Stability and fairness in sequencing games: optimistic approach and pessimistic scenarios

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Stability and fairness in sequencing games: optimistic approach and pessimistic scenarios

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Abstract

Sequencing deals with the problem of assigning slots to agents who are waiting for a service. We study sequencing problems as coalition form games defined in optimistic and pessimistic scenarios. Each agent's level of utility is his Shapley value payoff from the corresponding coalition form game. First, we show that while the core of the optimistic game is always empty, the Shapley value of the pessimistic game is an allocation in its core. Second, we impose the "generalized welfare lower bound" (GWLB) that ex-ante guarantees each agent a minimum level of utility. One of many application of GWLB is the "expected costs bound". It guarantees each agent his expected cost when all arrival orders are equally likely. We prove that the Shapley value payoffs (in both optimistic and pessimistic scenarios) satisfy GWLB if and only if it satisfies the expected costs bound (ECB).

Keywords: Sequencing, welfare lower bounds, core, cooperative game, Shapley value

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1 Introduction

There is a finite set of agents who are in need of a service. Each agent has one job to process and the service provider can serve one agent at a time. A job can not be interrupted once it starts getting processed. Each agent is identified by two parameters namely, his per unit time waiting cost and his job processing time. We allow both the parameters to vary across agents. An agent's urgency is defined as the ratio of his per unit time waiting cost to his job processing time. Each agent is assigned a position and makes a monetary contribution to the service provider (can either be a payment or reward). Preferences are defined over pairs consisting of a position and a monetary contribution. They are quasi-linear. A rule is a mapping that specifies for each problem of this type, an order in which agents are served and a list of monetary contributions. Efficiency entails minimizing the aggregate job completion cost by serving agents in a non-increasing ordering of their urgency indices.

This paper models sequencing problems as cooperative games. Each sequencing problem is associated to a coalition form game and the worth of a coalition is defined based on the priority of serving its members. First, we define the optimistic scenario where the members of a coalition are always served first. The optimistic worth of this coalition is the minimum job completion cost of serving its members, as if they are the first to arrive. Second, we define the pessimistic scenario where the members of a coalition are always served last. The pessimistic worth of this coalition is the minimum job completion cost of its members if they are the last to arrive and the non-coalitional members are served before them. We calcuate the Shapley value payoffs for both these games. This allows us to design (for both the optimistic and pessimistic scenarios) a list of equitable monetary contributions when agents are served efficiently.

An allocation rule is a *Shapley value rule* if it selects those allocations that assign to each agent, a utility level equal to his Shapley value payoff from a corresponding coalition form game. Banerjee et al. [3] introduce the "generalized welfare lower bound" in which each agent is guaranteed a minimum level of ex-post utility. This bound is a unifying representation of several specific lower bounds in the literature [1]. One specific application of this bound is the expected costs bound that guarantees each agent a utility level (in the absence of transfers) equal to his expected cost when all arrival orders are equally likely. We show that the Shapley value rule satisfies the generalized welfare lower bound if and only if it satisfies the expected costs bound.

We also study the set of core stable solutions of both the optimistic and pessimistic coalition form games. The core of the optimistic games is always empty. Whereas, the Shapley value of the pessimistic game belongs in its core.

2 Literature

The literature studies sequencing from both incentive and normative point of views [2]. Sequencing problems are a subclass of indivisible object allotment problems. This general class has been examined from the cooperative game point of view (see Abdulkadiroglu and Sonmez [1], Moulin [26]) as well as from the fair allocation perspective (see Alkan et al. [2], Cres and Moulin ([13], [12]); Tadenuma [31], Tadenuma and Thomson ([32], [33], [34]); Thomson [35]). Moulin [26] showed that in a rich class of problems of fair division with money, the Shapley value solution indeed has many normatively appealing properties.

Curiel et al. [14] are the first ones to study cooperation in sequencing situations. They introduce the 'equal gain splitting rule' in a cost saving game when customers are rearranged with the objective of minimizing the total cost. The equal gain splitting rule is shown to belong to the core of the corresponding coalition form game where the worth of a coalition is the maximal cost saving its members can ensure by rearranging themselves efficiently. They also calculate the Shapley value and the τ value of this game.

A special case of sequencing problems in which agents have identical job processing times

[1]See Moulin [25] and Yengin [36] (study identical costs bound (ICB)), Maniquet [19], Chun [6], Banerjee et. al [3], Kayi and Ramaekars [18], and Mitra [24] (study both identical costs bound (ICB) and expected costs bound (ECB)), Chun and Yengin [11] (study the k-welfare bounds)

[2]For sequencing problems with incentives, see Dolan [17], Mendelson and Whang [20], Suijs [30], Mitra ([22], [23]), De [15], Banerjee et al. [3]. For the normative studies, see Chun [5], Chun, Mitra and Mutuswami ([8], [9], [10]), Chun and Yengin [37], and De [16]

are the class of queueing problems. Maniquet [19] studies queueing problems as coalition form games and defines the worth of a coalition in an optimistic manner. Chun [6] takes an identical approach but defines the worth of a coalition in a pessimistic manner. The minimal (maximal) transfer rule has been derived by applying the Shapley value solution to the corresponding coalition form game and characterized axiomatically.

