sum_{j=0}^{J} \mu_{x_i}^j \frac{\partial U^j}{\partial x_i} = \frac{\partial MU}{\partial x_i} \qquad i=1,2,...,I \qquad \text{<7.12>}$$

the utility functions U^j, $j=1,2,...,J$ are called **aggregable**.

If $MU = \sum_{j=1}^{J} U^j$, the master utility function is called the overall utility function. If the master utility function MU is convex, x converges to the maximum value of MU under the constraint conditions Z^k, $k=1,2,...,K$.

7.8.2.2. Other constraint forces

Another type of constraint force that can occur, especially in the case of a constraint force describing a limited resource, is a constraint force that is centrally directed to the origin. We therefore refer to this as a "**central constraint force**".

$$f^Z(x_1(t),x_2(t)) = \begin{pmatrix} f_1^Z(x_1(t),x_2(t)) \\ f_2^Z(x_1(t),x_2(t)) \end{pmatrix} = \varphi(t) \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \qquad \text{<7.13>}$$

A model for this are constraint forces such as occur in theoretical biology in the derivation of the so-called replicator equation (Glötzl 2023a). In biology, this model assumption of a central constraint force is equivalent to the assumption that in the struggle for limited resources, equally high death rates are triggered for all species.

Let us illustrate this with an example. A typical dynamic in biology is the initially independent exponential growth of 2 species A and B with birth rates b_A, b_B .

$$\begin{aligned} n'_A &= b_A n_A \qquad b_A \text{ "growth rate"} \\ n'_B &= b_B n_B \qquad b_B \text{ "growth rate"} \end{aligned} \qquad \text{<7.14>}$$

A constraint typical for biology is, for example, the assumption of limited resources. This can be given, for example, by a limitation of the food supply or also by a limitation of the habitat. This results in the sum of the number of absolute frequencies of the different species remaining constant. This is formally described by the constraint condition

$$Z(n_1, n_2, ...) = \sum_i n_i - constant = 0$$

Assuming that the constraint condition triggers equally high death rates in both species, the differential algebraic equation system is obtained

$$\begin{aligned} n'_A &= b_A n_A - \varphi n_A \\ n'_B &= b_B n_B - \varphi n_B \\ Z(n_A, n_B) &= n_A + n_B - n = 0 \qquad n \text{ constant} \end{aligned} \qquad \text{<7.15>}$$

Assuming that A is twice as successful ("powerful") in the struggle for resources, the death rate for A would be half as high and thus the system of equations would be

$$\begin{aligned} n'_A &= b_A n_A - \varphi \frac{1}{2} n_A \\ n'_B &= b_B n_B - \varphi n_B \\ Z(n_A, n_B) &= n_A + n_B - n = 0 \qquad n \text{ constant} \end{aligned}$$

When applied to economic constraints, this can be interpreted as follows. Agents can have different powers to oppose constraints. For example, if raw materials are limited in total, it may be easier for some countries to obtain the necessary raw materials than for others.

In the most general case, different types of constraint forces can occur. Essential for the modeling is only that the constraint forces used must be linearly independent and multiplied by the respective Lagrange multiplier.

Note: In the case where not all constraint forces are vertical, x typically does not converge to the maximum value of MU under the constraints $Z^k, k = 1, 2, ..., K$, even if the master utility function is convex.

As a rule, it is sufficient to use purely vertical constraint forces. In the following, we will therefore always restrict ourselves to vertical constraint forces.

7.9. GCD model equations for the general case (utility functions and constraints also depend on antiderivatives and/or derivatives of x_1, x_2)

7.9.1. Constraints depend on antiderivatives and/or derivatives

So far, we have assumed that the constraints depend only on x. However, the constraints can also depend on the antiderivatives $X = (X_1, X_2, ..., X_I)$. This means, X_i is antiderivative of x_i, iff $X_i' = x_i$. The constraints can depend in principle, however, also on the time derivatives $x' = (x_1', x_2', ..., x_I')$. In physics it is valid (Flannery 2011), that the constraint force always results from derivative with respect to the highest time derivative of x, i.e.

If $Z(..., X_i, ...)$ then $f_i^Z = \dfrac{\partial Z}{\partial X_i}$ and

$$x_i' = \sum_{j=0}^{J} \mu_{x_i}^j \frac{\partial U^j}{\partial x_i} + \sum_{k=1}^{K} \lambda^k \frac{\partial Z^k(..., X_i, ...)}{\partial X_i} \qquad i = 1, 2, ...I \qquad \text{<7.16>}$$

If $Z(..., X_i, x_{i,} ...)$ then $f_i^Z = \dfrac{\partial Z}{\partial x_i}$ and

$$x_i' = \sum_{j=0}^{J} \mu_{x_i}^j \frac{\partial U^j}{\partial x_i} + \sum_{k=1}^{K} \lambda^k \frac{\partial Z^k(..., X_i, x_i, ...)}{\partial x_i} \qquad i = 1, 2, ...I \qquad \text{<7.17>}$$

If $Z(..., X_i, x_i, x_i', ...)$ then $f_i^Z = \dfrac{\partial Z}{\partial x_i'}$ and

$$x_i' = \sum_{j=0}^{J} \mu_{x_i}^j \frac{\partial U^j}{\partial x_i} + \sum_{k=1}^{K} \lambda^k \frac{\partial Z^k(..., X_i, x_i, x_i', ...)}{\partial x_i'} \qquad i = 1, 2, ...I \qquad \text{<7.18>}$$

We assume that this approach is also plausible in economics in the case of vertical constraints.

7.9.2. Utility functions depend on antiderivatives and/or derivatives

So far, we have assumed that utility functions only depend on x. But also, the utility functions can additionally depend on antiderivatives and derivatives of x. In these cases, both the antiderivatives $X = (X_1, X_2, ..., X_I)$ and the derivatives $x' = (x_1', x_2', ..., x_I')$ are to be considered as additional variables in their own right, i.e.

$$X = (X_1, X_2, ..., X_I) = (x_{I+1}, x_{I+2}, ..., x_{2I})$$
$$x' = (x_1', x_2', ..., x_I') = (x_{2I+1}', x_{2I+2}', ..., x_{3I}')$$

In that case, the following additional constraints must be used

$$x_{I+i}' - x_i = 0 \qquad i = 1, 2, ..., I$$
$$x_i' - x_{2I+i} = 0 \qquad i = 1, 2, ..., I$$

7.10. Market forces

The behaviour of x_i is given by the general GCD model equation (for vertical constraint forces) for x_i <7.11>

$$x_i' = \sum_{j=0}^{J} \mu_{x_i}^j \frac{\partial U^j}{\partial x_i} + \sum_{k=1}^{K} \lambda^k \frac{\partial Z^k}{\partial x_i} \qquad i = 1,2,...I \qquad \text{<7.19>}$$

The right-hand side of <7.19>

$$\sum_{j=0}^{J} \mu_{x_i}^j \frac{\partial U^j}{\partial x_i} + \sum_{k=1}^{K} \lambda^k \frac{\partial Z^k}{\partial x_i}$$

thus describes the **market forces** that lead to a change in x_i and is composed of 2 parts. The market forces that agents exert on x_i

$$\sum_{j=0}^{J} \mu_{x_i}^j \frac{\partial U^j}{\partial x_i}$$

and the market forces that the constraints Z^k exert on x_i. These are just the constraint forces

$$f^{Z^k}(x) = \lambda^k \frac{\partial Z^k}{\partial x_i} \qquad k = 1,2,...,K$$

If for a particular i it holds that $\frac{\partial U^j(x)}{\partial x_i} = 0$, i.e. that the utility functions do not depend on $\mu_{x_1}^j$, or that the power factors $\mu_{x_1}^j = 0$, the general GCD model equation (for vertical constraint forces) reduces for x_i , to

$$x_i' = \sum_{k=1}^{K} \lambda^k \frac{\partial Z^k}{\partial x_i} \qquad i = 1,2,...I$$

In this case, the behaviour of x_i is determined exclusively by the constraint forces. Therefore, the constraint forces can also be called **"pure" market forces**,

7.11. Algebraically defined variables

So far we have assumed that the behavioural equations for all variables are given by differential equations in the form <7.11> to <7.18>. We therefore call these variables differentially determined variables. In the models, however, also variables are possible, with which the behavioural equations are not determined by a differential equation, but by an algebraic equation, e.g. by the assumption of a certain production function

$$Y(t) = \beta L^{\alpha} K^{(1-\alpha)}$$

or a specific rule for determining the amount of household income tax.

$$T^H(t) = 0.3wL$$

We call these variables algebraically defined variables. These algebraic behavioural equations can often be seen as limit values of differential equations with infinitely large power factors. For example, the behaviour of the government in collecting income tax could be described by the following behaviour. It aims to collect 30% of the wage income of the household as a tax. If the tax paid is less than this, e.g. through tax evasion, the government will try to increase the collection of the tax. This behaviour can be modeled in the following way, for example:

Let $U^G(T^H) = -\frac{1}{2}(0.3 - T^H)^2$ be the utility function of the Government G and Z any constraint, then results the behavioural equation

$$T^{H\,\prime} = \mu^H_{T^H}\frac{\partial U^G}{\partial T^H} + \lambda\frac{\partial Z}{\partial T^H} = \mu^H_{T^H}(0.3wL - T^H) + \lambda\frac{\partial Z}{\partial T^H}$$

If the government has infinite power to prevent tax evasion, this results in

$$T^{H\,\prime} = \mu^H_{T^H}(0.3wL - T^H) + \lambda\frac{\partial Z}{\partial T^H} \qquad \Leftrightarrow$$

$$\Leftrightarrow \qquad \frac{T^{H\,\prime}}{\mu^H_{T^H}} = (0.3wL - T^H) + \frac{\lambda}{\mu^H_{T^H}}\frac{\partial Z}{\partial T^H}$$

$$for\ \mu^H_{T^H} \to \infty\ results$$

$$0 = (0.3wL - T^H) \qquad \Leftrightarrow$$

$$T^H = 0.3wL$$

The algebraic behavioural equation $T^H = 0.3wL$ can thus be interpreted as a differential behavioural equation with infinite power of the government.

In case of occurrence of algebraically defined variables, when forming partial derivatives of utility functions and constraints with respect to the differentially defined variables, it must be taken into account that the algebraic variables occurring in utility functions and constraints may also depend on differentially defined variables. It is best to insert the algebraically defined variables into the utility functions and constraints before the differential equations are formed.

7.12. Numerical solutions

In most cases, the differential algebraic systems of equations cannot be solved analytically, but only numerically.

7.12.1. Initial values

In ordinary differential equation systems of the 1st order, the initial values for all variables are freely selectable. In contrast to ordinary differential equation systems, not all initial values of the variables are freely selectable in differential algebraic equation systems. The reason for this is that the initial values must satisfy the differential equations and also the constraints.

If there are no time derivatives in the constraints and there are K linearly independent constraints, only $I-K$ initial values can be chosen freely. The other initial values result from the solution of the system of equations of the constraints. However, if the constraints are nonlinear an analytical solution is often not possible. In many practical applications, however, the situation is much more complex, especially if time derivatives of variables also occur in the constraints.

In the usual numerical programs for solving differential-algebraic equations, an algorithm is therefore built in, which calculates from a sufficiently large number of initial values, other possible initial values, which approximately fulfill the system of equations up to a certain tolerance. One therefore needs an understanding of the model and a certain amount of experience to determine suitable initial values.

7.12.2. Parameter selection

The parameters of a GCD model cannot be chosen arbitrarily either. For the system of equations, a solution does not have to exist for every combination of parameters or be stable over a longer period of time. Therefore, one also needs an understanding of the model and a certain experience for the selection of the values for the individual parameters.

7.12.3. Numerical solution methods

We make use of two solution methods within the framework of MATHEMTICA, namely NDSolve and Modelica. Since differential algebraic systems of equations have a much higher overall complexity than ordinary differential systems of equations, many different methods of numerical procedures are available in NDSolve.

By default, it is usually sufficient to use:

Method→Automatic

Sometimes you need:

Method→{"EquationSimplification"->"Residual"}

Sometimes one needs:

Method→{IndexReduction→ Automatic }

Sometimes one needs:

Method→{IndexReduction→{True, ConstraintMethod→Projection}}

May be in special cases also other methods must be used

For the stability of the solutions, one has to distinguish 2 cases:

- The model itself may become unstable after a certain time because, for example, certain variables become 0.
- The model is basically stable, but the numerical errors can lead to instabilities after a longer runtime.

8. Examples of possible utility functions

8.1. Household

For example, a household may have the following targets:

- **Consumption target**: he would like to consume. His desire to consume more is greater the less he is currently consuming or can consume, and his desire to consume even more is smaller the more he is already consuming.
- **Labour target**: he would like to work, but not too much and not too little.
- **Money management target** (cash management target): he always wants to have liquid funds, not too little, so that he can buy everything he wants to buy at the moment and not too much, because he does not get any interest for it and it would be more advantageous to lend the money to the bank against interest on savings. Therefore, the higher the interest on savings, the lower his money-holding target.
- **Receivables holding target** (savings target): he would like to hold assets in the form of receivables from the bank, the more the higher the savings interest.

The stated targets of the household can be expressed, for example, by the following utility function:

$$U^H(C^H, L^H, M^H, A^H) = \left(C^H\right)^{\gamma} - \left(\hat{L}^H - L^H\right)^2 - (\hat{M}^H - M^H)^2 + A^H$$

$$\begin{array}{lll}
\textit{Variable:} & C^H & \textit{consumption} \\
 & L^H & \textit{labour} \\
 & M^H & \textit{money holding (liquid assets)} \\
 & A^H & \textit{claims on bank (savings)} \\
 & & \\
\textit{Parameter:} & \gamma & 0 \le \gamma \le 1 \\
 & \hat{L}^H & \textit{targeted labour} \\
 & \hat{M}^H & \textit{targeted money holding,} \\
 & & \textit{possibly depending on the interest rate}
\end{array}$$

8.2. Firm

A firm can have the following targets, for example:

- **Profit target**: The greater the profit, the greater the utility.
- **Warehousing target:** Warehousing causes costs and should therefore be as low as possible; on the other hand, it must not be too low, otherwise fluctuations in demand cannot be compensated.

- **Investment target**: The interest in investing depends (also!) on the level of interest rates on loans. If lending rates are 0 (or even negative due to possible investment incentives), as much is invested as is organisationally feasible. If lending rates rise, correspondingly less is invested.

The stated targets of the firm can be expressed, for example, in the following utility function.

$$U^F = profit^F - (\hat{S} - S)^2 - \left(invmax\left(1 - \theta(r + r_D)\right) - inv\right)^2$$

whereby the following „algebraically"defined variables are used

$$Y := \quad \beta L^a K^{1-a}$$

$$profit^F := pY - wL - (r + r_D)(-D^F) - DP =$$

$$= p\beta L^a K^{1-a} - wL - (r + r_D)(-D^F) - DP$$

$$invmax := inv\ K$$

The following gives the dependence of the utility function on the “differentially” defined variables:

$$U^F(p,L,K,w,D^F,DP,S,inv) =$$

$$= profit^F - (\hat{S} - S)^2 - \left(invmax\left(1 - \theta(r_{leit} + r_D)\right) - inv\right)^2 =$$

$$= p\beta L^a K^{1-a} - wL - (r_{leit} + r_D)(-D^F) - DP -$$

$$-(\hat{S} - S)^2 - \left(inv\ K\left(1 - \theta(r_{leit} + r_D)\right) - inv\right)^2$$

whereby the following „differentially"defined variables are used

p	price
L	*labour*
K	*capital*
w	*wages*
D^F	loans payable
DP	depreciation
S	inventories
inv	Net investment

whereby the following “algebraially” defined variables are used:

Y	*total output, Cobb - Douglas function*
$profit^F$	*profit*
$invmax$	*maximum net investment, when credit interest rates = 0*

and whereby the following "parameters" are used:

α	$Cobb-Douglas\ parameter$
β	technology *factor*
r_{leit}	central bank prime rate
r_D	ending rate premium on central bank base rate
$\hat{S}$	stock-keeping target
inv	maximal net investment factor
θ	factor for the interest rate dependency of the investments

Note: Note that the constraint $0 = K' - inv$ must apply to the variables K and inv in the sense of chapter 7.9.2.

8.3. Bank

For example, a bank may have the following target:

Profit target: The greater the profit, the greater the utility.

The stated target of the bank can be expressed, for example, in the following utility function.

$$U^B = profit^B$$

„Algebraically" defined variable

$$profit^B = +(r_{leit} + r_D).(-D^F) + (r_{leit} + r_D).(-D^G) - \\ -r_{leit} A^{ZB} - (r_{leit} + r_A) A^H$$

insert in U^B

$$U^B(D^F, D^G, A^{ZB}, A^H) = +(r_{leit} + r_D).(-D^F) + +(r_{leit} + r_D).(-D^G) - \\ - r_{leit} A^{ZB} - (r_{leit} + r_A) A^H$$

„Differentially" defined variables:

D^F	loans payable of firm
D^G	loans payable of government
A^{ZB}	loans receivable of central bank
A^H	loans receivable of household (Savings deposits)

„Parameters" are:

r_{leit}	central bank prime rate
r_D	lending rate premium on central bank interest rates
r_A	Savings interest surcharge on central bank interest rates

8.4. Central bank

The FED (Federal Reserve) has 3 targets:

- **Inflation target**: Inflation should be as close as possible to 2%.
- **Full employment target**: i.e., there should be neither unemployment nor overemployment due to overheating of the economy.
- **Target for the long-term interest rate**: moderate long-term interest rate. For the sake of simplicity, we will not consider this target any further in the following.

The first two targets can be modelled within the framework of the GCD models in the following two ways: by means of corresponding utility functions or by prescribing the setting of the prime interest rate by means of the so-called Taylor rule.

8.4.1. Utility function of a central bank

The full employment target can be expressed analogously to the utility function of the household by the term

$$-\left(\hat{L}-L\right)^2$$

in the utility function of the central bank. In contrast to the household, however, the central bank has no direct influence on employment, but only an indirect influence through its interest rate policy or its money supply policy. This means

$$m_L^{ZB} = 0 \qquad \textit{in contrast to } \; m_L^H \neq 0$$

$$m_{r_{leit}}^{ZB} > 0 \qquad \textit{Influence on the central bank base rate } r_{leit}$$

$$m_{N^{ZB}}^{ZB} > 0 \qquad \textit{Influence on money creation } N^{ZB}$$

A central bank can try to achieve the target of inflation in 2 different ways. Through interest rate policy (we characterise this by $\delta = 1$) or through money creation policy (we characterise this by $\delta = 0$). This behaviour of the central bank can be described by the following term in the utility function

$$\left(-\delta\, r_{leit} + (1-\delta)N^{ZB}\right)(\hat{p} - \frac{ps}{p})$$

with $0 \le \delta \le 1$

$\delta = 1$ *(pure interest rate policy)*

$\delta = 0$ *(pure money creation policy)*

r_{leit}	*central bank base rate*
N^{ZB}	*money creation* (*flow variable*!)
$\hat{p}$	*inflation target*
p	*price*
ps	*temporal price change* *(due to constraint 0 = ps - p')* *because of chapter* 3.9.2

It should be noted that the central bank has no direct influence on the price p, but can again only influence p and ps indirectly via the central bank base rate and money creation. This means

$\mu_p^{ZB} = 0$	*Influence on the price* p
$\mu_{ps}^{ZB} = 0$	*Influence on the change of the price* ps
$\mu_{r_{leit}}^{ZB} > 0$	*Influence on the central bank base rate* r_{leit}
$\mu_{N^{ZB}}^{ZB} > 0$	*Influence on money creation* N^{ZB}

The utility function

$$U^{ZB} = \left(-\delta\, r + (1-\delta)N^{ZB}\right)(\hat{p} - \frac{ps}{p}) - \left(\hat{L} - L\right)^2$$

with constraint 0 = ps - p'

because of chapter 3.9.2

leads (in addition to the other terms from the utility functions of other agents and the constraints) in the general GCD - model equations <3.11> to

$$r' = \mu_r^{ZB} \frac{\partial U^{ZB}}{\partial r} + ... = -\mu_r^{ZB}\, \delta\, (\hat{p} - \frac{ps}{p})$$

$$N^{ZB}{}' = \mu_{N^{ZB}}^{ZB} \frac{\partial U^{ZB}}{\partial N^{ZB}} + ... = +\mu_{N^{ZB}}^{ZB} (1-\delta)(\hat{p} - \frac{ps}{p})$$

$$p' = \mu_p^{ZB} \frac{\partial U^{ZB}}{\partial p} + ... = 0 + ... \qquad \textit{because of } \mu_p^{ZB} = 0$$

$$ps' = \mu_{ps}^{ZB} \frac{\partial U^{ZB}}{\partial ps} + ... = 0 + ... \qquad \textit{because of } \mu_{ps}^{ZB} = 0$$

$$L' = \mu_L^{ZB} \frac{\partial U^{ZB}}{\partial L} + ... = 0 + ... \qquad \textit{because of } \mu_L^{ZB} = 0$$

The term $-\mu_r^{ZB}\,\delta(\hat{p}-\frac{ps}{p})$ means: If the central bank pursues an interest rate policy ($\delta = 1\; bzw.\; \delta > 0$), it exerts a force on the interest rate r such that r grows (i.e. $r' > 0$), if the actual inflation is greater than the targeted inflation $\frac{ps}{p}$. The same is true in reverse.

The term $+\mu_{N^{ZB}}^{ZB}(1-\delta)(\hat{p}-\frac{ps}{p})$ means: If the central bank pursues an interest rate policy ($\delta = 0\; bzw.\; \delta < 1$), it exerts a force on the interest rate r such that r grows (i.e. $r' > 0$), if the actual inflation is smaller than the targeted inflation $\frac{ps}{p}$. The same is true in reverse.

8.4.2. Taylor rule

The **Taylor rule** is a monetary policy rule for setting the central bank base rate by a central bank. It reads:

$$\boxed{\begin{aligned} \textit{base rate} &= \text{real equilibrium interest rate} + \text{inflation} + \\ &+\sigma_1 \,\text{inflation gap} + \sigma_2 \,\text{growth rate gap} \end{aligned}} \quad \text{<8.1>}$$

Thereby, the weighting factors σ_1, σ_2 are derived from the actual behaviour of the central bank. If both gaps are equal to 0, the Taylor rule is equivalent to Fisher's rule

$$\textit{base rate} = \text{real equilibrium interest rate} + \text{inflation} \qquad \text{<8.2>}$$

We make the following simplifying assumptions:

Assumption 1: The economy is in equilibrium; therefore, it is reasonable to assume that the real equilibrium interest rate is equal to the real growth rate

$$\frac{Y'}{Y}.$$

Assumption 2: Full employment of the economy prevails exactly when the actual labour L is equal to the targeted labour $\hat{L}$, i.e.

$$\textit{Production at full employment } \hat{Y} = \beta K^{(1-\alpha)}\hat{L}^{\alpha}$$

$$\textit{growth rate at full employment} = \frac{\hat{Y}'}{Y}$$

If $\hat{p}$ denotes the targeted inflation rate, this results in

$$r_{leit} = \frac{Y'}{Y} + \frac{p'}{p} + \sigma_1(\frac{p'}{p} - \hat{p}) + \sigma_2(\frac{Y'}{Y} - \frac{\hat{Y}'}{\hat{Y}}) \qquad \text{<8.3>}$$

Interpretation: The interest rate is higher if the inflation rate $\frac{p'}{p}$ is higher than the target inflation rate $\hat{p}$ and/or the growth rate $\frac{Y'}{Y}$ is higher than the (target) growth rate at full employment.

If one inserts and simplifies one obtains

$$r_{leit} = \frac{p'}{p} + \sigma_1\left(\frac{p'}{p} - \hat{p}\right) + (1-\alpha)\frac{K'}{K} + (1+\sigma_2)\alpha\frac{L'}{L}$$ <8.4>

In terms of the GCD methodology, the Taylor rule sets the value of the policy rate as an algebraically defined variable. If the central bank acts only according to the Taylor rule, it does not act in the sense of optimising a utility function, but according to empirical values that have proven themselves in the past. In this case, one can therefore set the utility function of the central bank equal to 0.

8.4.3. Modified Taylor rule: Consideration of the interest rate premium on the key interest rate

The Fischer rule does not actually refer to the central bank's base interest rate, but to the lending rate. This consists of the base interest rate plus a premium. In economic equilibrium, this results in

$$\begin{aligned} &\textit{Loan interest rate} = \textit{base rate} + \textit{premium} = \\ &\qquad = \textit{growth rate} + \textit{inflation} \end{aligned}$$ <8.5>

Under these assumptions, this results in the **modified Taylor rule**

$$\boxed{\begin{aligned} &\text{base rate} = \\ &= \text{growth rate} - \textit{premium} + \text{inflation} + \\ &\qquad + \sigma_1\,\textit{inflation gap} + \sigma_2\,\textit{growth gap} \end{aligned}}$$ <8.6>

8.5. Government

The government pursues the following targets, for example.

- **Government expenditure target**: Government expenditure serves to fulfil government tasks and is often also referred to as government consumption. For simplicity's sake, we assume that the government behaves like a household. Its desire to consume even more is smaller the more it consumes anyway.
- **Government debt target**: e.g., target government debt in the sense of the Maastricht criteria (60% of GDP).
- **Employment target**: The government has the target of full employment, as does the Fed in the USA.
- **Tax ratio target**: for the sake of simplicity, we will not discuss this further below.
- **Growth target**: for the sake of simplicity, we will not discuss this further below.

The stated targets of the government can be expressed, for example, in the following utility function.

$$U^G = (C^G)^{\gamma_G} - (\hat{D}^G Y - D^G)^2 - (\hat{L} - L)^2$$

Where the „algebraically"defined variable Y is used

$$Y := \beta L^a K^{1\text{-}a}$$

Insert and you get the dependence of the "differentially" defined variables, i.e. the variaables defined by equation <3.7>

$$U^G(C^G, L, K, D^G) = (C^G)^{\gamma_G} - (\hat{D}^G \beta L^a K^{1\text{-}a} - D^G)^2 - (\hat{L} - L)^2$$

$$\begin{array}{ll} \textit{with parameters} & \\ \gamma_G & \textit{Cobb}-\textit{Douglas parameter} \\ & \textit{for governmental consumption} \\ \hat{D}^G = -0.6 & \textit{Maastricht factor} \\ \hat{L} & \textit{targeted labour} \end{array}$$

9. What insights can be gained from the modeling of GCD macro models

9.1. Practical insights: Causes and pattern of business cycles, analysis of measures to achieve economic policy targets

The simplest macroeconomic model imaginable consists of 2 agents: 1 company that produces 1 good and 1 household that works for the company and buys or consumes this good.

Even this simplest macroeconomic model shows that under certain assumptions about the power relations between household and firm and assumptions about the other parameters of the model, business cycles occur. This means that the individual variables show an approximately cyclical behaviour and the phase shifts between the individual variables remain approximately the same.

In chapter 11 we present and analyse this simple model and present some basic results.

As an example for measures to achieve economic policy targets in model B1, B2 and C1, C2 we analyse in a simple way the different effects for possible central bank policies: monetary supply policy, interest policy or behaviour in the sense of the Taylor rule.

The most **important tasks** that need to be done in the future to be able to use GCD models for practical problems in economics are:

a) Adjustment of parameters to describe real circumstances and comparison of model results with real business cycle trends.

b) Extend GCD models to multiple households, firms, and goods, and in particular to commodity and financial markets. For a first approach see (Richters 2021)

c) In the long run, develop a more complex, real-world model to enable better economic forecasting and test measures to achieve economic policy targets.

d) Elaborate GCD models with economic shocks in detail.

e) Elaborate GCD models with intertemporal utility functions in detail.

9.2. Theoretical insight: Different macroeconomic theories differ in their assumptions of different power factors

A. Sen has shown in (A. K. Sen 1963) that

- the basic neoclassical model of macroeconomics
- the macroeconomic model of Kaldor
- the macroeconomic model of Johansen
- and the Keynesian model

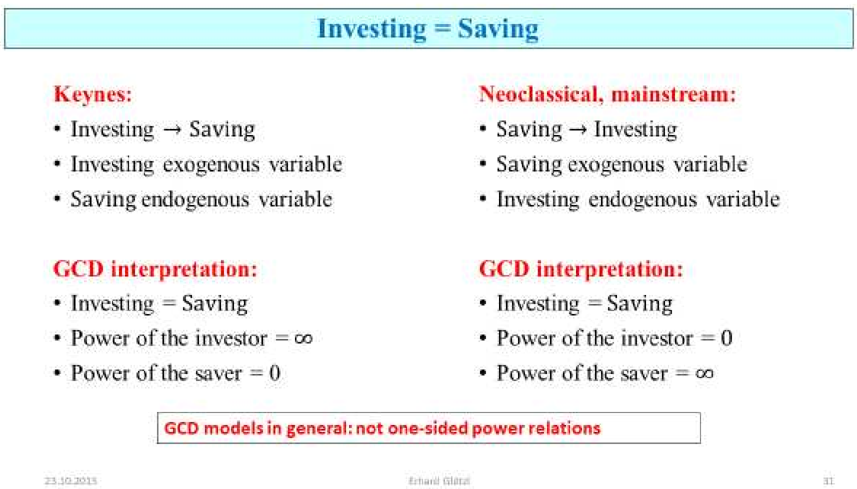
differ only in their assumptions about which variables are exogenous and which variables are endogenous.

In the methodology of the GCD models it holds:

The variable x is exogenously determined $\Leftrightarrow$ There is an agent A with $\mu_x^A = \infty$
The variable x is endogenously determined $\Leftrightarrow$ For all agents $\mu_x^{A_i} = 0$

This means that the economic models described by Sen always assume one-sided power relations. Since in the GCD models the power factors can assume all values between 0 and ∞, i.e. that also not one-sided power relations are possible, all hybrid forms of economic theories can also be modeled within the framework of GCD models. This means that a continuous transition from one economic theory to another economic theory can be represented by the continuous transition of the various power factors from $0 \to \infty$ or. $\infty \to 0$. Since one-sided power relations hardly ever occur in reality, reality can therefore be better described with GCD models. In chapter 18 we describe in detail examples of corresponding theories and the corresponding models.

We show, for example, that even the theoretical assumptions about the causal relationship between "saving" and "investing", which differ from a neoclassical and a Keynesian perspective, can be understood as assumptions about one-sided power relations from the perspective of GCD models:



In Chapter 18.2 we describe the corresponding models and their interpretation as GCD models in detail.

10. The open-source programme "GCDconfigurator”

In order to facilitate the concrete application to any complex GCD models (with non-intertemporal utility functions), we have written the open-source program "GCDconfigurator", with which any GCD model can be programmed very comfortably and solved numerically with the help of MATHEMATICA.

Essentially, it is sufficient to enter the following:

- The algebraically defined variables
- The utility functions for each agent
- The constraints

The output is the time evolution of all variables depending on the freely variable size of the power factors, the other parameters and the initial conditions.

The programme requires the installation of JAVA and MATHEMATICA. It can be downloaded from GitHub with the corresponding instructions (Glötzl und Binter 2022) under

https://github.com/lbinter/gcd

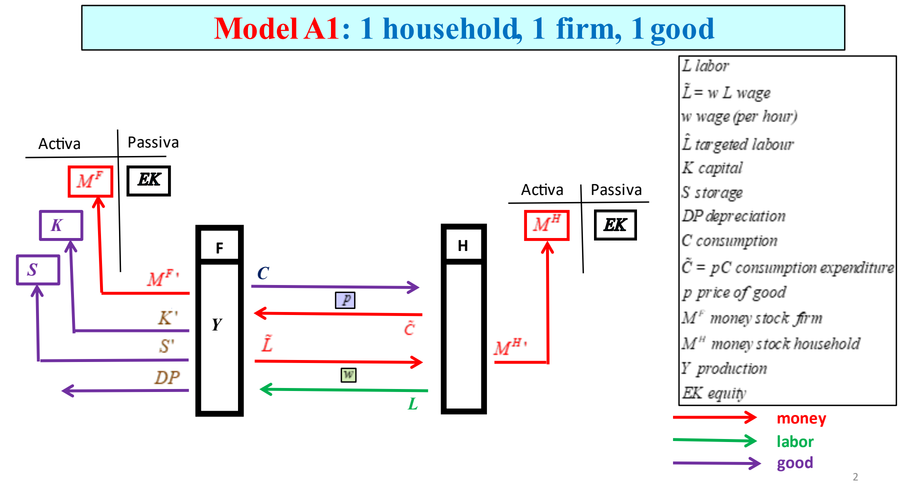
It allows in the 1st step to set up the GCD equation system in a convenient way just from the specification of the utility functions, constraints and initial conditions. In the 2nd step, the program enables the calculation of the solutions using MATHEMATICA. The results are calculated and plotted graphically as a time evolution of the variables, where the individual parameters can be varied in a convenient way.

All MATHEMATICA program codes used for calculations of the various GCD models can be downloaded under

https://www.dropbox.com/sh/npis47xjqkecggv/AAAMzCVhmhDYIIhoB5MfATFya?dl=0

11. Model A1, (1 household, 1 firm, 1 good, without interest)

11.1. Overview of the setup



Model A1: basic equations

algebraically defined variables

$Y(L,K) = \quad \beta L^{\alpha} K^{1-\alpha}$ "production function"

$DP(K) = \quad \widehat{dp}\, K$ "depreciation"

utility functions

$U^H(C,L,MH) = \quad C^{\gamma} - \left(\hat{L} - L\right)^2 - (\hat{M}^H - M^H)^2$ "utility function household"

$U^F(Y,L,S) = \quad pY - wL - (\hat{S} - S)^2$ "utility function firm"

constraints

$Z^H = 0 = wL - pC - M^{H}{'}$ for money of household H

$Z^F = 0 = pC - wL - M^{F}{'}$ for money of firm F

$Z_1 = 0 = Y(L,K) - C - K' - S' - DP$ for good 1 of firm F

With the aid of the GCDconfigurator programme, the differential-algebraic equation system of the A1 model is calculated from this:

Model A1: diff.-alg. equation system

```
uF[t] == -(sdach - s[t])^2 - l[t] × w[t] + p[t] × y[t]
uH[t] == cH[t]^γ - (ldach - l[t])^2 - (mHdach - mH[t])^2

dp[t] == dpdach k[t]
inv[t] == k'[t]
y[t] == β k[t]^(1-α) l[t]^α

cH'[t] == γ μHcH cH[t]^(-1+γ) + p[t] λ1[t] - p[t] λ2[t] - λ3[t]
k'[t] == (1 - α) β μFk k[t]^(-α) l[t]^α p[t] - λ3[t]
l'[t] == 2 μHl (ldach - l[t]) + μFl (α β k[t]^(1-α) l[t]^(-1+α) p[t] - w[t]) - w[t] λ1[t] +
   w[t] λ2[t] + α β k[t]^(1-α) l[t]^(-1+α) λ3[t]
mF'[t] == -λ1[t]
mH'[t] == 2 μHmH (mHdach - mH[t]) - λ2[t]
p'[t] == β μFp k[t]^(1-α) l[t]^α + cH[t] λ1[t] - cH[t] λ2[t]
s'[t] == 2 μFs (sdach - s[t]) - λ3[t]
w'[t] == -μFw l[t] - l[t] λ1[t] + l[t] λ2[t]

0 == cH[t] × p[t] - l[t] × w[t] - mF'[t]
0 == -cH[t] × p[t] + l[t] × w[t] - mH'[t]
0 == -cH[t] - dpdach k[t] + β k[t]^(1-α) l[t]^α - k'[t] - s'[t]

cH[0] == k0^(1-α) l0^α β
k[0] == k0
l[0] == l0
mF[0] == mF0
mH[0] == mH0
p[0] == p0
s[0] == s0
w[0] == w0
```

11.2. Description of the A1 model in detail

The one good serves as both a consumption good and an investment good. We assume that vertical constraint forces occur.

Since the target is first to show the principle, we choose the production function and the utility functions as simple as possible.

We choose a simple Cobb-Douglas production function as the production function, and the goods excreted per year (depreciation) are proportional to the capital stock. This results in the 2 necessary algebraically defined variables. They are necessary because they occur in the utility functions or constraints.

$$\begin{aligned} &Y(L,K) = \beta L^{\alpha} K^{1-\alpha} && \beta > 0,\ 0 < \alpha < 1 \\ &DP(K) = dp\, K && 0 \le dp \le 1 \end{aligned} \qquad \text{<11.1>}$$

In addition, one can be interested, for example, in net investment, for which one defines as a further algebraically defined variable

$$inv(K) = K' \qquad \text{<11.2>}$$

Households want to consume with decreasing marginal utility. Consumption of consumer goods C leads to a utility for households in the amount of C^{γ} with $0 < \gamma < 1$. They strive for a desired working time $\hat{L}$. Deviations from the desired working time $\hat{L}$ lead to a reduction of utility by $(L-\hat{L})^2$. In addition, households aim to keep cash in the amount of $\hat{M}^H$. Deviations from the desired cash position $\hat{M}^H$ lead to a reduction in utility by $(\hat{M}^H - M^H)^2$. This leads to the **utility function for the household**

$$U^H = C^\gamma - (\hat{L}-L)^2 - (\hat{M}^H - M^H)^2 \qquad 0<\gamma<1 \qquad \text{<11.3>}$$

For the company, in the simplest case, the utility initially consists of the goods produced, which are valued at the selling price, i.e. pY . The produced goods are used for:

C Sales = Consumption

S' change in inventory

K' changes in productive capital stock

In principle, it would be possible to weight the utility of these uses differently. For the sake of simplicity, we will refrain from doing so. Therefore, this utility is reduced by the cost of labor and the cost of storage, which we evaluate through the deviations from the planned inventory. For simplicity, we assume that holding money in cash has no influence on the utility. This leads to the **utility function for the firm**

$$\begin{aligned} U^F &= pY(L,K) - wL - (\hat{S}-S)^2 = \\ &= p\beta L^\alpha K^{1-\alpha} - wL - (\hat{S}-S)^2 \end{aligned} \qquad \text{<11.4>}$$

From the model graph, it can be seen that the following **constraints** must be satisfied:

$$\begin{aligned} &Z_1 = 0 = wL - pC - M^H\,' && \textit{for money of household } H \\ &Z_2 = 0 = pC - wL - M^F\,' && \textit{for money of firm } F \\ &Z_3 = 0 = Y(L,K) - C - K' - S' && \textit{for good } 1 \textit{ of firm } F \end{aligned} \qquad \text{<11.5>}$$

According to the methodology of GCD models, the interest or desire of households to change consumption is the greater the more the utility changes when consumption changes, i.e., the interest is proportional to $\frac{\partial U^H}{\partial C}$. However, the interest in changing consumption does not in itself lead to an actual change in consumption, because the household must also have the power or opportunity to actually implement its desire to change consumption. For example, a household cannot or can only partially enforce its additional consumption wish, e.g., to go to the cinema or on holiday, because it is in quarantine or the borders are closed. This restriction of the possibility to enforce his or her consumption change wishes is described by a (possibly time-dependent) "power factor" μ_C^H . Analogously, the firm could have an interest $\frac{\partial U^F}{\partial C}$ and power μ_C^F to influence consumption. In the specific case $\frac{\partial U^F}{\partial C} = 0$. This results in the following behavioural equation for the **ex-ante planned change in consumption**

$$C' = \mu_C^H \frac{\partial U^H}{\partial C} + \mu_C^F \frac{\partial U^F}{\partial C} = \mu_C^H \gamma C^{\gamma-1} \qquad \text{<11.6>}$$

The same considerations apply to labour L as to consumption. Even the household's wish to increase or reduce working time does not in itself lead to an actual change in working time, because the household must also have the power or possibility to actually implement its wish to change. For example, a household might not be able to enforce its wish to increase working time, or only partially, because it is on short-time working or unemployed, or it might not be able to enforce its wish to reduce working time because it is contractually obliged to work overtime. This restriction of the possibility to enforce his wishes for a change in working time is also described by a (possibly time-dependent) power factor, which we denote with μ_L^H. The same applies to the firm's ability to influence working time.

Therefore, the behavioural equation for the **ex-ante planned change in working time** is as follows

$$L' = \mu_L^H \frac{\partial U^H}{\partial L} + \mu_L^F \frac{\partial U^F}{\partial L} = 2\mu_L^H (\hat{L} - L) + \mu_L^F (p\beta\alpha L^{\alpha-1}K^{1-\alpha} - w)$$

The ex-ante behavioural equations for the other variables result analogously.

However, the plans of the 2 agents household and firm to change consumption C, labour L and the other variables cannot be enforced independently of each other, because the constraints

$$\begin{aligned} Z_1 &= 0 = wL - pC - M^{H}{}' && \textit{for money of household } H \\ Z_2 &= 0 = pC - wL - M^{F}{}' && \textit{for money of firm } F \\ Z_3 &= 0 = Y(L,K) - C - K' - S' - DP && \textit{for good } 1 \textit{ of firm } F \end{aligned}$$

<11.7>

lead to constraint forces, which we assume are vertical constraint forces. The constraint force for the change in consumption therefore results in

$$\lambda_1 \frac{\partial Z_1}{\partial C} + \lambda_2 \frac{\partial Z_2}{\partial C} + \lambda_3 \frac{\partial Z_3}{\partial C} = -\lambda_1 p + \lambda_2 p - \lambda_3$$

The behavioural equation for the actual **ex-post change in consumption** is therefore

$$\begin{aligned} C' &= \mu_C^H \frac{\partial U^H}{\partial C} + \lambda_1 \frac{\partial Z_1}{\partial C} + \lambda_2 \frac{\partial Z_2}{\partial C} + \lambda_3 \frac{\partial Z_3}{\partial C} = \\ &= \mu_C^H \gamma C^{\gamma-1} - \lambda_1 p + \lambda_2 p - \lambda_3 \end{aligned} \qquad \text{<11.8>}$$

Analogously, the actual **ex-post change in labour** is as follows

$$\begin{aligned} L' &= \mu_L^H \frac{\partial U^H}{\partial L} + \mu_L^F \frac{\partial U^F}{\partial L} + \lambda_1 \frac{\partial Z_1}{\partial L} + \lambda_2 \frac{\partial Z_2}{\partial L} + \lambda_3 \frac{\partial Z_3}{\partial L} = \\ &= 2\mu_L^H (\hat{L} - L) + \mu_L^F (p\beta\alpha L^{\alpha-1}K^{1-\alpha} - w) + \\ &\quad + \lambda_1 w - \lambda_2 w + \lambda_3 \alpha\beta L^{\alpha-1}K^{1-\alpha} \end{aligned}$$

This also applies analogously to the company's investments. In the case of the company, too, the actual implementation of ex-ante planned investment increases can be prevented by real restrictions, e.g. by interruptions in supply chains. In the same way, a desired reduction in investment may not be possible to the desired extent because the project is a large-scale project of many years' duration. These restrictions can in turn be described by a (possibly time-dependent) power factor μ_K^B. This results in the following behavioural equation for the actual ex-post change in capital

$$\begin{aligned} K' &= \mu_K^F \frac{\partial U^F}{\partial K} + \lambda_1 \frac{\partial Z_1}{\partial K} + \lambda_2 \frac{\partial Z_2}{\partial K} + \lambda_3 \frac{\partial Z_3}{\partial K'} = \\ &= \mu_K^F p \beta (1-\alpha) L^\alpha K^{-\alpha} - \lambda_3 \end{aligned} \qquad \text{<11.9>}$$

Note that we have to use $\frac{\partial Z_3}{\partial K'}$ instead of $\frac{\partial Z_3}{\partial K}$ because the constraint forces are always derived from the highest time derivative of the variables (see chapter 7.9.1 and (Flannery 2011)).

