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Does Space Matter? The Case of the Housing Expenditure Cap*

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Abstract

In our evaluation of the housing expenditure share cap, a macroprudential policy, we discover the importance of modeling space. The spatial considerations allow households to sort into segmented housing markets based on income. Our model generates the observed negative relationship between housing expenditure share and income. More importantly, the cap policy causes a more considerable reduction in housing costs for low-income families than for high-income families in a spatial model. Depending on the assumption of households' preference, this mechanism leads to a minor increase or even a modest decrease in welfare inequality in a spatial model than in a spaceless model.

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1 Introduction

This paper argues that spatial considerations are essential for modeling housing markets and evaluating housing-related policies. We illustrate this point through a cap policy on the ratio of housing expenditures to disposable income, referred to as the housing expenditure share henceforth. We document some stylized facts regarding housing expenditure shares across different income classes. We can generate them through a homothetic CES utility function in a canonical monocentric city model or a "spaceless" model with non-homothetic preference. Yet, subject to the same expenditure cap policy, these two models have divergent welfare inequality predictions.

Our research is partly motivated by the booming housing research in recent years. After the 2008 Global Financial Crisis, many became aware that an increase in house prices could translate into an "excessive" amount of credit allocated to the real estate sector and increase the risk of having a financial crisis.[1] In response, many countries have implemented macroprudential policies, which help control the growth of house prices and housing-related credit.[2] According to IMF (2018), "...141 countries reported a total of 1,313 macroprudential measures, for an average of measures per country of 9.3" (p.6). The scale and diversity of all these macroprudential measures are enormous. It is almost impossible to assess all different macroprudential policies in a unifying framework.[3] To complement the literature on mortgage debt and risk-taking, this paper focuses on a policy that caps the housing expenditure share, a common concern of many policymakers.[4]

This type of cap policy has been explicitly and implicitly adopted in many contexts. For instance, in some affordable housing programs, the U.S. Department of Housing and Urban Development (HUD) imposes that rent "does not exceed 30 percent of the adjusted income of a family whose annual income equals 65 percent of the median income for the

[1]For a review of the literature, see Leung and Ng (2019), Leung (2023), among others.

[2]See, e.g., Akinci and Olmstead-Rumsey (2018), Carreras, Davis, and Piggott (2018), Claessens, Ghosh, and Mihet (2014), and Gambacorta and Murcia (2020).

[3]Empirical works suggest that the relationship between macroprudential policies and inequality may be non-monotonic and depend on several factors (Biljanovska et al., 2023; Carpantier, Olivera, and Van Kerm, 2018; Casiraghi et al., 2018; Guerello, 2018). Favilukis, Mabille, and Van Nieuwerburgh (2019) combine the life-cycle dynamics with the spatial choice between two communities. They calibrate their model to match specific features of New York City.

[4]For instance, OECD (2019) states that "...Another common measure for housing affordability used here is the 'housing cost overburden rate', which measures the proportion of households or population that spend more than 40% of their disposable income on housing cost (in line with Eurostat methodology)." USA President Joe Biden also said that "every American in every zip code should have access to housing that is affordable - taking up no more than 30% of income so they have money left over to meet other needs." See Biden (2020) for more details.

area, as determined by HUD."[5] Among macroprudential policies implemented in many countries, caps are imposed on the debt servicing ratio in various types of stress tests (IMF, 2014).[6] In Appendix B, we show through an extended version of the model that this is equivalent to imposing a cap on the housing expenditure share in our static model.

However, the empirical evidence for the implications of such caps for housing markets and welfare is mixed. According to Biljanovska et al. (2023), many studies based on aggregate data generate statistically insignificant results, and studies based on micro-data tend to suggest more significant effects of macroprudential policies. One potential explanation is that some researchers assume away the spatial dimension of the macro-prudential policies, i.e., there is typically only one housing market in the model (e.g., Alpanda, Cateau, and Meh, 2018).[7] In reality, the housing markets are segmented even within the same city. This paper shows that models differ in spatial considerations and preference choices can deliver very different policy implications.

