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24 August 2023

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Abstract

The accuracy of predictions of price and return probabilities substantially determines the reliability of asset pricing and portfolio theories. We develop successive approximations that link up predictions of the market-based probabilities of price and return for the whole stock market with predictions of price and return probabilities for stocks of a particular company and show that economic complexity limits the accuracy of any forecasts. The economic origin of the restrictions lies in the fact that the predictions of the *m*-th statistical moments of price and return require descriptions of the economic variables composed by sums of the *m*-th powers of economic or market transactions during an averaging time interval. The attempts to predict the *n*-th statistical moments of price and return of stocks that are under the action of a single risk result in estimates of the *n*-dimensional risk rating vectors for economic agents. In turn, the risk rating vectors play the role of coordinates for the description of the evolution of economic variables. The lack of a model description of the economic variables composed by sums of the 2-d and higher powers of market transactions causes that, in the coming years, the accuracy of the forecasts will be limited at best by the first two statistical moments of price and return, which determine Gaussian distributions. One can ignore existing barriers and limits but cannot overcome or resolve them. That significantly reduces the reliability and veracity of modern asset pricing and portfolio theories. Our results could be essential and fruitful for the largest investors and banks, economic and financial authorities, and market participants.

Keywords: price and return, market trade, risk ratings, statistical moments, probability predictions

JEL: C0, E4, F3, G1, G12

This research received no support, specific grants, or financial assistance from funding agencies in the public, commercial, or nonprofit sectors. We welcome valuable offers of grants, support, and positions.

1. Introduction

One can consider the asset-pricing puzzle a two-part problem. The first one describes the asset pricing and portfolio models under the assumption that predictions of price and return probabilities are given. The second one derives the forecasts of price and return probabilities, or only their average values and volatilities at horizon *T*, using various models, which are mostly not based on the first part. However, the market-based choice of the asset price probability and its prediction at horizon *T* hides essential difficulties of an economic nature, which significantly limit the accuracy of any price probability forecasts as well as the reliability of the asset pricing and portfolio theories.

In the recent decades, the asset price theories have obtained many important results, and references (Markowitz, 1952; Sharpe, 1964; Fama, 1965; Merton, 1973; Fama, 1990; Cochrane, 2001; Fama and French, 2015; Campbell, 2018) present only a millesimal part of the current studies. Actually, the foundation of modern asset pricing theories remains rather peculiar and based on "highly restrictive and undoubtedly unrealistic assumptions.” (Sharpe, 1964), and Sharpe’s opinion that “…the proper test of a theory *is not the realism of its assumptions* but the acceptability of its implications…” 60 years later, it is generally accepted and still in use.

The predictions of average price, return, and volatilities establish a separate, important field of research that is covered by as numerous studies as the asset pricing theories themselves. The price forecasting is considered within economic forecasting (Diebold, 1999; Snowberg, Wolfers, and Zitzewitz, 2012), time series analysis (Davis, 1941; Brockwell and Davis, 2002), Monte-Carlo simulations (McLeish, 2005), and, in the last decade, intensive studies of machine learning and AI methods for evaluating stock price predictions (Cao et al., 2021; Kelly and Xiu, 2023). We refer to these papers to indicate some forecasting methods only, but don’t consider here any reasonable review of that broad and important research.

However, the current models of stock price and price probability forecasts, as well as asset pricing theories, may omit important factors and interrelations that determine the dependence of market-based price probability on market trade randomness. In turn, the predictions of market trade stochasticity should be based on general economic models that link market trade evolution with economic development. These rather complex problems cannot be resolved in one paper, but we make an attempt to figure out significant factors and relations that vitally impact the predictability of price and return probabilities.

In our paper, we present a pure theoretical consideration of the market-based price and return probabilities in a general economic context and highlight the economic obstacles that essentially limit the accuracy of forecasts of the probabilities at any horizon T. It is well known (Shephard, 1991; Shiryaev, 1999; Shreve, 2004) that the probability measure, characteristic function, and a set of statistical moments present almost equal descriptions of a random variable. In this paper, we consider a finite number of statistical moments as a tool for approximate predictions of price and return probabilities. We study the economic obstacles that vastly limit our ability to consider and predict many statistical moments and significantly restrict the accuracy of any forecasts of price and return probabilities.

To forecast the statistical moments of price and return of stocks of a particular company, one should have estimates of similar forecasts of other stocks traded on the market. Obviously, predictions of economic variables are impossible without knowledge, forecasts, and estimates of their economic environment.

To derive the forecasts of the economic environment, we consider consecutive approximations of the market-based statistical moments of price and return, which are determined by the statistical moments of market trade values and volumes. We start with a description of the market-based statistical moments of price and returns of stocks of individual companies. Further, we consider risk ratings, which take continuous numeric values. We show that the description of the m-th statistical moments of trade values and volumes for $m=1,2,..n,$ depends on the assessments of the risk ratings of the traded companies. To describe the n-th statistical moments of trading stocks of a company, which is under the action of a single risk, one should estimate the risk ratings in the form of a n-dimensional vector $\boldsymbol{x} = (x_1,...x_n)$. Each component x_m, $m=1,2,..n$ of the risk vector $\boldsymbol{x}$ is estimated using economic variables of the m-th order, which are determined by the sums of the m-th powers of economic or financial transactions. We show that the usage of risk coordinates $\boldsymbol{x}=(x_1,...x_n)$ allows making a transition from the description of statistical moments of stocks of individual companies to the description of *collective* statistical moments of stocks of companies with risk ratings in the neighborhood of a vector $\boldsymbol{x}=(x_1,...x_n)$. That transition describes continuous economic media approximation and models the price and return statistical moments of stocks as functions of time t and risk vector $\boldsymbol{x}$. Subsequent approximations describe the statistical moments of *collective* trade, price, and returns of all stocks traded at the market as functions of time t only.

To derive these successive approximations, one should average market trade time series over sequential time intervals $\varepsilon << \Delta \leq \Delta_x \leq \Delta_x$. As ε we denote the constant period between market trades, which could be a second or even a fraction of a second. To assess the statistical moments of market trades, price, and return of stocks of a particular company during a reasonable time interval Δ, one should average initial market transactions over Δ. To derive a continuous economic media approximation and describe price and return statistical moments as functions of time t and coordinates $\boldsymbol{x}$, one should sum market trades with stocks of companies with risk coordinates in the neighborhood of vector $\boldsymbol{x}$ of the economic domain and average the time series over the interval Δ_x. To derive the statistical moments of price and return of the whole stock market, one should sum trades with stocks of all companies on the market and average the time series over Δ_m. We derive the equations, which describe the dynamics of additive economic variables as functions of $(t,\boldsymbol{x})$ and the equations of *collective* variables of the whole market as functions of time t only. The slow evolution in time of the trade statistical moments of the whole stock market should serve as an economic environment for the description of the price statistical moments in the continuous economic media approximation as functions of $(t,\boldsymbol{x})$. The solutions of the equations of the continuous economic media approximation at horizon T establish the ground for the estimates of the price and return statistical moments of stocks of a particular company. To do that, an investor should estimate the possible risk ratings $\boldsymbol{x}_q=(x_{q1},...x_{qn})$ of a company q at horizon T. Then an investor could map the anticipated assessment of risk at the forecast of the *collective* trade statistical moments as functions of $(t,\boldsymbol{x})$ and estimate the statistical moments of return of stocks of a company q at horizon T. That completely determines the statistical moments of price at the same horizon.

The origin of the economic obstacles, which limit the predictive capacity and accuracy of the price and return probabilities, lies in the fact that one can approximate only a finite number n of the market-based statistical moments. Each m-th statistical moment for $m=1,2,..n$ of price or return is determined by the m-th statistical moments of the market trade values and volumes averaged over the time interval Δ, Δ_x or Δ_m. Each averaging interval contains only a finite number of market trades, and that limits the number of statistical moments that can be estimated. To increase the accuracy of the current price probability, one can enlarge the averaging time interval to assess more statistical moments. However, that ambition is vastly limited for two reasons. First, the increase in the time-averaging interval reduces the ability to make prompt investment decisions. The second, and much more significant economic barrier, is determined by the irremovable growth of the complexity of forecasting of each extra trade statistical moment. As we show in sections 3 and 5, each extra m-th statistical moment of price or return, which can increase the accuracy of predicted probabilities, creates one additional tough problem in the economic description. We show that the sums of the m-th powers of market trade values and volumes, as well as the sums of the m-th powers of any economic or financial trades made during the averaging interval, determine the economic variables of the *m*-th order. For example, conventional economic investments during Δ are determined by the sums (without duplication) of the first power of investment transactions made by the agents during Δ. Similar to that, we call "*m*-th investments" the sums of the *m*-th powers of the investment transactions during Δ. We denote such economic variables as the variables of the *m*-th order. The mutual interactions of the economic variables of the *m*-th order establish economic problems almost similar to the complex description of conventional macroeconomic theories, which model relations between the variables of the first order. In turn, the assessments of the risk ratings of companies, must use the economic variables of the *m*-th order, which result in the derivation of components x_m of the risk vector $\boldsymbol{x}=(x_1,\dots x_n)$.

The complexity of economic modeling of the *m*-th order and the strong rise of uncertainty driven by the estimates of each additional component x_m of the risk vector $\boldsymbol{x}=(x_1,\dots x_n)$, greatly limit the number of statistical moments that could be predicted, which significantly diminish the accuracy of the price and return probabilities forecasts.

Actually, the stocks traded on modern markets are mostly subject to several risks. The estimates of ratings of several risks $j=1,2,..J$, which use the economic variables of the *m*-th order $m=1,\dots n$, produce the risk ratings of a company as matrix variables $\boldsymbol{x}=(x_{mj})$, $m=1,..n;$ $j=1,2,..J$. Here, J determines the number of risks, and n determines the number of statistical moments of the price and return that are predicted by the model. Numerous risks, which impact stocks trading, additionally increase the complexity of economic modeling and reduce the accuracy of predictions of price and return probabilities.

Eventually, our findings can be briefly stated as follows: The current markets provide a lot of trading data with a period ε, which allows to assess "today" many – 10, 20 or more - statistical moments of the market trade, price, and return during any reasonable averaging interval that can be equal hours or days. That helps "today" approximate the probabilities of the price and return with high accuracy. However, the predictions of the statistical moments at horizon T meet the irremovable barriers of economic complexity. The predictions of the 2-d statistical moments of price and return need forecasts of the 2-d statistical moments of market trade. In turn, that requires the economic theory of the 2-d order. The predictions of the 3-d, 4-th, etc., statistical moments require corresponding economic models of the 3-d and 4-th orders, which are even more complex. The assessments of the risk ratings determined by the economic variables of the 2-d, 3-d, or 4-th order increase the model complexity further. In the coming years, in the best scenario, the accuracy of the forecasts will be limited by the 2-d statistical moments, and hence the market-based price probability forecasts will be limited by

Gaussian distributions or their extensions (see App. A). The ignorance of the limits driven by the existing economic complexity may allow one to derive forecasts of the price and return statistical moments and probabilities, but with such high uncertainty that it makes predictions useless.

The rest of the paper is organized as follows: In section 2, we consider the market-based statistical moments of price and return. In section 3, we discuss the vector nature of ratings for a single risk $\boldsymbol{x}=(x_1,...x_n)$. Further, we discuss the statistical moments of returns of stocks of companies with risk ratings in the neighborhood of a risk vector $\boldsymbol{x}$ and the statistical moments of stocks of all companies traded on the whole market. In section 4, we consider the statistical moments of stock price, determined by the statistical moments of returns. In section 5, we discuss the complexities that would face any investor in his attempts to forecast the price and return probabilities of stocks of a company using his own predictions and the *collective* economic environment of the market in the continuous economic media approximation. Section 6: Conclusion. In Appendix A, we present simple approximations of characteristic functions and probability measures by a finite number of statistical moments. In Appendix B, we introduce the main notions and equations that describe the dynamics of the *collective* economic variables in the continuous economic media approximation and in the approximation of the whole market.

