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Abstract

The stochastic volatility (SV) model has been one of the most popular models for latent stock return volatility. Extensions of the SV model focus on either improving volatility inference or modeling higher moments of the return distribution. This study investigates which extension can better improve return density forecasts. By examining various specifications with S&P 500 daily returns for nearly 20 years, we find that a more accurate capture of volatility dynamics with realized volatility and implied volatility is more important than modeling higher moments for a conventional SV model in terms of both density and tail forecasts. The accuracy of volatility estimation and forecasts should be the precondition for higher moments extensions.

JEL Classification: C11, C15, C22, C52

Keywords: Stochastic volatility, realized volatility, implied volatility, MCMC, density forecast

1 Introduction

Although the importance of heteroskedasticity for financial time series has been well documented (Nelson 1991; Schwert and Seguin 1990), modeling and forecasting volatility accurately could be difficult due to the unobservability of the true volatility process. To infer latent volatility, Taylor (1982) proposes the stochastic volatility (SV) model to estimate latent log volatility by observed daily stock returns. Extensions to the SV model fall into two categories. One stream of literature, focusing on modeling higher moments in addition to SV dynamics, extends the SV model with asymmetry, fat tails, and other distributional features (Barndorff-Nielsen 1997; Liesenfeld and Jung 2000; Chib et al. 2002; Jensen and Maheu 2010; Yu 2012; Kalli et al. 2013). Another stream of literature, focusing on a better capture of latent volatility, introduces the *ex-post* realized volatility (RV) or the *ex-ante* implied volatility (IV) to traditional volatility models (Koopman et al. 2005; Takahashi et al. 2009; Asai et al. 2017), to facilitate the estimation or prediction of latent volatility.

However, it remains unclear which approach, volatility or higher moments, makes a greater improvement in return density forecasts. This study contributes to the literature by comparing various model specifications to answer this question. We find that in an SV model, incorporating RV or IV information significantly outperforms the fat-tailed SV-t (Chib et al. 2002) and the semiparametric SV-DPM (Jensen and Maheu 2010) models in both return density forecasts and tail forecasts, suggesting that a more accurate capture of heteroskedasticity should be prioritized than modeling higher moments.

The existing literature augments the SV model with RV and IV, as they exploit the additional information for volatility inference contained in high-frequency returns and option prices. RV is a consistent estimator of latent volatility estimated from high-frequency returns (Andersen and Bollerslev 1998; Barndorff-Nielsen and Shephard 2001), and IV, calculated from option prices, is a risk-neutral expectation of future volatility and predicts future realized volatility (e.g. Christensen and Prabhala 1998; Blair et al. 2001; Busch et al. 2011; Kambouroudis et al. 2021). Koopman et al. (2005) is among the first to incorporate RV and

IV separately as an explanatory variable in the volatility process of the SV model. Applications of the augmentation approach can be found in Becker et al. (2007) and Kambouroudis et al. (2016). Since RV is a direct measure of latent volatility, Takahashi et al. (2009) propose the realized stochastic volatility (RVSV) model that jointly models RV and returns[1]. Joint modeling of RV and returns[2] improves estimates of latent volatility and allows for further extension of the SV model (Asai et al. 2017; Hansen et al. 2012; Koopman and Scharth 2012; Shirota et al. 2014; Zhang and Zhao 2023).

We adopt both the methods of Koopman et al. (2005) and Takahashi et al. (2009) to incorporate RV and IV into the SV framework and test the return density forecast performance against the Bayesian semiparametric and fat-tailed SV models that focus more on modeling higher moments. We confirm that RV and IV contain important and not mutually exclusive information to improve volatility forecasts in stochastic volatility modeling. Except for those jointly model IV and returns, models that embed RV and/or IV information have more accurate S&P 500 return volatility forecasts as well as significantly better return density forecasts and tail forecasts. Empirical results suggest that, for better return density forecasts of the SV model, improving the accuracy of volatility estimation and prediction should be prioritized and sophisticated specifications involving higher moments are recommended to be conditional on the accurate estimation of the latent volatility.

This study is organized as follows. Section 2 illustrates the specification of econometric models. Section 3 describes the source and statistical features of the data. Section 4 explains the algorithms to estimate and forecast each model. Section 5 discusses empirical results for in-sample estimation and out-of-sample forecasts. And Section 6 concludes the study.

[1]IV contains variance risk premia (see e.g. Bollerslev et al. 2009; Carr and Wu 2009) and is not a consistent estimator of true volatility. As shown in Section 5, jointly modeling IV and return estimates a volatility process almost indistinguishable from IV, and is dominated by RVSV in out-of-sample forecasts. However, IV is an effective predictor of the volatility process in the SV model and significantly improves out-of-sample performance.

[2]A potential argument, given the consitency of RV, is that we could directly modeling and forecasting RV (e.g. Corsi 2009). However, for daily stock returns, RV prediction is not necessarily equivalent to volatility prediction due to microstructure noise (Andersen and Teräsvirta 2009; Buccheri and Corsi 2021) and non-trading hour bias (Hansen et al. 2012; Lyócsa and Todorova 2020). Return density forecasts of SV models embody volatility (and higher moment) prediction by directly utilizing daily returns.