The equations of behaviour for M^H, M^F, S, p, w are derived analogously. In sum, this results in the model equations

differentiell behavioural equations

$$\begin{aligned} C' &= \mu_C^H \frac{\partial U^H}{\partial C} + \mu_C^F \frac{\partial U^F}{\partial C} + \lambda_1 \frac{\partial Z_1}{\partial C} + \lambda_2 \frac{\partial Z_2}{\partial C} + \lambda_3 \frac{\partial Z_3}{\partial C} = \\ &= \mu_C^H \gamma C^{\gamma-1} - \lambda_1 p + \lambda_2 p - \lambda_3 \\ L' &= \mu_L^H \frac{\partial U^H}{\partial L} + \mu_L^F \frac{\partial U^F}{\partial L} + \lambda_1 \frac{\partial Z_1}{\partial L} + \lambda_2 \frac{\partial Z_2}{\partial L} + \lambda_3 \frac{\partial Z_3}{\partial L} = \\ &= \mu_L^H (\hat{L} - L) + \lambda_1 w - \lambda_2 w + \lambda_3 \alpha \beta L^{\alpha-1} K^{1-\alpha} \\ K' &= \mu_K^H \frac{\partial U^H}{\partial K} + \mu_K^F \frac{\partial U^F}{\partial K} + \lambda_1 \frac{\partial Z_1}{\partial K} + \lambda_2 \frac{\partial Z_2}{\partial K} + \lambda_3 \frac{\partial Z_3}{\partial K'} = \\ &= \mu_K^F p \beta (1-\alpha) L^\alpha K^{-\alpha} - \lambda_3 \\ M^{H}{}' &= \mu_{M^H}^H \frac{\partial U^H}{\partial M^H} + \mu_{M^H}^F \frac{\partial U^F}{\partial M^H} + \lambda_1 \frac{\partial Z_1}{\partial M^{H}{}'} + \lambda_2 \frac{\partial Z_2}{\partial M^H} + \lambda_3 \frac{\partial Z_3}{\partial M^H} = \\ &= 2\mu_{M^H}^H \left(\hat{M}^H - M^H\right) - \lambda^H \\ M^{F}{}' &= \mu_{M^F}^H \frac{\partial U^H}{\partial M^F} + \mu_{M^F}^F \frac{\partial U^F}{\partial M^F} + \lambda^H \frac{\partial Z^H}{\partial M^{F}{}'} + \lambda^B \frac{\partial Z^B}{\partial M^F} + \lambda_1 \frac{\partial Z_1}{\partial M^F} = \\ &= -\lambda_2 \\ S' &= \mu_S^H \frac{\partial U^H}{\partial S} + \mu_S^F \frac{\partial U^F}{\partial S} + \lambda_1 \frac{\partial Z_1}{\partial S} + \lambda_2 \frac{\partial Z_2}{\partial S} + \lambda_3 \frac{\partial Z_3}{\partial S'} = \\ &= \mu_S^F \left(\hat{S} - S\right) - \lambda_3 \\ p' &= \mu_p^H \frac{\partial U^H}{\partial p} + \mu_p^F \frac{\partial U^F}{\partial p} + \lambda_1 \frac{\partial Z_1}{\partial p} + \lambda_2 \frac{\partial Z_2}{\partial p} + \lambda_3 \frac{\partial Z_3}{\partial p} = \\ &= \mu_p^F \beta K^{1-\alpha} L^\alpha - \lambda_1 c + \lambda_2 c \\ w' &= \mu_w^H \frac{\partial U^H}{\partial w} + \mu_w^F \frac{\partial U^F}{\partial w} + \lambda_1 \frac{\partial Z_1}{\partial w} + \lambda_2 \frac{\partial Z_2}{\partial w} + \lambda_3 \frac{\partial Z_3}{\partial w} = \\ &= -\mu_w^F L + \lambda_1 L - \lambda_2 L \end{aligned}$$

Or written in a clearer way

differentiell behavioural equations

$$C' = \mu_C^H \gamma C^{\gamma-1} - \lambda_1 p + \lambda_2 p - \lambda_3$$

$$L' = 2\mu_L^H (\hat{L} - L) + \mu_L^F \left(\alpha\beta K^{1-\alpha} L^{-1+\alpha} p - w\right) + \lambda_1 w - \lambda_2 w + \lambda_3 \alpha\beta K^{1-\alpha} L^{-1+\alpha}$$

$$K' = \mu_K^F \beta (1-\alpha) L^{\alpha} K^{-\alpha} p - \lambda_3$$

$$M^{H}{}' = 2\mu_{M^H}^H \left(\hat{M}^H - M^H\right) - \lambda_1$$

$$M^{F}{}' = -\lambda_2$$

$$S' = \mu_S^F \left(\hat{S} - S\right) - \lambda_3$$

$$p' = \mu_p^F \beta K^{1-\alpha} L^{\alpha} - \lambda_1 c + \lambda_2 c$$

$$w' = -\mu_w^F L + \lambda_1 L - \lambda_2 L$$

11.3. Calculation results of model A1

Depending on the choice of parameters, the system converges to a stationary state (see figure 1) or the system describes the occurrence of business cycles (see figure 2). A change in the parameters usually only changes the frequency and amplitude of the business cycle fluctuations. This means that the qualitative sequence of business cycles over a wide range of parameters is independent of the specific choice of parameters. For example, it can be seen that the minima or maxima of the variables typically occur in the following order (see figure 2):

	Minima	**Maxima**
1	Profit	Price
2	Price	Profit
3	Investment	Employment
4	Employment	Investment
5	BIP	BIP
6	Capital	Money stock of the company
7	Money stock of the company	Storage goods
8	Storage goods	Capital
9	Consumption	Consumption
10	Wages	Wages
11	Money stock of the household	Money stock of the household

Existing business cycle theories each assume certain cause-and-effect relationships between different variables. In contrast, in GCD models, business cycle fluctuations can only be explained by assumptions

- on the behaviour or utility functions of agents
- and about the balance of power between the agents.

In this context, the following remark seems **important**: In economics, there is usually a very complex interplay of the various variables. This complex interaction can be modeled well by systems of differential equations. However, the complex behaviour of differential equation systems cannot usually be described by simple cause-effect relationships. **Simple cause-effect relationships are therefore generally not suitable for correctly reflecting economic interactions.**

Figure 1: model A1

https://www.dropbox.com/s/4yeu7j077yiis65/Modell%20A1%20Version%2012.ndsolve.nb?dl=0

Figure 2: model A1, business cycle analysis

https://www.dropbox.com/s/evx09cjv18d2k7a/Modell%20A1%20Version%207%2C%20Konjunkturanalyse%20V6.ndsolve.nb?dl=0

Figure 1: model A1

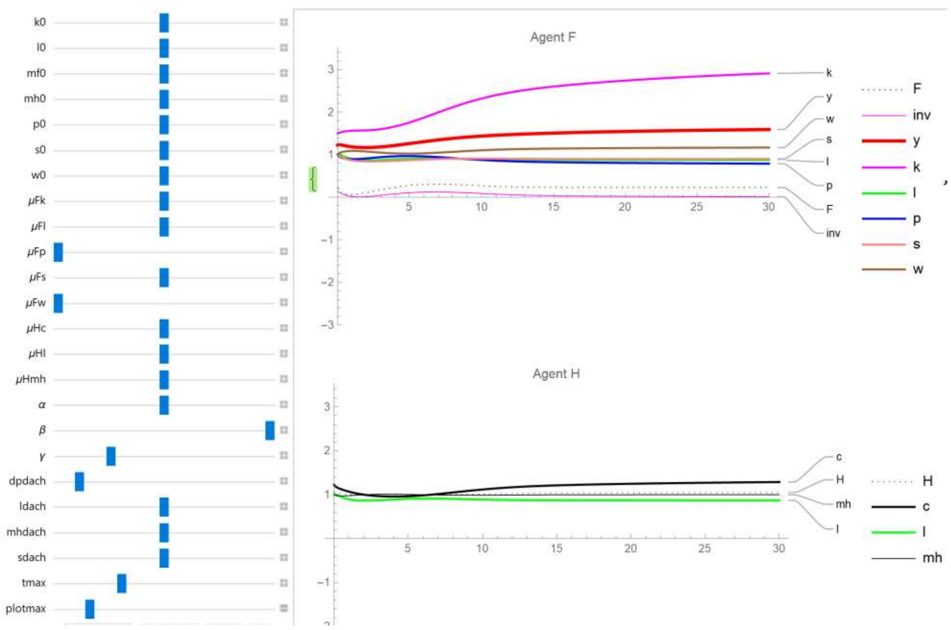
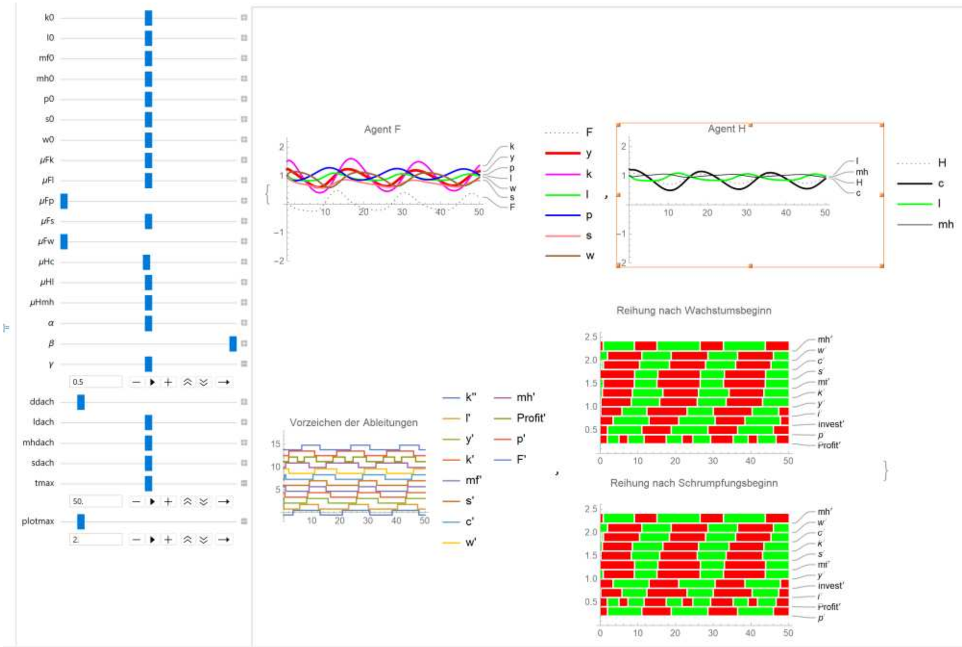
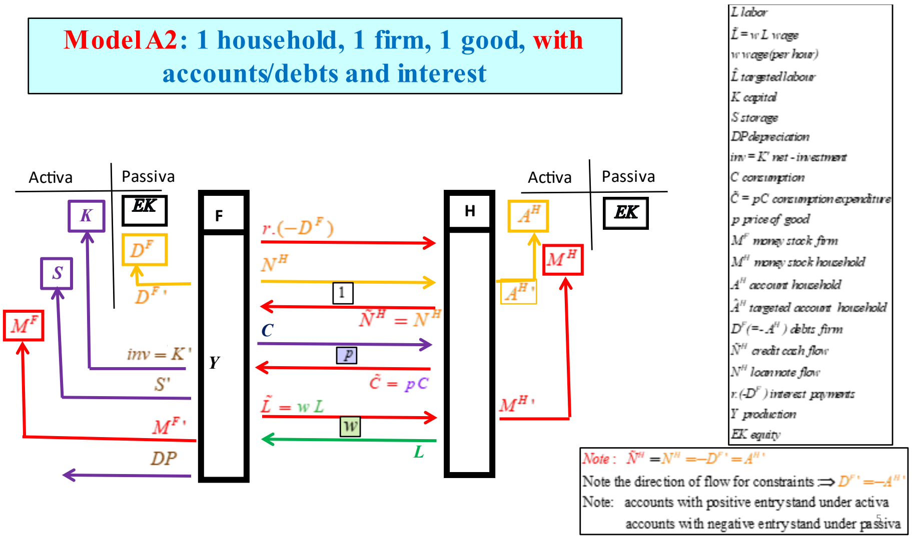


Figure 2: model A1, business cycle analysis



12. Model A2: 1 household, 1 firm, 1 good, with accounts/debts and interest

12.1. Overview of the setup



Model A2: basic equations

algebraically defined variables

$Y(L,K) = \quad \beta L^{\alpha} K^{1-\alpha}$ "production function"

$DP(K) = \quad \widehat{dp} K$ "depreciation"

utility functions

$U^H(C,L,M^H) = \quad C^{\gamma} - \left(\hat{L} - L\right)^2 - (\hat{M}^H - M^H)^2 + rA^H$ "utility function household"

$U^F(Y,L,S) = \quad pY - wL - (\hat{S} - S)^2 - r(-D^F)$ "utility function firm"

constraints

$Z_1 = 0 = wL - pC + rA^H - N^H - M^{H\prime}$ for money flow of household H

$Z_2 = 0 = pC - wL - r(-D^F) + N^H - M^{F\prime}$ for money flow of firm F

$Z_3 = 0 = Y(L,K) - C - S' - DP - K'$ for good 1 flow of firm F

$Z^H = 0 = N^H - A^{H\prime}$ for accounts / debts flow of H

$Z^F = 0 = -N^H - D^{F\prime}$ for accounts / debts flow of F

6

Assuming vertical constraints, the differential-algebraic equation system of model A2 is calculated from this with the help of the GCDconfigurator.

Model A2: diff.-alg. equation system for vertical constraints

```
uF[t] == r dF[t] - (sdach - s[t])^2 - l[t] × w[t] + p[t] × y[t]
uH[t] == r aH[t] + cH[t]^γ - (ldach - l[t])^2 - (mHdach - mH[t])^2

dp[t] == dpdach k[t]
inv[t] == k'[t]
y[t] == β k[t]^(1-α) l[t]^α

aH'[t] == r μHaH + r λ2[t] - λ5[t]
cH'[t] == γ μHcH cH[t]^(-1+γ) + p[t] λ1[t] - p[t] λ2[t] - λ3[t]
dF'[t] == r μFdF + r λ1[t] - λ4[t]
k'[t] == (1 - α) β μFk k[t]^-α l[t]^α p[t] - λ3[t]
l'[t] == 2 μHl (ldach - l[t]) + μFl (α β k[t]^(1-α) l[t]^(-1+α) p[t] - w[t]) - w[t] λ1[t] +
   w[t] λ2[t] + α β k[t]^(1-α) l[t]^(-1+α) λ3[t]
mF'[t] == -λ1[t]
mH'[t] == 2 μHmH (mHdach - mH[t]) - λ2[t]
nH'[t] == λ1[t] - λ2[t] - λ4[t] + λ5[t]
p'[t] == β μFp k[t]^(1-α) l[t]^α + cH[t] λ1[t] - cH[t] λ2[t]
s'[t] == 2 μFs (sdach - s[t]) - λ3[t]
w'[t] == -μFw l[t] - l[t] λ1[t] + l[t] λ2[t]

0 == r dF[t] + nH[t] + cH[t] × p[t] - l[t] × w[t] - mF'[t]
0 == r aH[t] - nH[t] - cH[t] × p[t] + l[t] × w[t] - mH'[t]
0 == -cH[t] - dpdach k[t] + β k[t]^(1-α) l[t]^α - k'[t] - s'[t]
0 == -nH[t] - dF'[t]
0 == nH[t] - aH'[t]

aH[0] == aH0
cH[0] == k0^(1-α) l0^α β
dF[0] == dF0
k[0] == k0
l[0] == l0
mF[0] == mF0
mH[0] == mH0
nH[0] == nH0
p[0] == p0
s[0] == s0
w[0] == w0
```

12.2. Systematic derivation of constraints from the model graph

Using the A2 model, we show how to systematically derive the relevant constraints.

Arrows represent flows. In model A2 there are 3 different flows.

- The flow of the good (violet)
- The flw of money (red)
- The flow of debt notes when money is given as credit (light brown)
- The flow of labour (green)

Each flow leads to a decrease in the corresponding balance sheet item (stock) in the balance sheet of the agent from which the flow originates and to an increase in the corresponding balance sheet item (stock) in the balance sheet of the agent to which the flow goes.

In addition, there are source terms, such as production by the company, or sinks, such as actual consumption of consumer goods by the household. This sink for consumer goods at home is not shown in the graph for the sake of clarity and because it leads to a trivial constraint under the assumption that everything is consumed immediately.

Thus, for each agent and each flow there is a constraint in the form

$$\boxed{\textit{inflow} - \textit{outflow} - \textit{stock change} = 0 \qquad \text{<12.1>}}$$

e.g., this results in a constraint Z_2 for the flow of money at the firm

$$Z_2 = 0 = pC - wL - rA^H + N - M^{F}{}'$$

When considering the direction of flow and the sign of variables on the liabilities side of the balance sheet (passive), one must **respect the convention** we use, namely that entries on the liabilities side of the balance sheet have a negative sign. This results, for example, in a constraint on the flow of debt notes in the company

$$Z^F = 0 = -N - D^{F}{}'$$

For interpretation: if the bank gives the company a loan of $\tilde{N} = 10$ €, this means that

- $\tilde{N} = +10$ money (red arrow) flows from the bank to the firm
- $N = \tilde{N} = +10$ debt notes flow from the firm to the bank (light brown arrow) if a debt note is issued for every euro
- that the debt increases and thus, due to the sign convention, the debt account on the liabilities side is reduced by 10, i.e. $D^{F'} = -10$

This results in

- debt note inflow to the firm = 0
- outflow of debt notes to the bank $N = 10$
- outflow of debt notes to the balance sheet $D^{F'} = -10$

$$\begin{aligned} Z^F &= \textit{promissory note inflow} - \\ &\quad - \textit{outflow of promissory notes to the bank} - \\ &\quad - \textit{outflow of promissory notes to the balance sheet} = \\ &= 0 - N - D^{F}{}' = 0 - 10 - (-10) = 0 \end{aligned}$$

If C denotes the inflow of consumption goods to the household and $\vec{C}$ denotes actual consumption and hence the destruction of consumption goods, then, assuming immediate consumption, the following applies:

$$\vec{C} = C.$$

Under the given assumption this is nothing else but the algebraically given behavioural equation for actual consumption $\vec{C}$. The constraint for the flow of consumption to the household $0 = C - \vec{C}$ is therefore equivalent to the algebraic definition equation of $\vec{C}$. Since $\vec{C}$ does not occur in the utility functions, this constraint is superfluous.

Analogously, the following constraints therefore arise:

$$Z_1 = 0 = wL - pC + rA^H - N^H - M^{H\prime}$$

for money of household H

$$Z_2 = 0 = pC - wL - r(-D^F) + N^H - M^{F\prime}$$

for money of firm F

$$Z_3 = 0 = Y(L,K) - C - S' - DP - K'$$

for good 1 *of firm F*

$$Z^H = 0 = N^H - A^{H\prime}$$

for receivables / liabilities of household H

$$Z^F = 0 = -N^H - D^{F\prime}$$

12.3. Systematic derivation of constraints from the transaction matrices

The constraints can also be derived from the transaction matrices used to describe SFC models. It should be noted that this always results in linearly dependent constraints that can be omitted.

The relevant constraints are marked in red.

Transaction matrices of model A2

money	constraint→	Z_1	Z_2	$Z_{\text{money balance}}$
	agent→	H	F	
	stock→	M^H	M^F	
flow↓	wage	$+\tilde{L} = +w.L$	$-\tilde{L} = -w.L$	0
	consumption	$-\tilde{C} = -p.C$	$+\tilde{C} = +p.C$	0
	credit	$-\tilde{N} = -1.N$	$+\tilde{N} = +1.N$	0
	interest	$+\tilde{Z} = +r.A^H$	$-\tilde{Z} = -r.A^H$	0
	sum	$\sum = M^{H\prime}$	$\sum = M^{F\prime}$	$\sum\sum = M^{H\prime} + M^{F\prime}$

$$Z_1 = 0 = wL - pC - N + rA^H - M^{H\prime}$$

$$Z_2 = 0 = -wL + pC + N - rA^H - M^{F\prime}$$

$Z_{\text{money balance}} = 0 = M^{H\prime} + M^{F\prime}$ linearly dependent on Z_1 and Z_2

debt note	constraint→	Z_3	Z_4	$Z_{\text{debt note balance}}$
	agent→	H	F	
	stock→	A^H	D^F	
flow↓	credit	$+N$	$-N$	0

	sum	$\sum = A^{H\prime}$	$\sum = D^{F\prime}$	$\sum\sum = 0$

$Z_3 = 0 = N - A^{H\prime}$

$Z_4 = 0 = N - D^{F\prime}$

$Z_{\text{debt note balance}} = 0 = A^{H\prime} + D^{F\prime}$ linearly dependent on Z_3 and Z_4

In the case of the good, we consider the following stocks:

K "Capital"
S "Storage goods"
CS "Consumption stock" (all goods consumed by the household)

good	constraint→	Z_5	Z_6	Z_7	$Z_{\text{good balance}}$
	agent→	H	F	F	
	stock→	CS	K	S	
flow↓	production		$+Y$		$+Y$
	storage goods		$-S'$	$+S'$	0
	depreciation		$-DP$		$-DP$
	Consumption goods	$+C$	$-C$		0
	use of C	$-C$			$-C$
	sum	$\sum - CS' = 0$	$\sum - K' = 0$	$\sum - S' = 0$	$\sum\sum - CS' - K' - S' = 0$

$Z_5 = 0 = C - C - CS'$ trivial

$Z_6 = 0 = Y - S' - DP - C - K'$

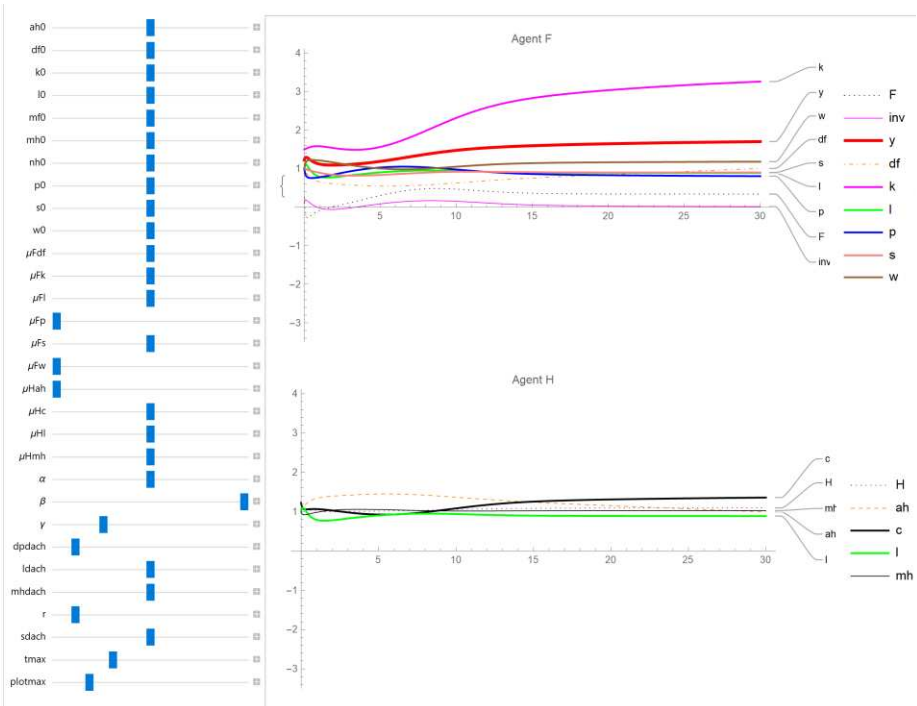
$Z_7 = 0 = S' - S'$ trivial

$Z_{\text{good balance}} = 0 = -CS' - K' - S' + Y - DP - C)$ linearly dependent

No non-trivial constraint arises for the labour L. Therefore, only the constraints coloured red remain. These are the same as those that resulted from the model graph in chapter 12.2.

12.4. Calculation results of model A2

https://www.dropbox.com/s/qe3qettcg714ztb/Modell%20A2%20Version%208.ndsolve.nb?dl=0



13. Model B1, (1 household, 1 firm, 1 good, 1 banking system), Interest rate policy versus monetary policy

13.1. Overview of the setup

The target of models B1 and B2 is to model the money creation process by the central bank in a simplified way.

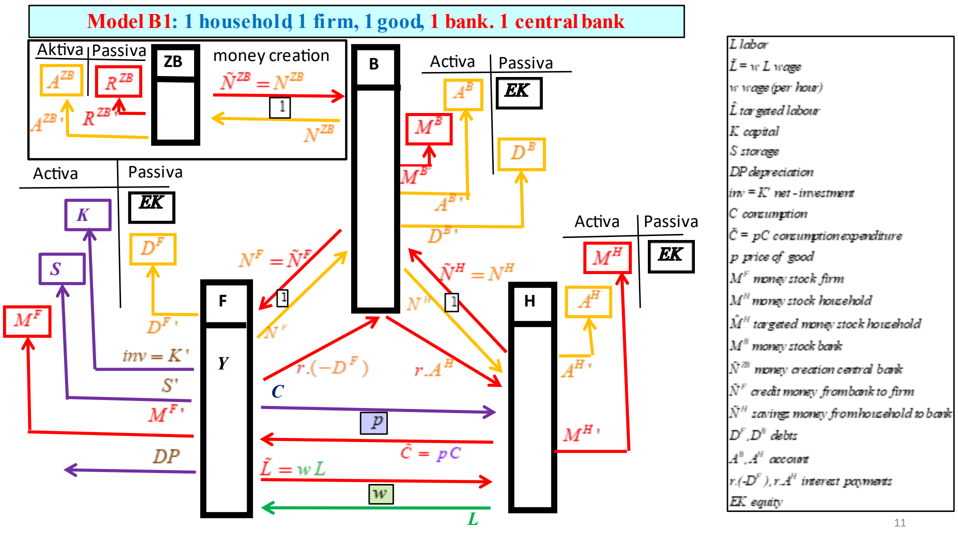
In model B1, the central bank is seen as an endogenous money creator and the bank is seen as an endogenous credit creator. The central bank's target is to keep inflation $\frac{p'}{p}$ at the target inflation $\hat{p} = 0.02$ i.e. 2% by means of interest rate policy ($\delta = 1$) and monetary-supply policy($\delta = 0$).

In this model B1, the central bank's interest rate policy is still modeled in a very simplified way. We assume that the policy rate is constant 0 (banks do not pay interest to the central bank) and that the central bank can, however, influence the interest rate directly. That the policy rate is constant 0 is possible and does not cause the bank to borrow arbitrarily from the central bank, since the bank is assumed to have a constant 0 utility function. This means that the bank has no particular interest in lending to firms or receiving savings deposits from households. Thus, the bank lends endogenously and accepts savings deposits endogenously.

In model B2, we will model the behaviour of the central bank according to the Taylor rule.

All these simplifying restrictions regarding money creation, we will still keep in models C1, C2. This is because in models C1, C2, we are concerned with modeling the government.

It is only in the much more comprehensive model D2 that we will largely abandon the restrictions on the modeling of money creation and the modeling of the government.



Pay attention when establishing the constraints:

(1) Claims A have a positive sign, liabilities D have a negative sign

(2) Banks' equity capital is 0. They do not make profits.

Model B1 : basic equations

algebraically defined variables

$Y(L,K) = \quad \beta L^{\alpha} K^{1-\alpha}$ "production function"

$DP(K) = \quad dp\, K$ "depreciation"

utility functions

$U^H(C,L,M^H) = \quad C^{\gamma} - (\hat{L} - L)^2 - (\hat{M}^H - M^H)^2 + r\,A^H$ "utility function household"

$U^F(Y,L,S) = \quad pY - w\,L - (\hat{S} - S)^2 - r(-D^F)$ "utility function firm"

$U^B = 0$ "utility functionbank"

$U^{ZB}(r,p,N^B) = \quad \left(-\delta r + (1-\delta) N^{ZB}\right)\left(\hat{p} - \frac{p'}{p}\right)$ "utility function central bank"

constraints

$Z_1 = 0 = wL - pC + r\,A^H - N^H - M^H{}'$ for money flow of household H

$Z_2 = 0 = -wL + pC - r(-D^F) + N^F - M^F{}'$ for money flow of firm F

$Z_3 = 0 = N^{ZB} - N^F + r(-D^F) - r\,A^H + N^H - M^B{}'$ for money flow of bank B

$Z_4 = 0 = -N^{ZB} - R^{ZB}{}'$ for money flow of central bank ZB

$Z_5 = 0 = Y(L,K) - C - S' - DP - K'$ for flow of good 1 of firm F

$Z_6 = 0 = N^H - A^H{}'$ for accounts / debts flow of household H

$Z_7 = 0 = -N^F - D^F{}'$ for accounts / debts flow of firm F

$Z_8 = 0 = -N^{ZB} + N^F - N^H - D^B{}' - A^B{}'$ for accounts / debts flow of of bank B

$Z_9 = 0 = N^{ZB} - A^{ZB}{}'$ for accounts / debts flow of central bank ZB

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Model B1 : diff.-alg. equation system

$$uB[t] = 0$$
$$uF[t] = dF[t] \times r[t] - (sdach - s[t])^2 - l[t] \times w[t] + p[t] \times y[t]$$
$$uH[t] = cH[t]^{\gamma} - (ldach - l[t])^2 - (mHdach - mH[t])^2 + aH[t] \times r[t]$$
$$uZB[t] = \left(pdach - \frac{ps[t]}{p[t]}\right)((1-\delta)\, nZB[t] - \delta\, r[t])$$

$$dp[t] = dpdach\, k[t]$$
$$inflation[t] = \frac{ps[t]}{p[t]}$$
$$inv[t] = k'[t]$$
$$y[t] = \beta\, k[t]^{1-\alpha}\, l[t]^{\alpha}$$

$$aB'[t] = -\lambda_2[t]$$
$$aH'[t] = \mu HaH\, r[t] - \lambda_4[t] - r[t]\,\lambda_6[t] + r[t]\,\lambda_8[t]$$
$$aZB'[t] = -\lambda_5[t]$$
$$cH'[t] = \gamma\, \mu HcH\, cH[t]^{-1+\gamma} + p[t]\,\lambda_7[t] - p[t]\,\lambda_8[t] - \lambda_9[t]$$
$$dB'[t] = -\lambda_2[t]$$
$$dF'[t] = \mu FdF\, r[t] - \lambda_1[t] - r[t]\,\lambda_6[t] + r[t]\,\lambda_7[t]$$
$$k'[t] = (1-\alpha)\,\beta\,\mu Fk\, k[t]^{-\alpha}\, l[t]^{\alpha}\, p[t] - \lambda_9[t]$$
$$l'[t] = 2\,\mu Hl\,(ldach - l[t]) + \mu Fl\left(\alpha\,\beta\, k[t]^{1-\alpha}\, l[t]^{-1+\alpha}\, p[t] - w[t]\right) - w[t]\,\lambda_7[t] + w[t]\,\lambda_8[t] + \alpha\,\beta\, k[t]^{1-\alpha}\, l[t]^{-1+\alpha}\,\lambda_9[t]$$
$$mB'[t] = -\lambda_6[t]$$
$$mF'[t] = -\lambda_7[t]$$
$$mH'[t] = 2\,\mu HmH\,(mHdach - mH[t]) - \lambda_8[t]$$
$$nF'[t] = -\lambda_1[t] + \lambda_2[t] - \lambda_6[t] + \lambda_7[t]$$
$$nH'[t] = -\lambda_2[t] + \lambda_4[t] + \lambda_6[t] - \lambda_8[t]$$
$$nZB'[t] = (1-\delta)\,\mu ZBnZB\left(pdach - \frac{ps[t]}{p[t]}\right) - \lambda_2[t] - \lambda_3[t] + \lambda_5[t] + \lambda_6[t]$$
$$p'[t] = \beta\,\mu Fp\, k[t]^{1-\alpha}\, l[t]^{\alpha} + \frac{\mu ZBp\, ps[t]\,((1-\delta)\, nZB[t] - \delta\, r[t])}{p[t]^2} + cH[t]\,\lambda_7[t] - cH[t]\,\lambda_8[t] - \lambda_{10}[t]$$
$$ps'[t] = -\frac{\mu ZBps\,((1-\delta)\, nZB[t] - \delta\, r[t])}{p[t]} + \lambda_{10}[t]$$
$$r'[t] = \mu Hr\, aH[t] + \mu Fr\, dF[t] - \delta\,\mu ZBr\left(pdach - \frac{ps[t]}{p[t]}\right) + (-aH[t] - dF[t])\,\lambda_6[t] + dF[t]\,\lambda_7[t] + aH[t]\,\lambda_8[t]$$
$$rZB'[t] = -\lambda_3[t]$$
$$s'[t] = 2\,\mu Fs\,(sdach - s[t]) - \lambda_9[t]$$
$$w'[t] = -\mu Fw\, l[t] - l[t]\,\lambda_7[t] + l[t]\,\lambda_8[t]$$

$$0 = -nF[t] - dF'[t]$$
$$0 = nF[t] - nH[t] - nZB[t] - aB'[t] - dB'[t]$$
$$0 = -nZB[t] - rZB'[t]$$
$$0 = nH[t] - aH'[t]$$
$$0 = nZB[t] - aZB'[t]$$
$$0 = -nF[t] + nH[t] + nZB[t] - aH[t] \times r[t] - dF[t] \times r[t] - mB'[t]$$
$$0 = nF[t] + cH[t] \times p[t] + dF[t] \times r[t] - l[t] \times w[t] - mF'[t]$$
$$0 = -nH[t] - cH[t] \times p[t] + aH[t] \times r[t] + l[t] \times w[t] - mH'[t]$$
$$0 = -cH[t] - dpdach\, k[t] + \beta\, k[t]^{1-\alpha}\, l[t]^{\alpha} - k'[t] - s'[t]$$
$$0 = ps[t] - p'[t]$$

$$aB[0] = aB0$$
$$aH[0] = aH0$$
$$aZB[0] = aZB0$$
$$cH[0] = \frac{1}{2}\, k0^{1-\alpha}\, l0^{\alpha}\,\beta$$
$$dB[0] = dB0$$
$$dF[0] = dF0$$
$$k[0] = k0$$
$$l[0] = l0$$
$$mB[0] = mB0$$
$$mF[0] = mF0$$
$$mH[0] = mH0$$
$$nF[0] = nF0$$
$$nH[0] = nH0$$
$$nZB[0] = nZB0$$
$$p[0] = \frac{2\, k0^{-1+\alpha}\, l0^{-\alpha}\,(-nH0 + aH0\, r0 + l0\, w0)}{\beta}$$
$$ps[0] = ps0$$
$$r[0] = r0$$
$$rZB[0] = rZB0$$
$$s[0] = s0$$
$$w[0] = w0$$

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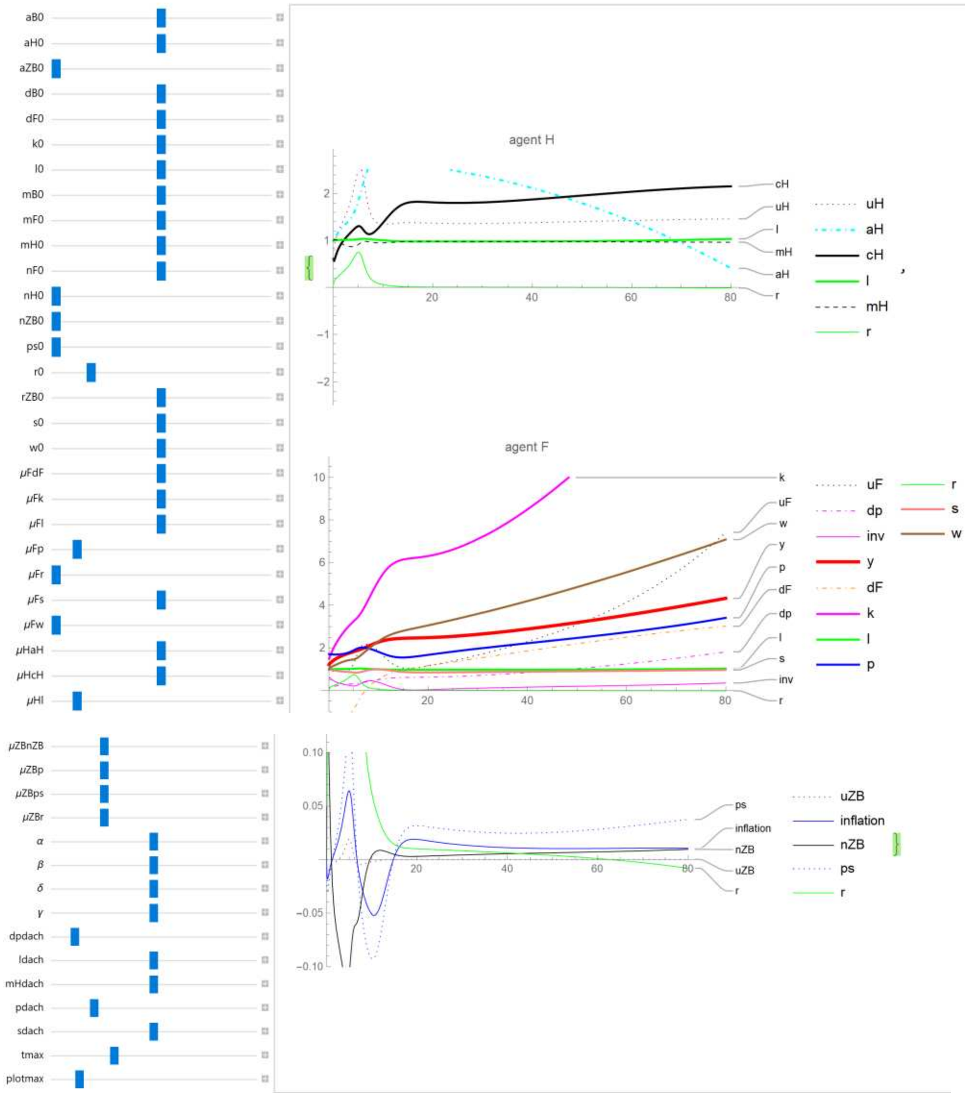
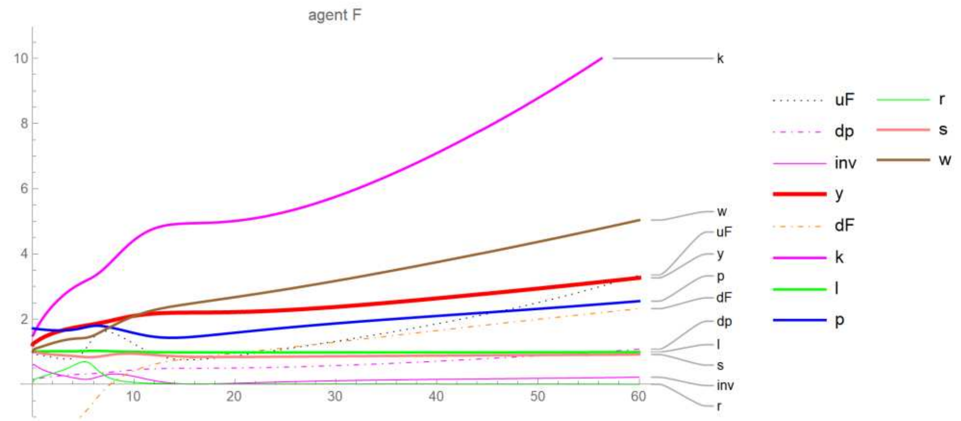
13.2. Calculation results of model B1

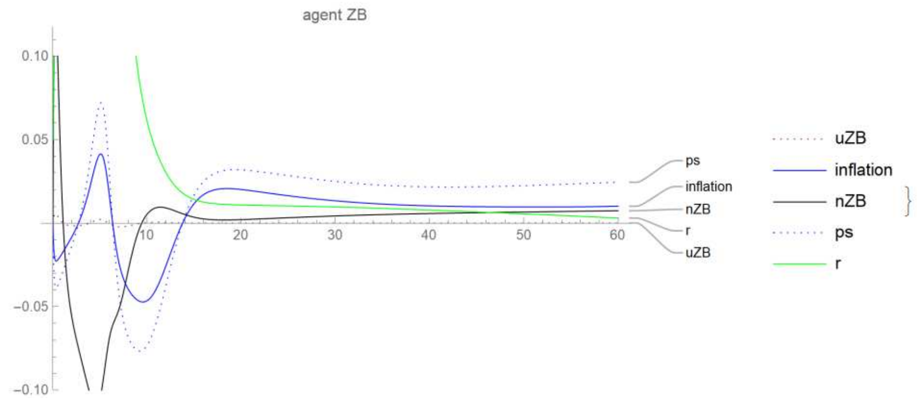
https://www.dropbox.com/s/rbtt9da2x8xm40n/Modell%20B1%20Version%207.ndsolve.nb?dl=0



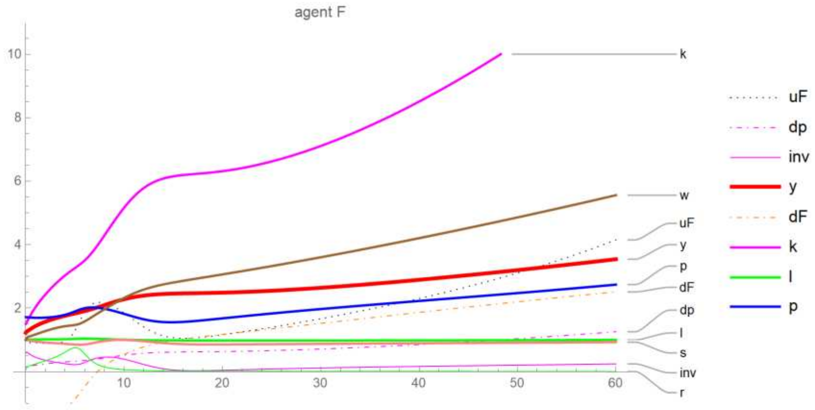
<u>Comparison of</u>

pure money supply policy $\delta = 0$





mixed money supply-interest rate policy $\delta = 0.5$



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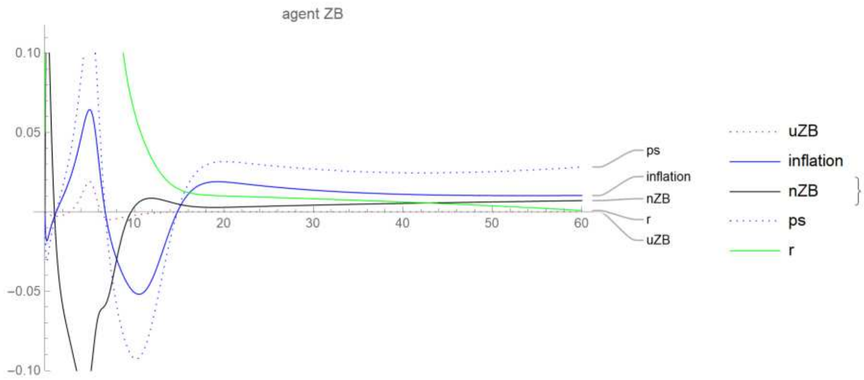
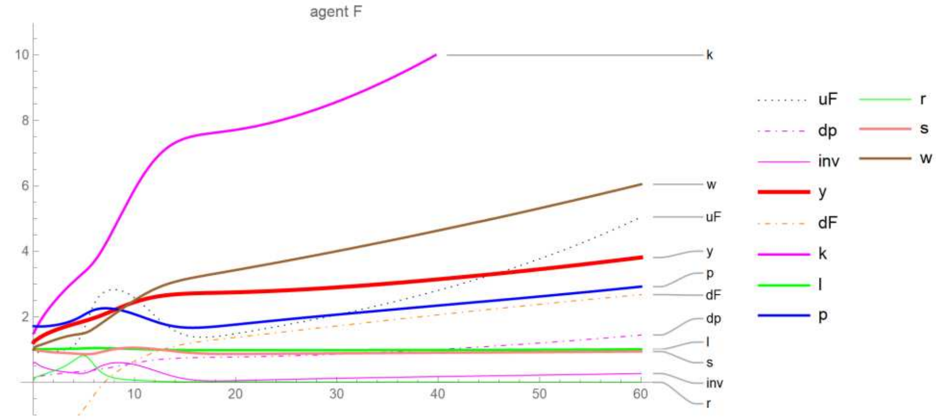


pure interest rate policy $\delta = 1$



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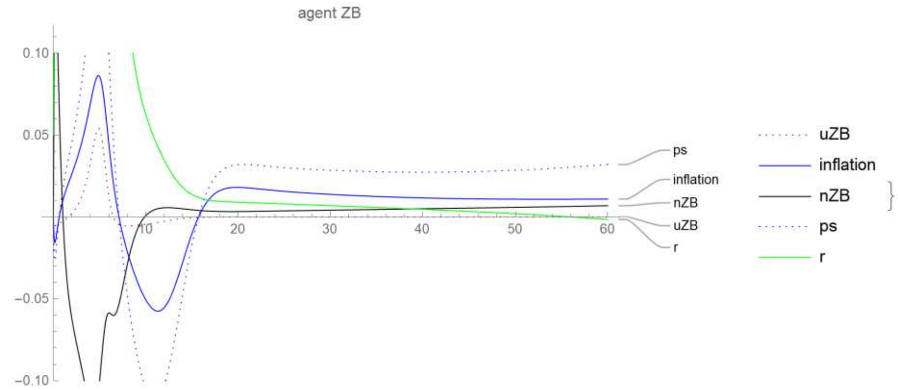

14. Model B2, (1 household, 1 firm, 1 good, 1 bank, 1 central bank) Taylor rule

14.1. Set up

Model B2 differs from model B1 only in the assumption that the central bank acts according to the Taylor rule.

