Three Remarks On Asset Pricing

Olkhov, Victor

Independent

24 July 2021

Three Remarks On Asset Pricing

Victor Olkhov

Independent, Moscow, Russia

victor.olkhov@gmail.com

ORCID: 0000-0003-0944-5113

Abstract

This paper considers the theoretical framework of the consumption-based asset-pricing model and derives successive approximations of the modified basic pricing equation using the Taylor series expansions of the investor's utility function during the averaging time interval. For linear and quadratic Taylor approximations, we derive new expressions for the mean asset price, mean payoff, volatility, skewness, and the amount of an asset that delivers maximum to the investor's utility. We introduce a new market-based approach to price probability determined by statistical moments of market trade values and volumes. We show that market-based price probability results in zero correlations between the time series of the *n*-th power of price p^n and trade volume U^n but doesn't cause statistical independence. We derive a correlation between the time series of prices p and the squares of trade volumes U^2. The market-based approach describes the impact of the size of the trade values and volumes on price statistical moments and probability. Predictions of the market-based price probability at horizon T should match forecasts of the statistical moments of the trade values and volumes at the same horizon T. Market-based price probability emphasizes direct dependence on the random properties of market trades.

Keywords : asset pricing; volatility; market-based price probability; market trades

JEL: G12

This research received no support, specific grant or financial assistance from funding agencies in the public, commercial or nonprofit sectors. We welcome valuable offers of grants, support and positions.

1. Introduction

Predictions of asset prices are the most desired results for investor and, at the same time, the most complex problems of economic and financial theory. The literature on asset pricing is huge and boundless, and our references present not historical review but only our personal preferences. We mention only a tiny part of the endless publications on asset pricing, starting with CAPM by Sharpe (1964), which was followed by various modifications such as Intertemporal CAPM by Merton (1973), the Arbitrage Theory of Capital Asset Pricing by Ross (1976), the consumption-based asset pricing model described by Duffie and Zame (1989), Cochrane (2001), Campbell (2002), and many others. In his paper, Cochrane (2001) demonstrates that the consumption-based asset-pricing framework gives a unified approach for the description of most variations of pricing models. We consider Cochrane's statement and methods at the roots of the consumption-based model as the key tools of current pricing theories.

One may consider most pricing theories as fruitful derivations of CAPM, which mostly have certain common assumptions, foundations, and limitations. Sharpe (1964) mentioned that his assumptions in the foundation of CAPM, such as "common pure rate of interest" and "homogeneity of investor expectations: investors are assumed to agree on the prospects of various investments" are "highly restrictive and undoubtedly unrealistic assumptions". CAPM and consumption-based models are based on the assumption of general market equilibrium, and they use utility functions that model investors' market decisions. The maximum condition of the investor's utility function results in the basic pricing equation that describes most current results in pricing models. These and other initial assumptions are the basis of modern asset pricing models. On the one hand, these assumptions support the description of definite relations between current asset prices, expected payoffs, discount factors, etc. On the other hand, "highly restrictive and undoubtedly unrealistic assumptions" carry the threat of failures and inconsistencies between predictions of pricing theories and real market price dynamics.

We don't study how conventional assumptions in the foundation of the pricing models impact their predictions or limit their applications. Instead, we take the consumption-based frame and consider how few remarks generated by market trade reality could impact the consequences and performance of asset pricing. As our main reference to the consumption-based model, we chose Cochrane (2001). If one agrees with Cochrane's statement that the consumption-based model and the basic pricing equation describe the results of most variants of pricing theories, then our remarks, approximations, and results make sense and could be applied to other asset pricing theories.

Our pure theoretical paper considers the frame of the consumption-based asset pricing model as a general economic problem and investigates its compliance with major economic issues. Actually, any economic and financial model describes processes and relations only as an approximation that captures certain averaging, smoothness, and coarsening of the economic reality. Thus, our first remark concerns the importance of considering a particular time averaging interval Δ as a determining factor in pricing models. Indeed, current stock markets support initial time axis division determined by the time series of the trades performed at moments t_i with a time shift $\varepsilon = t_i - t_{i-1}$ between trades. For simplicity in this paper, we consider the time shift ε as a constant. As usual, the time shift ε is sufficiently small and can be equal to 1 second or even a fraction of a second. That is not much use for modeling asset prices at a horizon of 1 month, quarter, or year. However, records of market trade time series with time shifts determine the initial market time axis division and define the discrete nature of all initial market trade data. To evaluate any reasonable pricing model at a given horizon T that can be equal to a month, quarter, year, and so on, one should choose the averaging time scale Δ, which should obey $\varepsilon << \Delta < T$. The choice of the averaging time interval Δ is a key factor in any pricing model. It determines the scale of averaging of the initial trade time series and thus performs the transition from the initial market time division multiple of ε to the averaged time division multiple of Δ. Below, we show how the choice of the averaging interval results in a modification of the consumption-based utility function and the basic pricing equation.

Our second remark indicates that the choice of the averaging interval Δ allows expanding the utility function and the basic pricing equation into Taylor series near average values of price and payoff and then averaging the fluctuating terms of the series during. Mathematical expectations of linear and quadratic Taylor approximations of the basic pricing equation by price and payoff variations during Δ give new expressions of the mean price and payoff, their volatilities, skewness, and other factors. Actually, even linear Taylor expansion demonstrates that the famous statement "price equals expected discounted payoff," with which Cochrane (2001) and Brunnermeier (2015) begin their papers, describes only markets with zero price volatility during current and "next" periods, which makes almost no economic sense. As we show in Sec. 4, the Taylor expansion of the modified basic pricing equation determines relations between mean price and price volatility during the current period and mean payoff and payoff volatility during the "next" period. Further in Section 4, we derive additional relations that extend the results of the consumption-based model.

Our third remark concerns the economic origin, definition, approximations, and forecasting of the asset price probability as the major problem of any pricing model and financial economics as a whole. Furthermore, we consider the assessments and predictions of the finite number of price statistical moments, which establish the basis for approximations of price probability and its forecasts, as the principal and most complex problems of financial economics and the key problem of any asset pricing model in particular. We introduce the market-based probability of asset price, which is determined by statistical moments of market trade values and volumes during Δ. Conventional treatment considers a frequency-based assessment of price probability, which is proportional to the number of trades at price p. Actually, any particular market trade at time t_i is determined by its trade value $C(t_i)$, volume $U(t_i)$, and price $p(t_i)$, which follow a trivial equation:

$$C(t_i) = p(t_i)U(t_i) \tag{1.1}$$

One should mention that it is impossible to independently define the probabilities of three variables: trade value $C(t_i)$, volume $U(t_i)$, and price $p(t_i)$ – those match equation (1.1). We consider trade value $C(t_i)$ and volume $U(t_i)$ time series as major random variables during the interval Δ, which completely determine random price $p(t_i)$ properties. It is well known that the properties of a random variable can be equally described by its probability measure, characteristic function, or set of statistical moments (Shephard, 1991; Shiryaev, 1999; Klyatskin, 2005). The random market trade time series during Δ determines assessments of the n-th statistical moments of trade value $C(t;n)$ (5.7) and volume $U(t;n)$ (5.8), and we use them to define the n-th statistical moments $p(t;n)$ (5.10) of price. We compare the frequency-based and the market-based *n-th* statistical moments of price and explain why the conventional frequency-based treatment of price probability makes too little economic sense.

We propose that readers become familiar with Cochrane (2001) and refer to his monograph for any notions or clarifications. In Sec. 2, we briefly recall the main notions of asset pricing according to Cochrane (2001). In Sec. 3, we consider remarks on the averaging interval Δ and explain the necessity for modification of the consumption-based basic pricing equation. In Sec. 4, we discuss the Taylor series expansion of the utility functions and derive successive approximations of the modified basic equation in linear and quadratic approximations by the price and payoff variations. In Sec. 5, we introduce the market-based price statistical moments and briefly consider their implications for asset pricing. Sec. 6: Conclusion. In App. A, we collect some calculations that define the maximum of an investor's utility. In App. B, we present simple approximations of the price characteristic function.

Equation (4.5) means equation 5 in Sec. 4, and (A.2) notes equation 2 in Appendix A. We assume that readers are familiar with the basic notions of probability, statistical moments, characteristic functions, etc.