Mishra and Rangarajan [21] also characterize the Shapley value solution for sequencing games in the optimistic case. [7] explores the consistency and monotonicity axioms in sequencing problems and studies how the maximal transfer rule responds to changes in waiting cost and processing time. Moulin [27] studies scheduling problems in which agent have identical waiting costs but differ in their job processing times. The server can monitor the length of the job but not the identity of the user; thus leading to merging, splitting or partially transferring jobs to offer cooperative strategic opportunities. He shows that the Shapley value solution is merge proof, but not split proof.

3 The model

A finite set of agents $N = \{1, 2, \ldots, n\}$ are in need of a service. A facility provider processes their jobs but can only do so one job at a time. For each $i \in N$, agent i is identified by a pair of parameters $(\theta_i, l_i) \in \mathcal{R}^2_{++}$ where θ_i is his per unit time waiting cost and l_i is his job processing time. Let L_i denote the job completion time for agent i and τ_i be his consumption of money. Agent i has quasi linear preferences defined over $\mathcal{R}_{++} \times \mathcal{R}$. They are assumed to be continuous and strictly monotone with respect to money. Given (L_i, τ_i), agent i gets a utility of $u_i(L_i, \tau_i) = -\theta_i L_i + \tau_i$ where $-\theta_i L_i$ is his cost of job completion and τ_i is the amount of money he either pays or receives. A *sequencing problem* with agent set N is a list $\Omega = (\theta, l)$ where $\theta = (\theta_1, \ldots, \theta_n)$ is the vector of per unit waiting costs and $l = (l_1, \ldots, l_n)$ is the vector of job processing times. The set of all sequencing problems is given by $\mathcal{S}^N$.

An order is a bijection $\sigma : N \to N$ that assigns a position to each agent $i \in N$. For instance, if $\sigma_i = 3$ then i occupies the third position. Let Σ be the set of all possible serving orders on N. For a given order $\sigma \in \Sigma$, let $P_i(\sigma) = \{j \in N \setminus \{i\} \mid \sigma_j < \sigma_i\}$ be the set of predecessors of i and $F_i(\sigma) = \{j \in N \setminus \{i\} \mid \sigma_j > \sigma_i\}$ be the set of successors. Let $\tau \in \mathcal{R}^N$ specify each agent's consumption of money. For $\Omega \in \mathcal{S}^N$, an **allocation** is a pair $(\sigma, \tau) \in \Sigma \times \mathcal{R}^N$. An **allocation rule** ψ associates to each $\Omega \in \mathcal{S}^N$ a non-empty subset $\psi(\Omega)$ of allocations. Given $\sigma \in \Sigma$, the job completion time for agent i is $L_i(\sigma) = \sum_{j \in P_i(\sigma)} l_j + l_i$ and his cost of job completion is $C_i(\sigma) = \theta_i L_i(\sigma)$. For each $\Omega \in \mathcal{S}^N$ and each (σ, τ), utility of agent i is $u_i(\sigma, \tau_i) = -\theta_i L_i(\sigma) + \tau_i$.

The total cost of job completion is $C(\sigma) = \sum_{i \in N} \theta_i L_i(\sigma)$. Let the ratio of the waiting cost to the processing time of agent i, given by θ_i / l_i, be agent i's **urgency index**. For $\Omega \in \mathcal{S}^N$, an order $\sigma \in \Sigma$ is **efficient**[3] if $\sigma \in \operatorname{argmin}_{\sigma} C(\sigma)$. It is well known that efficiency can be achieved if and only if for each pair $i, j \in N$, $\theta_i / l_i > \theta_j / l_j \Rightarrow \sigma_i < \sigma_j$ (Smith [29]). In case of a tie between two agents $i, j \in N$, we first take the linear order $1 < 2 < \ldots < n$ on the set N and then pick the order σ with $\sigma_i < \sigma_j$ iff $i < j$. When we deal with a strict subset of agents $S \subset N$, the order σ is restricted to S and written as $\sigma_S \in \Sigma^S$ where Σ^S is the set of all possible orderings of agents in S.

4 Sequencing games - optimistic scenario

Let $\mathcal{C}^N$ be the set of all coalitional games with the agent set N. To analyse sequencing problems as cooperative games, we associate to each $\Omega \in \mathcal{S}^N$ a coalitional game $v \in \mathcal{C}^N$. The worth of a coalition can be calculated in two ways depending on whether its members are served first (optimistic scenario) or last (pessimistic scenario). The former has been studied by Maniquet [19]. In the optimistic definition, for each $S \subseteq N$, the worth $v^{Opt}(S)$ is the minimum job completion cost of S when the members of S are served first. Formally, for each $\Omega \in \mathcal{S}^N$ and each $S \subseteq N$,

$$v^{Opt}(S) = -\sum_{i \in S} \theta_i (l_i + \sum_{j \in P_i(\sigma_S)} l_j) = -\sum_{i \in S} \theta_i L_i(\sigma_S). \tag{1}$$

[3] In general, we define efficiency for an allocation. However, due to the assumption of quasi linear preferences it meaningful to speak of the efficiency of an ordering.

where $\sigma_S \in \Sigma^S$ is an efficient order on S when they are served before $N \backslash S$.

