# A Classical Marxian Two-Sector Endogenous Cycle Model: Integrating Marx, Dutt, and Goodwin

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**A Classical Marxian Two-Sector Endogenous Cycle Model: Integrating Marx, Dutt, and Goodwin**

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**Abstract**

This paper introduces a Classical Marxian Two-Sector Endogenous Cycle (CMTSEC) model, merging Dutt's (1988) two-sector model of Classical convergence with labor dynamics inspired by Goodwin (1967) and an endogenous labor supply inspired by Harris (1983). Empirical support fortifies these assumptions. Utilizing the Hopf bifurcation theorem and numerical simulations, we demonstrate the model's capacity to produce stable limit cycles encompassing wage share, employment rate, and sectoral capital distribution. Notably, sectoral profit rates exhibit cyclic fluctuations, prompting a reevaluation of long-run equilibrium. The model underscores the role of investment sensitivity to sectoral profit rate disparities in determining cycle stability. Hence, the CMTSEC model extends Goodwin's (1967) endogenous cycle model, encapsulating the conflict between capital and labor while delving into the intricate dynamics of capitalist reproduction in a two-sector economy.

**Keywords:** two-sector model, labor market dynamics, endogenous cycles, sensitivity of investment to profit rate differentials, long-run equilibrium

**JEL codes:** C61, E11, E32, O41

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## 1. Introduction

The Marxian model formulated by Goodwin (1967) to describe the emergence of endogenous cycles as a result of the class struggle between capitalists and workers has received substantial attention. As highlighted by Azevedo Araujo et al. (2019), Goodwin's model has undergone extensive examination and expansion, encompassing a wide range of dimensions. These include discussions on inflation, fiscal policy, its relationship with neoclassical growth, efficiency wages, effective demand, induced technical change, disequilibrium, financial instability, open economy considerations, two-sector economies, among others.

In the context of two-sector economies, the work of Sato (1985) stands as a significant reference. Sato's analysis suggests that the existence of endogenous cycles hinges upon the disparity in the capital/labor ratio between the sectors responsible for producing consumption and capital goods. Indeed, Sato concludes that either endogenous cyclical fluctuations do not exist, or if they do, they remain short-lived, particularly when the capital/labor ratio within the sector producing consumption goods surpasses that of the sector engaged in capital goods production. Sato employs this insight to rationalize the proposition that within a two-sector economy, Goodwin's model may not necessarily capture the antagonism between capital and labor.

This paper aims to contribute to the existing literature by presenting an alternative extension of Goodwin's model within a two-sector framework. Diverging from Sato's (1985) approach, the proposed extension suggests the existence of persistent endogenous cycles arising from the class struggle, maintaining their presence even in the long run, irrespective of specific constraints in the disparity in capital/labor ratios between economic sectors. Instead, the stability and cyclical patterns of the model hinge on distinct factors: the labor productivity within the capital goods sector, the influence of the real wage on the (endogenous) labor supply, and the responsiveness of investment to differentials in sectoral profit rates. Specifically, we employ the existence part of the Hopf bifurcation theorem for three-dimensional dynamical systems to validate that our two-sector model generates stable limit cycles when investment's sensitivity to profit rate disparities approaches a specific critical value. This result is observed under the condition of sufficiently elevated productivity in the production of capital goods and a substantial influence of the real wage on the labor supply (Appendix 1).

The construction of the introduced two-sector model in this paper involves the following process. Initially, we establish a foundational framework by adopting as a baseline the two-sector model of Classical convergence formulated by Dutt (1988). This model serves as a useful starting point since it offers insights into the adjustment of prices to ensure overall equilibrium across productive sectors in the short run, while the sectoral distribution of capital adjusts in the long run depending on sectoral profit rate differentials. Subsequently, we expand upon Dutt's two-sector model by incorporating endogenous dynamics within the labor market, drawing inspiration from Goodwin's (1967) model. More specifically, we elucidate the intrinsic linkage between the employment rate, the wage share, and the sectoral distribution of capital over the long run. This nexus arises from the premise suggested by Marx (1976) that wage rate dynamics are endogenous since they depend on the allocation of labor supply between the employed and the unemployed, a distribution profoundly shaped by capital distribution across productive sectors. Additionally, we incorporate the notion of endogenous labor supply, which is driven by the growth rate of the real wage. This assumption bears a close connection to the Marxian framework put forth by Harris (1983) for the analytical examination of capitalist dynamics. To substantiate these assumptions regarding endogenous wage rate and labor supply, we provide empirical estimations for the US economy using the bounds-testing procedure outlined by Pesaran et al. (2001) (Appendix 2).

The rest of the paper is organized as follows. Section 2 introduces the two-sector model proposed in this paper in a three-stage approach. Firstly, we present Dutt's (1988) two-sector model of Classical convergence, maintaining its original assumptions, notation, and definitions of short-run and long-run equilibrium. Secondly, we explore implications of Dutt's concept of long-run convergence on the employment rate. Thirdly, we augment the model by integrating endogenous dynamics within the labor market. This includes incorporating equations describing the endogenous wage rate and endogenous labor supply, allowing us to formulate a three-dimensional dynamical system capable of generating stable limit cycles. We designate this model as a Classical Marxian Two-Sector Endogenous Cycle model (CMTSEC model). Section 3 presents and discusses numerical simulations of the CMTSEC model, exploring some of its economic implications. This analysis includes a redefinition of the long-run equilibrium, considering the cyclical fluctuations of sectoral rates of profit. Lastly, Section 4 provides conclusions and suggestions for future research.

## 2. The model

### 2.1. Convergence in Dutt's (1988) Classical-Marxian Two-Sector Model

In this section, we describe the original version of the two-sector model of Classical convergence formulated by Dutt (1988). The model involves two sectors: sector 1 produces consumption goods, and sector 2 creates investment goods. Production relies on labor and fixed capital without depreciation, using a fixed coefficient production function. Firms maintain post-installation capital stock and hire workers under short-term contracts. The society is divided into workers earning wages and capitalists owning firms' capital, extracting profits. Capitalists can save a fraction $s$ $(0 < s < 1)$ of their income, while workers do not save. Wages are paid after production, not affecting capitalists' calculations of their rate of profit. Both sectors share the same real wage, determined externally. Capitalists invest their savings and define their desired growth rates of capital depending on the difference between the sectoral rates of profit. Short-term output remains constant due to fixed capital stock, but long-term growth syncs with sector-specific investment rates.

Given these assumptions, Dutt (1988) presents two price-cost equations:

$$1 = pa_{21}r_1 + Va_{01} \quad (1)$$

$$p = pa_{22}r_2 + Va_{02} \quad (2)$$

where:

$$p = \frac{P_2}{P_1} \quad (3)$$

$$a_{0i} = \frac{L_i}{X_i}, \qquad i = 1,2 \quad (4)$$

$$a_{2i} = \frac{K_i}{X_i}, \qquad i = 1,2 \quad (5)$$

$$k = \frac{K_1}{K_2} \quad (6)$$

The term $P_i$ represents the price associated with the product of sector $i$, $p$ stands as the relative price of the capital good expressed in terms of the consumption good, $r_i$ is the profit rate of sector $i$, $L_i$ is the effective labor employed within sector $i$, $K_i$ represents the capital stock within sector $i$, $a_{0i}$ and $a_{2i}$ are the labor-output and capital-output ratios, respectively, $V$ is the real wage, and $k$ represents the sectoral distribution of capital.

Now, define the demand for each good $D_i$ in real terms as:

$$D_1 = V(a_{01}X_1 + a_{02}X_2) + (1 - s)p(r_1K_1 + r_2K_2) \quad (7)$$

$$D_2 = s(r_1K_1 + r_2K_2) \quad (8)$$

Regarding the dynamics of capital accumulation, Dutt (1988) assumes that each sector establishes its desired capital growth rate $g^i$ through the following behavioral equation:

$$g^i = g^j + \mu(r_i - r_j), \qquad i = 1,2, \qquad j = 1,2, \qquad i \neq j \quad (9)$$

where $\mu > 0$ represents the sensitivity of investment differentials to profit rate differentials.

In the short run, the relative price $p$ adjusts to discrepancies between demand and supply, following the subsequent behavioral equation:[2]

$$\dot{p} = f(D_2 - X_2) \quad (10)$$

where $f$ is a sign-preserving function. By substituting equations (1), (2), (5), (6), and (8) into (10) we get the following dynamic equation for the relative price $p$:[3]

$$\dot{p} = f\left(\frac{K_2}{pa_{21}a_{22}}\{s[a_{22}(1 - Va_{01})k - Va_{02}a_{21}] - pa_{21}(1 - s)\}\right) \quad (11)$$

Given this equation, Dutt (1988) characterizes short-run equilibrium as a state in which demand and supply balance is achieved for sector 2, leading to the stabilization of the relative price:

$$D_2 = X_2 \quad \rightarrow \quad \dot{p} = 0 \quad (12)$$

Equation (12) guarantees the demand-supply balance for sector 1 ($D_1 = X_1$).[4] Substituting (12) into (11) gives the short-run equilibrium level of the relative price, which is equal to:

$$p = \frac{s[a_{22}(1 - Va_{01})k - Va_{02}a_{21}]}{a_{21}(1 - s)} \quad (13)$$

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[2] For any variable $x$, $\dot{x} = dx/dt$ represents its time derivative, and $\dot{x}/x$ represents its growth rate.

[3] A notebook in Wolfram Mathematica that contains all mathematical deductions and numerical simulations presented in this paper is available as supplementary material. For further details, please contact the author.

[4] To verify this result we can make $D_2 = X_2$, then we substitute this equality into (8) and combine the result with (1), (2), (5), and (7) to get $D_1 = X_1$.