This paper is not for novices, and we believe that readers already know or can find on their own the definitions, terms, and models that are not given in the text. We assume that readers are familiar with conventional issues in economic theory, asset pricing and portfolio theories, risk assessment, the basics of probability theory, statistical moments, characteristic functions, partial differential equations, etc. We use the roman letters *A, B,* and *d* to denote scalars and bold ***B, P,*** to denote vectors and matrices. Reference (3.5) means equation 5 in section 3.

2. Market-based statistical moments of price and return

We assume that market trades of stocks of a company are made at a time t_i with a constant interval ε between trades:

$$\varepsilon - const \quad ; \quad t_i = t_0 + i\varepsilon \quad ; \quad i = 0,1,... \tag{2.1}$$

The trade time series at t_i introduces the initial market time axis division multiple of ε (2.1). As initial data, we consider the time series of trade values $C(t_i)$ and volumes $U(t_i)$ with stocks at times t_i, which determine trade price $p(t_i)$ due to a trivial equation:

$$C(t_i) = p(t_i)U(t_i) \quad ; \quad i = 0,1,... \tag{2.2}$$

Equation (2.2) defines the market trade price $p(t_i)$ of stocks of an individual company at t_i. The initial time axis division ε can be equal to a second or even a fraction of a second. The time series of the trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ are very irregular and of little help for predictions of the stock price at a time horizon T that can be equal to weeks, months, or years. One can consider market time series as random variables during any reasonable time interval $\Delta>>\varepsilon$. For simplicity, we take Δ as a multiple of ε (2.3) with N terms of the time series t_i inside Δ. To develop a pricing model at the horizon $T>>\varepsilon$ one should average the initial random market time series over the interval Δ (2.3):

$$\Delta = N\varepsilon \quad ; \quad N \gg 1 \; ; \quad \varepsilon \ll \Delta < T \tag{2.3}$$

After averaging market time series over Δ (2.3), one obtains more smooth data that can be more useful for forecasting at the horizon T. Averaged time series introduce a transition from the initial market time axis division that is multiple of ε to a new one, a rougher time axis division multiple of Δ. Market trades with stocks of any company determine three initial time series of the financial variables that should be taken into account by any pricing model: the trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ (2.2). It is impossible to independently define the probabilities of the trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ that match equation (2.2). The given probabilities of the trade value $C(t_i)$ and volume $U(t_i)$ determine the probability of the price $p(t_i)$ (2.2). We consider the random time series of the trade values $C(t_i)$ and volumes $U(t_i)$ as the primary, which completely determine the stochasticity of the market price $p(t_i)$. To support this statement, we refer to Fox et al. (2017), which provides the perfect methodology for estimating national accounts on basis of the aggregation of additive economic variables as the ground for the definition of non-additive variables such as price, inflation, bank rates, etc. We follow Fox et al. (2017) and consider the additive random variables determined by the time series of trade values $C(t_i)$ and volumes $U(t_i)$ as the basis for describing stochastic properties of the stock price and return.

Assume that the averaging interval Δ defines the time axis division t_k, $k=0,1,..$ multiple of Δ:

$$\Delta_k = \left[t_k - \frac{\Delta}{2}; t_k + \frac{\Delta}{2}\right] \quad ; \quad t_k = t_0 + \Delta \cdot k \quad ; \quad k = 0, 1, 2, \ldots \tag{2.4}$$

For convenience, we renumber the initial trade time series t_i (2.1; 2.3) and note them as t_{ik}, which belong to interval Δ_k (2.4):

$$t_k - \frac{\Delta}{2} \le t_{ik} \le t_k + \frac{\Delta}{2} \quad ; \quad t_{i+1,k} - t_{ik} = \varepsilon \quad ; \quad t_{i,k+1} - t_{ik} = \Delta \quad ; \quad i = 1, .. N \tag{2.5}$$

Thus, we consider N terms of the time series t_{ik} in each interval Δ_k (2.4). That allows equally assessing the statistical moments of the market trade value $C(t_k;n)$ and volume $U(t_k;n)$ in each averaging interval Δ_k as (2.6):

$$C(t_k;n) \equiv E[C^n(t_{ik})] \sim \frac{1}{N}\sum_{i=1}^{N} C^n(t_{ik}) \quad ; \quad U(t_k;n) \equiv E[U^n(t_{ik})] \sim \frac{1}{N}\sum_{i=1}^{N} U^n(t_{ik}) \quad (2.6)$$

Actually, the initial time series t_{ik} in each interval Δ_k (2.5) presents discrete data, as there is no data available between the trades with period ε. However, averaging the initial discrete time series during each interval Δ_k results in a continuous time model. One can use a moving average or other smoothing procedure to define (2.6) as a continuous time model. We use the notation t_k in (2.6) to outline the particular time interval Δ_k (2.5), which results in definition (2.6). We use the symbol ~ to underline that relations (2.6) define only assessments of mathematical expectation *E[..]* by a finite number *N* of terms of the time series that belong to the interval Δ_k (2.5). The *n*-th power of equation (2.2) at time t_{ik} gives:

$$C^n(t_{ik}) = p^n(t_{ik})\, U^n(t_{ik}) \quad ; \quad n = 1,2,\ldots \quad (2.7)$$

Equation (2.7) introduces the price *n-th* statistical moments (Olkhov, 2021a; 2022a; 2023a) in a way that has parallels to the definition of volume weighted average price (VWAP) (Berkowitz et al., 1983; Duffie and Dworczak, 2018):

$$p(t_k;n) \equiv E[p^n(t_{ik})] = \frac{1}{\sum_{i=1}^{N} U^n(t_{ik})} \sum_{i=1}^{N} p^n(t_{ik}) U^n(t_{ik}) \quad (2.8)$$

Relations (2.6 - 2.8) give:

$$p(t_k;n) = \frac{\sum_{i=1}^{N} C^n(t_{ik})}{\sum_{i=1}^{N} U^n(t_{ik})} = \frac{C(t_k;n)}{U(t_k;n)} \quad (2.9)$$

$$C(t_k;n) = p(t_k;n)\, U(t_k;n) \quad (2.10)$$

The first statistical moment of price $p(t_k;1)$ completely coincides with VWAP. It is well known that one can equally describe a random variable by its probability measure, characteristic function, or set of statistical moments (Shephard, 1991; Shiryaev, 1999; Shreve, 2004). Thus, the market-based statistical moments of price (2.8-2.10) completely describe the price as a random variable during Δ_k. A finite number *N* of the marker trades during any reasonable interval Δ results in that one can assess only a finite number of statistical moments of the trade value $C(t_k;n)$ and volume $U(t_k;n)$ (2.6). Hence, one can assess only a finite number of statistical moments of price (2.8-2.10). The finite number of price statistical moments $p(t_k;n)$ approximates the market-based price characteristic function and probability (Appendix A). The market-based price statistical moments (2.8-2.10) differ from the frequency-based price statistical moments $\pi(t_k;n)$:

$$\pi(t_k;n) \sim \frac{1}{N}\sum_{i=1}^{N} p^n(t_{ik}) = \frac{1}{N}\sum_{i=1}^{N} \frac{C^n(t_{ik})}{U^n(t_{ik})}$$

determined by probability $P(p) \sim m(p)/N$, which is proportional to the number $m(p)$ of trades at a price *p* if the total number of trades equals *N* during the interval Δ. The economic nature of these distinctions between the conventional frequency-based and the market-based price statistical moments is as follows: The market-based price statistical moments (2.8-2.10) take into account the impact of the size of the market's trade values and volumes. With growth of *n,* the price *n*-th statistical moments (2.8-2.10) more and more take into account the contribution of the market trades with huge values and volumes. Contrary to that, the frequency-based statistical moments $\pi(t_k;n)$ describe the price statistics determined by frequencies of price only. The market-based price statistical moments $p(t_k;n)$ equal conventional frequency-based statistical moments $\pi(t_k;n)$ only if all trade volumes $U(t_k)$ equal 1. Such an approximation is far from the reality of financial market trade. The market-based probability of stock price and return was described by Olkhov(2021a; 2022a; 2023a; 2023b), and we refer there for further details. Predictions of price probability on the horizon $t_K = T$

$$\Delta_K = \left[t_K - \frac{\Delta}{2}; t_K + \frac{\Delta}{2}\right] \quad ; \quad T = t_K = t_0 + \Delta \cdot K \tag{2.11}$$

require forecasting statistical moments of the trade values and volumes (2.6) at the same horizon $t_K = T$. The more trade statistical moments that can be predicted, the higher the accuracy of the market-based probability of the stock price.

To describe statistical moments of stock return when an investor sells stocks at time t_{ik} during Δ_k (2.4; 2.5), which he previously purchased at time t_{ik}-ξ one should consider the trade equation (2.2) during interval Δ_k (2.5) as follows:

$$C(t_{ik}) = p(t_{ik})U(t_{ik}) = \frac{p(t_{ik})}{p(t_{ik} - \xi)}\, p(t_{ik} - \xi)U(t_{ik}) = r(t_{ik}, \xi)S(t_{ik}, \xi)$$

We denote here return $r(t_{ik},\xi)$ as the ratio of price $p(t_{ik})$ at moment t_{ik} (2.5) to price $p(t_{ik}-\xi)$ in the past at a time $t_{i,k}$ -ξ. For convenience, we take the time shift ξ as a multiple of ε:

$$S(t_{ik}, \xi) \equiv p(t_{ik} - \xi)U(t_{ik}) \quad ; \quad r(t_{ik}, \xi) \equiv \frac{p(t_{ik})}{p(t_{ik} - \xi)} \quad ; \quad \xi = \varepsilon j \tag{2.12}$$

To simplify notations, we denote time shift ξ without index j (2.12). We denote $S(t_{ik},\xi)$ as the past value of the volume $U(t_{ik})$ of stocks at a time t_{ik}-ξ at a price $p(t_{ik}-\xi)$. Using (2.2; 2.12), we present the return trade equation (2.13):

$$C(t_{ik}) = r(t_{ik}, \xi)S(t_{ik}, \xi) \tag{2.13}$$

Similar to (2.6), we introduce statistical moments of the past value $S(t_{ik},\xi)$ (2.12) determined by the volume $U(t_{ik})$ of stocks at the price $p(t_{ik}-\xi)$:

$$S(t_k, \xi; n) \equiv E[S^n(t_{ik}, \xi)] \sim \frac{1}{N}\sum_{i=1}^{N} S^n(t_{ik}, \xi) = \frac{1}{N}\sum_{i=1}^{N} p^n(t_{ik} - \xi)U^n(t_{ik}) = \tag{2.14}$$

The n-th power of (2.13) gives

$$C^n(t_{i,k}) = r^n(t_{ik}, \xi)S^n(t_{ik}, \xi) \tag{2.15}$$

Similar to (2.8-2.10), we define the n-th statistical moments of return $r(t_k,\xi;n)$ of stocks at the time t_k with respect to the time shift ξ in the past at t_k-ξ as:

$$r(t_k,\xi;n) \equiv E[r^n(t_{ik},\xi)] = \frac{1}{\sum_{i=1}^{N} S^n(t_{ik},\xi)} \sum_{i=1}^{N} r^n(t_{ik},\xi) S^n(t_{ik},\xi) \tag{2.16}$$

$$r(t_k,\xi;n) = \frac{\sum_{i=1}^{N} C^n(t_{ik})}{\sum_{i=1}^{N} S^n(t_{ik},\xi)} = \frac{C(t_k;n)}{S(t_k,\xi;n)} \tag{2.17}$$

$$C(t_k;n) = r(t_k,\xi;n) S(t_k,\xi;n) \tag{2.18}$$

A finite number of statistical moments of return $r(t_k,\xi;n)$ determine the approximations of the return characteristic function and probability measure alike to approximations of the market-based price probability (Appendix A). For further details on the market-based probability of stock return, we refer to Olkhov (2023a; 2023b).