2. Brief Notations

In this Sec. we briefly remind main notations and assumptions of asset pricing used by Cochrane (2001). The consumption-based basic pricing equation has form:

$$p = E[m\,x] \tag{2.1}$$

In (2.1), p denotes the asset price at date t, $x=p_{t+1}+d_{t+1}$ – payoff, p_{t+1} - price and d_{t+1} - dividends at date $t+1$, m - stochastic discount factor and $E[..]$ – mathematical expectation at day $t+1$ made by the forecast under the information available at date t. Cochrane (2001) considers equation

(2.1) in various forms to show that most asset pricing models can be described by similar equations. For convenience, we briefly reproduce the derivation of the consumption-based basic pricing equation (2.1). Cochrane models investors by a utility function $W(c_t; c_{t+1})$ defined over current c_t and future c_{t+1} values of consumption at dates t and $t+1$.

$$W(c_t;\ c_{t+1}) = w(c_t) + \beta E[w(c_{t+1})] \tag{2.2}$$

$$c_t = e_t - p\xi \quad ; \quad c_{t+1} = e_{t+1} + x\xi \tag{2.3}$$

$$x = p_{t+1} + d_{t+1} \tag{2.4}$$

In (2.2), $w(c_t)$ and $w(c_{t+1})$ are utility functions at dates t and $t+1$; in (2.3), e_t and e_{t+1} "denotes the original consumption level (if the investor bought none of the asset), and ξ denotes the amount of the asset he chooses to buy" (Cochrane, 2001). Cochrane calls β as "subjective discount factor that captures impatience of future consumption". The first-order maximum condition for (2.2) by the amount of assets ξ is fulfilled by putting the derivative of (2.2) by ξ equals zero (Cochrane, 2001):

$$max_{\xi}\, W(c_t;\ c_{t+1}) \leftrightarrow \frac{\partial}{\partial \xi} W(c_t;\ c_{t+1}) = 0 \tag{2.5}$$

From (2.2-2.5) one obtains:

$$p = \beta E\left[\frac{w'(c_{t+1})}{w'(c_t)}\, x\right] = E[mx] \quad ; \quad m = \beta \frac{w'(c_{t+1})}{w'(c_t)} \quad ; \quad w'(c) \equiv \frac{d}{dc} w(c) \tag{2.6}$$

and (2.6) reproduces (2.1) for m (2.6). We refer Cochrane (2001) for any further details.

3. Remarks on Time Scales

We start with simple remarks on the averaging of economic and financial time series. Any economic or financial model, and asset pricing in particular, approximates real processes by averaging them over a certain time interval Δ. To describe market asset pricing, one should take into account that market trade time series are the only source of price variations. The interval ε between market transactions can be very small and can be equal to 1 second or even a fraction of a second. Initial market price time series $p(t_i)$ with time-shift ε are very irregular and not very useful for modelling and forecasting asset prices at any reasonable time horizon T that can be equal to a week, month, year, etc. To derive a reasonable description of asset prices, one should chose an averaging interval Δ and smooth variations of market prices during Δ. The choice of an averaging interval Δ is a very important challenge for each investor. The choice of a long interval Δ, which equals weeks or months, would result in smooth dynamics and stable predictions of the averaged variables but would limit the capacity to take investment decisions "this hour" or "today". Short averaging interval Δ, such as hours or days, improve ability to make "this hour" investment decisions, but average variables could be under the impact of multiple perturbations with periods equal to days or weeks. Different averaging intervals cause different random properties of variables and different models, that describe the evolution of averaged variables.

To perform a transition from the initial market trade time axis division, which is multiple of ε one should choose a time interval Δ such as $\varepsilon<<\Delta<T$ and average price time series $p(t_i)$ during Δ. The time shift $\Delta = t(k) - t(k\text{-}1)$ of averaged prices $p(t(k))$ at times $t(k)$ introduces a new division of the time axis division multiple of Δ. One can consider averaging intervals Δ_k as (3.1):

$$\Delta_k = \left[\, t(k) - \frac{\Delta}{2}\,; t(k) + \frac{\Delta}{2} \right] \;\; ; \;\; t(k) = t(0) + k\,\Delta \;\; ; \;\; k = 0,\ 1, 2, .. \qquad (3.1)$$

We take the duration of each averaging interval Δ_k equal Δ. One can consider time $t=t(0)$ as the moment "today" and the "next day" at time $t+1$ as $t(K)$ for some K>>1. What is most important: time axis division "today" at t and the "next-day" at $t+1$ must be the same. Indeed, time axis divisions can't be measured "today" in hours and "next-day" in weeks. Utility (2.2) "today" at moment t and the "next-day" at $t+1$ should have the same time axis divisions. Averaging any time series at the "next-day" at $t+1$ during the interval Δ undoubtedly implies averaging "today" at date t during an equal time interval Δ and vice versa. Thus, if the utility (2.2) is averaged at $t+1$ during the interval Δ, then the utility (2.2) also should be averaged at date t during the same interval Δ and (2.2) should take the form:

$$W(c_t;\ c_{t+1}) = E_t[w(c_t)] + \beta E[w(c_{t+1})] \qquad (3.2)$$

We denote $E_t[..]$ in (3.2) as the mathematical expectation "today" at date t during Δ. It does not matter how one considers the market price time series "today" – as random or as irregular. Mathematical expectation $E_t[..]$ performs smoothing of the random or irregular time series via aggregating data during Δ under a particular probability measure. Mathematical expectations $E_t[..]$ at t and $E[..]$ at $t+1$ during the same averaging intervals Δ establish identical time division of the problem at dates t and $t+1$ in (3.2). Hence, relations similar to (2.5; 2.6) should cause modification of the basic pricing equation (2.1; 2.6) in the form (3.3):

$$E_t[p\, w'(c_{\square})] = \beta E\ [x\, w'(c_{t+1})] \tag{3.3}$$

Cochrane (2001) takes the "subjective discount factor" β as non-random, and we follow his assumption. Mathematical expectation $E_t[..]$ averages $pw'(c_t)$ over random price p fluctuations during Δ "today". On the right side, $E[xw'(c_{t+1})]$ averages $xw'(c_{t+1})$ over random payoff fluctuations during Δ "next day" on the basis of data available at date t "today".

4. Remarks on Taylor series

Relation (2.5) presents the first-order condition for the amount of assets ξ_{max} that delivers the maximum to the investor's utility (2.2) or (3.2). Let us choose the averaging interval Δ and take the price p at date t during Δ and the payoff x at date $t+1$ during Δ as:

$$p\ =\ p_0 + \delta p\ ;\quad x = x_0 + \delta x\ ;\quad E_t[p] = p_0\ ;\ E[x] = x_0 \tag{4.1}$$

$$E_t[\delta p] = E[\delta x] = 0\ ;\ \sigma^2(p) = E_t[\delta^2 p]\ ;\ \sigma^2(x) = E[\delta^2 x] \tag{4.2}$$

Relations (4.1; 4.2) denote the average price p_0 and its volatility $\sigma^2(p)$ at date t and the average payoff x_0 and its volatility $\sigma^2(x)$ at date $t+1$. We consider δp and δx as random fluctuations of price and payoff during Δ. We indicate that we consider averaging during Δ as averaging of a random variable or as smoothing of an irregular variable. Thus, $E_t[p]$ – at date t smooths the random or irregular price p (4.1) during Δ and $E[x]$ – averages the random payoff x (4.1) during Δ at date $t+1$. We present the derivatives of utility functions in (3.3) by Taylor series in a linear approximation by δp and δx during Δ:

$$w'(c_t) = w'(c_{t;0}) - \xi w''(c_{t;0})\delta p\quad ;\quad w'(c_{t+1}) = w'(c_{t+1;0}) + \xi w''(c_{t+1;0})\delta x \tag{4.3}$$

$$c_{t;0} = e_t - p_0\xi\quad ;\quad c_{t+1;0} = e_{t+1} + x_0\xi$$

Now substitute (4.3) into (3.3), and due to (4.2), obtain the equation (4.4):

$$w'(c_{t;0})p_0 - \xi w''(c_{t;0})\sigma^2(p) = \beta w'(c_{t+1;0})x_0 + \beta\xi w''(c_{t+1;0})\sigma^2(x) \tag{4.4}$$

Taylor series are simple mathematical tools, and Cochrane (2001) also used them. We underline: Taylor series and (4.1-4.4) are determined by the duration of Δ. The change of Δ can implies a change of the mean price p_0, the mean payoff x_0 and their volatilities $\sigma^2(p)$, $\sigma^2(x)$ (4.2). Equation (4.4) is a linear approximation of the price and payoff fluctuations of the first-order max conditions (2.5) and assesses the root ξ_{max} that delivers maximum to the utility $W(c_t;c_{t+1})$ (3.2):

$$\xi_{max} = \frac{w'(c_{t;0})p_0 - \beta w'(c_{t+1;0})x_0}{w''(c_{t;0})\sigma^2(p) + \beta w''(c_{t+1;0})\,\sigma^2(x)} \tag{4.5}$$

We note that (4.5) is not an "exact" solution for ξ_{max} as derivatives of utilities w' and w'' also depend on ξ_{max} as it follows from (4.3). However, (4.5) gives an assessment of ξ_{max} in a linear approximation by Taylor series δp and δx averaged during Δ. Let us highlight that the ξ_{max} (4.5) depends on the price volatility $\sigma^2(p)$ at date t and on the forecast of payoff volatility $\sigma^2(x)$ at date $t+1$ (4.2).