This equilibrium level of the relative price is positive and stable if $k$ is sufficiently high to satisfy:

$$k > \frac{Va_{02}a_{21}}{a_{22}(1 - Va_{01})} \quad (14)$$

Upon substituting (13) into (1) and (2), we can deduce the values of the rates of profit that prevail in the short-run equilibrium:

$$r_1 = \frac{(1 - s)(1 - Va_{01})}{s[a_{22}(1 - Va_{01})k - Va_{02}a_{21}]} \quad (15)$$

$$r_2 = \frac{sa_{22}(1 - Va_{01})k - Va_{02}a_{21}}{a_{22}s[a_{22}(1 - Va_{01})k - Va_{02}a_{21}]} \quad (16)$$

In the context of long-run dynamics, Dutt (1988) examines the evolution of the sectoral distribution of capital $k$ as a result of capital accumulation within individual sectors and the mobility of capital across sectors. To elucidate this intuition, we can apply logarithms and time derivatives of equation (6), yielding:

$$\frac{\dot{k}}{k} = g^1 - g^2 \quad (17)$$

where $g^i = \dot{K_i}/K_i$. Substituting (9) into (17) gives:

$$\frac{\dot{k}}{k} = \mu(r_1 - r_2) \quad (18)$$

And substituting (1) and (2) into (18) results in:

$$\frac{\dot{k}}{k} = \mu\left[\frac{a_{22}(1 - Va_{01}) + Va_{02}a_{21} - a_{21}p}{pa_{21}a_{22}}\right] \quad (19)$$

Assuming the economy has attained its short-term equilibrium, we can substitute the equilibrium value of the relative price $p$, as defined by equation (13), into equation (19) to yield:

$$\frac{\dot{k}}{k} = \mu\left[\frac{a_{22}(1 - Va_{01})(1 - s) + Va_{02}a_{21} - sa_{22}(1 - Va_{01})k}{a_{22}s[a_{22}(1 - Va_{01})k - Va_{02}a_{21}]}\right] \quad (20)$$

Based on equation (20), Dutt (1988) characterizes long-run equilibrium as a state in which the sectoral distribution of capital stabilizes ($\dot{k} = 0$). Consequently, the equilibrium value of $k$ in the long run, as defined by this premise, equals:

$$k = \frac{a_{22}(1 - Va_{01})(1 - s) + Va_{02}a_{21}}{sa_{22}(1 - Va_{01})} \quad (21)$$

The stability of this long-run equilibrium value for $k$ is assured, as equation (21) aligns with the inequality presented in expression (14). As a result, any increase in $k$ will diminish the right-hand side of the dynamic equation (20).

### 2.2. Long-Run Convergence Implications for the Employment Rate

While the original version of the two-sector model formulated by Dutt (1988) does not incorporate it within its mathematical framework, its conception of long-run dynamics not only entails shifts in the sectoral distribution of capital $k$ but also entails significant dynamics in the labor market. To illustrate this point, let us define the aggregate employment rate $l$ as:

$$l = \frac{L_1 + L_2}{N} \quad (22)$$

where $N$ represents the labor supply. Substituting equations (4), (5), and (6) into (22) gives:

$$l = \left(\frac{K_2}{N}\right)\left(\frac{a_{01}a_{22}k + a_{02}a_{21}}{a_{21}a_{22}}\right) \quad (23)$$

Here we define the growth rate of the labor supply as:

$$\frac{\dot{N}}{N} = n \quad (24)$$

Through the utilization of logarithms and time derivatives on equation (23), and subsequent substitution of (24) into the result, and under the assumption of constant technical conditions (where coefficients $a_{0i}$ and $a_{2i}$ remain constant), the growth rate of the aggregate employment rate $l$ can be formulated as follows:

$$\frac{\dot{l}}{l} = g^2 - n + \frac{a_{01}a_{22}\dot{k}}{a_{02}a_{21} + a_{01}a_{22}k} \quad (25)$$

where $g^2 = \dot{K}_2/K_2$. Assuming the economy currently resides within its short-run equilibrium, wherein demand-supply balance prevails for both sectors 1 and 2, a consequential outcome is that aggregate investment equals aggregate savings, as indicated by:

$$g^1K_1 + g^2K_2 = s(r_1K_1 + r_2K_2) \quad (26)$$

Substituting (6), (15), (16), and (17) into (26), and solving for $g^2$ gives:

$$g^2 = \left(\frac{1}{1+k}\right)\left(\frac{1}{a_{22}} - \dot{k}\right) \quad (27)$$

Substituting (27) into (25) results in:

$$\frac{\dot{l}}{l} = \left(\frac{1}{1+k}\right)\left[\frac{1}{a_{22}} + \frac{(a_{01}a_{22} - a_{02}a_{21})\dot{k}}{a_{02}a_{21} + a_{01}a_{22}k}\right] - n \quad (28)$$

Within the framework of the long-run equilibrium concept proposed by Dutt (1988), wherein $\dot{k} = 0$, the incorporation of (21) into equation (28) reveals that the sole condition leading to the stabilization of the employment rate in the long run ($\dot{l} = 0$) occurs when the growth rate of the labor supply $n$ assumes the value:

$$n = \frac{s(1 - Va_{01})}{a_{22}(1 - Va_{01}) + Va_{02}a_{21}} \quad (29)$$

Otherwise, if $n$ lies below (above) the threshold outlined by equation (29), the employment rate $l$ will experience a permanent increase (decrease). However, viewed through a Marxian standpoint, a sustained alteration in the long-term employment rate appears at odds with the premise of a constant real wage $V$, as Marx (1976) contends that the wage rate dynamics hinge upon the distribution of labor supply between employed and unemployed individuals.[5] To integrate the Marxian insight regarding the influence of the employment rate on the real wage with Dutt's (1988) two-sector model of Classical convergence, we can enhance the model by incorporating supplementary assumptions derived from the Marxian literature on capitalist reproduction and endogenous cycles.

### 2.3. Endogenous Dynamics of the Labor Market and a Three-Dimensional Dynamic System

The original version of Dutt's (1988) two-sector model introduced in preceding sections operated under the assumption of a constant real wage $V$. In this section, we deviate from this premise and explore an alternative scenario where $V$ is endogenous and adjusts in response to changes in the employment rate $l$. The link between the employment rate and the real wage can be substantiated from a Marxian perspective, as it echoes the influence of the reserve army of labor in the bargaining power of the working class.[6] In this vein, for the sake of simplicity, we align with Goodwin's (1967) perspective, presuming that the growth rate of the real wage $V$ is a function of the aggregate employment rate $l$, as expressed through the subsequent behavioral equation:

$$\frac{\dot{V}}{V} = -\gamma + \rho l \quad (30)$$

where $\gamma$ represents an autonomous tendency of the real wage to fall and $\rho$ is the effect of the aggregate employment rate on the growth rate of $V$.

Following Goodwin (1967), to elucidate the distributive implications entailed by the dynamics of an endogenous real wage, we define the aggregate wage share as:

$$\omega = \frac{V(a_{01}X_1 + a_{02}X_2)}{X_1 + pX_2} \quad (31)$$

Assuming the relative price $p$ has attained its short-run equilibrium value, we can derive the following result by substituting equations (5), (6), and (13) into equation (31):

$$V = \frac{a_{22}k\omega}{(a_{01}a_{22}k + a_{02}a_{21})(1 - s + s\omega)} \quad (32)$$

Taking logarithms and time derivatives of (32) gives:

$$\frac{\dot{V}}{V} = \left(\frac{a_{02}a_{21}}{a_{02}a_{21} + a_{01}a_{22}k}\right)\left(\frac{\dot{k}}{k}\right) + \left(\frac{1 - s}{1 - s + s\omega}\right)\left(\frac{\dot{\omega}}{\omega}\right) \quad (33)$$

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[5] Here we assume the absence of underemployed workers. Nevertheless, it's worth noting that Marx (1976) encompassed within his concept of the reserve army of labor not solely the unemployed, but also workers experiencing underemployment with varying attributes. Delving into the nuances of this consideration is a topic reserved for future discussions.

[6] For a deeper discussion about the Marxian understanding of the interplay between the real wage and the employment rate, particularly concerning the dynamics of bargaining power within the framework of an aggregated Marxian model of endogenous cycles, refer to Cajas Guijarro and Vera (2022).

Substituting (30) into (33) results in:

$$\frac{\dot{\omega}}{\omega}=\left(\frac{1-s+s\omega}{1-s}\right)\left[-\gamma+\rho l-\left(\frac{a_{02}a_{21}}{a_{02}a_{21}+a_{01}a_{22}k}\right)\left(\frac{\dot{k}}{k}\right)\right] \quad (34)$$

Equation (34) emphasizes the reliance of the wage share growth rate on the evolving dynamics of the sectoral distribution of capital $k$. Indeed, a feedback effect emerges between these variables. Substituting (32) into (20) further reveals that the growth rate of $k$ is similarly contingent on the wage share $\omega$:

$$\dot{k}=\mu\left\{\frac{a_{01}a_{22}k(1-s-sk)(1-\omega)+a_{02}a_{21}[1-s(1-\omega)(1+k)+k\omega]}{sa_{22}(a_{02}a_{21}+a_{01}a_{22}k)(1-\omega)}\right\} \quad (35)$$

An additional facet that can be integrated into the model to enhance our depiction of the labor market is the consideration of endogenous labor supply. Drawing inspiration from Harris's (1983) presentation of capital accumulation dynamics within a Marxian framework, we postulate that the growth rate of the labor supply $n$ can itself be an endogenous variable, characterized by the subsequent behavioral equation:[7]

$$n=n_0+\zeta\left(\frac{\dot{V}}{V}\right) \quad (36)$$

In equation (36), $n_0$ denotes the exogenously driven propensity of the labor supply to grow, encompassing, for instance, demographic dynamics. In tandem with this, $\zeta$ represents the positive impact of the real wage's growth rate on the labor supply. This construct finds its rationale in the concept that intensified capitalist accumulation, materializing as an escalation in the real wage, generates incentives for the influx of new labor force participants. This trend may include migrant laborers from both less developed capitalist societies and non-capitalist contexts.[8]

Substituting equations (30) and (36) into (28) gives the following dynamic equation for the employment rate $l$:

$$\frac{\dot{l}}{l}=\left(\frac{1}{1+k}\right)\left[\frac{1}{a_{22}}+\frac{(a_{01}a_{22}-a_{02}a_{21})\dot{k}}{a_{02}a_{21}+a_{01}a_{22}k}\right]-(n_0-\zeta\gamma)-\zeta\rho l \quad (37)$$

Equations (34), (35), and (37) establish a three-dimensional dynamical system encompassing the state variables $l$, $\omega$, and $k$. For the sake of notation simplicity within this system, we introduce the definitions of capital/labor ratio $m_i$ and (labor) productivity $q_i$ within sector $i$ as follows:

$$m_i=\frac{K_i}{L_i}=\frac{a_{2i}}{a_{0i}}, \qquad i=1{,}2 \quad (38)$$

[7] In his original formulation, Harris (1983, equation 8) postulates that the growth rate of the labor supply depends on the growth rate of the aggregate capital stock.