To substantiate our claim, we proceed in several steps. We start by establishing three stylized facts in the United States that guide our modeling exercise and are relevant to the welfare implication of the cap policy. The facts are (1) the approximate constancy in the working hours across income groups, (2) the negative correlation between the share of housing-related expenditure and income, and (3) the positive association between commute time and distance with income. Simultaneously satisfying these three facts imposes stringent restrictions on the equilibrium model. For instance, a spaceless model needs the nonhomothetic preference to produce the observed negative relationship between housing expenditure share and income. In addition, because there is only one housing market, this spaceless model is silent on the observed spatial correlation between household income and residential locations. In contrast, we show that a canonical monocentric city model can simultaneously account for all three stylized facts even with homothetic preference (CES) (Brueckner, 1987; Mills, 1972; Muth, 1969).

Welfare implications from government policies such as an expenditure cap are dra-

[5] According to the California Department of Housing and Community Development, "State statutory limits are based on federal limits set and periodically revised by the U.S. Department of Housing and Urban Development (HUD) for the Section 8 Housing Choice Voucher Program. The comparable federal limit, more widely used, is 30 percent of gross income, with variations." For more details, see https://www.hcd.ca.gov/grants-and-funding/income-limits.

[6] For instance, the Hong Kong Monetary Authority reports that a household could borrow a mortgage loan of up to 157 times the households monthly income. After the stress-testing requirement came into effect on 15 September 2010, the maximum amount that the household could borrow was less than 80% of the situation before the HKMA's recent prudential measures were introduced (HKMA, 2010). Thus, the macroprudential measures effectively put a ceiling on the income share of housing-related expenditures for households without additional financial resources.

[7] There are exceptions. For instance, Acharya et al., 2022 show that caps on loan-to-value and loan-to-income ratios lead banks to reallocate mortgage credit from low- to high-income households and from urban to rural communities and change the relative house prices among different places as a result. This paper generates similar results in a structural model.

matically different in a spatial model than in a model without space. If preferences are homothetic, a welfare cap reduces welfare inequality in a spatial model but increases it in a model without space. This is because, under the homothetic preference, the expenditure share on housing is independent of the income level. In a spaceless model, lower-income households are forced to compete with higher-income agents in the unified housing market. As a result, lower-income families spend a larger fraction of their income on housing. Under the expenditure cap, lower-income households are forced to spend less on accommodation, reducing their welfare. At the same time, the housing expenditure cap is non-binding for the higher-income agents, and hence, the impact on their welfare is minimal. In a spatial model, each "location" in a monocentric city is a geographically segregated housing market, and agents self-select into different submarkets. Under the cap, each submarket's "market price" adjusts differently. Low-income families enjoy disproportionately more significant reductions in endogenous equilibrium rent than high-income families. In our baseline model, the welfare gain from these rent reductions entirely offsets the welfare loss due to being constrained by the cap.

On the other hand, if preferences are sufficiently non-homothetic, welfare inequality increases in both models but less so in the model with space. This is because agents endowed with non-homothetic preferences allocate different portions of their incomes to various goods. Under our calibration, lower-income agents spend a larger percentage of their income on housing and, hence, are more likely to be constrained by the cap policy. In the *non-homothetic spaceless* model, the higher-income agents would be "benefited" by the expenditure cap as the lower-income competitors are restricted to spending on housing. With non-homothetic preference, the higher-income agents would spend a smaller portion of their income on housing, reducing the housing rent, which partly offset the adverse effect of the cap policy on the lower-income households. In the *homothetic monocentric city* model, economic agents are self-selected into different "submarkets" to begin with. Hence, lower-income households cannot benefit from the demand reduction effect of the higher-income agents.