It is clear that sequential iterations may give more accurate approximations of ξ_{max}. Nevertheless, our approach and (4.5) give a new look at the basic equation (2.6; 3.3). If one follows the standard derivation of (2.6) (Cochrane, 2001) and neglects the averaging at date t in the left side (3.3), then (2.6; 4.5) give

$$\xi_{max} = \frac{w'(c_t)p - \beta w'(c_{t+1;0})x_0}{\beta w''(c_{t+1;0})\sigma^2(x)} \tag{4.6}$$

Relations (4.6) show that even the standard form of the basic equation (2.6) hides the dependence of the amount of assets ξ_{max} on the payoff volatility $\sigma^2(x)$ at date $t+1$. If one has an independent assessment of ξ_{max} then one can present (4.6) in a way similar to the basic equation (2.6):

$$p = \frac{w'(c_{t+1;0})}{w'(c_t)}\beta x_0 + \xi_{max}\frac{w''(c_{t+1;0})}{w'(c_t)}\beta\sigma^2(x) \tag{4.7}$$

Otherwise, if there are no independent assessments of ξ_{max}, then one should consider (4.6) as the solution of the first order maximum condition (2.5), which presents the root ξ_{max} of the amount of assets, determined for the given values in the right hand of (4.6). In that case, the basic pricing equations (2.1; 2.6; 4.7) make almost no sense, as the value of ξ_{max} in (4.7) is not determined. We consider this misstep – using the maximum condition (2.5) to determine the basic pricing equation (2.1; 4.7) instead of defining ξ_{max} as a root of the maximum condition (2.5) - a significant oversight of the consumption-based asset pricing model, which requires essential clarifications. One can transform (4.7) similar to (2.6):

$$p = m_0 x_0 + \xi_{max} m_1 \sigma^2(x) \tag{4.8}$$

$$m_0 = \frac{w'(c_{t+1;0})}{w'(c_t)}\beta \ ; \ m_1 = \frac{w''(c_{t+1;0})}{w'(c_t)}\beta \tag{4.9}$$

For the given ξ_{max} equation (4.8) in a linear approximation by Taylor series describes the dependence of the price p at date t (3.1) on the mean discount factors m_0 and m_1 (4.9), the mean payoff x_0 (4.1), and the payoff volatility $\sigma^2(x)$ during Δ. Let us stress that while the mean discount factor $m_0>0$, the mean discount factor $m_1<0$ because utility $w'(c_t)>0$ and $w''(c_t)<0$ for all t. Hence, irremovable payoff volatility $\sigma^2(x)$ at day $t+1$ states that price p at day t always less than discounted mean payoff x_0:

$$p < m_0 x_0 \quad ; \quad \xi_{max} m_1 \sigma^2(x) < 0$$

One can consider (4.8) as a linear Taylor expansion of (2.1; 2.6). However, equation (4.4) presents the dependence of mean price p_0 at day t on price volatility $\sigma^2(p)$ at day t, mean payoff x_0 and payoff volatility $\sigma^2(x)$ at day $t+1$. That definitely enlarges the conventional statement that "price equals expected discounted payoff". We indicate that (4.6-4.9) makes sense for the given value of ξ_{max}. As the price p in (4.8) should be positive, hence ξ_{max} should obey inequality (4.10):

$$0 < \xi_{max} < -\frac{w'(c_{t+1;0})}{w''(c_{t+1;0})}\frac{x_0}{\sigma^2(x)} \tag{4.10}$$

For the conventional power utility (A.2) (Cochrane, 2001), from (4.3) obtain for (4.10):

$$w(c) = \frac{1}{1-\alpha}c^{1-\alpha} \quad ; \quad \frac{w'(c)}{w''(c)} = -\frac{c}{\alpha} \quad ; \quad 0 < \alpha \leq 1$$

inequality (4.10) valid always if

$$\alpha\, \sigma^2(x) < x_0^2$$

For this approximation (4.10) limits the value of ξ_{max}. For (4.4; 4.5) obtain equations similar to (4.8; 4.9):

$$m_0 = \frac{w'(c_{t+1;0})}{w'(c_{t;0})}\beta > 0\ ; \ m_1 = \frac{w''(c_{t+1;0})}{w'(c_{t;0})}\beta < 0\ ; \ m_2 = \frac{w''(c_{t;0})}{w'(c_{t;0})} < 0 \tag{4.11}$$

$$p_0 = m_0 x_0 + \xi_{max}[m_1\sigma^2(x) + m_2\sigma^2(p)] \tag{4.12}$$

We use the same notions m_0, m_1 to denote the discount factors, taking into account the replacement of $w'(c_t)$ in (4.9) by $w'(c_t;0)$ in (4.11; 4.12). The modified basic equation (4.12) at date t describes the dependence of the mean price p_0 on the price volatility $\sigma^2(p)$ at date t, the mean payoff x_0 and the payoff volatility $\sigma^2(x)$ at date $t+1$ averaged during Δ.

Equation (4.12) illustrates the well-known practice that high volatility $\sigma^2(p)$ of the price at date t and a forecast of high volatility $\sigma^2(x)$ of payoff at date $t+1$ may cause a decline in the mean price p_0 at date t.

4.1 The Idiosyncratic Risk

Here we follow (Cochrane, 2001) and briefly consider the usage of the Taylor series for his example of the idiosyncratic risk for which the payoff x in (2.6) is not correlated with the discount factor m at moment $t+1$:

$$cov(m,x) = 0 \tag{4.13}$$

In this case equation (2.6) takes form:

$$p = E[mx] = E[m]E[x] + cov(m,x) = E[m]x_0 = \frac{x_0}{R_f} \tag{4.14}$$

The risk-free rate R_f in (4.14) is known ahead (Cochrane, 2001). Taking into account (4.3) in a linear approximation by δx Taylor series for the derivative of the utility $w'(c_t+1)$:

$$w'(c_{t+1}) = w'(c_{t+1;0}) + w''(c_{t+1;0})\xi\delta x \tag{4.15}$$

Hence, the discount factor m (2.6) takes form:

$$m = \beta\frac{w'(c_{t+1})}{w'(c_t)} = \frac{\beta}{w'(c_t)}\left[\Box'(c_{t+1;0}) + w''(c_{t+1;0})\xi\delta x\right]$$

$$E[m] = \bar{m} = \beta\frac{w'(c_{t+1;0})}{w'(c_t)} \quad ; \quad \beta E\left[\frac{w'(c_{t+1})}{w'(c_t)}\right]x_0 = \frac{x_0}{R_f} \quad ; \quad E[w'(c_{t+1})x] = 0$$

$$\delta m = m - \bar{m} = \frac{\beta}{w'(c_t)}w''(c_{t+1;0})\xi\delta x$$

Hence, (4.13) implies:

$$cov(m,x) = E[\delta m\delta x] = \beta\frac{w''(c_{t+1;0})}{w'(c_t)}\xi_{max}\sigma^2(x) = 0 \tag{4.16}$$

That causes zero payoff volatility $\sigma^2(x)=0$. Of course, zero payoff volatility does not model market reality, but (4.16) reflects the restrictions of the linear approximation (4.15). To overcome this discrepancy, take into account the Taylor series up to the second power by $\delta^2 x$:

$$w'(c_{t+1}) = w'(c_{t+1;0}) + w''(c_{t+1;0})\xi\delta x + \frac{1}{2}w'''(c_{t+1;0})\xi^2\delta^2 x \tag{4.17}$$

$$m = \beta\frac{w'(c_{t+1})}{w'(c_t)} = \frac{\beta}{w'(c_t)}\left[w'(c_{t+1;0}) + w''(c_{t+1;0})\xi\delta x + \frac{1}{2}w'''(c_{t+1;0})\xi^2\delta^2 x\right] \tag{4.18}$$

For this case, the mean discount factor $E[m]$ takes the form:

$$E[\square] = \bar{m} = \frac{\beta}{w'(c_t)}[w'(c_{t+1;0}) + \frac{1}{2}w'''(c_{t+1;0})\xi^2\sigma^2(x)] \qquad (4.19)$$

and variations of the discount factor δm:

$$\delta m = m - \bar{m} = \frac{\beta}{w'(c_t)}\ [w''(c_{t+1;0})\xi\delta x + \frac{1}{2}w'''(c_{t+1;0})\xi^2\{\delta^2 x - \sigma^2(x)\}$$

Thus the Taylor series approximation up to the second power by $\delta^2 x$ gives:

$$cov(m,x) = E[\delta m\delta x] = \left[w''(c_{t+1;0})\xi\sigma^2(x) + \frac{1}{2}w'''(c_{t+1;0})\xi^2\,\gamma^3(x)\right] = 0 \qquad (4.20)$$

$$\gamma^3(x) = E[\delta^3 x] \quad ; \quad Sk(x) = \frac{\gamma^3(x)}{\sigma^3(x)} \qquad (4.21)$$

