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ABSTRACT

This article presents a new model of collateralization. We study the economic impact of collateralization on the plumbing of financial system. The model gives an integrated view of different collateral arrangements. We show that the effect of collateral on asset prices is significant. Our study shows that a poorly designed collateral agreement can actually increase credit risk. We find evidence that collateral posting regimes that are originally designed and utilized for contracts subject to bilateral credit risk (e.g., a swap) may not work properly for contracts subject to multilateral credit risk (e.g., a CDS) in the presence of default correlations. These findings contradict the prevailing beliefs in financial markets about collateralization.

Key Words: collateralization, collateral posting, credit support annex, credit risk modeling, the plumbing of financial system, derivatives valuation subject to credit risk.

JEL Classification: E44, G21, G12, G24, G32, G33, G18, G28

Collateralization is an essential element in the so-called plumbing of the financial system that is the Achilles' heel of the global financial structure. It allows financial institutions to reduce economic capital and credit risk, free up lines of credit, and expand their range of counterparties. All contribute to the growth of financial markets. The benefits are broadly acknowledged and affect dealers and end users, as well as the financial system generally.

The posting of collateral is regulated by the Credit Support Annex (CSA) that specifies a variety of terms including the threshold, the independent amount, and the minimum transfer amount, etc. The threshold is the unsecured credit exposure that a party is willing to bear. The minimum transfer amount is the smallest amount of collateral that can be transferred. The independent amount plays the same role as the initial margin in a collateral agreement and can be regarded as a negative threshold (over-collateralization). That is the reason why people in the financial industry often refer to collateralized contracts (or instruments or products) as CSA contracts and non-collateralized contracts as non-CSA contracts.

The use of collateral in the financial markets has increased sharply over the past decade, yet it has received surprisingly little attention in the finance literature. Collateral management is often carried out in an ad-hoc manner, without reference to an analytical framework. Very little academic research has been done to quantitatively assess the economic implications of collateralization. Such a quantitative analysis is the primary contribution of this paper.

Johannes and Sundaresan (2007) analyze collateralized interest rate swaps (IRS) and predict that observed swap rates should stand above the rates implied by the portfolio of forward contracts and below the rates obtained from equating a swap with a portfolio of future contracts. They make several assumptions and simplifications: First, the default events of both counterparties are conditionally independent. Second, both counterparties have the same credit quality. Finally, swaps are fully collateralized.

Ottonello, etc. (2022) study the design of macroprudential policies based on quantitative collateral-constraint models and find the desirability macroprudential policies critically depends on the specific form of collateral used in debt contracts. Bianchi, etc. (2020) develop a quantitative model that focuses on collateral inefficiencies arising from prices that affect borrowing limits and individual agents not internalizing such price effects.

Devereux, etc. (2019) analyze how predictions of collateral-constraint models very with different timing assumptions. Du, etc. (2023) investigate how market participants price and manage counterparty credit risk using confidential trade repository data on single-name CDS transactions.

According to the International Swap Dealers Association (ISDA) Margin Survey (ISDA (2010)), the collateralized percentages for the credit derivatives (97%) and fixed income derivatives (84%) markets are substantially higher, whereas for the equity (68%), foreign exchange (63%), and commodities (62%) markets the levels are lower. These differences reflect the riskiness of the underlying trades. Some markets such as equity or foreign exchange are spot or very short-dated and thus present lower risk that is not practical or economic to secure with collateral. Other markets, such as commodities, use collateral selectively but may employ other forms of credit protection such as letters of credit instead. Since not all instruments need to be secured by collateral, a counterparty portfolio consisting of various transactions in different markets is most probably subject to partial-collateralization because some transactions are collateralized but others are not.

Upon default and early termination, the values due under the ISDA Master Agreement are determined. These amounts are then netted and a single net payment is made. All of the collateral on hand would be available to satisfy this total amount, up to the full value of that collateral. In other words, the collateral to be posted is calculated on the basis of the aggregated value of the portfolio, but not on the basis of any individual transaction. Since a counterparty portfolio is, in most cases, subject to partial-collateralization, studying partial-collateralization is even more important than studying full-collateralization.

One of the central tenets of modern financial economics is the necessity of some trade-off between risk and expected value. This paper addresses several essential questions concerning the posting of collateral. First, how does collateralization affect expected asset prices? To answer this question, we develop a comprehensive analytical framework for pricing collateralized financial contracts. To the best of our knowledge, this is the first study that attempts to examine the economic significance and implications of different (partial-, full-, and over-; unilateral and bilateral) collateral arrangements in a unified way.

Credit risk may be unilateral, bilateral, or multilateral. Some financial instruments, such as, debt products (e.g., loans, bills, notes, bonds, etc.), by nature contain only unilateral credit risk because only the default risk of one party appears to be relevant. Whereas some other instruments, such as, over the counter (OTC) derivatives, securities financing transactions (SFT), and credit derivatives, bear bilateral or multilateral credit risk because two or more parties are susceptible to default risk.

From the perspective of collateral obligations, collateral arrangements can be unilateral or bilateral. In unilateral arrangements, only one predefined counterparty has the right to call for collateral. Unilateral agreements are generally used when the other counterparty is much less creditworthy. In bilateral arrangements, on the other hand, both counterparties have the right to call for collateral. Bilateral agreements become increasingly popular.

In this paper, we will focus on the three most common cases: i) unilateral collateralization against unilateral credit risk, ii) bilateral collateralization against bilateral credit risk, and iii) bilateral collateralization against multilateral credit risk; but the general methodology is equally applicable to other situations as well. Our analysis shows that the posting of collateral indeed has a significant effect on the pricing of assets.

The amount of collateral is determined by a discontinuous and state-dependent indicator function that is the root cause of the complexity of collateralized valuation. We find that collateralized contracts normally have backward recursive natures and require backward induction valuations.

Second, how does collateralization affect risk, and does it always reduce credit risk? To answer this question, we conduct an extensive quantitative study and reveal some important findings, e.g., although many people in financial markets believe that collateralization can always mitigate credit risk, we show that a poorly designed and analyzed collateral arrangement can actually increase credit risk.

Collateralization provides protection in the event of a default, since the collateral taker has recourse to the collateral asset and can thus make good some or all of the loss suffered. The collateral amount should be greater than the recovery value at default. Otherwise collateralization becomes meaningless, because the non-default party would rather receive the recovery value than take the collateral when a default occurs. Equivalently, the value of a collateralized portfolio should exceed the value of the same portfolio without collateralization. Based on this principle, we derive an upper bound on the collateral threshold. If the real collateral threshold is less than this upper bound, the collateral arrangement can improve default recovery and mitigate credit risk as intended. If the real collateral threshold exceeds this upper bound, the collateral arrangement can actually deteriorate default recovery and aggravate credit risk. This is a perfect example of how good intentions may turn into bad outcomes. These results further emphasize the importance of carefully designing and quantifying collateral arrangements.

Third, can full-collateralization eliminate counterparty risk completely? The answer depends on what type of credit risk one may encounter. We find that full-collateralization can get rid of counterparty risk entirely for contracts subject to unilateral or bilateral default risk, e.g., an IRS. This result is consistent with the current market practice in which market participants commonly assume fully collateralized swaps are risk-free and it is common to build models of swap rates assuming that swaps are free of counterparty risk. However, we may not reach the same conclusion for contracts subject to multilateral credit risk, e.g., a credit default swap (CDS).

A CDS is a trilateral defaultable contract where the three parties are the protection buyer, the protection seller, and the reference entity. A CDS contract is normally used to transfer the credit risk of the reference entity between two counterparties. The contract reduces the credit risk of the reference entity but gives rise to counterparty risk. The risk circularity that transfers one type of risk (reference credit risk) into another (counterparty credit risk) within the CDS market is a concern for financial stability. The role of CDS in the 2007-2010 financial crisis has been heavily criticized. The total notional amount of outstanding CDS contracts fell from $62.2 trillion at the end of 2007 to $26.3 trillion in the middle of 2010. Following the financial crisis, almost all CDS contracts are fully collateralized. People believe that fully collateralized CDS contracts would guarantee that there should be no risk of failure to pay.

Collateral posting regimes are originally designed and utilized for contracts subject to bilateral credit risk (e.g., an IRS), but there are many reasons to be concerned about the success of collateral posting in offsetting the risks of contracts subject to multilateral credit risk (e.g., a CDS). First, the values of CDS contracts tend to move very suddenly with big jumps, whereas IRS prices are far smoother and less volatile than CDS prices. Second, CDS spreads/premia can widen very rapidly. The amount of collateral that one party is required to provide at short notice may, in some cases, be close to the notional amount of the CDS and may therefore exceed that party's short-term liquidity capacity, thereby triggering a liquidity crisis. Third, CDS contracts have many more risk factors than IRS contracts have. In this paper, we provide a profound analysis of the role of collateral in the CDS market and find that full-collateralization actually cannot neutralize counterparty risk completely for a CDS in the presence of default correlations. These findings contradict a prevailing belief in financial markets that full-collateralization can always eliminate counterparty risk utterly.