We explore the relevance of this key mechanism and the robustness of our welfare results in monocentric cities with a variety of alternative model settings. In Appendix B, we consider 1) a multi-period setting and 2) a setting with imperfect spatial sorting while keeping the utility function homothetic (CES). We show that our main conclusion that a housing expenditure cap has negligible effects on welfare inequality remains under these settings. In Section 5, we investigate the role of homothetic preference for our welfare results by considering a *non-homothetic monocentric city* model.[8] We find that, while the cap policy still aggravates welfare inequality in this model, the magnitude of this effect is smaller than that in the *non-homothetic spaceless* model because of the

[8]After all, while many existing models adopt homothetic preference (Epple and Romer, 1991; Nechyba, 2000, 2003), there are also examples where non-homothetic preferences are employed (Albouy, Ehrlich, and Liu, 2016; Wang and Xie, 2022).

general equilibrium effect. Thus, we conclude that the key mechanism discussed above is always present with its importance depending on households' preferences. We discuss the implications of this finding for future research in the conclusion (Section 6) and provide a comprehensive summary of the value of modeling space in Section 5.3.

This paper builds on and complements several strands of the literature. On top of the few we briefly discussed above, the first literature that inspires this paper is the one on housing affordability and the evaluation of related government policies (Ben-Shahar, Gabriel, and Oliner, 2020; Gabriel and Painter, 2020; Quigley and Raphael, 2004). They typically employ a rich micro-data set and adopt a reduced-form estimation approach. The second literature concerns the design of macroprudential policies, such as Buch, Vogel, and Weigert (2018), Gadanecz and Jayaram (2017), etc. Third, this paper is related to the literature on how loan-to-value ratio policy (LTV), which is a different kind of macroprudential policy, would affect the housing market, such as Aastveity, Juelsrud, and Wold (2020), Armstrong, Skilling, and Yao (2019), Bekkum et al. (2019), Laufer and Tzur-Ilan (2021), Tzur-Ilan (2019). Fourth, this paper is related to the literature that emphasizes the housing market heterogeneity at the city-level or regional-level (Beraja et al., 2019; Fratantoni and Schuh, 2003; Leung and Teo, 2011; Piazzesi, Schneider, and Stroebel, 2020; Sun and Tsang, 2018).

Our paper complements the literature in different ways. First, we build a structural model and hence complement the literature based on the reduced-form approach.[9] Second, we abstract from the macroeconomic risk of the housing market and the dynamic considerations that the previous literature has studied. Instead, our baseline model focuses on a static environment where ex-ante heterogeneous agents would choose among infinitely many locations within a city. Thus, rather than investigating the effects of the cap policy on financial and housing market stability, this paper examines the importance of spatial considerations for understanding the cap policy's implications on welfare inequality.

2 Empirical Facts

This section documents several crucial empirical findings that motivate the mechanism we explore in this paper. The first stylized fact is that work hours are roughly constant concerning income; the hours worked on average are around 8 hours for all income groups. Table 2 shows the evidence for 2017.

Second, we document that the housing expenditure share is decreasing with income. Table 1 shows the average annual before-tax income, total expenditure, expenditure on housing, and the fraction of total expenditure spent on housing for each before-tax income

[9] There is an emerging view that the reduced form approach and structural model approach are complementary to each other (Heckman, 2010; Todd and Wolpin, 2023). A merit of the "structural" approach is facilitating counterfactual analysis (Wolpin, 2013).

decile of the population in 2016.[10]

A comparison of the first two rows of Table 1 shows that the pre-tax income and the post-tax (and transfer) expenditure can differ significantly, especially for lower-income households. This observation is also consistent with recent research that pre-tax and post-tax in the U.S. can be very different (Auten and Splinter, 2019; Splinter, 2020). Hence, we measure the housing expenditure share (ratio of housing expenditures to disposable income) by the fraction of total expenditure spent on housing throughout the paper because it considers the effect of progressive taxes and transfers on disposable income. This is conceptually valid because total expenditure equals disposable income in the static models we will present in later sections. As shown in the fifth row of Table 1, the housing expenditure share decreases from 0.41 to 0.30 in a roughly monotonic manner.