[8] In his examination of the reserve army of labor, Marx (1976) incorporates within the category of the 'latent reserve' those migrant laborers who transition from subsistence agriculture to the industrial sector, propelled by the prospect of monetary remuneration.

$$q_i = \frac{L_i}{X_i} = \frac{1}{a_{0i}}, \qquad i = 1{,}2 \quad (39)$$

With these definitions in place, within the steady state ($\dot{k} = \dot{l} = \dot{\omega} = 0$), the system delineated by equations (34), (35), and (37) reveals a non-trivial equilibrium point $(k^*, \omega^*, l^*)$ given by:

$$k^* = \frac{Z_1}{m_2 n_0} \quad (40)$$

$$\omega^* = \frac{Z_2 Z_3}{Z_4} \quad (41)$$

$$l^* = \frac{\gamma}{\rho} \quad (42)$$

where:

$$Z_1 = q_2 - m_2 n_0, \qquad Z_2 = s q_2 - m_2 n_0, \qquad Z_3 = q_2 - n_0(m_2 - m_1)$$

$$Z_4 = s q_2^2 - n_0(m_2 - m_1)(q_2 + Z_2)$$

Appendix 1 analytically proves that the equilibrium point $(k^*, \omega^*, l^*)$ exhibits positive values and local stability under specific conditions. This stability requires a sufficiently high labor productivity in sector 2 ($q_2$) and a strong influence of the growth rate of the real wage on labor supply ($\zeta$). Additionally, investment sensitivity to profit rate differentials ($\mu$) must be sufficiently large to satisfy the following condition:

$$\mu > \mu^{HB} = \frac{m_2(1-s)(Z_2 Z_3 Z_5 - \gamma \zeta Z_7)}{Z_1 Z_7} \quad (43)$$

where:

$$Z_5 = s q_2 - n_0(m_2 - m_1), \qquad Z_7 = m_1 q_2 s \zeta Z_3 - (m_2 - m_1) Z_2 Z_5$$

Thus, when the capital goods sector exhibits high labor productivity, labor supply responds significantly to rising real wages, and capital accumulation has a pronounced sensitivity to sectoral profit rate differentials, the three-dimensional dynamical system represented by equations (34), (35), and (37) establishes a relevantly stable economic equilibrium.

Furthermore, in Appendix 1, the existence part of the Hopf bifurcation theorem for three-dimensional dynamical systems is used to prove that when the productivity in sector 2 ($q_2$) and the impact of the real wage on labor supply ($\zeta$) are sufficiently high, the three-dimensional dynamical system introduced in this paper generates limit cycles in proximity to its equilibrium point as $\mu$ approaches the critical value $\mu^{HB}$ defined in equation (43). Notably, this outcome remains unaffected by the disparity in capital/labor ratios between sectors 2 and 1 ($m_2 - m_1$). This distinction sets apart the present model from Sato's (1988) two-sector formulation, where either endogenous cycles do not exist, or they dissipate over time based on the divergence in capital intensity between sectors 1 and 2. For this reason, we designate the three-dimensional model defined by equations (34), (35), and (37) as a Classical Marxian Two-Sector Endogenous Cycle model (CMTSEC model).

## 3. Numerical Simulations and Discussion

To further explore the dynamics of the CMTSEC model, this section employs numerical simulations to illustrate the enduring presence of endogenous limit cycles across different levels of sectoral capital intensity. This result has been analytically demonstrated in the previous section and Appendix 1. To establish the parameter values required for our numerical simulations, we define an illustrative baseline according to the following criteria. Firstly, for the parameters related to the real wage Phillips curve ($\gamma$, $\rho$), the autonomous tendency of labor supply growth ($n_0$), the effect of the real wage on labor supply ($\zeta$), and the savings rate ($s$) (equivalent to the accumulation rate), we use the values estimated in Appendix 2 for the US economy from 1960 to 2019. We consider these estimations to offer empirical support for the assumptions regarding the endogenous wage rate and labor supply, as represented by equations (30) and (36), respectively.[9] Secondly, for the sake of simplicity, we normalize productivity and the capital/labor ratio in sector 1 ($q_1 = m_1 = 1$), and productivity in sector 2 ($q_2$), while we vary the capital/labor ratio in sector 2 ($m_2$).

Figure 1 presents a first numerical simulation of the CMTSEC model that illustrates the existence of stable limit cycles when the sector producing capital goods (sector 2) has a larger capital/labor ratio than the sector producing consumption goods (sector 1), i.e., $m_2 > m_1$ (Case 1).[10] In this simulation, we identify clockwise cycles in the phase plane formed by the wage share ($\omega$) and the employment rate ($l$), resembling the distributive cycles identified by Goodwin (1967). Additionally, we observe counterclockwise cycles in the planes formed by each of these state variables and the sectoral distribution of capital ($k$). In particular, the limit cycle that emerges in the $\omega - k$ phase plane appears to have a 'positive slope,' meaning that $\omega$ and $k$ tend to move in the same direction during each stage of the cycle. This outcome aligns with economic intuition: a higher (lower) wage share implies that workers receive a larger (smaller) share of production for consumption, expanding (contracting) the demand for goods produced by Sector 1 relative to Sector 2. Consequently, Sector 1 accumulates capital at a faster (slower) rate than Sector 2, resulting in an increase (decrease) in the sectoral distribution of capital ($k = K_1/K_2$). It is worth noting that the same qualitative results are obtained when we simulate the model with $m_1 > m_2$, *ceteris paribus*, as illustrated in Figure 2 (Case 2). This finding reinforces a critical point: within the context of the CMTSEC model proposed in this paper, disparities in capital/labor ratios between sectors may not be the primary determinants of cycle existence, stability, and direction.

The dynamics portrayed in Figures 1 and 2 motivate a reassessment of the concept of long-run equilibrium. In contrast to Dutt's (1988) definition, we can reinterpret long-run equilibrium as a state in which the sectoral distribution of capital cyclically responds to fluctuations generated by distributive cycles emerging from the class struggle between capitalists and workers. Over time, these distributive cycles not only induce persistent variations in the sectoral distribution of capital but can also lead to fluctuations in the sectoral rates of profit. To illustrate this concept, we can

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[9] Although we estimate the parameters $\gamma$, $\rho$, $n_0$, $\zeta$, and $s$ for the US economy in Appendix 2, it is important to emphasize that the baseline formulated in this paper serves the sole purpose of illustrating the theoretical results presented in Section 2. It is far from representing a comprehensive estimation of the model presented in this paper for the US economy. At best, this baseline can be regarded as a preliminary approximation that lays the groundwork for future discussion.

[10] More precisely, numerical simulations indicate the presence of a supercritical Hopf bifurcation: a scenario where a stable limit cycle encircles an unstable equilibrium point.

express the sectoral rates of profit in terms of the state variables $k$, $\omega$, and $l$, utilizing equations (15), (16), (32), (38), and (39):

$$r_1 = \frac{q_2\{(1-s)[m_1 + m_2(1-\omega)k] + m_1 s\omega\}}{m_2 sk(m_1 + m_2 k)(1-\omega)} \quad (44)$$

$$r_2 = \frac{q_2\{s[m_1 + m_2(1-\omega)k] - m_1(1+s)\omega\}}{m_2 s(m_1 + m_2 k)(1-\omega)} \quad (45)$$

In cases where the CMTSEC model generates limit cycles, the persistent cyclical fluctuations of the state variables $k$, $\omega$, and $l$ cause persistent fluctuations in the sectoral rates of profit, $r_1$ and $r_2$. This phenomenon is suggested by the numerical simulations of equations (44) and (45) presented in Figures 3 and 4, where we observe clockwise limit cycles in the plane $r_1 - r_2$.[11] To identify the value around which these sectoral profit rates fluctuate, we can utilize equation (18), recognizing that in the long-run steady state ($\dot{k} = \dot{\omega} = \dot{l} = 0$), the sectoral rates of profit become equal to an equilibrium rate, $r_1 = r_2 = r^*$. To determine this equilibrium rate, we can substitute the equilibrium point $(k^*, \omega^*, l^*)$ as defined in equations (40), (41), and (42) into either equation (44) or equation (45), yielding the following result:

$$r^* = \frac{n_0}{s} \quad (46)$$

Therefore, in the presence of a limit cycle, over the long run, the sectoral profit rates $r_1$ and $r_2$ continuously vary around a long-run equilibrium rate $r^*$. This equilibrium rate is equal to the ratio between the autonomous tendency of labor supply to grow ($n_0$) and the savings-accumulation rate ($s$). Drawing inspiration from Marx (1978, 1981), we can interpret this equilibrium rate, $r^*$, as a long-term center-of-gravity around which the sectoral rates consistently oscillate. These oscillations emerge from a complex interplay between distributive cycles linked to class struggle and the choices made by capitalists regarding the allocation of their capital, driven by their perceptions of sector-specific profitability.

Once we have illustrated the capacity of the CMTSEC model to produce persistent and stable limit cycles, we can delve into a more detailed discussion regarding the impact of investment sensitivity to profit rate differentials ($\mu$) on the model's dynamics. On one hand, when the sensitivity $\mu$ sufficiently exceeds the critical value $\mu^{HB}$ ($\mu \gg \mu^{HB}$), the CMTSEC model generates damped oscillations that gradually converge towards the equilibrium point, as depicted in the numerical simulation of the sectoral rates of profit presented in Figure 5A. Conversely, when the sensitivity $\mu$ is significantly lower than the critical value $\mu^{HB}$ ($\mu \ll \mu^{HB}$), the CMTSEC model produces unstable oscillations with an amplitude that increases until the model experiences a crash, as illustrated by the simulation of the sectoral rates of profit in Figure 5B.

The dynamics illustrated in Figure 5 underscore the significance of the capitalist class's power to allocate their capital between Sectors 1 and 2, contingent upon profit rate disparities (as represented

[11] Another relevant pattern observed in the numerical simulations presented in Figures 3 and 4 is the tendency for the sector producing capital goods to exhibit a more volatile profit rate than the sector producing consumption goods; specifically, $r_2$ appears to be more volatile than $r_1$. This observed pattern deserves future research since it seems to align with findings in other two-sector models within the literature, such as Murakami (2018).

by the investment sensitivity $\mu$). This factor emerges as pivotal in determining the stability of the endogenous cycles generated by the interplay between capital flows and distributive conflict within the CMTSEC model. When the capitalist class possesses substantial power over capital movements ($\mu \gg \mu^{HB}$), the resultant interaction tends to yield more stable endogenous cycles. Conversely, when their power to transfer capital between sectors is limited ($\mu \ll \mu^{HB}$), it leads to instability, potentially culminating in a structural crisis. Here, a structural crisis signifies a form of crisis that can only be resolved through external adjustments to the model's structural parameters (Cajas Guijarro and Vera 2022, p. 573).