2 Model Specifications

The basic SV model with normal innovation (SV-N) is specified as

$$r_t = \exp(h_t/2)z_t, \quad z_t \sim N(0,1), \tag{1a}$$

$$h_t = \alpha + \delta h_{t-1} + u_t, \quad u_t \sim N(0, \sigma_h^2). \tag{1b}$$

Table 1 lists different specifications to incorporate RV and IV information in the SV model. RV and IV could be jointly modeled with returns, as additional measures of latent log volatility h_t, or incorporated into the volatility process (Eq. (1b)) as predictors of h_t.

[Insert Table 1 about here.]

We categorize the incorporation of RV and IV into three scenarios: with both RV and IV, which consists of three models: RVSV-IV, SV-RVIV, RVIVSV[3]; with RV only, which consists of two models: RVSV, SV-RV; and with IV only, which consists of two models: IVSV, SV-IV. RV or IV in front of SV indicates joint modeling with returns, and those after SV with a hyphen indicate incorporated as an explanatory variable for h_t. In comparison, we adopt two benchmarks for higher-moment modeling: a Student t extension (SV-t) that replaces z_t in (1a) with $\epsilon_t \sim t(\nu)$ and captures potential fat tails in the return distribution, and a Bayesian semiparametric model (SV-DPM) that, in addition to the SV, approximates the unknown distribution of returns nonparametrically (Jensen and Maheu 2010).[4]

[3]IVSV-RV is omitted because jointly model return and IV does not perform well as shown in Section 5.

[4]The SV-DPM model can be specified as:

$$\begin{aligned}
r_t &= \sigma_t \exp(h_t/2)z_t, \quad z_t \sim N(0,1), \\
h_t &= \delta h_{t-1} + u_t, \quad u_t \sim N(0, \sigma_h^2), \\
\sigma_t^2 &\sim G, \quad G \sim DP(G_0, \alpha_0),
\end{aligned}$$

where G_0 is essentially the prior distribution of σ_t^2.

3 Data

We retrieve the daily close price of S&P 500 index (SPX) ranging from January 5, 2004 to December 31, 2021, from Bloomberg. Daily return is calculated as the logarithmic difference of daily close prices, scaled by 100. RVs are obtained from the Oxford-Mannstitute's Quantitative Finance Realized Library and scaled by 10,000. IVs are computed by squaring the VIX index from the Chicago Board Options Exchange (CBOE) and scaled by 1/252[5].

[Insert Table 2 about here.]

Table 2 provides the descriptive statistics. The SPX returns are slightly skewed to the left but heavily leptokurtic. Both log RV and log IV are slightly skewed to the right and slightly leptokurtic.

4 Estimation and Prediction Algorithm

We apply the Bayesian Markov chain Monte Carlo (MCMC) algorithm to estimate the models in Section 2. In each MCMC iteration,

1. $\mu | r_{1:T}, h_{1:T}$,
2. $\nu | r_{1:T}, h_{1:T}$ (for SV-t),
3. $a_{RV}, \sigma^2_{RV} | RV_{1:T}, h_{1:T}$ (if necessary),
4. $a_{IV}, \sigma^2_{IV} | IV_{1:T}, h_{1:T}$ (if necessary),
5. $\alpha, \delta, \beta, \gamma, \sigma^2_h | h_{1:T}, RV_{1:T}, IV_{1:T}$ (sample β and/or γ if necessary),
6. $h_t | h_{-t}, \dots$ for $t = 1, \dots, T$.

Step 1, 3, 4 and 5 can be easily drawn with the conjugate Gibbs sampler, while the Student-t degree of freedom parameter ν in step 2 is drawn with random-walk Metropolis-Hastings (MH). In step 6, taking the latent volatility process $h_t, t = 1, 2, ...T$ as parameters, we draw

[5] The VIX index represents the annualized standard deviations of returns in percentage. Scaling by 1/252 yields the daily implied volatility.

the samples of h_t using the single-move independent MH of Kim et al. (1998) with some adjustments to the proposal distribution.[6] Empirically, the average acceptance rate for each model is greater than 95%. Let $\mathcal{I}_t = \{r_{1:t}, RV_{1:t}, IV_{1:t}\}$ be the information set at time t. Collect M MCMC samples after dropping some iterations of burn-ins, and the posterior moment of a given function $g(\cdot)$ can be computed from those MCMC posterior draws:

$$\mathrm{E}\left[g(\theta)|\mathcal{I}_T\right] \approx \frac{1}{M}\sum_{i=1}^{M} g\left[\theta^{(i)}\right],$$

where $\theta^{(i)}$ is the draw for parameter θ in the ith iteration.