Table 1: Average Total Expenditures and Housing Expenditures by Income Decile

Item	Lowest 10 percent	Second 10 percent	Third 10 percent	Fourth 10 percent	Fifth 10 percent
(1) Income before taxes	\$6,502	\$16,229	\$24,432	\$33,499	\$43,931
(2) Average annual expenditures	\$23,588	\$26,675	\$34,221	\$39,308	\$43,975
(3) Average annual expenditures on housing	\$9,567	\$10,961	\$12,829	\$14,271	\$15,511
(4) Average annual expenditures on transportation	\$1,037	\$1,218	\$1,530	\$1,848	\$2,139
(3)/(2)	0.4056	0.4109	0.3749	0.3631	0.3527
Item	**Sixth 10 percent**	**Seventh 10 percent**	**Eighth 10 percent**	**Ninth 10 percent**	**Tenth 10 percent**
(1) Income before taxes	\$57,192	\$73,568	\$94,739	\$127,268	\$269,644
(2) Average annual expenditures	\$51,351	\$59,395	\$70,411	\$87,432	\$136,873
(3) Average annual expenditures on housing	\$17,119	\$19,285	\$22,085	\$26,719	\$40,547
(4) Average annual expenditures on transportation	\$2,450	\$2,776	\$3,220	\$3,802	\$5,305
(3)/(2)	0.3334	0.3247	0.3137	0.3056	0.2962

Source: Consumer Expenditure Survey, 2016; U.S. Bureau of Labor Statistics.

The third critical empirical finding is that more highly paid workers live farther away from their workplace. Using the 2010 American Community Survey (ACS) data, we first

[10]In Table 1, we use pre-tax total income. Table 2 is based on a different dataset; hence, we use annual wage, which is the labor income. In the United States, mortgage applications are often based on labor income and the stable part of the capital income. Leung and Tang, 2023 employ labor income only.

Table 2: Hours Worked on an Average Day by Weekly Earnings

Income Range	$0 - $590	$591 - $920	$921 - $1,440	$1,441 and higher
Hours	7.87	8.19	8.28	8.12

Source: American Time Use Survey, 2017; U.S. Bureau of Labor Statistics.

compute households' average commute time and distance in each annual wage and salary income distribution decile. We find that both commute time and distance are monotonically increasing with income. Recognizing that the differences in average commute time and distance across income groups might be partially driven by income-based geographic sorting, we also use a fixed-effects model to estimate the effect of income on commute time and distance.[11] Using decennial Panel data from the Census and the ACS from 1980 - 2010, we find that a 1,000 (2017) dollars increase in annual wage and salary income is associated with a 0.0359-minute increase in commute time, with this coefficient being significant at a 1% level. In 2017, the average commute speed is 23.42 miles per hour.[12] Therefore, in 2017, a 1,000-dollar increase in annual wage and salary income is associated with a 0.0140-mile increase in commute distance on average.[13] Another supporting evidence is that, as shown in the fourth row of Table 1, the average annual expenditures on transpiration increase with pre-tax income.

This paper develops a canonical monocentric model with an extended CES utility function to reproduce these empirical observations. More specifically, the utility function combines numeraire consumption goods, housing, and leisure to produce the final utility. While leaving the details for later sections, we provide some basic intuitions here. First, we show that wealthy families would reside in relatively remote areas with low housing/land prices. Intuitively, this is because the pecuniary component of commute cost is "less expensive" for more affluent households in a relative sense. Standard consumer theory predicts that the fraction of total expenditure spent on one commodity would increase with the price of this commodity if the elasticity of substitution between this commodity and other commodities is smaller than 1. Therefore, combining the features that (1) the unit price of housing is lower for more affluent households and (2) the elasticity of substitution between housing and numeraire consumption goods is greater

[11]We do not control for household characteristics such as age and family size in the regression. Consistent with our treatment of the association between income and housing expenditure share, we are trying to capture the "total" association between income and commute time, including the part that arises through these household characteristics.