For a game $v \in \mathcal{C}^N$, the **burden** imposed by agent $i \in N$ on each $S \subseteq N \backslash \{i\}$ is defined by $v(S \cup \{i\}) - v(S)$. For $v^{Opt} \in \mathcal{C}^N$ we get,

$$v^{Opt}(S \cup \{i\}) - v^{Opt}(S) = -\theta_i(l_i + \sum_{j \in P_i(\sigma_{S \cup \{i\}})} l_j) - l_i \sum_{j \in F_i(\sigma_{S \cup \{i\}})} \theta_j$$

where $\sigma_{S \cup \{i\}} \in \Sigma^{S \cup \{i\}}$ is an efficient ordering over $S \cup \{i\}$. The burden imposed by i is the sum of his individual waiting cost and the cost imposed on his successors in $S \cup \{i\}$.

The Shapley value of an agent is the average burden he imposes on all coalitions when all possible permutations of the grand coalition are considered. In other words, it is the expected value of i's burden on each coalition when all possible orders are equally likely. For each $\Omega \in \mathcal{S}^N$ and each $i \in N$, the Shapley payoff assigned to agent i is

$$Sh_i(v^{Opt}) = \sum_{S \subseteq N \backslash \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} [v^{Opt}(S \cup \{i\}) - v^{Opt}(S)].$$

Using the primitives (θ, l) of the model, the following lemma gives us the Shapley payoff of agent $i \in N$.

Lemma 1. Let σ be an efficient ordering on N. For each $i \in N$, the Shapley value of i in v^{Opt} is

$$Sh_i(v^{Opt}) = -\theta_i l_i - \sum_{j \in P_i(\sigma)} \theta_i l_j / 2 - \sum_{j \in F_i(\sigma)} \theta_j l_i / 2. \tag{2}$$

For a set of agents N, we first define a game $u_T \in \mathcal{C}^N$ on a coalition $T \subseteq N$ before proving the lemma.

Definition 1. Let $T \subseteq N$. The **unanimity** game on T is the game (N, u_T) defined by setting $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise.

Remark 1. A coalitional game $v \in \mathcal{C}^N$ can be uniquely expressed as a linear combination of unanimity games, i.e., for each $v \in \mathcal{C}^N$ and each $S \subseteq N$ there exists $(\delta_S)_{S \subseteq N}$, such that, $v =$

$\sum_{S\subseteq N}\delta_S u_S$. For each $S \subseteq N$, δ_S is the dividend of S in v that is defined as follows: if $|S| = 1$, $\delta_S = v(S)$ and if $|S| > 1$, $\delta_S = v(S) - \sum\limits_{\substack{T\subset S\\ T\neq S}} \delta_T$.

Claim 1. Consider the game $v^{Opt} \in \mathcal{C}^N$. For each $S \subseteq N$, the dividends δ_S are

$$\delta_S = \begin{cases} -\theta_i l_i & \text{if } |S| = 1 \\ -min_{i,j\in S}\{\theta_i/l_i, \theta_j/l_j\} l_i l_j & \text{if } |S| = 2 \\ 0 & \text{if } |S| \geq 3 \end{cases} \tag{3}$$

Proof.

- Case 1: When $|S| = 1$, let $S = \{i\}$. We have $\delta_{\{i\}} = v^{Opt}(i) = -\theta_i l_i$.
- Case 2: When $|S| = 2$, let $S = \{i, j\}$ and $\theta_i/l_i \geq \theta_j/l_j$. We then have $\delta_{\{i,j\}} = v^{Opt}\{i, j\} - \delta_{\{i\}} - \delta_{\{j\}} = -\theta_j l_i = -min\{\theta_i/l_i, \theta_j/l_j\} l_i l_j$.
- Case 3: If $|S| = 3$, let $S = \{i, j, k\}$ and without loss of generality $\theta_i/l_i \geq \theta_j/l_j \geq \theta_k/l_k$. Let $\delta_{\{i,j,k\}} = v^{Opt}\{i, j, k\} - \delta_{\{i,j\}} - \delta_{\{j,k\}} - \delta_{\{i,k\}} - \delta_{\{i\}} - \delta_{\{j\}} - \delta_{\{k\}} = -\theta_j l_i - \theta_k(l_i + l_j) + \theta_j l_i + \theta_k l_j + \theta_k l_i = 0$.

By induction on the size of the coalition S, let us assume $\delta_{S'} = 0$ where $3 \leq |S'| \leq |S|$. Without loss of generality, let $S = \{1, 2, \ldots s\}$ be such that $\theta_1/l_1 \geq \theta_2/l_2 \geq \ldots \geq \theta_s/l_s$. Using the induction hypothesis, $\delta_S = v^{Opt}(S) - \sum_{T\subset S;|T|=2}\delta_T - \sum_{T\subset S;|T|=1}\delta_T = -\sum_{j\in S}\theta_j L_j(\sigma_S) + \sum_{j\in S}\theta_j(\sum_{m\in P_j(\sigma_S)} l_m) + \sum_{j\in S}\theta_j l_j = 0$. This proves the claim and we can now show Lemma 1.