Fourth, how can one adjust a collateral arrangement when things go contrary to his wishes? For example, if the collateral threshold exceeds the upper bound, the collateral arrangement actually becomes harmful. There are two ways to reduce the collateral threshold: i) reducing the number of non-CSA transactions in a portfolio or ii) collateralizing entire counterparty relationships rather than particular products and then decreasing the threshold accordingly. The first solution may require multiple master agreements between two parties. Although it is common to use a number of different master agreements to govern a trading relationship with the same party under different jurisdictions, the use of separate master agreements under the same jurisdiction may not be a good practice since it may create some legal uncertainty. The second solution not only offers more flexibility to accomplish a desired collateral arrangement but also promotes greater operational and capital efficiency. The results are consistent with the recent practice where there is a trend in the privately negotiated derivative markets towards collateralizing entire counterparty relationships rather than particular products.

Fifth, what is the time-variation in the impact of different collateral arrangements? What is the actual variation across the different assets? Empirically, we find strong evidence that collateralization affects swap rates and CDS premia. The effects are time varying. The difference (spread) between the partially collateralized asset and the risk-free asset reflects the cost of bearing unsecured credit risk, whereas the difference between the over collateralized asset and the risk-free asset represents the benefit of taking over-secured credit risk. The cost or benefit increases as counterparty credit quality deteriorates. When counterparty risk is low, fully collateralized assets, partially collateralized assets, and over collateralized assets are almost coincident. However, when counterparty risk soars, the differences between differently collateralized assets surge, and then reach the peaks during the financial crisis. These empirical results are in line with economic intuition and corroborate our theoretical analysis.

Finally, we study the impact of counterparty risk on collateralized swap rates and collateralized CDS premia. Although there is a significant relation between counterparty risk and the cost of collateralized borrowing, we show that the effect on collateralized swap rates is relatively small. For a 10-year partially collateralized IRS, an increase in the floating-rate payer's credit spread of 100 basis points (bps) translates into a 2 bps decline in the swap rate, while a rise in the default correlation of 0.1 (1000 bps) only results in a 0.1 bps decrease in the swap rate. We also show that the impact of a dealer's credit risk on a CDS premium is small, whereas the effect of the default correlation between a dealer and a reference entity on a CDS premium is substantial. For a 5-year partially collateralized CDS, the CDS premium decreases by a 0.2 bps for every 100 bps that the dealer's credit spread increases, whereas the CDS premium declines 21 bps for an increase in the default correlation of 0.1.

It is worth noting that the impact of the default correlation between the dealer and the reference entity on the collateralized CDS premium is much more significant than that on the non-collateralized CDS premium (e.g., an increase in the default correlation of 0.1 maps into a 21 bps decline in the collateralized CDS premium, but only into an 8.3 bps decrease in the non-collateralized CDS premium). These results clearly support our theoretical prediction that collateral arrangements that are originally designed and utilized for contracts subject to bilateral credit risk (e.g., an IRS) may not function correctly for contracts subject to multilateral credit risk (e.g., a CDS) in the presence of default correlations.

The remainder of this paper is organized as follows: Section 1 presents a simple example to illustrate the basic ideas. Section 2 discusses unilateral collateralization against unilateral credit risk. Section 3 elaborates bilateral collateralization against bilateral credit risk. Section 4 describes bilateral collateralization against multilateral credit risk. The conclusions and discussion are provided in Section 5. All proofs and some detailed derivations are contained in the appendices.

1. A Simple Example

Consider a generic financial contract that promises to pay a $X_T > 0$ from party *B* to party *A* at maturity date *T*, and nothing before date *T*. The contract can be a transaction or a portfolio. We assume that party *A* is default-free, whereas party *B* is defaultable. This is a unilateral credit risk case where only the default risk of party *B* is relevant. Let valuation date be *t* where $t < T$. In this paper, all calculations are from the perspective of party *A*.

The risk free value of the financial contract is given by

$$V^F(t) = E[D(t,T)X_T] = E\left[\exp\left[-\int_t^T r(u)du\right]X_T\right] \tag{1}$$

where $E[]$ denotes the risk-neutral expectation, $D(t,T)$ denotes the risk-free discount factor at t for the maturity T, and $r(u)$ denotes the risk-free short rate at time u ($t \le u \le T$).

1.1 Without a collateral agreement

Assume that party A and party B do not have a CSA agreement. The binomial default rule considers only two possible states: default or survival. For the discrete one-period (t, T) economy, at the end of the period the financial contract either survives with the survival probability $p(t,T)$ or defaults with the default probability $q(t,T)$ where $q(t,T) + p(t,T) = 1$. The survival payoff is equal to the promised payoff X_T itself and the default payoff is a fraction of the promised payoff given by φX_T , where φ denotes the default recovery rate. Therefore, the risky value or the non-CSA value of the contract is the discounted expectation of the payoffs and is given by

$$V^{NC}(t) = E\left[D(t,T)\left(q(t,T)\varphi\, X_T + p(t,T)X_T\right)\right] \tag{2}$$

If we assume that default probabilities, interest rates, and recovery rates are uncorrelated, the equation (2) can be further expressed as,

$$V^{NC}(t) = \left[\bar{p}(t,T) + \bar{q}(t,T)\varphi\right]E\left[D(t,T)X_T\right] = \left[\bar{p}(t,T) + \bar{q}(t,T)\varphi\right]V^F(t) \tag{3}$$

where $\bar{p}(t,T) = E\left[p(t,T)\right]$ and $\bar{q}(t,T) = E\left[q(t,T)\right] = 1 - \bar{p}(t,T)$ denote the expected survival probability and the expected default probability respectively. This formula therefore accounts for the fundamental value, the default and survival probabilities, and the recovery at default.

The difference between the risk-free value and the risky value is called the *credit value adjustment* (CVA). The CVA reflects the market value of counterparty risk or the cost of protection required to hedge the credit risk of counterparties and is given by

$$CVA(t) = V^F(t) - V^{NC}(t) = \bar{q}(t,T)\left[1-\varphi\right]V^F(t) \tag{4}$$

Since the recovery rate is always less than 1, we have $CVA(t) > 0$ or $V^{NC}(t) < V^F$, i.e., the risky value is less than the risk-free value. An intuitive explanation is that the credit risk or the potential default loss makes the financial contract less valuable.

1.2 With a collateral agreement

Next, assume that there is a CSA agreement between parties *A* and *B* in which only party *B* is required to deliver collateral when the mark-to-market (MTM) value arises over the collateral threshold *H*. This is a case of unilateral collateralization against unilateral credit risk.

Conceptually, collateralization may reduce the default probabilities of firms as their leverage has been reduced. This effect, however, is very difficult to measure. Therefore, in this paper, we follow the general market consensus (see ISDA (2005)): collateral does not turn a bad counterparty into a good one – it will have no effect on your counterparty's default probability and will not improve the counterparty's credit rating.

Under a CSA agreement, the collateral is called as soon as the MTM value rises above the given collateral threshold H , or more precisely, above the threshold amount plus the minimum transfer amount, where $H > 0$ corresponds to partial-collateralization, $H = 0$ corresponds to full-collateralization, and $H < 0$ corresponds to over-collateralization. If the value of the contract to party *A* is less than the collateral threshold, no collateral is required from party *B*. If the value of the contract to party *A* is greater than the threshold, the required collateral is equal to the difference between the value and the threshold. Thus, the collateral amount posted at time *t* is given by

$$C(t) = \begin{cases} V^C(t) - H & \text{if } V^C(t) > H \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

where $V^C(t)$ is the value of the collateralized contract at time *t*.

Let us first consider the case of $V^C(t) > H$ where $C(t) = V^C(t) - H$. The major benefit of collateralization should be viewed as an improved recovery in the event of a counterparty default. At time *T*, if the contract survives, the survival value is the promised payoff X_T and the collateral taker returns the collateral to the collateral provider. If the contract defaults, the collateral taker possesses the collateral and the default payoff is the future value of the collateral, i.e., $C(t)/D(t,T)$. Here we consider the time value of money only. It can be seen from this, that collateral does not have any bearing on the survival payoff; instead, it takes effect on the default payoff only. The CSA value of the contract is the discounted expectation of the payoffs and is given by

$$\begin{aligned} V^{C}(t) &= E\left[D(t,T)\left(q(t,T)C(t)/D(t,T)+p(t,T)X_{T}\right)\right] \\ &= \bar{q}(t,T)\left(V^{C}(t)-H\right)+\bar{p}(t,T)E\left[D(t,T)X_{T}\right] \end{aligned} \tag{6}$$

or equivalently,

$$V^{C}(t)=E\left[D(t,T)X_{T}\right]-H\bar{q}(t,T)/\bar{p}(t,T)=V^{F}(t)-H\bar{o}(t,T) \tag{7}$$

where $\bar{o}(t,T)=\bar{q}(t,T)/\bar{p}(t,T)$ denotes the default odds that are expressed as the ratio of the default probability to the survival probability. Since the default odds are an expression of relative default probabilities, their value range can be from 0 to positive infinity. As we can see from equation (7), *the collateralization has a significant effect on the value of the contract as the threshold H moves away from zero*.

The default odds are just an alternative way of expressing the likelihood of default. In gambling, the odds do not represent the true chances that the event will occur, but are the amounts that the bookmaker will pay out on the winning bets. We may think of $H\bar{o}(t,T)$ as the cost/payout of bearing the unsecured credit risk. Equation (7) says that the value of the collateralized contract is equal to the risk-free value minus the cost of taking the unsecured credit risk. This formula therefore accounts for the fundamental value, the relative default probability, and the collateralization. One sanity check is that the recovery rate should not appear in collateralized valuation because the non-default party has recourse to the collateral in the event of a default.