Relations (2.9; 2.10) on the price statistical moments $p(t_k;n)$ and relations (2.17; 2.18) on the statistical moments of stock return $r(t_k,\xi;n)$ establish mutual relations between them:

$$p(t_k;n) U(t_k;n) = r(t_k,\xi;n) S(t_k,\xi;n) \tag{2.19}$$

$$p(t_k;n) = r(t_k,\xi;n) \frac{S(t_k,\xi;n)}{U(t_k;n)} \tag{2.20}$$

Relations (2.19; 2.20) prove that one can define the price statistical moments, characteristic function, and price probability measure using the statistical moments of stock return $r(t_k,\xi;n)$ and vice versa. To forecast the price probability, it is sufficient to predict the probability of return, and vice versa. These problems complement each other completely. One can forecast return statistical moments, and that is enough to predict price statistical moments.

However, the predictions of the market-based statistical moments of price and return of stock of a company are impossible without knowledge of the financial and market "environment": the estimates of the price and return of other stocks traded on the market. To describe the statistical moments of numerous stocks traded at the NYSE or Nasdaq, one should distribute different stocks by certain parameters to distinguish them from each other. As a parameter that helps develop a distribution of different stocks, we select the risk ratings of their issuer companies. In the next section, we explain how assessments of the risk ratings of issuer companies traded on the market introduce the notion of risk coordinates in the economic domain and describe the market-based statistical moments of the stock return as functions of risk coordinates.

3. The market-based statistical moments of stock return as functions of risk

In this section, we describe the statistical moments of stock return as functions of risk ratings, which we consider their coordinates in the economic domain (Olkhov, 2016a-2020). Up now, the major risk agencies, such as Fitch, Moody's, and S&P, assess the risk ratings of the majority of stocks, major banks, and corporations (Metz and Cantor, 2007; Chane-Kon et al., 2010; Kraemer and Vazza, 2012). Risk agencies use the letter notations *AAA, AA, BB,* and *C* to designate the risk rate. Each rating agency has its own letter grade system to protect and promote its business. However, more than 80 years ago, Durand (1941) proposed the use of numerical risk grades. Indeed, risk ratings are conditional terms that are used in economics and finance as helpful tools for sustainable management, investment, and economic modeling. Primarily, the risk notations should support the problems of economic modeling but not the promotion of a particular business. The use of numeric risk grades can support a unified methodology for risk assessments by different agencies and opens up wide opportunities for economic and financial modeling. We take the idea of numeric risk grades proposed by Durand (1941) and complement it by introducing continuous numeric risk grades. The notions of the most secure and the most risky grades are entirely contingent, and the symbol *AAA* can easily be replaced by a numeric value. We take the most secure risk grade to be equal to 0 and the most risky grade to be equal to 1. Thus, we replace the letter-based risk grade symbols *AAA, BB,* and *CC* by continuous numeric risk grades that fill the unit interval [0,1], which we call the economic domain. If one considers the economic system under the action of J risks, then the numeric values of agents' risk grades fill the economic domain as a unit cube $[0,1]^J$ of R^J. The description of the economic agents by their risk coordinates in the economic domain gives great advantages for economic and financial modeling and reveals hidden economic factors and processes that impact economic evolution and forecasting. Continuous numeric risk grades distribute market stocks by their risk coordinates. That helps develop a description of the statistical moments of market trades, prices, and returns, taking into account the risk coordinates of the stocks of a company. For simplicity, at the first stage, we consider market trades under the action of a single risk. Even the description of the statistical moments under action of a single risk requires an essential modification of the risk rating notion.

3.1 Risk ratings as vector variables

Here we reconsider and extend the notion of risk ratings in such a manner that it can match the description of a set of statistical moments as functions of risk. We don't discuss here a particular methodology for the assessment of numeric risk ratings and consider it a worthy task for the risk rating agencies. However, the complexity of financial markets and the description of statistical moments of stock prices and returns as a tool for approximations of price and return probabilities require significant complication of the risk rating notion. We consider the conventional assessments of risk ratings (Metz and Cantor, 2007; Chane-Kon et al., 2010; Kraemer and Vazza, 2012) to be a zero approximation. Actually, the substitution of the conventional letter designations of risk ratings by the proposed numeric continuous risk ratings is like opening the hidden Pandora's box of economic complexity. Indeed, one can ask a simple question: should the numeric risk rating of an economic agent be a scalar or a vector? Such a question is almost impossible for literal notations and thus could be a surprise for researchers. Actually, the idea of describing the *n*-th statistical moments of the market trade value $C(t_k;n)$ and volume $U(t_k;n)$ (2.6), statistical moments of price $p(t_k;n)$ (2.8-2.10), and the return $r(t_k,\xi;n)$ (2.16-2.18) of stocks of a company as functions of the company's risk rating x is very simple. The risk rating x can serve as the coordinates to describe the evolution of a company's economic and financial variables and statistical moments in particular. However, to identify the *m-th* statistical moments of return $r(t_k,\xi;m)$ (2.16) for $m=1,2,\dots n,$ one should assess the component x_m using the economic variables of the *m-th* order. The risk rating of a company for the economic model that describes n statistical moments takes the form of a n-dimensional vector $\boldsymbol{x}=(x_1,\dots x_n)$.

As an example, compare two statistical moments of the trade value of two companies a and b: $C_a(t_k;1)$, $C_a(t_k;2)$ and $C_b(t_k;1)$, $C_b(t_k;2)$. Assume that $C_a(t_k;1)=C_b(t_k;1)$. Then it is reasonable to assume that the risk coordinate x_1 of the company a, which identifies $C_a(t_k,x_1;1)$ is almost equal to the risk coordinate y_1 of the company b, which identifies $C_b(t_k,y_1;1)$, and $x_1 \sim y_1$. However, if the second statistical moments are very different and $C_a(t_k;2) >> C_b(t_k;2)$, then to describe such a case, one needs additional variables x_2 and y_2 to take $C_a(t_k,x_2;2) >> C_b(t_k,y_2;2)$. Thus, to describe a set of two statistical moments, one should use a vector $\boldsymbol{x}=(x_1, x_2)$. The description of n statistical moments requires a n-dimensional vector $\boldsymbol{x}=(x_1,\dots x_n)$.

The origin of the complexity of risk ratings for a company relates to the complication of the economic model, which takes into account a finite set $m=1,..n$ of the statistical moments of the market trade values $C(t_k;n)$ and volumes $U(t_k;n)$ (2.6). Indeed, conventional economic models describe macroeconomic variables, which are composed of the sums of the first powers of trade values and volumes. For example, macroeconomic investments, credits, or consumption are determined by the sums (without doubling) of investment, credit, or consumption transactions made by all agents of the entire economy during an interval Δ. One can consider an investment made by an agent at time t_i during an interval Δ as a market transaction, and the sum of investments made by all agents during an interval Δ defines macroeconomic investments as a macroeconomic variable. We underline that the sums are taken over the first powers of investment transactions. To a certain extent, conventional economic models describe the relations and mutual dependence between the macroeconomic variables, composed by the sums of the first powers of market transactions. For convenience, we note them as the 1st order economic variables and call conventional macroeconomics the 1st order economic theory. Assessments of the risks of a company using economic variables of the 1st order, generate the x_1 component of the risk rating vector $\boldsymbol{x}=(x_1,...x_n)$. However, the sums of squares of the market trades, and the squares of investment, credit, or consumption transactions made by agents introduce the macroeconomic variables of the 2-d order. For example, the sums of squares of the investments made by all agents during the interval Δ define the investments of the 2-d order. The sums of squares of credits during Δ define the credits of the 2-d order. We consider the description of the mutual dependence between macroeconomic variables of the *2-d* order as the *2-d* order economic theory (Olkhov, 2021b; 2021c; 2022b). The description of the *2-d* order macroeconomic variables and the description of the *2-d* statistical moments of the market trade value $C(t_k;2)$ and volume $U(t_k;2)$ (2.6), and the 2-d statistical moments of the price $p(t_k;2)$ and return $r(t_k,\xi;2)$ averaged during Δ establish the unified problem. Predictions of the *2-d* statistical moments of the price $p(t_k;2)$ and return $r(t_k,\xi;2)$ depend on forecasting of the *2-d* order variables. The use of the 2-d order economic variables to assess the risk ratings of a company generates ratings x_2, which serve as coordinates of the 2-d statistical moments of price, return, trade values, etc. Consequently, the sums of *n-th* order market trade values, investment transactions, etc., during Δ determine variables of the *n*-th order. Risk assessments based on the economic variables of the *n*-th order generate risk ratings x_n as the *n*-th component of the risk rating vector $\boldsymbol{x}=(x_1,...x_n)$. Risk assessments depend on the complexity of the economic model. The simplest case is presented by conventional economic models, which are based on the economic variables composed by the sums of the first powers of economic or financial transactions during Δ. Such models can assess the average market price and return as functions of x_1. However, predictions of the market-based volatilities of price and return require forecasts of the 2-d statistical moments of the market trade value $C(t_k;2)$ and volume $U(t_k;2)$ (2.6) and the 2-d order economic theory. The predictions of the set of *m-th* statistical moments $m=1,...n$ of price $p(t_k;m)$ (2.8-2.10) and return $r(t_k,\xi;m)$ (2.16-2.18) that approximate price and return probabilities require economic models of the *m-th* order for $m=1,2,..n$. The assessments of risks based on the economic variables of the *m-th* order generate x_m components of the risk rating vector $\boldsymbol{x}=(x_1,...x_n)$. As we show below, predictions of the risk rating vector $\boldsymbol{x}=(x_1,...x_n)$ of a company on the horizon T determine the finite number of $m=1,..n$ of statistical moments of return $r(t_k,\xi,x_j;m)$ and define approximations of return and price probabilities.

3.2 Statistical moments of the return as functions of time and risk coordinates

In this section, we consider the statistical moments (2.6) of the market trade values $C(t_k,x_{qm};m)$ and volumes $U(t_k,x_{qm};m)$, the statistical moments $S(t_k,\xi,x_{qm};m)$ (2.14), the statistical moments of price $p(t_k,x_{qm};m)$ (2.8-2.10) and return $r(t_k,\xi,x_{qm};m)$ (2.16-2.18) of the stocks of a company q as functions of x_{qm} component of the risk vector $\boldsymbol{x}_q=(x_{q1},\ldots x_{qn})$ for $m=1,2,..n$. We consider the transition from the description of the statistical moments of the stocks of a particular company as functions of its risk coordinates $\boldsymbol{x}_q=(x_{q1},\ldots x_{qn})$ to the description of the *collective* statistical moments of the stocks of all companies, which have risk coordinates in the neighborhood of the vector $\boldsymbol{x}=(x_1,\ldots x_n)$. As the economic domain for the model, which describes $m=1,2,..n$ statistical moments of the market trades, price, and return, we take (3.1):

$$\boldsymbol{x} \in [0,1]^n \; ; \quad \boldsymbol{x} = (x_1, \ldots x_n) \quad ; \quad 0 \le x_m \le 1 \; ; \quad m = 1, \ldots n \tag{3.1}$$

In 2022, the NYSE traded around 2500 stocks and the Nasdaq traded almost 3600 stocks of domestic and international companies (Statista, 2023). Let us study the stock market, which trades $Q>>1$ stocks of companies, and assume that the stocks are traded under the action of a single risk. A huge number $Q>>1$ of companies traded on the market causes that a lot of different companies $q=1,..Q(\boldsymbol{x})$ can have coordinates $\boldsymbol{x}_q$ in the neighborhood of the vector $\boldsymbol{x}$ of the economic domain. Choose a scale $d<1$, which defines a small volume $dV(\boldsymbol{x})$ in the economic domain (3.1):

$$0 < d < 1 \; ; \; dV(\boldsymbol{x}) \sim d^n \; ; \; \boldsymbol{x}_q \in dV(\boldsymbol{x}) \leftrightarrow x_i - \frac{d}{2} \le x_{qi} \le x_i + \frac{d}{2} \; ; \; \boldsymbol{x} = (x_1 \ldots x_n) \tag{3.2}$$

The choice of the scale d allows at time t_{ik} sum the trade values and volumes of all the stocks of companies q, $q=1,..Q(\boldsymbol{x})$ which have the risk coordinates $\boldsymbol{x}_q$ (3.1) inside a volume $dV(\boldsymbol{x})$ (3.2) near vector $\boldsymbol{x}$ of the economic domain.