Proof. The Shapley value of player $i \in N$ in the game v is given by $SV_i(v) = \sum\limits_{\substack{S\subseteq N\\ i\in S}} \frac{\delta_S}{|S|}$. By substituting Eq. (3) in this expression, we obtain

$$\begin{aligned} Sh_i(v^{Opt}) &= -\theta_i l_i - \frac{1}{2}\sum_{j\in N\setminus\{i\}} min\{\theta_i/l_i, \theta_j/l_j\} l_i l_j \\ &= -\theta_i l_i - \sum_{j\in P_i(\sigma)} \theta_i l_j/2 - \sum_{j\in F_i(\sigma)} \theta_j l_i/2. \end{aligned}$$

The desired conclusion. $\square$

Remark 2. For each $\Omega \in \mathcal{S}^N$ and each (σ, τ) where $\sigma \in \Sigma$ is an efficient ordering on N, the utility of each agent is his Shapley payoff from v^{Opt} (in Lemma 1). Agent i's consumption of money is

$$\tau_i = \sum_{j \in P_i(\sigma)} \theta_i l_j/2 - \sum_{j \in F_i(\sigma)} \theta_j l_i/2.$$

4.1 Core

This section provides insights on the existence of the core in the defined coalitional form games. We first provide a few preliminary definitions to understand the nature of allocations in the core and use the necessary and sufficient condition provided by Bondareva [4] and Shapley [28], for the core of a game to be non-empty. Let G^N denote the set of all coalitional form games with the player set N.

Given $v \in G^N$, we define an *outcome* of the game (a payoff vector) as an n-dimensional vector $x = (x_1, x_2, \ldots, x_n)$. Let, for each coalition $S \subseteq N$, $x(S)$ be the sum of individual payoffs assigned to the members of S. A payoff vector $x \in \mathcal{R}^n$ is *individually rational* for N if for each $i \in N$, we have $x_i \geq v(\{i\})$. It is *totally rational* if $x(N) = v(N)$. An *imputation* is a pay-off vector that is both individually and totally rational. The *core* of v is the set of all those imputations that satisfy $x(S) \geq v(S)$ for all non-empty coalitions $S \subset N$. The core of the game v is denoted by $C(v)$.

Definition 2. A collection $\Phi = \{T_1, T_2, \ldots T_k\} \subseteq 2^N$ of non-empty coalitions is *balanced* if for each $i \in N$, there exist positive numbers λ_j, $j \in \{1, 2, \ldots, k\}$, such that $\sum_{\substack{j \in \{1,2,\ldots,k\} \\ i \ni T_j}} \lambda_j = 1$.

Theorem 1. The core of the optimistic game v^{Opt} is always empty, that is $C(v^{Opt}) = \phi$.

Proof. We first show that the core of the game v^{Opt} is empty.

Let $\Phi = \{T_1, \ldots, T_k\}$ be a balanced family with corresponding balancing weights $\{\lambda_{T_j}\}_{T_j \in \Phi}$. The core of a game v is non-empty if and only if for all balanced collections $\Phi = \{T_1, \ldots, T_k\}$ and their corresponding balancing weights $\{\lambda_{T_j}\}_{T_j \in \Phi}$, the inequality $\sum_{T_j \in \Phi} \lambda_{T_j} v(T_j) \leq v(N)$ holds. For the game v^{Opt}, it follows that for each $T_j \subseteq N$, we have $v^{Opt}(T_j) = -\sum_{i \in T_j} \theta_i L_i(\sigma^*(\theta_{T_j}))$.

The left hand side of the above inequality can be expressed as,

$$
\begin{aligned}
\sum_{T_j \in \Phi} \lambda_{T_j} v^{Opt}(T_j) &= -\sum_{T_j \in \Phi} \lambda_{T_j} \Big[\sum_{i \in T_j} \theta_i L_i(\sigma^*(\theta_{T_j})) \Big] \\
&= -\sum_{T_j \in \Phi} \lambda_{T_j} \Big[\sum_{i \in T_j} \theta_i \Big(l_i + \sum_{k \in P_i(\sigma^*(\theta_N)) \cap T_j} l_k \Big) \Big] \\
&= -\sum_{i \in N} \theta_i \Big(\sum_{\substack{T_j \in \Phi \\ T_j \ni i}} \lambda_{T_j} \Big) l_i - \sum_{i \in N} \theta_i \Big[\sum_{k \in P_i(\sigma^*(\theta_N))} \Big(\sum_{\substack{T_j \ni i \\ T_j \ni k}} \lambda_{T_j} \Big) l_k \Big] \\
&= -\sum_{i \in N} \theta_i \Big(l_i + \sum_{k \in P_i(\sigma^*(\theta_N))} \Big(\sum_{\substack{T_j \ni i \\ T_j \ni k}} \lambda_{T_j} \Big) l_k \Big) \\
&> \sum_{i \in N} \theta_i \Big(l_i + \sum_{k \in P_i(\sigma^*(\theta_N))} l_k \Big) \\
&= v(N)
\end{aligned}
$$

This inequality established the result.

□

5 Sequencing games: pessimistic scenario

The worth of a coalition is defined using the pessimistic approach in Chun [6]. The worth of $S \subseteq N$ is denoted by $v^{Pes}(S)$ and is defined by taking the sum of its members' job completion cost in an efficient ordering provided the members of S are served after the members of $N \backslash S$. In other words, the members of S are the last to be served in the queue. Formally,

$$
v^{Pes}(S) = -\sum_{i \in S} \theta_i (L_i(\sigma_S) + \sum_{k \in N \backslash S} l_k) = v^{Opt}(S) - \sum_{i \in S} \theta_i (\sum_{k \in N \backslash S} l_k) \tag{4}
$$

where $\sigma_S \in \Sigma^S$ is an efficient ordering on S.