The difference between the CSA value and the non-CSA value reflects the benefit (a measure of how much collateralization may reduce credit risk) of the collateral arrangement, given by

$$BNT=V^{C}(t)-V^{NC}(t)=\bar{q}(t,T)(1-\varphi)V^{F}(t)-H\bar{o}(t,T)=CVA(t)-H\bar{o}(t,T) \tag{8}$$

We discuss the following three cases: Case 1: $H=0$ corresponds to full-collateralization. According to equation (7), we have $V^{C}(t)=V^{F}(t)$ when $H=0$, i.e., the CSA value under full-collateralization is equal to the risk-free value. In other words, *full-collateralization can eliminate the* *unilateral credit risk completely*. This is consistent with the market practice where market participants commonly assume fully collateralized contracts are risk-free. Full-collateralization is increasingly popular at the transaction level. From equation (8), we obtain $BNT = CVA$ when $H = 0$, i.e., the benefit of full-collateralization is exactly equal to the CVA. Said differently, *full-collateralization can entirely cover the cost of protection required to hedge unilateral credit risk.*

Case 2: $H > 0$ corresponds to partial-collateralization. According to equation (7), we get $V^C(t) < V^F(t)$ when $H > 0$, i.e., the CSA value under partial-collateralization is less than the risk-free value. From equation (8), we have $BNT < CVA$ when $H > 0$, i.e., the benefit of partial-collateralization can not completely offset the cost of hedging the credit risk. Partial-collateralization that reflects the risk tolerance and commercial intent of the firms is commonly seen at the portfolio level, because some products in the portfolio are collateralized and others are not.

Case 3: $H < 0$ corresponds to over-collateralization. According to equation (7), we have $V^C(t) > V^F(t)$ when $H < 0$. It is worth noting that the CSA value under over-collateralization is actually greater than the risk-free value. Over-collateralization is typically a one-way obligation for an end user to post additional collateral to a dealer, primarily as a cushion to guard against the residual credit risks (e.g., the replacement cost may continue to increase during the close-out period). Although both parties are subject to these residual credit risks, typically only the dealer is protected against them. This market practice has been developed based on the role that dealers play in the market and their relative credit standing.

Since collateral is used to improve recovery and thus mitigate credit risk, the collateral amount, in principal, should be greater than the recovery value at default. Otherwise, collateralization loses legitimacy, because the non-default party would rather receive the recovery value than take the collateral. Equivalently, the CSA value should be greater than the non-CSA value of the same portfolio, i.e.,

$$V^C(t) \geq V^{NC}(t) \tag{9}$$

According to equations (3) and (7), inequality (9) can be further expressed as

$$V^F(t) - H\bar{q}(t,T)/\bar{p}(t,T) \geq [\bar{p}(t,T) + \bar{q}(t,T)\varphi]V^F(t) \tag{10}$$

or equivalently,

$$H \leq H^U := \bar{p}(t,T)(1-\varphi)V^F(t) \tag{11}$$

Inequality (11) gives an upper bound on *H*. If we regard $\bar{q}(t,T)(1-\varphi)V^F(t)$ as the expected default loss (the loss multiplied by the default probability), then we can think of $\bar{p}(t,T)(1-\varphi)V^F(t)$ as the unexpected/complement default loss. Inequality (11) tells us that a well-designed collateral threshold should be less than the unexpected default loss.

If $H < H^U$, we have $V^C(t) > V^{NC}(t)$ (the CSA value is greater than the non-CSA value) and *BNT* > 0 (the benefit of the collateralization is positive) according to equation (8). This is a risk improvement situation where the collateral arrangement reduces credit risk. If $H > H^U$, we get $V^C(t) < V^{NC}(t)$ (the CSA value is less than the non-CSA value) and *BNT* < 0 (the benefit of the collateralization is negative). This is a risk deterioration situation where the collateral arrangement actually aggravates credit risk. Obviously, good intentions in this case turn out bad results. If $H = H^U$, we obtain $V^C(t) = V^{NC}(t)$ (the CSA value is equal to the non-CSA value) and *BNT* = 0 (the benefit of the collateralization is zero). This is a breakeven situation: no harm, no benefit. The above discussion further emphasizes the importance of quantifying collateralization.

Then, let us consider the case of $V^C(t) < H$ where $C(t) = 0$. At time *T*, if the contract survives, the survival payoff is the promised payoff X_T. If the contract defaults, the default payoff is 0 (the collateral amount is zero). The CSA value of the contract is the discounted expectation of the payoffs and is given by

$$V^C(t) = E[D(t,T)(C(t)q(t,T)/D(t,T) + p(t,T)X_T)] = \bar{p}(t,T)E[D(t,T)X_T] \tag{12}$$

We may think of $\bar{p}(t,T)E[D(t,T)X_T]$ as the expected survival value, which is independent of *H*. Equation (12) says that when *H* is greater than the expected survival value, the CSA value becomes irrelevant to H and is equal to the expected survival value itself. Consequently, the benefit of the collateral arrangement is given by

$$BNT = V^{C}(t) - V^{NC}(t) = -\bar{q}(t,T)\varphi E[D(t,T)X_T] = -\bar{q}(t,T)\varphi V^{F}(t) \tag{13}$$

Equation (13) leads to $BNT < 0$ and $V^{C}(t) < V^{NC}(t)$. Similar to the risk deterioration situation above, the collateral arrangement in this case actually increases credit risk.

1.3 **Numerical results**

We choose a very simple zero-coupon bond with a one-year maturity and a $1000 principal amount. We assume that i) there is a constant interest rate (continuously compounded) $r = 0.02$; ii) the issuer (party *B*) has a constant recovery rate of 60%; iii) the one-year survival probability of party *B* is 0.99; and iv) only party *B* is required to deliver collateral.

The risk-free value can be easily calculated as $980.2 according to equation (1); the non-CSA value as $976.3 according to equation (3); the upper bound as 388.2 according to equation (11); and the expected survival value as $970.4 according to equation (12). We can also compute the CSA value for any given collateral threshold *H* based on equation (7) or equation (12). Figure 1 plots the relationship between the collateral threshold and the CSA value. The x-axis represents the collateral threshold *H* and the y-axis represents the CSA value of the contract. Point A (0, $980.2) in this diagram corresponds to the full-collateralization where $H = 0$ and $V^{C}(t) = V^{F}(t) = 980.2$ (demonstrating that full-collateralization can eliminate unilateral credit risk completely). Point B (388.2, $976.3) illustrates the upper bound case where $H = H^{U} = 388.2$ and $V^{C}(t) = V^{NC}(t) = 976.3$. Point C (970.4, $970.4) represents the boundary situation where $V^{C}(t) = H = 970.4$.

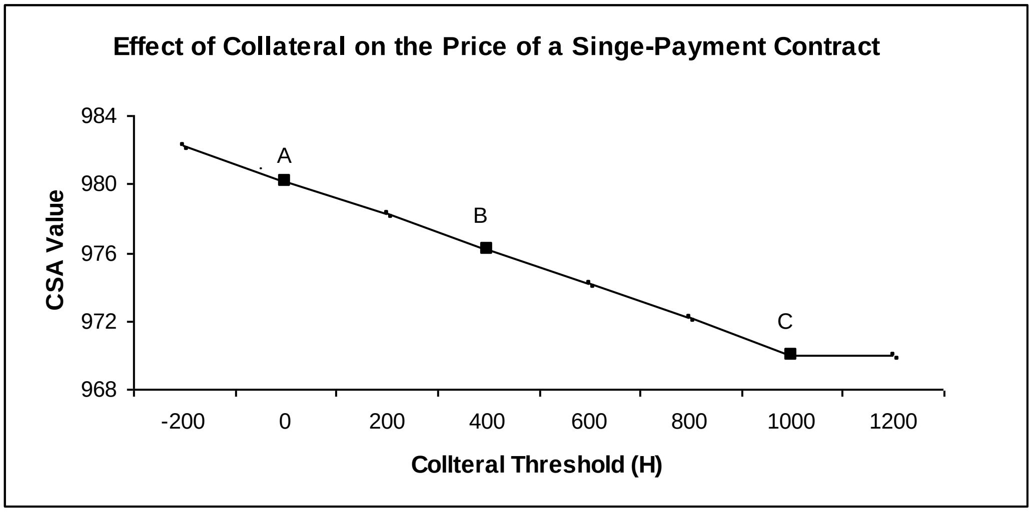

Figure 1. CSA Value of a Single Payment Contract vs. Collateral Threshold

This diagram illustrates the relationship between the CSA value and the collateral threshold. Point A (0, \$980.2) corresponds to the full-collateralization where the CSA value is equal to the risk-free value. Point B (388.2, \$976.3) represents the upper bound case where the CSA value is equal to the non-CSA value. Point C (970.4, \$970.4) illustrates the situation where the CSA value is equal to *H*.

The line in Figure 1 is divided into four parts by points A, B, and C. The line segment on the left side of point A corresponds to the over-collateralization where $H < 0$ and $V^C(t) > V^F(t)$. The line segment AB represents the risk improvement partial-collateralization where $0 < H < H^U$ and $V^C(t) > V^{NC}(t)$. The collateral arrangement in this case reduces credit risk as intended. The line segment BC exhibits the risk deterioration partial-collateralization where $H > H^U$ and $V^C(t) < V^{NC}(t)$. The collateral arrangement in this case actually increases credit risk. The line segment on the right side of Point C is flat, i.e., the CSA value is irrelevant to *H* when *H* is too big. It is worth to note that the value of a contract can not be less than the expected survival value.