In terms of the GCD methodology, the Taylor rule sets the value of the policy rate as an algebraically defined variable (see chapter 8.4.2).

If $\hat{p}$ denotes the target inflation rate, this results in

$$r = \frac{Y'}{Y} + \frac{p'}{p} + \sigma_1 \left(\frac{p'}{p} - \hat{p}\right) + \sigma_2 \left(\frac{Y'}{Y} - \frac{\hat{Y}'}{\hat{Y}}\right)$$

(For simplicity we write r instead of r_{leit}).

If you insert and simplify you get

$$r = \frac{p'}{p} + \sigma_1 \left(\frac{p'}{p} - \hat{p}\right) + (1-\alpha)\frac{K'}{K} + (1+\sigma_2)\alpha\frac{L'}{L} \qquad \text{<14.1>}$$

If the central bank acts only according to the Taylor rule, it does not act in the sense of optimizing a utility function, but according to empirical values that have proven their worth in the past. In this case, therefore, the utility function of the central bank can be set equal to 0.

Model B2 : basic equations for standard Taylor rule

algebraically defined variables

$Y(L,K) = \quad \beta L^{\alpha} K^{1-\alpha}$ "production function"

$DP(K) = \quad \widehat{dp}\, K$ "depreciation"

$r = \frac{p'}{p} + \sigma_1 (\frac{p'}{p} - \hat{p}) + (1-\alpha)\frac{K'}{K} + (1+\sigma_2)\alpha \frac{L'}{L}$ "standard Taylor rule"

utility functions

$U^H(C,L,M^H) = \quad C^{\gamma} - \left(\hat{L} - L\right)^2 - (\hat{M}^H - M^H)^2 + r A^H$ "utility function household"

$U^F(Y,L,S) = \quad pY - wL - (\hat{S} - S)^2 - r(-D^F)$ "utility function firm"

$U^B = 0$ "utility function bank"

$U^{ZB} = \ 0$ "utility function central bank"

constraints

$Z_1 = 0 = wL - pC + rA^H - N^H - M^{H\,\prime}$ for money flow of household H

$Z_2 = 0 = -wL + pC - r(-D^F) + N^F - M^{F\,\prime}$ for money flow of firm F

$Z_3 = 0 = N^{ZB} - N^F + r(-D^F) - rA^H + N^H - M^{B\,\prime}$ for money flow of bank B

$Z_4 = 0 = -N^{ZB} - R^{ZB\,\prime}$ for money flow of central bank ZB

$Z_5 = 0 = Y(L,K) - C - S' - DP - K'$ for flow of good 1 of firm F

$Z_6 = 0 = N^H - A^{H\,\prime}$ for accounts / debts flow of household H

$Z_7 = 0 = -N^F - D^{F\,\prime}$ for accounts / debts flow of firm F

$Z_8 = 0 = -N^{ZB} + N^F - N^H - D^{B\,\prime} - A^{B\,\prime}$ for accounts / debts flow of of bank B

$Z_9 = 0 = N^{ZB} - A^{ZB\,\prime}$ for accounts / debts flow of central bank ZB

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Model B2 : diff.-alg. equation system standard Taylor rule

```
uB[t] == 0
uF[t] == dF[t] × r[t] - (sdach - s[t])^2 - l[t] × w[t] + p[t] × y[t]
uH[t] == cH[t]^γ - (ldach - l[t])^2 - (mHdach - mH[t])^2 + aH[t] × r[t]
uZB[t] == 0

dp[t] == dpdach k[t]
inflation[t] == ps[t]/p[t]
inv[t] == k'[t]
r[t] == -pdach σ1 + σ1 ps[t]/p[t] - (-1+α) k'[t]/k[t] + α (1+σ2) l'[t]/l[t]
y[t] == β k[t]^(1-α) l[t]^α
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$$\mathrm{aB}'[t] == -\lambda_2[t]$$

$$\mathrm{aH}'[t] == -\lambda_4[t] + \lambda_6[t]\left(\mathrm{pdach}\,\sigma1 - \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} + \frac{(-1+\alpha)\,k'[t]}{k[t]} - \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right) + \mu\mathrm{HaH}\left(-\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right) + \lambda_8[t]\left(-\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)$$

$$\mathrm{aZB}'[t] == -\lambda_5[t]$$

$$\mathrm{cH}'[t] == \gamma\,\mu\mathrm{HcH}\,\mathrm{cH}[t]^{-1+\gamma} + p[t]\,\lambda_7[t] - p[t]\,\lambda_8[t] - \lambda_9[t]$$

$$\mathrm{dB}'[t] == -\lambda_2[t]$$

$$\mathrm{dF}'[t] == -\lambda_1[t] + \lambda_6[t]\left(\mathrm{pdach}\,\sigma1 - \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} + \frac{(-1+\alpha)\,k'[t]}{k[t]} - \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right) + \mu\mathrm{FdF}\left(-\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right) + \lambda_7[t]\left(-\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)$$

$$k'[t] = \mathrm{If}\left[\frac{(-1+\alpha)\,\mathrm{aH}[t]}{k[t]} + \frac{(-1+\alpha)\,\mathrm{dF}[t]}{k[t]} \neq 0,\ \frac{(-1+\alpha)\,\mathrm{aH}[t]}{k[t]} + \frac{(-1+\alpha)\,\mathrm{dF}[t]}{k[t]},\ -\frac{(-1+\alpha)\,\mathrm{aH}[t]\,k'[t]}{k[t]^2} - \frac{(-1+\alpha)\,\mathrm{dF}[t]\,k'[t]}{k[t]^2}\right]$$
$$\lambda_6[t] + \mathrm{If}\left[-\frac{(-1+\alpha)\,\mathrm{dF}[t]}{k[t]} \neq 0,\ -\frac{(-1+\alpha)\,\mathrm{dF}[t]}{k[t]},\ \frac{(-1+\alpha)\,\mathrm{dF}[t]\,k'[t]}{k[t]^2}\right]\lambda_7[t] +$$
$$\mathrm{If}\left[-\frac{(-1+\alpha)\,\mathrm{aH}[t]}{k[t]} \neq 0,\ -\frac{(-1+\alpha)\,\mathrm{aH}[t]}{k[t]},\ \frac{(-1+\alpha)\,\mathrm{aH}[t]\,k'[t]}{k[t]^2}\right]\lambda_8[t] - \lambda_9[t] + \frac{(-1+\alpha)\,\mu\mathrm{Hk}\,\mathrm{aH}[t]\,k'[t]}{k[t]^2} +$$
$$\mu\mathrm{Fk}\left((1-\alpha)\,\beta\,k[t]^{-\alpha}\,l[t]^{\alpha}\,p[t] + \frac{(-1+\alpha)\,\mathrm{dF}[t]\,k'[t]}{k[t]^2}\right)$$

$$l'[t] ==$$
$$\mathrm{If}\left[-\frac{\alpha\,(1+\sigma2)\,\mathrm{aH}[t]}{l[t]} - \frac{\alpha\,(1+\sigma2)\,\mathrm{dF}[t]}{l[t]} \neq 0,\ -\frac{\alpha\,(1+\sigma2)\,\mathrm{aH}[t]}{l[t]} - \frac{\alpha\,(1+\sigma2)\,\mathrm{dF}[t]}{l[t]},\ \frac{\alpha\,(1+\sigma2)\,\mathrm{aH}[t]\,l'[t]}{l[t]^2} + \frac{\alpha\,(1+\sigma2)\,\mathrm{dF}[t]\,l'[t]}{l[t]^2}\right]\lambda_6[t] +$$
$$\mathrm{If}\left[\frac{\alpha\,(1+\sigma2)\,\mathrm{dF}[t]}{l[t]} \neq 0,\ \frac{\alpha\,(1+\sigma2)\,\mathrm{dF}[t]}{l[t]},\ -w[t] - \frac{\alpha\,(1+\sigma2)\,\mathrm{dF}[t]\,l'[t]}{l[t]^2}\right]\lambda_7[t] +$$
$$\mathrm{If}\left[\frac{\alpha\,(1+\sigma2)\,\mathrm{aH}[t]}{l[t]} \neq 0,\ \frac{\alpha\,(1+\sigma2)\,\mathrm{aH}[t]}{l[t]},\ w[t] - \frac{\alpha\,(1+\sigma2)\,\mathrm{aH}[t]\,l'[t]}{l[t]^2}\right]\lambda_8[t] + \alpha\,\beta\,k[t]^{1-\alpha}\,l[t]^{-1+\alpha}\,\lambda_9[t] +$$
$$\mu\mathrm{Hl}\left(2\,(\mathrm{ldach} - l[t]) - \frac{\alpha\,(1+\sigma2)\,\mathrm{aH}[t]\,l'[t]}{l[t]^2}\right) +$$
$$\mu\mathrm{Fl}\left(\alpha\,\beta\,k[t]^{1-\alpha}\,l[t]^{-1+\alpha}\,p[t] - w[t] - \frac{\alpha\,(1+\sigma2)\,\mathrm{dF}[t]\,l'[t]}{l[t]^2}\right)$$

$$\mathrm{mB}'[t] == -\lambda_6[t]$$

$$\mathrm{mF}'[t] == -\lambda_7[t]$$

$$\mathrm{mH}'[t] == 2\,\mu\mathrm{HmH}\,(\mathrm{mHdach} - \mathrm{mH}[t]) - \lambda_8[t]$$

$$\mathrm{nF}'[t] == -\lambda_1[t] + \lambda_2[t] - \lambda_6[t] + \lambda_7[t]$$

$$\mathrm{nH}'[t] == -\lambda_2[t] + \lambda_4[t] + \lambda_6[t] - \lambda_8[t]$$

$$\mathrm{nZB}'[t] == -\lambda_2[t] - \lambda_3[t] + \lambda_5[t] + \lambda_6[t]$$

$$p'[t] == -\frac{\mu\mathrm{Hp}\,\sigma1\,\mathrm{aH}[t]\times\mathrm{ps}[t]}{p[t]^2} + \mu\mathrm{Fp}\left(\beta\,k[t]^{1-\alpha}\,l[t]^{\alpha} - \frac{\sigma1\,\mathrm{dF}[t]\times\mathrm{ps}[t]}{p[t]^2}\right) + \left(\frac{\sigma1\,\mathrm{aH}[t]\times\mathrm{ps}[t]}{p[t]^2} + \frac{\sigma1\,\mathrm{dF}[t]\times\mathrm{ps}[t]}{p[t]^2}\right)\lambda_6[t] + \left(\mathrm{cH}[t] - \frac{\sigma1\,\mathrm{dF}[t]\times\mathrm{ps}[t]}{p[t]^2}\right)\lambda_7[t] + \left(-\mathrm{cH}[t] - \frac{\sigma1\,\mathrm{aH}[t]\times\mathrm{ps}[t]}{p[t]^2}\right)\lambda_8[t] - \lambda_{10}[t]$$

$$\mathrm{ps}'[t] == \frac{\mu\mathrm{Hps}\,\sigma1\,\mathrm{aH}[t]}{p[t]} + \frac{\mu\mathrm{Fps}\,\sigma1\,\mathrm{dF}[t]}{p[t]} + \left(-\frac{\sigma1\,\mathrm{aH}[t]}{p[t]} - \frac{\sigma1\,\mathrm{dF}[t]}{p[t]}\right)\lambda_6[t] + {} + \frac{\sigma1\,\mathrm{dF}[t]\,\lambda_7[t]}{p[t]} + \frac{\sigma1\,\mathrm{aH}[t]\,\lambda_8[t]}{p[t]} + \lambda_{10}[t]$$

$$\mathrm{rZB}'[t] == -\lambda_3[t]$$

$$s'[t] == 2\,\mu\mathrm{Fs}\,(\mathrm{sdach} - s[t]) - \lambda_9[t]$$

$$w'[t] == -\mu\mathrm{Fw}\,l[t] - l[t]\,\lambda_7[t] + l[t]\,\lambda_8[t]$$

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$$0 == -\mathrm{nF}[t] - \mathrm{dF}'[t]$$

$$0 == \mathrm{nF}[t] - \mathrm{nH}[t] - \mathrm{nZB}[t] - \mathrm{aB}'[t] - \mathrm{dB}'[t]$$

$$0 == -\mathrm{nZB}[t] - \mathrm{rZB}'[t]$$

$$0 == \mathrm{nH}[t] - \mathrm{aH}'[t]$$

$$0 == \mathrm{nZB}[t] - \mathrm{aZB}'[t]$$

$$0 == -\mathrm{nF}[t] + \mathrm{nH}[t] + \mathrm{nZB}[t] - \mathrm{aH}[t]\left(-\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right) - \mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right) - \mathrm{mB}'[t]$$

$$0 == \mathrm{nF}[t] + \mathrm{cH}[t]\times p[t] - l[t]\times w[t] + \mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right) - \mathrm{mF}'[t]$$

$$0 == -\mathrm{nH}[t] - \mathrm{cH}[t]\times p[t] + l[t]\times w[t] + \mathrm{aH}[t]\left(-\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right) - \mathrm{mH}'[t]$$

$$0 == -\mathrm{cH}[t] - \mathrm{dpdach}\,k[t] + \beta\,k[t]^{1-\alpha}\,l[t]^{\alpha} - k'[t] - s'[t]$$

$$0 == \mathrm{ps}[t] - p'[t]$$

$$\mathrm{aB}[0] == \mathrm{aB0}$$
$$\mathrm{aH}[0] == \mathrm{aH0}$$
$$\mathrm{aZB}[0] == \mathrm{aZB0}$$
$$\mathrm{cH}[0] == \tfrac{1}{2}\,\mathrm{k0}^{1-\alpha}\,\mathrm{l0}^{\alpha}\,\beta$$
$$\mathrm{dB}[0] == \mathrm{dB0}$$
$$\mathrm{dF}[0] == \mathrm{dF0}$$
$$k[0] == \mathrm{k0}$$
$$l[0] == \mathrm{l0}$$
$$\mathrm{mB}[0] == \mathrm{mB0}$$
$$\mathrm{mF}[0] == \mathrm{mF0}$$
$$\mathrm{mH}[0] == \mathrm{mH0}$$
$$\mathrm{nF}[0] == \mathrm{nF0}$$
$$\mathrm{nH}[0] == \mathrm{nH0}$$
$$\mathrm{nZB}[0] == \mathrm{nZB0}$$
$$p[0] == \mathrm{p0}$$
$$\mathrm{ps}[0] == \mathrm{ps0}$$
$$\mathrm{rZB}[0] == \mathrm{rZB0}$$
$$s[0] == \mathrm{s0}$$
$$w[0] == \mathrm{w0}$$

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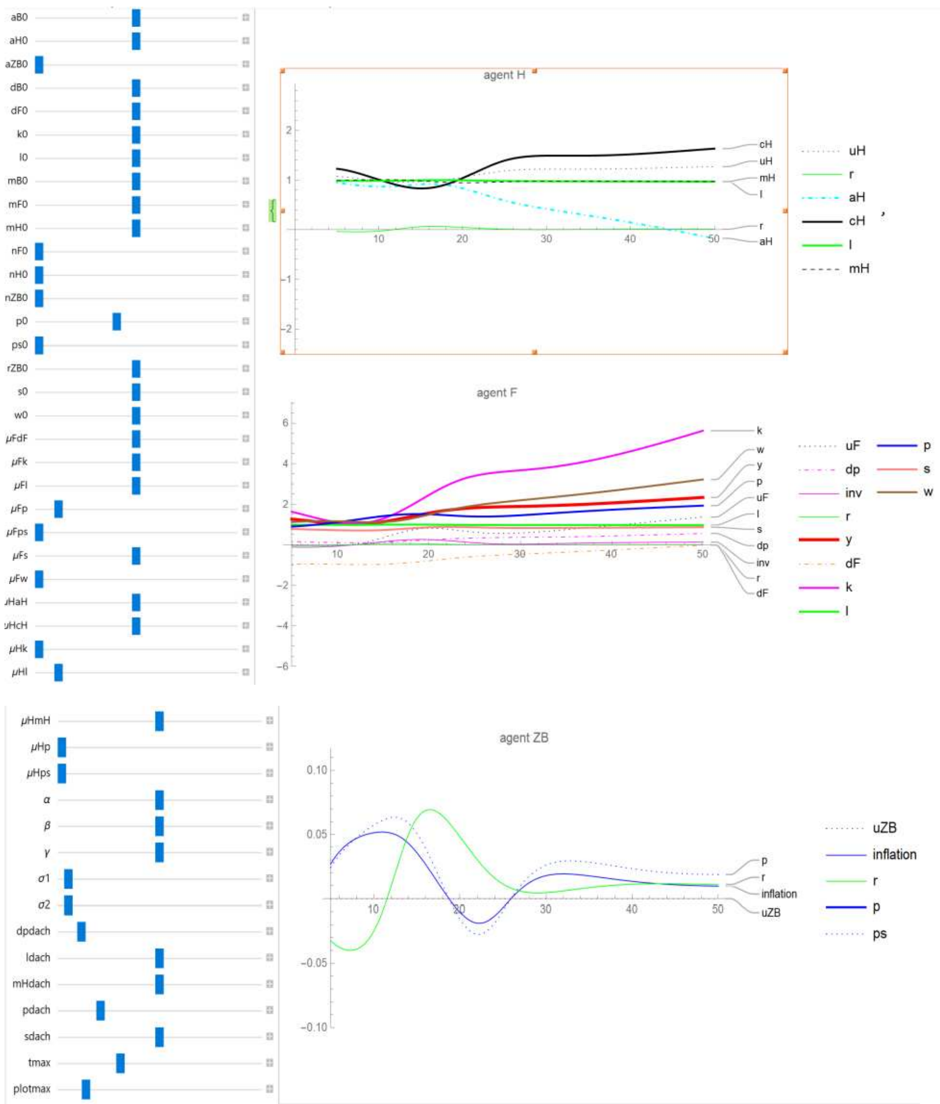
14.2. Calculation results of model B2

https://www.dropbox.com/s/5rj153v24swq7m7/Modell%20B2%20Version%203.ndsolve.nb?dl=0

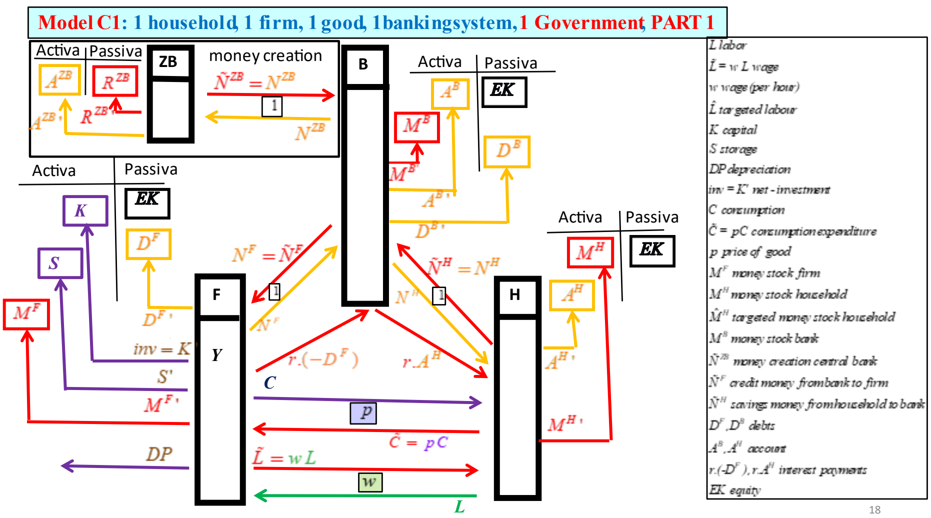


15. Model C1, (1 household, 1 firm, 1 good, 1 banking system, 1 government) interest rate policy versus money supply policy

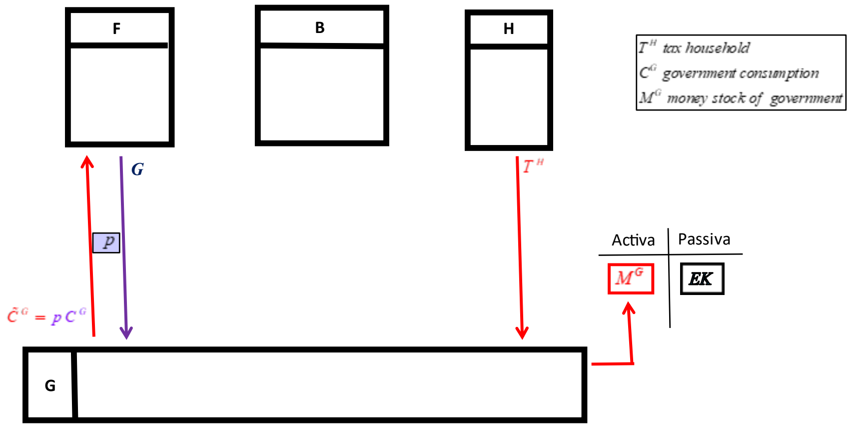
15.1. Set up

The target of model C1 is to extend model B1 by the government G as an agent in a simple form.

The government has a utility function analogous to that of the household. It collects an income tax from the household, which either flows to its money stock M^G or is used for government consumption C^G.



Model C1: 1 household, 1 firm, 1 good, 1 bankingsystem, 1 Government, PART 2



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Model C1 : basic equations

algebraically defined variables

$Y(L,K) =$	$\beta L^{\alpha} K^{1-\alpha}$	"production function"
$DP(K) =$	$\overline{dp}K$	"depreciation"
$T^H(w,L) =$	$\tau^H wL$	"income tax household"

utility functions

$U^H = C^{1/2} - (\hat{L} - L)^2 - (\hat{M}^H - M^H)^2 + rA^H$	"utility function household"
$U^F = pY - wL - (\hat{S} - S)^2 - r(-D^F)$	"utility function firm"
$U^B = 0$	"utility function bank"
$U^{ZB} = \left(-\delta r + (1-\delta)N^{ZB}\right)(\hat{p} - \frac{ps}{p})$	"utility function central bank"
$U^G = G^{1/2}$	"utility function government"

constraints

$Z_1 = 0 = wL - pC + rA^H - N^H - T^H - M^{H\prime}$	for money flow of household H
$Z_2 = 0 = -wL + pC - r(-D^F) + N^F + pG - M^{F\prime}$	for money flow of firm F
$Z_3 = 0 = N^{ZB} - N^F + r(-D^F) - rA^H + N^H - M^{B\prime}$	for money flow of bank B
$Z_4 = 0 = -N^{ZB} - R^{ZB\prime}$	for money flow of central bank ZB
$Z_5 = 0 = Y(L,K) - C - G - S' - DP - K'$	for flow of good 1 of firm F
$Z_6 = 0 = N^H - A^{H\prime}$	for accounts / debts flow of household H
$Z_7 = 0 = -N^F - D^{F\prime}$	for accounts / debts flow of firm F
$Z_8 = 0 = -N^{ZB} + N^F - N^H - D^{B\prime} - A^{B\prime}$	for accounts / debts flow of of bank B
$Z_9 = 0 = N^{ZB} - A^{ZB\prime}$	for accounts / debts flow of central bank ZB
$Z_{10} = 0 = -pG + T^H - M^{G\prime}$	for money flow of government
$Z_{11} = 0 = ps - p'$	because "no derivation in utility function of ZB"

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ModelC1: diff.-alg. equation system

```
uB[t] == 0
uF[t] == dF[t] × r[t] - (sdach - s[t])^2 - l[t] × w[t] + p[t] × y[t]
uG[t] == cG[t]^γG
uH[t] == cH[t]^γH - (ldach - l[t])^2 - (mHdach - mH[t])^2 + aH[t] × r[t]
uZB[t] == (pdach - ps[t]/p[t]) ((1 - δ) nZB[t] - δ r[t])

cGschlange[t] == cG[t] × p[t]
dp[t] == dpdach k[t]
inflation[t] == ps[t]/p[t]
inv[t] == k'[t]
taxH[t] == τH l[t] × w[t]
y[t] == β k[t]^(1-α) l[t]^α

aB'[t] == -λ8[t]
aH'[t] == μHaH r[t] + r[t] λ1[t] - r[t] λ3[t] - λ6[t]
aZB'[t] == -λ9[t]
cG'[t] == γG μGcG cG[t]^(-1+γG) + p[t] λ2[t] - λ5[t] - p[t] λ10[t]
cH'[t] == γH μHcH cH[t]^(-1+γH) - p[t] λ1[t] + p[t] λ2[t] - λ5[t]
dB'[t] == -λ8[t]
dF'[t] == μFdF r[t] + r[t] λ2[t] - r[t] λ3[t] - λ7[t]
k'[t] == (1 - α) β μFk k[t]^-α l[t]^α p[t] - λ5[t]
l'[t] == 2 μHl (ldach - l[t]) + μFl (α β k[t]^(1-α) l[t]^(-1+α) p[t] - w[t]) + (w[t] - τH w[t]) λ1[t] -
   w[t] λ2[t] + α β k[t]^(1-α) l[t]^(-1+α) λ5[t] + τH w[t] λ10[t]
mB'[t] == -λ3[t]
mF'[t] == -λ2[t]
mG'[t] == -λ10[t]
mH'[t] == 2 μHmH (mHdach - mH[t]) - λ1[t]
nF'[t] == λ2[t] - λ3[t] - λ7[t] + λ8[t]
nH'[t] == -λ1[t] + λ3[t] + λ6[t] - λ8[t]
nZB'[t] == (1 - δ) μZBnZB (pdach - ps[t]/p[t]) + λ3[t] - λ4[t] - λ8[t] + λ9[t]
p'[t] == β μFp k[t]^(1-α) l[t]^α + μZBp ps[t] ((1-δ) nZB[t] - δ r[t]) / p[t]^2 - cH[t] λ1[t] + (cG[t] + cH[t]) λ2[t] -
   cG[t] λ10[t] - λ11[t]
ps'[t] == -μZBps ((1-δ) nZB[t] - δ r[t]) / p[t] + λ11[t]
r'[t] == μHr aH[t] + μFr dF[t] - δ μZBr (pdach - ps[t]/p[t]) + aH[t] λ1[t] + dF[t] λ2[t] +
   (-aH[t] - dF[t]) λ3[t]
rZB'[t] == -λ4[t]
s'[t] == 2 μFs (sdach - s[t]) - λ5[t]
w'[t] == -μFw l[t] + (l[t] - τH l[t]) λ1[t] - l[t] λ2[t] + τH l[t] λ10[t]

0 == -nH[t] - cH[t] × p[t] + aH[t] × r[t] + l[t] × w[t] - τH l[t] × w[t] - mH'[t]
0 == nF[t] + cG[t] × p[t] + cH[t] × p[t] + dF[t] × r[t] - l[t] × w[t] - mF'[t]
0 == -nF[t] + nH[t] + nZB[t] - aH[t] × r[t] - dF[t] × r[t] - mB'[t]
0 == -nZB[t] - rZB'[t]
0 == -cG[t] - cH[t] - dpdach k[t] + β k[t]^(1-α) l[t]^α - k'[t] - s'[t]
0 == nH[t] - aH'[t]
0 == -nF[t] - dF'[t]
0 == nF[t] - nH[t] - nZB[t] - aB'[t] - dB'[t]
0 == nZB[t] - aZB'[t]
0 == -cG[t] × p[t] + τH l[t] × w[t] - mG'[t]
0 == ps[t] - p'[t]