Because each company traded on the market is described by the individual risk vector $\boldsymbol{x}_q=(x_{q1},\ldots x_{qn})$, we denote its trade values $C(t_{ik},\boldsymbol{x}_q)$ and volumes $U(t_{ik},\boldsymbol{x}_q)$ as functions of $\boldsymbol{x}_q$. However, we assume that the sums of the m-th powers of trade values $C^m(t_{ik},\boldsymbol{x}_q)$ and volumes $U^m(t_{ik},\boldsymbol{x}_q)$ depend on the m-th components x_m of the vector $\boldsymbol{x}=(x_1,\ldots x_n)$. For $m=1,2,..n$ we note the sums of the m-th powers of values $C^m(t_{ik},\boldsymbol{x}_q)$ and volumes $U^m(t_{ik},\boldsymbol{x}_q)$ inside $dV(\boldsymbol{x})$ (3.2) as:

$$C(t_{ik}, x_m; m) = \sum_{x_q \in dV(x)} C^m(t_{ik}, \boldsymbol{x}_q) \quad ; \quad U(t_{ik}, x_m; m) = \sum_{x_q \in dV(x)} U^m(t_{ik}, \boldsymbol{x}_q) \tag{3.3}$$

The value $C(t_{ik},x_m;m)$ (3.3) at a time t_{ik} equals the sum of the m-th powers of trade values with the stocks of different companies q, which have coordinates $\boldsymbol{x}_q$ inside a small volume $dV(\boldsymbol{x})$ (3.2) near the vector $\boldsymbol{x}$ of the economic domain (3.1). The volume $U(t_{ik},x_m;m)$ (3. 3) equals the corresponding sum of the m-th powers of all trade volumes at a time t_{ik}. Simple relations

(3.3) for $m=1,2,..n$ transfer the description of the trade values $C^m(t_{ik},\boldsymbol{x}_q)$ and volumes $U^m(t_{ik},\boldsymbol{x}_q)$ as functions of coordinates $\boldsymbol{x}_q$ of the stocks of a company q to the description of the *collective* trade values $C(t_{ik},x_m;m)$ and volumes $U(t_{ik},x_m;m)$ as functions of coordinates $\boldsymbol{x}$. We introduce a similar definition of the sums of the m-th powers of values $S^m(t_{ik},\xi,\boldsymbol{x}_q)$:

$$S(t_{ik},\xi,x_m;m) = \sum_{x_q \in dV(x)} S^m\left(t_{ik},\xi,\boldsymbol{x}_q\right) \tag{3.4}$$

The sums on the left side of (3.3; 3.4) are functions of x_m components of coordinates $\boldsymbol{x}$. The relations (3.3; 3.4) roughen the description of the market trade with the stocks of companies $q=1,..Q$ and transfer it to the description of the *collective* trade values and volumes as functions of components x_m of the vector $\boldsymbol{x}$. We take the return equation (2.15) as:

$$C(t_{ik},x_m;m) = r(t_{ik},\xi,x_m;m)S(t_{ik},\xi,x_m;m) \quad ; \quad m = 1,2,..n \tag{3.5}$$

Equation (3.5) at a time t_{ik} for the time shift ξ (2.12) introduces a new notion: the *collective* m-th return $r(t_{ik},\xi,x_m;m)$ as a function of x_m. The economic meaning of (3.5) is that it determines the *collective* return $r(t_{ik},\xi,x_m;m)$ as a ratio of the sums of the *m-th* powers of values $C(t_{ik},x_m;m)$ (3.3) of the stocks of all companies in the neighborhood of the vector $\boldsymbol{x}$, which were sold at a time t_{ik} to the sum of the *m-th* powers of their values $S(t_{ik},\xi,x_m;m)$ (3.4) in the past at time t_{ik}-ξ. Equation (3.5) transfers the description of the values and returns of the stocks of a company q with coordinates $\boldsymbol{x}_q$ to the description of the *collective* market trade values $C(t_{ik},x_m;m)$ (3.3), $S(t_{ik},\xi,x_m;m)$ (3.4), and return $r(t_{ik},\xi,x_m;m)$ (3.5), as functions of x_m. That is the first step to replacing consideration of the financial properties of stocks of a particular company with a description of the *collective* market-based financial properties as a function of coordinates $\boldsymbol{x}$. However, the return $r(t_{ik},\xi,x_m;m)$ determined by (3.5), depends on the time t_{ik} (2.5) of a particular trade. The interval between trades equals ε (2.1; 2.3), which can be equal to a second or even a fraction of a second. Such frequencies of trades result in high irregularities in the trade values and cause irregularities or randomness of the return. To derive a regular and smooth description of return as a function of time and coordinates, one should average the return $r(t_{ik},\xi,x_m;m)$ (3.5) over the averaging interval. The choice of the time averaging interval is not a simple problem. We have already described the time interval ε (2.1; 2.3), which is determined by the frequency of the market trades. We introduced the interval Δ (2.3-2.5), which determines the averaging of the trade values and volumes of the individual stocks of a particular company. We assume that Δ is the same for all Q stocks traded on the whole market. The trade values $C(t_{ik},x_m;m)$ (3.3) and $S(t_{ik},\xi,x_m;m)$ (3.4) for $m=1,2,..n$ are determined by the sums of corresponding variables of the individual stocks in the neighborhood $dV(\boldsymbol{x})$ (3.2) of $\boldsymbol{x}$. We assume that the time averaging interval $\Delta \leq \Delta_x$ and take that Δ_x is the same for all points $\boldsymbol{x}$ in the economic domain. The choice of the averaging interval Δ_x introduces a new time axis division with time series τ_k, which describe the collective trades averaged over Δ_x. For simplicity, we take Δ_x as a multiple of Δ and (2.3)

$$\Delta_x = k_x \Delta = k_x N\varepsilon \quad ; \quad k_x = 1,2,.. \quad ; \quad \Delta = N\varepsilon \tag{3.6}$$

$$\tau_k = t_0 + k\,\Delta_x \quad ; \quad k = 1,.. \quad ; \quad \tau_k - \frac{\Delta_x}{2} \le t_{ik} \le \tau_k + \frac{\Delta_x}{2} \quad ; \quad i = 0,1,\dots k_x N \tag{3.7}$$

We mention that in (3.7), the time series t_{ik} represents the renumbered times t_i (2.1) of the initial time series of market trades. Similar to (2.4; 2.5) we renumber the initial time series t_i (2.1) so that each interval Δ_x (3.6; 3.7) contains k_xN terms of the market trades (3.3-3.5) at t_{ik}. The choice of the interval Δ_x helps average the sums of the trade values $C(t_{ik},x_m;m)$ (3.3). We determine the m-th statistical moments of the trade values $C(\tau_k,x_m;m)$ at time τ_k averaged over Δ_x as (3.8; 3.9):

$$C(\tau_k, x_m; m) \equiv E[C(t_{ik}, x_m; m)] \sim \frac{1}{k_x N} \sum_{i=1}^{k_x N} C(t_{ik}, x_m; m) = \frac{1}{k_x N} \sum_{i=1}^{k_x N} \sum_{x_q \in dV(x)} C^m(t_{ik}, \boldsymbol{x}_q) \tag{3.8}$$

If one changes the order of sums in (3.8) then:

$$C(\tau_k, x_m; m) \sim \sum_{x_q \in dV(x)} \frac{1}{k_x N} \sum_{i=1}^{k_x N} C^m(t_{ik}, \boldsymbol{x}_q) = \sum_{x_q \in dV(x)} C(\tau_k, x_{qm}; m) \quad ; \quad m = 1,2,..n \tag{3.9}$$

$$C(\tau_k, x_{qm}; m) = \frac{1}{k_x N} \sum_{i=1}^{k_x N} C^m(t_{ik}, \boldsymbol{x}_q)$$

Above relations denote the m-th statistical moment of the trade values of the stocks of company q averaged over Δ_x. The variable x_{qm} defines the m-th component of the vector $\boldsymbol{x}_q=(x_1,\dots x_n)$. Thus, $C(\tau_k,x_m;m)$ in (3.8; 3.9) for $m=1,2,..n$ equals the sum of the m-th statistical moments of the trade values of stocks of all companies q with coordinates $\boldsymbol{x}_q$ in the $dV(\boldsymbol{x})$ (3.2) averaged over Δ_x (3.6; 3.7). We denote $C(\tau_k,x_m;m)$ in (3.8; 3.9) as the m-th statistical moment of the trade value. The similar meaning has the m-th statistical moment of the trade volume $U(\tau_k,x_m;m)$ in (3.10; 3.11):

$$U(\tau_k, x_m; m) \equiv E[U(t_{ik}, x_m; m)] \sim \frac{1}{k_x N} \sum_{i=1}^{k_x N} U(t_{ik}, x_m; m) \quad ; \quad m = 1,2,\dots n \tag{3.10}$$

$$U(\tau_k, x_m; m) \sim \sum_{x_q \in dV(x)} \frac{1}{k_x N} \sum_{i=1}^{k_x N} U^m(t_{ik}, \boldsymbol{x}_q) = \sum_{x_q \in dV(x)} U(\tau_k, x_{qm}; m) \tag{3.11}$$

Relations (3.8-3.11) define the m-th statistical moments of the market trade values $C(\tau_k,x_m;m)$ and volumes $U(\tau_k,x_m;m)$ as functions of time τ_k and components x_m of the vector $\boldsymbol{x}=(x_1,\dots x_n)$. Similar considerations determine the m-th statistical moments $S(\tau_k,\xi,x_m;m)$ of (3.4) at x_m:

$$S(\tau_k, \xi, x_m; m) \equiv E[S(t_{ik}, \xi, x_m; m)] \sim \frac{1}{k_x N} \sum_{i=1}^{k_x N} S(t_{ik}, \xi, x_m; m) = \frac{1}{k_x N} \sum_{i=1}^{k_x N} \sum_{x_q \in dV(x)} S^m(t_{ik}, \xi, \boldsymbol{x}_q) \tag{3.12}$$

$$S(\tau_k, \xi, x_m; m) \sim \sum_{x_q \in dV(x)} \frac{1}{k_x N} \sum_{i=1}^{k_x N} S^m(t_{ik}, \xi, \boldsymbol{x}_q) = \sum_{x_q \in dV(x)} S(\tau_k, x_{qm}; m) \tag{3.13}$$

We highlight that we use the symbol ~ to show that (3.8-3.13) are the assessments of the statistical moments by a finite number k_xN of terms of time series. We consider the return equation (3.5) and determine the m-th statistical moments of return $r(\tau_k,\xi,x_m;m)$ at τ_k (3.7):

$$r(\tau_k,\xi,x_m;m) \equiv E\left[r^m\left(t_{ik},\xi,x_{qm};\right)\right] = \frac{\sum_{i=1}^{k_xN} r(t_{ik},\xi,x_m;m)S(t_{ik},\xi,x_m;m)}{\sum_{i=1}^{k_xN} S(t_{ik},\xi,x_m;m)} = \frac{\sum_{x_q\in dV(x)} C(\tau_k,x_{qm};m)}{\sum_{x_q\in dV(x)} S(\tau_k,\xi,x_{qm};m)} \quad (3.14)$$

$$C(\tau_k,x_m;m) = r(\tau_k,\xi,x_m;m)S(\tau_k,\xi,x_m;m) \quad ; \quad m=1,2,..n \quad (3.15)$$