Sk(x) – denotes normalized payoff skewness at date *t+1,* treated as the measure of asymmetry of the probability distribution during *Δ*. For approximation (4.18) from (4.20; 4.21), obtain relations on the skewness *Sk(x)* and ξ_{max}:

$$\xi_{ma\square}\ Sk(x)\sigma(x) = -2\frac{w''(c_{t+1;0})}{w'''(c_{t+1;0})} \qquad (4.22)$$

For the conventional power utility (A.2)

$$w(c) = \frac{1}{1-\alpha}c^{1-\alpha}$$

and (4.3) relations (4.22) take the form

$$\xi_{max} = \frac{2e_{t+1}}{(1+\alpha)Sk(x)\sigma(x)-2x_0} \qquad (4.23)$$

It is assumed that the second derivative of utility $w''(c_{t+1})<0$ always negative and the third derivative $w'''(c_{t+1})>0$ is positive, and hence the right side in (4.22) is positive. Hence, to get a positive ξ_{max} for (4.23) for the power utility (A.2), the payoff skewness *Sk(x)* should obey inequality (4.24) that defines the lower limit of the payoff skewness *Sk(x):*

$$Sk(x) > \frac{2x_0}{(1+\alpha)\sigma(x)} \qquad (4.24)$$

In (4.14), R_f denotes the risk-free rate. Hence, (4.19; 4.22; 4.24) define relations:

$$\frac{\beta}{w'(c_t)}\left[w'(c_{t+1;0}) + \frac{1}{2}w'''(c_{t+1;0}){\xi_{max}}^2\sigma^2(x)\right] = \frac{1}{R_f}$$

$$\frac{1}{2}{\xi_{max}}^2\sigma^2(x) = \frac{1}{\beta R_f}\frac{w'(c_t)}{w'''(c_{t+1;0})} - \frac{w'(c_{t+1;0})}{w'''(c_{t+1;0})}$$

$$Sk^2(x) = \frac{R_f}{1-m_0 R_f}\frac{m_1^2}{m_3} > \frac{4x_0^2}{(1+\alpha)^2\sigma^2(x)} \quad ; \quad m_0 < 1/R_f$$

$$\frac{\sigma^2(x)}{4x_0^2} > \frac{m_3}{m_1^2}\frac{1-m_0R_f}{(1+\alpha)^2 R_f} \qquad (4.25)$$

Inequality (4.25) establishes the lower limit on the payoff volatility $\sigma^2(x)$ normalized by the square of the mean payoff x_0^2. The lower limit on the right side of (4.25) is determined by the discount factors (4.26), the risk-free rate R_f, and the conventional power utility factor α (A.2).

$$m_0 = \beta \frac{w'(c_{t+1;0})}{w'(c_t)} \;\; ; \; m_1 = \beta \frac{w''(c_{t+1;0})}{w'(c_t)} \;\; ; \;\; m_3 = \beta \frac{w'''(c_{t+1;0})}{w'(c_t)} \tag{4.26}$$

The coefficients in (4.26) differ a little from (4.1), as (4.26) takes the denominator $w'(c_t)$ instead of $w'(c_{t;0})$ in (4.11), but we use the same letters to avoid extra notations. The similar calculations for (3.2; 3.3) describe both the price volatility $\sigma^2(p)$ and price skewness $Sk(p)$ at date t and the payoff volatility $\sigma^2(x)$ and payoff skewness $Sk(x)$ at date $t+1$. Further approximations by the Taylor series of the utility derivative $w'(c_t)$ up to $\delta^3 p$ and $w'(c_{t+1})$ up to $\delta^3 x$ similar to (4.17) could give assessments of kurtosis of the price probability at date t and the kurtosis of the payoff probability at date $t+1$ estimated during interval Δ.

4.2 The Utility Maximum

Relations (2.5) define the first-order condition that determines the amount of asset ξ_{max} that delivers the max to the utility $W(c_t;c_{t+1})$ (2.2; 3.2). To confirm that function $W(c_t;c_{t+1})$ has max at ξ_{max} , the first order condition (2.5) must be supplemented by condition:

$$\frac{\partial^2}{\partial \xi^2} W(c_t; c_{t+1}) < 0 \tag{4.27}$$

The use of (4.27) has interesting consequences. From (2.2–2.4) and (4.27), obtain:

$$p^2 > -\frac{\beta}{w''(c_t)} E[x^2 \, w''(c_{t+1}) \,] \tag{4.28}$$

Take the linear Taylor series expansion of the second derivative of the utility $w''(c_{t+1})$ by δx

$$w''(c_{t+1}) = w''\big(c_{t+1;0}\big) + w'''\big(c_{t+1;0}\big)\xi\delta x$$

Then (4.28) takes the form:

$$p^2 > -\beta \frac{w''(c_{t+1;0})}{w''(c_t)} [x_0{}^2 + \sigma^2(x)] - \beta \frac{w'''(c_{t+1;0})}{w''(c_t)} \xi_{max} \, [2x_0\sigma^2(x) + \gamma^3(x)] \tag{4.29}$$

For the power utility (A.2), (see App.A) obtain relations on (4.27; 4.29). If the payoff volatility $\sigma^2(x)$ multiplied by factor $(1+2\alpha)$ is less then the mean payoff x_0^2 (4.30; A.5):

$$(1 + 2\alpha)\sigma^2(x) < x_0{}^2 \quad\quad ; \quad\quad \frac{1}{3} \le \frac{1}{1+2\alpha} < 1 \tag{4.30}$$

Then (4.29) is always valid. If payoff volatility $\sigma^2(x)$ is high (A.6)

$$(1 + 2\alpha)\sigma^2(x) > \; x_0{}^2$$

Then (4.29) valid only for ξ_{max} (A.6):

$$\xi_{max} < \frac{e_{t+1}[{x_0}^2+\sigma^2(x)]}{x_0\,[(1+2\alpha)\sigma^2(x)-{x_0}^2]}$$

However, this upper limit for ξ_{max} can be high enough. The same but more complex considerations can be presented for (3.2).

$$E_t[p^2 w''(c_t)] < -E[\beta x^2\, w''(c_{t+1})\,]$$

5. Remarks on the Price Probability

There are at least two different approaches to the definition of asset price probability. We note the first one, and the most conventional, as the frequency-based approach. However, we believe that the nature of economic and market relations determines the different ways to consider price probability. We call it the market-based price probability.

The usual treatment of price probability "is based on the probabilistic approach and using A. N. Kolmogorov's axiomatic of probability theory, which is generally accepted now" (Shiryaev, 1999). The conventional definition of the price probability is based on the frequency of trades at a price *p* during the averaging interval *Δ*. The economic foundation of such a choice is simple: it is assumed that each of *N* trades during *Δ* has equal probability ~ *1/N*. If there are *m(p)* trades at the price *p* then the probability *P(p)* of the price *p* during *Δ* is assessed as *m(p)/N*. The use of the frequency of the particular event is an absolutely correct, general, and conventional approach to probability definition. The conventional frequency-based approach to price probability checks how almost all standard probability measures (Walck, 2007; Forbes et al, 2011) fit the description of the market's random price. Parameters, which define standard probabilities, permit calibrating each in a manner that increases plausibility and consistency with the observed random price time series. For different assets and markets, different standard probabilities are tested and applied to fit and predict the random price dynamics as well as possible.

However, one may ask a simple question: Does the conventional frequency-based approach to price probability fit random market pricing? Indeed, the asset price is a result of the market trade, and it seems reasonable that the market trade randomness should conduct the price stochasticity. We propose a new definition of the market-based price probability that is different from the conventional frequency-based probability and is entirely determined by the statistical moments of the market trade values and volumes.

Let us remind ourselves that almost 30 years ago, the volume weighted average price (VWAP) was introduced and is widely used now (Berkowitz et al, 1988; Buryak and Guo, 2014; Busseti and Boyd, 2015; Duffie and Dworczak, 2018; CME Group, 2020). The definition of the VWAP $p(t;1)$ that matches equation (1.1) during Δ is follows. Assume that during Δ (5.3), there are N market trades at moments t_i, $i=1,...N$. Let's denote $E[..]$ as a mathematical expectation. Then the VWAP $p(t;1)$ (5.1) that match (1.1) during Δ (5.3) at moment t equals

$$p(t;1) \equiv E[p(t_i)] = \frac{1}{\sum_{i=1}^{N} U(t_i)} \sum_{i=1}^{N} p(t_i) U(t_i) \equiv \frac{C_\Sigma(t;1)}{U_\Sigma(t;1)} \quad (5.1)$$

$$C_\Sigma(t;1) \equiv \sum_{i=1}^{N} C(t_i) \equiv \sum_{i=1}^{N} p(t_i)\, U(t_i) \quad ; \quad U_\Sigma(t;1) \equiv \sum_{i=1}^{N} U(t_i) \quad (5.2)$$

$$\Delta = \left[t - \frac{\Delta}{2}, t + \frac{\Delta}{2}\right] \quad ; \quad t_i \in \Delta\,, \; i = 1, \dots N \quad (5.3)$$

We consider the time series of the trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ as random variables during Δ (5.3). Equation (1.1) at moment t_i defines the price $p(t_i)$ of market trade value $C(t_i)$ and volume $U(t_i)$. The sum $C_\Sigma(t;1)$ of values $C(t_i)$ (5.2) and the sum $U_\Sigma(t;1)$ of volumes $U(t_i)$ (5.2) of N trades during Δ (5.3) define the VWAP $p(t;1)$ (5.1).