Given this interpretation of the influence of $\mu$ on the stability of cycles, we can examine the relationship between the critical value of the investment sensitivity $\mu^{HB}$ and other parameters of the model. This relationship is explored in Figure 6 for both Case 1 ($m_2 > m_1$) and Case 2 ($m_1 > m_2$). As observed in the figure, the critical value $\mu^{HB}$ decreases (indicating relatively 'more stable' cycles for a given $\mu$) when Sector 1 exhibits a higher capital/labor ratio ($m_1$), when there are elevated autonomous tendencies in real wage and labor supply growth ($\gamma$, $n_0$), and when the effect of the real wage on labor supply ($\zeta$) is larger. Conversely, $\mu^{HB}$ increases (suggesting relatively 'more unstable' cycles for a given $\mu$) when Sector 2 has higher productivity ($q_2$) and a greater capital/labor ratio ($m_2$). Notably, $\mu^{HB}$ remains constant with respect to productivity in Sector 1 ($q_1$) and the effect of the employment rate on the real wage ($\rho$).

Concerning the savings-accumulation rate ($s$), we identify a peculiar pattern. The rate $s$ consistently elevates the critical value $\mu^{HB}$ in Case 1. However, in Case 2, a concave relationship between these variables emerges. This result implies that the impact of relatively high levels of the savings rate ($s$) on cycle stability hinges upon the disparity in capital intensity between sectors 1 and 2. Specifically, when the sector involved in capital goods production (Sector 2) exhibits greater capital intensity than the sector engaged in consumption goods production (Sector 1), elevated levels of $s$ tend to destabilize cycles for a given $\mu$. Conversely, when Sector 1 boasts higher capital intensity compared to Sector 2, heightened levels of $s$ tend to fortify cycle stability for a given $\mu$ when $s$ goes beyond a certain value. It is important to emphasize that while we identify this qualitative divergence between Cases 1 and 2, necessitating further discussion, this divergence does not undermine the main finding of this paper. Specifically, the difference in capital/labor ratios between Sectors 1 and 2 is not the primary determinant governing the existence and duration of endogenous cycles in a two-sector economy that integrates insights from Goodwin's (1967) model and Classical-Marxian perspectives on capitalist expanded reproduction.

## 4. Conclusion

This paper has introduced a Classical Marxian Two-Sector Endogenous Cycle (CMTSEC) model. This model is a synthesis of the two-sector Classical convergence model formulated by Dutt (1988) and some endogenous dynamics of the labor market. Specifically, we incorporate an assumption of an endogenous real wage inspired by Goodwin's (1967) model of endogenous distributive cycles, and an assumption of endogenous labor supply inspired by Harris's (1983) formulation of capitalist dynamics. Empirical evidence sustaining both assumptions concerning endogenous real wage and labor supply is provided (see Appendix 2). The CMTSEC model presents an alternative extension of Goodwin's model within a two-sector framework, demonstrating persistent endogenous cycles arising from class struggle, even in the long run, regardless of specific constraints on the disparity in capital/labor ratios between sectors producing capital and consumption goods.

In this regard, utilizing the existence part of the Hopf bifurcation theorem for three-dimensional dynamic systems, we demonstrate that the CMTSEC model has the capacity to generate persistent and stable limit cycles (see Appendix 1). These cycles emerge from a complex interaction between distributive conflict linked to class struggle and capitalists' choices regarding the allocation of their capital, driven by their perceptions of sector-specific profitability. In this context, variables like the employment rate, wage share, and sectoral capital distribution are intricately interconnected, defining a three-dimensional dynamical system. Consequently, even sectoral rates of profit exhibit persistent cyclical movements. Following Marx, we propose conceiving long-run equilibrium as a state wherein sectoral profit rates cyclically fluctuate around an equilibrium rate, serving as a long-term center-of-gravity.

Our theoretical analysis and numerical simulations of the CMTSEC model underscore the critical role played by investment sensitivity to sectoral profit rate differentials ($\mu$) in shaping the nature and stability of cycles. The power of the capitalist class to reallocate capital based on profit rate differentials, as represented by $\mu$, significantly influences cycle stability. Robust power in this regard leads to more stable endogenous cycles. Conversely, weak power to move capital between sectors can result in unstable cycles that may culminate in structural crises. Therefore, the stability of cycles is influenced by a complex set of parameters, including autonomous tendencies in real wage and labor supply growth, the effect of the real wage on labor supply, and sectoral productivity.

Additionally, this paper may serve as a base for future research in various directions. Firstly, we suggest extending the CMTSEC model to encompass technical change, at least in the form of increasing labor productivity, aligning it more closely with Goodwin's (1967) original model of endogenous distributive cycles. Secondly, we advocate for a more in-depth empirical validation of the CMTSEC model, complementing the results presented in Appendix 2 with empirical estimations of labor productivities, capital/labor ratios, and their dynamics within the sectors producing consumption and capital goods. Thirdly, we propose considering more complex dynamics associated with the labor market, encompassing different conditions of employment and underemployment as mentioned by Marx in his discussion of the reserve army of labor and its different categories (latent, floating, and stagnant reserve). This may require, at least, the inclusion of more intricate dynamic equations for labor supply. Fourthly, we recommend comparing the results provided by the CMTSEC model with other two-sector models, particularly regarding the difference in economic volatility between the sector producing capital goods (potentially more volatile) and the sector producing consumption goods (potentially less volatile). Fifthly, extending the CMTSEC model to incorporate additional Marxian insights, such as the labor value theory or the tendency of the general rate of profit to decline, could result in a more comprehensive sectoral model of endogenous cycles, with the work of Nikolaos et al. (2022) serving as a relevant reference. Finally, we suggest constructing a more complex two-sector model that combines the endogenous labor market dynamics considered in this paper with the existence of market power, excess capacity utilization, and sectoral investment functions. In this context, Dutt's (1988) two-sector model of convergence with monopoly power may serve as a useful starting point.

In conclusion, the Classical Marxian Two-Sector Endogenous Cycle model presented in this paper advances our comprehension of the intricate dynamics governing the expanded reproduction of capital and the role played by the reserve army of labor—concepts initially elucidated by Marx in the foundational volumes of Capital. This model captures the interaction between capital movements and distributive conflict, concurrently yielding three-dimensional, stable, endogenous cycles for the wage share, the employment rate, and the sectoral distribution of capital. The model achieves these outcomes without needing specific constraints tied to differences in capital intensity across sectors. Therefore, it stands as an alternative extension of Goodwin's (1967) model that captures the antagonism between capital and labor considering, at the same time, the complexity of capitalist reproduction within the context of a two-sector economy.

## Disclosure statement

No potential conflict of interest was reported by the author(s).

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### Appendix 1. Local Stability and Existence of a Hopf bifurcation

By combining equations (34), (35), and (37), we formulate a dynamical system that can be expressed in a generalized form as follows:

$$\dot{k} = F_k(k,\omega,l), \qquad \dot{\omega} = F_\omega(k,\omega,l), \qquad \dot{l} = F_l(k,\omega,l)$$

where $F_k$, $F_\omega$, and $F_l$ are functions that depend on the state variables $k$, $\omega$, and $l$. In the steady state ($\dot{k} = \dot{\omega} = \dot{l} = 0$) this system has a non-trivial equilibrium point $(k^*, \omega^*, l^*)$ given by:

$$k^* = \frac{Z_1}{m_2 n_0}, \qquad \omega^* = \frac{Z_2 Z_3}{Z_4}, \qquad l^* = \frac{\gamma}{\rho}$$

where:

$$Z_1 = q_2 - m_2 n_0, \qquad Z_2 = s q_2 - m_2 n_0, \qquad Z_3 = q_2 - n_0(m_2 - m_1)$$

$$Z_4 = s q_2^2 - n_0(m_2 - m_1)(q_2 + Z_2)$$

This equilibrium point assumes positive values ($k^* > 0, \omega^* > 0, l^* > 0$) when all of $Z_1$, $Z_2$, $Z_3$, and $Z_4$ are positive. This condition is satisfied if $q_2$ is sufficiently large.

Upon linearizing the dynamical system around its equilibrium point $(k^*, \omega^*, l^*)$, the resulting Jacobian matrix $J$ is equal to:

$$J = \begin{bmatrix} -\dfrac{\mu Z_6}{m_2 s Z_3} & \dfrac{Z_4^2 \mu}{m_2^2 m_1 n_0^2 s Z_3} & 0 \\ \dfrac{n_0^2 \mu m_1 Z_2 Z_5 Z_6}{s(1-s) Z_3 Z_4^2} & -\dfrac{Z_2 Z_5 \mu}{m_2 s (1-s) Z_3} & \dfrac{Z_1 Z_2 Z_3 Z_5 \rho}{(1-s) Z_4^2} \\ -\dfrac{n_0^2 \gamma}{q_2 \rho}\left[m_2 + \dfrac{\mu Z_6 (m_2 - m_1)}{Z_3^2 s}\right] & \dfrac{\gamma \mu Z_4^2 (m_2 - m_1)}{m_1 m_2 q_2 Z_3^2 s \rho} & -\gamma\zeta \end{bmatrix}$$

where:

$$Z_5 = s q_2 - n_0(m_2 - m_1), \qquad Z_6 = s q_2^2 - n_0^2 m_2 (m_2 - m_1)$$

The characteristic equation of the Jacobian $J$ is $\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$, where $\lambda$ represents the eigenvalue of $J$. To ensure the local stability of the dynamical system around its non-trivial equilibrium point, it is required that all eigenvalues $\lambda$ possess negative real components. The fulfillment of this stability prerequisite, according to the Routh-Hurwitz criteria, requires the positivity of $b_1$, $b_2$, and $b_3$, alongside the condition $y = b_1 b_2 - b_3 > 0$.