Lemma 2. Let σ be an efficient ordering over N. For each $i \in N$, the Shapley payoff of an agent i in game $v^{Pes} \in \mathcal{C}^N$ is

$$Sh_i(v^{Pes}) = -\theta_i(l_i + \sum_{j\neq i} l_j) + \sum_{j\in P_i(\sigma)} \theta_j l_i/2 + \sum_{j\in F_i(\sigma)} \theta_i l_j/2. \quad (5)$$

Proof.

Claim 2. Consider the game $v^{Pes} \in \mathcal{C}^N$. For each $S \subseteq N$, the dividends δ_S are ,

$$\delta_S = \begin{cases} -\theta_i(l_i + \sum_{j\neq i} l_j) & \text{if } |S| = 1 \\ max_{i,j\in S}\{\theta_i/l_i, \theta_j/l_j\} l_i l_j & \text{if } |S| = 2 \\ 0 & \text{if } |S| \geq 3 \end{cases} \quad (6)$$

Proof

- When $|S| = 1$, let $S = \{i\}$. We have $\delta_{\{i\}} = v^{Pes}(\{i\}) = -\theta_i(l_i + \sum_{j\neq i} l_j)$.
- If $|S| = 2$, let us $S = \{i, j\}$ and without loss of generality suppose that $\theta_i/l_i \geq \theta_j/l_j$. We then have $\delta_{\{i,j\}} = v^{Pes}\{i, j\} - \delta_{\{i\}} - \delta_{\{j\}} = \theta_i l_j = max\{\theta_i/l_i, \theta_j/l_j\} l_i l_j$.
- If $|S| = 3$ and let $S = \{i, j, k\}$ and without loss of generality let $\theta_i/l_i \geq \theta_j/l_j \geq \theta_k/l_k$. We define $\delta_{\{i,j,k\}} = v^{Pes}\{i, j, k\} - \delta_{\{i,j\}} - \delta_{\{j,k\}} - \delta_{\{i,k\}} - \delta_{\{i\}} - \delta_{\{j\}} - \delta_{\{k\}} = -\theta_i(l_j + l_k) - \theta_j(l_i + l_k) - \theta_k(l_i + l_j) + \theta_i(l_j + l_k) + \theta_j(l_i + l_k) + \theta_k(l_i + l_j) = 0$.

By induction on the size of the coalition S, let us assume $\delta_{S'} = 0$ where $3 \leq |S'| \leq |S|$. Without loss of generality, let $S = \{1, 2, \dots s\}$ be such that $\theta_1/l_1 \geq \theta_2/l_2 \geq \dots \geq \theta_s/l_s$. By induction hypothesis, $\delta_S = v^{Pes}(S) - \sum_{T\subset S;|T|=2} \delta_T - \sum_{T\subset S;|T|=1} \delta_T = -\sum_{i\in S} \theta_i(L_i(\sigma_S) + \sum_{k\in N\setminus S} l_k) - \sum_{i\in S} \theta_i(\sum_{j\in F_i(\sigma_S)} l_j) + \sum_{i\in S} \theta_i(l_i + \sum_{j\neq i} l_j)$. The term $\sum_{j\neq i} l_j$ in the last expression can be written as, $\sum_{j\in S\setminus\{i\}} l_j + \sum_{j\in N\setminus S} l_j$. Further, the expression $\sum_{j\in S\setminus\{i\}} l_j$ can be expressed as $\sum_{j\in P_i(\sigma_S)} l_j + \sum_{j\in F_i(\sigma_S)} l_j$. We prove the claim by rewriting $\sum_{j\neq i} l_j$ in terms of these expressions. $\square$

The Shapley value of player $i \in N$ in the game v is given by $SV_i(v) = \sum\limits_{\substack{S\subseteq N\\ i\in S}} \frac{\Delta_v(S)}{|S|}$. By substituting Eq. (6) in this expression, we obtain

$$\begin{aligned} Sh_i(v^{Pes}) &= -\theta_i(l_i + \sum_{j\neq i} l_j) + \frac{1}{2}\sum_{j\in N\setminus\{i\}} max\{\theta_i/l_i, \theta_j/l_j\} l_i l_j \\ &= -\theta_i \sum_{j\in N} l_j + \sum_{j\in P_i(\sigma)} \theta_j l_i/2 + \sum_{j\in F_i(\sigma)} \theta_i l_j/2 \end{aligned}.$$

This gives us the desired conclusion. $\square$

Remark 3. For each $\Omega \in \mathcal{S}^N$and each (σ, τ) where $\sigma \in \Sigma$ is an efficient ordering on N, each agent's utility is his Shapley payoff from the game v^{Pes} (given by Lemma 2). The consumption of money by each agent $i \in N$ is

$$\tau_i = \sum_{j\in P_i(\sigma^*(\theta_N))} \theta_j l_i/2 - \sum_{j\in F_i(\sigma^*(\theta_N))} \theta_i l_j/2.$$

5.1 Core

This section studies the existence of core in the above coalition form games. We show that the Shapley value belongs to the core of the game v^{Pes}.

Theorem 2. The Shapley value of the game v^{Pes} belongs to its core, that is, $Sh(v^{Pes}) \in C(v^{Pes})$.