This example is very simple, but it shows several essential features of pricing a collateralized contract and quantifying a collateral arrangement. It also provides the intuition for the more general results. In the following sections, we will develop the idea illustrated in this example into a comprehensive quantitative framework in a more rigorous manner.

2. How Does Unilateral Collateralization Affect Unilateral Credit Risk?

We consider a filtered probability space (Ω , $\mathcal{F}$, $\{\mathcal{F}_t\}_{t\geq 0}$, $\mathcal{P}$) satisfying the usual conditions, where Ω denotes a sample space, $\mathcal{F}$ denotes a σ -algebra, $\mathcal{P}$ denotes a probability measure, and $\{\mathcal{F}_t\}_{t\geq 0}$ denotes a filtration.

In the reduced-form framework, the stopping (or default) time τ_i of firm *i* is modeled as a Cox arrival process (a doubly stochastic Poisson process) whose first jump occurs at default and is defined by,

$$\tau_i = \inf\left\{ t : \int_0^t h_i(\Gamma_s) ds \geq \Delta_i \right\} \tag{14}$$

where $h_i(\Gamma_s)$ denotes the stochastic hazard rate or arrival intensity dependent on an exogenous common state Γ_s , and Δ_i is a unit exponential random variable independent of Γ_s . Dependence between the default times is only introduced by the dependence of the intensity $h_i(\Gamma_s)$ on a common process Γ_s . Consequently, conditional on the path of Γ_s , defaults are independent, which is the reason why this setup is also often called the conditional independence setup.

It is well-known that the survival probability from time *t* to *s* in this framework is defined by

$$p_i(t,s) := P_i(\tau > s \mid \tau > t, \Gamma) = \exp\left(-\int_t^s h_i(u) du \right) \tag{15a}$$

The default probability for the period (*t*, *s*) in this framework is defined by

$$q_i(t,s) := P_i(\tau \leq s \mid \tau > t, \Gamma) = 1 - p_i(t,s) = 1 - \exp\left(-\int_t^s h_i(u) du \right) \tag{15b}$$

Applying the law of iterated expectations, we express the expected survival probability for the period (*t*, *s*) as

$$\bar{p}_i(t,s) = E\left[\exp\left(-\int_t^s h_i(u) du \right) \middle| \mathcal{F}_t \right] \tag{16a}$$

where $E\{\bullet|\mathcal{F}_t\}$ is the expectation conditional on the $\mathcal{F}_t$. The expected default probability for the period (*t*, *s*) is expressed as

$$\bar{q}_i(t,s) = 1 - \bar{p}_i(t,s) = 1 - E\left[\exp\left(-\int_t^s h_i(u)du\right)\middle|\mathcal{F}_t\right] \tag{16b}$$

Two parties are denoted as *A* and *B*. The unilateral credit risk assumes that only one party is defaultable and the other one is default-free. In this section, we assume that party *A* is default-free, whereas party *B* is defaultable. The unilateral CSA agreement between parties *A* and *B* only requires party *B* to deliver collateral when the MTM value arises over the collateral threshold *H*. For reasons that will become clear shortly, we focus initially on single payment cases.

Suppose that a contract has *m* cash flows. Let the *m* cash flows be represented as $X_i > 0$ with payment dates T_i, where *i* = 1,…,*m*.

Extending equation (22) from one-period to multiple-periods, we derive the following proposition:

Proposition 2: *The non-CSA value of the multiple payment contract subject to unilateral credit risk is given by*

$$V^{NC}(t) = \sum_{i=1}^{m} E\left[\left(\prod_{j=0}^{i-1} G(T_j, T_{j+1})\right) X_i \middle| \mathcal{F}_t\right] \tag{26a}$$

where $t = T_0$ *and*

$$G(T_j, T_{j+1}) = D(T_j, T_{j+1})\left[p(T_j, T_{j+1}) + q(T_j, T_{j+1})\varphi(T_{j+1})\right] \tag{26b}$$

Proof: See the Appendix.

This is a closed-form solution. Proposition 2 says that for a non-CSA contract subject to unilateral credit risk, we can evaluate each payoff separately and sum the corresponding results. In other words, payoffs in this case can be treated as independent.

Using a similar derivation as in Proposition 2, we can extend Proposition 1 from one-period to multiple-periods. The CSA value of the multiple payment contract subject to unilateral credit risk is given by

$$V^C(t) = \sum_{i=1}^{m} E\left[\prod_{j=0}^{i-1}\left(K(T_j, T_{j+1})\right)X_i \mid \mathcal{F}_t\right] - \sum_{i=0}^{m-1} E\left[\prod_{j=0}^{i-1}\left(K(T_j, T_{j+1})\right)J(T_i, T_{i+1}) \mid \mathcal{F}_t\right] \quad (27a)$$

where

$$K(T_j, T_{j+1}) = p(T_j, T_{j+1})D(T_j, T_{j+1})\left[1 + 1_{H<L(T_j,T_{j+1})}\,\bar{q}(T_j, T_{j+1}) / \bar{p}(T_j, T_{j+1})\right] \quad (27b)$$

$$J(T_j, T_{j+1}) = 1_{H<L(T_j,T_{j+1})}\,H\bar{q}(T_j, T_{j+1}) / \bar{p}(T_j, T_{j+1}) = 1_{H<L(T_j,T_{j+1})}\,H\bar{o}(T_j, T_{j+1}) \quad (27c)$$

$$L(T_j, T_{j+1}) = E\left[K(T_j, T_{j+1})\left(V^C(T_{j+1}) + X_{j+1}\right) \mid \mathcal{F}_t\right] \quad (27d)$$

The valuation in equation (27) is complex. The intermediate values are vital to determine the final price. For a payment period, the current price has a dependence on the future price. Only on the final payment date T_m, the value of the contract and the maximum amount of information needed to determine the $L(T_{m-1}, T_m)$, $K(T_{m-1}, T_m)$ and $J(T_{m-1}, T_m)$ are revealed. This type of problem can be best solved by working backwards in time, with the later value feeding into the earlier ones, so that the process builds on itself in a recursive fashion, which is referred to as *backward induction*. The most popular backward induction valuation algorithms are lattice/tree and regression-based Monte Carlo.

The benefit of the collateral arrangement is given by $BNT = V^C(t) - V^{NC}(t)$. We can solve the upper bound H^U numerically according to the boundary condition $BNT \geq 0$. A carefully designed collateral arrangement should meet $H \leq H^U$.

3. How Does Bilateral Collateralization Affect Bilateral Credit Risk?

Bilateral credit risk or counterparty risk arises in connection with OTC derivatives and SFT, since both contract parties are exposed to default risk. The hypothesis of a unilateral counterparty risk has been seen in the past as a practical estimate for modeling contracts between major financial institutions and their clients. But the realization that even the most prestigious investment banks could go bankrupt has shattered the foundations for resorting to unilateral models. The clients of banks are nowadays prone to question such an assumption and are willing to ask for suitable adjustments of contractual terms in order to gain a better security on their financial contracts, as well as on their collaterals, in the event of a counterparty default.

Bilateral collateral arrangements enable the counterparties to pass collateral between each other to cover the MTM exposure of the specified contracts. Under a two-way agreement, the collateralization obligation is mutual and applicable to both counterparties.

There is ample evidence that corporate defaults are correlated. For example, companies in the same geographical region or producing the same type of products tend to be affected similarly by external events and as a result may experience financial difficulties at the same time. Default correlation refers to the tendency for two firms to default at the same time. Capturing default correlation in counterparty risk is critical.

Two counterparties are denoted as *A* and *B*. The binomial default rule considers only two possible states: default or survival. Therefore, the default indicator Y_j for party *j* (*j*=*A*, *B*) follows a Bernoulli distribution, which takes value 1 with default probability q_j, and value 0 with survival probability p_j, i.e., $P\{Y_j = 0\} = p_j$ and $P\{Y_j = 1\} = q_j$. The marginal default distributions can be determined by the reduced-form models. The joint distributions of a bivariate Bernoulli variable can be easily obtained via the marginal distributions by introducing extra correlations.

Consider a pair of random variables (Y_A, Y_B) that has a bivariate Bernoulli distribution. The joint probability representations are given by

$$p_{00} := P(Y_A = 0, Y_B = 0) = p_A p_B + \sigma_{AB} \quad (28a)$$

$$p_{01} := P(Y_A = 0, Y_B = 1) = p_A q_B - \sigma_{AB} \quad (28b)$$

$$p_{10} := P(Y_A = 1, Y_B = 0) = q_A p_B - \sigma_{AB} \quad (28c)$$

$$p_{11} := P(Y_A = 1, Y_B = 1) = q_A q_B + \sigma_{AB} \quad (28d)$$

where $\sigma_{AB} := E[(Y_A - q_A)(Y_B - q_B)] = \rho_{AB}\sigma_A\sigma_B = \rho_{AB}\sqrt{q_A p_A q_B p_B}$, $E(Y_j) = q_j \times 1 + p_j \times 0 = q_j$, and $\sigma_j^2 := E\left[(Y_j - q_j)^2\right] = q_j^2 p_j + p_j^2 q_j = p_j q_j$; ρ_{AB} denotes the default correlation coefficient of *A* and *B*.

Let valuation date be t. Consider a financial contract that promises to pay a X_T from party B to party A at maturity date T, and nothing before date T where $T > t$. The payoff X_T may be positive or negative, i.e. the contract may be either an asset or a liability to each party.