To evaluate density forecasts, we compute the log-predictive likelihoods for each model. Define Θ as the set of all parameters. For any model $\mathcal{M}_A$, its out-of-sample log-predictive likelihoods can be estimated as

$$\begin{aligned}\log PL_A = \log p(r_{t+1:T}|\mathcal{I}_t, \mathcal{M}_A) &= \sum_{l=t+1}^{T} \log p(r_l|\mathcal{I}_{l-1}, \mathcal{M}_A) \\ &\approx \sum_{l=t+1}^{T} \log\left[\frac{1}{M}\sum_{i=1}^{M} p(r_l|\mathcal{I}_{l-1}, \Theta^{(i)}, \mathcal{M}_A)\right]\end{aligned}$$

where $\Theta^{(i)}$ is the draw for the parameter set Θ in the ith iteration. The model with greater log-predictive likelihoods has more accurate density forecasts.

5 Empirical Results

Priors for μ, a_{RV}, a_{IV}, α, δ, β, γ are commonly assumed as $N(0,1)$, and σ^2_{RV}, σ^2_{IV}, σ^2_h are assumed as $IG(0.25, 0.25)$. The prior for the degree of freedom parameter ν in the SV-t model is assumed as $U(2, 100)$, and for the concentration parameter α_0 in the SV-DPM is $Gamma(2, 8)$. In each estimation, the first 5,000 iterations are discarded as burn-ins, and the next 10,000 iterations are recorded as MCMC samples.

[6] See Appendix A for details.

5.1 In-sample Estimates

[Insert Table 3 about here.]

Table 3 shows the in-sample posterior estimates for each parameter in each model. The SV autoregressive parameter δ is close to 1 when no volatility measure is included in the volatility equation Eq. (1b) (RVIVSV, RVSV, IVSV, SV-t, SV-DPM and SV-N), while it reduces drastically otherwise (RVSV-IV, SV-RVIV, SV-RV, SV-IV). Regarding the volatility-to-volatility parameter σ_h^2, incorporating IV (RV) increases σ_h^2 from around 0.06 to about 0.3 (0.2). The joint modeling of RV and return can also increase σ_h^2, while the joint modeling of IV and return decreases σ_h^2.

[Insert Figure 1 about here.]

Figure 1 plots the posterior means of the latent log volatility h_t over time. The top left plot is the log RV, which is a rough process, suggesting high volatility-of-volatility. Both SV-N and SV-t produce a smooth h_t process[7]. By incorporating RV and IV, the estimated h_t process can be much rougher than SV-N, depending on the specifications. In general, the rougher the h_t process, the closer it is to the log RV. The lower part of Table 3 confirms this pattern. RVSV-IV has the roughest h_t estimates and the lowest in-sample error, compared to RV (QLIKE of 0.0529)[8]. Joint modeling of returns and IV (RVIVSV and IVSV) can greatly reduce the average width of the 0.95 density interval for h_t but with the highest estimation error of 0.27 for QLIKE[9]. Apart from the inaccurate RVIVSV and IVSV, RVSV-IV has the narrowest average 0.95 density interval. In summary, RVSV-IV, SV-RVIV and RVSV model can estimate the in-sample volatility h_t closest to the log RV process with a 0.95 density interval narrower than SV-DPM, and the SV-RV and SV-IV model have an estimation error similar to SV-DPM but a narrower 0.95 density interval.

[7] h_t from SV-N is approximately the same as that from SV-t.

[8] Given the microstructure noise of RV, we use the QLIKE loss function from Patton (2011) to measure the accuracy of the estimated or predicted volatility compared to RV: $QLIKE = \frac{\exp(h_t)}{RV_t} - \log \frac{\exp(h_t)}{RV_t} - 1$

[9] In fact, joint modeling of returns and IV procduce a estimated h_t that almost identical to log IV, since the estimated σ_{IV}^2 are close to zero in Table 3.

5.2 Out-of-Sample Forecasts

[Insert Table 4 about here.]

A recursive out-of-sample prediction is performed with the first year of the data as the initial training sample. Table 4 reports the performance of the out-of-sample denisty and point forecasts for each model. Except for those jointly model IV and returns, all models that incorporate RV and/or IV information significantly outperform the Bayesian semiparametric SV-DPM and fat-tailed SV-t model in both density forecasts, with log Bayes factor greater than 147[10], and volatility forecasts, with a minimum QLIKE score improvement of about 0.05. Joint modeling of IV and returns improves the return density forecasts from the SV-DPM by a small but significant amount, while the improvement from volatility forecasts is marginal except for the RVIVSV model. The RVSV-IV model performs significantly better than all other models with the highest log-predictive likelihoods of -5298.977 and the lowest QLIKE score of 0.2096. The results suggest that an effort to predict the second moment accurately is more important than modeling the higher moments in a flexible way.

[Insert Figure 2 about here.]

Figure 2 plots the cumulative log Bayes factors of the well-performing models compared to the SV-DPM model. All lines are approximately upward-sloping, showing that the improvement relative to the SV-DPM is not a consequence of extreme observations.

Incorporating both RV and IV information can also improve tail forecasts. Taylor (2019) and Patton et al. (2019) separately propose a scoring rule that evaluates the predictive value-at-risk and expected shortfall jointly. Table 5 shows the scores of these two measures; the lower the scores, the better. Interestingly, the Bayesian semiparametric SV-DPM model and the fat-tailed SV-t model fail to beat the models with additional information incorporated in the volatility dynamics at all three levels of risk in both scoring rules, except for the

[10] Log Bayes factor of model $\mathcal{M}_A$ against model $\mathcal{M}_B$ is $\log BF_{AB} = \log PL_A - \log PL_B$, and a log Bayes factor greater than 5 is considered as strong evidence in favour of $\mathcal{M}_A$.