To justify our definitions of the statistical moments of the trade values $C(\tau_k,x_m;m)$ (3.8-3.9), volumes $U(\tau_k,x_m;m)$ of "sale" (3.10-3.11) and values of "purchase" $S(\tau_k,\xi,x_m;m)$ (3.12; 3.13), which define the m-th statistical moments of return $r(\tau_k,\xi,x_m;m)$ (3.14; 3.15), we refer to the famous work by Markowitz (1952). If one follows Markowitz, one can assume that an "investor" has a portfolio of $q=1,..Q(\boldsymbol{x})$ stocks, which are determined by their coordinates $\boldsymbol{x}_q$. An "investor" purchased these stocks in the past at t_{ik}-ξ and "today" at t_{ik} sells them. The volume of the sale $U(t_{ik},\boldsymbol{x}_q)$ at t_{ik} is the same as the volume of the purchase at t_{ik}-ξ. Thus, the values of a single purchase $S(t_{ik},\xi,x_{q1})$ and the values of a single sale $C(t_{ik},x_{q1})$ determine the return $r(t_{ik},\xi,x_{q1})$ of stock q at t_{ik} at point $\boldsymbol{x}_q$ as (3.16):

$$C\left(t_{ik},x_{q1}\right) = r\left(t_{ik},\xi,x_{q1}\right)S\left(t_{ik},\xi,x_{q1}\right) \quad (3.16)$$

The total value of purchase $S_\Sigma(\tau_k,\xi,x_{q1})$ at τ_k-ξ and the total value of sale $C_\Sigma(\tau_k,x_{q1})$ at τ_k determine the total values of an "investor's" portfolio q at τ_k-ξ and now at τ_k during interval Δ_x (3.6). The ratio of the portfolio's values define the return $r(\tau_k,\xi,x_{q1};1)$ of the portfolio q as:

$$C_\Sigma\left(\tau_k,x_{q1}\right) = r\left(\tau_k,\xi,x_{q1};1\right)S_\Sigma\left(\tau_k,\xi,x_{q1};1\right) \quad (3.17)$$

$$C_\Sigma\left(\tau_k,x_{q1};1\right) = \sum_{i=1}^{k_xN} C\left(t_{ik},\boldsymbol{x}_q\right) \quad ; \quad S_\Sigma\left(\tau_k,\xi,x_{q1};1\right) = \sum_{i=1}^{k_xN} S\left(t_{ik},\xi,\boldsymbol{x}_q\right) \quad (3.18)$$

Due to Markowitz (1952), the return $r(\tau_k,\xi,x_1;1)$ of the portfolio formed with $q=1,..Q(\boldsymbol{x})$ stocks at $\boldsymbol{x}$ at a time τ_k should be weighed by the "purchase" values $S_\Sigma(\tau_k,\xi,x_{q1})$:

$$S_\Sigma\left(\tau_k,\xi,x_{q1};1\right) = \sum_{x_q\in dV(x)} S_\Sigma\left(\tau_k,\xi,x_{q1};1\right) = \sum_{x_q\in dV(x)} \sum_{i=1}^{k_xN} S\left(t_{ik},\xi,x_{q1}\right) \quad (3.19)$$

$$C_\Sigma(\tau_k,x_1;1) = \sum_{x_q\in dV(x)} C_\Sigma\left(\tau_k,x_{q1};1\right) = \sum_{x_q\in dV(x)} \sum_{i=1}^{k_xN} C\left(t_{ik},x_{q1}\right) \quad (3.20)$$

From (3.19; 3.20) obtain the return $r(\tau_k,\xi,x_1;1)$ of the portfolio at $\boldsymbol{x}$ (3.21;3.22):

$$C_\Sigma(\tau_k,x_1;1) = r(\tau_k,\xi,x_1;1)S_\Sigma(\tau_k,\xi,x_1;1) \quad (3.21)$$

$$r(\tau_k,\xi,x_1;1) = \frac{\sum_{x_q\in dV(x)} r(\tau_k,\xi,x_{q1};1)S_\Sigma(\tau_k,\xi,x_{q1};1)}{\sum_{x_q\in dV(x)} S_\Sigma(\tau_k,\xi,x_{q1};1)} = \frac{\sum_{x_q\in dV(x)} C_\Sigma(\tau_k,x_{q1};1)}{\sum_{x_q\in dV(x)} S_\Sigma(\tau_k,\xi,x_{q1};1)} \quad (3.22)$$

From (3.22) and (3.9; 3.13) obtain the return $r(\tau_k,\xi,x_1;1)$ of the portfolio through statistical moments of the "purchased" $S(\tau_k,\xi, x_1;1)$ (3.12; 3.13) and "sold" $C(\tau_k, x_1;1)$ (3.8; 3.9) values

$$r(\tau_k,\xi,x_1;1) = \frac{\sum_{x_q\in dV(x)} C_\Sigma(\tau_k,x_{q1};1)}{\sum_{x_q\in dV(x)} S_\Sigma(\tau_k,\xi,x_{q1};1)} = \frac{\sum_{x_q\in dV(x)} C(\tau_k,x_{q1};1)}{\sum_{x_q\in dV(x)} S(\tau_k,\xi,x_{q1};1)} = \frac{C(\tau_k,x_1;1)}{S(\tau_k,\xi,x_1;1)} \quad (3.23)$$

Our definition of the m-th statistical moments of the return $r(\tau_k,\xi,x_m;m)$ (3.14; 3.15) at a point $\boldsymbol{x}$ for $m=1$ completely coincides with the definition of the portfolio return (3.21-3.23) by Markowitz (1952). For $m=2,3,..n$ the market-based *m-th* statistical moments of the "purchased" values $S(\tau_k,\xi,x_m;m)$ (3.12; 3.13) and "sold" values $C(\tau_k,x_m;m)$ (3.8; 3.9), trade volumes $U(\tau_k,x_m;m)$ (3.10; 3.11) and the *m-th* statistical moments of return $r(\tau_k,\xi,x_m;m)$ (3.14; 3.15) give a direct extension of Markowitz' approach for the *m-th* statistical moments of the portfolio return.

3.4 The statistical moments of the return of the whole stock market

To define the market-based statistical moments of return of the whole market, one should sum in (3.3) the stocks of all companies, traded on the market. We define the *collective m-th* trade values $C(t_{ik};m)$, the m-th volumes $U(t_{ik};m)$, and the m-th "purchased" values $S(t_{ik},\xi;m)$:

$$C(t_{ik};m) = \sum_{x_q} C^m(t_{ik},\boldsymbol{x}_q) \quad ; \quad U(t_{ik};m) = \sum_{x_q} U^m(t_{ik},\boldsymbol{x}_q) \tag{3.24}$$

$$S(t_{ik},\xi;m) = \sum_{x_q} S^m(t_{ik},\xi,\boldsymbol{x}_q) \quad ; \quad m = 1,2,\dots n \tag{3.25}$$

Relations in (3.24; 3.25) denote sums of trade values and volumes of stocks over all Q companies traded on the whole stock market. The *collective* m-th return $r(t_{ik},\xi;m)$ of all stocks on the market at time t_{ik} takes the form:

$$C(t_{ik};m) = r(t_{ik},\xi;m)S(t_{ik},\xi;m) \tag{3.26}$$

To smooth variations of the return at a time t_{ik}, one should choose the market's averaging interval Δ_m. The *collective* values and volumes (3.24; 3.25) of the stock market have a characteristic time of change longer than the interval Δ_x selected for the averaging of market trades and returns at the point $\boldsymbol{x}$ of the economic domain. For simplicity, we take the market averaging interval Δ_m as:

$$\Delta_x \leq \Delta_m \quad ; \quad \Delta_m = k_m\Delta_x = k_m k_x \Delta = k_m k_x N\varepsilon \quad ; \quad k_x, k_m = 1,2,.. \tag{3.27}$$

The market averaging interval Δ_m introduces a new time axis division multiple of Δ_m

$$\mu_k = t_0 + k\,\Delta_m \quad ; \quad k = 1,.. \quad ; \quad \mu_k - \frac{\Delta_m}{2} \leq t_{ik} \leq \mu_k + \frac{\Delta_m}{2} \quad ; \quad i = 0,1,\dots k_m k_x N \tag{3.28}$$

Similar to (3.8-3.15) for $m=1,2,\dots n$ obtain the m-th statistical moments of the trade values, volumes, and returns of the whole market as follows:

$$C(\mu_k;m) \equiv E[C(t_{ik};m)] \sim \frac{1}{k_m k_x N} \sum_{i=1}^{k_m k_x N} C(t_{ik};m) = \frac{1}{k_m k_x N} \sum_{i=1}^{k_m k_x N} \sum_{x_q} C^m(t_{ik},\boldsymbol{x}_q) \tag{3.29}$$

$$C(\mu_k;m) \sim \sum_{x_q} \frac{1}{k_m k_x N} \sum_{i=1}^{k_m k_x N} C^m(t_{ik},\boldsymbol{x}_q) = \sum_{x_q} C(\mu_k, x_{qm};m) \tag{3.30}$$

$$U(\mu_k;m) \equiv E[U(t_{ik};m)] \sim \frac{1}{k_m k_x N} \sum_{i=1}^{k_m k_x N} U(t_{ik};m) \tag{3.31}$$

$$U(\mu_k;m) \sim \sum_{x_q} \frac{1}{k_m k_x N} \sum_{i=1}^{k_m k_x N} U^m(t_{ik},\boldsymbol{x}_q) = \sum_{x_q} U(\mu_k, x_{qm};m) \tag{3.32}$$

$$S(\mu_k,\xi;m)\sim\frac{1}{k_m k_x N}\sum_{i=1}^{k_m k_x N}S(t_{ik},\xi;m)=\frac{1}{k_m k_x N}\sum_{i=1}^{k_m k_x N}\sum_{\boldsymbol{x}_q}S^m(t_{ik},\xi,\boldsymbol{x}_q) \quad (3.33)$$

$$S(\mu_k,\xi;m)\equiv E[S(t_{ik},\xi;m)]\sim\sum_{\boldsymbol{x}_q}\frac{1}{k_m k_x N}\sum_{i=1}^{k_m k_x N}S^m(t_{ik},\xi,\boldsymbol{x}_q)=\sum_{\boldsymbol{x}_q}S(\mu_k,x_{qm};m) \quad (3.34)$$

From (3.29-3.34) obtain for the m-th statistical moments of return $r(\mu_k,\xi;m)$ of the whole market at a time μ_k averaged during the interval Δ_m (3.27; 3.28):

$$r(\mu_k,\xi;m)\equiv E[r(t_{ik},\xi;m)]=\frac{\sum_{i=1}^{k_x N}r(t_{ik},\xi;m)S(t_{ik},\xi;m)}{\sum_{i=1}^{k_x N}S(t_{ik},\xi;m)}=\frac{\sum_{x_q}C(\mu_k,x_{qm};m)}{\sum_{x_q}S(\mu_k,x_{qm};m)} \quad (3.35)$$

$$C(\mu_k;m)=r(\mu_k,\xi;m)S(\mu_k,\xi;m) \quad (3.36)$$

At the end of this section, we highlight the importance of the four consecutive time axis divisions determined by the four time intervals $\varepsilon << \Delta \le \Delta_x \le \Delta_m$. The smallest interval ε is determined by the frequency of market trading. It introduces the initial market trade time series at t_i (2.1) for the further averaging procedures. The scale Δ determines the time averaging interval for the assessments of statistical moments of market trade and return of stock of a particular company. For simplicity, we assume that Δ is the same for all stocks, traded at the market and that Δ is multiply of ε, so that $\Delta=N\varepsilon$ (2.3). The interval Δ introduces a new time axis division t_k (2.4; 2.5) multiply of Δ. The sum of trades with stocks with risk coordinates $\boldsymbol{x}_q$ in the neighborhood of point $\boldsymbol{x}$ of the economic domain transfers the description of the statistical moments of market trade and return of stocks of particular companies to the description of the statistical moments of *collective* market trade and return as functions of coordinates $\boldsymbol{x}$. The *collective* trade value, volume, and return of stocks with risk coordinates $\boldsymbol{x}_q$ in the neighborhood of point $\boldsymbol{x}$ change more slowly than the trade value, volume, and return of the individual stocks. Hence, the effective averaging of the time series of the *collective* trade value, volume, and return near point $\boldsymbol{x}$ can require a time interval Δ_x that is longer than the interval Δ of the individual stocks. For convenience, we take the time scale Δ_x as a multiple of Δ. We take $\Delta_x=k_x\,\Delta=k_x\,N\varepsilon$ (3.6) and (3.7) Δ_x introduces a new time axis division τ_k (3.7) as multiple of Δ_x. The time series τ_k (3.7) describe the statistical moments of trade value, volume, and return averaged over Δ_x. Finally, the collective trade and return of the whole stock market determine the market interval Δ_m. The change in trade of the whole market is slower, than the change of *collective* trade of stocks near point $\boldsymbol{x}$. Thus, the market interval Δ_m can be longer than Δ_x. We take market interval Δ_m, which determines time averaging of the *collective* trades of stocks of all companies on the whole market, as $\Delta_m=k_m\,\Delta_x = k_m\,k_x\,\Delta=k_m k_x N\varepsilon$ (3.27). The market interval Δ_m introduces the market time axis division μ_k (3.28), which determines the time series of the statistical moments of trade value, volume, and return of all stocks traded at the whole market. These four time series describe the financial problems of the stock market with different accuracy. The different choices of time intervals result in different approximations of financial markets.