We hope that readers are able distinguish the difference between the notations of consumption c_t (2.2; 2.3) and utility U (2.2) in Sections 2-4 and trade value $C(t_i)$ and volume $U(t_i)$ (5.1) in the current Section.

It is obvious that VWAP (5.1) can be equally determined (5.4) by the mean value $C(t;1)$ (5.5) and the mean volume $U(t;1)$ (5.6) of N trades during Δ:

$$C(t;1) = p(t;1)\, U(t;1) \quad (5.4)$$

The mean trade value $C(t;1)$ and volume $U(t;1)$ are assessed by the finite number N of trades during Δ (5.3) through the conventional frequency-based approach:

$$C(t;1) \equiv E[C(t_i)] \sim \frac{1}{N} \sum_{i=1}^{N} C(t_i) \quad (5.5)$$

$$U(t;1) \equiv E[U(t_i)] \sim \frac{1}{N} \sum_{i=1}^{N} U(t_i) \quad (5.6)$$

The notion ~ indicates that (5.5; 5.6) give only assessments of mean trade value *C(t;1)* and mean volume *U(t;1)* by a finite number *N* of trades during *Δ* (5.3). VWAP *p(t;1)* (5.4) is a coefficient between the mean value *C(t;1)* (5.5) and the mean volume *U(t;1)* (5.6).

Actually, the trade equation (1.1) imposes constraints on the probabilities of the trade value $C(t_i)$, volume $U(t_i)$ and price $p(t_i)$ time series. Given the probabilities of trade value $C(t_i)$ and volume $U(t_i)$ time series during *Δ,* that match (1.1) should determine the price probability. However, VWAP *p(t;1)* and relations (5.1-5.6) are not sufficient to define all random properties of price as a random variable during *Δ* (5.3). Actually, it is well known that properties of a random variable can be equally described by probability measure, characteristic function, and a set of statistical moments (Shephard, 1991; Shiryaev, 1999; Klyatskin, 2005). To approximate the properties of the market trade value and volume as random variables during *Δ* (5.3), one could assess their *n*-th statistical moments of the trade value *C(t;n)* and volume *U(t;n)*:

$$C(t;n) \equiv E[C^n(t_i)] \sim \frac{1}{N}\sum_{i=1}^{N} C^n(t_i) \tag{5.7}$$

$$U(t;n) \equiv E[U^n(t_i)] \sim \frac{1}{N}\sum_{i=1}^{N} U^n(t_i) \tag{5.8}$$

We mention that the *n*-th power of (1.1) for each particular trade at the time t_i gives:

$$C^n(t_i) = p^n(t_i)U^n(t_i) \ \ ; \ \ n = 1, 2, \ldots \tag{5.9}$$

We use (5.7-5.9) to determine the *n*-th statistical moments *p(t;n)* of price for *n=1,2,3,…* via the *n*-th statistical moments of the trade value *C(t;n)* (5.7) and volume *U(t;n)* (5.8). We extend the definition of the VWAP (5.1; 5.2) and use (5.7; 5.8; 5.11) to introduce the *n*-th statistical moment *p(t;n)* of price in a way similar to VWAP (5.1) as the *n*-th power volume averaged:

$$p(t;n) \equiv E[p^n(t_i)] = \frac{1}{\sum_{i=1}^{N} U^n(t_i)} \sum_{i=1}^{N} p^n(t_i)U^n(t_i) = \frac{C_\Sigma(t;n)}{U_\Sigma(t;n)} = \frac{C(t;n)}{U(t;n)} \tag{5.10}$$

$$C_\Sigma(t;n) \equiv \sum_{i=1}^{N} C^n(t_i) = \sum_{i=1}^{N} p^n(t_i)U^n(t_i) \ \ ; \ \ U_\Sigma(t;n) \equiv \sum_{i=1}^{N} U^n(t_i) \tag{5.11}$$

We highlight that definitions (5.10) use equation (5.9) and that results expression (5.12) of price *n*-th statistical moments *p(t;n)* through *n*-th statistical moments of the market trade value *C(t;n)* and volume *U(t;n)*:

$$C(t;n) = p(t;n)U(t;n) \tag{5.12}$$

Definitions of price n-th statistical moments $p(t;n)$ (5.10; 5.12) for all $n=1,2,\ldots$ match equation (5.9) for the n-th power of price $p^n(t_i)$ at time t_i during Δ (5.3). It is important that the n-th statistical moments $p(t;n)$ of price (5.10; 5.12) for all $n=1,2,\ldots$ completely determine the properties of market price as a random variable during Δ (5.3).

Let us outline important unnoticed consequences of the VWAP $p(t;1)$ (5.1) and similar consequences of our definition of price n-th statistical moments $p(t;n)$ (5.10; 5.12). The definition of VWAP $p(t;1)$ (5.1) results in zero correlations between the time series of price $p(t_i)$ and trade volume $U(t_i)$ during Δ (5.3). Indeed, from (1.1; 5.1; 5.5; 5.6) obtain:

$$E[C(t_i)] \sim \frac{1}{N}\sum_{i=1}^{N} C(t_i) = \frac{1}{N}\sum_{i=1}^{N} p(t_i)U(t_i) \sim E[p(t_i)U(t_i)] \sim$$

$$\sim \frac{1}{\sum_{i=1}^{N} U(t_i)}\sum_{i=1}^{N} p(t_i)U(t_i) \cdot \frac{1}{N}\sum_{i=1}^{N} U(t_i) \sim E[p(t_i)]E[U(t_i)] \tag{5.13}$$

Hence, from (5.13) obtain the correlation $corr\{p(t_i)U(t_i)\}$ between time series of price $p(t_i)$ and trade volume $U(t_i)$, which are averaged during Δ (5.3):

$$corr\{p(t_i)U(t_i)\} \equiv E[p(t_i)U(t_i)] - E[p(t_i)]E[U(t_i)] = 0 \tag{5.14}$$

Zero correlations (5.14) between price-volume time series are determined by using VWAP (5.1) and average volume $U(t;1)$ (5.6). However, many publications detect positive or negative correlations between price and trading volume (Tauchen and Pitts, 1983; Karpoff, 1987; Campbell et al., 1993; Llorente et al., 2001; DeFusco et al., 2017). These papers describe correlations determined by the frequency-based definition of price probability. Assessments of correlations between any time series follow definitions of its averaging procedures. The use of different probabilities causes different results in correlations. The use of VWAP (5.1; 5.2; 5.13; 5.14) states that there are no correlations between trade volume and price.

Our definitions of price n-th statistical moments $p(t;n)$ (5.7-5.12) for all $n=1,2,3,..$ cause zero correlations $corr\{p^n(t_i)U^n(t_i)\}$ between time series of the n-th power of price $p^n(t_i)$ and volume $U^n(t_i)$ over Δ (5.3). One can easily reproduce (5.13; 5.14) for $n=1,2,3,\ldots$:

$$E[C^n(t_i)] \sim \frac{1}{N}\sum_{i=1}^{N} C^n(t_{\square}) \sim E[p^n(t_i)U^n(t_i)] \sim \frac{1}{N}\sum_{i=1}^{N} p^n(t_i)U^n(t_i) =$$

$$= \frac{1}{\sum_{i=1}^{N} U^n(t_i)} \sum_{i=1}^{N} p^n(t_i) U^n(t_i) \cdot \frac{1}{N} \sum_{i=1}^{N} U^n(t_i) \sim E[p^n(t_i)] E[U^n(t_i)] \quad (5.15)$$

$$corr\{p^n(t_i)U^n(t_i)\} \equiv E[p^n(t_i)U^n(t_i)] - E[p^n(t_i)]E[U^n(t_i)] = 0 \quad (5.16)$$

Thus, the market-based definition of price *n*-th statistical moments *p(t;n)* (5.7-5.12) causes zero correlations between time series of the *n*-th power of price $p^n(t_i)$ and volume $U^n(t_i)$ during *Δ* but doesn't imply statistical independence between time series of $p(t_i)$ and volume $U(t_i)$. For example we derive a correlation $corr\{p(t_i)U^2(t_i)\}$ between time series of price $p(t_i)$ and squares of trade volumes $U^2(t_i)$ during *Δ:*

$$E[p(t_i)U^2(t_i)] \equiv E[C(t_i)U(t_i)] = E[C(t_i)]E[U(t_i)] + corr\{C(t_i)U(t_i)\}$$

$$E[p(t_i)U^2(t_i)] = E[p(t_i)]E[U^2(t_i)] + corr\{p(t_i)U^2(t_i)\}$$

$$corr\{p(t_i)U^2(t_i)\} = E[C(t_i)U(t_i)] - p(t;1)U(t;2)$$

Thus, from above (5.4-5.6; 5.13), one easily obtains:

$$corr\{p(t_i)U^2(t_i)\} = corr\{C(t_i)U(t_i)\} - p(t;1)\sigma^2(U) \quad (5.17)$$