IVSV model at the risk level of 1%, and the gains are more and more prominent when moving further to the left tail. It shows that better volatility modeling is more desirable than modeling higher moments in tail forecasts, especially in the far left tail of the return distribution, where a rarer but worse risk event may occur.

6 Conclusion

This study investigates whether modeling volatility more accurately or modeling higher moments should be prioritized in stock return density forecasts. We find that by incorporating the *ex-post* volatility measure RV and the *ex-ante* volatility measure IV into the SV model, volatility dynamics is captured more accurately both in-sample and out-of-sample. More importantly, both return density forecasts and tail forecasts can be significantly improved compared to the flexible Bayesian semiparametric model SV-DPM and the fat-tailed model SV-t model. However, the results do not deny the value of modeling higher moments. Our study suggests that future study of higher-moment extension of the SV model should be based on the precondition that the volatility dynamics is accurately and sufficiently captured.

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Table 1: Stochastic Volatility Models with RV and/or IV

Model	Specification
Return equation	
	$r_t = \exp(h_t/2)z_t, \quad z_t \sim N(0,1)$
With both RV and IV	
RVSV-IV	$\log RV_t = a_{RV} + h_t + \epsilon_t^{RV}, \quad \epsilon_t^{RV} \sim N(0, \sigma_{RV}^2)$ $h_t = \alpha + \delta h_{t-1} + \gamma \log IV_{t-1} + u_t, \quad u_t \sim N(0, \sigma_h^2)$
SV-RVIV	$h_t = \alpha + \delta h_{t-1} + \beta \log RV_{t-1} + \gamma \log IV_{t-1} + u_t, \quad u_t \sim N(0, \sigma_h^2)$
RVIVSV	$\log RV_t = a_{RV} + h_t + \epsilon_t^{RV}, \quad \epsilon_t^{RV} \sim N(0, \sigma_{RV}^2)$ $\log IV_t = a_{IV} + h_t + \epsilon_t^{IV}, \quad \epsilon_t^{IV} \sim N(0, \sigma_{IV}^2)$ $h_t = \alpha + \delta h_{t-1} + u_t, \quad u_t \sim N(0, \sigma_h^2)$
With RV only	
RVSV	$\log RV_t = a_{RV} + h_t + \epsilon_t^{RV}, \quad \epsilon_t^{RV} \sim N(0, \sigma_{RV}^2)$ $h_t = \alpha + \delta h_{t-1} + u_t, \quad u_t \sim N(0, \sigma_h^2)$
SV-RV	$h_t = \alpha + \delta h_{t-1} + \beta \log RV_{t-1} + u_t, \quad u_t \sim N(0, \sigma_h^2)$
With IV only	
IVSV	$\log IV_t = a_{IV} + h_t + \epsilon_t^{IV}, \quad \epsilon_t^{IV} \sim N(0, \sigma_{IV}^2)$ $h_t = \alpha + \delta h_{t-1} + u_t, \quad u_t \sim N(0, \sigma_h^2)$
SV-IV	$h_t = \alpha + \delta h_{t-1} + \gamma \log IV_{t-1} + u_t, \quad u_t \sim N(0, \sigma_h^2)$

1. Each model lists the difference in specification and will not be complete unless it is coupled with the common return equation.

Table 2: Descriptive Statistics for SPX Returns

	Mean	Median	StdDev	Skewness	Ex.Kurtosis	Min	Max
r	0.0323	0.0735	1.1947	-0.5474***	14.1882***	-12.6703	10.6420
$\log RV$	-0.8555	-1.0024	1.1555	0.6250***	0.7870***	-4.4063	4.3500
$\log IV$	0.1910	0.0455	0.7453	1.0540***	1.2351***	-1.1041	3.3008

1. r is the log returns for SPX in percentage. Source: CRSP.
2. $\log RV$ is the log realized variance for r. Source: Oxford-Mannstitute's Quantitative Finance Realized Library.
3. $\log IV$ is the log implied variance for r. Source: CBOE.
4. Data start from January 5, 2004 to December 31, 2021 with 4517 observations.
5. *** indicates that the p-value of the D'Agostino test (skewness) or the Anscombe-Glynn test (kurtosis) is less than 0.01.