aB[0] == aB0
aH[0] == aH0
aZB[0] == aZB0
cG[0] == l0 w0 τH / p0
cH[0] == (-dpdach k0 p0 + k0^(1-α) l0^α p0 β - l0 w0 τH) / p0
dB[0] == dB0
dF[0] == -(-dpdach k0 p0 - l0 w0 + k0^(1-α) l0^α p0 β) / r0
k[0] == k0
l[0] == l0
mB[0] == mB0
mF[0] == mF0
mG[0] == mG0
mH[0] == mH0
nF[0] == nF0
nH[0] == nH0
nZB[0] == nZB0
p[0] == p0
ps[0] == ps0
r[0] == r0
rZB[0] == -mB0 - mF0 - mG0 - mH0
s[0] == s0
w[0] == w0
```

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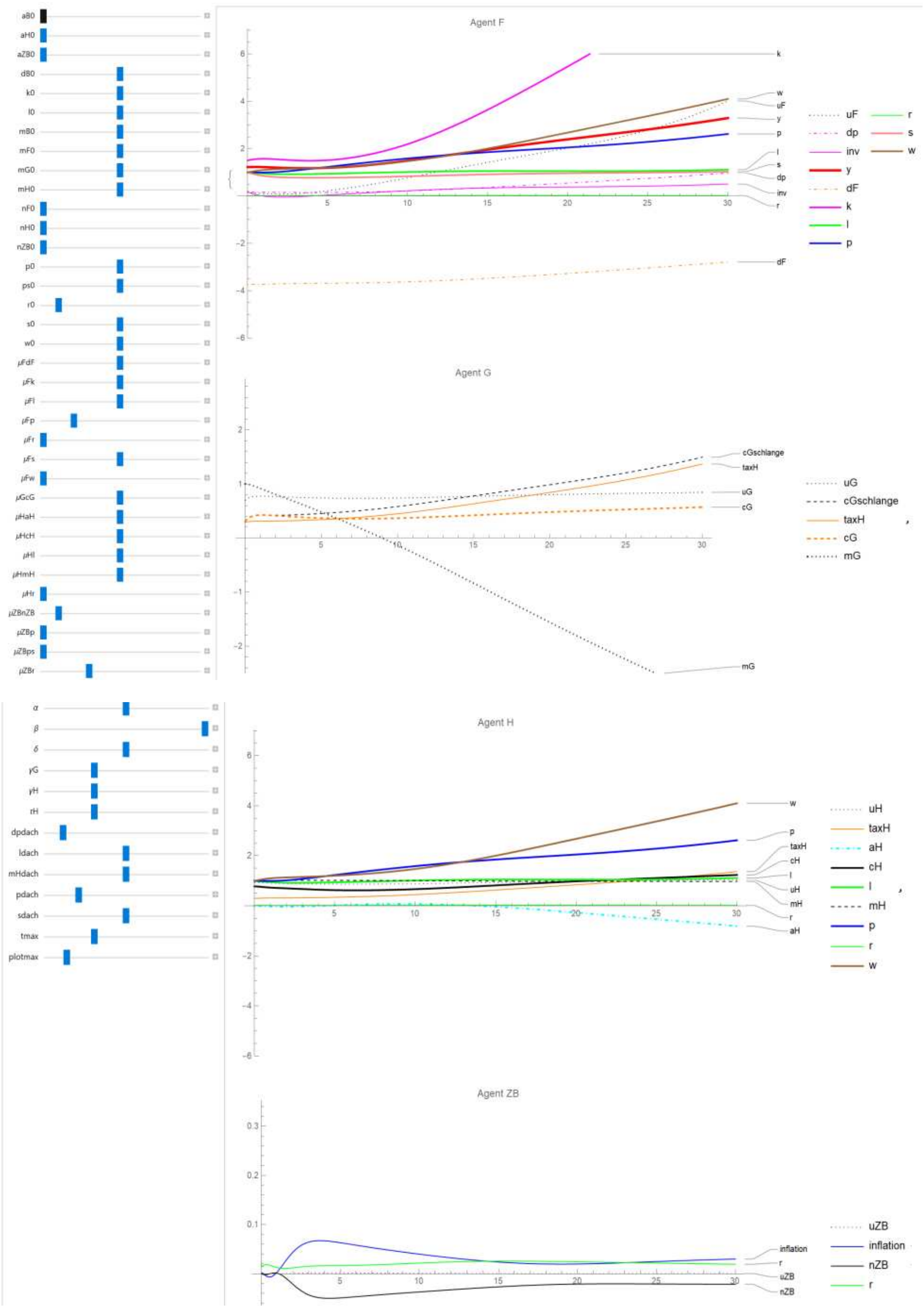
15.2. Calculation results of model C1

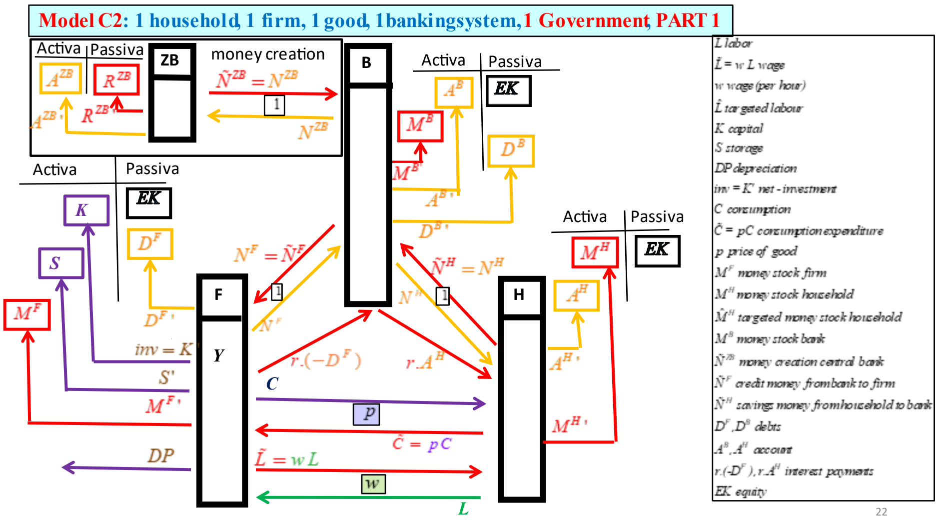
https://www.dropbox.com/s/vmg7wbyghwbg2h7/Modell%20C1%20Version%2019.ndsolve.nb?dl=0

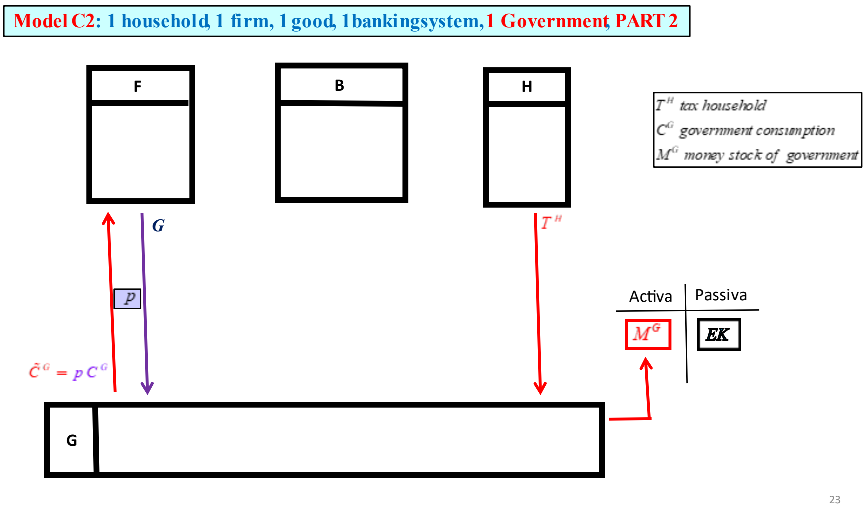


16. Model C2, (1 household, 1 firm, 1 good, 1 banking system, 1 government)), standard Taylor rule

16.1. Set up

Model C2 corresponds to the extension of model B2 by the agent government in the sense of model C1 and corresponds to model C1 with the change that the central bank acts in the sense of the standard Taylor rule.





Model C2 : basic equations

algebraically defined variables

$Y(L,K) =$	$\beta L^{\alpha} K^{1-\alpha}$	*"production function"*
$DP(K) =$	$\overline{dp} K$	*"depreciation"*
$T^{H}(w,L) =$	$\tau^{H} w L$	*"income tax household"*

$r = \frac{p'}{p} + \sigma_1 (\frac{p'}{p} - \hat{p}) + (1-\alpha)\frac{K'}{K} + (1+\sigma_2)\alpha \frac{L'}{L}$ *"standard Taylor rule"*

utility functions

$U^{H} = C^{\gamma_H} - \left(\hat{L} - L\right)^2 - (\hat{M}^{H} - M^{H})^2 + r A^{H}$ *"utility function household"*

$U^{F} = pY - wL - (\hat{S} - S)^2 - r(-D^{F})$ *"utility function firm"*

$U^{B} = 0$ *"utility function bank"*

$U^{ZB} = 0$ *"utility function central bank"*

$U^{S} = G^{\gamma_G}$ *"utility function government"*

constraints

$Z_1 = 0 = wL - pC + rA^{H} - N^{H} - T^{H} - M^{H\,\prime}$ *for money flow of household H*

$Z_2 = 0 = -wL + pC - r(-D^{F}) + N^{F} + pG - M^{F\,\prime}$ *for money flow of firm F*

$Z_3 = 0 = N^{ZB} - N^{F} + r(-D^{F}) - rA^{H} + N^{H} - M^{B\,\prime}$ *for money flow of bank B*

$Z_4 = 0 = -N^{ZB} - R^{ZB\,\prime}$ *for money flow of central bank ZB*

$Z_5 = 0 = Y(L,K) - C - G - S' - DP - K'$ *for flow of good 1 of firm F*

$Z_6 = 0 = N^{H} - A^{H\,\prime}$ *for accounts / debts flow of household H*

$Z_7 = 0 = -N^{F} - D^{F\,\prime}$ *for accounts / debts flow of firm F*

$Z_8 = 0 = -N^{ZB} + N^{F} - N^{H} - D^{B\,\prime} - A^{B\,\prime}$ *for accounts / debts flow of of bank B*

$Z_9 = 0 = N^{ZB} - A^{ZB\,\prime}$ *for accounts / debts flow of central bank ZB*

$Z_{10} = 0 = -pG + T^{H} - M^{G\,\prime}$ *for money flow of government* 24

$Z_{11} = 0 = ps - p'$ because *"no derivation in utility function of ZB"*

Model C2: diff.-alg. equation system

$$\mathtt{uB[t]} == 0$$

$$\mathtt{uF[t]} == \mathtt{dF[t]} \times \mathtt{r[t]} - (\mathtt{sdach} - \mathtt{s[t]})^2 - \mathtt{l[t]} \times \mathtt{w[t]} + \mathtt{p[t]} \times \mathtt{y[t]}$$

$$\mathtt{uG[t]} == \mathtt{cG[t]}^{\gamma G}$$

$$\mathtt{uH[t]} == \mathtt{cH[t]}^{\gamma H} - (\mathtt{ldach} - \mathtt{l[t]})^2 - (\mathtt{mHdach} - \mathtt{mH[t]})^2 + \mathtt{aH[t]} \times \mathtt{r[t]}$$

$$\mathtt{uZB[t]} == 0$$

$$\mathtt{cGschlange[t]} == \mathtt{cG[t]} \times \mathtt{p[t]}$$

$$\mathtt{dp[t]} == \mathtt{dpdach}\ \mathtt{k[t]}$$

$$\mathtt{inflation[t]} == \frac{\mathtt{ps[t]}}{\mathtt{p[t]}}$$

$$\mathtt{inv[t]} == \mathtt{k'[t]}$$

$$\mathtt{r[t]} == -\mathtt{pdach}\ \sigma 1 + \frac{\sigma 1\ \mathtt{ps[t]}}{\mathtt{p[t]}} - \frac{(-1+\alpha)\ \mathtt{k'[t]}}{\mathtt{k[t]}} + \frac{\alpha\ (1+\sigma 2)\ \mathtt{l'[t]}}{\mathtt{l[t]}}$$

$$\mathtt{taxH[t]} == \tau \mathtt{H}\ \mathtt{l[t]} \times \mathtt{w[t]}$$

$$\mathtt{y[t]} == \beta\ \mathtt{k[t]}^{1-\alpha}\ \mathtt{l[t]}^{\alpha}$$

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$$\text{aB}'[t] == -\lambda_8[t]$$

$$\text{aH}'[t] == -\lambda_6[t] + \lambda_3[t]\left(\text{pdach}\,\sigma1 - \frac{\sigma1\,\text{ps}[t]}{\text{p}[t]} + \frac{(-1+\alpha)\,\text{k}'[t]}{\text{k}[t]} - \frac{\alpha\,(1+\sigma2)\,\text{l}'[t]}{\text{l}[t]}\right) +$$
$$\mu\text{HaH}\left(-\text{pdach}\,\sigma1 + \frac{\sigma1\,\text{ps}[t]}{\text{p}[t]} - \frac{(-1+\alpha)\,\text{k}'[t]}{\text{k}[t]} + \frac{\alpha\,(1+\sigma2)\,\text{l}'[t]}{\text{l}[t]}\right) +$$
$$\lambda_1[t]\left(-\text{pdach}\,\sigma1 + \frac{\sigma1\,\text{ps}[t]}{\text{p}[t]} - \frac{(-1+\alpha)\,\text{k}'[t]}{\text{k}[t]} + \frac{\alpha\,(1+\sigma2)\,\text{l}'[t]}{\text{l}[t]}\right)$$

$$\text{aZB}'[t] == -\lambda_9[t]$$

$$\text{cG}'[t] == \gamma\text{G}\,\mu\text{GcG}\,\text{cG}[t]^{-1+\gamma\text{G}} + \text{p}[t]\,\lambda_2[t] - \lambda_5[t] - \text{p}[t]\,\lambda_{10}[t]$$

$$\text{cH}'[t] == \gamma\text{H}\,\mu\text{HcH}\,\text{cH}[t]^{-1+\gamma\text{H}} - \text{p}[t]\,\lambda_1[t] + \text{p}[t]\,\lambda_2[t] - \lambda_5[t]$$

$$\text{dB}'[t] == -\lambda_8[t]$$

$$\text{dF}'[t] == -\lambda_7[t] + \lambda_3[t]\left(\text{pdach}\,\sigma1 - \frac{\sigma1\,\text{ps}[t]}{\text{p}[t]} + \frac{(-1+\alpha)\,\text{k}'[t]}{\text{k}[t]} - \frac{\alpha\,(1+\sigma2)\,\text{l}'[t]}{\text{l}[t]}\right) +$$
$$\mu\text{FdF}\left(-\text{pdach}\,\sigma1 + \frac{\sigma1\,\text{ps}[t]}{\text{p}[t]} - \frac{(-1+\alpha)\,\text{k}'[t]}{\text{k}[t]} + \frac{\alpha\,(1+\sigma2)\,\text{l}'[t]}{\text{l}[t]}\right) +$$
$$\lambda_2[t]\left(-\text{pdach}\,\sigma1 + \frac{\sigma1\,\text{ps}[t]}{\text{p}[t]} - \frac{(-1+\alpha)\,\text{k}'[t]}{\text{k}[t]} + \frac{\alpha\,(1+\sigma2)\,\text{l}'[t]}{\text{l}[t]}\right)$$

$$\text{k}'[t] == \text{If}\left[-\frac{(-1+\alpha)\,\text{aH}[t]}{\text{k}[t]} \neq 0,\ -\frac{(-1+\alpha)\,\text{aH}[t]}{\text{k}[t]},\ \frac{(-1+\alpha)\,\text{aH}[t]\,\text{k}'[t]}{\text{k}[t]^2}\right]\lambda_1[t] +$$
$$\text{If}\left[-\frac{(-1+\alpha)\,\text{dF}[t]}{\text{k}[t]} \neq 0,\ -\frac{(-1+\alpha)\,\text{dF}[t]}{\text{k}[t]},\ \frac{(-1+\alpha)\,\text{dF}[t]\,\text{k}'[t]}{\text{k}[t]^2}\right]\lambda_2[t] +$$
$$\text{If}\left[\frac{(-1+\alpha)\,\text{aH}[t]}{\text{k}[t]} + \frac{(-1+\alpha)\,\text{dF}[t]}{\text{k}[t]} \neq 0,\ \frac{(-1+\alpha)\,\text{aH}[t]}{\text{k}[t]} + \frac{(-1+\alpha)\,\text{dF}[t]}{\text{k}[t]},\ -\frac{(-1+\alpha)\,\text{aH}[t]\,\text{k}'[t]}{\text{k}[t]^2} - \frac{(-1+\alpha)\,\text{dF}[t]\,\text{k}'[t]}{\text{k}[t]^2}\right]$$
$$\lambda_3[t] - \lambda_5[t] + \frac{(-1+\alpha)\,\mu\text{Hk}\,\text{aH}[t]\,\text{k}'[t]}{\text{k}[t]^2} + \mu\text{Fk}\left((1-\alpha)\,\beta\,\text{k}[t]^{-\alpha}\,\text{l}[t]^{\alpha}\,\text{p}[t] + \frac{(-1+\alpha)\,\text{dF}[t]\,\text{k}'[t]}{\text{k}[t]^2}\right)$$

$$\text{l}'[t] == \text{If}\left[\frac{\alpha\,(1+\sigma2)\,\text{aH}[t]}{\text{l}[t]} \neq 0,\ \frac{\alpha\,(1+\sigma2)\,\text{aH}[t]}{\text{l}[t]},\ \text{w}[t] - \tau\text{H}\,\text{w}[t] - \frac{\alpha\,(1+\sigma2)\,\text{aH}[t]\,\text{l}'[t]}{\text{l}[t]^2}\right]\lambda_1[t] +$$
$$\text{If}\left[\frac{\alpha\,(1+\sigma2)\,\text{dF}[t]}{\text{l}[t]} \neq 0,\ \frac{\alpha\,(1+\sigma2)\,\text{dF}[t]}{\text{l}[t]},\ -\text{w}[t] - \frac{\alpha\,(1+\sigma2)\,\text{dF}[t]\,\text{l}'[t]}{\text{l}[t]^2}\right]\lambda_2[t] +$$
$$\text{If}\left[-\frac{\alpha\,(1+\sigma2)\,\text{aH}[t]}{\text{l}[t]} - \frac{\alpha\,(1+\sigma2)\,\text{dF}[t]}{\text{l}[t]} \neq 0,\ -\frac{\alpha\,(1+\sigma2)\,\text{aH}[t]}{\text{l}[t]} - \frac{\alpha\,(1+\sigma2)\,\text{dF}[t]}{\text{l}[t]},\right.$$
$$\left.\frac{\alpha\,(1+\sigma2)\,\text{aH}[t]\,\text{l}'[t]}{\text{l}[t]^2} + \frac{\alpha\,(1+\sigma2)\,\text{dF}[t]\,\text{l}'[t]}{\text{l}[t]^2}\right]\lambda_3[t] + \alpha\,\beta\,\text{k}[t]^{1-\alpha}\,\text{l}[t]^{-1+\alpha}\,\lambda_5[t] + \tau\text{H}\,\text{w}[t]\,\lambda_{10}[t] +$$
$$\mu\text{Hl}\left(2\,(\text{ldach} - \text{l}[t]) - \frac{\alpha\,(1+\sigma2)\,\text{aH}[t]\,\text{l}'[t]}{\text{l}[t]^2}\right) +$$
$$\mu\text{Fl}\left(\alpha\,\beta\,\text{k}[t]^{1-\alpha}\,\text{l}[t]^{-1+\alpha}\,\text{p}[t] - \text{w}[t] - \frac{\alpha\,(1+\sigma2)\,\text{dF}[t]\,\text{l}'[t]}{\text{l}[t]^2}\right)$$

$$\text{mB}'[t] == -\lambda_3[t]$$
$$\text{mF}'[t] == -\lambda_2[t]$$
$$\text{mG}'[t] == -\lambda_{10}[t]$$
$$\text{mH}'[t] == 2\,\mu\text{HmH}\,(\text{mHdach} - \text{mH}[t]) - \lambda_1[t]$$
$$\text{nF}'[t] == \lambda_2[t] - \lambda_3[t] - \lambda_7[t] + \lambda_8[t]$$
$$\text{nH}'[t] == -\lambda_1[t] + \lambda_3[t] + \lambda_6[t] - \lambda_8[t]$$
$$\text{nZB}'[t] == \lambda_3[t] - \lambda_4[t] - \lambda_8[t] + \lambda_9[t]$$

$$\text{p}'[t] == -\frac{\mu\text{Hp}\,\sigma1\,\text{aH}[t]\times\text{ps}[t]}{\text{p}[t]^2} + \mu\text{Fp}\left(\beta\,\text{k}[t]^{1-\alpha}\,\text{l}[t]^{\alpha} - \frac{\sigma1\,\text{dF}[t]\times\text{ps}[t]}{\text{p}[t]^2}\right) +$$
$$\left(-\text{cH}[t] - \frac{\sigma1\,\text{aH}[t]\times\text{ps}[t]}{\text{p}[t]^2}\right)\lambda_1[t] + \left(\text{cG}[t] + \text{cH}[t] - \frac{\sigma1\,\text{dF}[t]\times\text{ps}[t]}{\text{p}[t]^2}\right)\lambda_2[t] +$$
$$\left(\frac{\sigma1\,\text{aH}[t]\times\text{ps}[t]}{\text{p}[t]^2} + \frac{\sigma1\,\text{dF}[t]\times\text{ps}[t]}{\text{p}[t]^2}\right)\lambda_3[t] - \text{cG}[t]\,\lambda_{10}[t] - \lambda_{11}[t]$$

$$\text{ps}'[t] == \frac{\mu\text{Hps}\,\sigma1\,\text{aH}[t]}{\text{p}[t]} + \frac{\mu\text{Fps}\,\sigma1\,\text{dF}[t]}{\text{p}[t]} + \frac{\sigma1\,\text{aH}[t]\,\lambda_1[t]}{\text{p}[t]} + \frac{\sigma1\,\text{dF}[t]\,\lambda_2[t]}{\text{p}[t]} +$$
$$\left(-\frac{\sigma1\,\text{aH}[t]}{\text{p}[t]} - \frac{\sigma1\,\text{dF}[t]}{\text{p}[t]}\right)\lambda_3[t] + \lambda_{11}[t]$$

$$\text{rZB}'[t] == -\lambda_4[t]$$
$$\text{s}'[t] == 2\,\mu\text{Fs}\,(\text{sdach} - \text{s}[t]) - \lambda_5[t]$$
$$\text{w}'[t] == -\mu\text{Fw}\,\text{l}[t] + (\text{l}[t] - \tau\text{H}\,\text{l}[t])\,\lambda_1[t] - \text{l}[t]\,\lambda_2[t] + \tau\text{H}\,\text{l}[t]\,\lambda_{10}[t]$$

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$$0 == -\text{nH}[t] - \text{cH}[t]\times\text{p}[t] + \text{l}[t]\times\text{w}[t] - \tau\text{H}\,\text{l}[t]\times\text{w}[t] +$$
$$\text{aH}[t]\left(-\text{pdach}\,\sigma1 + \frac{\sigma1\,\text{ps}[t]}{\text{p}[t]} - \frac{(-1+\alpha)\,\text{k}'[t]}{\text{k}[t]} + \frac{\alpha\,(1+\sigma2)\,\text{l}'[t]}{\text{l}[t]}\right) - \text{mH}'[t]$$

$$0 == \text{nF}[t] + \text{cG}[t]\times\text{p}[t] + \text{cH}[t]\times\text{p}[t] - \text{l}[t]\times\text{w}[t] +$$
$$\text{dF}[t]\left(-\text{pdach}\,\sigma1 + \frac{\sigma1\,\text{ps}[t]}{\text{p}[t]} - \frac{(-1+\alpha)\,\text{k}'[t]}{\text{k}[t]} + \frac{\alpha\,(1+\sigma2)\,\text{l}'[t]}{\text{l}[t]}\right) - \text{mF}'[t]$$

$$0 == -\text{nF}[t] + \text{nH}[t] + \text{nZB}[t] - \text{aH}[t]\left(-\text{pdach}\,\sigma1 + \frac{\sigma1\,\text{ps}[t]}{\text{p}[t]} - \frac{(-1+\alpha)\,\text{k}'[t]}{\text{k}[t]} + \frac{\alpha\,(1+\sigma2)\,\text{l}'[t]}{\text{l}[t]}\right) -$$
$$\text{dF}[t]\left(-\text{pdach}\,\sigma1 + \frac{\sigma1\,\text{ps}[t]}{\text{p}[t]} - \frac{(-1+\alpha)\,\text{k}'[t]}{\text{k}[t]} + \frac{\alpha\,(1+\sigma2)\,\text{l}'[t]}{\text{l}[t]}\right) - \text{mB}'[t]$$

$$0 == -\text{nZB}[t] - \text{rZB}'[t]$$
$$0 == -\text{cG}[t] - \text{cH}[t] - \text{dpdach}\,\text{k}[t] + \beta\,\text{k}[t]^{1-\alpha}\,\text{l}[t]^{\alpha} - \text{k}'[t] - \text{s}'[t]$$
$$0 == \text{nH}[t] - \text{aH}'[t]$$
$$0 == -\text{nF}[t] - \text{dF}'[t]$$
$$0 == \text{nF}[t] - \text{nH}[t] - \text{nZB}[t] - \text{aB}'[t] - \text{dB}'[t]$$
$$0 == \text{nZB}[t] - \text{aZB}'[t]$$
$$0 == -\text{cG}[t]\times\text{p}[t] + \tau\text{H}\,\text{l}[t]\times\text{w}[t] - \text{mG}'[t]$$
$$0 == \text{ps}[t] - \text{p}'[t]$$

$$\text{aB}[0] == \text{aB0}$$
$$\text{aH}[0] == \text{aH0}$$
$$\text{aZB}[0] == \text{aZB0}$$
$$\text{cG}[0] == \frac{\text{l0}\,\text{w0}\,\tau\text{H}}{\text{p0}}$$
$$\text{cH}[0] == \frac{-\text{dpdach}\,\text{k0}\,\text{p0} + \text{k0}^{1-\alpha}\,\text{l0}^{\alpha}\,\text{p0}\,\beta - \text{l0}\,\text{w0}\,\tau\text{H}}{\text{p0}}$$
$$\text{dB}[0] == \text{dB0}$$
$$\text{dF}[0] == \text{dpdach}\,\text{k0}\,\text{p0} + \text{l0}\,\text{w0} - \text{k0}^{1-\alpha}\,\text{l0}^{\alpha}\,\text{p0}\,\beta$$
$$\text{k}[0] == \text{k0}$$
$$\text{l}[0] == \text{l0}$$
$$\text{mB}[0] == \text{mB0}$$
$$\text{mF}[0] == \text{mF0}$$
$$\text{mG}[0] == \text{mG0}$$
$$\text{mH}[0] == \text{mH0}$$
$$\text{nF}[0] == \text{nF0}$$
$$\text{nH}[0] == \text{nH0}$$
$$\text{nZB}[0] == \text{nZB0}$$
$$\text{p}[0] == \text{p0}$$
$$\text{ps}[0] == \text{ps0}$$
$$\text{rZB}[0] == -\text{mB0} - \text{mF0} - \text{mG0} - \text{mH0}$$
$$\text{s}[0] == \text{s0}$$
$$\text{w}[0] == \text{w0}$$

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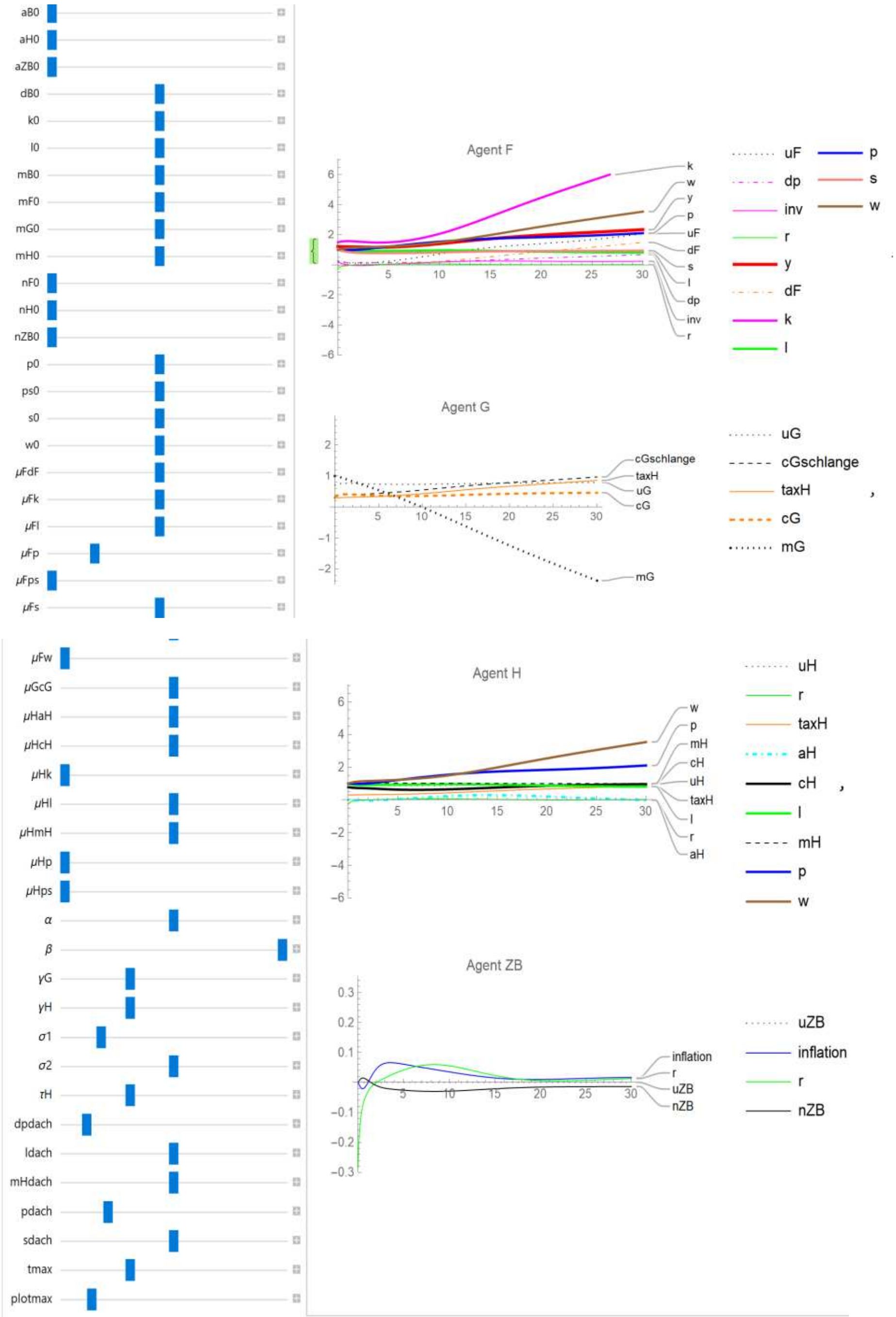
16.2. Calculation results of model C2

https://www.dropbox.com/s/tuytrcb4rh1r0vh/Modell%20C2%20Version%201%20%28Taylor-Regel%29.ndsolve.nb?dl=0



17. Model D2, comprehensive model

17.1. Setup

We extend the C2 model in the following aspects:

- The interest rate is formulated in more detail: in central bank policy rate, premium for lending rates, premium for savings rates r_{leit}, r_D, r_A

- The central bank distributes the profit to the government, the bank distributes the profit to the household, the firm distributes a part of the profit to the household

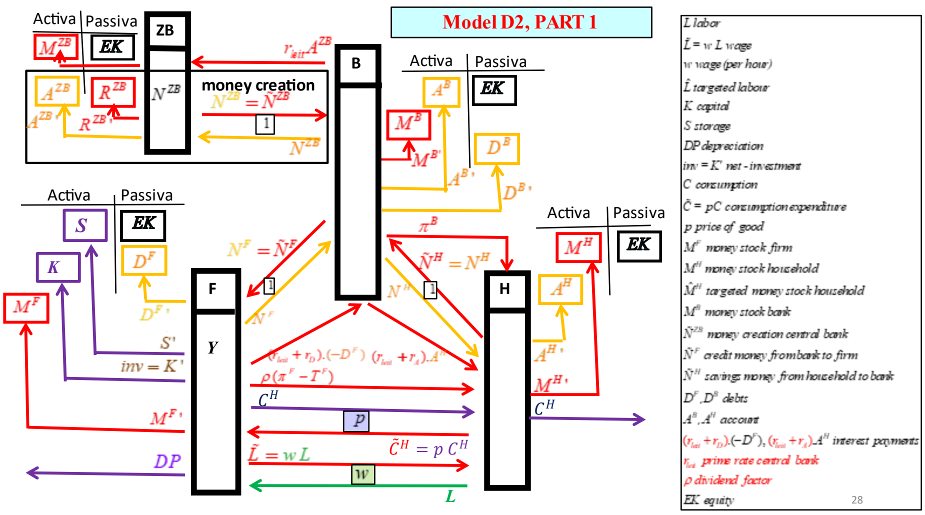
- Taxes are composed of income and property taxes for household and firm.

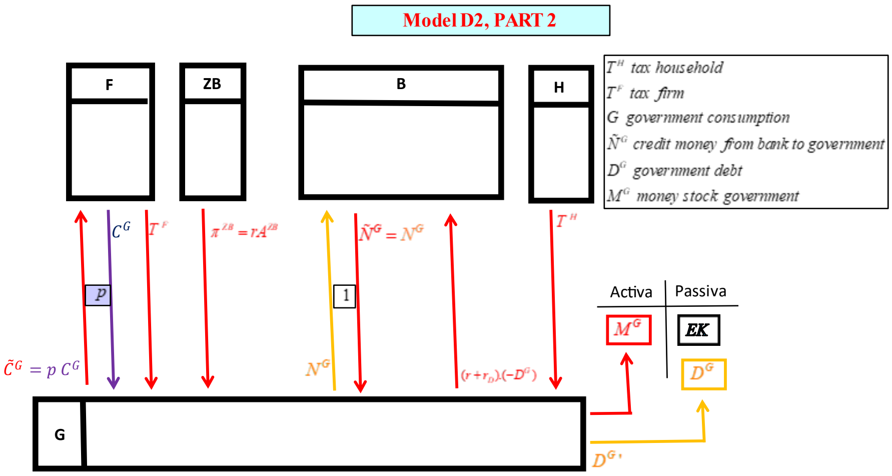
- The government aims to achieve a debt level of 60% of GDP in line with the Maastricht criterion.

- The central bank acts according to the modified Taylor rule

- The target level of the firm's investment and the target level of the household's money stock are interest rate dependent.

- The targeted level of the government's money stock is not interest rate dependent





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Model D2 : basic equations

algebraically defined variables

$DP = \bar{dp}\,K$	"depreciation"
$inflation = \dfrac{p'}{p}$	"inflation"
$inv = K'$	"net investment"
$invmax = 0.1\,K$	"maximal investment (inv if $r_{(...)} + r_D = 0$)"
$r = -r_D + \dfrac{p'}{p} + \sigma_1\left(\dfrac{p'}{p} - \hat{p}\right) + (1-\alpha)\dfrac{K'}{K} + (1+\sigma_2)\alpha\dfrac{L'}{L}$	"modified Taylor rule"
$T^H = \tau^H(wL + (r+r_A)A^H) + \upsilon^H(A^H + M^H)$	"income tax + asset tax H"
$T^F = \tau^F\left(p\beta L^\alpha K^{1-\alpha} - wL - (r+r_D)(-D^F) - DP\right) + \upsilon^F\left(M^F + S + K + D^F\right)$	"income tax + asset tax F"
$\pi^B = profit^B = +(r+r_D).(-D^F) + (r+r_D).(-D^G) - rA^{ZB} - (r+r_A).A^H$	"profit before taxes B"
$\pi^F = profit^F = p\beta L^\alpha K^{1-\alpha} - wL - (r+r_D)(-D^F) - DP$	"profit before taxes F"
$\pi^{ZB} = profit^{ZB} = r\,A^{ZB}$	"profit ZB"
$Y = \beta L^\alpha K^{1-\alpha}$	"production function"

Nutzenfunktionen

$U^B = profit^B$	"utility function bank B"
$U^F = profit^F - (\hat{S} - S)^2 - \left(invmax\left(1-\eta(r+r_D)\right) - inv\right)^2$	"utility function firm F"
$U^G = (C^G)^{\gamma_G} - (\hat{D}^G Y - D^G)^2$	"utility function government G"
$U^H = (C^H)^{\gamma_H} - \left(\hat{L} - L\right)^2 - (mhmax\left(1-\theta(r+r_A)\right) - M^H)^2 + (r+r_A)A^H$	"utility function H"
$U^{ZB} = 0$	"utility function central bank ZB"

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constraints

$$Z_1 = 0 = -r\,A^{ZB} + N^{ZB} - N^F + (r + r_D)(-D^F) + (r + r_D).(-D^G)$$
$$-(r + r_A)A^H + N^H - \pi^B - N^G - M^{B\,\prime} \qquad \textit{money flow B}$$
$$Z_2 = 0 = -wL + pC^H - \rho(\pi^F - T^F) - (r + r_D)(-D^F)$$
$$+N^F + pC^G - T^F - M^{F\,\prime} \qquad \textit{money flow F}$$
$$Z_3 = 0 = -pC^G + T^F + \pi^{ZB} + N^G - (r + r_D).(-D^G) + T^H - M^{G\,\prime} \qquad \textit{money flow G}$$
$$Z_4 = 0 = wL - pC^H + \rho(\pi^F - T^F) + (r + r_A)A^H - N^H + \pi^B - T^H - M^{H\,\prime} \qquad \textit{money flow H}$$
$$Z_5 = 0 = r\,A^{ZB} - \pi^{ZB} - M^{ZB\,\prime} \qquad \textit{money flow ZB}$$
$$Z_6 = 0 = -N^{ZB} - R^{ZB\,\prime} \qquad \textit{reserve flow ZB}$$
$$Z_7 = 0 = Y - DP - C^H - C^G - K' - S' \qquad \textit{flow of good F}$$
$$Z_8 = 0 = -N^{ZB} + N^F - N^H + N^G - A^{B\,\prime} - D^{B\,\prime} \qquad \textit{accounts / debts flow B}$$
$$Z_9 = 0 = -N^F - D^{F\,\prime} \qquad \textit{accounts / debts flow F}$$
$$Z_{10} = 0 = -N^G - D^{G\,\prime} \qquad \textit{accounts / debts flow G}$$
$$Z_{11} = 0 = N^H - A^{H\,\prime} \qquad \textit{accounts / debts flow H}$$
$$Z_{12} = 0 = N^{ZB} - A^{ZB\,\prime} \qquad \textit{accounts / debts flow ZB}$$ [31]

$$\mathrm{uB}[t] == \mathrm{profitB}[t]$$
$$\mathrm{uF}[t] == \mathrm{profitF}[t] - (-\mathrm{inv}[t] + \mathrm{invmax}[t]\,(1 - \eta\,(\mathrm{rd}[t] + \mathrm{rleit}[t])))^2 - (\mathrm{sdach} - \mathrm{s}[t])^2$$
$$\mathrm{uG}[t] == \mathrm{cG}[t]^{\gamma G} - (\mathrm{mGdach} - \mathrm{mG}[t])^2 - (-\mathrm{dG}[t] + \mathrm{dGdach}\,\mathrm{y}[t])^2$$
$$\mathrm{uH}[t] == \mathrm{cH}[t]^{\gamma H} - (\mathrm{ldach} - \mathrm{l}[t])^2 + \mathrm{aH}[t]\,(\mathrm{ra}[t] + \mathrm{rleit}[t]) - (-\mathrm{mH}[t] + \mathrm{mHmax}\,(1 - \theta\,(\mathrm{ra}[t] + \mathrm{rleit}[t])))^2$$
$$\mathrm{uZB}[t] == 0$$

$$\mathrm{dp}[t] == \mathrm{dpdach}\,\mathrm{k}[t]$$
$$\mathrm{inflation}[t] == \frac{\mathrm{p}'[t]}{\mathrm{p}[t]}$$
$$\mathrm{inv}[t] == \mathrm{k}'[t]$$
$$\mathrm{invmax}[t] == 0.1\grave{}\ \mathrm{k}[t]$$
$$\mathrm{profitB}[t] == -\mathrm{aZB}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right) - \mathrm{aH}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} + \mathrm{ra}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right) - \mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} + \mathrm{rd}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right) - \mathrm{dG}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} + \mathrm{rd}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right)$$
$$\mathrm{profitF}[t] == -\mathrm{dpdach}\,\mathrm{k}[t] + \beta\,\mathrm{k}[t]^{1-\alpha}\,\mathrm{l}[t]^{\alpha}\,\mathrm{p}[t] - \mathrm{l}[t] \times \mathrm{w}[t] + \mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} + \mathrm{rd}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right)$$
$$\mathrm{profitZB}[t] == \mathrm{aZB}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right)$$
$$\mathrm{rleit}[t] == -\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} - \mathrm{rd}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}$$
$$\mathrm{tF}[t] == \nu\mathrm{F}\,(\mathrm{dF}[t] + \mathrm{k}[t] + \mathrm{mF}[t] + \mathrm{s}[t]) + \tau\mathrm{F}\left(-\mathrm{dpdach}\,\mathrm{k}[t] + \beta\,\mathrm{k}[t]^{1-\alpha}\,\mathrm{l}[t]^{\alpha}\,\mathrm{p}[t] - \mathrm{l}[t] \times \mathrm{w}[t] + \mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} + \mathrm{rd}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right)\right)$$
$$\mathrm{tH}[t] == \nu\mathrm{H}\,(\mathrm{aH}[t] + \mathrm{mH}[t]) + \tau\mathrm{H}\left(\mathrm{l}[t] \times \mathrm{w}[t] + \mathrm{aH}[t]\left(-\mathrm{pdach}\,\sigma 1 + \frac{\sigma 1\,\mathrm{ps}[t]}{\mathrm{p}[t]} + \mathrm{ra}[t] - \frac{(-1+\alpha)\,\mathrm{k}'[t]}{\mathrm{k}[t]} + \frac{\alpha\,(1+\sigma 2)\,\mathrm{l}'[t]}{\mathrm{l}[t]}\right)\right)$$ [31]
$$\mathrm{y}[t] == \beta\,\mathrm{k}[t]^{1-\alpha}\,\mathrm{l}[t]^{\alpha}$$

$$\mathrm{uB}[t] == \mathrm{profitB}[t]$$
$$\mathrm{uF}[t] == \mathrm{profitF}[t] - (-\mathrm{inv}[t] + \mathrm{invmax}[t]\,(1 - \eta\,(\mathrm{rd}[t] + \mathrm{rleit}[t])))^2 - (\mathrm{sdach} - s[t])^2$$
$$\mathrm{uG}[t] == \mathrm{cG}[t]^{\gamma G} - (\mathrm{mGdach} - \mathrm{mG}[t])^2 - (-\mathrm{dG}[t] + \mathrm{dGdach}\,y[t])^2$$
$$\mathrm{uH}[t] == \mathrm{cH}[t]^{\gamma H} - (\mathrm{ldach} - l[t])^2 + \mathrm{aH}[t]\,(\mathrm{ra}[t] + \mathrm{rleit}[t]) - (-\mathrm{mH}[t] + \mathrm{mHmax}\,(1 - \theta\,(\mathrm{ra}[t] + \mathrm{rleit}[t])))^2$$
$$\mathrm{uZB}[t] == 0$$

$$\mathrm{dp}[t] == \mathrm{dpdach}\,k[t]$$
$$\mathrm{inflation}[t] == \frac{p'[t]}{p[t]}$$
$$\mathrm{inv}[t] == k'[t]$$
$$\mathrm{invmax}[t] == 0.1\text{`}\,k[t]$$
$$\mathrm{profitB}[t] == -\mathrm{aZB}[t]\left(-\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right) - \mathrm{aH}[t]\left(-\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} + \mathrm{ra}[t] - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right) - \mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} + \mathrm{rd}[t] - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right) - \mathrm{dG}[t]\left(-\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} + \mathrm{rd}[t] - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)$$
$$\mathrm{profitF}[t] == -\mathrm{dpdach}\,k[t] + \beta\,k[t]^{1-\alpha}\,l[t]^{\alpha}\,p[t] - l[t]\times w[t] + \mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} + \mathrm{rd}[t] - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)$$
$$\mathrm{profitZB}[t] == \mathrm{aZB}[t]\left(-\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)$$
$$\mathrm{rleit}[t] == -\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} - \mathrm{rd}[t] - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}$$
$$\mathrm{tF}[t] == \nu F\,(\mathrm{dF}[t] + k[t] + \mathrm{mF}[t] + s[t]) + \tau F\left(-\mathrm{dpdach}\,k[t] + \beta\,k[t]^{1-\alpha}\,l[t]^{\alpha}\,p[t] - l[t]\times w[t] + \mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} + \mathrm{rd}[t] - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)\right)$$
$$\mathrm{tH}[t] == \nu H\,(\mathrm{aH}[t] + \mathrm{mH}[t]) + \tau H\left(l[t]\times w[t] + \mathrm{aH}[t]\left(-\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} + \mathrm{ra}[t] - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)\right)$$
$$y[t] == \beta\,k[t]^{1-\alpha}\,l[t]^{\alpha}$$

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$$\mathrm{uB}[t] == \mathrm{profitB}[t]$$
$$\mathrm{uF}[t] == \mathrm{profitF}[t] - (-\mathrm{inv}[t] + \mathrm{invmax}[t]\,(1 - \eta\,(\mathrm{rd}[t] + \mathrm{rleit}[t])))^2 - (\mathrm{sdach} - s[t])^2$$
$$\mathrm{uG}[t] == \mathrm{cG}[t]^{\gamma G} - (\mathrm{mGdach} - \mathrm{mG}[t])^2 - (-\mathrm{dG}[t] + \mathrm{dGdach}\,y[t])^2$$
$$\mathrm{uH}[t] == \mathrm{cH}[t]^{\gamma H} - (\mathrm{ldach} - l[t])^2 + \mathrm{aH}[t]\,(\mathrm{ra}[t] + \mathrm{rleit}[t]) - (-\mathrm{mH}[t] + \mathrm{mHmax}\,(1 - \theta\,(\mathrm{ra}[t] + \mathrm{rleit}[t])))^2$$
$$\mathrm{uZB}[t] == 0$$

$$\mathrm{dp}[t] == \mathrm{dpdach}\,k[t]$$
$$\mathrm{inflation}[t] == \frac{p'[t]}{p[t]}$$
$$\mathrm{inv}[t] == k'[t]$$
$$\mathrm{invmax}[t] == 0.1\text{`}\,k[t]$$
$$\mathrm{profitB}[t] == -\mathrm{aZB}[t]\left(-\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right) - \mathrm{aH}[t]\left(-\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} + \mathrm{ra}[t] - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right) - \mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} + \mathrm{rd}[t] - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right) - \mathrm{dG}[t]\left(-\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} + \mathrm{rd}[t] - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)$$
$$\mathrm{profitF}[t] == -\mathrm{dpdach}\,k[t] + \beta\,k[t]^{1-\alpha}\,l[t]^{\alpha}\,p[t] - l[t]\times w[t] + \mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} + \mathrm{rd}[t] - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)$$
$$\mathrm{profitZB}[t] == \mathrm{aZB}[t]\left(-\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)$$
$$\mathrm{rleit}[t] == -\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} - \mathrm{rd}[t] - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}$$
$$\mathrm{tF}[t] == \nu F\,(\mathrm{dF}[t] + k[t] + \mathrm{mF}[t] + s[t]) + \tau F\left(-\mathrm{dpdach}\,k[t] + \beta\,k[t]^{1-\alpha}\,l[t]^{\alpha}\,p[t] - l[t]\times w[t] + \mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} + \mathrm{rd}[t] - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)\right)$$
$$\mathrm{tH}[t] == \nu H\,(\mathrm{aH}[t] + \mathrm{mH}[t]) + \tau H\left(l[t]\times w[t] + \mathrm{aH}[t]\left(-\mathrm{pdach}\,\sigma1 + \frac{\sigma1\,\mathrm{ps}[t]}{p[t]} + \mathrm{ra}[t] - \frac{(-1+\alpha)\,k'[t]}{k[t]} + \frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)\right)$$
$$y[t] == \beta\,k[t]^{1-\alpha}\,l[t]^{\alpha}$$

$$
\begin{aligned}
&\mathrm{ps}'[t] == \mu\mathrm{Bps}\left(-\frac{\sigma1\,\mathrm{aH}[t]}{p[t]}-\frac{\sigma1\,\mathrm{aZB}[t]}{p[t]}-\frac{\sigma1\,\mathrm{dF}[t]}{p[t]}-\frac{\sigma1\,\mathrm{dG}[t]}{p[t]}\right)+\\
&\quad\left(\rho\left(\frac{\sigma1\,\mathrm{dF}[t]}{p[t]}-\frac{\sigma1\,\tau\mathrm{F}\,\mathrm{dF}[t]}{p[t]}\right)-\frac{\sigma1\,\tau\mathrm{H}\,\mathrm{aH}[t]}{p[t]}-\frac{\sigma1\,\mathrm{aZB}[t]}{p[t]}-\frac{\sigma1\,\mathrm{dF}[t]}{p[t]}-\frac{\sigma1\,\mathrm{dG}[t]}{p[t]}\right)\lambda_2[t]+\\
&\quad\left(\frac{\sigma1\,\tau\mathrm{H}\,\mathrm{aH}[t]}{p[t]}+\frac{\sigma1\,\mathrm{aZB}[t]}{p[t]}+\frac{\sigma1\,\tau\mathrm{F}\,\mathrm{dF}[t]}{p[t]}+\frac{\sigma1\,\mathrm{dG}[t]}{p[t]}\right)\lambda_7[t]+\\
&\quad\left(-\rho\left(\frac{\sigma1\,\mathrm{dF}[t]}{p[t]}-\frac{\sigma1\,\tau\mathrm{F}\,\mathrm{dF}[t]}{p[t]}\right)+\frac{\sigma1\,\mathrm{dF}[t]}{p[t]}-\frac{\sigma1\,\tau\mathrm{F}\,\mathrm{dF}[t]}{p[t]}\right)\lambda_9[t]+\lambda_{13}[t]+\\
&\quad\mu\mathrm{Fps}\left(\frac{\sigma1\,\mathrm{dF}[t]}{p[t]}+\frac{0.2\grave{}\;\eta\,\sigma1\,k[t]\left(-k'[t]+0.1\grave{}\;k[t]\left(1-\eta\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)\right)\right)}{p[t]}\right)+\\
&\quad\mu\mathrm{Hps}\left(\frac{\sigma1\,\mathrm{aH}[t]}{p[t]}+\frac{2\,\mathrm{mHmax}\,\theta\,\sigma1\left(-\mathrm{mH}[t]+\mathrm{mHmax}\left(1-\theta\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}+\mathrm{ra}[t]-\mathrm{rd}[t]-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)\right)\right)}{p[t]}\right)
\end{aligned}
$$

$$\mathrm{rZB}'[t] == -\lambda_6[t]$$

$$
\begin{aligned}
&\mathrm{ra}'[t] == -\mu\mathrm{Bra}\,\mathrm{aH}[t]-\tau\mathrm{H}\,\mathrm{aH}[t]\,\lambda_2[t]+\tau\mathrm{H}\,\mathrm{aH}[t]\,\lambda_7[t]+\\
&\quad\mu\mathrm{Hra}\\
&\quad\left(\mathrm{aH}[t]+\right.\\
&\qquad 2\,\mathrm{mHmax}\,\theta\\
&\qquad\left.\left(-\mathrm{mH}[t]+\mathrm{mHmax}\left(1-\theta\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}+\mathrm{ra}[t]-\mathrm{rd}[t]-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)\right)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{rd}'[t] == \mu\mathrm{Frd}\,\mathrm{dF}[t]+\mu\mathrm{Brd}\,(-\mathrm{dF}[t]-\mathrm{dG}[t])-\mathrm{aZB}[t]\,\lambda_1[t]+\\
&\quad(-\mathrm{aH}[t]-\mathrm{dF}[t]+\rho\,(\mathrm{dF}[t]-\tau\mathrm{F}\,\mathrm{dF}[t])-\mathrm{dG}[t])\,\lambda_2[t]+\tau\mathrm{F}\,\mathrm{dF}[t]\,\lambda_7[t]+\\
&\quad(\mathrm{aH}[t]+\mathrm{aZB}[t]+\mathrm{dF}[t]+\mathrm{dG}[t])\,\lambda_8[t]+(-\tau\mathrm{F}\,\mathrm{dF}[t]-\rho\,(\mathrm{dF}[t]-\tau\mathrm{F}\,\mathrm{dF}[t]))\,\lambda_9[t]+\\
&\quad\mu\mathrm{Hrd}\\
&\quad\left(-\mathrm{aH}[t]-\right.\\
&\qquad 2\,\mathrm{mHmax}\,\theta\\
&\qquad\left.\left(-\mathrm{mH}[t]+\mathrm{mHmax}\left(1-\theta\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}+\mathrm{ra}[t]-\mathrm{rd}[t]-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)\right)\right)\right)
\end{aligned}
$$

$$\mathrm{s}'[t] == 2\,\mu\mathrm{Fs}\,(\mathrm{sdach}-\mathrm{s}[t])-\nu\mathrm{F}\,\rho\,\lambda_2[t]+\nu\mathrm{F}\,\lambda_7[t]+(-\nu\mathrm{F}+\nu\mathrm{F}\,\rho)\,\lambda_9[t]-\lambda_{12}[t]$$

$$
\begin{aligned}
&\mathrm{w}'[t] == -\mu\mathrm{Fw}\,l[t]+(l[t]-\tau\mathrm{H}\,l[t]+\rho\,(-l[t]+\tau\mathrm{F}\,l[t]))\,\lambda_2[t]+(-\tau\mathrm{F}\,l[t]+\tau\mathrm{H}\,l[t])\,\lambda_7[t]+\\
&\quad(-l[t]+\tau\mathrm{F}\,l[t]-\rho\,(-l[t]+\tau\mathrm{F}\,l[t]))\,\lambda_9[t]
\end{aligned}
$$

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$$
\begin{aligned}
&0 == -\mathrm{aZB}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)+\\
&\quad\mathrm{aZB}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}-\mathrm{rd}[t]-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)-\mathrm{mZB}'[t]
\end{aligned}
$$

$$
\begin{aligned}
&0 == -\nu\mathrm{H}\,(\mathrm{aH}[t]+\mathrm{mH}[t])-\mathrm{nH}[t]-\mathrm{cH}[t]\times p[t]+l[t]\times \mathrm{w}[t]-\\
&\quad\mathrm{aZB}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)-\\
&\quad\mathrm{aH}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}+\mathrm{ra}[t]-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)+\\
&\quad\mathrm{aH}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}+\mathrm{ra}[t]-\mathrm{rd}[t]-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)-\\
&\quad\mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}+\mathrm{rd}[t]-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)-\\
&\quad\mathrm{dG}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}+\mathrm{rd}[t]-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)-\\
&\quad\tau\mathrm{H}\left(l[t]\times \mathrm{w}[t]+\mathrm{aH}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}+\mathrm{ra}[t]-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)\right)+\\
&\quad\rho\left(-\mathrm{dpdach}\,k[t]+\beta\,k[t]^{1-\alpha}\,l[t]^{\alpha}\,p[t]-\nu\mathrm{F}\,(\mathrm{dF}[t]+k[t]+\mathrm{mF}[t]+\mathrm{s}[t])-l[t]\times \mathrm{w}[t]+\right.\\
&\qquad\mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}+\mathrm{rd}[t]-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)-\\
&\qquad\tau\mathrm{F}\left(-\mathrm{dpdach}\,k[t]+\beta\,k[t]^{1-\alpha}\,l[t]^{\alpha}\,p[t]-l[t]\times \mathrm{w}[t]+\right.\\
&\qquad\quad\left.\left.\mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}+\mathrm{rd}[t]-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)\right)\right)-\mathrm{mH}'[t]
\end{aligned}
$$

$$0 == -\mathrm{nF}[t]-\mathrm{dF}'[t]$$

$$0 == -\mathrm{nG}[t]-\mathrm{dG}'[t]$$

$$0 == \mathrm{nF}[t]+\mathrm{nG}[t]-\mathrm{nH}[t]-\mathrm{nZB}[t]-\mathrm{aB}'[t]-\mathrm{dB}'[t]$$

$$0 == -\mathrm{nZB}[t]-\mathrm{rZB}'[t]$$

$$
\begin{aligned}
&0 == \nu\mathrm{H}\,(\mathrm{aH}[t]+\mathrm{mH}[t])+\mathrm{nG}[t]-\mathrm{cG}[t]\times p[t]+\nu\mathrm{F}\,(\mathrm{dF}[t]+k[t]+\mathrm{mF}[t]+\mathrm{s}[t])+\\
&\quad\mathrm{aZB}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)+\\
&\quad\mathrm{dG}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)+\\
&\quad\tau\mathrm{H}\left(l[t]\times \mathrm{w}[t]+\mathrm{aH}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}+\mathrm{ra}[t]-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)\right)+\\
&\quad\tau\mathrm{F}\left(-\mathrm{dpdach}\,k[t]+\beta\,k[t]^{1-\alpha}\,l[t]^{\alpha}\,p[t]-l[t]\times \mathrm{w}[t]+\right.\\
&\qquad\left.\mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}+\mathrm{rd}[t]-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)\right)-\mathrm{mG}'[t]
\end{aligned}
$$

$$
\begin{aligned}
&0 == -\mathrm{nF}[t]-\mathrm{nG}[t]+\mathrm{nH}[t]+\mathrm{nZB}[t]+\mathrm{aZB}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)-\\
&\quad\mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)-\\
&\quad\mathrm{dG}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)+\\
&\quad\mathrm{aH}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}+\mathrm{ra}[t]-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)-\\
&\quad\mathrm{aZB}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}-\mathrm{rd}[t]-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)-\\
&\quad\mathrm{aH}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}+\mathrm{ra}[t]-\mathrm{rd}[t]-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)+\\
&\quad\mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}+\mathrm{rd}[t]-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)+\\
&\quad\mathrm{dG}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}+\mathrm{rd}[t]-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)-\mathrm{mB}'[t]
\end{aligned}
$$

$$
\begin{aligned}
&0 == \mathrm{nF}[t]+\mathrm{cG}[t]\times p[t]+\mathrm{cH}[t]\times p[t]-\nu\mathrm{F}\,(\mathrm{dF}[t]+k[t]+\mathrm{mF}[t]+\mathrm{s}[t])-l[t]\times \mathrm{w}[t]+\\
&\quad\mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)-\\
&\quad\tau\mathrm{F}\left(-\mathrm{dpdach}\,k[t]+\beta\,k[t]^{1-\alpha}\,l[t]^{\alpha}\,p[t]-l[t]\times \mathrm{w}[t]+\right.\\
&\qquad\left.\mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}+\mathrm{rd}[t]-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)\right)-\\
&\quad\rho\left(-\mathrm{dpdach}\,k[t]+\beta\,k[t]^{1-\alpha}\,l[t]^{\alpha}\,p[t]-\nu\mathrm{F}\,(\mathrm{dF}[t]+k[t]+\mathrm{mF}[t]+\mathrm{s}[t])-l[t]\times \mathrm{w}[t]+\right.\\
&\qquad\mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}+\mathrm{rd}[t]-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)-\\
&\qquad\tau\mathrm{F}\left(-\mathrm{dpdach}\,k[t]+\beta\,k[t]^{1-\alpha}\,l[t]^{\alpha}\,p[t]-l[t]\times \mathrm{w}[t]+\right.\\
&\qquad\quad\left.\left.\mathrm{dF}[t]\left(-\mathrm{pdach}\,\sigma1+\frac{\sigma1\,\mathrm{ps}[t]}{p[t]}+\mathrm{rd}[t]-\frac{(-1+\alpha)\,k'[t]}{k[t]}+\frac{\alpha\,(1+\sigma2)\,l'[t]}{l[t]}\right)\right)\right)-\mathrm{mF}'[t]
\end{aligned}
$$

$$0 == \mathrm{nH}[t]-\mathrm{aH}'[t]$$

$$0 == \mathrm{nZB}[t]-\mathrm{aZB}'[t]$$

$$0 == -\mathrm{cG}[t]-\mathrm{cH}[t]-\mathrm{dpdach}\,k[t]+\beta\,k[t]^{1-\alpha}\,l[t]^{\alpha}-k'[t]-\mathrm{s}'[t]$$

$$0 == \mathrm{ps}[t]-p'[t]$$

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```
aB[0] == aB0
aH[0] == aH0
aZB[0] == aZB0
cG[0] == cG0
cH[0] == cH0
dB[0] == dB0
dF[0] == dF0
dG[0] == dG0
k[0] == k0
l[0] == l0
mB[0] == mB0
mF[0] == mF0
mG[0] == mG0
mG[0] == mG0
mH[0] == mH0
mZB[0] == mZB0
nF[0] == nF0
nG[0] == nG0
nH[0] == nH0
nZB[0] == nZB0
p[0] == p0
ps[0] == ps0
rZB[0] == -mB0 - mF0 - mG0 - mH0
ra[0] == ra0
rd[0] == rd0
s[0] == s0
w[0] == w0
```