4. The statistical moments of stock price as functions of risks

Relations (2.19; 2.20) tie up the statistical moments of the price and return of stocks of a company *q*. The results of section 3 define the statistical moments of the *collective* price of stocks of companies with risk coordinates near point $\boldsymbol{x}$ and the statistical moments of the price of the whole market. From (2.19; 2.20) and (3.8-3.15), the *m*-th statistical moments $p(\tau_k,x_m;m)$ of the *collective* price of stocks at $\boldsymbol{x}$ at a time τ_k averaged during Δ_x take the form:

$$p(\tau_k, x_m; m) = r(\tau_k, \xi, x_m; m)\frac{S(\tau_k,\xi,x_m;m)}{U(\tau_k,x_m;m)} = \frac{C(\tau_k,x_m;m)}{U(\tau_k,x_m;m)} \quad (4.1)$$

$$C(\tau_k, x_m; m) = p(\tau_k, x_m; m)U(\tau_k, x_m; m) \quad (4.2)$$

From (3.29 – 3.36), the *m*-th statistical moments $p(\mu_k;m)$ of the *collective* price of all stocks traded at the market at the time μ_k averaged during Δ_m (3.27; 3.28) take the form:

$$p(\mu_k; m) = r(\mu_k, \xi; m)\frac{S(\mu_k,\xi;m)}{U(\mu_k;m)} = \frac{C(\tau_k;m)}{U(\tau_k;m)} \quad (4.3)$$

$$C(\mu_k; m) = p(\mu_k; m)U(\mu_k; m) \quad (4.4)$$

The finite sets of *n* statistical moments of price $p(\tau_k,x_m;m)$ (4.1; 4.2) and $p(\mu_k;m)$ (4.3; 4.4) for $m=1,..n$ define the *n*-approximations of the price characteristic functions and probabilities (App.A).

5. Economic evolution and prediction of statistical moments of stock return

The continuous economic media approximation presents the transition from the description of the economic variables of individual agents to the description of the *collective* variables as functions of risk coordinates $\boldsymbol{x}$ (Olkhov, 2016a – 2021b). As the *collective* variables, one can consider the sums of the *m*-th powers of market trade values $C_\Sigma(t_k,x_{qm};m)$, volumes $U_\Sigma(t_k,x_{qm};m)$, and "purchased" values $S_\Sigma(t_k,\xi,x_{qm};m)$ during the interval Δ_k (2.4).

$$C_\Sigma(t_k, x_{qm}; m) = \sum_{i=1}^{N} C^m(t_{i,k}, \boldsymbol{x}_q) = N \cdot C(t_k, x_{qm}; m) \quad (5.1)$$

$$U_\Sigma(t_k, x_{qm}; m) = \sum_{i=1}^{N} U^m(t_{i,k}, \boldsymbol{x}_q) = N \cdot U(t_k, x_{qm}; m) \quad (5.2)$$

$$S_\Sigma(t_k, \xi, x_{qm}; m) = \sum_{i=1}^{N} S^n(t_{ik}, \xi, \boldsymbol{x}_q) = N \cdot S(t_k, \xi, x_{qm}; m) \quad (5.3)$$

The sums $C_\Sigma(t_k,x_{mq};m)$ (5.1), $U_\Sigma(t_k,x_{mq};m)$ (5.2), and $S_\Sigma(t_k,\xi,x_{mq};m)$ (5.3), for $m=1,2,…n$ define the *m*-th statistical moments of price $p(t_k,x_{qm};m)$ (2.8-2.10) or (4.1; 4.2) and return $r(t_k,\xi,x_{qm};m)$ (2.16-2.18) in the same way:

$$r(t_k, \xi, x_{qm}; m) = \frac{C_\Sigma(t_k,x_m;m)}{S_\Sigma(t_k,\xi,x_{qm};m)} \quad ; \quad p(t_k, x_m; m) = \frac{C_\Sigma(t_k,x_m;m)}{U_\Sigma(t_k,x_m;m)} \quad (5.4)$$

We use relations that are similar to (5.1-5.3) to define the sums $C_\Sigma(\tau_k,x_m;m)$ (5.5) from (3.8), $U_\Sigma(\tau_k,x_m;m)$ (5.6) from (3.10), and $S_\Sigma(\tau_k,\xi,x_m;m)$ (5.6) from (3.12):

$$C_\Sigma(\tau_k,x_m;m) = \sum_{i=1}^{k_xN} C(t_{ik},\boldsymbol{x};m) = \sum_{i=1}^{k_xN} \sum_{\boldsymbol{x}_q \in dV(\boldsymbol{x})} C^m\big(t_{ik},\boldsymbol{x}_q\big) \quad (5.5)$$

$$U_\Sigma(\tau_k,x_m;m) = \sum_{i=1}^{k_xN} U(t_{ik},\boldsymbol{x};m) = \sum_{i=1}^{k_xN} \sum_{\boldsymbol{x}_q \in dV(\boldsymbol{x})} U^m\big(t_{ik},\boldsymbol{x}_q\big) \quad (5.6)$$

$$S_\Sigma(\tau_k,\xi,x_m;m) = \sum_{i=1}^{k_xN} S(t_{ik},\xi,\boldsymbol{x};n) = \sum_{i=1}^{k_xN} \sum_{\boldsymbol{x}_q \in dV(\boldsymbol{x})} S^m\big(t_{ik},\xi,\boldsymbol{x}_q\big) \quad (5.7)$$

$$r(\tau_k,\xi,x_m;m) = \frac{C_\Sigma(\tau_k,x_m;m)}{S_\Sigma(\tau_k,\xi,x_m;m)} \quad ; \quad p(\tau_k,x_m;m) = \frac{C_\Sigma(\tau_k,x_m;m)}{U_\Sigma(\tau_k,x_m;m)} \quad (5.8)$$

For the case of the whole stock market, the similar notations define the sums $C_\Sigma(\mu_k;m)$ (5.9) from (3.30), $U_\Sigma(\mu_k;m)$ (5.10) from (3.31), and $S_\Sigma(\mu_k,\xi;m)$ (5.11) from (3.33):

$$C_\Sigma(\mu_k;m) = \sum_{i=1}^{k_mk_xN} C(t_{ik};m) = \sum_{i=1}^{k_mk_xN} \sum_{\boldsymbol{x}_q} C^m\big(t_{ik},\boldsymbol{x}_q\big) \quad (5.9)$$

$$U_\Sigma(\mu_k;m) = \sum_{i=1}^{k_mk_xN} U(t_{ik};m) = \sum_{i=1}^{k_mk_xN} \sum_{\boldsymbol{x}_q} U^m\big(t_{ik},\boldsymbol{x}_q\big) \quad (5.10)$$

$$S_\Sigma(\mu_k,\xi;m) = \sum_{i=1}^{k_mk_xN} S(t_{ik},\xi;m) = \sum_{i=1}^{k_mk_xN} \sum_{\boldsymbol{x}_q} S^m\big(t_{ik},\xi,\boldsymbol{x}_q\big) \quad (5.11)$$

$$r\big(\mu_k,\xi;m\big) = \frac{C_\Sigma(\mu_k;m)}{S_\Sigma(\mu_k,\xi;m)} \quad ; \quad p\big(\mu_k;m\big) = \frac{C_\Sigma(\mu_k;m)}{U_\Sigma(\mu_k;m)} \quad (5.12)$$

The evolution of the *collective* additive variables such as the sums (5.1-5.3; 5.5-5.7; 5.9-5.11) can be described by the equations, which are somewhat similar to the equations of flows of fluids (App.B). However, we highlight that the nature of economic evolution has nothing in common with physical hydrodynamics, and we believe any direct comparisons make almost no sense. The dynamics of the ratios of additive economic variables describe the evolution of non-additive variables, like the *m*-th statistical moments of return and price. The change in agents' risk ratings due to economic, financial, and other factors causes the motion of agents in the economic domain. Each agent carries its additive economic variables. The c*ollective* motion of agents in the economic domain generates the flows of agents' additive economic and financial variables. The equations of motion (App.B) describe the dynamics of the collective additive economic variables and their flows as functions of time and risk coordinates (Olkhov, 2019; 2020).

An investor, who is seeking to forecast the probability of price and return of a company *q* at horizon *T* should follow the path we described above, but in reverse order. On that path investors will face irresistible economic obstacles, which limit the accuracy of any forecasts of the market-based probabilities of price and return.

Any amount of economic, financial, or market data "today" can help assess only approximations of current probabilities of price and return. The accuracy of these approximations determines the possible errors in forecasts, and, hence, the probable financial losses. Any assessments of the current probabilities of price and return of stocks of a company are determined by the choice of the time interval Δ, which averages the initial irregular market time series with a period ε between trades. The interval $\Delta >> \varepsilon$ determines the assessments of the statistical moments of the market trade value $C(t_k;n)$ and volume $U(t_k;n)$ (2.6). In turn, the assessments of the trade statistical moments determine the price $p(t_k;n)$ (2.8-2.10) and return $r(t_k,\xi;n)$ (2.16-2.18) statistical moments, which are linked by the relations (2.19; 2.20). The more statistical moments an investor can assess during the interval Δ, the greater the accuracy of the price and return probability he could get "today." However, a long averaging interval Δ equal to days, weeks, or months could increase the number of statistical moments but would raise the uncertainty of current, "this hour" investor's decisions. The choice of the duration of the averaging interval is an important problem that should be solved by each investor.

It is well known that any predictions of economic variables for an individual company require forecasts of the economic and market environment. In section 3, we describe the statistical moments (3. 8; 3.10; 3.12; 3.30; 3.31; 3.33) of market trade and the sums of market trades (5.1-5.3; 5.5-5.7; 5.9-5.11) as functions of risk coordinates in the economic domain. The main advantage of our approach lies in the fact that we perform successive transitions from the description of the trade statistical moments of an individual company to the description of the statistical moments of the *collective* trades at point $\boldsymbol{x}$ and then to the description of the statistical moments of trades of all companies on the whole market.

To predict the m-th statistical moments of price and return at the horizon T, an investor should forecast the sums of the m-th trade values $C_\Sigma(t;m)$, volumes $U_\Sigma(t;m)$, and values $S_\Sigma(t,\xi;m)$ on the whole market. Each extra $m=2,3,..n$ requires the development of the additional economic model of the m-th order to describe the interactions between economic variables of the *m-th* order. In simple words, to increase the accuracy of the price or return probability forecast with an extra m-th statistical moment, one should develop additional *m-th* order economic theory, each comparable with the existing conventional macroeconomic theory.

It is the first "impossible task" that an investor should consider on his way to estimating the economic environment that can help him forecast price and return probabilities. The forecasts of the whole stock market are useful but are too uncertain for predictions of the return and price probabilities of a particular company. An investor should refine the forecast and extend the description of market trade statistical moments from the model of the whole stock market to the continuous economic approximation of the market trades at point $\boldsymbol{x}$. Using the slow variations of the market trade sums (5.9-5.11) or the statistical moments (3.39-3.44) of the whole market as an economic environment, an investor should forecast the dynamics of the trade sums (5.5-5.7) or the trade statistical moments (3.18-3.21) as functions of $\boldsymbol{x}$.