Table 3: In-Sample Posterior Estimates

	RVSV-IV	SV-RVIV	RVIVSV	RVSV	SV-RV	IVSV	SV-IV	SV-DPM	SV-t	SV-N
α	-0.3210	-0.2837	-0.0051	-0.0302	0.1808	-0.0049	-0.7965	-	-0.0128	-0.0119
	(-0.3653, -0.2770)	(-0.4235, -0.1522)	(-0.0093, -0.0009)	(-0.0431, -0.0178)	(0.1400, 0.2257)	(-0.0090, -0.0008)	(-0.9394, -0.6522)		(-0.0217, -0.0045)	(-0.0210, -0.0035)
δ	0.5873	0.0096	0.9851	0.9424	0.4589	0.9856	-0.0473	0.9849	0.9749	0.9722
	(0.5394, 0.6368)	(-0.1350, 0.1597)	(0.9800, 0.9902)	(0.9302, 0.9540)	(0.3628, 0.5461)	(0.9806, 0.9905)	(-0.2175, 0.1228)	(0.9786, 0.9906)	(0.9649, 0.9837)	(0.9625, 0.9811)
β	-	0.4415	-	-	0.5068	-	-	-	-	-
		(0.3614, 0.5182)			(0.4281, 0.5952)					
γ	0.5040	0.7598	-	-	-	-	1.4027	-	-	-
	(0.4390, 0.5675)	(0.5588, 0.9713)					(1.1678, 1.6353)			
σ_h^2	0.2629	0.3660	0.0165	0.1316	0.2006	0.0159	0.3720	0.0551	0.0580	0.0648
	(0.2312, 0.2955)	(0.2896, 0.4539)	(0.0155, 0.0176)	(0.1151, 0.1498)	(0.1470, 0.2666)	(0.0149, 0.0170)	(0.2952, 0.4492)	(0.0410, 0.0704)	(0.0419, 0.0758)	(0.0483, 0.0839)
a_{RV}	-0.3111	-	-0.5125	-0.3336	-	-	-	-	-	-
	(-0.3542, -0.2705)		(-0.5536, -0.4734)	(-0.3798, -0.2912)						
σ_{RV}^2	0.0839	-	0.4620	0.1725	-	-	-	-	-	-
	(0.0620, 0.1055)		(0.4431, 0.4820)	(0.1583, 0.1874)						
a_{IV}	-	-	0.5339	-	-	0.5328	-	-	-	-
			(0.4992, 0.5671)			(0.4875, 0.5654)				
σ_{IV}^2	-	-	0.0036	-	-	0.0039	-	-	-	-
			(0.0031, 0.0041)			(0.0034, 0.0044)				
ν	-	-	-	-	-	-	-	-	24.6865	-
									(11.9085, 59.8367)	
α_0	-	-	-	-	-	-	-	0.3005	-	-
								(0.0768, 0.6677)		
K	-	-	-	-	-	-	-	3.9471	-	-
								(2.0000, 7.0000)		
QLIKE h	**0.0529**	0.1785	0.2691	0.0895	0.1975	0.2720	0.2088	0.1945	0.2042	0.2148
0.95DI width	0.9569	2.2190	**0.2128**	1.0210	1.8426	0.2238	2.2358	2.9270	1.7067	1.7074

1. The table reports the posterior mean and the 0.95 density interval for each parameter.
2. The lower part of the table reports the QLIKE loss function of Patton (2011) and the average width of the 0.95 density interval (0.95DI width) of the estimated volatility $\exp(h_t)$ ($\sigma_t^2 \exp(h_t)$ for SV-DPM) compared to the realized volatility RV_t. The lower the QLIKE, the more accurate the volatility estimates.

Table 4: Out-of-Sample Density and Point Forecasts

Model	log PL	log BF	QLIKE
RVSV-IV	**-5298.977**	212.4142	**0.2096**
SV-RVIV	-5359.476	151.9145	0.2497
RVIVSV	-5491.637	19.7541	0.3191
RVSV	-5354.552	156.8383	0.2173
SV-RV	-5347.470	163.9210	0.2272
IVSV	-5494.956	16.4352	0.3112
SV-IV	-5363.520	147.8712	0.2635
SV-DPM	-5511.391	—	0.3155
SV-t	-5519.601	-8.2104	0.3134
SV-N	-5527.284	-15.8932	0.3207

1. Data start on January 3, 2004. The out-of-sample period is from January 4, 2005 to December 31, 2021.
2. Log BF denotes the log-predictive likelihood (log PL) of the model in the first column minus that of the SV-DPM. The higher the value, the greater the improvement in density forecasts from the SV-DPM.
3. QLIKE denotes the loss function for volatility forecasts of Patton (2011), evaluated at the predicted volatility and compared to the realized volatility RV_t. The lower the QLIKE, the more accurate the volatility forecasts.

Table 5: Out-of-Sample Tail Forecasts

Risk level (α)	1%		5%		10%	
Scoring rule	Taylor	PZC	Taylor	PZC	Taylor	PZC
RVSV-IV	**2.1322**	**1.1097**	**1.7955**	**0.7268**	**1.6278**	**0.5012**
SV-RVIV	2.2647	1.2407	1.8145	0.7449	1.6348	0.5078
RVIVSV	2.2872	1.2651	1.8519	0.7851	1.6721	0.5484
RVSV	2.2767	1.2519	1.8280	0.7566	1.6435	0.5140
SV-RV	2.2159	1.1915	1.8152	0.7446	1.6373	0.5088
IVSV	2.3279	1.3054	1.8601	0.7929	1.6769	0.5528
SV-IV	2.1656	1.1449	1.8184	0.7523	1.6480	0.5246
SV-DPM	2.3283	1.3078	1.9074	0.8410	1.7124	0.5886
SV-t	2.3207	1.2997	1.9009	0.8339	1.7052	0.5806
SV-N	2.3651	1.3439	1.9012	0.8340	1.7057	0.5809

1. This table reports the scoring rule of Taylor (2019), denoted as "Taylor" and Patton et al. (2019), denoted as "PZC", which both evaluate the value-at-risk and expected shortfall forecasts jointly. The lower the score, the better the tail forecasts.