Correlation $corr\{C(t_i)U(t_i)\}$ (5.17) between time series of trade value and volume could be assessed by (5.5; 5.6) and (5.17.1):

$$corr\{C(t_i)U(t_i)\} \equiv E[C(t_i)U(t_i)] - E[C(t_i)]E[U(t_i)]$$

$$E[C(t_i)U(t_i)] \sim \frac{1}{N} \sum_{i=1}^{N} C(t_i)\, U(t_i) \quad (5.17.1)$$

In (5.17), we denote as $\sigma^2(U)$ the volatility of the trade volume (5.18):

$$\sigma^2(U) \equiv U(t;2) - U^2(t;1) \quad (5.18)$$

It is obvious that the market-based price statistical moments *p(t;n)* (5.10; 5.12) differ from the statistical moments *π(t;n)* generated by frequency-based price probability *P(p)* (5.19):

$$P(p) \sim \frac{m(p)}{N} \quad ; \quad \pi(t;n) \sim \frac{1}{N} \sum_{i=1}^{N} p^n(t_i) \quad (5.19)$$

$$\pi(t;n) \sim \frac{1}{N} \sum_{i=1}^{N} p^n(t_i) = \frac{1}{N} \sum_{i=1}^{N} \frac{C^n(t_i)}{U^n(t_i)} \neq \frac{\sum_{i=1}^{N} C^n(t_i)}{\sum_{i=1}^{N} U^n(t_i)} = \frac{C_\Sigma(t;n)}{U_\Sigma(t;n)} = \frac{C(t;n)}{U(t;n)} = p(t;n) \quad (5.20)$$

The difference between the frequency-based *π(t;n)* and the market-based *p(t;n)* price statistical moments determines the economic distinctions between the two approaches to the definition of the price probability. Statistical moments *π(t;n)* equal *p(t;n)* only if all trade volumes equal unit $U(t_i)=1$ during *Δ* (5.3).

To clarify the economic origin of the difference between two assessments of the price statistical moments determined by the market-based approach (5.10; 5.12) and the conventional frequency-based approach (5.19; 5.20), we mention, that in a general case, the *n*-th statistical moments $p_\mu(t;n)$ of the given price time series $p(t_i)$, $i=1,..N$ during interval Δ (5.3) can be assessed via weighted functions $\mu_i(t;n)$:

$$\mu_i(t;n) \geq 0 \quad ; \quad \sum_{i=1}^{N} \mu_i(t;n) = 1 \quad ; \quad p_\mu(t;n) = \sum_{i=1}^{N} \mu_i(t;n)\, p^n(t_i) \tag{5.21}$$

The frequency-based price statistical moments (5.19) correspond to all $\mu_i(t;n)=1/N$ and the market-based statistical moments (5.10) of price take $\mu_i(t;n)$ as (5.22):

$$\mu_i(t;n) = \frac{1}{\sum_{i=1}^{N} U^n(t_i)} U^n(t_i) \tag{5.22}$$

The frequency-based approach to price probability with all $\mu_i(t;n)=1/N$ assumes that the price probabilities of all *N* market trades are equal to *1/N*. The market-based approach proposes that only the *n*-th statistical moments of the trade values (5.7) and volumes (5.8) are determined using the frequency-based approach, with their probabilities equal to *1/N.* However, the market-based price probability reveals that price statistical moments (5.10; 5.21; 5,22) depend on the statistical moments of the trade values and volumes. With increasing *n,* the *n*-th statistical moments of price $p(t;n)$ more and more reveal the impact of huge trade values and volumes. The market-based approach supports a simple market rule: the price of trade with a value of $100 million is much more significant for the market than the price of trade with a value of $10. Hence, the price probabilities of these two trades must be different. The market-based price probability describes that important relationship.

From equations (1.1; 5.9) and due to the *n*-th statistical moments of trade values and volumes (5.7; 5.8), one obtains that the *n*-th statistical moments $p_\mu(t;n)$ of price determined by weighted functions $\mu_i(t;n)$ (5.21) cause correlations $corr_\mu\{p^n(t_i)U^n(t_i)\}$ (5.23) between *n*-th powers of price $p^n(t_i)$ and trade volume $U^n(t_i)$ time series:

$$corr_\mu\{p^n(t_i)U^n(t_i)\} = \frac{1}{N}\sum_{i=1}^{N} p^n(t_i)U^n(t_i) - \sum_{i=1}^{N} \mu_i(t;n)\, p^n(t_i) \frac{1}{N}\sum_{i=1}^{N} U^n(t_i) \tag{5.23}$$

For $n=1$ relations (5.23) and $\mu_i(t;n)=1/N$ determine the frequency-based price-volume $corr\{p(t_i)U(t_i)\}$ correlation during Δ (5.3):

$$corr\{p(t_i)U(t_i)\} = \frac{1}{N}\sum_{i=1}^{N} p(t_i)U(t_i) - \frac{1}{N}\sum_{i=1}^{N} p(t_i)\frac{1}{N}\sum_{i=1}^{N} U(t_i)$$

This form of correlation corresponds to frequency-based price probability (5.19) and was studied by (Tauchen and Pitts, 1983; Karpoff, 1987; Campbell et al., 1993; Llorente et al., 2001; DeFusco et al., 2017). However, market-based price statistical moments (5.7; 5.8), which are described by (5.22), result in zero correlations (5.16) for all n=1,2,…

Now let us consider the approximations of price characteristic functions and probability measures by finite sets of price statistical moments. The set of price *n*-th statistical moments $p(t;n)$ (5.10; 5.12) for all $n=1,2,3,\ldots$ determines the Taylor series of the price characteristic function $F(t;x)$ (Shephard, 1991; Shiryaev, 1999; Klyatskin, 2005):

$$F(t;x) = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} p(t;n)\, x^n \qquad (5.24)$$

In (5.24), i denotes the imaginary unit $i^2 = -1$. However, any records of market trades during Δ (5.3) assess only a finite number of statistical moments of the trade value $C(t;n)$ (5.7) and volume $U(t;n)$ (5.8). Hence, one can assess only a finite number of price statistical moments $p(t;n)$ (5.10; 5.12). In App.B, we consider simple successive approximations of the price characteristic function $F_K(t;x)$ that take into account the finite number K of the Taylor series terms (5.24) and corresponding K-approximations of the price probability measure $\eta_K(t;p)$ derived as Fourier transforms of the characteristic function $F_K(t;x)$:

$$\eta_K(t;p) = \frac{1}{\sqrt{2\pi}} \int dx\, F_K(t;x)\, exp(-ixp) \qquad (5.25)$$

Relations (5.25) define successive approximations of the price probability measure $\eta_K(t;p)$. Assessments of the finite number K of market trade and price statistical moments result in the conclusion that one can only forecast approximations of the price characteristic function or price probability measure that match the finite number K of price statistical moments $p(t;n)$ (5.10; 5.12).

$$p(t;n) = \frac{d^n}{(i)^n dx^n} F_K(t;x)|_{x=0} = \int dp\, \eta_K(t;p) p^n \quad ; \quad n \le K \qquad (5.26)$$

Any hypothesis on the form of the price probability measure $\eta_K(t;p)$ during Δ (5.3) and predictions of the price probability at horizon T should match relations (5.10; 5.12; 5.26) at $t+T$. Thus, one should predict K statistical moments of the trade value $C(t;n)$ (5.7) and volume *U(t;n)* (5.8) at *t+T* for *n≤K*. That equals the prediction of the *K*-approximations of the market trade probabilities at horizon *T*. In simple words, the accuracy of price probability predictions depends on the precision of forecasts of market trade statistical moments. For example, consider the market-based price volatility $\sigma^2(t;p)$ (Olkhov, 2020):

$$\sigma^2(t;p) \equiv E\left[\left(p(t_i) - p(t;1)\right)^2\right] = p(t;2) - p^2(t;1) = \frac{C(t;2)}{U(t;2)} - \frac{C^2(t;1)}{U^2(t;1)} \tag{5.27}$$

From (5.7; 5.8; 5.11), one can express market-based price volatility $\sigma^2(t;p)$ as:

$$\sigma^2(t;p) = \frac{C(t;2)}{U(t;2)} - \frac{C^2(t;1)}{U^2(t;1)} = \frac{C_\Sigma(t;2)}{U_\Sigma(t;2)} - \frac{C_\Sigma^2(t;1)}{U_\Sigma^2(t;1)} \tag{5.28}$$

Prediction of the price volatility $\sigma^2(t;p)$ at horizon *T* during *Δ* requires forecasts of the market trade statistical moments *C(t;1), C(t;2)* (5,7) and *U(t;1), U(t;2)* (5.8) at the same horizon *T*. The accuracy of the price probability forecasts is determined by the accuracy of the market trade probability predictions. In simple words, to predict price probability, one should be able to predict market trade values and trade volumes probabilities, which is almost the same as "predicting the future of the entire economy".