The coefficients $b_i$ depend on the trace $\mathrm{T}$, the determinant $\Delta$, and the minors of $J$, as expressed by the following equations:

$$b_1 = -\mathrm{T} = \gamma\zeta + \frac{Z_1 \mu}{m_2(1-s)}$$

$$b_2 = \begin{vmatrix} J_{22} & J_{23} \\ J_{32} & J_{33} \end{vmatrix} + \begin{vmatrix} J_{11} & J_{13} \\ J_{31} & J_{33} \end{vmatrix} + \begin{vmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{vmatrix} = \frac{\gamma \mu Z_1 Z_7}{m_1 m_2 q_2 Z_3 s (1-s)}$$

$$b_3 = -\Delta = \frac{\gamma\mu Z_1 Z_2 Z_5}{m_1 m_2 q_2 s(1-s)}$$

where:

$$Z_7 = m_1 q_2 s\zeta Z_3 - (m_2 - m_1) Z_2 Z_5$$

Assuming a significant magnitude for $\zeta$, we can deduce that $Z_7$ becomes positive.

Substituting coefficients $b_1$, $b_2$, and $b_3$ into $y = b_1 b_2 - b_3$ gives:

$$y = \frac{Z_1\gamma\mu}{m_1 m_2 q_2 s(1-s)}\left\{\frac{[\gamma\zeta m_2(1-s) + Z_1\mu]Z_7}{m_2(1-s)Z_3} - Z_2 Z_5\right\}$$

The term $y$ is positive if $\mu$ is sufficiently high to satisfy the following condition:

$$\mu > \mu^{HB} = \frac{m_2(1-s)(Z_2 Z_3 Z_5 - \gamma\zeta Z_7)}{Z_1 Z_7}$$

In summary, the dynamical system derived from equations (34), (35), and (37) demonstrates a positive and locally stable equilibrium point when $q_2$, $\zeta$, and $\mu$ are sufficiently large.

Addressing the existence of a Hopf bifurcation within the dynamical system, following Liu (1994), it is necessary to prove two conditions for confirming the presence of such a bifurcation: (HB1) The coefficients $b_1$, $b_2$, and $b_3$ must be positive. (HB2) If we designate $\mu$ as the bifurcation parameter, then we must identify a critical value $\mu^{HB}$ for which the following holds:

$$y(\mu^{HB}) = 0, \qquad \left.\frac{dy}{d\mu}\right|_{\mu=\mu^{HB}} \neq 0$$

Condition (HB1) is fulfilled when $q_2$ and $\zeta$ are sufficiently large. Concerning condition (HB2), firstly, it can be immediately proved that substituting $\mu^{HB}$ into $y$ results in zero without changing the positivity of $b_1$, $b_2$, and $b_3$. Secondly, upon differentiating $y$ with respect to $\mu$ and substituting $\mu^{HB}$ into the outcome, we obtain:

$$\left.\frac{dy}{d\mu}\right|_{\mu=\mu^{HB}} = -\frac{Z_1\gamma(\gamma\zeta Z_7 - Z_2 Z_3 Z_5)}{m_1 m_2 q_2 s(1-s) Z_3}$$

Since there is no justification to assume that this derivative is null, we conclude that condition (HB2) is satisfied. Consequently, the model undergoes a Hopf bifurcation, signifying the emergence of limit cycles near its equilibrium point $(k^*, \omega^*, l^*)$ when $\mu$ approaches the critical value $\mu^{HB}$.

## Appendix 2. Data Construction and Econometric Estimation for Equations (30) and (36)

To conduct numerical simulations of the CMTSEC model, we estimate the real wage Phillips curve, as described in equation (30), and the dynamic labor supply equation, as specified in equation (36), utilizing annual data from the AMECO database[12] covering the period from 1960 to 2019 for the US economy. Our estimation procedure closely aligns with the approach outlined by Grasselli and Maheshwari (2018).

Regarding data compilation, in line with the approach of Grasselli and Maheshwari (2018), we utilize the AMECO database to formulate the real wage bill in the overall economy $W$ as:

$$W = \left(1 + \frac{\text{Self Employed}}{\text{Total Employees}}\right)\left(\frac{\text{Compensation of Employees}}{\text{GDP Deflator}}\right)$$

Similarly, we define total employment $L$ and total labor force $N$ as follows:

$$L = \text{Total employees} + \text{Self employed}$$

$$N = \text{Total employment} + \text{Total unemployed}$$

Using these definitions, we obtain the real wage $V$ and the employment rate $l$ through the following expressions:

$$V = \frac{W}{L} = \left(\frac{1}{\text{Total Employees}}\right)\left(\frac{\text{Compensation of Employees}}{\text{GDP Deflator}}\right)$$

$$l = \frac{L}{N} = \frac{\text{Total employees} + \text{Self employed}}{\text{Total employment} + \text{Total unemployed}}$$

After obtaining the values for $V$, $l$, and $N$, we proceed to estimate discrete-time versions of equations (30) and (36) using 'level models' for the long run, as shown below:

$$\Delta \ln V_t = -\gamma + \rho l_t + e_{Vt} \quad (A1)$$

$$\Delta \ln N_t = n_0 + \zeta \Delta \ln V_t + e_{Nt} \quad (A2)$$

Here, $\Delta \ln V_t = \ln V_t - \ln V_{t-1}$ and $\Delta \ln N_t = \ln N_t - \ln N_{t-1}$ are discrete approximations of the growth rates of $N$ and $V$, respectively, while $e_{Vt}$ and $e_{Nt}$ represent error terms.

Following Grasselli and Maheshwari (2018), we estimate the parameters $\gamma$, $\rho$, $n_0$, and $\zeta$ using the long-run multipliers obtained from the Autoregressive Distributive Lag (ARDL) estimator as proposed in the bounds-testing procedure by Pesaran et al. (2001). This approach requires that the variables included in equations (A1) and (A2) ($\Delta \ln V_t$, $l_t$, and $\Delta \ln N_t$) are either stationary ($I(0)$) or, at most, integrated of order one ($I(1)$). The results of the Augmented Dickey-Fuller (ADF) tests, presented in Table A1, support this condition, indicating that $\Delta \ln V_t$ and $l_t$ can be assumed to be $I(1)$, while $\Delta \ln N_t$ can be considered stationary.

[12] AMECO: Annual macro-economic database of the European Commission's Directorate General for Economic and Financial Affairs. Data retrieved in tabular format on March 25, 2023, from https://ec.europa.eu/info/business-economy-euro/indicators-statistics/economic-databases/macro-economic-database-ameco/download-annual-data-set-macro-economic-database-ameco_en

Upon establishing that the variables included in equations (A1) and (A2) exhibit either $I(0)$ or $I(1)$ characteristics, we proceed to construct the following Unrestricted Error Correction Models (UECM):

$$\Delta(\Delta \ln V_t) = b_0 + b_1 \Delta \ln V_{t-1} + b_2 l_{t-1} + \sum_{i=1}^{p_V - 1} \psi_{V,i}\, \Delta(\Delta \ln V_{t-i}) + \sum_{i=0}^{q_V - 1} \gamma_{l,i} \Delta l_{t-i} + \varepsilon_{Vt} \quad (A3)$$

$$\Delta(\Delta \ln N_t) = c_0 + c_1 \Delta \ln N_{t-1} + c_2 \Delta \ln V_{t-1} + \sum_{i=1}^{p_N - 1} \psi_{N,i}\, \Delta(\Delta \ln N_{t-i}) + \sum_{i=0}^{q_N - 1} \gamma_{V,i}\, \Delta(\Delta \ln V_{t-i}) + \phi_N D + \varepsilon_{Nt} \quad (A4)$$

These models can be estimated through Ordinary Least Squares (OLS), and the appropriate lag lengths $p_V$, $q_V$, $p_N$, and $q_N$ can be determined using the Akaike Information Criterion (AIC). To estimate equations (A3) and (A4) for the US economy from 1960 to 2019, we employed the R package *ARDL* developed by Natsiopoulos and Tzeremes (2022). The results of these estimations are presented in Tables A2 and A3. In the case of equation (A4), we introduced a vector of dummy variables, denoted as $D = \{D_{1970}, D_{1980}, D_{1990}, D_{2000}, D_{2010}\}$, where each variable $D_j$ assumes the value of 1 for years $t \geq j$ and 0 otherwise. This was done to account for potential shifts or changes in the US labor supply at intervals of 10 years.[13] Table A2 reveals that $l_{t-1}$, $\Delta l_t$, and $\Delta l_{t-1}$ exert a statistically significant influence on $\Delta(\Delta \ln V_t)$ at the 95% confidence level. Similarly, Table A3 shows that $\Delta(\Delta \ln V_t)$ and $\Delta(\Delta \ln V_{t-2})$ significantly impact $\Delta(\Delta \ln N_t)$ at the 95% confidence level. These findings provide empirical support for the existence of a statistically meaningful relationship among the variables initially considered in equations (A1) and (A2).

After estimating equations (A3) and (A4), we require to examine the residuals of these models to ensure robust inference and avoid spurious correlations. In this regard, Table A4 reveals that, at a 95% confidence level, we do not find evidence to reject several crucial assumptions. These include the absence of serial correlation (Breusch-Godfrey test), homoskedasticity (Breusch-Pagan test), the presence of ARCH effects, conformity with normality (Jarque Bera test), no functional form misspecification (RESET test), and model stability (CUSUM OLS and recursive residuals).

After verifying the residuals, we proceed with the bounds-testing procedure as proposed by Pesaran et al. (2001). Specifically, we conduct an F test to examine the null hypothesis of no long-run relationship ($H_0: b_1 = b_2 = 0$ for equation (A3) and $H_0: c_1 = c_2 = 0$ for equation (A4)), along with a t test to assess the null hypothesis of the existence of a degenerate case ($H_0: b_1 = 0$ for equation (A3) and $H_0: c_1 = 0$ for equation (A4)). The results of these tests are detailed in Table A5, and we find compelling evidence to reject all null hypotheses for equations (A3) and (A4) at the 95% confidence level. Consequently, our findings support the premise that equations (A3) and (A4) indeed capture meaningful long-run relationships among the variables under consideration.

Having established the potential presence of long-run relationships, we proceed to estimate the long-run coefficients $\hat{\gamma}$, $\hat{\rho}$, $\hat{n}_0$, and $\hat{\zeta}$. These estimates are detailed in Table A6, demonstrating that all coefficients exhibit the anticipated signs and hold statistical significance at the 95% confidence

[13] For the growth rate of the real wage, we did not find any significant effect from the dummy variables included in vector $D$. Consequently, we opted to exclude these variables from the estimation of equation (A3).

level. Hence, we employ these estimations to establish a baseline for the numerical simulations of the CMTSEC model, as mentioned in Section 3.