Proof. To prove that $Sh(v^{Pes}) \in C(v^{Pes})$, we first show that the allocation $(Sh_1(v^{Pes}), \ldots, Sh_n(v^{Pes}))$ is an imputation. Using equation (4) we can write,

$$
\begin{aligned}
v^{Pes}(\{i\}) &= -\theta_i(l_i + \sum_{j\in N\setminus\{i\}} l_j) \\
&= -\theta_i \sum_{j\in N} l_j \\
&< \sum_{j\in P_i(\sigma^*(\theta))} \theta_j l_i/2 + \sum_{j\in F_i(\sigma^*(\theta))} \theta_i l_j/2 - \theta_i \sum_{j\in N} l_j \\
&= Sh_i(v^{Pes})
\end{aligned}
$$

Further,

$$
\begin{aligned}
Sh_i(v^{Pes}) &= -\sum_{i\in N}[\theta_i \sum_{j\in N} s_j - \sum_{j\in P_i(\sigma^*(\theta_N))} \theta_j s_i/2 - \sum_{j\in F(\sigma^*(\theta_N))} \theta_i s_j/2] \\
&= -\sum_{i\in N}[\theta_i(s_i + \sum_{j\in P_i(\sigma^*(\theta_N))} s_j + \sum_{j\in F_i(\sigma^*(\theta_N))} s_j) - \sum_{j\in P_i(\sigma^*(\theta_N))} \theta_j s_i/2 - \sum_{j\in F(\sigma^*(\theta_N))} \theta_i s_j/2] \\
&= -\sum_{i\in N}[\theta_i(s_i + \sum_{j\in P_i(\sigma^*(\theta_N))} s_j) + \sum_{j\in F_i(\sigma^*(\theta_N))} \theta_i s_j - \sum_{j\in P_i(\sigma^*(\theta_N))} \theta_j s_i/2 - \sum_{j\in F(\sigma^*(\theta_N))} \theta_i s_j/2] \\
&= -\sum_{i\in N}[\theta_i(s_i + \sum_{j\in P_i(\sigma^*(\theta_N))} s_j) + \sum_{j\in F_i(\sigma^*(\theta_N))} \theta_i s_j/2 - \sum_{j\in P_i(\sigma^*(\theta_N))} \theta_j s_i/2] \\
&= -\sum_{i\in N} \theta_i(s_i + \sum_{j\in P_i(\sigma^*(\theta_N))} s_j) - \sum_{i\in N}(\sum_{j\in F_i(\sigma^*(\theta_N))} \theta_i s_j)/2 + \sum_{i\in N}(\sum_{j\in P_i(\sigma^*(\theta_N))} \theta_j s_i)/2 \\
&= -\sum_{i\in N} \theta_i(s_i + \sum_{j\in P_i(\sigma^*(\theta_N))} s_j) \\
&= v^{Pes}(N).
\end{aligned}
$$

The next step is to prove that for all non-empty coalition $S \subset N$, we have $\sum_{i\in S} Sh_i(v^{Pes}) \geq$

$v^{Pes}(S)$. For any given coalition S we have the following,

$$\begin{aligned}
\sum_{i\in S} Sh_i(v^{Pes}) &= -\sum_{i\in S}\theta_i\Big(\sum_{j\in N}s_j\Big)+\sum_{i\in S}s_i\Big(\sum_{j\in P_i(\sigma^*(\theta_N))}\theta_j\Big)\Big/2+\sum_{i\in S}\theta_i\Big(\sum_{j\in F_i(\sigma^*(\theta_N))}s_j\Big)\Big/2\\
&= -\sum_{i\in S}\theta_i S_i(\sigma^*(\theta_N))+\sum_{i\in S}s_i\Big(\sum_{j\in P_i(\sigma^*(\theta_N))}\theta_j\Big)\Big/2-\sum_{i\in S}\theta_i\Big(\sum_{j\in F_i(\sigma^*(\theta_N))}s_j\Big)\Big/2\\
&= -\sum_{i\in S}\theta_i\Big(s_i+\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\in S}}s_j+\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\notin S}}s_j\Big)+\sum_{i\in S}s_i\Big(\sum_{j\in P_i(\sigma^*(\theta_N))}\theta_j\Big)\Big/2\\
&\qquad -\sum_{i\in S}\theta_i\Big(\sum_{j\in F_i(\sigma^*(\theta_N))}s_j\Big)\Big/2\\
&= -\sum_{i\in S}\theta_i\Big(s_i+\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\in S}}s_j\Big)-\sum_{i\in S}\theta_i\Big(\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\notin S}}s_j\Big)-\sum_{i\in S}\theta_i\Big(\sum_{\substack{j\in F_i(\sigma^*(\theta_N))\\ j\notin S}}s_j\Big)\Big/2\\
&\quad -\sum_{i\in S}\theta_i\Big(\sum_{\substack{j\in F_i(\sigma^*(\theta_N))\\ j\in S}}s_j\Big)\Big/2+\sum_{i\in S}s_i\Big(\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\in S}}\theta_j\Big)\Big/2+\sum_{i\in S}s_i\Big(\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\notin S}}\theta_j\Big)\Big/2\\
&= -\sum_{i\in S}\theta_i\Big(s_i+\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\in S}}s_j\Big)-\sum_{i\in S}\theta_i\Big(\sum_{j\in N\setminus S}s_j\Big)\Big/2\\
&\qquad +\Big[\sum_{i\in S}s_i\Big(\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\notin S}}\theta_j\Big)-\sum_{i\in S}\theta_i\Big(\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\notin S}}s_j\Big)\Big]\Big/2
\end{aligned}$$

Claim 3. For any $i\in S$, the term $\Big[\sum_{i\in S}s_i\Big(\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\notin S}}\theta_j\Big)-\sum_{i\in S}\theta_i\Big(\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\notin S}}s_j\Big)\Big]\Big/2\geq 0$

Proof. Without loss of generality, let us assume that $\theta_1/s_1\geq\ldots\geq\theta_n/s_n$. By the definition of outcome efficiency, for any $i\in N$ and for any $j\in P_i(\sigma*(\theta))$, where $\sigma^*(\theta)$ is an efficient ordering of agents in a non-increasing order of their urgency indices, we have $\theta_j/s_j\geq\theta_i/s_i$ implying $s_i\theta_j\geq s_j\theta_i$. We can thus say, $\sum_{i\in S}\Big[\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\notin S}}\Big(s_i\theta_j-\theta_i s_j\Big)\Big]\geq 0$ by the above argument.