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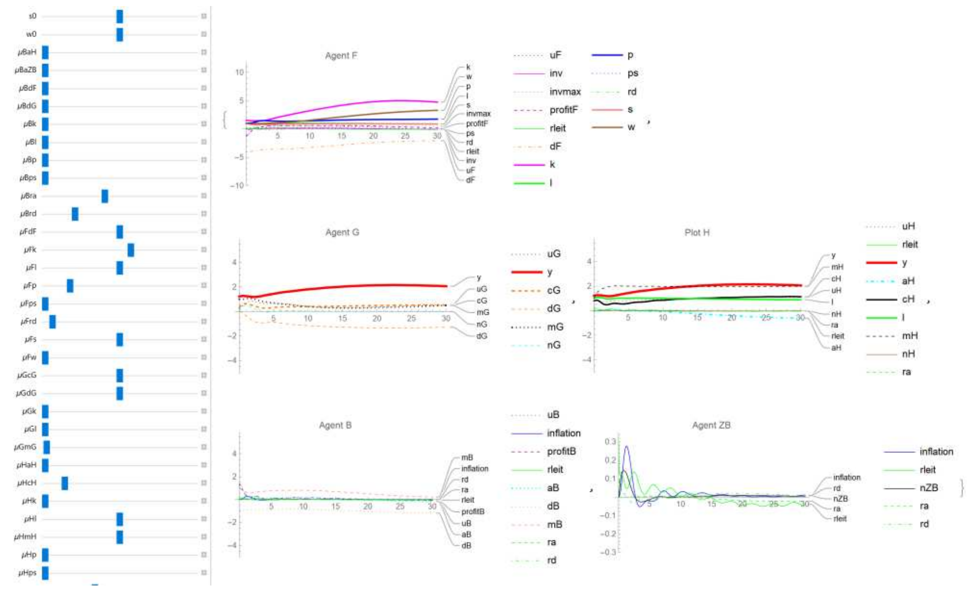
17.2. Calculation results of model D2

https://www.dropbox.com/s/xf95y9seacer2lh/Modell%20D2%20Version%2011.ndsolve.nb?dl=0



18. Different economic theories differ only by different assumptions about the power of agents

18.1. Basic idea

The basic idea of GCD models can also be formulated in the following way: With GCD models, the supposedly insurmountable opposition of different economic models can be eliminated in the sense that they can be understood as versions of a single model that differ from each other only by different one-sided power relations or adjustment speeds. On the other hand, GCD models offer the possibility of better representing reality, because mixed power relations usually correspond better to reality than one-sided power relations.

This is illustrated by the following 2 examples.

18.2. Savings → Investment (Neoclassics) or Investment → Savings (Keynes)

18.2.1. Problem description

The two economic schools of neoclassical economics and Keynesianism differ diametrically in their assumptions about the variables "saving" and "investing".

In the **Keynesian sense**, investing is an exogenous variable, saving is an endogenous variable and the cause-effect relationship applies

$$Investing \Rightarrow Saving$$

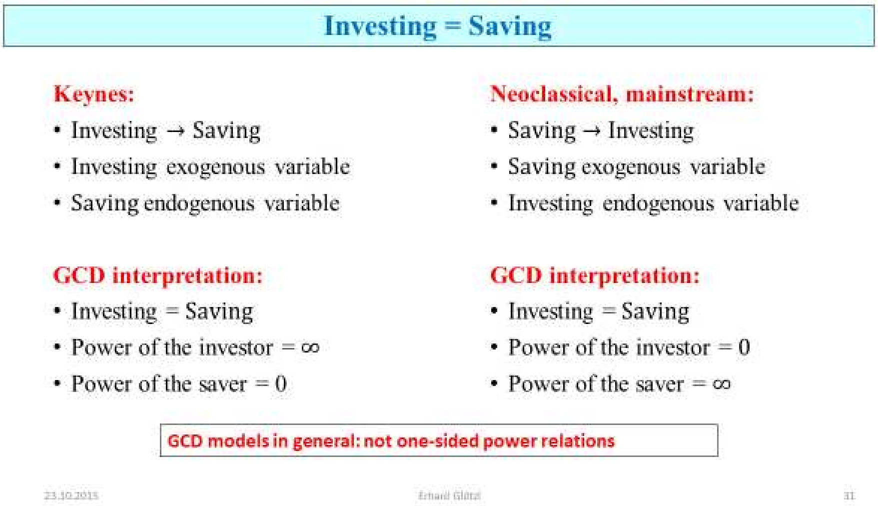
In the **neoclassical sense**, the opposite is true: investing is an endogenous variable, saving is an exogenous variable and the cause-effect relationship applies.

$$Saving \Rightarrow Investing$$

From the perspective of the GCD models, these seemingly insurmountable opposites can be overcome and resolved in the following sense. The statement that saving and investing must always be the same corresponds to an accounting identity that results from the definition of saving and investing. The two economic schools differ only in the different assumptions about the power of savers and investors.

The Keynesian cause-effect relationship results from the assumption that the power of investors is ∞ and the power of savers is 0. The neoclassical cause-effect relationship results from the opposite assumption that the power of investors is 0 and the power of savers is ∞.

In reality, however, these one-sided power relations do not usually occur, but rather mixed power relations. Therefore, reality can be better described with GCD models than with Keynesian or neoclassical models.



The model equations for the **Keynesian model** are

$$I = \hat{I}$$
$$S = I$$

The model equations for the **neoclassical model** are

$$S = \hat{S}$$
$$I = S$$

Furthermore, with the assumed master utility function

$$MU = \frac{1}{2}(\hat{I} - I)^2 + \frac{1}{2}(\hat{S} - S)^2$$

the **general equilibrium model** can be formulated as maximising MU under the constraint $Z(I,S) = I - S = 0$ in the following way:

$$0 = \frac{\partial MU}{\partial I} + \lambda \frac{\partial Z}{\partial I} = (\hat{I} - I) + \lambda$$
$$0 = \frac{\partial MU}{\partial S} + \lambda \frac{\partial Z}{\partial S} = (\hat{S} - S) - \lambda$$
$$0 = I - S$$

All these 3 models can be understood as special cases of the following GCD model:

utility functions

$$U^F = \frac{1}{2}(\hat{I} - I)^2 \qquad F\ firm,\ I\ investment,\ \hat{I}\ targeted\ investment$$

$$U^H = \frac{1}{2}(\hat{S} - S)^2 \qquad H\ household,\ S\ savings,\ \hat{S}\ targeted\ savings$$

constraints

$$0 = I - S$$

basic GCD - equations

$$(a) \qquad I' = \mu_I^F(\hat{I} - I) + \lambda$$

$$(b) \qquad S' = \mu_S^H(\hat{S} - S) - \lambda$$

$$(c) \qquad 0 = I - S$$

From this GCD model we get the Keynesian model with the assumptions

$$\mu_I^F = \infty$$

$$\mu_S^H = 0 \qquad (oder\ 0 \le \mu_S^H < \infty)$$

because it follows from equation (a)

$$I' = \mu_I^F(\hat{I} - I) + \lambda \Rightarrow \frac{I'}{\mu_I^F} = (\hat{I} - I) + \frac{\lambda}{\mu_I^F}$$

$$\Rightarrow wegen\ \mu_I^F = \infty \qquad 0 = (\hat{I} - I) + 0$$

$$\Rightarrow I = \hat{I}$$

and from equation (c)

$$S = I$$

Equation (b) is not needed. It would also be possible $0 \le \mu_S^H < \infty$.

Similarly, the neoclassical model results with the assumptions

$$\mu_I^F = 0 \qquad (oder\ 0 \le \mu_I^F < \infty)$$

$$\mu_S^H = \infty$$

and the general equilibrium model with the assumptions

$$I' = 0 \quad Annahme\ des\ stationären\ Gleichgewichts$$

$$S' = 0 \quad Annahme\ des\ stationären\ Gleichgewichts$$

$$\mu_I^F = 1$$

$$\mu_S^H = 1$$

GCD ~ mixed power **Keynesian,Neoclassic ~ one sided power**
Constraint GE ~ stationary

$GCD - Model$ "$Lagrange - Closure$"	$Keynes$	$Neoclassic$	$Constraint\ GE$
$I' = \mu_I^B(IF - I) + \lambda$	$I = IF$	$\times\times\times\times$	$0 = \frac{\partial MU}{\partial I} + \lambda = (IF - I) + \lambda$
$S' = \mu_S^H(SF - S) - \lambda$	$\times\times\times\times$	$S = SF$	$0 = \frac{\partial MU}{\partial S} - \lambda = (SF - S) - \lambda$
$I - S = 0$	$S = I$	$I = S$	$I - S = 0$
μ_I^B $Power\ of\ Business$	$\mu_I^B = \infty$	$\mu_I^B = 0$	$\mu_I^B = 1$
μ_S^H $Power\ of\ Households$	$\mu_S^H = 0$	$\mu_S^H = \infty$	$\mu_S^H = 1$
			$MU = \frac{1}{2}((IF - I)^2 + (SF - S)^2)$

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18.2.2. Formally analogous problems

Completely analogous to the accounting identity $I = S$, in a closed economy the accounting identity applies that the sum of the accounts A (receivables) is always equal to the sum of the debts D (liabilities), i.e. $A = D$ or with the convention used in this paper for the negative sign of liabilities $A = -D$. The development of these quantities over time depends on the one hand on the interests of the sum of creditors and the sum of debtors, and on the other hand on their power to enforce these interests[2] (Glötzl 1999; 2015).

The two models (investing/saving and liabilities/receivables) are not only formally mathematically completely equivalent to each other, but they are also formally completely equivalent to the movement on an inclined straight line inclined at 45 degrees, which is described by the constraint $x_1 = x_2$ (see chapter 3.3 and (Glötzl 2015)).

---

[2] (Glötzl 1999; 2009; 2023b) describes the "fundamental paradox of the monetary economy". It states that in an economy where credit is measured in monetary units, the power of the sum of creditors to increase their acounts is always greater than the power of the sum of debtors to reduce their debts. In other words, it describes the "powerlessness" of debtors relative to the "power of creditors”". These power relations are ultimately the cause of debt traps and the constant growth of accounts and debts.

Similar Models

Model	variables		constraint condition
Inclined plane	x1	x2	x1=x2
Investment versus Saving	I	S	I=S
Creditors versus Debitors	R	D	R=D

18.2.3. Calculations

The GCD equation system is given by:

$$uF[t] == -\frac{1}{2}(idach - inv[t])^2$$
$$uH[t] == -\frac{1}{2}(sdach - spar[t])^2$$

$$inv'[t] == \mu Finv\,(idach - inv[t]) + \lambda_1[t]$$
$$spar'[t] == \mu Hspar\,(sdach - spar[t]) - \lambda_1[t]$$

$$0 == inv[t] - spar[t]$$

$$inv[0] == inv0$$
$$spar[0] == inv0$$

We assume that investors want to invest 4 units and savers want to save 2 units, i.e.

$$idach = 4$$
$$sdach = 2$$

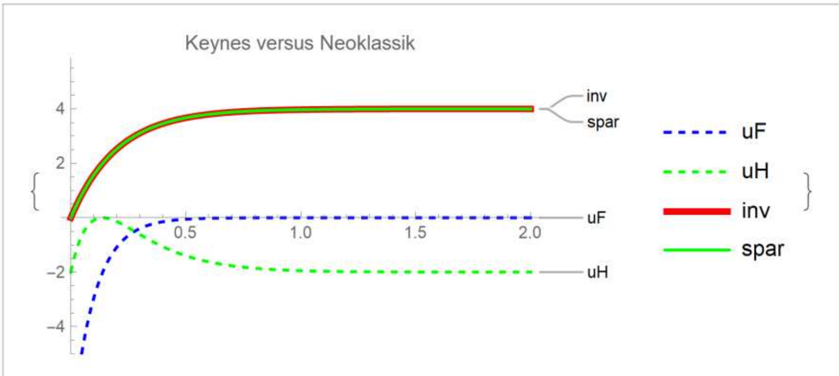
At the time $t = 0$ neither investing nor saving occurs, i.e.

$$inv[0] = spar[0] = 0$$

The following numerical calculations show the different behaviour for the different assumptions about the power factors.

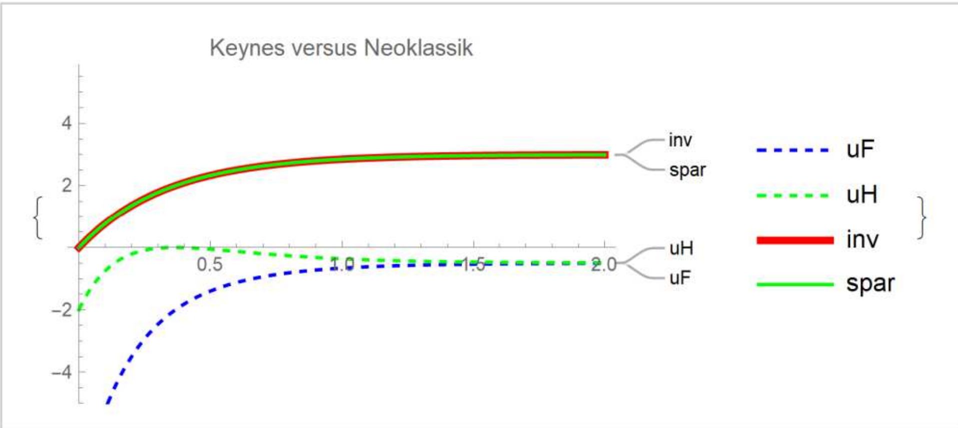
https://www.dropbox.com/s/mq6s03sbunlmzob/Keynes%20Version%206.ndsolve.nb?dl=0

Keynes $\mu Finv = \infty$ (*approximated by* $\mu Finv = 10$), $\mu Hspar = 0$



Investing (= saving) converge against the firm's targeted investments.

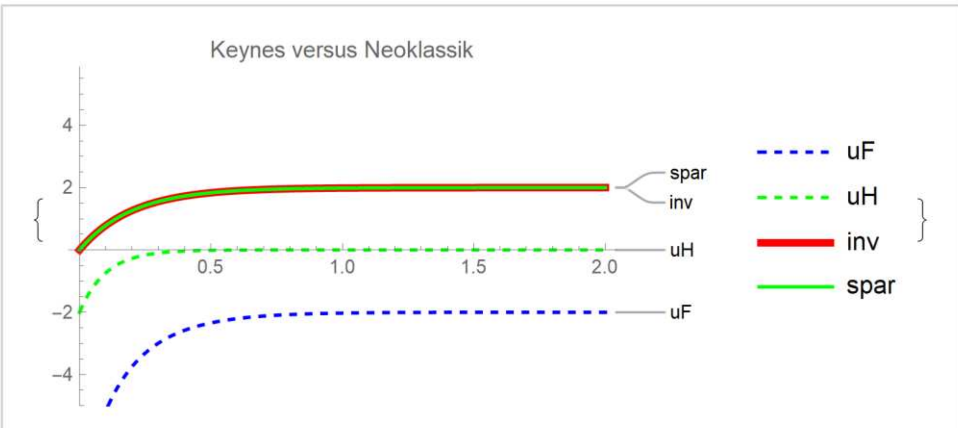
equal power $\mu Finv = 3$ $\mu Hspar = 3$



Investing(=saving) converges to a mixture of the investment targeted by the firm and the saving targeted by the household. The speed of convergence, depends on the level of the power factors, because the power factors can always be interpreted as speed-adjustment factors (see also chapter 7.7).

Neoclassics $\mu Finv = 0$ $\mu Hspar = \infty$

$(approximated\ by\ \mu Hspar = 10)$



Investing (=saving) converge against the saving targeted by the household.

18.2.4. On the relationship of "drop closure", "Lagrange closure", GCD and general equilibrium GE

We explain the relationship first with the simple example above and then in the following chapter 18.3 with the models of A. Sen (A. K. Sen 1963). More detailed information can be found in (Glötzl 2015).

Based on the utility functions for F and H

$$U^F = -\frac{1}{2}(\hat{I} - I)^2 \qquad F\ Firma,\ I\ Investieren,$$

$$\hat{I}\ angestrebtes\ Investieren$$

$$U^H = -\frac{1}{2}(\hat{S} - S)^2 \qquad H\ Haushalt,\ S\ Sparen,$$

$$\hat{S}\ angestrebtes\ Sparen$$

the ex-ante behavioural equations (i.e., the behavioural equations without considering the constraint $0 = I - S$) are as follows

$$\begin{aligned} &(a) \qquad I' = \mu_I^F (\hat{I} - I)^\wedge 2 \\ &(b) \qquad S' = \mu_S^H (\hat{S} - S)^\wedge 2 \end{aligned} \qquad \text{<18.1>}$$

This system of equations has 2 variables (S, I) and 2 equations. It is therefore solvable with appropriate initial conditions.

However, these ex-ante solutions do not describe the reality, because they usually do not fulfill the constraint $0 = I - S$ which has to be fulfilled.

If the constraint is added to the ex-ante system of equations, the following is obtained

$$\begin{aligned} &(a) \qquad I' = \mu_I^F \frac{1}{2}(\hat{I} - I) \\ &(b) \qquad S' = \mu_S^H \frac{1}{2}(\hat{S} - S) \\ &(c) \qquad I = S \end{aligned} \qquad \text{<18.2>}$$

This system of equations consists of 3 equations for 2 variables and is therefore usually not solvable. A method with which this system of equations is changed in such a way that it becomes solvable is called a closure method.

18.2.4.1. Drop closure

In the simplest case, one omits so many equations until the system of equations becomes solvable. This basic procedure is used by A. Sen in (A. K. Sen 1963)).

If in the case of equation system <18.2> equation (a) is omitted, the result is

$$(b) \qquad S' = \mu_S^H \frac{1}{2}(\hat{S} - S)$$
$$(c) \qquad I = S \qquad \text{<18.3>}$$

which corresponds exactly to the neoclassical approach and, in equilibrium ($S' = 0$)

$$(b) \qquad S = \hat{S}$$
$$(c) \qquad I = S$$

results.

If we omit (b), we get

$$(a) \qquad I' = \mu_I^F \frac{1}{2}(\hat{I} - I)$$
$$(c) \qquad I = S \qquad \text{<18.4>}$$

which is exactly in line with the Keynesian approach and in equilibrium ($I' = 0$)

$$(a) \qquad I = \hat{I}$$
$$(c) \qquad I = S$$

results.

18.2.4.2. Lagrange Closure, GCD, general equilibrium

In the case of Lagrange Closure, the opposite approach is taken: equations are not omitted, but new additional variables are introduced until the system of equations becomes solvable. In the concrete case, one adds the Lagrange multiplier λ as a new additional variable to the variables and the constraint forces to the behaviour equations in the sense of the GCD methodology. This results in the GCD equation system, which is usually solvable.

$$(a) \qquad I' = \mu_I^F \frac{1}{2}(\hat{I} - I) + \lambda \frac{\partial Z}{\partial I} = \mu_I^F \frac{1}{2}(\hat{I} - I) + \lambda$$
$$(b) \qquad S' = \mu_S^H \frac{1}{2}(\hat{S} - S) + \lambda \frac{\partial Z}{\partial S} = \mu_S^H \frac{1}{2}(\hat{S} - S) - \lambda \qquad \text{<18.5>}$$
$$(c) \qquad Z = 0 = I - S$$

We show that in the Keynesian case, because of $\mu_S^H = 0$ this system of equations <18.5> transforms to the system of equations

$$(a) \qquad I' = \mu_I^F \frac{1}{2}(\hat{I} - I) + \lambda$$
$$(c) \qquad Z = 0 = I - S \qquad \text{<18.6>}$$

This means that $\mu_S^H = 0$ leads to (b) becoming linearly dependent on (a) and (c) and can therefore be omitted in the sense of drop closure. This is discussed in more detail in (Glötzl 2015).

In the case of the general equilibrium $I' = 0$, because of $\mu_I^F = \infty$ it follows that

$(a) \qquad I = \hat{I}$

$(c) \qquad I = S$

Proof:

Because of $\mu_S^H = 0$ and because of (c), it follows from <18.5>

$(a) \qquad I' == \mu_I^F \frac{1}{2}(\hat{I} - I) + \lambda$

$(b1) \qquad S' == -\lambda$

$(c) \qquad Z = 0 = I - S$

$(d) \qquad Z' = 0 = I' - S'$

If we apply (d) in (b1), we get

$(a) \qquad I' = \mu_I^F \frac{1}{2}(\hat{I} - I) + \lambda$

$(b2) \qquad I' = -\lambda$

$(c) \qquad Z = 0 = I - S$

$(d) \qquad Z' = 0 = I' - S'$

From (a) and (b2) we get

$$\lambda = -\frac{1}{2}\mu_I^F \frac{1}{2}(\hat{I} - I)$$

Inserting into (a) and (b1) results in

$(a) \qquad I' = \frac{1}{2}\mu_I^F \frac{1}{2}(\hat{I} - I)$

$(b) \qquad S' = \frac{1}{2}\mu_I^F \frac{1}{2}(\hat{I} - I)$

$(c) \qquad Z = 0 = I - S$

$(d) \qquad Z' = 0 = I' - S'$

Thus, equation (b) is linearly dependent on (a) and (d) and can therefore be omitted.

In the case of the general equilibrium ($I' = 0$), this results in the following equations because of $\mu_I^F = \infty$

$(a) \qquad I = \hat{I}$

$(c) \qquad I = S$

by bringing μ_I^F to the left side at first.

Summary: The Keynesian model results from the GCD model both by drop-closure, by omitting equation (b), and by setting the power factor $\mu_S^H = 0$. The power factor μ_I^F need only be $\mu_I^F > 0$, it can also be $\mu_I^F = \infty$. The magnitude of μ_I^F only determines the speed of convergence. For the neoclassical model, everything applies correspondingly.

18.3. A. Sen: different economic theories differ in their assumptions about the endogeneity or exogeneity of different variables.

18.3.1. Problem description

In 1963, Amartya Sen showed that neoclassical and Keynesian models can often be derived from the same system of equations and essentially differ only in which behavioural equations are dropped (A. K. Sen 1963). This also corresponds to a decision on the direction of causality within the model.

Similarly to the previous chapter, all models examined by Sen can be understood as special cases of a single GCD model and dropping certain equations is equivalent to assuming different one-sided power relations. Again, it is true that in reality, these one-sided power relations do not usually occur, but rather mixed power relations. Therefore, reality can be better described with GCD models than with the models cited by Sen.

The original system of equations of Sen is

(1)	$Y(L,K) = \beta L^{a} K^{1\text{-}a}$	*we assume a Cobb – Douglas production function*	
(2)	$w = \dfrac{\partial Y}{\partial L}$	*w wages, L labour*	
(3)	$Y = P + wL$	*P profit*	
(4)	$I = S_L\, L + S_P\, P$	S_L *Savings share of employment income*	<18.7>
		S_P *Savings share of profit*	
(5)	$I = i_1 + i_2 Y$	*we assume this standard investment function*	

For clarity, we also introduce the variable S for saving and the constraints $0 = I - S$ and $I = K'$. Implicitly, Sen assumes that L is exogenously given by $L = \hat{L}$. This yields the system of equations (14.8) which is equivalent to (14.7). We write it in our methodology as follows *algebraically defined variables*

(1) $Y(L,K)=\beta L^{a}K^{1-a}$

behavioural equations

(2) $w=\dfrac{\partial Y}{\partial L}$

(4) $S=S_{L}\ L+S_{P}\,P$

(5) $I=i_1+i_2\,Y$ <18.8>

(6) $L=\hat{L}$

constraints

(3) $0=Y-P-wL$

(7) $0=I-S$

(8) $0=I-K'$

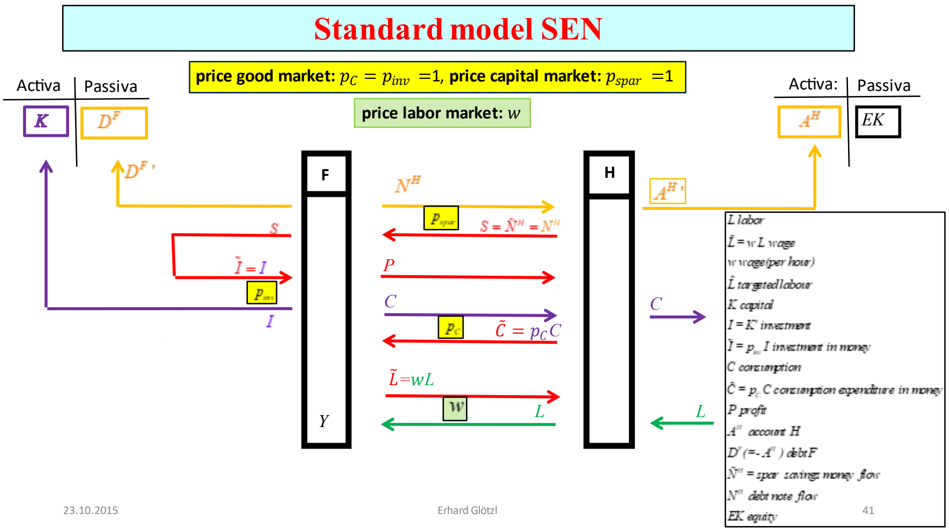
This system of equations consists of 8 equations for the 7 variables

$Y,\ L,\ K,\ w,\ S, P, I$

and is therefore generally not solvable. Sen shows that by dropping different equations (drop closure) different solvable economic models result:

omitting (5) *results in the neoclassical model*
omitting (2) *results in the Kaldor model*
(Neo - Keynesianisches Modell) <18.9>
omitting (4) *results in the Johansen model*
omitting (6) *results in the Keynesian model*
of the General Theory

We show below that the system of equations <18.8> and the various models <18.9> arise from a single GCD model through a specific choice of power factors in each case.



The variables N^H, D^F, A^H are only listed for the sake of completeness. They are omitted in the following (as by Sen), because due to the assumption $p_{spar} = 1$ immediately $spar = p_{spar} N^H = N^H$ is valid.

Standard model SEN, neoclassic, Kaldor, Johansen, Keynes, GCD

SEN	neoclassic	Kaldor	Johansen	Keynes	GCD - Modell
overdetermined	drop (5)	drop (2)	drop (4)	drop(6)	"Lagrange – Closure"
7 variables : Y, L, K, w, S, P, I					
8 equations					
algebraically defined variables					
(1) $Y(L,K) = \beta L^a K^{1-a}$					
behavioural equations					
(2) $w = \frac{\partial Y}{\partial L}$	$\mu_w^F = \infty$	$\mu_w^F = 0$	$\mu_w^F = \infty$	$\mu_w^F = \infty$	$w' = \mu_w^F (\frac{\partial Y}{\partial L} - w) - \lambda_1 L$
(4) $S = \hat{S}_L L + \hat{S}_P P$	$\mu_S^H = \infty$	$\mu_S^H = \infty$	$\mu_S^H = 0$	$\mu_S^H = \infty$	$S' = \mu_S^H (s_L wL + s_P P - S) - \lambda_2$
(5) $I = i_1 + i_2 Y$	$\mu_I^F = 0$	$\mu_I^F = \infty$	$\mu_I^F = \infty$	$\mu_I^F = \infty$	$I' = \mu_I^F (i_1 + i_2 Y - I) + \lambda_2 + \lambda_3$
(6) $L = \hat{L}$	$\mu_L^H = \infty$	$\mu_L^H = \infty$	$\mu_L^H = \infty$	$\mu_L^H = 0,$	$L' = \mu_L^H (\hat{L} - L) - .... + \lambda_1 (\frac{\partial Y}{\partial L} - w)$
constraints					
(3) $0 = Y - P - wL$					
(7) $0 = I - S$					
(8) $0 = I - K'$					

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The GCD model SEN results from the following basic equations:

GCD-Modell SEN : Grundgleichungen

algebraically defined variables

$Y(L,K)=\beta L^{\alpha}K^{1-\alpha}$ "*production function*"

utility functions

$U^{H}=-\frac{1}{2}\left(\hat{L}-L\right)^{2}-\frac{1}{2}(\hat{S}_{L}L+\hat{S}_{P}P-S)^{2}$ "*utility function household*"

$U^{F}=-\frac{1}{2}(i_{1}+i_{2}Y-I)^{2}-\frac{1}{2}(\frac{\partial Y}{\partial L}-w)^{2}$ "*utility function firm*"

constraints

(3) $0=Y-P-wL$

(7) $0=I-S$

(8) $0=I-K'$

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The GCD equations are then

```
uF[t] == -1/2 (α β k[t]^(1-α) l[t]^(-1+α) - w[t])^2 - 1/2 (i1 - inv[t] + i2 y[t])^2
uH[t] == -1/2 (ldach - l[t])^2 - 1/2 (spardachl l[t] + spardachprofit profit[t] - spar[t])^2

c[t] == -spar[t] + l[t] × w[t]
y[t] == β k[t]^(1-α) l[t]^α

inv′[t] == μFinv (i1 - inv[t] + i2 β k[t]^(1-α) l[t]^α) + λ2[t] + λ3[t]
k′[t] ==
  μFk (-i2 (1 - α) β k[t]^-α l[t]^α (i1 - inv[t] + i2 β k[t]^(1-α) l[t]^α) -
      (1 - α) α β k[t]^-α l[t]^(-1+α) (α β k[t]^(1-α) l[t]^(-1+α) - w[t])) + (1 - α) β k[t]^-α l[t]^α λ1[t] - λ3[t]
l′[t] == μHl (ldach - l[t] - spardachl (spardachl l[t] + spardachprofit profit[t] - spar[t])) +
   μFl (-i2 α β k[t]^(1-α) l[t]^(-1+α) (i1 - inv[t] + i2 β k[t]^(1-α) l[t]^α) -
      (-1 + α) α β k[t]^(1-α) l[t]^(-2+α) (α β k[t]^(1-α) l[t]^(-1+α) - w[t])) + (α β k[t]^(1-α) l[t]^(-1+α) - w[t]) λ1[t]
profit′[t] == -spardachprofit μHprofit (spardachl l[t] + spardachprofit profit[t] - spar[t]) - λ1[t]
spar′[t] == μHspar (spardachl l[t] + spardachprofit profit[t] - spar[t]) - λ2[t]
w′[t] == μFw (α β k[t]^(1-α) l[t]^(-1+α) - w[t]) - l[t] λ1[t]

0 == β k[t]^(1-α) l[t]^α - profit[t] - l[t] × w[t]
0 == inv[t] - spar[t]
0 == inv[t] - k′[t]

inv[0] == inv0
k[0] == k0
l[0] == l0
profit[0] == -l0 w0 + k0^(1-α) l0^α β
spar[0] == inv0
w[0] == w0
```

Dividing the differential equations for w,S,I,L by $\mu_{w}^{F},\mu_{S}^{H},\mu_{I}^{F},\mu_{L}^{H}$ in each case and setting

$$\mu_{w}^{F}=\infty$$

$$\mu_{S}^{H}=\infty$$

$$\mu_{I}^{F}=\infty$$

$$\mu_{L}^{H}=\infty$$

we get the 4 behavioural equations of the standard model SEN

(2) $w = \dfrac{\partial Y}{\partial L}$

(4) $S = S_L\ L + S_P\ P$

(5) $I = i_1 + i_2 Y$

(6) $L = \hat{L}$

In addition one has the differential behavioural equations for K and P.

If one sets individual power factors equal to 0, this leads in an analogous way, as it was shown in chapter 18.2.4.1, to the fact that the corresponding differential equation becomes linearly dependent on the others and can therefore be omitted. More details can also be found in (Glötzl 2015).

18.3.2. Calculation results

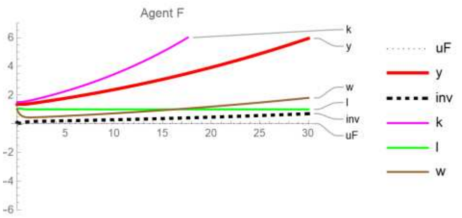
https://www.dropbox.com/s/p0280ndhlb946lg/Modell%20SEN%20Version%2011.ndsolve.nb?dl=0

In order to solve the differential-algebraic GCD equation system with NDSolve one has to use the method

Method→{IndexReduction→Automatic}

$\mu = \infty$ is always approximated by $\mu = 6$.