Finally, regarding the savings rate, $s$, in our model, where aggregate savings equal aggregate investment, we consider $s$ as equivalent to an accumulation rate. Hence, we adopt the following definition of the accumulation rate, as proposed by Grasselli and Maheshwari (2018):

$$s = \frac{\text{Gross capital formation}}{Y - W}$$

where:

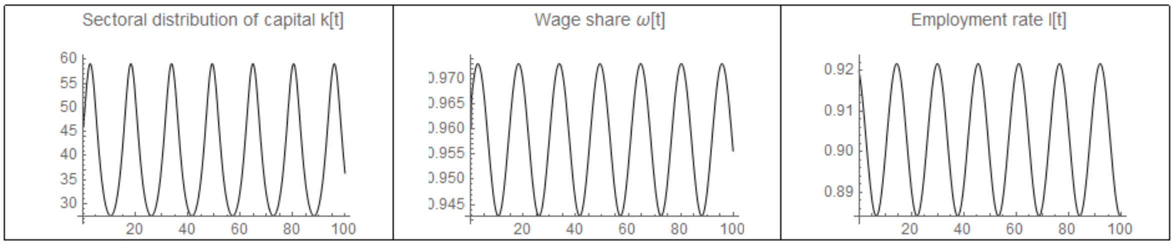
$$Y = \frac{\text{GDP at current prices} - \text{net taxes on production and imports}}{\text{GDP Deflator}}$$

Applying this definition to the AMECO database, we can estimate the value $\hat{s}$ by calculating the historical average for the US economy from 1960 to 2019, yielding:

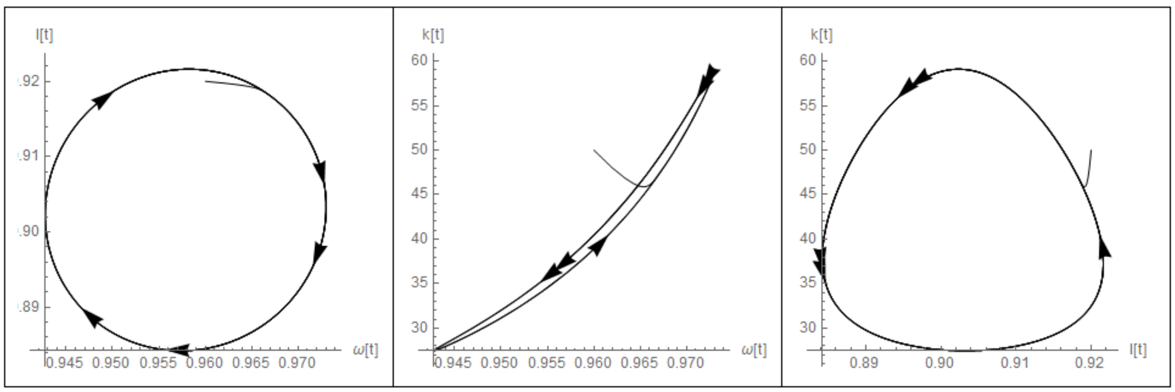
$$\hat{s} = 0.5626761$$

**Figure 1. Simulation of Limit Cycles in State Variables with $m_2 > m_1$ (Case 1)**

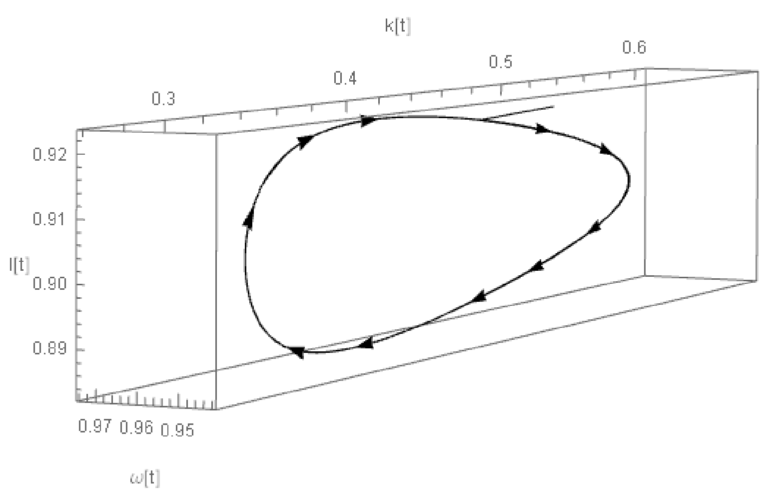
1A. Time series




1B. Two-dimensional parametric plots



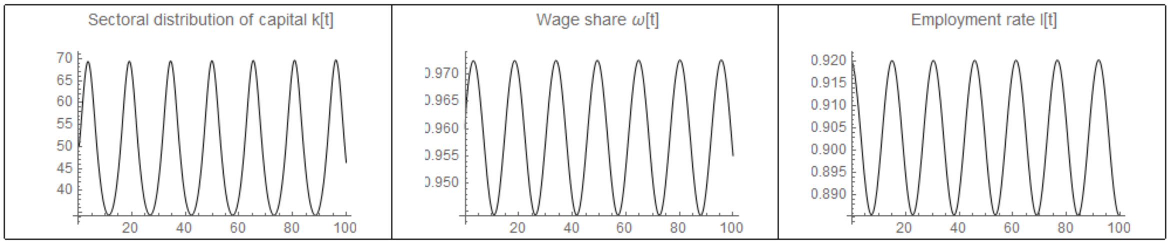

1C. Three-dimensional parametric plot



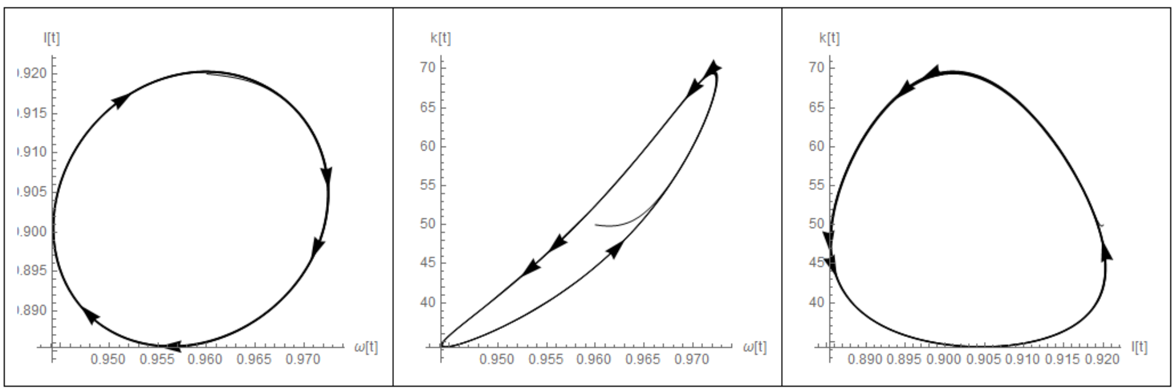

Note: Simulation of the CMTSEC model with parameter values $q_1 = 1, q_2 = 1, m_1 = 1, m_2 = 1.1, s = 0.5626761, \gamma = 0.3072315, \rho = 0.3403163, n_0 = 0.02357968, \zeta = 0.08431691, \mu = \mu^{HB} \approx 8.5738$ and initial conditions $k_0 = 50, \omega_0 = 0.96, l_0 = 0.92$. Equilibrium point: $k^* = 37.554, \omega^* = 0.957822, l^* = 0.902782$. Figure 1C adjusts variable $k$ to improve plot visualization.