2. The data and the out-of-sample period are the same as in Table 4.

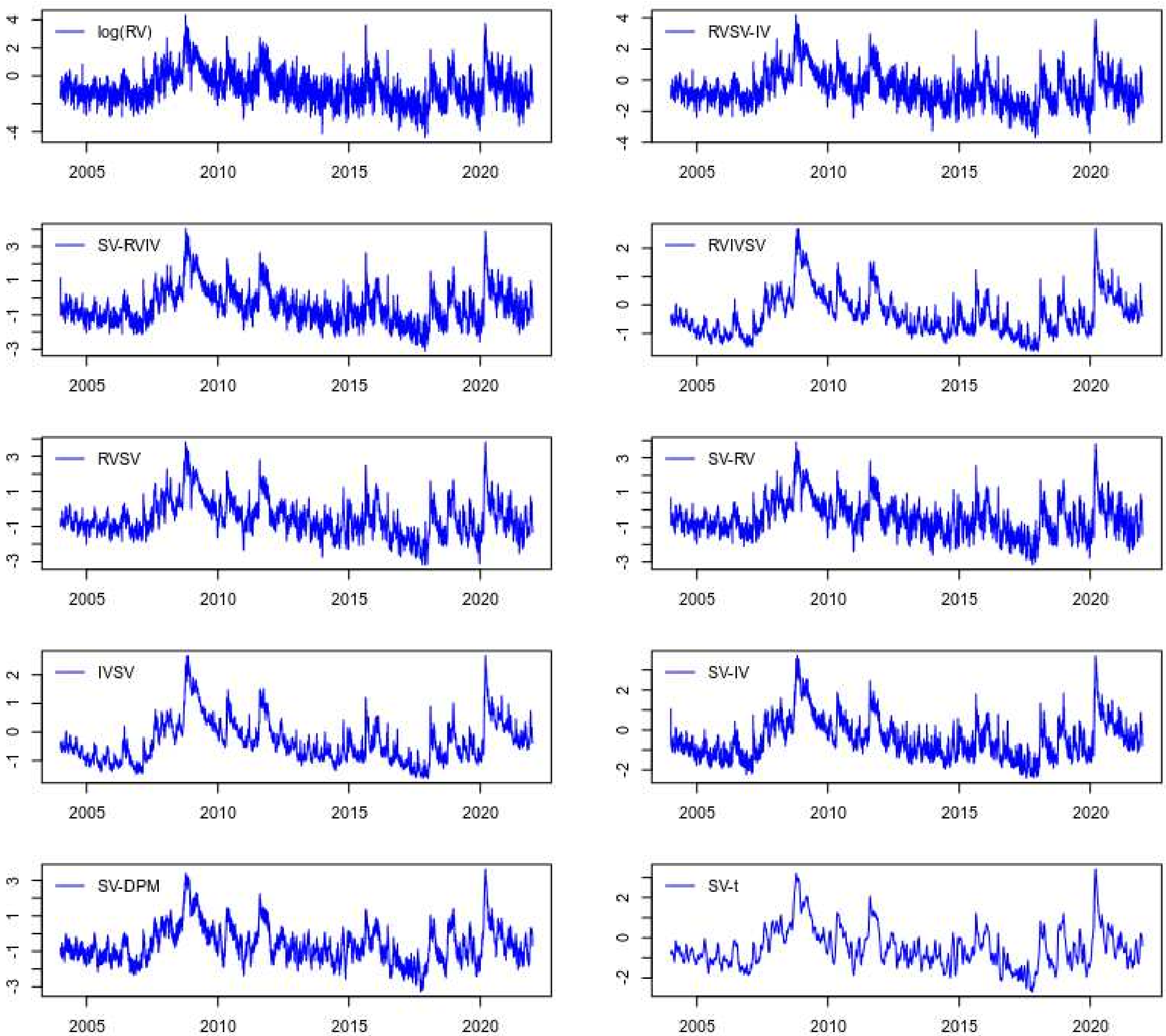

Figure 1: In-Sample Posterior Means for h_t

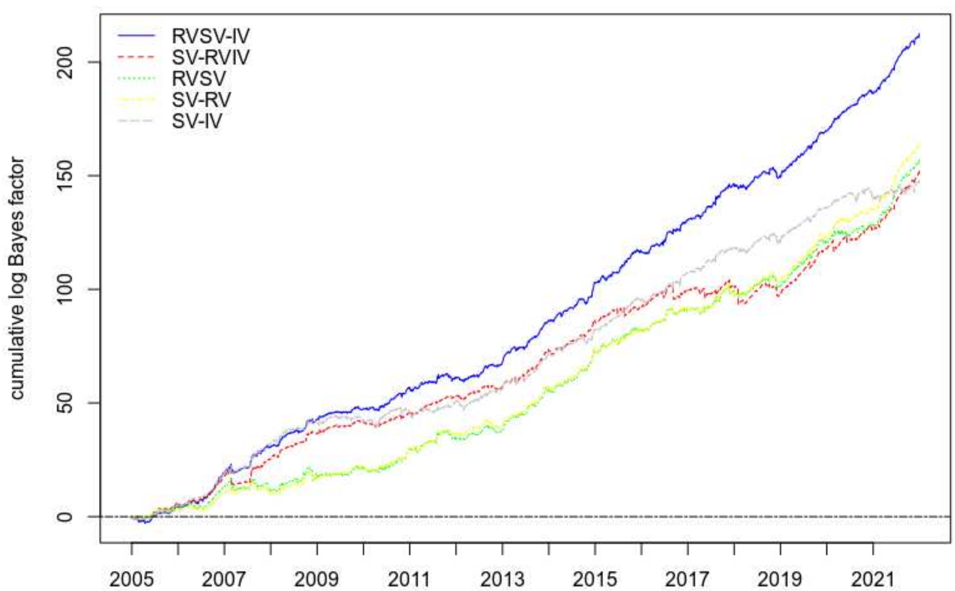

Figure 2: Cumulative Log Bayes Factors of RVSV-IV against Selected Models

Appendix A Proposal Distributions for SV Volatility

Let $h_{-t} = (h_1, \ldots, h_{t-1}, h_{t+1}, \ldots, h_T)'$ and θ be the set of all parameters, the posterior distribution of $h_t | h_{-t}, r_{1:T}, RV_{1:T}, IV_{1:T}, \theta$ for different models are:

$$\begin{aligned}
p_{\text{RVSV-IV}}(h_t|h_{-t},\dots) &\propto p(r_t|\mu,h_t)p(RV_t|h_t,\dots)p(h_t|h_{t-1},IV_{t-1},\dots)p(h_{t+1}|h_t,IV_t,\dots),\\
p_{\text{SV-RVIV}}(h_t|h_{-t},\dots) &\propto p(r_t|\mu,h_t)p(h_t|h_{t-1},RV_{t-1},IV_{t-1},\dots)p(h_{t+1}|h_t,RV_t,IV_t,\dots),\\
p_{\text{RVIVSV}}(h_t|h_{-t},\dots) &\propto p(r_t|\mu,h_t)p(RV_t|h_t,\dots)p(IV_t|h_t,\dots)p(h_t|h_{t-1},\dots)p(h_{t+1}|h_t,\dots),\\
p_{\text{RVSV}}(h_t|h_{-t},\dots) &\propto p(r_t|\mu,h_t)p(RV_t|h_t,\dots)p(h_t|h_{t-1},\dots)p(h_{t+1}|h_t,\dots),\\