Let us first consider the case without a CSA. If $V(t+\Delta t) \geq 0$, there are a total of four possible states at time T: i) Both A and B survive with probability p_{00}. The contract value is equal to the payoff X_T. ii) A defaults but B survives with probability p_{10}. The contract value is a fraction of the payoff given by $\overline{\varphi}_B(T)X_T$ where $\overline{\varphi}_B$ represents the non-default recovery rate. $\overline{\varphi}_B$ =0 represents the one-way settlement rule, while $\overline{\varphi}_B$ =1 represents the two-way settlement rule. iii) A survives but B defaults with probability p_{01}. The contract value is a fraction of the payoff $\varphi_B(T)X_T$, where φ_B represents the default recovery rate. iv) Both A and B default with probability p_{11}. The contract value is a fraction of the payoff given by $\varphi_{AB}(T)X_T$, where φ_{AB} denotes the joint recovery rate when both parties A and B default simultaneously. A similar logic applies to the case of $V(t+\Delta t) < 0$. Therefore, the non-CSA value is the discounted expectation of the payoffs and is given by

$$\begin{aligned} V^{NC}(t) &= E\Big\{D(t,T)\Big[1_{X_T \geq 0}\langle p_{00}(t,T) + \varphi_B(T)p_{01}(t,T) + \overline{\varphi}_B(T)p_{10}(t,T) + \varphi_{AB}(T)p_{11}(t,T)\rangle X_T \big| \mathcal{F}_t \\ &\quad + 1_{X_T<0}\langle p_{00}(t,T) + \overline{\varphi}_A(T)p_{01}(t,T) + \varphi_A(T)p_{10}(t,T) + \varphi_{AB}(T)p_{11}(t,T)\rangle X_T \big| \mathcal{F}_t \Big]\Big\} \\ &= E\big(\Phi(t,T)X_T \big| \mathcal{F}_t\big) = E\big[D(t,T)\big(1_{X_T \geq 0}\phi_B(t,T) + 1_{X_T<0}\phi_A(t,T)\big)X_T \big| \mathcal{F}_t\big] \end{aligned} \tag{29a}$$

where

$$\begin{aligned} \phi_B(t,T) = {} & p_B(t,T)p_A(t,T) + \varphi_B(T)q_B(t,T)p_A(t,T) + \overline{\varphi}_B(T)p_B(t,T)q_A(t,T) \\ & + \varphi_{AB}(T)q_B(t,T)q_A(t,T) + \sigma_{AB}(t,T)\big(1 - \varphi_B(T) - \overline{\varphi}_B(T) + \varphi_{AB}(T)\big) \end{aligned} \tag{29b}$$

$$\begin{aligned} \phi_A(t,T) = {} & p_B(t,T)p_A(t,T) + \varphi_A(T)q_A(t,T)p_B(t,T) + \overline{\varphi}_A(T)p_A(t,T)q_B(t,T) \\ & + \varphi_{AB}(T)q_B(t,T)q_A(t,T) + \sigma_{AB}(t,T)\big(1 - \varphi_A(T) - \overline{\varphi}_A(T) + \varphi_{AB}(T)\big) \end{aligned} \tag{29c}$$

We may think of $\Phi(t,T)$ as the bilaterally risk-adjusted discount factor. Equation (29) tells us that the non-CSA price of a single payment contract subject to bilateral credit risk can be expressed as the present value of the payoff discounted by a bilaterally risk-adjusted discount factor that has a switching-type dependence on the sign of the payoff.

Next, we study the impact of collateral on this contract. The collateral amount at *t* is given by

$$C(t) = \begin{cases} V^C(t) - H_B & if\ V^C(t) > H_B \\ 0 & if\ H_A \le V^C(t) \le H_B \\ V^C(t) - H_A & if\ V^C(t) < H_A \end{cases} \tag{30a}$$

or

$$C(t) = 1_{V(t)>H_B}\left(V^C(t) - H_B\right) + 1_{V(t)<H_A}\left(V^C(t) - H_A\right) \tag{30b}$$

where $H_B \ge 0$ and $H_A \le 0$ are the collateral thresholds for parties *B* and *A*.

At time *T* there are several possible states: i) Both *A* and *B* survive with probability p_{00}. The value of contract is equal to the payoff X_T. ii) Either or both parties *A* and *B* default. The contract value is the future value of the collateral, i.e., $C(t)/D(t,T)$. Therefore, we have the following proposition.

Proposition 3*: The bilateral CSA value of the single payment contract subject to bilateral credit risk is given by*

$$V^C(t) = E\left[M(t,T)X_T \mid \mathcal{F}_t\right] - Q(t,T) \tag{31a}$$

where

$$M(t,T) = D(t,T)p_{00}(t,T)\left[1_{\beta(t,T)>H_B} / \bar{p}_{00}(t,T) + 1_{H_A \le \beta(t,T) \le H_B} + 1_{\beta(t,T)<H_A} / \bar{p}_{00}(t,T)\right] \tag{31b}$$

$$Q(t,T) = 1_{\beta(t,T)>H_B} H_B\left(1 - \bar{p}_{00}(t,T)\right)/\bar{p}_{00}(t,T) + 1_{\beta(t,T)<H_A} H_A\left(1 - \bar{p}_{00}(t,T)\right)/\bar{p}_{00}(t,T) \tag{31c}$$

where $\beta(t,T) = E\left[D(t,T)p_{00}(t,T)X_T \middle| \mathcal{F}_t\right]$, *and* $\bar{p}_{00}(t,T) = E\left[p_{00}(t,T) \middle| \mathcal{F}_t\right]$.

Proof: See the Appendix.

We may think of $M(t,T)$ as the bilaterally CSA adjusted discount factor and $Q(t,T)$ as the cost of bearing the unsecured credit risk. Proposition 3 tells us that the value of the bilaterally collateralized contract is equal to the present value of the payoff discounted by the bilaterally CSA adjusted discount factors minus the cost of taking the unsecured counterparty risk.

If we assume that default probabilities, interest rates, and recovery rates are uncorrelated, Proposition 3 can be further expressed as:

$$V^C(t) = \hat{O}(t,T)E\left[D(t,T)X_T \mid \mathcal{F}_t\right] - Q(t,T) = \hat{O}(t,T)V^F(t) - Q(t,T) \tag{32a}$$

where

$$\hat{O}(t,T) = 1_{\hat{\beta}(t,T)>H_B} + 1_{H_A \le \hat{\beta}(t,T) \le H_B}\, \overline{p}_{00}(t,T) + 1_{\hat{\beta}(t,T)<H_A} \tag{32b}$$

$$\hat{\beta}(t,T) = \overline{p}_{00}(t,T)E\left[D(t,T)X_T \middle| \mathcal{F}_t\right] = \overline{p}_{00}(t,T)V^F(t) \tag{32c}$$

In particular, if $H_A = H_B = 0$ (corresponding to bilateral full-collateralization), we have $\hat{O}(t,T) = 1$, $Q(t,T) = 0$, and $V^C(t) = V^F(t)$ according to equation (32), i.e., under full-collateralization the bilateral CSA value of the contract is equal to the risk-free value. In other words, *bilateral full-collateralization can completely eliminate bilateral credit risk.*

4. How Does Bilateral Collateralization Affect Multilateral Credit Risk?

The interest in the financial industry for the modeling and pricing of multilateral defaultable contracts arises mainly in two respects: in the management of credit risk at a portfolio level and in the valuation of credit derivatives.

Let us first discuss the three-party case. A CDS is a contract subject to trilateral credit risk where the three defaultable parties are counterparties *A*, *B* and reference entity *F*. CDS contracts are the most popular form of credit derivatives and are also the most important building blocks in the credit market. An understanding of credit derivatives therefore must be underpinned by a full understanding of the CDS.

A CDS is a contract that provides insurance against the risk of a default by the reference entity. The buyer of the CDS makes periodic payments to the seller until the end of the life of the CDS or until a credit event occurs. A credit event usually requires a final accrual payment by the buyer and a loss protection payment by the seller. The protection payment is equal to the difference between par and the price of the cheapest to deliver (CTD) asset of the reference entity on the face value of the protection.

Normally, a CDS is used to transfer the credit risk of a reference entity from one party to the other. The contract reduces the credit risk of the reference entity but gives rise to another form of risk:

counterparty risk. The counterparty risk has become the main concern in the CDS market. During the recent financial crisis, CDS was blamed for playing a pivotal role in the collapse of Lehman Brothers and the disintegration of AIG. The use of collateral for CDS has significantly increased since then. Indeed, almost all CDS contracts (>97%) are fully collateralized according to the ISDA Margin Survey (see ISDA (2010)). Can full-collateralization really guarantee that there is no risk of failure to pay in the CDS market?