Neoclassical model $\mu_w^F = \infty,\ \mu_S^H = \infty,\ \mu_I^F = 0,\ \mu_L^H = \infty$



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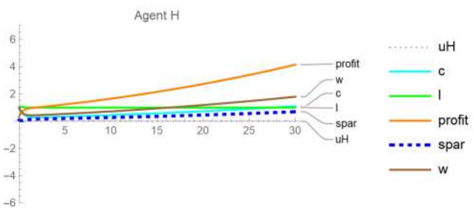
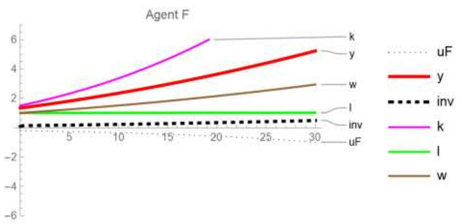


Kaldor model $\mu_w^F = 0,\ \mu_S^H = \infty,\ \mu_I^F = \infty,\ \mu_L^H = \infty$



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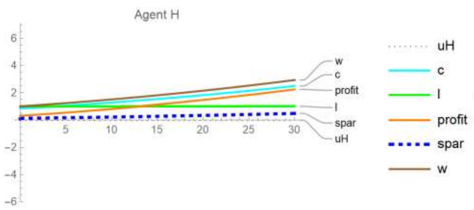
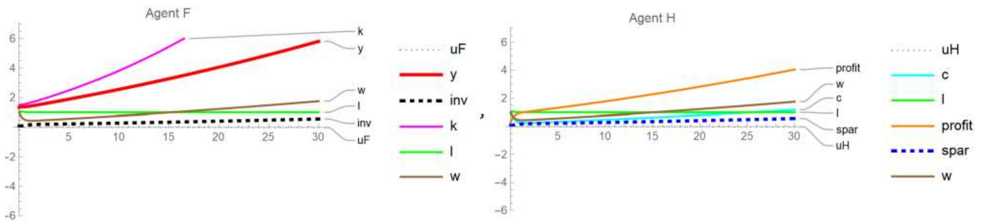
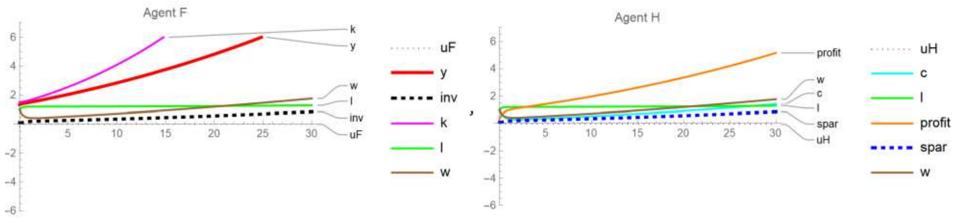


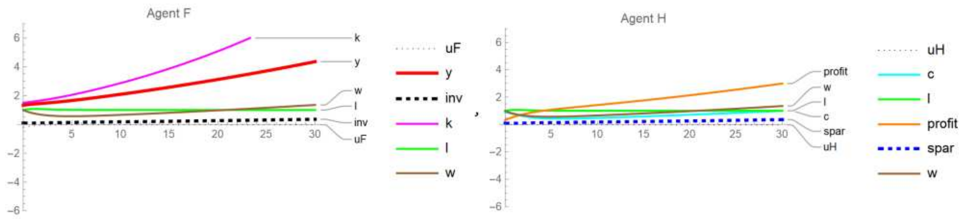
Johansen model $\mu_w^F = \infty,\ \mu_S^H = 0,\ \mu_I^F = \infty,\ \mu_L^H = \infty$



Keynes model $\mu_w^F = \infty,\ \mu_S^H = \infty,\ \mu_I^F = \infty,\ \mu_L^H = 0$



GCD-Modell with mixed parameters $\mu_w^F = 1,\ \mu_S^H = 1,\ \mu_I^F = 1,\ \mu_L^H = 1$



18.3.3. On the relationship between GCD models, General Constrained Equilibrium models (GCE model) and DSGE models.

A general equilibrium model can only start from 1 master utility function to be maximized (note: multiple utility functions cannot be maximized at the same time, they have to be combined to 1 master utility function, e.g. by weighting). A possible master utility function for a general constrained equilibrium (GCE) model would be:

$$\hat{U} = U^H + U^F$$
$$= -\frac{1}{2}\left(\hat{L} - L\right)^2 - \frac{1}{2}(S_L\, L + S_P P - S)^2 - \frac{1}{2}(i_1 + i_2 Y - I)^2 - \frac{1}{2}\left(\frac{\partial Y}{\partial L} - w\right)^2$$

With the algebraically defined variable

$$Y(L,K) = \beta L^{\alpha} K^{1-\alpha}$$

this results in

$$\hat{U}(L,P,S,K,I,w) = -\frac{1}{2}\left(\hat{L}-L\right)^2 - \frac{1}{2}(S_L\,L + S_P P - S)^2 -$$
$$-\frac{1}{2}(i_1 + i_2\beta\,L^\alpha K^{1-\alpha} - I)^2 - \frac{1}{2}(\beta\,\alpha L^{\alpha-1}K^{1-\alpha} - w)^2$$

The constraints remain the same:

$$Z_1 = 0 = Y - P - w\,L$$
$$Z_2 = 0 = I - S$$
$$Z_3 = 0 = I - K'$$

A maximum under constraints can only exist if the "first order" conditions are fulfilled, i.e.

$$\begin{aligned}
0 &= \frac{\partial \hat{U}}{\partial L} + \lambda_1\frac{\partial Z_1}{\partial L} + \lambda_2\frac{\partial Z_2}{\partial L} + \lambda_3\frac{\partial Z_3}{\partial L} = \frac{\partial U^H}{\partial L} + \frac{\partial U^F}{\partial L} + \lambda_1\frac{\partial Z_1}{\partial L} + \lambda_2\frac{\partial Z_2}{\partial L} + \lambda_3\frac{\partial Z_3}{\partial L}\\
0 &= \frac{\partial \hat{U}}{\partial P} + \lambda_1\frac{\partial Z_1}{\partial P} + \lambda_2\frac{\partial Z_2}{\partial P} + \lambda_3\frac{\partial Z_3}{\partial P} = \frac{\partial U^H}{\partial P} + \frac{\partial U^F}{\partial P} + \lambda_1\frac{\partial Z_1}{\partial P} + \lambda_2\frac{\partial Z_2}{\partial P} + \lambda_3\frac{\partial Z_3}{\partial P}\\
0 &= \frac{\partial \hat{U}}{\partial S} + \lambda_1\frac{\partial Z_1}{\partial S} + \lambda_2\frac{\partial Z_2}{\partial S} + \lambda_3\frac{\partial Z_3}{\partial S} = \frac{\partial U^H}{\partial S} + \frac{\partial U^F}{\partial S} + \lambda_1\frac{\partial Z_1}{\partial S} + \lambda_2\frac{\partial Z_2}{\partial S} + \lambda_3\frac{\partial Z_3}{\partial S}\\
0 &= \frac{\partial \hat{U}}{\partial K} + \lambda_1\frac{\partial Z_1}{\partial K} + \lambda_2\frac{\partial Z_2}{\partial K} + \lambda_3\frac{\partial Z_3}{\partial K'} = \frac{\partial U^H}{\partial K} + \frac{\partial U^F}{\partial K} + \lambda_1\frac{\partial Z_1}{\partial K} + \lambda_2\frac{\partial Z_2}{\partial K} + \lambda_3\frac{\partial Z_3}{\partial K'}\\
0 &= \frac{\partial \hat{U}}{\partial I} + \lambda_1\frac{\partial Z_1}{\partial I} + \lambda_2\frac{\partial Z_2}{\partial I} + \lambda_3\frac{\partial Z_3}{\partial I} = \frac{\partial U^H}{\partial I} + \frac{\partial U^F}{\partial I} + \lambda_1\frac{\partial Z_1}{\partial I} + \lambda_2\frac{\partial Z_2}{\partial I} + \lambda_3\frac{\partial Z_3}{\partial I}\\
0 &= \frac{\partial \hat{U}}{\partial w} + \lambda_1\frac{\partial Z_1}{\partial w} + \lambda_2\frac{\partial Z_2}{\partial w} + \lambda_3\frac{\partial Z_3}{\partial w} = \frac{\partial U^H}{\partial w} + \frac{\partial U^F}{\partial w} + \lambda_1\frac{\partial Z_1}{\partial w} + \lambda_2\frac{\partial Z_2}{\partial w} + \lambda_3\frac{\partial Z_3}{\partial w}\\
Z_1 &= 0 = Y - P - w\,L\\
Z_2 &= 0 = I - S\\
Z_3 &= 0 = I - K'
\end{aligned}$$

<18.10>

This system of equations <18.10> is obviously identical to the GCD system of equations in steady state, i.e. for

$$L' = P' = S' = K' = I' = w' = 0$$

Note: This does not apply in general, but only if the utility functions can be aggregated to a master utility function (see chapter 4.3, chapter 7.6 and chapter 7.8.2).

In contrast to GCE models, in DSGE models in particular (apart from the stochastic terms) not a master utility function is maximized under constraints, but rather the master utility function discounted by the discount rate β is maximized

$$\hat{U}_\beta(t) = \int_0^t e^{-\beta t}\hat{U}(t)dt \rightarrow max \quad under\ constraints$$

For holonomic constraints this problem can be solved by the variational problem with the Lagrange function

$$\hat{U}^{Z}_{\beta}(t)=\int_{0}^{t} e^{-\beta t}\left(\hat{U}(t)+\sum \lambda_j \frac{\partial Z_j}{\partial x_i}\right)dt \to \max$$

This leads to the corresponding Euler equations that describe the dynamics of the DSGE model.

Note: Without going into more detail here, we would like to point out the following: If the constraints are neither holonomic nor integrable nor linear, the two problems

(1) $\hat{U}_{\beta}(t)=\int_{0}^{t} e^{-\beta t}\hat{U}(t)dt \to \max$ under constraints

(2) $\hat{U}^{Z}_{\beta}(t)=\int_{0}^{t} e^{-\beta t}\left(\hat{U}(t)+\sum \lambda_j \frac{\partial Z_j}{\partial x_i}\right)dt \to \max$

are different and lead to different Euler equations and thus different dynamics. The dynamics to (1) is called "vakonomic mechanics". For more details see (Glötzl 2018).

19. Obesity or consumption/environment model

In section 7.9.2 we referred to the special case where a utility function depends on variables $x = \{x_1, x_2, ..., x_I\}$ as well as on their antiderivatives $X = (X_1, X_2, ..., X_I)$ and/or the derivatives $x' = (x_1', x_2', ..., x_I')$ of these variables. In these cases, both the antiderivatives $X = (X_1, X_2, ..., X_I)$ and the derivatives $x' = (x_1', x_2', ..., x_I')$ are to be regarded as additional variables of their own and appropriate constraints are to be added describing the relations between antiderivatives, functions and derivatives of the function.

We will describe this situation using a simple example with only one variable x, where the utility function depends on a flow variable x as well as on some stock variable X.

A good illustrative example is that we all like to eat but do not want to be fat. Here, x describes the flow variable eat and X the stock variable, which describes the body weight.

utility function

$$U(x, X) = -(\hat{x} - x)^2 - (\hat{X} - X)^2 \qquad oder \qquad U = x^\gamma - (\hat{X} - X)^2$$

constraint

$$0 = X' - x + \sigma X$$

The utility function describes the decreasing marginal benefit of eating and the increasing marginal cost of body weight. The constraint describes that eating increases weight and decreases it at the rate σ due to natural weight loss. If the parameter $\sigma = 0$, then the constraint just describes the direct stock-flow relationship $X' = x$ between X and x.

This results in the following GCD equation system

$$uA[t] == x[t]^{\gamma 1} - xx[t]^{\gamma 2}$$

$$x'[t] == \gamma 1\, \mu Ax\, x[t]^{-1+\gamma 1} - \lambda_1[t]$$
$$xx'[t] == -\gamma 2\, \mu Axx\, xx[t]^{-1+\gamma 2} + \lambda_1[t]$$

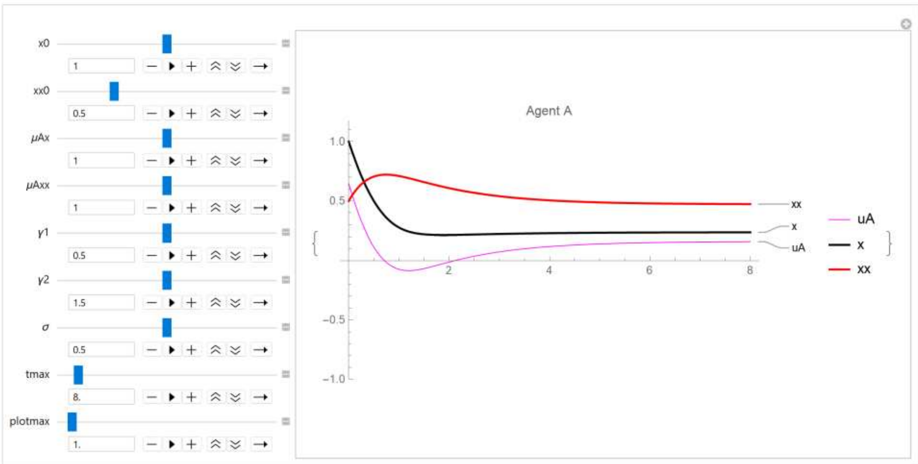
$$0 == -x[t] + \sigma\, xx[t] + xx'[t]$$

$$x[0] == x0$$
$$xx[0] == xx0$$

https://www.dropbox.com/s/7cj6lflrc9qybgg/Modell%20Fresssack%20Version%202.ndsolve.nb?dl=0

The result of the calculation is, for example:

(In the plot stands xx for the stock variable X)



We give this simple example mainly because this contradictory behaviour of flow variable and stock variable is also relevant in many environmental problems. For example, the following other interpretations are also possible:

	Flow variable x	Stock variable X
Land consumption	Building	Built-up area
Waste	Production	Total waste
Plastic packaging	Consumption	Plastic waste in the sea

Carbon dioxide	Fossil fuel combustion	Carbon dioxide concentration in the air

Furthermore, this simple model serves as an example for a model in which the stock and flow variables occur simultaneously in the utility function. As already explained in chapter 7.9, in this case a separate variable must be introduced for the stock variable and the flow variable. The relationship between the two is described by a constraint $X' = x - \sigma X$. If the parameter $\sigma = 0$, then the constraint just describes the direct stock-flow relationship $X' = x$ between X and x.

D. Supply, demand and price shock models

20. Modeling of supply, demand and price shocks

20.1. 2 different types of shocks

Basically, a shock can lead to 2 fundamentally different types of shocks:

(1) Variable shock

All model parameters remain unchanged, but at the time of the shock t_s one or more model variables $V \in \{C, L, K, M^H, M^F, S, p, w\}$ abruptly change by the factor f_V from V to $f_V V$

$$V(t_s) \rightarrow f_V V(t_s)$$

Interpretation: The basic behaviour of all agents remains the same, but an external event suddenly changes the value of a variable (e.g. the price of energy). The system restarts, as it were, with this new value as the starting value.

(2) Model shock

All variables remain unchanged, but one or more model parameters or power factors $\pi \in \{\alpha, \beta, \gamma, \mu_C^H, \mu_L^H, \mu_L^F, \mu_K^F, \mu_{M^H}^H, \mu_S^F, \mu_p^F, \mu_w^F\}$ are no longer constant but change over time, i.e. $\pi \rightarrow \pi(t)$. For the sake of simplicity, we describe the temporal behaviour of such a parameter $\pi(t)$ by multiplication with a sawtooth curve:

$$\pi(t) = \pi\,\sigma(t)$$

where the sawtooth curve is defined by

$$\begin{aligned} &t_s \ \textit{time of the shock} \\ &f \ \textit{shock factor} \\ &d \ \textit{duration of the linearly decreasing shock effects} \end{aligned}$$

$$\sigma(t) = \begin{cases} 1 & \textit{for } t < t_s \\ f & \textit{for } t = t_s \\ \textit{linear from } f \textit{ to } 1 & \textit{for } t_s < t < t_s + d \\ 1 & \textit{for } t_s + d_i^j \le t \end{cases}$$

20.2. Examples of demand shocks

For example, a demand shock N can have three different causes:

(1) Variable shock: A demand shock can be triggered by the fact that at the time of the demand shock t_N consumption C is reduced by a factor f_C^N from C to $f_C^N C$:

$$C \to f_C^N C$$

At the same time, the constraint Z_3 must always be fulfilled, even during the shock. This is always guaranteed by the numerical solution method for differential-algebraic equations of Mathematica NDSolve. In addition, of course, one can make any other assumptions, such as that production Y and investment K' remain the same and that everything that is consumed less $(1-f_N^C)C(t_N)$ is stored, i.e.

$$S'(t_A) \to S'(t_A) + (1-g_C^N)C(t_A)$$

The dynamic system then continues to develop with these new initial values.

(2) Model shock: The model parameter γ describes the consumption preference of the household. A demand shock can be triggered by the changes of γ over time according to a sawtooth curve:

$$\gamma \to \gamma(t) = \sigma(t)\gamma$$

(3) Model shock: The power factor μ_C^H describes the power of the household to actually enforce its consumption interests (e.g. due to quarantine measures). A demand shock can be triggered by a change of μ_C^H in time according to a sawtooth curve:

$$\mu_C^H \to \mu_C^H(t) = \sigma(t)\mu_C^H$$

20.3. Examples of supply shocks

For example, a supply shock A at the time can have the following causes:

(1) Model and variable shock: A supply shock could be triggered by the fact that the production function

$$Y(L,K) = \beta L^\alpha K^{1-\alpha}$$

changes over time according to a sawtooth curve with a shock factor f_β^A. This initially corresponds to a model shock, because this is described by the fact that the parameter β changes according to a sawtooth curve with a shock factor f_β^A:

$$\beta \to \beta(t) = \sigma(t)\beta$$

At the same time, production Y suddenly changes by the shock factor f_β^A at the time of the shock t_A:

$$Y(t_A) \to f_\beta^A Y(t_A)$$

Because of the constraint

$$Z_3 = 0 = Y(L,K) - C - K' - S'$$

at the time t_A, therefore, C, K', S' must also change so that the constraint is fulfilled. This leads to sudden changes in at least one of the variables or in all of them. This is always guaranteed by NDSolve. For example, one can also make additional more precise assumptions about the behaviour of the other variables, e.g. one could assume that at the time t_A also C, K', S'

change by the shock factor f_β^A, i.e.

$$C(t_A) \to f_\beta^A C(t_A)$$
$$K'(t_A) \to f_\beta^A K'(t_A)$$
$$S'(t_A) \to f_\beta^A S'(t_A)$$

and that the dynamic system can then adapt to these new starting conditions and the time-varying parameter

$$\beta(t) = \sigma_\beta^A(t)\beta$$

(2) Model shock: The model parameter α describes the labour intensity of production. A supply shock can be triggered by the changes of α over time according to a sawtooth curve

$$\alpha \to \alpha(t) = \sigma(t)\alpha$$

(3) Model shock: The power factor μ_K^F describes the power of the firm to actually enforce its investment interests (e.g. because of administrative regulations). A supply shock can be triggered by the fact that the power factor μ_K^F changes over time according to a sawtooth curve:

$$\mu_K^F \to \mu_K^F(t) = \sigma(t)\mu_K^F$$

20.4. Price shock

For example, a price shock P can be modeled by changing the price p at the time t_P by the factor f_p^P

$$p(t_P) \to f_p^P p(t_P)$$

This corresponds to a variable shock.

20.5. Policy shocks

In addition, a wide variety of fiscal and monetary policy measures that apply from certain time points can of course also be interpreted as economic policy shocks and modelled in the same way, e.g:

- government measures:
  - Tax reform
  - Increase or decrease in public debt
  - etc.
- Changes in central bank policy:
  - From money supply control to interest rate control
  - Change in the inflation target
  - Purchase programmes
  - etc.

21. Topics to be discussed

In the economy, a shock can occur for a number of reasons, e.g.

- sudden changes in raw material prices
- sudden changes in consumer behaviour due to quarantine regulations
- sudden production restrictions due to a disruption in the supply chain
- etc.

From an economic point of view, there are 2 fundamental topics related to shocks:

(1) Forecasting: How will the economic variables change?

(2) Evaluating countermeasures: What measures can be taken to overcome the shock as quickly as possible or with as little effort as possible?

Possible measures are, for example:

- Various forms of financial assistance from the government to firms
- Various forms of financial benefits to consumers
- Different ways of financing additional government expenditure
- short-time working models
- Central bank monetary policy measures
- Organisational measures, e.g. relieving companies of administrative regulations, extending opening hours in the retail sector, etc. Such organisational measures are expressed in the models by changes in power factors or other parameters.

The target of section C. is to show that GCD models are basically suitable for answering these 2 questions and that this can be done very easily and conveniently with the help of the open-source program GCDconfigurator. The additions necessary to incorporate shocks into a model programmed with GCDconfigurator are very easy to program.

In order to apply GCD models to real economic situations, they would of course have to be extended accordingly and adapted to the real conditions.

Using model A1 as an example, we show how special supply, demand and price shocks can be modeled and what effects they have on the further course of the economy.

Using model B1, we show how central bank measures have different effects on a price shock depending on whether the central bank pursues a monetary policy or an interest rate policy.

In order to make the effects clearly visible, the model calculations are carried out for very strong shocks of a magnitude that is unlikely to occur in reality.

22. Calculations with model A1 on various shocks

We model the following shocks

```
(*Price shock as variable shock,
 Variable shock for p[t] at time tnp, p[t] jumps to fpp x p[t] at time tnp*)
WhenEvent[t == tp, {p[t] → fpp p[t]}],
bei Ereignis

(*Demand shock as variable shock,
 Variable shock for C[t] at time tnc, C[t] jumps to fnc x c[t] at time tnc*)
WhenEvent[t == tnc, {c[t] → fnc c[t]}],
bei Ereignis

(* Demand shock as model shock (shock of power of household),
Model shock for μHc[t] at time tn, decays over the time period of dn,
σnμHc[t] a sawtooth curve,
σnμHc[0]=1, jumps at time tn to fnμHc, goes back linearly to 1 in the time period of dn,
μHc[t]=μHc x σnμHc[t] a sawtooth curve,
μHc[0]=μHc, jumps at time tn to μHc x fnμHc, goes back linearly to μHc in the time period of dn *)

σnμHc[t] == σnμHc0[t] + σnμHc1[t] (t - tn),
σnμHc0[0] == 1, σnμHc1[0] == 0,
WhenEvent[t == tn, {σnμHc0[t] → fnμHc, σnμHc1[t] → (1 - fnμHc) / dn}],
bei Ereignis
WhenEvent[t == tn + dn, {σnμHc0[t] → 1, σnμHc1[t] → 0}],
bei Ereignis

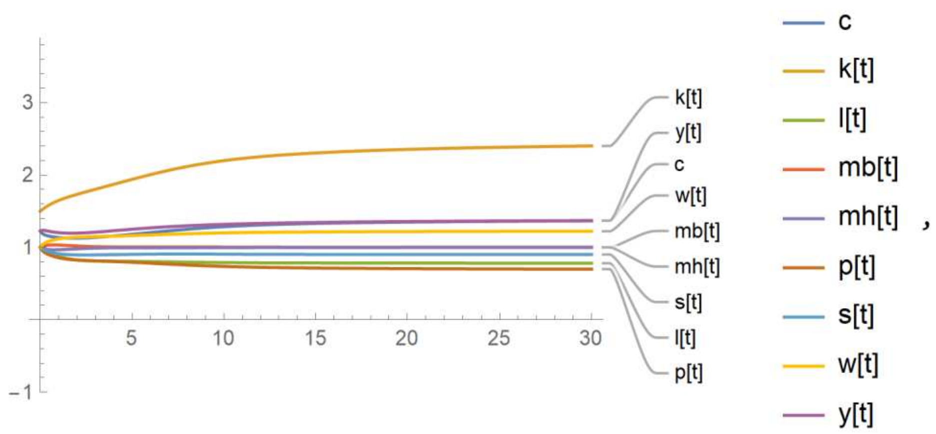
(* Supply shock as model shock (shock of power of firm),
Model shock for μFk[t] at time ta, decays over the time period of da,
σaμFk[t] a sawtooth curve,
σaμFk[0]=1, jumps at time tn to faμFk, goes back linearly to 1 in the time period of da,
μFk[t]=μFk x σaμFk[t] a sawtooth curve,
μFk[0]=μFk, jumps at time ta to μFk x faμFk, goes back linearly to μFk in the time period of da *)

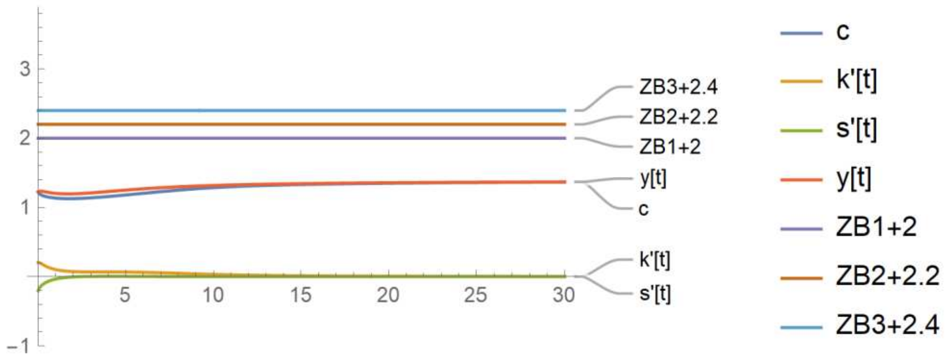

σaμFk[t] == σaμFk0[t] + σaμFk1[t] (t - ta),
σaμFk0[0] == 1, σaμFk1[0] == 0,
WhenEvent[t == ta, {σaμFk0[t] → faμFk, σaμFk1[t] → (1 - faμFk) / da}],
bei Ereignis
WhenEvent[t == ta + da, {σaμFk0[t] → 1, σaμFk1[t] → 0}],
bei Ereignis

(*Supply shock as model shock (technology shock),
Model shock for β[t] at time taβ, decays over the time period of daβ ,
σaβ[t] a sawtooth curve,
σaβ[0]=1, jumps at time taβ to faβ, goes back linearly to 1 in the time period of daβ,
β[t]=β x σaβ[t] a sawtooth curve,
β[0]=β, springt zum Zeitpunkt taβ auf β x faβ, goes back linearly to β in the time period of daβ *)
```

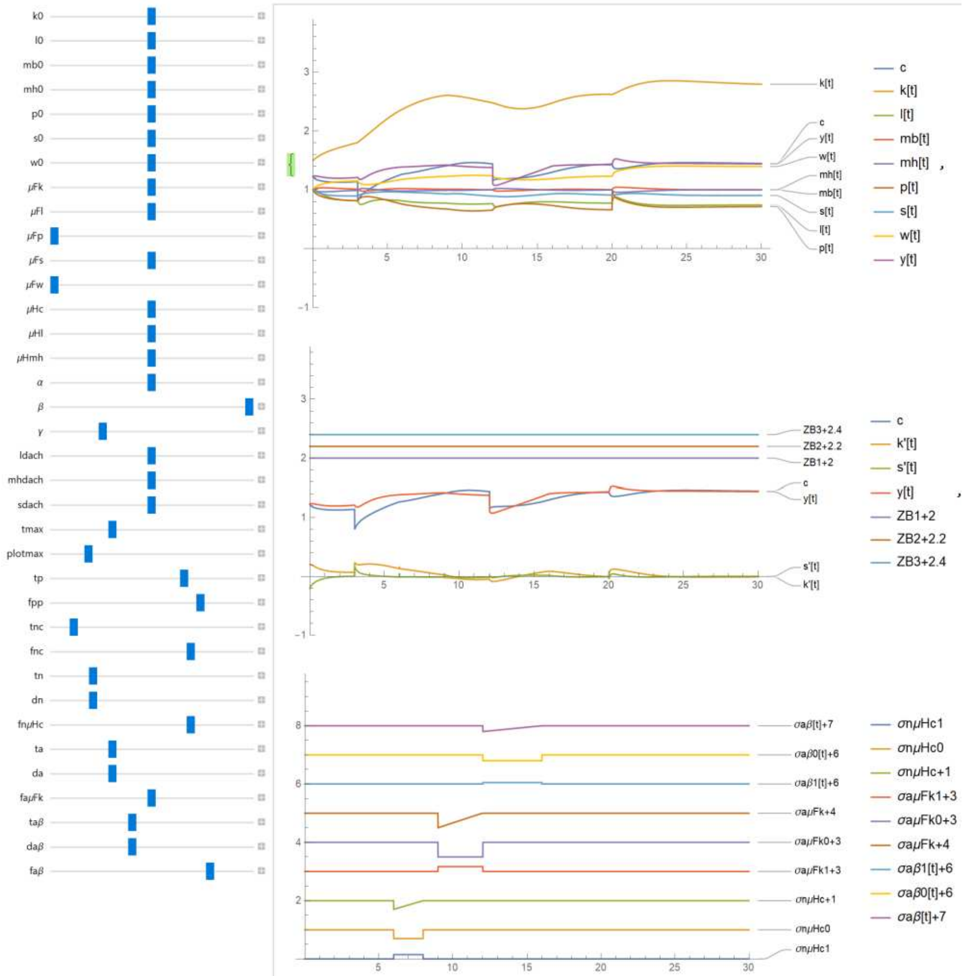
https://www.dropbox.com/s/dn6uxjb0l34qzwo/Modell%20A1schock%20Version%2024.nb?dl=0

With no shocks

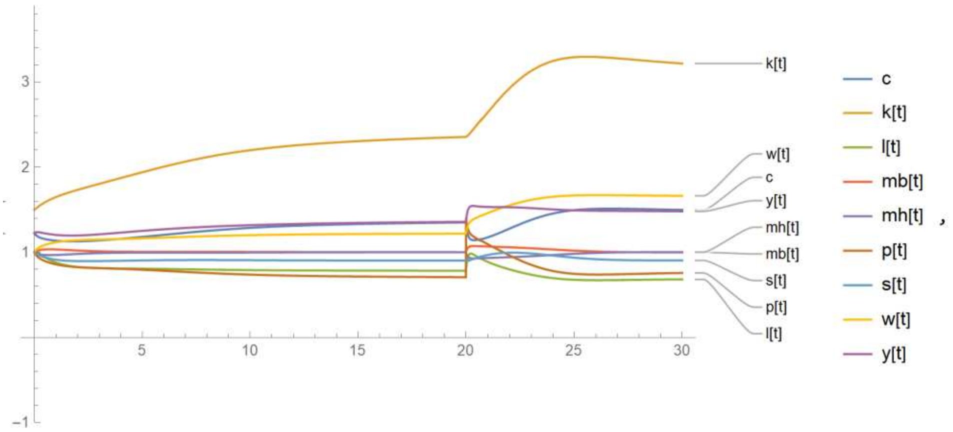

3
2
1
−1
5
10
15
20
25
30
k[t]
y[t]
c
w[t]
mb[t]
mh[t]
s[t]
l[t]
p[t]
c
k[t]
l[t]
mb[t]
mh[t] ,
p[t]
s[t]
w[t]
y[t]

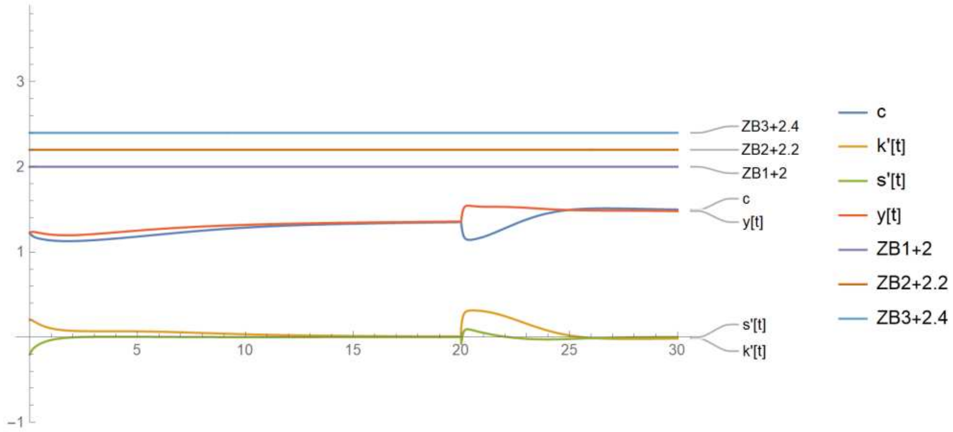

3
2
1
−1
5
10
15
20
25
30
ZB3+2.4
ZB2+2.2
ZB1+2
y[t]
c
k'[t]
s'[t]
c
k'[t]
s'[t]
y[t]
ZB1+2
ZB2+2.2
ZB3+2.4

With multiple shocks



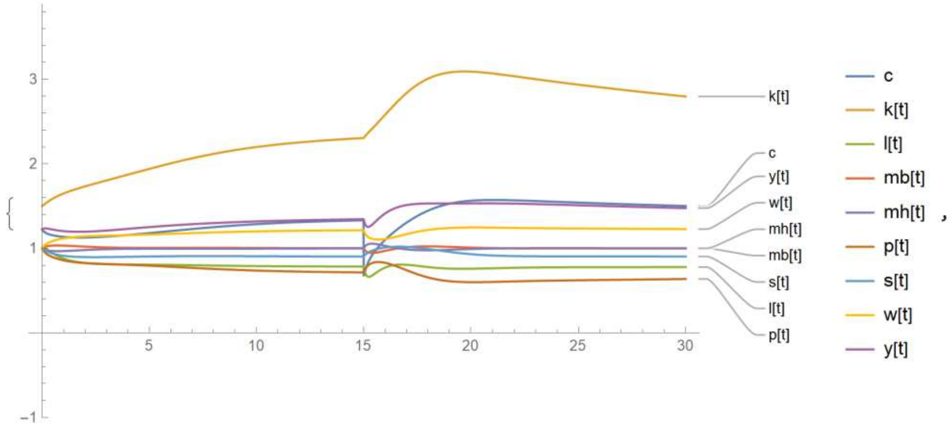
Price shock at $t = 20, \qquad p \to 2\,p$

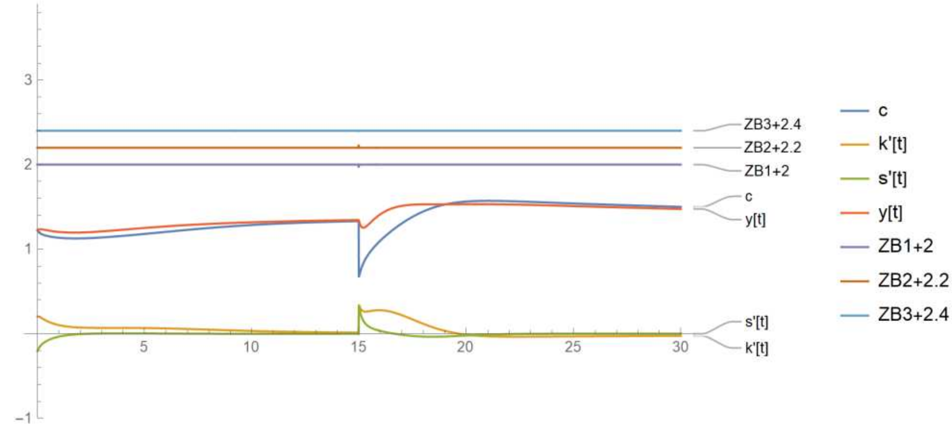




Demand shock due to variable shock (consumption shock)

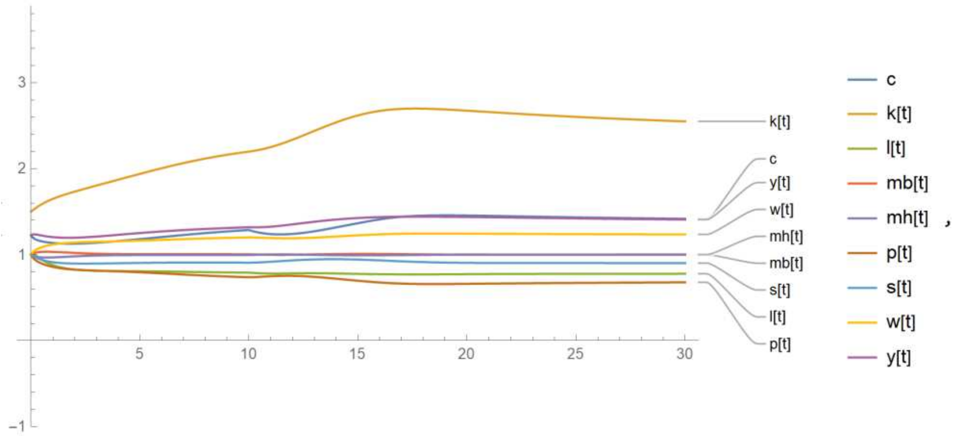
at $t = 15$ $\qquad$ $C \rightarrow 0.5\ C$

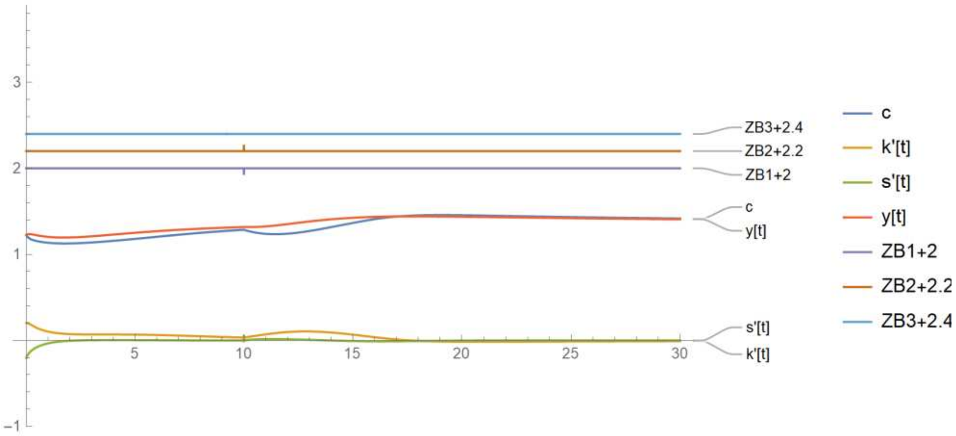




Demand shock due to model shock (shock to power of households)

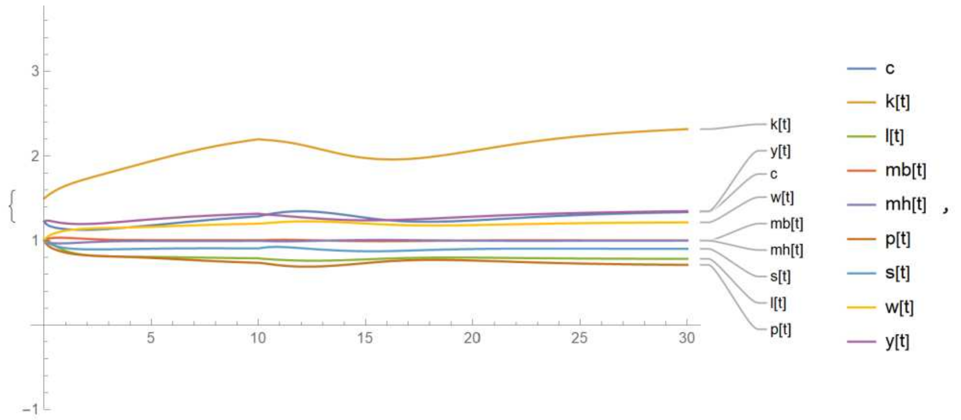
Demand shock $\mu_C^H \to 0.5\mu_C^H$ at $t=10$, thereafter, the power of the households increases again linearly to μ_C^H within the time period $dn=5$

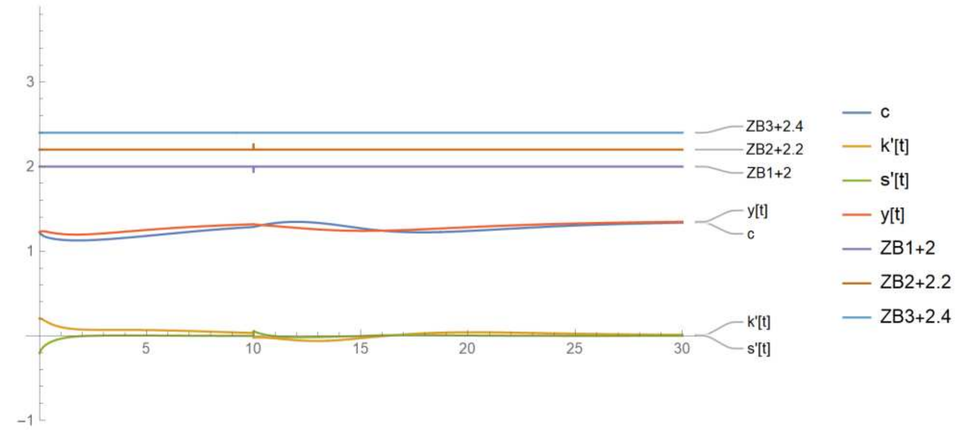


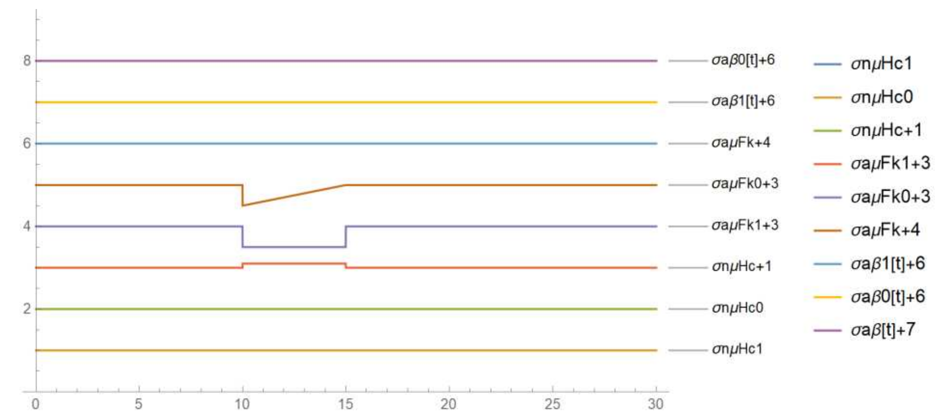


Supply shock due to model shock (shock to power of the firm)

Supply shock $\mu_K^F \rightarrow 0.5\mu_K^F$ at $t=10$, thereafter the power of the firm increases again linearly to μ_K^F within the time period $da=5$

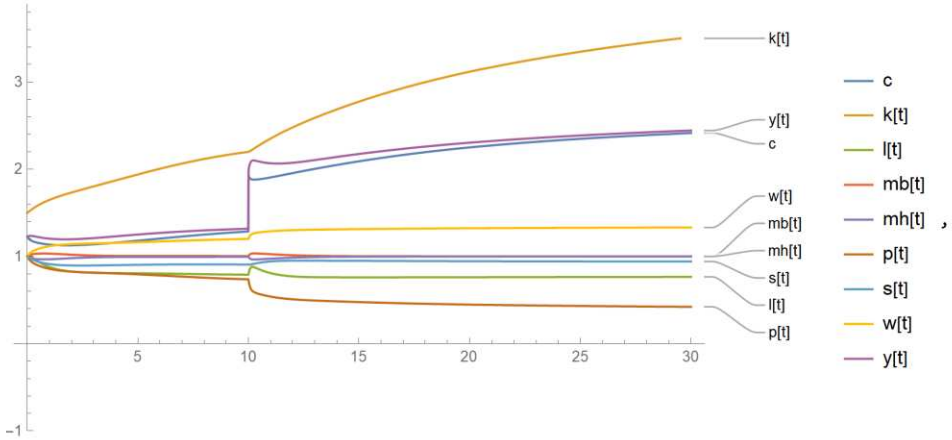


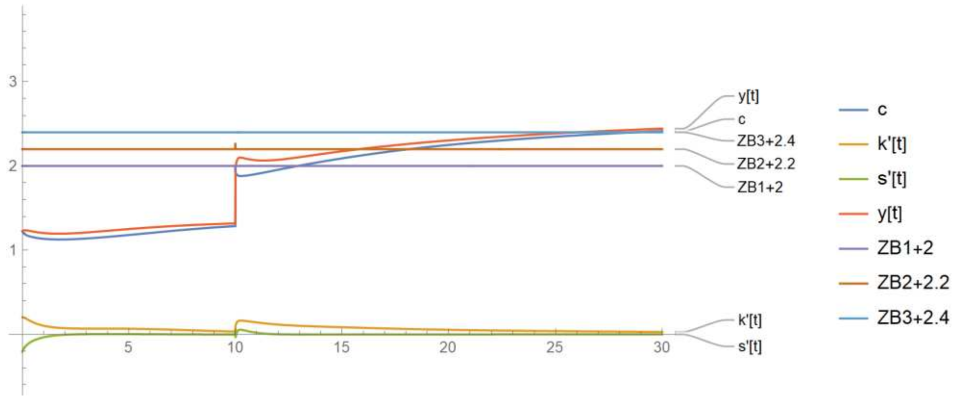


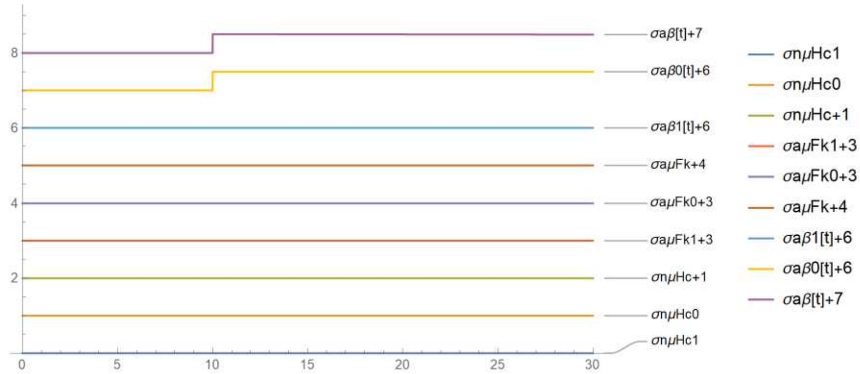


Supply shock due to model shock (technology jump)

Supply shock $\beta \rightarrow 1.5\beta$ at $t=10$ i.e. a technological jump occurs, the resulting increase in productivity is maintained permanently


k[t]
y[t]
c
w[t]
mb[t]
mh[t]
s[t]
l[t]
p[t]
c
k[t]
l[t]
mb[t]
mh[t]
p[t]
s[t]
w[t]
y[t]


y[t]
c
ZB3+2.4
ZB2+2.2
ZB1+2
k'[t]
s'[t]
c
k'[t]
s'[t]
y[t]
ZB1+2
ZB2+2.2
ZB3+2.4


σaβ[t]+7
σaβ0[t]+6
σaβ1[t]+6
σaμFk+4
σaμFk0+3
σaμFk1+3
σnμHc+1
σnμHc0
σnμHc1
σnμHc1
σnμHc0
σnμHc+1
σaμFk1+3
σaμFk0+3
σaμFk+4
σaβ1[t]+6
σaβ0[t]+6
σaβ[t]+7

23. Calculations with model B1 for central bank polices in case of inflation and deflation shock

23.1. Inflation and deflation shock as variable shock for the price

The simplest way to model an inflation respectively deflation shock is to model it as a variable shock for the price as shown in chapter 22.