Given the above claim, for any $S \subset N$ observe that,

$$\sum_{i \in S} Sh_i(v^{Pes}) - v^{Pes}(S) = \sum_{i \in S} \theta_i \Big(\sum_{j \in N \setminus S} s_j \Big)/2 + \Big[\sum_{i \in S} s_i \Big(\sum_{\substack{j \in P_i(\sigma^*(\theta_N)) \\ j \notin S}} \theta_j \Big) - \sum_{i \in S} \theta_i \Big(\sum_{\substack{j \in P_i(\sigma^*(\theta_N)) \\ j \notin S}} s_j \Big) \Big]/2$$
$$> 0$$

This completes the proof.

$\square$

6 Shapley value and GMWB

Definition 3. The *Shapley rule* is an allocation rule that serves agents in their efficient order and assigns transfers such that the payoff of an agent is his Shapley value in the corresponding coalition form game.

Definition 4 (Banerjee et.al [3]). An allocation rule ψ satisfies *expected costs bound* if for any $\Omega \in \mathcal{S}^N$ with $\frac{\theta_1}{s_1} = \frac{\theta_2}{s_2} = \ldots = \frac{\theta_n}{s_n}$, for each $(\sigma, \tau) \in \psi(\Omega)$ and for each $i \in N$ we have:

$$u_i \geq -\theta_i [s_i + \sum_{j \neq i} s_j/2]$$

This means the utility of each agent is atleast as much as their expected cost when no transfers are allowed and every possible ordering is equally likely.

Define $O_i : \mathcal{R}^n_{++} \rightarrow \mathcal{R}^n_{++}$. Given a sequencing problem $\Omega \in \mathcal{S}^N$, let $O_i(s)$ represent the welfare level of agent i and $O(s) := (O_1(s), \ldots, O_n(s)) \in \mathbb{R}^n$ be the welfare vector.

Definition 5 (Banerjee et.al [3]). An allocation rule ψ satisfies GWLB if for any $\Omega \in \mathcal{S}^N$ with an associated $O(s)$, for each $(\sigma, \tau) \in \psi(\Omega)$ and for each $i \in N$ we have:

$$\pi_i(\sigma, \tau_i) \geq -\theta_i O_i(s)$$

where,

$-\theta_i O_i(s)$ is the minimum utility guaranteed to i

The generalized welfare lower bound imposes a lower bound on each agent's utility function, in the form of a minimum guarantee. Banerjee et al. [3] have introduced this bound which is a unified and comprehensive representation of several specific lower bounds that have been previously examined in the literature. The expected costs bound is one application of this bound.

This section provides the necessary and sufficient condition for the Shapley value rule to satisfy the generalized welfare lower bound (GWLB) property.

Theorem 3. For both the optimistic and the pessimistic formulations, the Shapley value rule satisfies generalized welfare lower bound if and only if it satisfies expected costs bound.

Proof. Part A. The first part of the proof considers the corresponding characteristic form game under the optimistic approach given by v^{Opt}. The transfers are designed so that the utility of each agent $i \in N$ is given by the Shapley value $Sh_i(v^{Opt})$.

For a sequencing problem $\Omega \in \mathcal{S}^N$ with an associated $O(s)$, the utility of player $i \in N$ (corresponding to the Shapley value of the game v^{Opt}) will satisfy the GMWB property if, $Sh_i(v^{Opt}) \geq -\theta_i O_i(s)$ implying $-\theta_i s_i - \sum_{j \in P_i(\sigma^*(\theta_N))} \theta_i s_j/2 - \sum_{j \in F_i(\sigma^*(\theta_N))} \theta_j s_i/2 \geq -\theta_i O_i(s)$. Or, $\theta_i(O_i(s) - s_i) - \sum_{j \in P_i(\sigma^*(\theta_N))} \theta_i s_j/2 - \sum_{j \in F_i(\sigma^*(\theta_N))} \theta_j s_i/2 \geq 0$. Let, $O_i(s) = s_i + \sum_{j \neq i} s_j/2 + \epsilon_i$. Thus we have, $\theta_i \epsilon_i + \theta_i(\sum_{j \neq i} s_j/2) - \sum_{j \in P_i(\sigma^*(\theta_N))} \theta_i s_j/2 - \sum_{j \in F_i(\sigma^*(\theta_N))} \theta_j s_i/2 \geq 0$. This implies, $\sum_{j \in F_i(\sigma^*(\theta_N))} (\theta_i s_j - \theta_j s_i)/2 + \theta_i \epsilon_i \geq 0$. Or, $\sum_{j \in F_i(\sigma^*(\theta_N))} (u_i - u_j) + \theta_i \epsilon_i \geq 0$. We must have $\epsilon_i \geq 0$. This proves necessity.