The default indicator for party *j* (*j* = *A* or *B* or *F*) follows a Bernoulli distribution, which takes value 1 with default probability q_j, and value 0 with survival probability p_j. The joint probability representations of a trivariate Bernoulli distribution (see Teugels (1990) for details) are given by

$$p_{000} := P(Y_A = 0, Y_B = 0, Y_F = 0) = p_A p_B p_F + p_F \sigma_{AB} + p_B \sigma_{AF} + p_A \sigma_{BF} - \theta_{ABF} \tag{35a}$$

$$p_{100} := P(Y_A = 1, Y_B = 0, Y_F = 0) = q_A p_B p_F - p_F \sigma_{AB} - p_B \sigma_{AF} + q_A \sigma_{BF} + \theta_{ABF} \tag{35b}$$

$$p_{010} := P(Y_A = 0, Y_B = 1, Y_F = 0) = p_A q_B p_F - p_F \sigma_{AB} + q_B \sigma_{AF} - p_A \sigma_{BF} + \theta_{ABF} \tag{35c}$$

$$p_{001} := P(Y_A = 0, Y_B = 0, Y_F = 1) = p_A p_B q_F + q_F \sigma_{AB} - p_B \sigma_{AF} - p_A \sigma_{BF} + \theta_{ABF} \tag{35d}$$

$$p_{110} := P(Y_A = 1, Y_B = 1, Y_F = 0) = q_A q_B p_F + p_F \sigma_{AB} - q_B \sigma_{AF} - q_A \sigma_{BF} - \theta_{ABF} \tag{35e}$$

$$p_{101} := P(Y_A = 1, Y_B = 0, Y_F = 1) = q_A p_B q_F - q_F \sigma_{AB} + p_B \sigma_{AF} - q_A \sigma_{BF} - \theta_{ABF} \tag{35f}$$

$$p_{011} := P(Y_A = 0, Y_B = 1, Y_F = 1) = p_A q_B q_F - q_F \sigma_{AB} - q_B \sigma_{AF} + p_A \sigma_{BF} - \theta_{ABF} \tag{35g}$$

$$p_{111} := P(Y_A = 1, Y_B = 1, Y_F = 1) = q_A q_B q_F + q_F \sigma_{AB} + q_B \sigma_{AF} + q_A \sigma_{BF} + \theta_{ABF} \tag{35h}$$

where

$$\theta_{ABF} := E\big((Y_A - q_A)(Y_B - q_B)(Y_F - q_F)\big) \tag{35i}$$

We define a new statistic, *comrelation*, as a generalization of the concept of the correlation for three random variables as follows:

$$\zeta_{ABF} = \frac{E\left[(X_A - \mu_A)(X_B - \mu_B)(X_F - \mu_F)\right]}{\sqrt[3]{E\left|X_A - \mu_A\right|^3 \times E\left|X_B - \mu_B\right|^3 \times E\left|X_F - \mu_F\right|^3}} \tag{36}$$

where X_A, X_B, and X_F are three random variables; μ_A, μ_B, and μ_F are the means of X_A, X_B, and X_F.

According to the Holder inequality, we have

$$\left|E\left((X_A-\mu_A)(X_B-\mu_B)(X_F-\mu_F)\right)\right| \le E\left|(X_A-\mu_A)(X_B-\mu_B)(X_F-\mu_F)\right| \le \sqrt[3]{E\left|X_A-\mu_A\right|^3 \times E\left|X_B-\mu_B\right|^3 \times E\left|X_F-\mu_F\right|^3} \tag{37}$$

Obviously, the comrelation is in the range of [-1, 1]. Equation (35i) can be rewritten as

$$\theta_{ABF} := E\left((Y_A-q_A)(Y_B-q_B)(X_F-q_F)\right) = \zeta_{ABF}\sqrt[3]{E\left|Y_A-q_A\right|^3 \times E\left|Y_B-q_B\right|^3 \times E\left|Y_F-q_F\right|^3} = \zeta_{ABF}\sqrt[3]{\left|p_A q_A(p_A^2+q_A^2)p_B q_B(p_B^2+q_B^2)p_F q_F(p_F^2+q_F^2)\right|} \tag{38}$$

where $E\left|Y_j-q_j\right|^3 = \left|1-q_j\right|^3 q_j + \left|0-q_j\right|^3 p_j = p_j q_j(p_j^2+q_j^2)$, $E(Y_j) = q_j \times 1 + p_j \times 0 = q_j$, $j = A, B, F$.

Numerical and empirical results

In our study, we choose a 5-year CDS with a quarterly payment frequency and a $1,000,000 notional. Counterparty *A* buys a protection from counterparty *B* (dealer), i.e., party *A* pays a periodic premium to party *B* and, in exchange, receives a payoff if the reference entity *F* defaults.

Since the payoffs of a *CDS* are mainly determined by credit events, we need to characterize the evolution of the hazard rates. Here we choose the *Cox-Ingersoll-Ross* (CIR) model. The CIR process has been widely used in the literature of credit risk, given by

$$dh_t = a(b-h_t)dt + \sigma\sqrt{h_t}\,dW_t \tag{46}$$

where a denotes the mean reversion speed, b denotes the long-term mean, σ denotes the volatility, and W_t denotes the Brownian motion.

Table 4: Risk-Neutral Parameters for CIR Model

This table presents the risk-neutral parameters that are calibrated to the current market. 'A+100bps' represents a '100 bps' parallel shift in the A-rated *CDS* spreads.

Credit Quality	A	A+100bps	A+200bps	A+300bps
Long-Term Mean a	0.035	0.056	0.077	0.099
Mean Reverting Speed *b*	0.14	0.18	0.25	0.36

Volatility σ	0.022	0.028	0.039	0.056

The market data are shown in Table 1 (see https://finpricing.com/lib/IrBasisCurve.html). The calibrated parameters are shown in Table 4. The details of the calibration are beyond the scope of this paper.

We assume that i) the interest rates shown in table 1 are deterministic; ii) the reference entity F has an “A+200bps” credit quality; iii) both parties have a constant default recovery rate of 60%; iv) both parties have a constant non-default recovery rate of 100% (two-way settlement); and v) the joint recovery is 50%, i.e., $\varphi_{AB} = 0.5$.

First, we study the impact of counterparty risk on CDS premia. By definition, a breakeven CDS spread is a premium that makes the market value of a given CDS at inception zero. The effect of counterparty risk on the CDS premia is displayed in Table 5. We discuss the following three cases:

Case 1: There is no default correlation between the dealer and the reference entity. In this case, an increase in the dealer’s credit spread of 100 bps translates into a 0.8 bps decline in the CDS. Since a CDS contract could involve a very large payment by the protection seller to the protection buyer, people normally believe that the size of the effect of counterparty risk tends to be orders of magnitude larger than those for IRS. However, our model shows that without taking into account default correlations, the effect of counterparty risk on CDS premia is relatively small. This finding is in line with the findings of the empirical study of Arora et al (2010). They empirically find that an increase in the dealer’s credit spread of 645 bps only maps into 1 bps decrease on average in the dealer’s spread for selling credit protection.

Table 5: Impact of Counterparty Risk on CDS Premia

This table shows the impact of the dealer’s credit quality on the CDS premia. We assume that i) party A is risk-free and party B (dealer) is risky, and ii) $\rho_{AB} = \rho_{AF} = \theta_{ABF} = 0$. The default recovery rates for all parties are 0.6 and the joint recovery rate is 0.5. Reference entity F has an “A+200bps” credit quality. ‘A+200bps’ represents a ‘200 bps’ parallel shift in the A-rated *CDS* spreads.

Party *B*	A	A+100bps	A+200bps	A	A+100bps	A+200bps	A+100bps	A+100bps

Correlation ρ_{BF}	0	0	0	0.1	0.1	0.1	0.2	0.3
CDS premium	0.02694	0.02686	0.02677	0.02641	0.02603	0.02573	0.02519	0.02436

Case 2: There is a default correlation between the dealer and the reference entity (e.g., let $\rho_{BF} = 0.1$). We find that a rise in the dealer's credit spread of 100 bps results in a 3.4 bps decline in the CDS premium. The results indicate that CDS premia are more sensitive to counterparty risk in the presence of default correlations.

Case 3: We show the sensitivity of CDS premia to changes in the default correlation between the dealer and the reference entity: an increase in the default correlation of 0.1 (1000 bps) translates into an 8.3 bps decrease in the CDS premium. This is consistent with the economic intuition that a protection seller who is positively correlated with the reference entity (a wrong way risk) should charge a lower price for selling credit protection.

We repeat a similar exercise for the collateralized CDS premia where we assume that $H_A = 0$ and $H_B = 2000$. The results are presented in Table 6. Without taking into account the default correlation between the dealer and the reference entity, we find that an increase in the dealer's credit spread of 100 bps translates into a 0.2 bps decline in the collateralized CDS premium. However, if we take into account the default correlation (e.g., let $\rho_{BF} = 0.1$), a rise in the dealer's credit spread of 100 bps results in a 8 bps decline in the CDS premium. In addition, a hike in the default correlation of 0.1 causes a 21 bps decrease in the collateralized CDS premium.

Table 6: Impact of Counterparty Risk on Collateralized CDS Premia

This table displays the effect of the dealer's credit quality on the collateralized CDS premia where $H_A = 0$ and $H_B = 2000$. We assume that i) party A is risk-free and party *B* (dealer) is risky, and ii) $\rho_{AB} = \rho_{AF} = \theta_{ABF} = 0$. Reference entity *F* has an "A+200bps" credit quality. 'A+200bps' represents a '200 bps' parallel shift in the A-rated *CDS* spreads.

Party *B*	A	A+100bps	A+200bps	A	A+100bps	A+200bps	A+100bps	A+100bps

Correlation ρ_{BF}	0	0	0	0.1	0.1	0.1	0.2	0.3
CDS premium	0.02698	0.02696	0.02693	0.02563	0.02482	0.02403	0.02269	0.02056

It is worth noting that the impact of the default correlation on the collateralized CDS premium is much more substantial than that on the non-collateralized CDS premium. These numerical results clearly support our theoretical prediction that collateral arrangements that are originally designed and utilized for contracts subject to bilateral credit risk (e.g., an IRS) may not be suitable for contracts subject to multilateral credit risk (e.g., a CDS) in the presence of default correlations.