However, at that point, an investor faces additional hardships, including the problem of risk rating assessments. Each extra *m*-th trade statistical moment (3.39-3.44) generates a set of the economic variables of the *m*-th order. For each $m=1,2,..n$, the risk assessments using economic variables of the *m*-th order result in the definition of the *m*-th component x_m of the risk rating vector $\boldsymbol{x}=(x_1,...x_n)$. To describe the dynamics of *m-th* trade statistical moments (3.18-3.23) as functions of $\boldsymbol{x}$ for $m=1,2,...n$, an investor should assess the risk ratings using *m*-th order economic variables as a source for evaluating each *m*-th component x_m of the risk vector $\boldsymbol{x}=(x_1,...x_n)$. In the case of several risks $j=1,2,..J$, which can impact economic and market performance, the risk assessment becomes *J*-times more difficult, and risk ratings take the matrix form $\boldsymbol{x}=(x_{m,j})$, $m=1,..n; j=1,.2,..J$.

The complexity and uncertainty related to the assessments of risk ratings significantly limit the capacity of investors to forecast a lot of the trade's statistical moments. That limits the accuracy of the predictions of the market-based probabilities of stock price and return.

Now assume that an investor succeeds in accomplishing these two difficult models. An investor chooses the number *n* of the *m*-th trade statistical moments $m=1,2,..n$ at a horizon *T* and evaluates the dynamics of the *m*-th statistical moments of the whole stock market (3.39-3.44). Further, at a horizon *T* for the single risk, an investor carries out the description of the trade statistical moments (3.18-3.23) as functions of the risk vector $\boldsymbol{x}=(x_1,...x_n)$ in the economic domain $[0,1]^n$ using the slow dynamics of the statistical moments of the whole market as an economic environment. Now, using forecasts of the *m*-th statistical moments (3.18-3.23) as functions of risk $\boldsymbol{x}$ at a horizon *T,* an investor can evaluate the return statistical moments $r(T,\xi,x_m;m)$ (3.24; 3.25) or (5.8).

An investor can consider the *m-th* statistical moments of return as functions of x_m as good approximations of the corresponding *m*-th statistical moments of return of stocks of a company *q* with a risk component x_{qm} near x_m, such as $|x_m-x_{qm}|<<1$. It is reasonable, that the price statistical moments of stocks of companies *q* with vector risk coordinates $\boldsymbol{x}_q=(x_{q1},...x_{qn})$ in the neighborhood of vector $\boldsymbol{x}=(x_1,...x_n)$ could vary a lot from the collective price statistical moments at point $\boldsymbol{x}$. However, statistical moments $r(t_k,\xi,x_{qm};m)$ of return (3.9-3.11) of stocks of individual companies *q* in the neighborhood of vector $\boldsymbol{x}=(x_1,...x_n)$ should be almost the same as statistical moments of the collective return $r(\tau_k,\xi,x_m;m)$ (3.24-3.25) at point $\boldsymbol{x}$. If an investor can forecast at a horizon *T* the risk rating vector $\boldsymbol{x}_q=(x_{q1},...x_{qn})$ of a particular company *q,* then he can assess the statistical moments of return of its stocks. Due to relations (2.19; 2.2), the prediction of the statistical moments of return at a horizon T determines the statistical moments of price.

After these long and difficult calculations, an investor would be able to approximate the probabilities of the price and return of stocks of a company q at a horizon T. That may help to make a decision that concerns the investments in stocks of that individual company. However, even that rather complex model doesn't take into account the self-impact of investors' decisions to invest or not to invest in the stocks of a particular company. To develop such a model, one should describe the impact of the investment decisions of a particular investor on the assessments of the risk ratings of stocks of all companies on the stock market and on the evolution of the collective economic and financial variables as functions of $\boldsymbol{x}$. We consider such a problem an ambition for the next level.

Currently, the econometric data, modeling, and forecasting of the economic variables of the 2-d order that are composed by sums of squares of economic, financial, or market transactions are absent. The development of these models could help predict the price and return statistical moments of the 2-d order. That could allow the development of predictions of price and return probabilities in the Gaussian approximations. To increase the accuracy of the market-based probability predictions, one should develop economic models of the 3-d order and beyond.

6. Conclusion

This paper brings to the table the economic obstacles that interfere with and greatly limit any attempts to derive precise forecasts of the market-based probabilities of the price and return of stocks of a company. Actually, the accuracy of forecasts of the price and return probabilities is the core issue of asset pricing and portfolio theories, as it almost completely determines the possible variations of an investor's decisions. The above model gives the investor the possibility to compare the returns of stocks of different companies with different risk ratings and to make his own choice.

The investor should take into account that the number of statistical moments that can be predicted is the major factor that limits the accuracy of the forecasts of the price and return probabilities. One can ignore the complexity of forecasting the m-th statistical moment but cannot overcome or solve the problem. The predictions of the probability of return that don't take into account the dependence of each m-th statistical moment on the economic theory of the m-th order would have such high uncertainty that they are almost useless for investors.

Currently, the lack of any research on the *m*-th order economic theories for *m=2,3,..n* limits the possible predictions of the probabilities of market trade, price, and return by simple Gaussian approximations. Even the prediction of Gaussian probability requires forecasting the 2-d statistical moments of market trade values and volumes. That needs the *2*-d order economic theory, which is absent now. The prediction of the first two statistical moments determines the wide range of approximations of the characteristic functions (A.10) that can extend the random price and return properties beyond Gaussian distributions. Each step further, beyond Gaussian probabilities, needs a lot of econometric and theoretical studies.

We emphasize that the above rather complex model doesn't take into account a lot of extra factors that can significantly impact the description of market trades, statistical moments, and price probability. We mention the dependence of market trades on the *collective* expectations of the sellers and buyers of stocks, which for sure impact the evolution of the price probability. The consideration of these factors will increase the complexity of the model by many times. One can find approximations that take into account the impact of the *collective* expectations of the sellers and buyers on market trade (Olkhov, 2019).

We assume that a general look at the problem of the accuracy of the price and return probability predictions can generate research interest and further studies. However, the precise future of the market-based probabilities of stock price and return are reliably hidden from investors and researchers.

Appendix A

Approximations of the price characteristic function and probability by a finite set of statistical moments

Let us take the set of the price *m*-th statistical moments *p(t;m)*:

$$p(t;m) = \frac{C(t;m)}{U(t;m)} \quad ; \quad m = 1,2, \dots n \tag{A.1}$$

The set of statistical moments (A.1) determines the price characteristic function *F(t;x)* as a Taylor series:

$$F(t;x) = 1 + \sum_{m=1}^{\infty} \frac{i^m}{m!} p(t;m)\, x^m \tag{A.2}$$

In (A.2), i is an imaginary unit, and $i^2=-1$. The finite set of *the m-th* statistical moments *p(t;m)* of price (A.1) determines the *n*-approximation of the characteristic function $F_n(t;x)$

$$F_n(t;x) = 1 + \sum_{m=1}^{n} \frac{i^m}{m!} p(t;m)\, x^m \tag{A.3}$$

One can define the *n*-approximation of the price probability measure $\eta_n(t;p)$ as Fourier transforms of characteristic functions $F_n(t;x)$:

$$\eta_n(t;p) = \frac{1}{\sqrt{2\pi}} \int dx\, F_n(t;x) \exp(-ixp) \tag{A.4}$$

The relations between the statistical moments *p(t;m)*, *n*-approximations of the characteristic function $F_n(t;x)$, and the probability measure $\eta_n(t;p)$ are simple:

$$p(t;m) = \frac{d^m}{(i)^m dx^m} F_n(t;x)|_{x=0} = \int dp\, \eta_n(t;p) p^m \quad ; \quad m \leq n \tag{A.5}$$

To get the characteristic function that generates the same set of the price *m*-th statistical moments *p(t;m), m=1,2,..n* (A.1), and causes the smooth probability $\eta_n(t;p)$, we consider the characteristic functions in the form:

$$F_n(t;x) = \exp\left\{\sum_{m=1}^{n} \frac{i^m}{m!}\, a_m\, x^m - b\, x^{2k}\right\} \quad ; \quad 2k > n \; ; \; b > 0 \tag{A.6}$$

The coefficients a_m, *m=1,..n* are successively determined by the relations (A.5). The terms bx^{2k}, *b>0, 2k>n* don't impact the relations (A.5) for $m \leq n$ but guarantee the existence of the price probability measures $\eta_n(t;p)$ as Fourier transforms (A.4). The uncertainty of the coefficients *b>0* and *2k>n* in (A.6) underlines the well-known fact that the first *n* statistical moments don't exactly determine the characteristic function and probability measure of a random variable. The relations (A.6) describe the set of characteristic functions $F_n(t;x)$ with different *b>0* and *2k>n* and the corresponding set of probability measures $\eta_n(t;p)$ that match (A.4; A.5). Actually, one can take the approximation $F_n(t;x)$ (A.6):

$$F_n(t;x) = \exp\left\{\sum_{m=1}^{n} \frac{i^m}{m!}\, a_m\, x^m - Q(x)\right\} \; ; \; Q(x) = \sum_{m=n+1}^{2K} a_m\, x^m \; ; \; a_{2K} > 0 \tag{A.7}$$

Above approximations for any coefficients a_m, $m=n+1,...2K$, don't impact the first n statistical moments, and $a_{2K}>0$ guarantees the existence of the price probability measures $\eta_n(t;p)$ (A.4). For $n=2$ the approximation $F_2(t;x)$ describes the Gaussian probability $\eta_2(t;p)$:

$$F_2(t;x) = \exp\left\{i\,p(t;1)x - \frac{a_2}{2}x^2\right\} \tag{A.8}$$

It is easy to show that

$$p_2(t;2) = -\frac{d^2}{dx^2}F_2(t;x)|_{x=0} = a_2 + p^2(t;1) = p(t;2)$$

$$a_2 = p(t;2) - p^2(t;1) = \sigma^2(t;p)$$

The coefficient a_2 equals the market-based price volatility $\sigma^2(t;p)$ and the Fourier transform (A.4) for $F_2(t;x)$ gives the Gaussian price probability $\eta_2(t;p)$:

$$\eta_2(pt;) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma(p)}\exp\left\{-\frac{(p-p(t;1))^2}{2\sigma^2(t;p)}\right\} \tag{A.9}$$

The approximation (A.7) gives variations of the "Gaussian" characteristic function $F_{2Q}(t;x)$:

$$F_{2Q}(t;x) = \exp\left\{i\,p(t;1)x - \frac{\sigma^2(t;p)}{2}x^2 - Q(x)\right\} \tag{A.10}$$

$$Q(x) = \sum_{m=3}^{2K} a_m\, x^m \;\; ; \;\; a_{2K} > 0$$

For $n=3$ the approximation $F_3(t;x)$ has the form:

$$F_3(t;x) = \exp\left\{i\,p(t;1)x - \frac{\sigma^2(t;p)}{2}x^2 - i\,\frac{a_3}{6}x^3\right\}$$

$$p_3(t;3) = i\frac{d^3}{dx^3}F_3(t;x)|_{x=0} = a_3 + 3p(t;1)\sigma^2(t;p) + \; p^3(t;1) = p(t;3)$$

$$a_3 = p(t;3) - 3p(t;2)p(t;1) + 2\,p^3(t;1) = E\left[\left(p - p(t;1)\right)^3\right] = Sk(t;p)\sigma^3(t;p)$$

The coefficient a_3 depends on the price skewness $Sk(t;p)$, which describes the asymmetry of the price probability from the normal distribution. Even the Gaussian approximation $F_2(t;x)$, $\eta_2(t;p)$ (A.8; A.9) reveals the direct dependence of the price volatility $\sigma^2(t;p)$ on the *2-d* statistical moments of the trade value *C(t;2)* and volume *U(t;2)*. Thus, the prediction of price volatility $\sigma^2(t;p)$ for Gaussian $\eta_2(t;p)$ (A.9) should follow non-trivial forecasting of the statistical moments of the market trade value *C(t;2)* and volume *U(t;2)*.