6. Conclusion

Each economic theory and asset pricing in particular should directly indicate the time scales *Δ* of the model under consideration. Time series of the market trades with time shift *ε* introduce initial division of the time axis multiple of *ε*. Asset pricing models should take into account these initial data as the only source for averaged market time series. Any averaging of market time series presumes the usage of a particular time averaging interval Δ>>ε. Averaging of initial market time series during *Δ* introduces transition from initial time axis division multiple of *ε* to new division multiple of *Δ*. To consider utility function and price dynamics "today" and "next day", one should use the same time axis division "today" and "next day" and hence the same averaging interval *Δ*. Averaging of the investor's utility function "today" and "next day" introduces modifications to the investor's utility and basic pricing equation. The choice of interval *Δ* allows considering the Taylor series expansions of the modified investor's utility and basic pricing equation by price and payoff fluctuations and subsequent averaging of fluctuations. For linear and quadratic approximations of the basic pricing equation that give relations, that describe mean price, price volatility, mean payoff,

payoff volatility, and etc. In the linear Taylor approximation (4.12) presents dependence of mean price p_0 "today" during Δ, on price volatility $\sigma^2(p)$ "today"

$$p_0 = m_0 x_0 + \xi_{max}[m_1 \sigma^2(x) + m_2 \sigma^2(p)] \tag{6.1}$$

and on mean payoff x_0, payoff volatility $\sigma^2(x)$ "next day" and the amount of assets ξ_{max} that delivers max to the investor's utility and equals the root of the equation (3.3). On the one hand, (6.1) modifies the conventional statement "price equals expected discounted payoff" and demonstrates dependence on price volatility $\sigma^2(p)$ "today". On the other hand, (6.1) uncovers the direct dependence of the mean price p_0 "today" on the amount of assets ξ_{max} that delivers max to the investor's utility. That direct dependence doesn't add confidence in the impeccability of the consumption-based model's frame, and further argumentation is required to solve the troubles, that arise with the direct dependence of (6.1) on ξ_{max}.

The use of the averaging interval Δ as a mandatory factor in any financial model results in the introduction of the market-based probability of asset prices. Indeed, aggregations of market time series during Δ permit considering total values and volumes of market trades during Δ as important variables, which govern the variations of market price. As we show, n-th statistical moments of the trade value $C(t;n)$ (5.7) and volume $U(t;n)$ (5.8) project the impact of the size of the trading values and volumes on statistical moments of market price. With increasing n, the impact of trades with large values on market prices grows.

Time series of the performed market trades assess only the finite number K of statistical moments $C(t;n)$ (5.7) and $U(t;n)$ (5.8) and determine K-approximations of the market price probability. Any predictions of the price probability at horizon T should match forecasts of the n-th trade statistical moments for $n \leq K$ at the same horizon T.

We define the market-based price statistical moments $p(t;n)$ (5.10; 5.12) as extensions of VWAP (5.1). Their usage results in zero correlations $corr\{p^n(t_i)U^n(t_i)\}=0$ (5.14; 5.16) between time series of the n-th power of price $p^n(t_i)$ and trading volume $U^n(t_i)$. In particular, VWAP causes zero correlations (5.14) between the time series of price $p(t_i)$ and trading volume $U(t_i)$. That impacts studies on price-volume correlations based on the usage of a frequency-based approach to price probability. Zero correlations (5.16) between the n-th power of price $p^n(t_i)$ and trade volume $U^n(t_i)$ don't cause statistical independence between price and volume random variables during Δ (5.3). We derive an expression for the correlation $corr\{p(t_i)U^2(t_i)\}$ (5.17) between price and squares of trade volumes during Δ (5.3). This trinity – the averaging interval Δ, the Taylor series, and the market-based price probability can provide successive approximations for other versions of asset pricing, financial, and economic models. These methods were used to describe Value-at-Risk problems, volatility, option pricing, market-based price and payoff autocorrelations, and the market-based probability of stock returns (Olkhov, 2020-2023).

Appendix A.

Max of Utility

We start with (4.29):

$$p^2 > -\beta \frac{w''(c_{t+1;0})}{w''(c_t)} [x_0{}^2 + \sigma^2(x)] - \beta \frac{w'''(c_{t+1;0})}{w''(c_t)} \xi_{max} [2x_0\sigma^2(x) + \gamma^3(x)] \tag{A.1}$$

If the right side is negative, then it is always valid. If the right side is positive, then there exists a lower limit on the price *p*. For simplicity, neglect term $\gamma^3(x)$ to compare with $2x_0\sigma^2(x)$ and take the conventional power utility $w(c)$ (Cochrane, 2001) as:

$$w(c) = \frac{1}{1-\alpha} c^{1-\alpha} \tag{A.2}$$

Let us consider the case with the negative right side (A.1). Simple but long calculations give:

$$-\beta \frac{w''(c_{t+1;0})}{w''(c_t)} [x_0{}^2 + \sigma^2(x)] < \beta \frac{w'''(c_{t+1;0})}{w''(c_t)} \xi_{max}\, 2x_0\sigma^2(x)$$

$$\xi_{max}\, 2x_0\sigma^2(x) < -\frac{w'''(c_{t+1;0})}{w''(c_{t+1;0})} [x_0{}^2 + \sigma^2(x)] \tag{A.3}$$

Let us take into account (A.2) and for (A.3) obtain:

$$\frac{w''(c)}{w'''(c)} = -\frac{c}{1+\alpha} \quad ; \quad \xi_{max}\, 2x_0\sigma^2(x) < \frac{e_{t+1} + x_0\xi_{max}}{1+\alpha} [x_0{}^2 + \sigma^2(x)]$$

$$\xi_{max} x_0 [(1+2\alpha)\sigma^2(x) - x_0{}^2] < e_{t+1}[x_0{}^2 + \sigma^2(x)] \tag{A.4}$$

Inequality (A.4) determines that the right side (A.1) is negative in two cases. The left side of (A.4) is negative and

$$(1+2\alpha)\sigma^2(x) < x_0{}^2 \quad ; \quad \frac{1}{3} \le \frac{1}{1+2\alpha} < 1 \tag{A.5}$$

Inequality (A.5) describes small payoff volatility $\sigma^2(x)$. In this case, the right side of (A.1) is negative for all ξ_{max} and all price *p* and hence (4.27) that defines the max of utility (2.5) is valid. The left side of (A.4) is positive, and

$$(1+2\alpha)\sigma^2(x) > x_0{}^2 \quad ; \quad \xi_{max} < \frac{e_{t+1}[x_0{}^2 + \sigma^2(x)]}{x_0 [(1+2\alpha)\sigma^2(x) - x_0{}^2]} \tag{A.6}$$

This case describes high payoff volatility and the upper limit on ξ_{max} to utility (2.5). Take the positive right side in (A.1). Then (A.4) is replaced by the opposite inequality

$$\xi_{max} x_0 [(1+2\alpha)\sigma^2(x) - x_0{}^2] > e_{t+1}[x_0{}^2 + \sigma^2(x)] \tag{A.7}$$

It is valid for (A.6) only. (A.7) determines a lower limit on ξ_{max} to utility (2.5):

$$\xi_{max} > \frac{e_{t+1}[x_0{}^2 + \sigma^2(x)]}{x_0 [(1+2\alpha)\sigma^2(x) - x_0{}^2]}$$

Appendix B.

Approximations of the price characteristic function and probability measure

The Taylor series expansions of the market price characteristic function result in successive approximations of the characteristic function. The derivation of approximations is a self-standing problem, and here we present simple examples of such approximations only. We consider simple approximations of the price characteristic function $F_K(t;x)$ and the price probability measure $\eta_K(t;p)$ during Δ (5.3) that fit the obvious condition. As such, we consider the approximations $F_K(t;x)$ of the price characteristic function that match (B.1):

$$p(t;n) = \frac{C(t;n)}{U(t;n)} = \frac{d^n}{(i)^n dx^n} F_k(t;x)|_{x=0} = \int dp\, \eta_k(t;p) p^n \; ; \;\; n \le K \qquad \text{(B.1)}$$

Statistical moments determined by $F_K(t;x)$ for $n>K$ will be different from price statistical moments $p(t;n)$ (5.10; 5.12), but first K moments will be equal to $p(t;n)$.