Next, we discuss the impact of different collateral arrangements on CDS premia. We assume that i) party *A* has an 'A+100bps' credit quality and party *B* has an 'A' credit quality; ii) $\theta_{ABF} = 0.23$, $\rho_{AB} = \rho_{AF} = \rho_{BF} = 0.1$; and iii) $\varphi_{AB} = 0.6$. The CDS premium is supposed to be 0.025. The counterparty-risk-free value can be easily computed as $7953.3 based on equation (43). The non-CSA value can be calculated as $5992.8 according to equation (42). Since the main risk of CDS is that the seller of protection is unable to pay in the case of a credit event, we are more interested in the effects of collateral posted by party *B* on the CDS value.

We use equation (44) to price the collateralized CDS. The effects of collateral on CDS values are shown in Figure 7. The relationship is convex rather than linear**.** Point A (-3900, $7953.3) in this diagram represents the over-collateralization where $H_B = -3900$ and $V^C(t) = V^F(t) = 7953.3$. Point B (0, $7254.8) illustrates the full-collateralization where $H_B = 0$ and $V^C(t) = 7254.8 < V^F(t)$ (demonstrating that full-collateralization can not completely eliminate counterparty risk for a CDS). Point C (12100, $5992.8) exhibits the upper bound case where $H_B = 12100$ and $V^C(t) = V^{NC}(t) = 5992.8$.

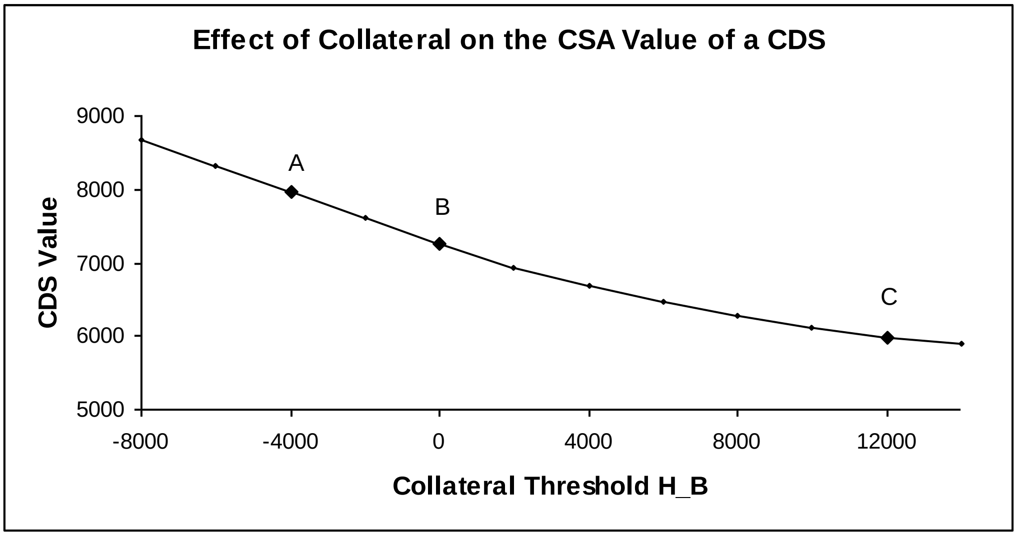

Figure 7. CSA Value of a CDS vs. Collateral Threshold H_B

This diagram illustrates the relationship between the CSA Value of a CDS and the collateral threshold H_B where $H_A = 0$. Point A (-3900, $7953.3) corresponds to the over-collateralization where the CSA value is equal to the risk-free value. Point B (0, $7254.8) represents the full-collateralization. Point C (12100, $5992.8) exhibits the upper bound case where the CSA value is equal to the non-CSA value.

From Figure 7, we can draw the following conclusions: i) Full-collateralization in the CDS market can not eliminate counterparty risk completely, i.e., under full-collateralization the CSA value of a CDS is not equal to the counterparty-risk-free value; ii) Only certain over-collateralization can entirely neutralize counterparty risk; iii) If $H_B < -3900$, the CSA value exceeds the risk-free value; If $-3900 < H_B < 12100$, the CSA value is greater than the non-CSA value, and the collateral arrangement in this case reduces credit risk; If $H_B > 12100$, the CSA value is less than the non-CSA value, and the collateralization in this case actually deteriorates credit risk.

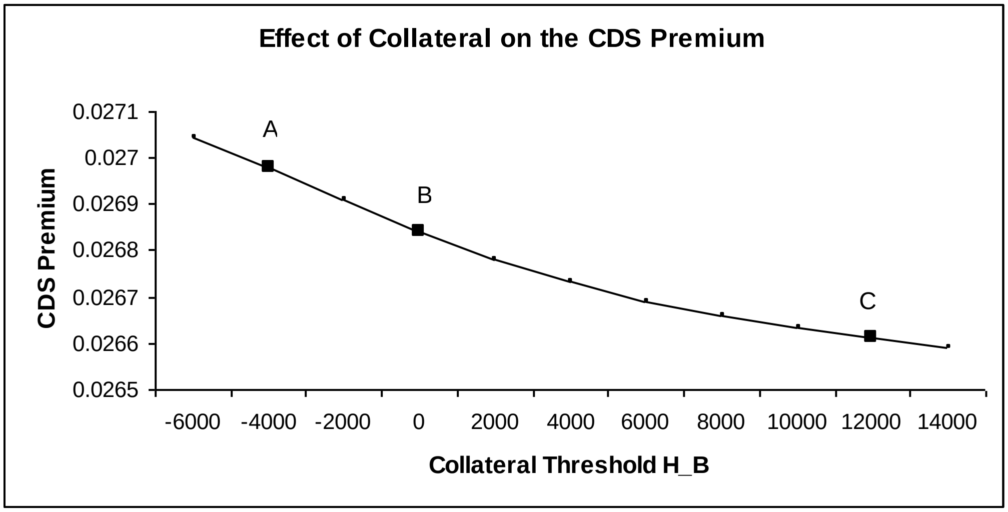

Figure 8. Collateralized CDS Premium vs. Collateral Threshold H_B

This diagram illustrates the relationship between the collateralized CDS premium and the collateral threshold H_B where $H_A = 0$. Point A (-3900, 0.0270) corresponds to the over-collateralization where the CSA value is equal to the risk-free value. Point B (0, 0.02684) represents the full-collateralization. Point C (12100, 0.02661) exhibits the upper bound case where the CSA value is equal to the non-CSA value.

We can further study the effects of collateral on CDS premia. We assume that the above CDS is a new trade and everything else remains the same. The 5-year counterparty-risk-free CDS premium is calculated as 0.0270. The non-CSA CDS premium is computed as 0.02661. The fully collateralized CDS premium is calculated as 0.02684. The effects of collateral on the CDS premia are shown in Figure 8.

Finally, using the same data as in the previous section, we conduct an empirical study. We assume that i) both party *B* (dealer) and reference entity *F* have a generic AA credit quality and party *A* is risk-free; and ii) $H_A = 0$. We discuss three collateral arrangements: $H_B = 0$ (full-collateralization), $H_B = 2000$ (partial-collateralization), or $H_B = -2000$ (over-collateralization).

We first consider the case where there is no correlation between the dealer and the reference entity. We have $\sigma_{AB} = \sigma_{AF} = \sigma_{BF} = \theta_{ABF} = 0$ since $\rho_{BF} = 0$ and $h_A = 0$. The time series plots of collateralized CDS spreads under different collateral arrangements are shown in Figure 9. The fully collateralized CDS spreads coincide with the market (counterparty-risk-free) CDS spreads. The difference between the partially collateralized CDS spread and the market CDS spread reflects the cost of bearing unsecured credit risk, whereas the difference between the over collateralized CDS spread and the market CDS spread represents the benefit of taking over-secured credit risk. The cost/benefit increases as the counterparty credit quality deteriorates.