**Figure 2. Simulation of Limit Cycles in State Variables with $m_1 > m_2$ (Case 2)**

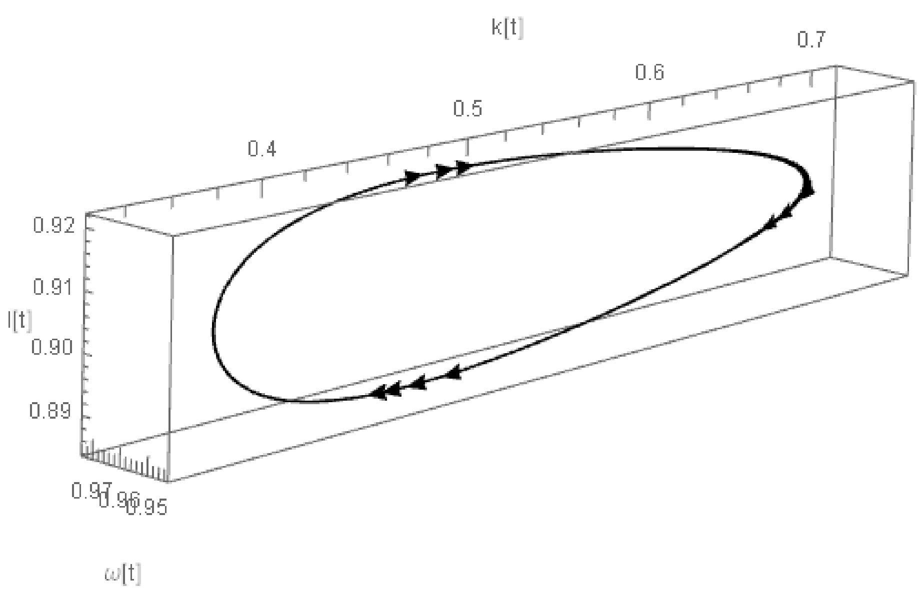
2A. Time series




2B. Two-dimensional parametric plots



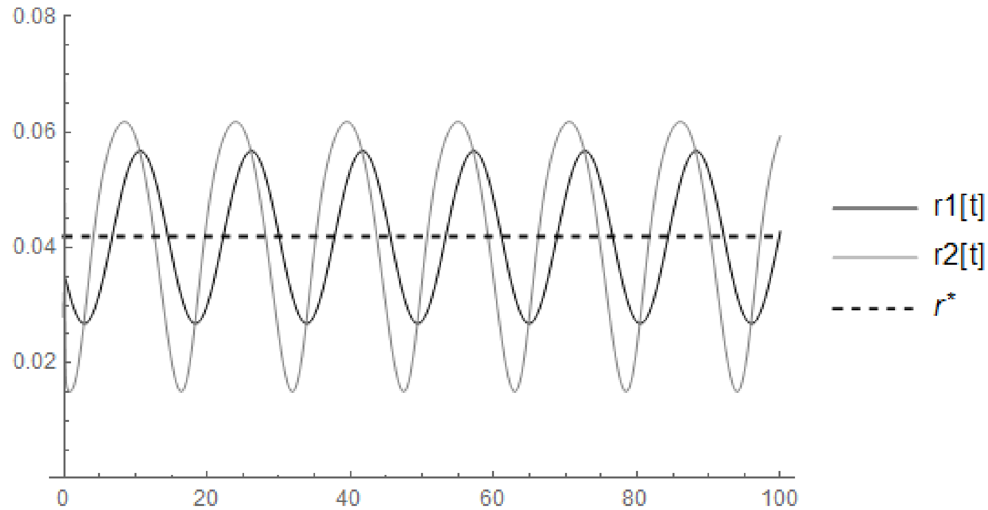

2C. Three-dimensional parametric plot



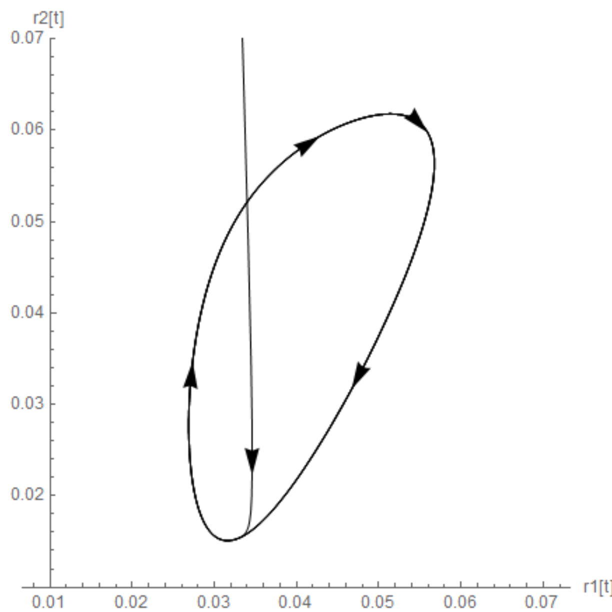

Note: Simulation of the CMTSEC model with parameter values $q_1 = 1, q_2 = 1, m_1 = 1, m_2 = 0.9, s = 0.5626761, \gamma = 0.3072315, \rho = 0.3403163, n_0 = 0.02357968, \zeta = 0.08431691, \mu = \mu^{HB} \approx 1.56755$ and initial conditions $k_0 = 50, \omega_0 = 0.96, l_0 = 0.92$ . Equilibrium point: $k^* = 46.1216, \omega^* = 0.958363, l^* = 0.902782$. Figure 1C adjusts variable $k$ to improve plot visualization.

**Figure 3. Simulation of Limit Cycles in Sectoral Rates of Profit with $m_2 > m_1$ (Case 1)**

3A. Time series



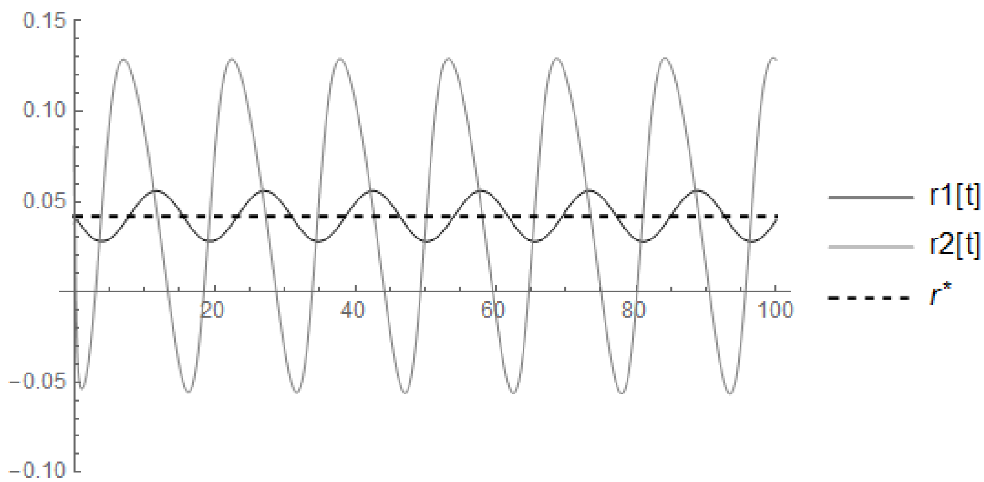

3B. Two-dimensional parametric plot



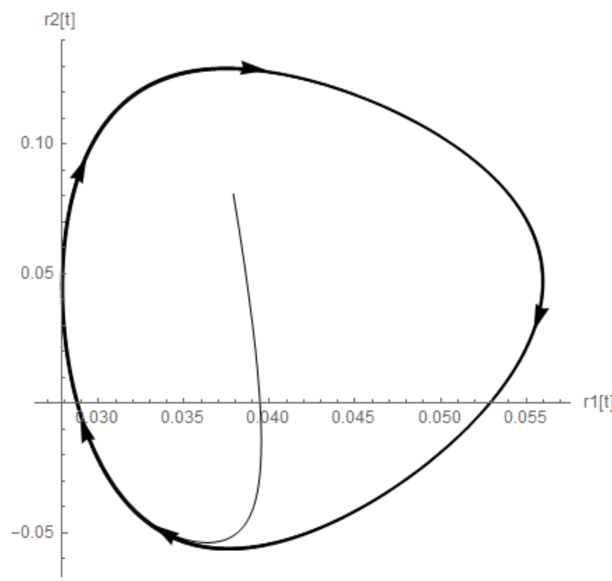

Note: Simulation of profit rates $r_1$, $r_2$, and $r^*$ defined by equations (44), (45), and (46), with parameter values $q_1 = 1, q_2 = 1, m_1 = 1, m_2 = 1.1, s = 0.5626761, \gamma = 0.3072315, \rho = 0.3403163, n_0 = 0.02357968, \zeta = 0.08431691, \mu = \mu^{HB} \approx 8.5738$ and initial conditions $k_0 = 50, \omega_0 = 0.96, l_0 = 0.92, r_{10} = 0.02797, r_{20} = 0.2166$. Equilibrium point: $r^* = 0.0419$.

**Figure 4. Simulation of Limit Cycles in Sectoral Rates of Profit with $m_1 > m_2$ (Case 2)**

4A. Time series



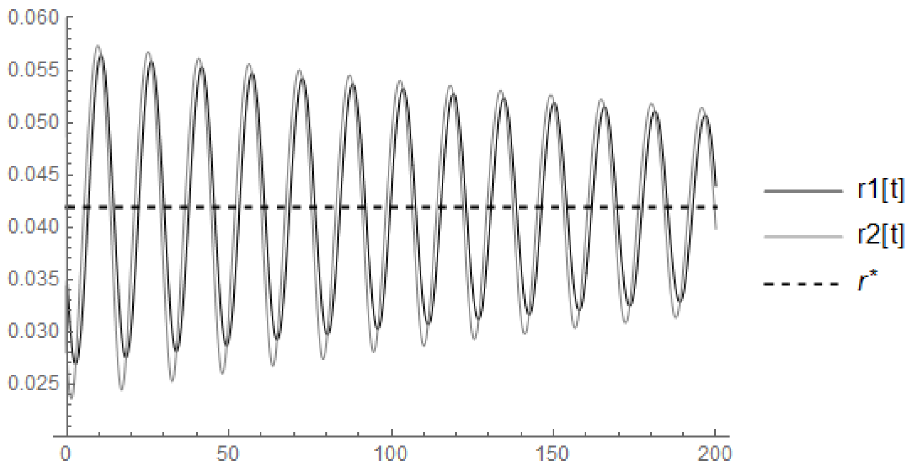

4B. Two-dimensional parametric plot



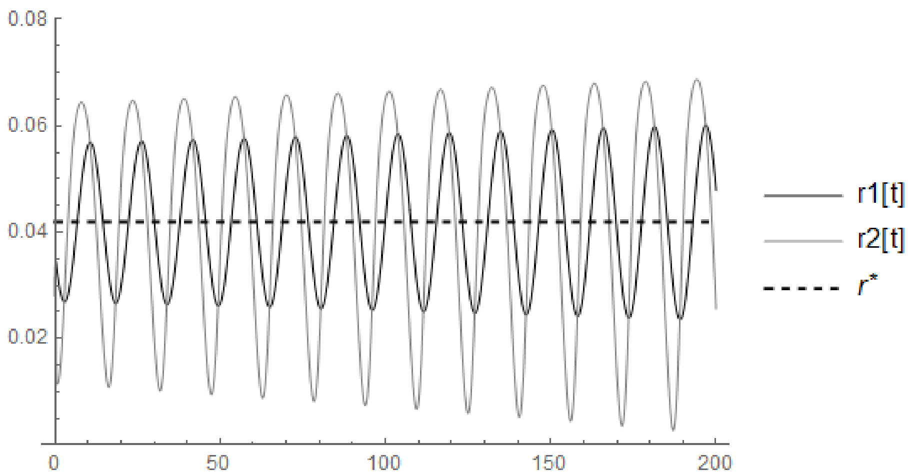

Note: Simulation of profit rates $r_1$, $r_2$, and $r^*$ defined by equations (44), (45), and (46), with parameter values $q_1 = 1, q_2 = 1, m_1 = 1, m_2 = 0.9, s = 0.5626761, \gamma = 0.3072315, \rho = 0.3403163, n_0 = 0.02357968, \zeta = 0.08431691, \mu = \mu^{HB} \approx 1.56755$ and initial conditions $k_0 = 50, \omega_0 = 0.96, l_0 = 0.92, r_{10} = 0.03787, r_{20} = 0.0808$. Equilibrium point: $r^* = 0.0419$.