p_{\text{SV-RV}}(h_t|h_{-t},\dots) &\propto p(r_t|\mu,h_t)p(h_t|h_{t-1},RV_{t-1},\dots)p(h_{t+1}|h_t,RV_t,\dots),\\
p_{\text{IVSV}}(h_t|h_{-t},\dots) &\propto p(r_t|\mu,h_t)p(IV_t|h_t,\dots)p(h_t|h_{t-1},\dots)p(h_{t+1}|h_t,\dots),\\
p_{\text{SV-IV}}(h_t|h_{-t},\dots) &\propto p(r_t|\mu,h_t)p(h_t|h_{t-1},IV_{t-1},\dots)p(h_{t+1}|h_t,IV_t,\dots).
\end{aligned}$$

Let $rv_t = \log RV_t$ and $iv_t = \log IV_t$, then note that

$$\begin{aligned}
p(RV_t|h_t,\dots) &= N\left(rv_t\middle|a_{RV}+h_t,\sigma^2_{RV}\right) = N\left(h_t\middle|rv_t-a_{RV},\sigma^2_{RV}\right),\\
p(IV_t|h_t,\dots) &= N\left(iv_t\middle|a_{IV}+h_t,\sigma^2_{IV}\right) = N\left(h_t\middle|iv_t-a_{IV},\sigma^2_{IV}\right),\\
p(h_t|h_{t-1},RV_{t-1},IV_{t-1},\dots) &= N\left(h_t\middle|\alpha+\delta h_{t-1}+\beta rv_{t-1}+\gamma iv_{t-1},\sigma^2_h\right),\\
p(h_t|h_{t-1},RV_{t-1},\dots) &= N\left(h_t\middle|\alpha+\delta h_{t-1}+\beta rv_{t-1},\sigma^2_h\right),\\
p(h_t|h_{t-1},IV_{t-1},\dots) &= N\left(h_t\middle|\alpha+\delta h_{t-1}+\gamma iv_{t-1},\sigma^2_h\right),\\
p(h_t|h_{t-1},\dots) &= N\left(h_t\middle|\alpha+\delta h_{t-1},\sigma^2_h\right).
\end{aligned}$$