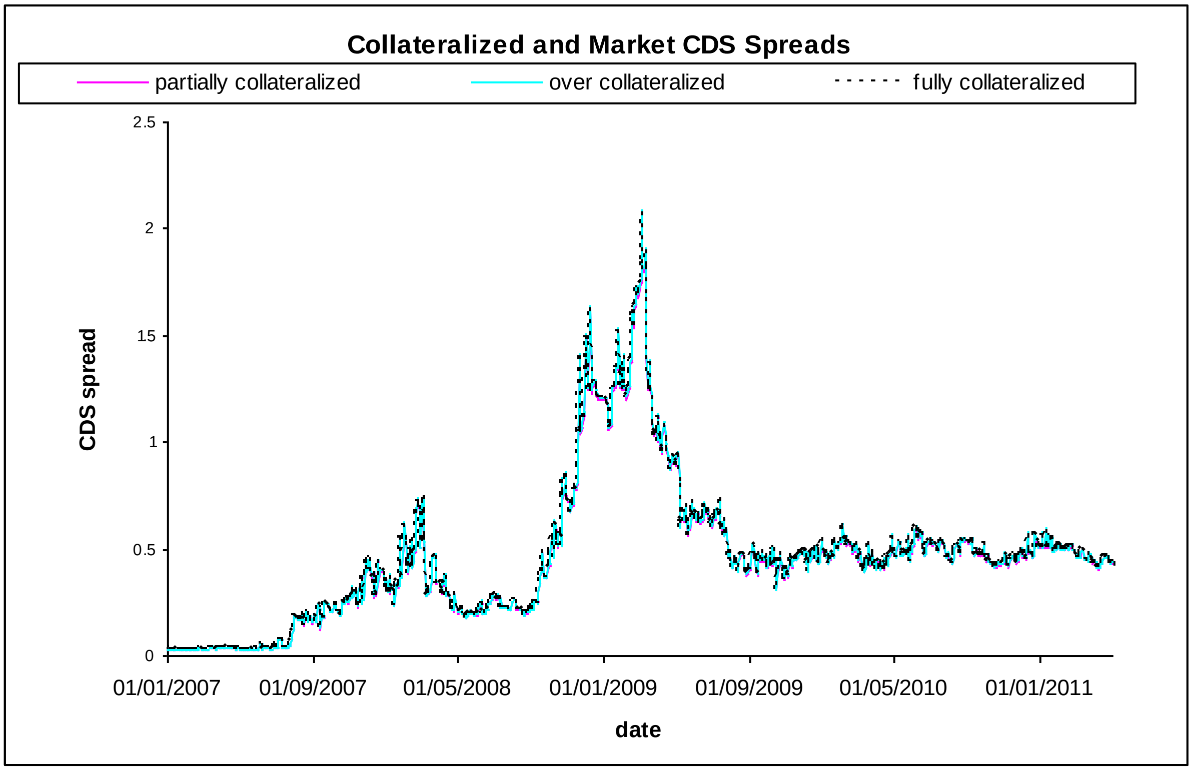

Figure 9. Time Series of Collateralized CDS Spreads (without Default Correlation)

This diagram illustrates the impact of collateralization on the time series of CDS spreads from January 1, 2007 to April 5, 2011, where the default correlation is zero, i.e., $\rho_{BF} = 0$. The three different arrangements are i) full-collateralization ($H_A = 0$ and $H_B = 0$), ii) partial-collateralization ($H_A = 0$ and $H_B = 2000$), and iii) over-collateralization ($H_A = 0$ and $H_B = -2000$). The fully collateralized CDS spreads coincide with the market CDS spreads.

Figure 9 exhibits significant time variations in the collateralized CDS spreads. Prior to July 2007, the CDS spreads are low and tranquil. As a result, the partially collateralized CDS spreads, the fully collateralized CDS spreads, and the over collateralized CDS spreads are almost coincident. But after July 2007, the CDS spreads rise dramatically. The difference between the partially collateralized CDS spread and the market CDS spread soars, and reaches a peak of -85 bps, while the difference between the over collateralized CDS spread and the market CDS spread surges, and hits a peak of 101 bps. These time variations are strong evidence that the cost/benefit of taking unsecured/over-secured credit risk is relative to the credit quality.

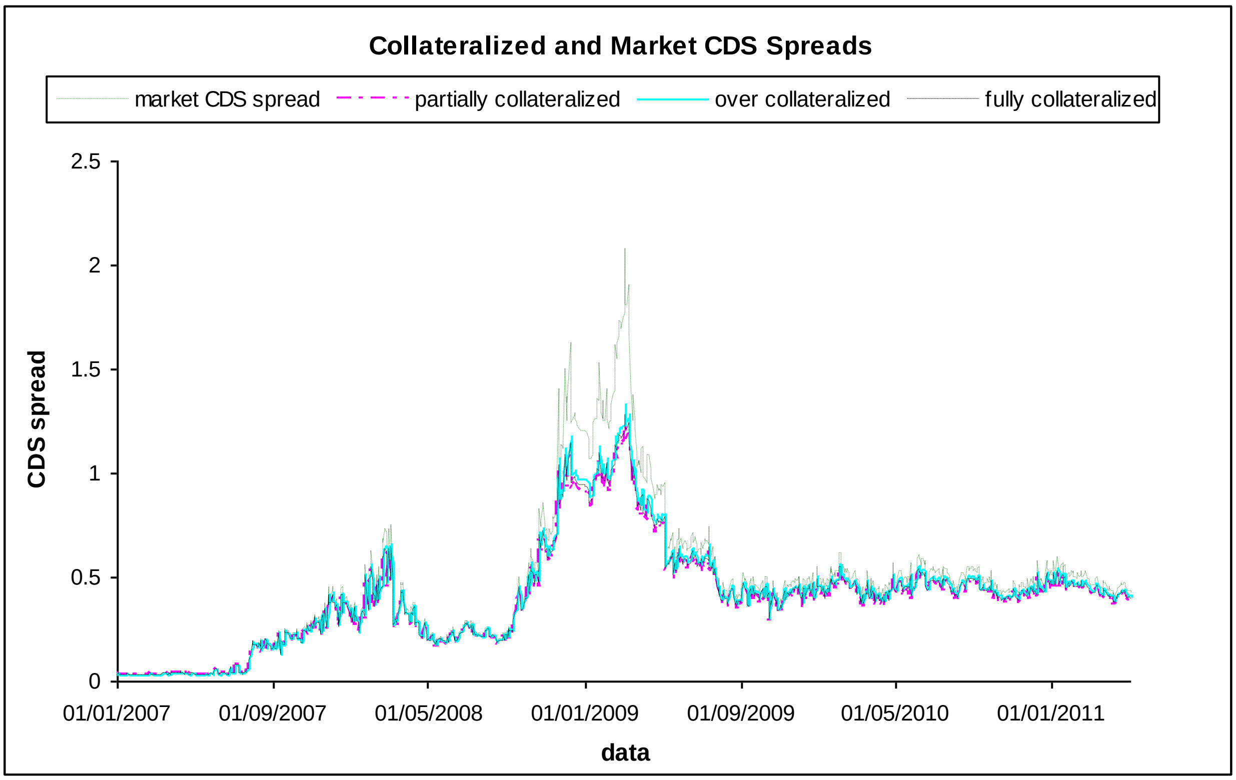

Figure 10. Time Series of Collateralized CDS Spreads (with Default Correlation)

This diagram illustrates the impact of collateralization on the time series of CDS spreads from January 1, 2007 to April 5, 2011 where the default correlation is not 0 (e.g., $\sigma_{BF} = 0.2$). The collateral arrangements are: full-collateralization ($H_A = 0$ and $H_B = 0$), partial-collateralization ($H_A = 0$ and $H_B = 2000$), and over-collateralization ($H_A = 0$ and $H_B = -2000$).

We then consider the case where the correlation between the dealer and the reference entity is not 0 (for instance, let $\sigma_{BF} = 0.2$). The empirical results are displayed in Figure 10. From the time series plots, we find that the default correlation has a substantial effect on the CDS spreads. During the financial crisis, the difference between the partially collateralized CDS spread and the market CDS spread reaches a peak of -8412 bps; the difference between the over collateralized CDS spread and the market CDS spread hits a record of -7378 bps; and the difference between the fully collateralized CDS spread and the market CDS spread rises to a crescendo of -7916 bps. In general, the party buying default protection should worry about the default correlation between the reference entity and the default protection seller.

5. **Conclusion and Discussion**

This article addresses a very important topic of the impact of collateralization on asset prices and risk management. This is the so called plumbing of financial system that affects many outcomes. To the best of our knowledge, our study is the first of its kind, attempting to provide a thorough quantitative analysis of the economic advantages and disadvantages of different collateral arrangements in a unified way.

The prevailing beliefs in financial markets are that collateralization can always mitigate credit risk and furthermore full-collateralization can eliminate credit risk completely. Our findings challenge this view. We find that a poorly designed collateral agreement can actually increase credit risk. We also find that although full-collateralization can eliminate counterparty risk completely for contracts subject to bilateral credit risk (e.g., an IRS), it can not get rid of counterparty risk entirely for contracts subject to multilateral credit risk (e.g., a CDS).

Empirically, we find strong evidence that collateralization affects swap rates and CDS premia. The effects are time varying. In particular, the effects of the default correlations between dealers and reference entities on collateralized CDS premia are substantial.

The results further emphasize the importance of carefully designing and quantifying collateral arrangements in order to make the right business decisions. These findings may be of interest to regulators, academics and practitioners.

Second, we assume that it is costless to post and maintain collateral. In fact, there are many costs inherent in a collateral process, mainly financial costs and operational costs. Financial costs include initial and ongoing legal expenses associated with the negotiation process and the development and maintenance of necessary documentation. Financial costs should also contain custodian charge fees for safekeeping of collateral and fees for delivery or receipt of collateral. The interest rate differential on cash investments may be an additional financial cost. Operational costs include system development and enhancement, operation maintenance, and any logistic support. In general, each counterparty absorbs the cost of holding the other's collateral, and these costs are generally understood to cancel each other out since the collateral terms will often be the same for either party. However, if the cost is not negligible, how can one measure and calibrate it?

Third, the practice of reusing posted collateral in another transaction has become extremely widespread and is generally referred to as "*rehypothecation*". Rehypothecation is a practice that occurs principally in the financial markets, where a bank or other broker-dealer reuses the collateral pledged by its clients as collateral for its own borrowing. Rehypothecation can generate a liquidity risk for the collateral provider through excess collateralization as a result of either a lag in collateral delivery or haircuts on securities posted as collateral. How can one model rehypothecation?

Finally, there is a time lag, called margin period of risk, which is the time period during which the institutions would execute a replacement trade. It is usually assumed to be the sum of call period and cure period. The call period is the time period that defines the frequency at which collateral is monitored and called for. The cure period is the time interval between the time when the counterparty ceases to post collateral and an early termination event being declared by the dealer. The margin period of risk exposes firms with additional exposure above the threshold. How does this time lag affect valuation and risk?

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