For any $i \in N$, it is given that $O_i(s) \geq s_i + \sum_{j \neq i} s_j/2$. The utility of player i is given by his Shapley value $Sh_i(v^{Opt})$. For any such player, consider the expression $Sh_i(v^{Opt}) + \theta_i O_i(s) = -\theta_i s_i - \sum_{j \in P_i(\sigma^*(\theta_N))} \theta_i s_j/2 - \sum_{j \in F_i(\sigma^*(\theta_N))} \theta_j s_i/2 + \theta_i s_i + \theta_i \sum_{j \neq i} s_j/2$. Since $\sigma^*(\theta_N)$ is an efficient ordering of the members of the grand coalition (N), then for any agent $i \in N$, if an agent $j \in F_i(\sigma^*(\theta_N))$ we must have $\theta_i/s_i \geq \theta_j/s_j$. This means, $Sh_i(v^{Opt}) + \theta_i O_i(s) = \sum_{j \in F_i(\sigma^*(\theta_N))} (\theta_i s_j - \theta_j s_i) \geq 0$. For any $i \in N$, we have $Sh_i(v^{Opt}) \geq -\theta_i O_i(s)$. This proves sufficiency.

Part B. We define the associated cooperative game (v^{Pes}) using the pessimistic approach. The utility of each player $i \in N$ corresponds to the Shapley value of this game, $Sh_i(v^{Pes})$.

For a given sequencing problem $\Omega \in \mathcal{S}^N$ with an associated $O(s)$, the utility of player $i \in N$ (corresponding to the Shapley value of the game v^{Pes}) will satisfy the GMWB property if, $Sh_i(v^{Pes}) \geq -\theta_i O_i(s)$ implying $-\theta_i(\sum_{j\in N} s_j) + \sum_{j\in P_i(\sigma^*(\theta_N))} \theta_j s_i/2 + \sum_{j\in F_i(\sigma^*(\theta_N))} \theta_i s_j/2 \geq -\theta_i O_i(s)$. Or, $\theta_i(O_i(s) - \sum_{j\in N} s_j) + \sum_{j\in P_i(\sigma^*(\theta_N))} \theta_j s_i/2 - \sum_{j\in F_i(\sigma^*(\theta_N))} \theta_i s_j/2 \geq 0$. Let, $O_i(s) = s_i + \sum_{j\neq i} s_j/2 + \epsilon_i$. Thus, $\theta_i\epsilon_i - \theta_i(\sum_{j\neq i} s_j/2) + \sum_{j\in P_i(\sigma^*(\theta_N))} \theta_j s_i/2 + \sum_{j\in F_i(\sigma^*(\theta_N))} \theta_i s_j/2 \geq 0$. This then implies, $\sum_{j\in P_i(\sigma^*(\theta_N))}(\theta_j s_i - \theta_i s_j)/2 + \theta_i\epsilon_i \geq 0$. Or, $\sum_{j\in P_i(\sigma^*(\theta_N))}(u_j - u_i) + \theta_i\epsilon_i \geq 0$. We must have $\epsilon_i \geq 0$. This shows the necessary part.

For any $i \in N$, it is given that $O_i(s) \geq s_i + \sum_{j\neq i} s_j/2$. The utility of player i is given by his Shapley value $Sh_i(v^{Pes})$. For any such player, consider the expression $Sh_i(v^{Pes}) + \theta_i O_i(s) = -\theta_i(\sum_{j\in N} s_j) + \sum_{j\in P_i(\sigma^*(\theta_N))} \theta_j s_i/2 + \sum_{j\in F_i(\sigma^*(\theta_N))} \theta_i s_j/2 + \theta_i s_i + \theta_i \sum_{j\neq i} s_j/2 = -\theta_i \sum_{j\neq i} s_j/2 + \sum_{j\in P_i(\sigma^*(\theta_N))} \theta_j s_i/2 + \sum_{j\in F_i(\sigma^*(\theta_N))} \theta_i s_j/2$. Since $\sigma^*(\theta_N)$ is an efficient ordering of the members of the grand coalition (N), then for any agent $i \in N$, if an agent $j \in P_i(\sigma^*(\theta_N))$ we must have $\theta_j/s_j \geq \theta_i/s_i$. This means, $Sh_i(v^{Opt}) + \theta_i O_i(s) = \sum_{j\in P_i(\sigma^*(\theta_N))}(\theta_j s_i - \theta_i s_j) \geq 0$. For any $i \in N$, we have $Sh_i(v^{Pes}) \geq -\theta_i O_i(s)$. This proves the sufficiency part. $\square$

7 Conclusion

This paper maps sequencing problems to cooperative games and adopts an optimistic and a pessimistic approach to define the worth of a coalition. We study two solution concepts: the core, which deals with stability of feasible allocations and the Shapley value, which assigns the outcome in a fair and an impartial manner. We observe that the core of of the optimistic game is always empty. For the pessimistic game, the Shapley value has been shown to be a core-stable allocation. Under both the scenarios, the consumption of money by each agent is such that, the utility of each individual corresponds to his Shapley payoff from the associated coalition form game. We show that it is both necessary and sufficient for the Shapley value rule to satisfy the expected costs bound for it to satisfy the generalized welfare lower bound. Expected costs bound guarantees each agent his expected cost when every possible order of arrival is equally likely.

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