and

$$\begin{aligned}
p(h_{t+1}|h_t,RV_t,IV_t,\dots) &= N\left(h_t\middle|(h_{t+1}-\alpha-\beta rv_t-\gamma iv_t)/\delta,\sigma^2_h/\delta^2\right),\\
p(h_{t+1}|h_t,RV_t,\dots) &= N\left(h_t\middle|(h_{t+1}-\alpha-\beta rv_t)/\delta,\sigma^2_h/\delta^2\right),\\
p(h_{t+1}|h_t,IV_t,\dots) &= N\left(h_t\middle|(h_{t+1}-\alpha-\gamma iv_t)/\delta,\sigma^2_h/\delta^2\right),\\
p(h_{t+1}|h_t,\dots) &= N\left(h_t\middle|(h_{t+1}-\alpha)/\delta,\sigma^2_h/\delta^2\right).
\end{aligned}$$

Therefore, the posterior of h_t for each model is proportional to $p(r_t|\mu, h_t)N(h_t|\mu_{\mathcal{M},t}, \sigma^2_{\mathcal{M}})$ where $\mathcal{M}$ represents a particular model and

$$\begin{aligned}
\mu_{\text{RVSV-IV},t} &= \sigma^2_{\text{RVSV-IV}} \left[\frac{(1-\delta)\alpha + \delta(h_{t-1} + h_{t+1}) - \gamma(\delta iv_t - iv_{t-1})}{\sigma^2_h} + \frac{rv_t - a_{RV}}{\sigma^2_{RV}} \right], \\
\sigma^2_{\text{RVSV-IV}} &= \frac{1}{(1+\delta^2)/\sigma^2_h + 1/\sigma^2_{RV}}, \\
\mu_{\text{SV-RVIV},t} &= \sigma^2_{\text{SV-RVIV}} \left[\frac{(1-\delta)\alpha + \delta(h_{t-1} + h_{t+1}) - \beta(\delta rv_t - rv_{t-1}) - \gamma(\delta iv_t - iv_{t-1})}{\sigma^2_h} \right], \\
\sigma^2_{\text{SV-RVIV}} &= \frac{1}{(1+\delta^2)/\sigma^2_h}, \\
\mu_{\text{RVIVSV},t} &= \sigma^2_{\text{RVIVSV}} \left[\frac{(1-\delta)\alpha + \delta(h_{t-1} + h_{t+1})}{\sigma^2_h} + \frac{rv_t - a_{RV}}{\sigma^2_{RV}} + \frac{iv_t - a_{IV}}{\sigma^2_{IV}} \right], \\
\sigma^2_{\text{RVIVSV}} &= \frac{1}{(1+\delta^2)/\sigma^2_h + 1/\sigma^2_{RV} + 1/\sigma^2_{IV}}, \\
\mu_{\text{RVSV},t} &= \sigma^2_{\text{RVSV}} \left[\frac{(1-\delta)\alpha + \delta(h_{t-1} + h_{t+1})}{\sigma^2_h} + \frac{rv_t - a_{RV}}{\sigma^2_{RV}} \right], \\
\sigma^2_{\text{RVSV}} &= \frac{1}{(1+\delta^2)/\sigma^2_h + 1/\sigma^2_{RV}}, \\
\mu_{\text{SV-RV},t} &= \sigma^2_{\text{SV-RV}} \left[\frac{(1-\delta)\alpha + \delta(h_{t-1} + h_{t+1}) - \beta(\delta rv_t - \log RV_{t-1})}{\sigma^2_h} \right], \\
\sigma^2_{\text{SV-RV}} &= \frac{1}{(1+\delta^2)/\sigma^2_h}, \\
\mu_{\text{IVSV},t} &= \sigma^2_{\text{IVSV}} \left[\frac{(1-\delta)\alpha + \delta(h_{t-1} + h_{t+1})}{\sigma^2_h} + \frac{iv_t - a_{IV}}{\sigma^2_{IV}} \right], \\
\sigma^2_{\text{IVSV}} &= \frac{1}{(1+\delta^2)/\sigma^2_h + 1/\sigma^2_{IV}}, \\
\mu_{\text{SV-IV},t} &= \sigma^2_{\text{SV-IV}} \left[\frac{(1-\delta)\alpha + \delta(h_{t-1} + h_{t+1}) - \gamma(\delta iv_t - \log IV_{t-1})}{\sigma^2_h} \right], \\
\sigma^2_{\text{SV-IV}} &= \frac{1}{(1+\delta^2)/\sigma^2_h}.
\end{aligned}$$

The initial value h_0 and the terminal value h_{T+1} are simulated from the SV process. Follow Kim et al. (1998), approximate $\exp(-h_t)$ with first-order Taylor expansion at $\mu_{\mathcal{M},t}$, then the independent proposal is

$$h_t|h_{-t}, \ldots \sim N\left(\mu_{\mathcal{M},t} + \frac{\sigma^2_{\mathcal{M}}}{2} \left[(r_t - \mu)^2 \exp(-\mu_{\mathcal{M},t}) - 1 \right], \sigma^2_{\mathcal{M}} \right).$$