Appendix B

Equations of motion in the economic domain

Here, we briefly consider the equations, which describe the dynamics of the *collective* trade values $C_\Sigma(\tau_k,x_m;m)$ (5.5) of stocks of companies with risk coordinates $\boldsymbol{x}_q$ in the neighborhood of $\boldsymbol{x}$ in the economic domain. We smooth the dependence on time τ_k and consider the model with continuous time τ. The sums of trade values (3.13) of agents with coordinates inside $dV(\boldsymbol{x})$ determine the continuous approximation of the trade values in the economic domain as a function of $\boldsymbol{x}$. The derivation of the equations of motion of the continuous economic media approximation almost exactly reproduces the derivation of the conventional widespread equations of continuous mechanics (Childress, 2009). The relations (5.5) determine $C_\Sigma(t,x_m;m)$ as the sum of the m-th power of trade values $C^m\ (t,x_{qm})$ of stocks of companies with risk coordinates x_{qm} inside a small volume $dV(\boldsymbol{x})$ so $|x_m-x_{qm}|<<1$ taken by all such agents during the interval Δ_x. We consider $C_\Sigma(t,x_m;m)$ as a function of continuous coordinate x_m in the economic domain [0,1]. To derive the equations on $C_\Sigma(t,x_m;m)$ as a function of t and x_m we should consider the function $C_\Sigma(t,x_m;m)$ similar to the continuous media, which can flow inside the economic domain. To explain the origin of such a flow, let us refer to the widespread assessments of the risk transition matrices of the largest banks and corporations published by major risk-rating agencies (Metz and Cantor, 2007; Moody's, 2009; Fitch, 2017; S&P, 2018). The risk transition matrices determine the probabilities a_{ij} that an agent with a risk rating x_i during a time interval T can change its rating to x_j. If one replaces the current conventional letter designations of the risk ratings with the numeric ones proposed by us, then it is easy to show that the transition matrices determine the motion of agents in the economic domain with a particular velocity (Olkhov, 2017b-2020). Indeed, the transition time T from rating x_i to rating x_j defines the interval l_{ij} and velocity υ_{ij} between x_i and x_j:

$$l_{ij} = x_j - x_i \quad ; \quad v_{ij} = \frac{l_{ij}}{T} \tag{B.1}$$

Taking probabilities a_{ij} of the transition from x_i to x_j during the time T with a velocity υ_{ij} (B.1) one assesses the mean velocity $\upsilon(t,x_i)$ of agent at point x_i:

$$v(t,x_i) = \sum_{j=1}^{K} v_{ij} a_{ij} = \frac{1}{T} \sum_{j=1}^{K} l_{ij} a_{ij} \quad ; \quad \sum_{j=1}^{K} a_{ij} = 1 \tag{B.2}$$

In (B.2), K means the number of the different numerical risk grades that defines the risk transition matrix KxK. Let us apply these relations to the description of the motion of stocks of a company q with velocity $\boldsymbol{\upsilon}(t,\boldsymbol{x}_q)$ in the economic domain. Each company q with a risk rating $\boldsymbol{x}_q$ at moment t_{ik} with velocity $\boldsymbol{\upsilon}(t_{ik},\boldsymbol{x}_q)$ carries its m-th power of trade value $C^m(t_{ik},\boldsymbol{x}_q)$.

Hence, similar to (5.5), one obtains the *collective* flow $P_C(\tau_k,x_m;m)$ and the *collective* velocity $\boldsymbol{v}_C(\tau_k,x_m;m)$ of sums of the m-th powers of the trade values as:

$$P_{\mathrm{C}}(\tau_k,x_m;m) = \sum_{i=1}^{k_xN} \sum_{\boldsymbol{x}_q \in dV(x)} C^m\big(t_{ik},\boldsymbol{x}_q\big)\, v\big(t_{ik},x_q\big) = C_\Sigma(\tau_k,x_m;m)v_{\mathrm{C}}(\tau_k,x_m;m) \quad \text{(B.3)}$$

The collective transport of the trade values defines a new notion: the *collective m*-th flows $P_C(\tau_k,x_m;m)$ (B.2) of the market trade values $C_\Sigma(\tau_k,x_m;m)$ (5.5) and their collective velocities $\upsilon_C(\tau_k,x_m;m)$. Below, we consider $C_\Sigma(t,x_m;m)$, $P_C(t,x_m;m)$ and $\upsilon_C(t,x_m;m)$ as the functions of continuous time t and variable x_m. To derive the equations, let us consider the change of $C_\Sigma(t,x_m;m)$ in a small interval $\delta X=[x_m,\ x_m+dx]$ during the time dt. Two factors determine its change in a small interval δX (Childress, 2009). The first one determines the change in time:

$$\delta X dt\ \frac{\partial}{\partial t} C_\Sigma(t,x_m;m)$$

The second factor determines the change of $C_\Sigma(t,x_m;m)$ due to the flows of $P_C(t,x_m;m)$ (B.2) in and out of the small interval δX. Indeed, the velocity $\upsilon_C(t,x_m;m)$ carries in and out the amount of $C_\Sigma(t,x_m;m)$ and that results in total change of $C_\Sigma(t,x_m;m)$ inside δX during dt as:

$$dt\ \ [\,P_{\mathrm{C}}(t,x_m+dx;m) - P_{\mathrm{C}}(t,x_m;m)] = \delta X dt \frac{\partial}{\partial x_m} P_{\mathrm{C}}(t,x_m;m)$$

As δX and dt are arbitrary small, one obtains the total change of $C_\Sigma(t,x_m;m)$ as

$$\frac{\partial}{\partial t} C_\Sigma(t,x_m;m) + \frac{\partial}{\partial x_m} P_{\mathrm{C}}(t,x_m;m) = F_C(t,x_m;m) \quad \text{(B.4)}$$

If one takes x_m as a vector $\boldsymbol{x}_m$, then (B.4), results in the well-known Gauss' theorem (Strauss 2008, p.179). To derive equations that describe the evolution of the flow $P_C(t,x_m;m)$ (B.2) one should repeat the same derivation and obtain:

$$\frac{\partial}{\partial t} P_{\mathrm{C}}(t,x_m;m) + \frac{\partial}{\partial x_m}[P_{\mathrm{C}}(t,x_m;m)v_{\mathrm{C}}(t,x_m;m)] = G_C(t,x_m;m) \quad \text{(B.5)}$$

The factors $F_C(t,x_m;m)$ and $G_C(t,x_m;m)$ in the right hand of (B.4; B.5) determine the impact of the economic environment on the evolution of the *collective* trade values $C_\Sigma(t,x_m;m)$ (5.5) and their flows $P_C(t,x_m;m)$ (B.2). These factors determine the economic model and the economic origin of the evolution of market trade. If one considers the market trades of stocks of companies that are under the action of w different risks, then x_m becomes the vector $\boldsymbol{x}_m=(x_{m1},\dots\ x_{mw})$. The flow $\boldsymbol{P}_C(t,\boldsymbol{x}_m;m)$, velocity $\boldsymbol{v}_C(t,\boldsymbol{x}_m;m)$ (B.2), and factor $\boldsymbol{G}_C(t,\boldsymbol{x}_m;m)$ (B.5), also become the vector variables, and equations (B.4; B.5) take the vector forms :

$$\frac{\partial}{\partial t} C_\Sigma(t,\boldsymbol{x}_m;m) + \nabla\cdot \boldsymbol{P}_{\mathrm{C}}(t,\boldsymbol{x}_m;m) = F_C(t,\boldsymbol{x}_m;m) \quad \text{(B.6)}$$

$$\frac{\partial}{\partial t} \boldsymbol{P}_{\mathrm{C}}(t,\boldsymbol{x}_m;m) + \nabla\cdot [\boldsymbol{P}_{\mathrm{C}}(t,\boldsymbol{x}_m;m)\boldsymbol{v}_{\mathrm{C}}(t,\boldsymbol{x}_m;m)] = \boldsymbol{G}_C(t,\boldsymbol{x}_m;m) \quad \text{(B.7)}$$

$$\nabla\cdot \boldsymbol{P}_{\mathrm{C}}(t,\boldsymbol{x}_m;m) \equiv \sum_{j=1}^{w} \frac{\partial}{\partial x_{mj}} P_{\mathrm{Cj}}(t,\boldsymbol{x}_m;m) = \sum_{j=1}^{w} \frac{\partial}{\partial x_{mj}} C_\Sigma(t,\boldsymbol{x}_m;m)v_{\mathrm{Cj}}(t,\boldsymbol{x}_m;m)$$

$$\nabla \cdot [\boldsymbol{P}_{\mathrm{C}}(t,\boldsymbol{x}_m;m)\boldsymbol{v}_{\mathrm{C}}(t,\boldsymbol{x}_m;m)] \equiv \sum_{j=1}^{w} \frac{\partial}{\partial x_{mj}} P_{Ci}(t,\boldsymbol{x}_m;m) v_{Cj}(t,\boldsymbol{x}_m;m) \quad ; \quad i = 1,..w$$

We call (B.4-B.7) equations of the continuous economic media approximation. The left side of (B.4-B.7) is the conventional standard form of the continuous media equations, which are derived and have been in use for almost a century by any textbook on the physics of fluids (Childress, 2009). The economic origin of the model, which differs from any hydrodynamic equations, is presented by the right side factors (B.4-B.7), which describe the economic and market nature of the continuous economic media approximation. Exactly, the choice of $F_C(t,\boldsymbol{x}_m;m)$ (B.6) and $\boldsymbol{G}_C(t,\boldsymbol{x}_m;m)$ (B.7) completely differs the model from any comparisons with the physics of fluids.

Equations of motion of the whole stock market

To derive the equations, which describe the evolution in time of the additive economic variables of the stock market, one can take the integrals of (B.6; B.7) by $d\boldsymbol{x}$ over the economic domain and obtain ordinary differential equations by time variable only:

$$\frac{\partial}{\partial t} C_\Sigma(t;m) = F_C(t;m) \quad ; \quad \frac{\partial}{\partial t} \boldsymbol{P}_{\mathrm{C}}(t;m) = \boldsymbol{G}_C(t;m) \tag{B.8}$$

$$C_\Sigma(t;m) = \int C_\Sigma(t,\boldsymbol{x}_m;m)\, d\boldsymbol{x} \quad ; \quad \boldsymbol{P}_{\mathrm{C}}(t;m) = \int \boldsymbol{P}_{\mathrm{C}}(t,\boldsymbol{x}_m;m)\, d\boldsymbol{x} \tag{B.9}$$

Equations (B.8) have a simple form, but their complexities are hidden by the right hand factors. The important consequences of the transition from equations (B.6; B.7) to equations (B.8) of the whole stock market are tied up with the additional economic variables that significantly impact the economic dynamics. As an example, we mention the mean risk linked to a particular economic variable. Let us take the sums of the *m*-th power of trade values $C_\Sigma(t,x_m;m)$ and consider the mean *m*-th risk $\boldsymbol{X}_C(t;m)$ determined as:

$$\boldsymbol{X}_C(t;m)\, C_\Sigma(t;m) = \int \boldsymbol{x}_m\, C_\Sigma(t,\boldsymbol{x}_m;m)\, d\boldsymbol{x}_m$$

For each $m=1,..n$, the vector $\boldsymbol{X}_C(t;m)=(x_C(t;1),\dots x_C(t;J))$ determines the mean risks of sums of the *m*-th power of trade values $C_\Sigma(t;m)$ in the unit cube $[0,1]^J$. The dynamics of its components $x_C(t;j)$ for $j=1,..J$ in time can be described as fluctuations in the interval [0,1]. The fluctuations of the mean risk $\boldsymbol{X}_C(t;m)$ of the sums of the *m*-th powers of trade values $C_\Sigma(t,x_m;m)$ describe the market cycles, which are similar to the business cycles, credit cycles, and so on. The mean risks of trade values $C_\Sigma(t,x_m;m)$ differ from the mean risks of the sums of the *m*-th powers of trade volumes $U_\Sigma(t,x_m;m)$ or mean risks linked with other collective economic variables. The hidden dynamics of mean risks describe important properties of market trade evolution that are almost completely missed by current economic models.

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