We suggest the approximation $F_K(t;x)$ of the price characteristic function $F(t;x)$ (5.24)

$$F_K(t;x) = exp\left\{\sum_{m=1}^{K} \frac{i^m}{m!}\, a_m\, x^m - b\, x^{2n}\right\} \;\; ; \;\; K = 1, 2, \dots ; \; K < 2n \; ; \; b > 0 \qquad \text{(B.2)}$$

For each approximation of $F_K(t;x)$, terms a_m in (B.2) depend on price statistical moments $p(t;m)$, $m \le K$ and match relations (B.1). The terms bx^{2n}, $b>0$, $2n>K$ don't impact relations (B.1) but guarantees the existence of the price probability measures $\eta_K(t;p)$ as Fourier transforms (5.25). The uncertainty and variability of the coefficient $b>0$ and power $2n>K$ in (B.2) underscores the well-known fact that the first k statistical moments don't explicitly determine the characteristic function and probability measure of a random variable. Relations (B.2) describe the set of characteristic functions $F_K(t;x)$ with different $b>0$ and $2n>K$ and the corresponding set of probability measures $\eta_K(t;p)$ that match (B.1; 5.25). For $K=1$ approximate price characteristic function $F_1(t;x)$ and measure $\eta_1(t;p)$ are trivial:

$$F_1(t;x) = exp\{i\, a_1 x\} \;\; ; \; p(t;1) = -i \frac{d}{dx} F_1(t;x)|_{x=0} = a_1 \qquad \text{(B.3)}$$

$$\eta_1(t;p) = \int dx \;\, A_1(xt;)\, exp - ipx = \delta\big(p - \; p(t;1)\big) \qquad \text{(B.4)}$$

For $K=2$ approximation $F_2(t;x)$ describes the Gaussian probability measure $\eta_2(t;p)$:

$$F_2(x;t) = exp\left\{i\, p(t;1)x - \frac{a_2}{2} x^2\right\} \qquad \text{(B.5)}$$

It is easy to show that

$$p_2(t;2) = -\frac{d^2}{dx^2}F_2(t;x)|_{x=0} = a_2 + p^2(t;1) = p(t;2)$$

$$a_2 = p(t;2) - p^2(t;1) = \sigma^2(t;p) \quad \text{(B.6)}$$

Coefficient a_2 equals price volatility $\sigma^2(t;p)$ (5.27) and the Fourier transform (5.25) for $F_2(t;x)$ gives Gaussian price probability measure $\eta_2(t;p)$:

$$\eta_2(pt;) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma(p)} exp\left\{-\frac{(p-p(t;1))^2}{2\sigma^2(t;p)}\right\} \quad \text{(B.7)}$$

For $K=3$ approximation $F_3(t;x)$ has form:

$$F_3(t;x) = exp\left\{i\,p(t;1)x - \frac{\sigma^2(t;p)}{2}x^2 - i\,\frac{a_3}{6}x^3\right\} \quad \text{(B.8)}$$

$$p_3(t;3) = i\frac{d^3}{dx^3}F_3(t;x)|_{x=0} = a_3 + 3p(t;1)\sigma^2(t;p) + p^3(t;1) = p(t;3)$$

$$a_3 = p(t;3) - 3p(t;2)p(t;1) + 2\,p^3(t;1)$$

$$a_3 = E\left[\left(p - p(t;1)\right)^3\right] = Sk(t;p)\sigma^3(t;p) \quad \text{(B.9)}$$

Coefficient a_3 (B.9) depends on price skewness $Sk(t;p)$, which describes the asymmetry of the price probability from the normal distribution. For the $K=4$ approximation $F_4(t;x)$ during Δ (5.3) depends on the choice of $b>0$ and power $2n>4$:

$$F_4(t;x) = exp\left\{i\,p(t;1)x - \frac{\sigma^2(t;p)}{2}x^2 - i\,\frac{a_3}{6}x^3 + \frac{a_4}{24}x^4 - bx^{2n}\right\}\ ;\ 2n > 4 \quad \text{(B.10)}$$

Simple, but long calculations give:

$$a_4 = p(t;4) - 4p(t;3)p(t;1) + 12p(t;2)p^2(t;1) - 6p^4(t;1) - 3p^2(t;2)$$

$$a_4 = E\left[\left(p(t_i) - p(t;1)\right)^4\right] - 3E^2\left[\left(p(t_i) - p(t;1)\right)^2\right]$$

Price kurtosis $Ku(p)$ (B.11) describes how the tails of the price probability measure $\eta_K(t;p)$ differ from the tails of a normal distribution.

$$Ku(p)\sigma_p^4(t;p) = E\left[\left(p(t_i) - p(t;1)\right)^4\right] \quad \text{(B.11)}$$

$$a_4 = [Ku(p) - 3]\sigma_p^4(t;p)$$

Even the simplest Gaussian approximation $F_2(t;x)$, $\eta_2(t;p)$ (B.5; B.7) highlights the direct dependence of price volatility $\sigma^2(t;p)$ (B.6; 5.27) on 2-d statistical moments of the trade value $C(t;2)$ and volume $U(t;2)$. Thus, prediction of price volatility $\sigma^2(t;p)$ for Gaussian measure $\eta^2(t;p)$ (B.9) should follow non-trivial forecasting of the statistical moments of the market trade value $C(t;2)$ and volume $U(t;2)$.

References

Berkowitz, S.A., Dennis, E., Logue, D.E., Noser, E.A. Jr., (1988). The Total Cost of Transactions on the NYSE, The Journal of Finance, 43, (1), 97-112

Brunnermeier, M.K, (2015). Asset pricing I: Pricing Models, FIN 501Princeton Univ., 1-159 https://markus.scholar.princeton.edu/sites/g/files/toruqf2651/files/markus/files/fin_501_lecture_notes_2014.pdf

Buryak, A., Guo, I. (2014). Effective And Simple VWAP Options Pricing Model, Intern. J. Theor. Applied Finance, 17, (6), 1450036, https://doi.org/10.1142/S0219024914500356

Busseti, E., Boyd, S. (2015). Volume Weighted Average Price Optimal Execution, 1-34, arXiv:1509.08503v1

Campbell, J.Y., Grossman, S.J. and J.Wang, (1993). Trading Volume and Serial Correlation in Stock Return. Quatr. Jour. Economics, 108 (4), 905-939

Campbell, J.Y. (2002). Consumption-Based Asset Pricing. Harvard Univ., Cambridge, Discussion Paper # 1974, 1-116

CME Group (2020) https://www.cmegroup.com/search.html?q=VWAP

Cochrane, J.H. (2001). Asset Pricing. Princeton Univ. Press, Princeton, N. Jersey, US

DeFusco, A.A., Nathanson, C.G. and E. Zwick, (2017). Speculative Dynamics of Prices and Volume, Cambridge, MA, NBER WP 23449, 1-74

Duffie, D., and W. Zame, (1989). The Consumption-Based Capital Asset Pricing Model, Econometrica, 57 (6), 1279-1297

Duffie, D., Dworczak, P. (2018). Robust Benchmark Design. NBER, WP 20540, 1-56

Forbes, C., Evans, M., Hastings, N., Peacock, B. (2011). Statistical Distributions. Wiley

Karpoff, J.M. (1987). The Relation Between Price Changes and Trading Volume: A Survey. The Journal of Financial and Quantitative Analysis, 22 (1), 109-126

Klyatskin, V.I. (2005). Stochastic Equations through the Eye of the Physicist, Elsevier B.V.

Llorente, G., Michaely R., Saar, G. and J. Wang. (2001). Dynamic Volume-Return Relation of Individual Stocks. NBER, WP 8312, Cambridge, MA., 1-55

Merton, R.C. (1973). An Intertemporal Capital Asset Pricing Model, Econometrica, 41, (5), 867-887

Olkhov, V., (2020). Volatility Depend on Market Trades and Macro Theory. MPRA, WP102434, 1-18

Olkhov, V., (2021a). To VaR, or Not to VaR, That is the Question. SSRN, WP 3770615, 1-14

Olkhov, V., (2021b). Classical Option Pricing and Some Steps Further, SSRN, WP 3587369, 1-16

Olkhov, V., (2022). Price and Payoff Autocorrelations in the Consumption-Based Asset Pricing Model, SSRN, WP 4050652, 1- 18

Olkhov, V., (2023). The Market-Based Probability of Stock Returns, SSRN, WP 4350975, 1-25

Ross, S. A., (1976). The Arbitrage Theory of Capital Asset Pricing, Jour. Economic Theory, 13, 341–360

Sharpe, W.F. (1964). Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk. The Journal of Finance, 19 (3), 425-442

Shephard, N.G. (1991). From Characteristic Function to Distribution Function: A Simple Framework for the Theory. Econometric Theory, 7 (4), 519-529

Shiryaev, A.N. (1999). Essentials Of Stochastic Finance: Facts, Models, Theory. World Sc. Pub., Singapore. 1-852

Tauchen, G.E., Pitts, M. (1983). The Price Variability-Volume Relationship On Speculative Markets, Econometrica, 51, (2), 485-505

Walck, C. (2007). Hand-book on statistical distributions. Univ.Stockholm, SUF–PFY/96–012. Publication place: Publisher, vol. 3, pp. 54–96