**Figure 5. Influence of Investment Sensitivity ($\mu$) on Sectoral Profit Rate Cycles**

5A. Damped oscillations with $\mu \gg \mu^{HB}$



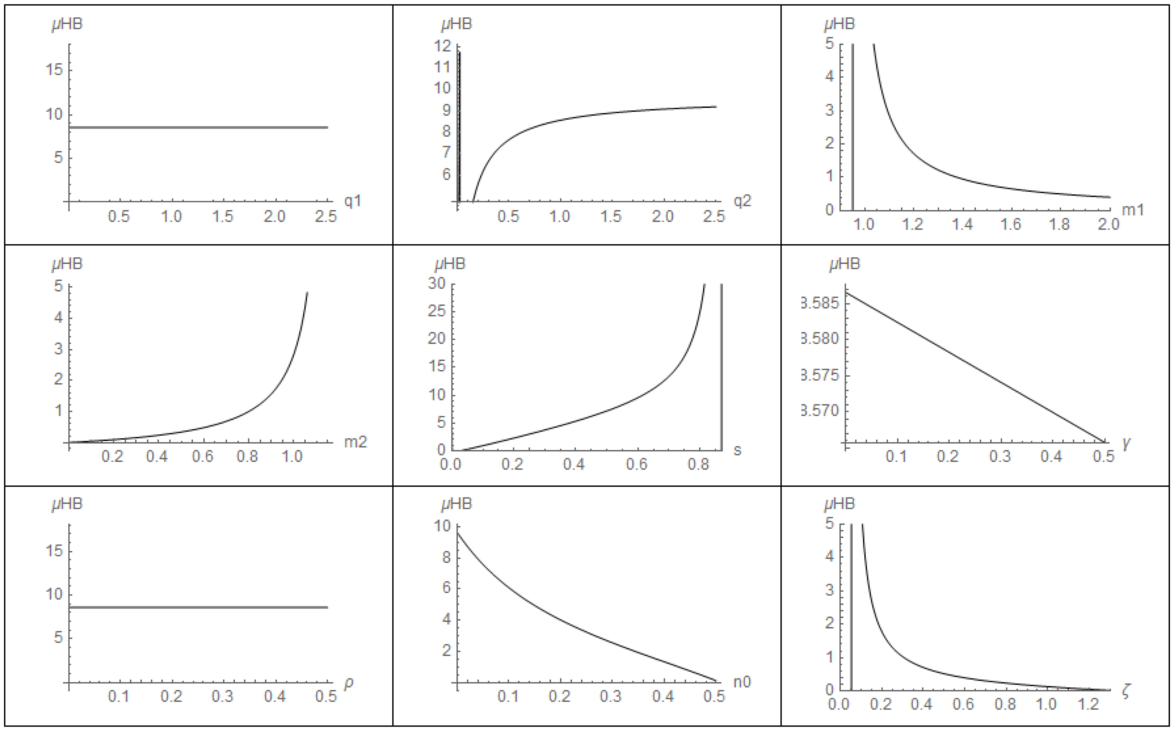

5B. Unstable oscillations with $\mu \ll \mu^{HB}$



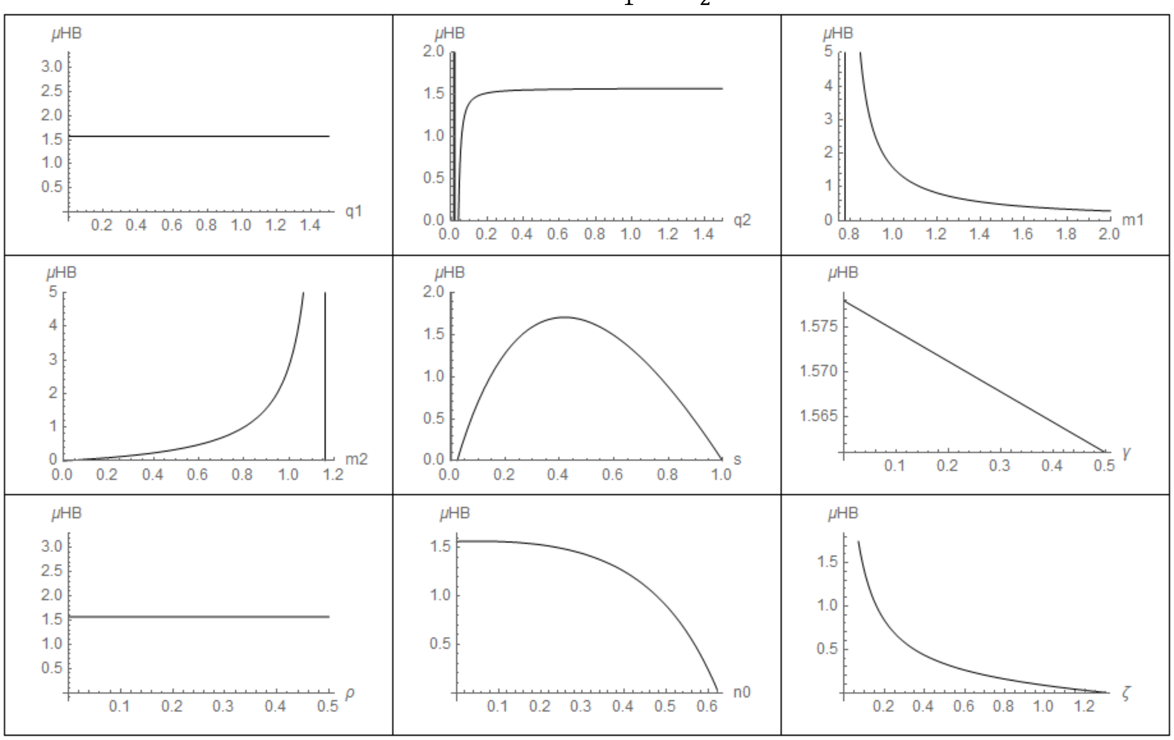

Note: Simulation of profit rates $r_1$, $r_2$, and $r^*$ defined by equations (44), (45), and (46), with parameter values $q_1 = 1, q_2 = 1, m_1 = 1, m_2 = 1.1, s = 0.5626761, \gamma = 0.3072315, \rho = 0.3403163, n_0 = 0.02357968, \zeta = 0.08431691, \mu^{HB} \approx 8.5738$ and initial conditions $k_0 = 50, \omega_0 = 0.96, l_0 = 0.92, r_{10} = 0.02797, r_{20} = 0.2166$. Equilibrium point: $r^* = 0.0419$. Figure 5A considers $\mu = 20$ and Figure 5B considers $\mu = 7$.

## Figure 6. Relationship between $\mu^{HB}$ and Parameters of the CMTSEC Model

6A. Case 1: $m_2 > m_1$




6B. Case 2: $m_1 > m_2$




Note: Each plot shows the estimated relationship between the critical value $\mu^{HB}$ and a specific parameter of the model, while keeping all other parameter values fixed at those listed in the caption of Figure 1 for Case 1 and the caption of Figure 2 for Case 2. Only positive values of $\mu^{HB}$ are considered.

**Table A1. Unit Root Tests (p-values)**

| Variables | $\Delta \ln V_t$ | $l_t$ | $\Delta \ln N_t$ |
|---|---|---|---|
| ADF test – Levels | 0.2979 | 0.3432 | <0.01 |
| ADF test – First difference | <0.01 | <0.01 | <0.01 |

**Table A2. Estimation of Equation (A3)**

| Regressor | Coefficient | Standard Error | t-value | p-value |
|---|---|---|---|---|
| Intercept | -0.18165 | 0.09484 | -1.915 | 0.060844 |
| $\Delta \ln V_{t-1}$ | -0.59125 | 0.12474 | -4.74 | 1.65E-05 |
| $l_{t-1}$ | 0.20121 | 0.10124 | 1.988 | 0.052043 |
| $\Delta l_t$ | 0.59442 | 0.15644 | 3.8 | 0.000375 |
| $\Delta l_{t-1}$ | -0.41863 | 0.15236 | -2.748 | 0.008185 |

**Table A3. Estimation of Equation (A4)**

| Regressor | Coefficient | Standard Error | t-value | p-value |
|---|---|---|---|---|
| Intercept | 0.016942 | 0.003827 | 4.427 | 6.24E-05 |
| $\Delta \ln N_{t-1}$ | -0.718493 | 0.123414 | -5.822 | 6.16E-07 |
| $\Delta \ln V_{t-1}$ | 0.060581 | 0.101045 | 0.6 | 0.55188 |
| $\Delta(\Delta \ln N_{t-1})$ | 0.069356 | 0.118216 | 0.587 | 0.56041 |
| $\Delta(\Delta \ln V_t)$ | 0.14551 | 0.064774 | 2.246 | 0.02975 |
| $\Delta(\Delta \ln V_{t-1})$ | 0.096125 | 0.076803 | 1.252 | 0.21733 |
| $\Delta(\Delta \ln V_{t-2})$ | 0.14635 | 0.063117 | 2.319 | 0.02512 |
| $D_{1970}$ | -0.000565 | 0.002793 | -0.202 | 0.84065 |
| $D_{1980}$ | -0.005543 | 0.002041 | -2.716 | 0.00941 |
| $D_{1990}$ | -0.004472 | 0.002122 | -2.107 | 0.04082 |
| $D_{2000}$ | -0.002665 | 0.002064 | -1.292 | 0.20323 |
| $D_{2010}$ | 0.000946 | 0.002214 | 0.427 | 0.67126 |

**Table A4. Tests for Residuals (p-values)**

| Tests | | Residuals of equation (A3) | Residuals of equation (A4) |
|---|---|---|---|
| Breusch-Godfrey | 1 lag | 0.6267 | 0.3187 |
| | 2 lags | 0.8569 | 0.3571 |
| | 3 lags | 0.9433 | 0.5513 |
| | 4 lags | 0.9828 | 0.09253 |
| | 5 lags | 0.9947 | 0.1383 |
| Breusch-Pagan | | 0.2705 | 0.8875 |
| ARCH LM | | 0.9449 | 0.8481 |
| Jarque Bera | | 0.05903 | 0.85 |
| RESET | | 0.3908 | 0.8022 |
| CUSUM OLS residuals | | 0.5746 | 0.9951 |
| CUSUM recursive residuals | | 0.3759 | 0.9163 |

**Table A5. Long-Run Relationship Tests Using the Procedure by Pesaran et al. (2001) (statistics and p-values)**

| | Equation (A3) | Equation (A4) |
|---|---|---|
| F test | 11.23492 (<0.01) | 16.9696 (<0.01) |
| t test | 4.74 (<0.01) | -5.8218 (<0.01) |

Note: Tests obtained for the case of unrestricted intercept and no trend.

**Table A6. Long-run Estimates for Equations (A1) and (A2)**

| | Dependent variable | Coefficient | Estimate | t-value | p-value |
|---|---|---|---|---|---|
| Equation (A1) | $\Delta \ln V_t$ | $-\hat{\gamma}$ | -0.3072315 | -2.10281 | 0.04024793 |
| | | $\hat{\rho}$ | 0.3403163 | 2.199351 | 0.03223718 |
| Equation (A2) | $\Delta \ln N_t$ | $\hat{n}_0$ | 0.02357968 | 5.2236888 | 4.59E-06 |
| | | $\hat{\zeta}$ | 0.08431691 | 0.6004583 | 5.51E-01 |

Note: Long-run multipliers estimated from equations (A3) and (A4)