For example, we use:

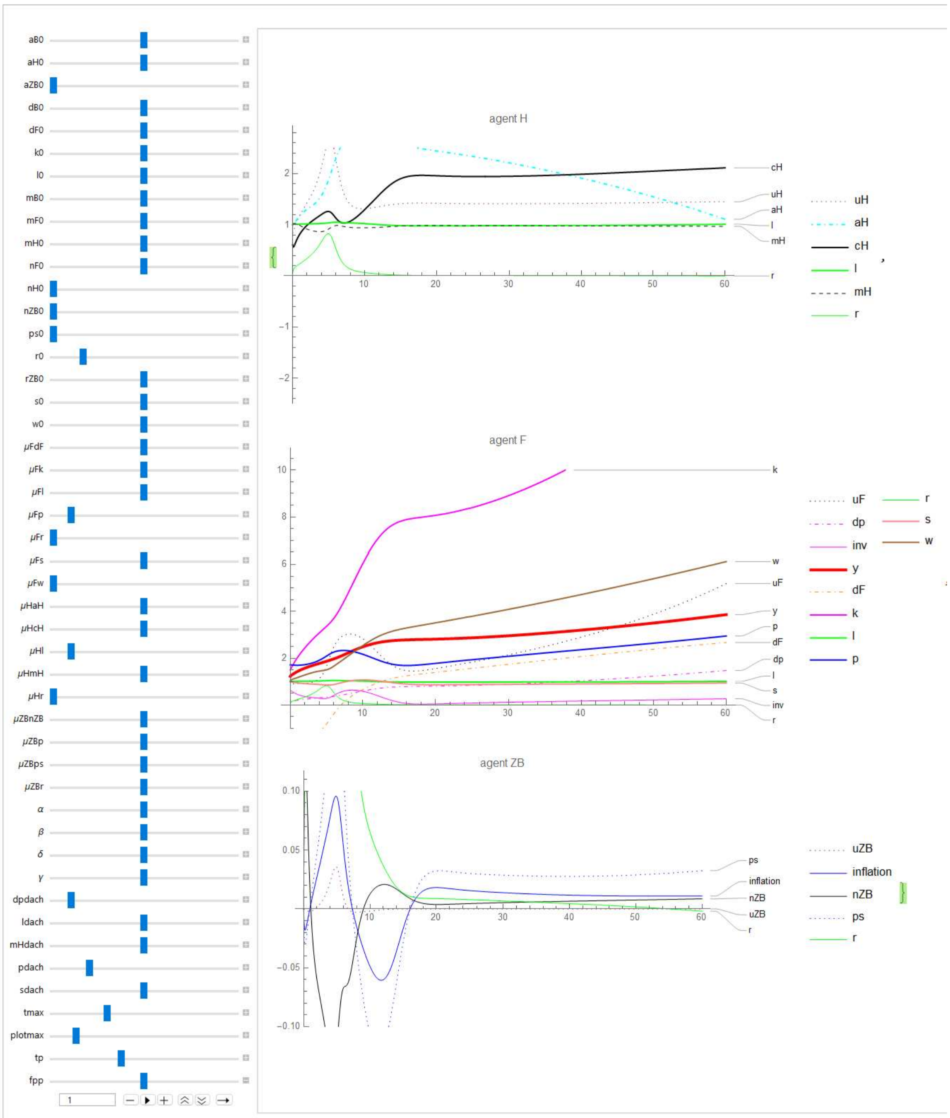
inflation shock: $p \to 1.5\,p$
deflation shock: $p \to 0.5\,p$
at time $t = 20$, because by this time the system has already settled in.

https://www.dropbox.com/s/2k4699r3b0aswsv/Modell%20B1%20SCHOCK%20Version%203.ndsolve.nb?dl=0

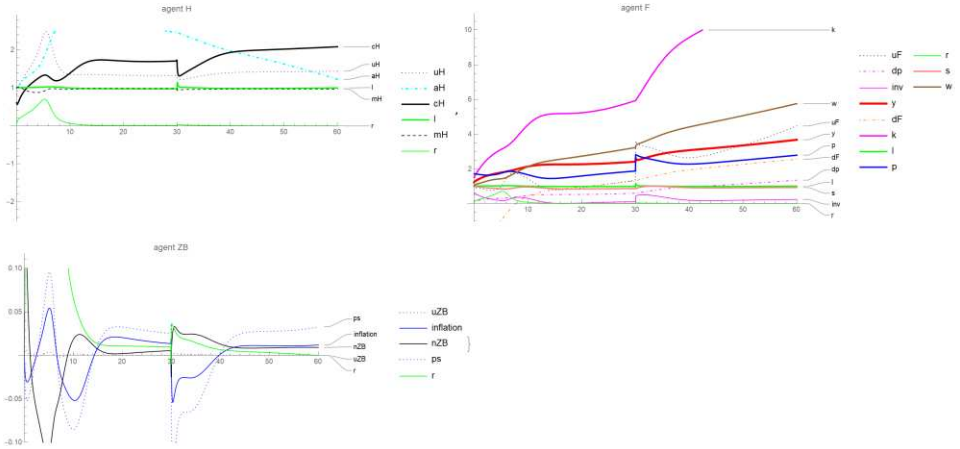
Certainly, one could also interpret these calculations in economic terms. But without prior adjustment of the models to real conditions, a real interpretation is not really serious. Therefore, we will not comment further on the calculated graphs.

As emphasized several times, the target of this book is to present the methodology of the GCD models in principle and to give an idea of what can be done with them and in what form. For application to concrete economic questions, the GCD models still need to be adapted to real conditions. This is one of the tasks that still has to be done in the future.

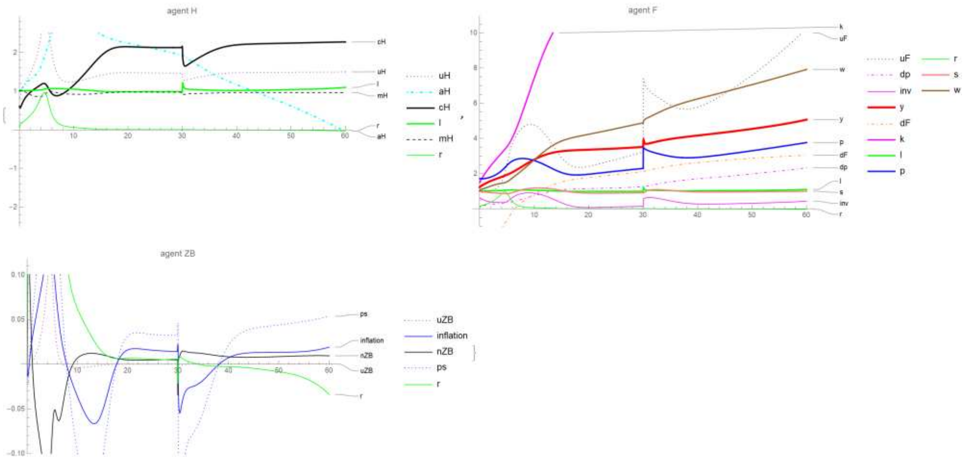
Model B1 without shock as in chapter 13.


aB0
aH0
aZB0
dB0
dF0
k0
l0
mB0
mF0
mH0
nF0
nH0
nZB0
ps0
r0
rZB0
s0
w0
μFdF
μFk
μFl
μFp
μFr
μFs
μFw
μHaH
μHcH
μHl
μHmH
μHr
μZBnZB
μZBp
μZBps
μZBr
α
β
δ
γ
dpdach
ldach
mHdach
pdach
sdach
tmax
plotmax
tp
fpp
agent H
uH
aH
cH
l
mH
r
agent F
uF
dp
inv
y
dF
k
l
p
r
s
w
agent ZB
uZB
inflation
nZB
ps
r

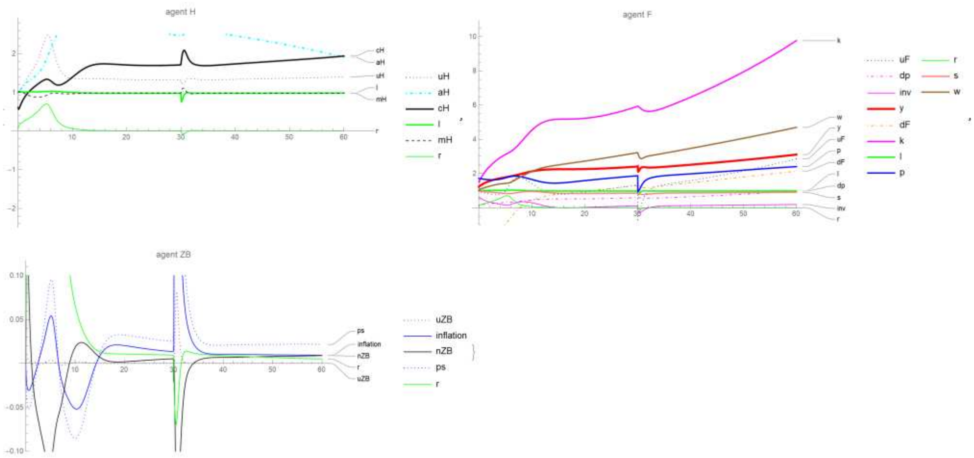
Inflation shock $p \to 1.5\,p$, pure money supply policy of central bank $\delta = 0$



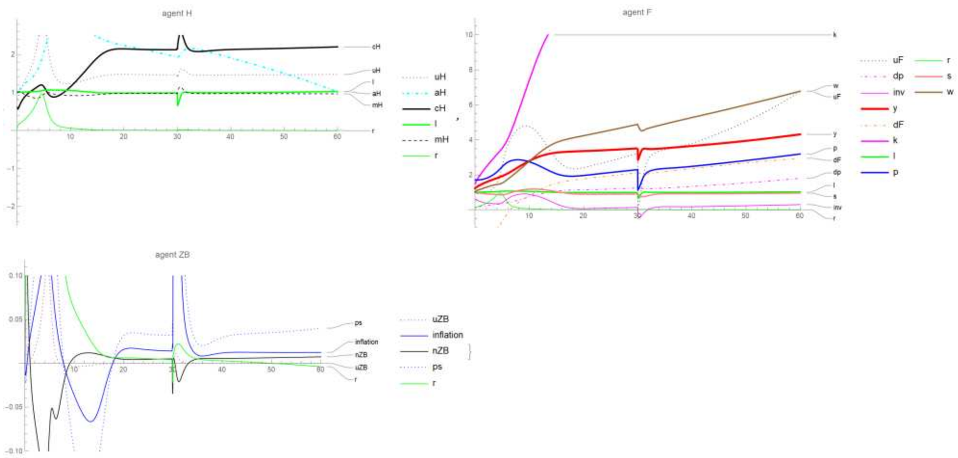
Inflation shock $p \to 1.5\,p$, pure interest policy of central bank $\delta = 1$



deflation shock $p \to 0.5\,p$, pure money supply policy of central bank $\delta = 0$



deflation shock $p \to 0.5\,p$, pure interest policy of central bank $\delta = 1$



23.2. Inflation and deflation shock as model shock

Another possibility to model an inflation or deflation shock would be:

Introduce an agent A who has some power μ_{ps}^{A} to influence the price change $ps = p'$. (Consider e.g. OPEC as agent, which has the intention and the power to influence the trend in oil prices). If A intends to increase the price p at time t_0 for 1 year this leads to an inflation or deflation shock which can be modeled in the following way:

$$U^A = -(ps - ps)^2$$
$$\mu_{ps}^{A}(t) = 0 \qquad for\ t \in [0, t_0]\ and\ \ t > t_0 + 1$$
$$\mu_{ps}^{A}(t) = 1 \qquad for\ t \in [t_0, t_0 + 1]$$

We give this as an example, but do not calculate this model in detail.

Obviously there are a lot of other possibilities to model price respectively inflation or deflation shocks.

E. GCD with intertemporal utility functions (IGCD models)

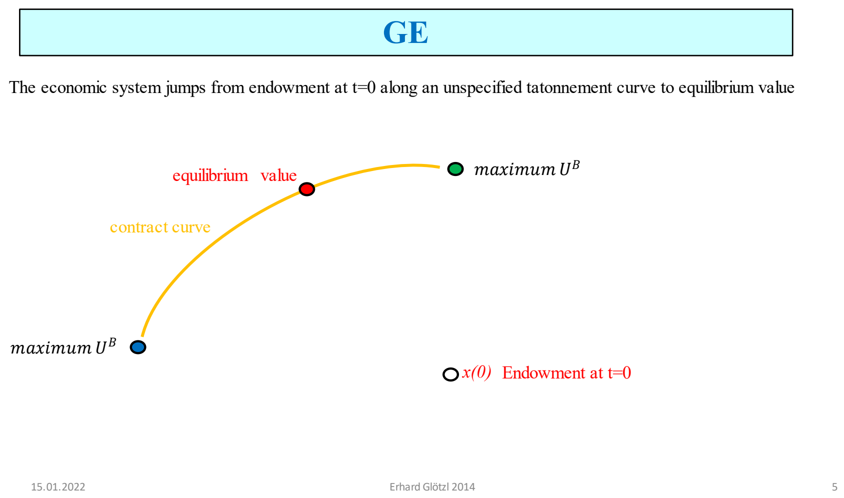
24. IGCD: Intertemporal General Constrained Dynamics

For sake of simplicity most is described for 2 agents A, B and 1 constraint Z .

24.1. Comparison of the basic ideas

24.1.1. GE (for non-intertemporal utility functions)

The economic system jumps from endowment at t = 0 along an unspecified tatonnement curve to equilibrium value as symbolically is shown in the following graphic. (see more in chapter 6).



24.1.2. GCD (for non-intertemporal utility functions)

The basic idea of the **GCD** method for **non-intertemporal** utility functions is that each agent tries to change the variables in the direction in which the change in its individual utility function is maximum at any given time. In other words, every agent tries to change the variables in the direction of the gradient of its individual utility function:

$$\begin{pmatrix} \dfrac{\partial U^A(x_1,x_2)}{\partial x_1} \\ \dfrac{\partial U^A(x_1,x_2)}{\partial x_2} \end{pmatrix} \quad \textit{resp.} \quad \begin{pmatrix} \dfrac{\partial U^B(x_1,x_2)}{\partial x_1} \\ \dfrac{\partial U^B(x_1,x_2)}{\partial x_2} \end{pmatrix}$$

His desire for change is limited by his power to enforce his interest. This is expressed by the power factors $(\mu_{x_1}^A, \mu_{x_1}^B, \mu_{x_2}^A, \mu_{x_2}^B)$. $\mu_{x_1}^A$ describes the power of the agent A to influence the variable x_1 and $\mu_{x_1}^A \frac{\partial U^A(x_1,x_2)}{\partial x_1}$ describes the effective force exerted by the agent on the change of the variable. This results in the effective forces

$$\begin{pmatrix} \mu_{x_1}^A \frac{\partial U^A(x_1,x_2)}{\partial x_1} \\ \mu_{x_2}^A \frac{\partial U^A(x_1,x_2)}{\partial x_2} \end{pmatrix} \quad \textit{resp.} \quad \begin{pmatrix} \mu_{x_1}^B \frac{\partial U^B(x_1,x_2)}{\partial x_1} \\ \mu_{x_2}^B \frac{\partial U^B(x_1,x_2)}{\partial x_2} \end{pmatrix}$$

Since normally the desires and the power of different agents are different, the system develops ex-ante according to the resultant of the two effective forces:

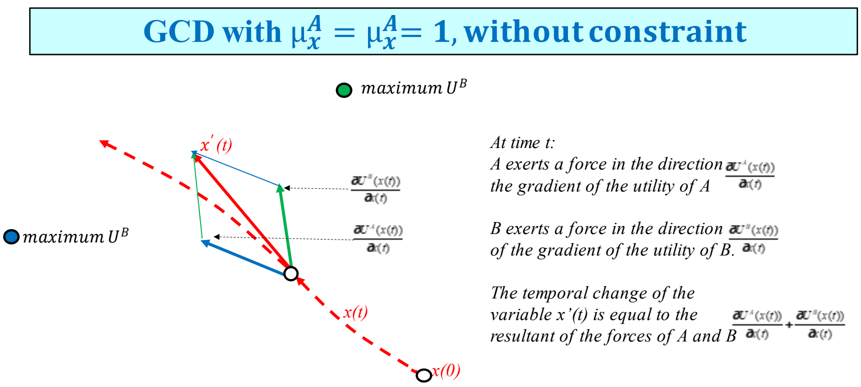
$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \mu_{x_1}^A \frac{\partial U^A(x_1,x_2)}{\partial x_1} \\ \mu_{x_2}^A \frac{\partial U^A(x_1,x_2)}{\partial x_2} \end{pmatrix} + \begin{pmatrix} \mu_{x_1}^B \frac{\partial U^B(x_1,x_2)}{\partial x_1} \\ \mu_{x_2}^B \frac{\partial U^B(x_1,x_2)}{\partial x_2} \end{pmatrix}$$

Considering the constraint Z, we obtain the GCD equation system for the ex-post dynamics:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \mu_{x_1}^A \frac{\partial U^A(x_1,x_2)}{\partial x_1} \\ \mu_{x_2}^A \frac{\partial U^A(x_1,x_2)}{\partial x_2} \end{pmatrix} + \begin{pmatrix} \mu_{x_1}^B \frac{\partial U^B(x_1,x_2)}{\partial x_1} \\ \mu_{x_2}^B \frac{\partial U^B(x_1,x_2)}{\partial x_2} \end{pmatrix} + \lambda \begin{pmatrix} \frac{\partial Z(x_1,x_2)}{\partial x_1} \\ \frac{\partial Z(x_1,x_2)}{\partial x_2} \end{pmatrix}$$

$$0 = Z(x_1, x_2)$$

The dynamics of a GCD model symbolically is shown in the following graphic.



24.1.3. GE for intertemporal utility functions

GE models are characterised by the fact that an objective function is maximised at the time. In GE models, in contrast to GCD models, it must therefore always be assumed that the individual utility functions can be aggregated to a master utility function MU , which then serves as an objective function, because maximisation is only ever possible for one objective function and not for several at the same time. In the case of non-intertemporal GE models, such as the Ramsey model or DSGE models, this objective function is the time integral over a master utility function MU discounted at a discount rate r, which is maximised. The model equations therefore result from the requirement

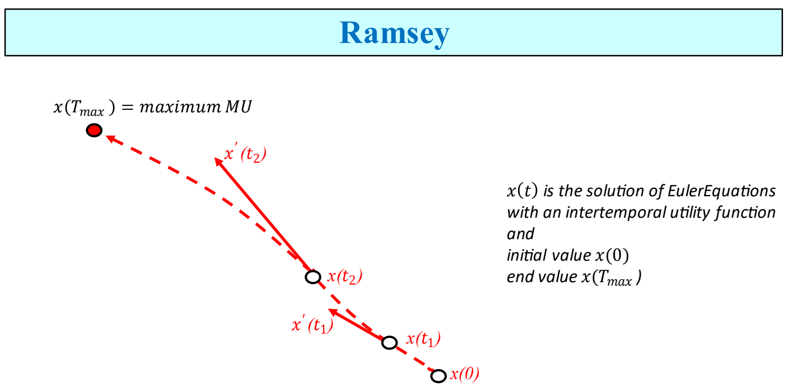
$$U^{int} = \int_0^\infty e^{-r\tau} MU(x_1(\tau), x_2(\tau))\, d\tau \to max$$

or, in the case of a constraint arising from the requirement

$$U^{int} = \int_0^\infty e^{-r\tau} \Big(MU(x_1(\tau), x_2(\tau)) - \lambda(\tau) Z(x_1(\tau), x_2(\tau)) \Big) d\tau \to max$$

These variation problems lead to the Euler-Lagrange equation system for ex-ante respectively ex-post dynamics. This is a differential equation system which the solutions for the intertemporal GE model must fulfill in any case. The Euler-Lagrange equation system thus describes the dynamics of an intertemporal GE model in the same way as the GCD equation system <7.7> does for a GCD model.

The dynamics of the Ramsey model symbolically is shown in the following graphic.



24.1.4. IGCD: GCD with intertemporal utility functions

The basic idea of the **GCD** method for **intertemporal** utility functions is that each agent solves its own variational problem at any given time t . In other words, each agent looks for the solution that maximises its individual intertemporal utility function at the time t :

$$U^{A\,int\,t} = \int_0^\infty e^{-r(t+\tau)} U^A(x_1^{A\,int\,t}(t+\tau), x_2^{A\,int\,t}(t+\tau))\, d\tau \rightarrow max$$

respectively

$$U^{A\,int\,t} = \int_0^\infty e^{-r(t+\tau)} U^B(x_1^{B\,int\,t}(t+\tau), x_2^{B\,int\,t}(t+\tau))\, d\tau \rightarrow max$$

Thus

$$(x_1^{A\,\mathrm{int}\,t}(t+\tau), x_2^{A\,\mathrm{int}\,t}(t+\tau))$$
$$(x_1^{B\,\mathrm{int}\,t}(t+\tau), x_2^{B\,\mathrm{int}\,t}(t+\tau))$$

denotes the solutions of the independent variational problems for A and B at time t, which depends on the future time τ.

In non-intertemporal GCD models the agents try to change the variables in the direction of the gradient of its utility functions,

$$\begin{pmatrix} \dfrac{\partial U^A(x_1,x_2)}{\partial x_1} \\ \dfrac{\partial U^A(x_1,x_2)}{\partial x_2} \end{pmatrix} \quad resp. \quad \begin{pmatrix} \dfrac{\partial U^B(x_1,x_2)}{\partial x_1} \\ \dfrac{\partial U^B(x_1,x_2)}{\partial x_2} \end{pmatrix}$$

which, taking their individual economic powers $\mu_{x_1}^A, \mu_{x_2}^A, \mu_{x_1}^B, \mu_{x_2}^B$ into account, leads to

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \mu_{x_1}^A \dfrac{\partial U^A(x_1,x_2)}{\partial x_1} \\ \mu_{x_2}^A \dfrac{\partial U^A(x_1,x_2)}{\partial x_2} \end{pmatrix} + \begin{pmatrix} \mu_{x_1}^B \dfrac{\partial U^B(x_1,x_2)}{\partial x_1} \\ \mu_{x_2}^B \dfrac{\partial U^B(x_1,x_2)}{\partial x_2} \end{pmatrix}$$

In intertemporal IGCD models the agents try to change the variables in the direction they assume to be optimal for their intertemporal utility, that is just the time derivative of their individual solutions

$$\begin{pmatrix} \left.\dfrac{d\, x_1^{A\,\mathrm{int}\,t}(t+\tau)}{d\tau}\right|_{\tau=0} \\ \left.\dfrac{d\, x_2^{A\,\mathrm{int}\,t}(t+\tau)}{d\tau}\right|_{\tau=0} \end{pmatrix} \quad resp. \quad \begin{pmatrix} \left.\dfrac{d\, x_1^{B\,\mathrm{int}\,t}(t+\tau)}{d\tau}\right|_{\tau=0} \\ \left.\dfrac{d\, x_2^{B\,\mathrm{int}\,t}(t+\tau)}{d\tau}\right|_{\tau=0} \end{pmatrix}$$

Assuming that their power to enforce their interests in such a way is proportional to their relative individual powers

$$\frac{\mu_1^A}{\mu_1^A + \mu_1^B}, \frac{\mu_2^A}{\mu_2^A + \mu_2^B}, \frac{\mu_1^B}{\mu_1^A + \mu_1^B}, \frac{\mu_2^B}{\mu_2^A + \mu_2^B}$$

leads to the **ex-ante IGCD equation (**for intertemporal utility functions**)**

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} \frac{\mu_1^A}{\mu_1^A+\mu_1^B} & \left.\frac{d\,x_1^{A\,\mathrm{int}\,t}(t+\tau)}{d\tau}\right|_{\tau=0} \\ \frac{\mu_2^A}{\mu_2^A+\mu_2^B} & \left.\frac{d\,x_2^{A\,\mathrm{int}\,t}(t+\tau)}{d\tau}\right|_{\tau=0} \end{pmatrix} + \begin{pmatrix} \frac{\mu_1^B}{\mu_1^A+\mu_1^B} & \left.\frac{d\,x_1^{B\,\mathrm{int}\,t}(t+\tau)}{d\tau}\right|_{\tau=0} \\ \frac{\mu_2^B}{\mu_2^A+\mu_2^B} & \left.\frac{d\,x_2^{B\,\mathrm{int}\,t}(t+\tau)}{d\tau}\right|_{\tau=0} \end{pmatrix} \qquad \text{<24.1>}$$

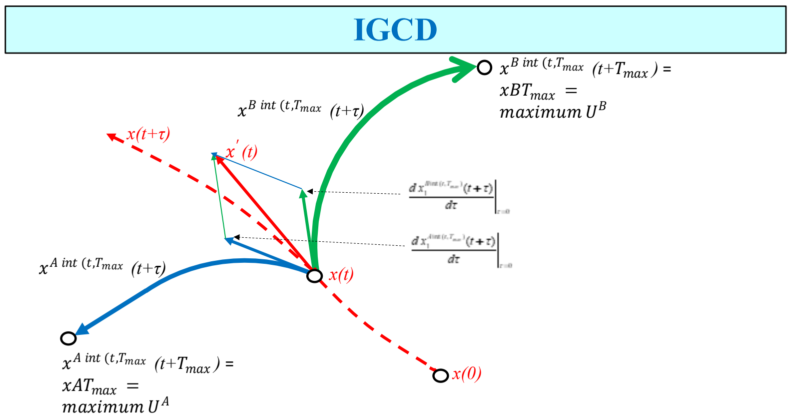
For sake of simplicity, we denote in the following also the solutions of the variational problems with constraint Z by

$$(x_1^{A\,\mathrm{int}\,t,Z}(t+\tau), x_2^{A\,\mathrm{int}\,t,Z}(t+\tau))$$
$$(x_1^{B\,\mathrm{int}\,t,Z}(t+\tau), x_2^{B\,\mathrm{int}\,t,Z}(t+\tau))$$

then the **ex-post IGCD equation (**for intertemporal utility functions**)** reads formally the same as <24.1>

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} \frac{\mu_1^A}{\mu_1^A+\mu_1^B} & \left.\frac{d\,x_1^{A\,\mathrm{int}\,t}(t+\tau)}{d\tau}\right|_{\tau=0} \\ \frac{\mu_2^A}{\mu_2^A+\mu_2^B} & \left.\frac{d\,x_2^{A\,\mathrm{int}\,t}(t+\tau)}{d\tau}\right|_{\tau=0} \end{pmatrix} + \begin{pmatrix} \frac{\mu_1^B}{\mu_1^A+\mu_1^B} & \left.\frac{d\,x_1^{B\,\mathrm{int}\,t}(t+\tau)}{d\tau}\right|_{\tau=0} \\ \frac{\mu_2^B}{\mu_2^A+\mu_2^B} & \left.\frac{d\,x_2^{B\,\mathrm{int}\,t}(t+\tau)}{d\tau}\right|_{\tau=0} \end{pmatrix} \qquad \text{<24.2>}$$

The dynamics of an IGCD model symbolically is shown in the following graphic.



24.2. Definition of IGCD in detail:

For the sake of clarity and simplicity, we rewrite the GCD-system of equations for two agents A,B with the **non-intertemporal** utility functions U^A, U^B, the 2 variables x_1, x_2 and the constraint Z.

$$x_1' = \mu_{x_1}^A \frac{\partial U^A}{\partial x_1} + \mu_{x_1}^B \frac{\partial U^B}{\partial x_1} + \lambda \frac{\partial Z}{\partial x_1}$$
$$x_2' = \mu_{x_{x_2}}^A \frac{\partial U^A}{\partial x_2} + \mu_{x_1}^B \frac{\partial U^B}{\partial x_2} + \lambda \frac{\partial Z}{\partial x_2}$$
$$0 = Z(x_1, x_1', x_2, x_2')$$

Designate T_{max} the end time and for each $t \in [0, T_{max}]$ designate $U^{A\,\mathrm{int}(t,T_{max})}$, $U^{B\,\mathrm{int}(t,T_{max})}$ the intertemporal utility functions of the two agents A, B for optimization in the period from t to T_{max} with discount rates r^A, r^B and describe $x_1(t+\tau), x_2(t+\tau)$ the time evolution of x_1, x_2 as a function of $\tau \in [t, T_{max}]$. The intertemporal utility functions are given by

$$U^{A\,\mathrm{int}(t,T_{max})}(x_1, x_2) =$$
$$= \int_0^{T_{max}} e^{-r^A(t+\tau)} \Big(U^A(x_1(t+\tau), x_2(t+\tau)) - \lambda Z(x_1(t+\tau), x_2(t+\tau)) \Big) d\tau$$
$$U^{B\,\mathrm{int}(t,T_{max})}(x_1, x_2) =$$
$$= \int_0^{T_{max}} e^{-r^B(t+\tau)} \Big(U^B(x_1(t+\tau), x_2(t+\tau)) - \lambda Z(x_1(t+\tau), x_2(t+\tau)) \Big) d\tau$$

At each point in time t, both agents **independently** try to maximise their intertemporal utilities under the constraint Z. The initial conditions must correspond to the values of the variables at the current time. The final condition is chosen by each agent individually according to his individual interest.

$$x_1^{A\,\mathrm{int}(t,T_{max})}(t) = x_1(t) \qquad x_1^{A\,\mathrm{int}(t,T_{max})}(T_{max}) = x1AT_{max}$$
$$x_2^{A\,\mathrm{int}(t,T_{max})}(t) = x_2(t) \qquad x_2^{A\,\mathrm{int}(t,T_{max})}(T_{max}) = x2AT_{max}$$

$$x_1^{B\,\mathrm{int}(t,T_{max})}(t) = x_1(t) \qquad x_1^{B\,\mathrm{int}(t,T_{max})}(T_{max}) = x1BT_{max}$$
$$x_2^{B\,\mathrm{int}(t,T_{max})}(t) = x_2(t) \qquad x_2^{B\,\mathrm{int}(t,T_{max})}(T_{max}) = x2BT_{max}$$

This gives for each fixed point in time t and for each agent for the period of time from t until T_{max} the intertemporal optimal solutions which are designated by

$$x_1^{A\,\mathrm{int}(t,T_{max})}(t+\tau),\ x_2^{A\,\mathrm{int}(t,T_{max})}(t+\tau)$$

respectively

$$x_1^{B\,\mathrm{int}(t,T_{max})}(t+\tau),\ x_2^{B\,\mathrm{int}(t,T_{max})}(t+\tau).$$

This solutions result from the Euler equations[3] of the two variation problems with constraints and with the corresponding initial and final conditions and thus for each fixed t and T_{max} are functions of $\tau \in [t, T_{max}]$:

[3] Be careful: to use “EulerEquations” in Mathematica correctly, one has to define $\vec{x}^t(\tau) := x(t+\tau)$ and use $\vec{x}^t(\tau)$ instead of $x(t+\tau)$

$$\text{EulerEquations}[e^{-r^A(t+\tau)}U^A(x_1^{A\,\text{int}(t,T_{max})}(t+\tau),$$
$$x_2^{A\,\text{int}(t,T_{max})}(t+\tau))+\lambda Z(x_1^{A\,\text{int}(t,T_{max})}(t+\tau),x_2^{A\,\text{int}(t,T_{max})}(t+\tau)),$$
$$\left\{x_1^{A\,\text{int}(t,T_{max})}(t+\tau),x_2^{A\,\text{int}(t,T_{max})}(t+\tau)\right\},\tau]$$

with intial and end values

$$x_1^{A\,\text{int}(t,T_{max})}(t)=x_1(t) \qquad x_1^{A\,\text{int}(t,T_{max})}(T_{max})=x1AT_{max}$$
$$x_2^{A\,\text{int}(t,T_{max})}(t)=x_2(t) \qquad x_2^{A\,\text{int}(t,T_{max})}(T_{max})=x2AT_{max}$$

$$\text{EulerEquations }[e^{-r^B(t+\tau)}U^B(x_1^{B\,\text{int}(t,T_{max})}(t+\tau),$$
$$x_2^{B\,\text{int}(t,T_{max})}(t+\tau))+\lambda Z(x_1^{B\,\text{int}(t,T_{max})}(t+\tau),x_2^{B\,\text{int}(t,T_{max})}(t+\tau)),$$
$$\left\{x_1^{B\,\text{int}(t,T_{max})}(t+\tau),x_2^{B\,\text{int}(t,T_{max})}(t+\tau)\right\},\tau]$$

with intial and end values

$$x_1^{B\,\text{int}(t,T_{max})}(t)=x_1(t) \qquad x_1^{B\,\text{int}(t,T_{max})}(T_{max})=x1BT_{max}$$
$$x_2^{B\,\text{int}(t,T_{max})}(t)=x_2(t) \qquad x_2^{B\,\text{int}(t,T_{max})}(T_{max})=x2BT_{max}$$

Typically, the constraint does not depend on $x_2'(t)$, i.e.

$$0=Z(x_1(t),x_1'(t),x_2(t))$$

and the variable $x_2(t)$ can be expressed as a function of $x_1(t)$ and inserted into the utility function. This is what we will always assume in the following, because this simplifies the problem considerably. This is explained using the Ramsey model as an example (see chapters 25.1, 25.2). It leads to the fact that the Lagrange multiplier $\lambda(t)$ drops out and the variational problem with constraint is simplified to a variational problem without constraint and the utility function only depends on $x_1(t)$,. The variational problem to be solved is then

$$\text{EulerEquations }[e^{-r^A(t+\tau)}\,U^A(x_1^{A\,\text{int}(t,T_{max})}(t+\tau)),$$
$$\{x_1^{A\,\text{int}(t,T_{max})}(t+\tau)\},\tau]$$

with initial and end values

$$x_1^{A\,\text{int}(t,T_{max})}(t)=x_1(t) \qquad x_1^{A\,\text{int}(t,T_{max})}(T_{max})=x1HT_{max}$$

$$\text{EulerEquations }[e^{-r^B(t+\tau)}\,U^B(x_1^{B\,\text{int}(t,T_{max})}(t+\tau)),$$
$$\{x_1^{B\,\text{int}(t,T_{max})}(t+\tau)\},\tau]$$

with initial and end values

$$x_1^{B\,\text{int}(t,T_{max})}(t)=x_1(t) \qquad x_1^{B\,\text{int}(t,T_{max})}(T_{max})=x1BT_{max}$$

The end values can be selected freely.

Assuming the trajectory of x_1 until t is $x_1(s)$ with $s<t$.

In order to follow its optimal path for the future, agent A must try to set the temporal change at time t equal to the temporal change of its intertemporal maximised trajectory, i.e.

$$x_1'(t)=\left.\frac{d\,x_1^{A\,\text{int}(t,T_{max})}(t+\tau)}{d\tau}\right|_{\tau=0}$$

But also, the agent B must try to set the temporal change $x_1(t)$ equal to the temporal change of his intertemporal maximized course, i.e.

$$x_1'(t) = \left.\frac{d\,x_1^{B\,\mathrm{int}(t,T_{max})}(t+\tau)}{d\tau}\right|_{\tau=0}$$

But these two wishes cannot both be fulfilled at the same time. The actual temporal change of $x_1(t)$ at the time t therefore results in retrospect on the one hand as a mixture of the wishes of A and B (weighted with their relative power relations) and on the other hand from the fact that the constraint at the time must also be fulfilled. This results in

$$\begin{aligned} x_1'(t) = {} & \frac{\mu_1^A}{\mu_1^A + \mu_1^B} \left.\frac{d\,x_1^{A\,\mathrm{int}(t,T_{max})}(t+\tau)}{d\tau}\right|_{\tau=0} + \\ & + \frac{\mu_1^B}{\mu_1^A + \mu_1^B} \left.\frac{d\,x_1^{B\,\mathrm{int}(t,T_{max})}(t+\tau)}{d\tau}\right|_{\tau=0} + \lambda(t)\frac{\partial Z(x_1(t), x_2(t))}{\partial x_1(t)} \end{aligned}$$

Since we have assumed the simplifying case and expressed $x_2(t)$ through $x_1(t)$, the constraint is always fulfilled and the last term falls away. This results in

$$x_1'(t) = \frac{\mu_1^A}{\mu_1^A + \mu_1^B} \left.\frac{d\,x_1^{A\,\mathrm{int}(t,T_{max})}(t+\tau)}{d\tau}\right|_{\tau=0} + \frac{\mu_1^B}{\mu_1^A + \mu_1^B} \left.\frac{d\,x_1^{B\,\mathrm{int}(t,T_{max})}(t+\tau)}{d\tau}\right|_{\tau=0}$$

This equation describes the temporal behaviour of x_1 as function of t . The initial condition $x_1(0)$ results from the model assumptions for the time $t = 0$.

Thus, taking into account the final values $x1AT_{max}, x1BT_{max}$ assumed by the agents for their variational problem and the initial value $x_1(0) = x10$, the following **IGCD** (intertemporal GCD) **equation system** results:

$$
\begin{array}{l}
\textit{behavioural equation for } x_1(t) \\
x_1'(t) = \dfrac{\mu_1^A}{\mu_1^A + \mu_1^B} \left. \dfrac{d\, x_1^{A\,\mathrm{int}(t,T_{max})}(t+\tau)}{d\tau} \right|_{\tau=0} + \dfrac{\mu_1^B}{\mu_1^A + \mu_1^B} \left. \dfrac{d\, x_1^{B\,\mathrm{int}(t,T_{max})}(t+\tau)}{d\tau} \right|_{\tau=0} \\
\textit{initial value for } x_1(t) \\
x_1(0) = x10 \\
\\
\textit{Euler equations for } x_1^{A\,\mathrm{int}(t,T_{max})}(\tau) \textit{ for A with initial and final values} \\
\mathrm{EulerEquations}\left[e^{-r^A(t+\tau)} U^A(x_1^{H\,\mathrm{int}(t,T_{max})}(t+\tau)), \left\{ x_1^{H\,\mathrm{int}(t,T_{max})}(t+\tau) \right\}, \tau \right] \\
x_1^{A\,\mathrm{int}(t,T_{max})}(t) = x_1(t) \qquad\qquad x_1^{A\,\mathrm{int}(t,T_{max})}(T_{max}) = x1AT_{max} \\
\textit{Euler equations for } x_1^{B\,\mathrm{int}(t,T_{max})}(\tau) \textit{ for B with initial and final values} \\
\mathrm{EulerEquations}\left[e^{-r^B(t+\tau)} U^B(x_1^{B\,\mathrm{int}(t,T_{max})}(t+\tau)), \left\{ x_1^{B\,\mathrm{int}(t,T_{max})}(t+\tau) \right\}, \tau \right] \\
x_1^{B\,\mathrm{int}(t,T_{max})}(t) = x_1(t) \qquad\qquad x_1^{B\,\mathrm{int}(t,T_{max})}(T_{max}) = x1BT_{max}
\end{array}
\qquad \text{<24.3>}
$$

With n variables and m constraints, the number of variables is reduced to $k = n - m$ variables $x_1, x_2, ..., x_k$ respectively $x_i,\ i = 1,2,...,k$. This results in the **IGCD (**intertemporal GCD) **equation system for 2 agents and** k **variables**

$$
\begin{array}{l}
\textit{for all } i = 1,2,...,k \\
\textit{behavioural equations for } x_1(t) \\
x_i'(t) = \dfrac{\mu_i^A}{\mu_i^A + \mu_i^B} \left. \dfrac{d\, x_i^{A\,\mathrm{int}(t,T_{max})}(t+\tau)}{d\tau} \right|_{\tau=0} + \dfrac{\mu_i^B}{\mu_i^A + \mu_i^B} \left. \dfrac{d\, x_i^{B\,\mathrm{int}(t,T_{max})}(t+\tau)}{d\tau} \right|_{\tau=0} \\
\textit{initial values for } x_i(t) \\
x_i(0) = xi0 \\
\\
\textit{Euler equations for } x_i^{A\,\mathrm{int}(t,T_{max})}(\tau) \textit{ for A with initial and final values} \\
\mathrm{EulerEquations}\ [e^{-r^A(t+\tau)} U^A(x_1^{A\,\mathrm{int}(t,T_{max})}(t+\tau), ..., x_k^{A\,\mathrm{int}\,T_{max}}(t+\tau)), \\
\qquad\qquad \left\{ (x_1^{A\,\mathrm{int}\,T_{max}}(t+\tau), ..., x_k^{A\,\mathrm{int}\,T_{max}}(t+\tau)) \right\}, \tau] \\
x_i^{A\,\mathrm{int}(t,T_{max})}(t) = x_i(t) \qquad\qquad x_i^{A\,\mathrm{int}(t,T_{max})}(T_{max}) = xiAT_{max} \\
\textit{Euler equations for } x_i^{B\,\mathrm{int}(t,T_{max})}(\tau) \textit{ for B with initial and final values} \\
\mathrm{EulerEquations}\ [e^{-r^B(t+\tau)} U^B(x_1^{B\,\mathrm{int}(t,T_{max})}(\tau), ..., x_k^{B\,\mathrm{int}\,T_{max}}(\tau)), \\
\qquad\qquad \left\{ (x_1^{B\,\mathrm{int}\,T_{max}}(\tau), ..., x_k^{B\,\mathrm{int}\,T_{max}}(\tau)) \right\}, \tau] \\
x_i^{B\,\mathrm{int}(t,T_{max})}(t) = x_i(t) \qquad\qquad x_i^{B\,\mathrm{int}(t,T_{max})}(T_{max}) = xiBT_{max}
\end{array}
\qquad \text{<24.4>}
$$

Note

(a) Up to now we have set fixed end values for the end time T_{max} for intertemporal optimisation. For other end conditions (e.g. "free" or "greater than") these conditions can be replaced by the corresponding so-called transversality conditions.

(b) The intertemporal optimisation for infinite time intervals can be approximated by large T_{max} .

24.3. Numerical solution

The differential equation systems <24.3> and <24.4> cannot be solved directly with NDSolve from Mathematica. For the numerical solution the interval $\left[0,T_{max}\right]$ must be divided into N intervals with the points in time $t_0 = 0, t_1, t_2, ..., t_N = T_{max}$. Proceed step by step as follows:

(1) Solve the Euler equations for the interval $\left[0,T_{max}\right]$ with initial and final values

$$x_i^{A\,\mathrm{int}\,(t_0,T_{max})}(t_0) = xi0 \qquad x_i^{A\,\mathrm{int}\,(t_0,T_{max})}(T_{max}) = xiAT_{max}$$
$$x_i^{B\,\mathrm{int}\,(t_0,T_{max})}(t_0) = xi0 \qquad x_i^{B\,\mathrm{int}\,(t_0,T_{max})}(T_{max}) = xiBT_{max}$$

(2) Calculate $x_i'(t_0)$

$$x_i'(t_0) = \frac{\mu_i^A}{\mu_i^A + \mu_i^B} \left.\frac{d\, x_i^{A\,\mathrm{int}\,(t_0,T_{max})}(t_0+\tau)}{d\tau}\right|_{\tau=0} + $$
$$+ \frac{\mu_i^B}{\mu_i^A + \mu_i^B} \left.\frac{d\, x_i^{B\,\mathrm{int}\,(t_0,T_{max})}(t_0+\tau)}{d\tau}\right|_{\tau=0}$$

(3) Calculate $x_i(t_1)$

either as a linear approximation:

$$x_i(t_1) = x_i(t_0) + x_i'(t_0)(t_1 - t_0)$$

or as an exponential approximation:

$$x_i(t_1) = x_i(t_0)\; e^{x_i'(t_0)\,(t_1 - t_0)}$$

(4) Solve the Euler equations for the interval $\left[t_1,T_{max}\right]$ with initial or final values

$$x_i^{A\,\mathrm{int}\,(t_1,T_{max})}(t_1) = x_i(t_1) \qquad x_i^{A\,\mathrm{int}\,(t_1,T_{max})}(T_{max}) = xiAT_{max}$$
$$x_i^{B\,\mathrm{int}\,(t_1,T_{max})}(t_1) = x_i(t_1) \qquad x_i^{B\,\mathrm{int}\,(t_1,T_{max})}(T_{max}) = xiBT_{max}$$

(5) Calculate $x_i'(t_1)$

$$x_i'(t_1) = \frac{\mu_i^A}{\mu_i^A + \mu_i^B} \left. \frac{d\, x_i^{A\,\mathrm{int}\,(t_1, T_{max})}(t_1 + \tau)}{d\tau} \right|_{\tau=0} + $$
$$+ \frac{\mu_i^B}{\mu_i^A + \mu_i^B} \left. \frac{d\, x_i^{B\,\mathrm{int}\,(t_1, T_{max})}(t_1 + \tau)}{d\tau} \right|_{\tau=0}$$

(6) Calculate $x_i(t_2)$

either as a linear approximation:

$$x_i(t_2) = x_i(t_1) + x_i'(t_1)(t_2 - t_1)$$

or as an exponential approximation:

$$x_i(t_2) = x_i(t_1)\, e^{x_i'(t_1)\,(t_2 - t_1)}$$

(7) Solve the Euler equations for the interval $\left[t_2, T_{max}\right]$ with initial or final values

$$x_i^{A\,\mathrm{int}\,(t_2, T_{max})}(t_2) = x_i(t_2) \qquad x_i^{A\,\mathrm{int}\,(t_2, T_{max})}(T_{max}) = xiAT_{max}$$
$$x_i^{B\,\mathrm{int}\,(t_2, T_{max})}(t_2) = x_i(t_2) \qquad x_i^{B\,\mathrm{int}\,(t_2, T_{max})}(T_{max}) = xiBT_{max}$$

(8) Etc.

24.4. The relationship between the dynamics of GCD models (with non-intertemporal utility functions) and the dynamics of GE models with intertemporal utility functions

24.4.1. Basic principles

In simplified terms, non-intertemporal GCD models behave at $t=0$ the same as intertemporal GE models, in which the future is increasingly devalued by shortening the optimisation period. It should be noted that intertemporal GE models require that the utility functions can be aggregated. Therefore, the relationship between these two models can only be established for utility functions that can be aggregated. For simplicity, we describe everything for 2 agents A,B, 2 goods (x_1,x_2) and 1 constraint

$$Z(x_1,x_2)=0$$

2 utility functions U^A,U^B are called aggregable if there is a utility function MU so that (see chapter 4.3)

$$\mu_{x_1}^H \frac{\partial U^H}{\partial x_1} + \mu_{x_1}^B \frac{\partial U^B}{\partial x_1} = \frac{\partial MU}{\partial x_1}$$
$$\mu_{x_2}^H \frac{\partial U^H}{\partial x_2} + \mu_{x_2}^B \frac{\partial U^B}{\partial x_2} = \frac{\partial MU}{\partial x_2}$$

The non-intertemporal GCD model is described ex-ante (i.e. without considering the constraints) by

$$x_1' = \frac{\partial MU}{\partial x_1}$$
$$x_2' = \frac{\partial MU}{\partial x_2}$$

and ex-post (i.e. taking into account the constraints) described by

$$x_1' = \frac{\partial MU}{\partial x_1} + \lambda \frac{\partial Z}{\partial x_1}$$
$$x_2' = \frac{\partial MU}{\partial x_2} + \lambda \frac{\partial Z}{\partial x_2}$$
$$0 = Z(x_1,x_2)$$

The GE non-intertemporal model is described ex-ante (i.e. without considering the constraints) by:

$$\int_0^{T_{max}} MU(x_1(t),x_2(t))\,dt \rightarrow max$$

and ex-post (i.e. with consideration of the constraint) described by

$$\int_0^{T_{max}} \left(MU(x_1(t),x_2(t)) + \lambda(t) Z(x_1(t),x_2(t)) \right) dt \to max$$

A necessary condition that must be fulfilled by x_1, x_2 such that the integrals become maximum are the Euler-Lagrange equations.

24.4.2. A non-intertemporal GCD model behaves at time t = 0 in the same way as an intertemporal GE model with a very short optimisation interval

Looking at the ex-ante behaviour of a GE model with a non-intertemporal utility function, it follows

$$\int_0^{T_{max}} e^{-rt} MU(x_1(t),x_2(t))\, dt \to max$$

Assume that $T_{\max}$ and r are very small. If one carries out a series expansion of $e^{-r\tau}$ and MU with respect to t at point $t=0$ one obtains the following

$$\int_0^{T_{max}} e^{-rt} MU(x_1(t),x_2(t))\, dt =$$

$$= \int_0^{T_{max}} (1 - rt + ...) \left(MU(x_1(0),x_2(0)) + \left.\frac{d\,MU}{dt}\right|_{t=0} .t + \frac{1}{2} \left.\frac{d^2 MU}{dt^2}\right|_{t=0} .t^2 + ... \right) dt \approx$$

$$\approx \int_0^{T_{max}} \left(MU(x_1(0),x_2(0)) + \left.\frac{d\,MU}{dt}\right|_{t=0} .t + ... \right) dt$$

because of the assumption r is small and for small T_{max} t is small

$$= T_{max}\, MU(x_1(0),x_2(0)) + \int_0^{T_{max}} \left.\frac{\partial MU(x_1(t),x_2(t))}{\partial x_1(t)}\right|_{x_1(t)=x_1(0)} \left.\frac{d\,x_1(t)}{dt}\right|_{t=0} .t\, dt +$$

$$+ \int_0^{T_{max}} \left.\frac{\partial MU(x_1(t),x_2(t))}{\partial x_2(t)}\right|_{x_2(t)=x_2(0)} \left.\frac{d\,x_2(t)}{dt}\right|_{t=0} .t\, dt + ...$$

$$\approx T_{max}\, MU(x_1(0),x_2(0)) + \left(\frac{\partial MU}{\partial x_1} x_1'\Big|_{t=0} + \frac{\partial MU}{\partial x_2} x_2'\Big|_{t=0} \right) \frac{T_{max}{}^2}{2} \qquad \textit{for small } T_{max}$$

The first term is constant, the second term becomes maximal exactly when the vector $(\frac{\partial MU}{\partial x_1}, \frac{\partial MU}{\partial x_2})$ and the vector (x_1', x_2') at the time $t=0$ point in the same direction, i.e. there is a $\mu \in \mathbb{R}$ such that

$$\left.\begin{pmatrix} x_1' \\ x_2' \end{pmatrix}\right|_{t=0} = \mu \left.\begin{pmatrix} \dfrac{\partial MU}{\partial x_1} \\ \dfrac{\partial MU}{\partial x_2} \end{pmatrix}\right|_{t=0} = \left.\begin{pmatrix} \dfrac{\partial \mu MU}{\partial x_1} \\ \dfrac{\partial \mu MU}{\partial x_2} \end{pmatrix}\right|_{t=0}$$

This means that it applies to small r and small T_{max}:

A GE model with an intertemporal utility function

$$U^{\text{int}(0,T_{\max})} = \int_0^{T_{max}} e^{-rt} MU(x_1(t), x_2(t))\, dt$$

behaves at the time $t = 0$ ex-ante (i.e. without considering the constraint) similar to a non-intertemporal GCD model with the utility function μMU.

25. The principles of IGCD are first presented using the Ramsey model as an example

25.1. The Ramsey model

The standard Ramsey model consists of 1 agent (household) that attempts to maximize the intertemporal utility of consumption C over the period from $t_0 = 0 \; to \; T_{\max}$. The utility function U^A is

$$U^A(C(t)) = C(t)^\gamma \qquad\qquad 0 \le \gamma \le 1$$

The constraint Z is given by

$$Z(C(t), K(t), K'(t)) = K(t)^{(1-\alpha)} - K'(t) - C(t) = 0$$

The intertemporal utility function of the household $U^{A\,\mathrm{int}}$ is given by

$$U^{A\,\mathrm{int}(0,T_{\max})}(C) = \int_0^{T_{\max}} e^{-r\,\tau}\, U^A(C(\tau))d\tau = \int_0^{T_{\max}} e^{-r\,\tau}\, C(\tau)^\gamma\, d\tau$$

Calculate $C(t)$ from the constraint and insert in U^A . This results in the variation problem

$$U^{A\,\mathrm{int}(0,T_{\max})}(K) = \int_0^{T_{\max}} e^{-r\,\tau} (K(\tau))^{(1-\alpha)} - K'(\tau))^\gamma\, d\tau \to \max$$

$$with\; K(0) = k0 \qquad\qquad K(T_{\max}) = kT_{\max}$$

The solution is obtained by solving the Euler equation with initial and final values:

$$\text{EulerEquations}\left[e^{-r\tau}(K(\tau)^{(1-\alpha)} - K'(\tau))^\gamma, \{K(\tau)\}, \tau\right]$$

$$with\; K(0) = k0 \qquad\qquad K(T_{\max}) = kT_{\max}$$

which result in the differential equation system to be solved

$$\begin{aligned} &0 = (-1+\alpha)K(\tau) + r\,K'(\tau)^{(1+\alpha)} + (-1+\alpha)(-2+\gamma)K(\tau)^\alpha K'(\tau) + \\ &\qquad + K(\tau)^{2\alpha}(-r\,K'(\tau) + (-1+\gamma)K''(\tau)) \\ &K(0) = k0 \\ &K(T_{\max}) = kT_{\max} \end{aligned} \qquad \text{<25.1>}$$

25.2. The Ramsey model (modeled with Lagrange function with constraint)

To model the standard Ramsey model, you can also use the Lagrange function with constraint and proceed as follows:

As before the standard Ramsey model consists of 1 agent (household) that attempts to maximize the intertemporal utility of consumption C over the period from $t_0 = 0$ to $T_{\max}$. The utility function U^A is

$$U^A(C(t)) = C(t)^\gamma \qquad\qquad 0 \le \gamma \le 1$$

The constraint Z is given by

$$Z(C(t), K(t), K'(t)) = K(t)^{(1-\alpha)} - K'(t) - C(t) = 0$$

The intertemporal utility function of the household $U^{A\,\text{int}}$ is given by

$$U^{A\,\text{int}(0,T_{\max})}(C) = \int_0^{T_{\max}} e^{-r\,\tau}\, U^A(C(\tau))d\tau = \int_0^{T_{\max}} e^{-r\,\tau}\, C(\tau)^\gamma\, d\tau$$

Instead of using the constraint, we use the Lagrange function with constraint. This results in the variation problem

$$\int_0^{T_{\max}} \left(e^{-r\tau}C(\tau)^\gamma + \lambda(\tau)(K(\tau)^{(1-\alpha)} - K'(\tau) - C(\tau))\right)d\tau \to \max$$

$$with\ K(0) = k0 \qquad\qquad K(T_{\max}) = kT_{\max}$$

This results in the Euler equation with initial and final values:

$$\text{EulerEquations}\ [\left(e^{-r\tau}C(\tau)^\gamma + \lambda(\tau)(K(\tau)^{(1-\alpha)} - K'(\tau) - C(\tau))\right),$$
$$\{C(\tau), K(\tau), \lambda(\tau)\}, \tau]$$

$$K(0) = k0$$
$$K(T_{\max}) = kT_{\max}$$

which result in the following differential equation system

$$e^{-r\tau}\gamma C(\tau)^{(-1+\gamma)} - \lambda(\tau) = 0$$
$$-(-1+\alpha)K(\tau)^{-\alpha}\lambda(\tau) + \lambda'(\tau) = 0$$
$$-C(\tau) + K(\tau)^{(1-\alpha)} - K'(\tau) = 0$$
$$K(0) = K0$$
$$K(T_{\max}) = KT_{\max}$$

The differentiation of the first and third equations and the addition of these equations to the equation system results in the differential equation system

$$
\begin{array}{ll}
(1) & e^{-r\tau}\gamma C(\tau)^{(-1+\gamma)} - \lambda(\tau) = 0 \\
(2) & -e^{-r\tau} r C(\tau)^{(-1+\gamma)} + e^{-r\tau}(-1+\gamma)\gamma C(\tau)^{(-2+\gamma)} C'(\tau) - \lambda'(\tau) = 0 \\
(3) & -(-1+\alpha)K(\tau)^{-\alpha}\lambda(\tau) + \lambda'(\tau) = 0 \\
(4) & -C(\tau) + K(\tau)^{(1-\alpha)} - K'(\tau) = 0 \\
(5) & -C'(\tau) + (1-\alpha)K(\tau)^{-\alpha}K'(\tau) - K''(\tau) = 0 \\
(6) & K(0) = K0 \\
(7) & K(T_{\max}) = KT_{\max}
\end{array}
\qquad \text{<25.2>}
$$

The solution of this (complicated) differential-algebraic equation is much more complicated than the solution of the equation system <25.1> in chapter 25.1.

To show that both systems of equations are equivalent, the following steps are taken:

Calculate $\lambda'(\tau)$ from (3) and leave out (3)

Insert $\lambda'(\tau)$ into (2)

Calculate $\lambda(\tau)$ from (1) and leave out (1)

Insert $\lambda(\tau)$ into (2)

Calculate $C(\tau)$ from (4) and leave out (4)

Insert $C(\tau)$ into (2)

Calculate $C'(\tau)$ from (5) and leave out (5)

Insert $C'(\tau)$ into (2)

Simplify under the condition $r > 0 \ \textit{and} \ K(\tau) - K(\tau)^{\alpha} K'(\tau) \neq 0$

This again results in <25.1>

$$
\begin{aligned}
0 = {} & (-1+\alpha)K(\tau) + r\,K'(\tau)^{(1+\alpha)} + (-1+\alpha)(-2+\gamma)K(\tau)^{\alpha}K'(\tau) + \\
& + K(\tau)^{2\alpha}(-r\,K'(\tau) + (-1+\gamma)K''(\tau))
\end{aligned}
$$

$$K(0) = k0$$

$$K(T_{\max}) = kT_{\max}$$

The conclusion from all this: It is much more convenient to use the constraint to calculate $C(\tau)$ and to eliminate $\lambda(\tau)$ and to use the Lagrange function without constraint than the Lagrange function with constraint.

25.3. The GCD Ramsey model (with non-intertemporal utility function)

The utility function U^A can be used not only to construct the (intertemporal) standard Ramsey model (see Chapter 25.1), but also to construct a (non-intertemporal) standard GCD model, which we call the GCD Ramsey model. This makes it possible to show the different dynamic behaviour of these two models (see chapter 25.5)

The utility function U^A and the constraints are the same as in the standard Ramsey model

$$U^A(C(t)) = C(t)^\gamma \qquad\qquad 0 \le \gamma \le 1$$
$$Z(C(t), K(t), K'(t)) = K(t)^{(1-\alpha)} - K'(t) - C(t) = 0$$

So we have 2 variables $C(t), K(t)$ and 1 constraint. The corresponding differential-algebraic GCD equation system consists of 2 differential equations (the behavioural equations for the 2 variables) and 1 algebraic equation (the constraint). For the sake of simplicity, we set all power factors to 1. The ex-ante behavioural equations describe that the household tries to change the consumption $C(t)$ along the partial derivation of $U^A(C(t).K(t))$ with respect to $C(t)$ and tries to change the capital $K(t)$ along the partial derivation of $U^A(C(t).K(t))$ with respect to $K(t)$. The ex-post equation for the behavioural equation for $C(t)$ is obtained by adding the constraint forces given by $\lambda(t)$ multiplied by the partial derivative from Z with respect to $C(t)$. In the same way, the ex-post behavioural equation is obtained for $K(t)$ by adding the constraint force given by $\lambda(t)$ multiplied by the partial derivative from Z with respect to $K(t)$. Together with the constraint, the differential algebraic GCD equation system to be solved is obtained:

$$C'(t) = \frac{\partial U^A(C(t), K(t))}{\partial C(t)} + \lambda(t)\frac{\partial Z(C(t), K(t), K'(t))}{\partial C(t)} = \gamma C(t)^{(\gamma-1)} - \lambda(t)$$
$$K'(t) = \frac{\partial U^A(C(t), K(t))}{\partial K(t)} + \lambda(t)\frac{\partial Z(C(t), K(t), K'(t))}{\partial K(t)} = 0 - \lambda(t)$$
$$0 = Z(C(t), K(t), K'(t)) = K(t)^{(1-\alpha)} - K'(t) - C(t)$$

This results in

$$C'(t) = \gamma C(t)^{(\gamma-1)} - \lambda(t)$$
$$K'(t) = -\lambda(t)$$
$$0 = K(t)^{(1-\alpha)} - K'(t) - C(t)$$

25.4. The IGCD Ramsey model (with intertemporal utility function)

The utility function U^A, the intertemporal utility function $U^{A\,\mathrm{int}}$ and the constraints are the same as in the standard Ramsey model

$$U^{A}(C(t)) = C(t)^{\gamma} \qquad 0 \le \gamma \le 1$$
$$Z(C(t), K(t), K'(t)) = K(t)^{(1-\alpha)} - K'(t) - C(t) = 0$$
$$U^{A\,\mathrm{int}(0,T_{\max})}(C) = \int_0^{T_{\max}} e^{-r\,\tau}\, U^{A}(C(t))d\tau = \int_0^{T_{\max}} e^{-r\,\tau}\, C(t)^{\gamma}\, d\tau$$

We calculate $C(\tau)$ from the constraint and insert it into U^{A}. This results in the intertemporal utility function $U^{A\,\mathrm{int}(0,T_{\max})}$

$$U^{A\,\mathrm{int}(0,T_{\max})}(K) = \int_0^{T_{\max}} e^{-r\,\tau}(K(t))^{(1-\alpha)} - K'(t))^{\gamma}\, d\tau \rightarrow \max$$
$$\textit{with } K(0) = k0 \qquad K(T_{\max}) = kT_{\max}$$

Since we only have 1 agent A and 1 variable $x_1 = K$, <24.3> reduces to

(1)	$K'(t) = \left. \dfrac{d\,K^{A\,\mathrm{int}(t,T_{max})}(\tau)}{d\tau} \right\|_{\tau=t}$
(2)	$k(0) = k0$
(3)	EulerEquations $[e^{-r\tau}\left(K^{A\,\mathrm{int}(t,T_{max1})}(t+\tau)^{(1-\alpha)} - K^{A\,\mathrm{int}(t,T_{max})}{}'(t+\tau)\right)^{\gamma},$
	$\left\{K^{A\,\mathrm{int}(t,T_{max1})}(\tau)\right\}, \tau]$
	with initial and final value
	$K^{A\,\mathrm{int}(t,T_{max})}(t) = K(t)$
	$K^{A\,\mathrm{int}(t,T_{max})}(T_{max}) = KT_{max}$

From the uniqueness theorem for differential equations, it follows that the **standard Ramsey model and the IGCD-Ramsey model have the same solutions** if they have the same initial values:

- Designate $K_R(t) = K_{R,(0,T_{\max})}(t)$ the solution of the classic Ramsey model and $K_G(t)$ the solution of the IGCD-Ramsey model with the initial condition $K_R(0) = K_G(0) = K0$ and the final condition $K_R(T_{\max}) = K_G(T_{\max}) = KT_{\max}$
- Let $t_a \in [0, T_{\max}]$ and designate $K_{R,(t_a,T_{\max})}(t)$ the solution of the classical Ramsey model with the initial condition $K_{R,(t_a,T_{\max})}(t_a) = K_R(t_a)$ and the final condition $K_{R,(t_a,T_{\max})}(T_{\max}) = KT_{\max}$
- then the following applies:

a) $\quad K_R(t_a + \tau) = K_{R,(t_a,T_{\max})}(t_a + \tau)$ for all $\tau \in [0, T_{\max} - t_a]$

Because a variational problem for a part of the whole interval gives the same solution as the variational problem for the whole interval, if the initial and final values correspond to the solution values of the variational problem for the whole interval.

$$b) \qquad K_R{}'(t_a) = K_{R,(t_a,T_{\max})}{}'(t_a) = \qquad \textit{because of } a)$$

$$= K_{G,(t_a,T_{\max})}{}'(t_a) = \qquad \textit{because } K_{R,(t_a,T_{\max})} = K_{G,(t_a,T_{\max})}$$

$$\textit{because of } (3)$$

$$= K_G{}'(t_a) \qquad \textit{because of } (1)$$

$$c) \qquad K_R{}'(t) = K_G{}'(t) \qquad \textit{because of } (b) \textit{ and because}$$

$$t_a \textit{ can be chosen arbitrary}$$

$$d) \qquad K_R(0) = K_G(0) \qquad \textit{because of preconditions}$$

$$e) \qquad K_R = K_G \qquad \textit{because of } c), d) \textit{ and}$$

$$\textit{the uniqueness theorem}$$

$$\textit{of differential equations}$$

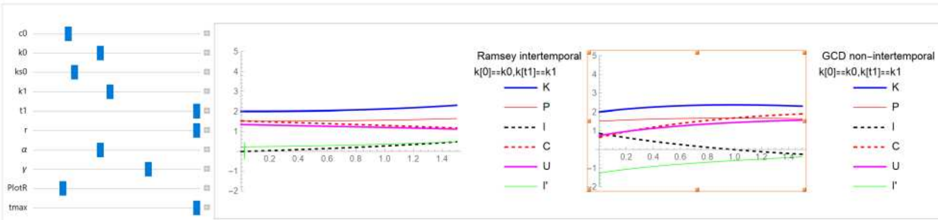
Of course, this only applies to this special case, where there is only 1 agent. Of course, this does not apply if there are several agents.

25.5. Numerical calculations and comparison of Ramsey model and GCD Ramsey model

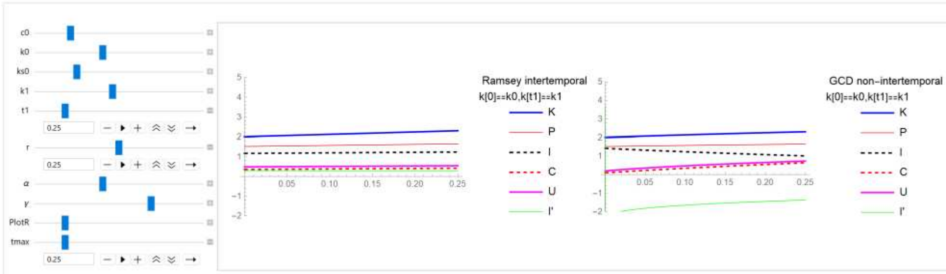
The Ramsey model is equivalent to a IGCD Ramsey model, because there is only 1 agent involved (see chapter 25.4). We therefore only compare the Ramsey model with the GCD Ramsey model.

https://www.dropbox.com/s/2sn0hh7tdry7wdp/Vergleich%20Ramsey%20klassisch%2C%20GCD%20klassisch%2C%20GCD%20intertemp%20Version%2010.nb?dl=0

Results of the calculations for “large” discount rate $r = 0.5$ and $T_{\max} = t_1 = 1.5$ you can find in the next graphic. It shows the difference in the dynamics of the standard (intertemporal) Ramsey model and the non-intertemporal GCD-Ramsey model.



A calculation for the “small” discount rate $r = 0.25$ and small $T_{\max} = t_1 = 0.25$ gives the expected result shown in 24.4.2.: If discount rate and optimisation interval are small, the two models (standard (intertemporal) Ramsey and non-intertemporal GCD-Ramsey) are similar.


c0
k0
ks0
k1
t1
0.25
r
0.25
α
γ
PlotR
tmax
0.25
Ramsey intertemporal
k[0]==k0,k[t1]==k1
K
P
I
C
U
I'
GCD non-intertemporal
k[0]==k0,k[t1]==k1
K
P
I
C
U
I'

26. Comparison of IGCD models with DSGE models

DSGE models require utility functions that can be aggregated to a master utility function. In the case of GCD and IGCD models, the utility functions do not need to be able to be aggregated.

Due to the use of a master utility function, DSGE models consist of only 1 variational problem and the economic system is in principle controlled by only 1 agent. The simplest DSGE model is the Ramsey model. In chapter 24.4 we showed for the Ramsey model that it is equivalent to the corresponding IGCD Ramsey model. As a main result, we therefore propose (that it should be possible to show) that (non-stochastic, expectation-free) DSGE models are in principle equivalent to (non-stochastic, expectation-free) IGCD models with only 1 agent.

But although this is not currently done, GCD and IGCD models can in principle be extended by stochasticity and expectations in the same way as DSGE models. Thus, DSGE models should in principle be equivalent to IGCD models with only 1 agent.

DSGE models are essentially equilibrium models. The dynamics in DSGE models arise from the maximisation of an intertemporal master utility function leading to the Euler-Lagrange equations. The dynamics after a shock is caused by the swing back to the equilibrium state. However, non-intertemporal GCD and (intertemporal) IGCD models are "true" dynamic models that can be formulated independently of any equilibrium states. Both can also be used to model economic shocks (Glötzl 2022).

In summarising, IGCD models can therefore be seen as a generalisation or alternative to DSGE models.

27. Model $A1^{int}$: IGCD model corresponding to model A1

We develop the IGCD model from the non-intertemporal GCD model A1. For this purpose, we develop the intertemporal utility functions from the utility functions for the A1 model and specify the system of differential equations for the corresponding IGCD model in accordance with the definition in Chapter 24.2.

27.1. Intertemporal utility functions

The algebraically defined equations, utility functions and the constraints of model A1 are unchanged:

algebraically defined variables

$$Y(L,K)=\beta L^{\alpha}K^{1-\alpha} \qquad \beta>0,\ 0<\alpha<1$$

utility function

$$U^{H}=C^{\gamma}-(L-\hat{L})^{2}-(M^{H}-\hat{M}^{H})^{2} \qquad 0<\gamma<1$$

$$U^{B}=pY(L,K)-wL-(S-\hat{S})^{2}=p\beta L^{\alpha}K^{1-\alpha}-wL-(S-\hat{S})^{2}$$

constraints

$$Z^{H}=0=wL-pC-M^{H}{}'$$

$$Z^{B}=0=pC-wL-M^{B}{}'$$

$$Z_{1}=0=Y(L,K)-C-K'-S'=\beta L^{\alpha}K^{1-\alpha}-C-K'-S'$$

We simplify in the following steps

- Calculate $wL=pC+M^{H}{}'$ from the constraint Z^{H} and put in U^{B} and in Z^{B}
- Calculate $C=\beta L^{\alpha}K^{1-\alpha}-K'-S'$ from the constraint Z_{1} and put in U^{H} and U^{B} and simplify.

This results in

$$U^{H}=(\beta L^{\alpha}K^{1-\alpha}-K'-S')^{\gamma}-(L-\hat{L})^{2}-(M^{H}-\hat{M}^{H})^{2}$$

$$U^{B}=K'+S'+M^{H}{}'-(S-\hat{S})^{2}$$

$$Z^{B}=0=M^{H}{}'+M^{B}{}'$$

The utility functions depend on the 4 variables

$$L,K,S,M^{H}$$

or their derivatives. The system is completely determined by these variables, because the variable M^B is completely determined by the variable M^H by $M^B{}' = -M^H{}'$ due to the constraint Z^B.

This gives the intertemporal utility functions

$$U^{H\,\mathrm{int}((t,T_{max})} =$$

$$= \int_t^{T_{max}} e^{-r^H(t+\tau)} \Big((\beta L^{\alpha}(t+\tau) K^{1-\alpha}(t+\tau) - K'(t+\tau) - S'(t+\tau))^{\gamma} - (L(t+\tau) - \hat{L})^2 - (M^H(t+\tau) - \hat{M}^H)^2 \Big) d\tau$$

$$U^{B\,\mathrm{int}(t,T_{max})} =$$

$$= \int_t^{T_{max}} e^{-r^B(t+\tau)} \Big(K'(t+\tau) + S'(t+\tau) + M^H{}'(t+\tau) - (S(t+\tau) - \hat{S})^2 \Big) d\tau$$

27.2. Intertemporal GCD-equations

The (intertemporal) IGCD equations are obtained according to **Fehler! Verweisquelle konnte nicht gefunden werden.**:

$$for\ all\ i = 1,2,3,4$$
$$denote$$
$$x_1 = L$$
$$x_2 = K$$
$$x_3 = S$$
$$x_4 = M^H$$
$$behavioural\ equations\ for\ x_i(t)$$
$$x_i'(t) = \frac{\mu_i^A}{\mu_i^A + \mu_i^B} \left. \frac{d\,x_i^{A\,\mathrm{int}(t,T_{max1})}(t+\tau)}{d\tau} \right|_{\tau=0} + \frac{\mu_i^B}{\mu_i^A + \mu_i^B} \left. \frac{d\,x_i^{B\,\mathrm{int}(t,T_{max})}(t+\tau)}{d\tau} \right|_{\tau=0}$$
$$initial\ values\ for\ x_i(t)$$
$$x_i(0) = xi0$$

$$Euler\ equations\ for\ A\ with\ \mathrm{int}ial\ and\ end\ values\ for\ x_i^{A\,\mathrm{int}(t,T_{max})}(\tau)$$
$$\text{EulerEquations}\ [\int_t^{T_{max}} e^{-r^A(t+\tau)} U^H(x_1^{A\,\mathrm{int}(t,T_{max})}(t+\tau),\ldots,x_4^{A\,\mathrm{int}(t,T_{max})}(t+\tau)),$$
$$\left\{x_1^{A\,\mathrm{int}(t,T_{max})}(\tau),\ldots,x_4^{A\,\mathrm{int}(t,T_{max1})}(\tau)\right\},\tau]$$
$$x_i^{A\,\mathrm{int}(t,T_{max1})}(t) = x_i(t) \qquad x_i^{A\,\mathrm{int}(t,T_{max})}(t_1) = xiAT_{max}$$

$$Euler\ equations\ for\ B\ with\ \mathrm{int}ial\ and\ end\ values\ for\ x_i^{B\,\mathrm{int}(t,T_{max})}(\tau)$$
$$\text{EulerEquations}\ [\int_t^{T_{max}} e^{-r^B}(t+\tau) U^B(x_1^{B\,\mathrm{int}(t,T_{max1})}(t+\tau),\ldots,x_4^{B\,\mathrm{int}(t,T_{max1})}(t+\tau)),$$
$$\left\{(x_1^{B\,\mathrm{int}(t,T_{max1})}(\tau),\ldots,x_4^{B\,\mathrm{int}(t,T_{max1})}(\tau))\right\},\tau]$$
$$x_i^{B\,\mathrm{int}(t,T_{max1})}(t) = x_i(t) \qquad x_i^{B\,\mathrm{int}(t,T_{max1})}(T_{max}) = xiBT_{max}$$

27.3. Numerical calculations

The numerical calculations can be performed as shown in chapter 24.3

F. Summary

28. Summary and conclusions

28.1. Principle of GCD

By using differential-algebraic equations in continuous time, the GCD approach extends existing analogies between classical mechanics and economics from constrained optimization to constrained dynamics.

28.2. Problem 8 by Stephen Smale

Problem 8 of the 18 problems published by Smale in 1998 (Smale 1991; 1997; 1998; Smale Institute 2003) is: introducing dynamics (adjustment of prices) in economic equilibrium theory (Arrow-Debreu equilibrium model). The problem arose from Smale's own involvement with mathematical economics.

GCD models describe the dynamics of economic systems away from equilibrium. They converge to the solutions of general equilibrium theory under certain conditions. They describe not only the dynamic adjustment of prices but also of all other economic variables and thus may represent a solution to S. Smale's problem 8.

The method is based on the standard method in physics for modeling dynamics under constraints.

28.3. GCD is a fundamentally new methodology for modeling economic systems and, in a certain sense, can be seen as a metatheory of economic modeling

Simplified, there are so far 4 basic groups of methods for modeling economic systems:

28.3.1. Neoclassical general equilibrium theory (GE, DSGE)

This is essentially based on the maximization of an (overall) utility function under constraints (overall utility maximization). The existence of an overall 1 utility function presupposes the aggregability of individual utility functions.

28.3.2. Post-Keynesian Models

These reject the use of individual utility functions and describe the aggregate variables via differential equations.

A special case of these are the Stock-Flow-Consistent (SFC) models.

28.3.3. Agent-Based Models (ABM)

These describe the behaviour of mostly many agents based on individual interactions among them.

28.3.4. The relation of the basic principles of GCD models to these types of economic models

- The dynamic evolution of the variables is determined in GCD models by the fact that each of the agents applies an individual force to these variables and the actual dynamics is determined by the resultant of these forces. These individual forces can be described (in most practical cases) as gradients of individual utility functions. The resulting dynamics can be called individual utility optimization as opposed to neoclassical overall utility maximization. A detailed discussion of the relationship between individual utility optimization and overall utility maximization can be found in (Glötzl 2023a).
- Note on post-Keynesian models: However, agents' forces do not necessarily arise as gradients of individual utility functions. Therefore, GCD models can also describe post-Keynesian models that cannot be described by utility functions. In principle, the forces (on the right-hand side of the differential equations of post-Keynesian models) can always be decomposed into a gradient component (resulting from a utility function) and a rotation component. This is called a Helmholtz decomposition, which is not only possible in 3 dimensions, as it usually occurs in physics, but is possible in arbitrary dimensions (Glötzl und Richters 2021b; 2021a)
- GCD models are always stock-flow consistent (SFC). But not only (economic) accounting identities, but also any other relations or conservation laws like the 1st law of chemistry (conservation of mass) or the 1st law of thermodynamics (conservation of energy) can be used as constraints.
- GCD models are always agent-based respectively microfounded

28.4. GCD models can be the bases for a new economic thinking in terms of: economic power, economic force, economic constraint force

Especially the concept of economic power is of fundamental importance for understanding economics (Rothschild 2002b). With GCD models, this concept can also be formally incorporated into economic models. In comparison with classical mechanics in physics, power factors correspond to the reciprocal of mass (Glötzl 2015). Conventional economic models usually describe one-sided power relations, which, however, rarely occur in reality. GCD models can also be used to better describe mixed power relations and thus reality.

GCD models can be the basis for a new theoretical understanding of e.g.:

- Economic growth
- Business cycles and economic crises
- Analogies between physics and economics

28.5. With the help of the GCD methodology, a formally clean definition of the terms ex-ante and ex-post is possible

28.6. Non-equilibrium dynamics

NCD models can be used to describe true disequilibrium dynamics. In particular, it is also possible to describe situations in which no equilibrium exists or situations in which the utility function is not concave.

28.7. Genuine competitive models

Apart from game-theoretic models, the other types of economic models mentioned cannot be used to describe genuine competition models, i.e. models in which the individual optimization strategy does not lead to an overall optimum. In reality, however, such situations, which are similar to the prisoner's dilemma, are very common. With GCD models, genuine competition models can be described very well.

28.8. Applications

GCD models and IGCD models can be used for many practical tasks such as economic forecasting, modeling the impacts of fiscal or monetary policy, modeling business cycle fluctuations and economic shocks.

28.9. GCD models are a generalisation and alternative to DSGE models

GCD models in principle can also be formulated with intertemporal utility functions called IGCD models (Glötzl 2023a). IGCD models can be seen as a generalisation or alternative to DSGE models.

28.10. What remains to be done in the future

a) Adjustment of parameters to describe real circumstances and comparison of model results with real business cycle trends.

b) Extend GCD models to multiple households, firms, and goods, and in particular to commodity and financial markets.

c) In the long run, develop a more complex, real-world model to enable better economic forecasting and test measures to achieve economic policy targets.

d) Elaborate GCD models with economic shocks in detail.

e) Elaborate IGCD models (with intertemporal utility functions) in detail.

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