[12]See National Household Travel Survey 2017, https://www.bts.gov/topics/national-household-travel-survey.

[13]While the magnitude of this association might seem small, it is close to what we find in our baseline model. In our baseline model, a 1,000-dollar increase in annual wage and salary income is associated with a 0.0175-mile increase in commute distance. In Appendix B, we consider an alternative model setting under which we can perfectly match this association by adding unobserved preference shocks to households' location decisions. We find quantitatively similar results in this alternative model.

than 1, we generate the pattern that the fraction of housing expenditure is decreasing in income. By the same token, as the elasticity of substitution between leisure and other commodities is unity, the work hours are roughly constant concerning income.

3 A Monocentric City Model: Baseline

We present a stylized static model that describes the resource allocation within a city. An absentee landlord owns all the land and allocates land through auction. The household that offers the highest bid acquires the land for each location if this highest bid exceeds an exogenous agricultural rent P_a.[14] Households then enjoy the housing service from their acquired land and commute to a location r miles away from their home to work. This paper does not explicitly model housing development and uses the terms land and housing interchangeably. After paying for the land, households spend the rest of their income on non-durable consumption goods and commutes.

In terms of spatial configuration, in this section, we consider a monocentric city model in which the city is built around a central business district (CBD), i.e., all households commute to the city center for work. There are infinitely many housing markets in this model. These markets are defined by the distance r to the CBD and have potentially different equilibrium rents. We first analytically show that this monocentric city model's spatial feature helps us generate the observed negative relationship between income and the housing expenditure share without non-homothetic preference. We then calibrate the baseline model to match the U.S. economy circa 2017.

3.1 Household's Problem

This subsection defines and solves a household's utility maximization problem. The solutions to the maximization problem have several relevant implications for the empirical findings in Section 2.

3.1.1 Utility Function, Budget Constraint, and Household Heterogeneity

A household's utility is determined by its consumption on numeraire good c, housing lot size h, and leisure l. Formally, we assume that its preference is *homothetic* and its utility is given by:

$$U(c,h,l) = l^{1-\alpha}[\theta c^{1-\rho} + (1-\theta)h^{1-\rho}]^{\frac{\alpha}{1-\rho}}. \quad (1)$$

Our specification extends the Cobb-Douglas utility function typically used in previous work on computable spatial equilibrium models (e.g. Hanushek and Yilmaz, 2007, 2013). We can interpret it as a two-step aggregator. First, housing h and consumption goods c are combined in a general CES manner with an elasticity of substitution of $\frac{1}{\rho}$. Then, their aggregate enters with leisure l in a Cobb-Douglas way to produce the final utility (Krusell et al., 2000; Ogaki and Reinhart, 1998).

[14] Alternatively, we could consider a city with exogenous boundaries, and the agricultural rent would be endogenously determined in equilibrium. Our main results are robust to this alternative specification.

We now describe the budget constraint faced by the household. Consider a household with an hourly wage w, located r miles away from the CBD.[15] The household allocates the $24 \times 7 = 168$ hours in a week to work, leisure (l hours), and commuting (br hours), where b is the time cost per mile of weekly round-trip commute. Hence, the total income of this household is given by:

$$Income = [168 - l - br]w. \tag{2}$$

The household pays for the consumption goods and housing rents with income. We normalize the price of the composite consumption goods to 1 and denote the rent for one unit of housing at location r as $P(r)$. To capture the pecuniary cost of commuting, we assume that the weekly round-trip commute costs ar dollars. Formally, the total expenditure is given by:

$$Expenditure = c + P(r)h + ar. \tag{3}$$

As usual, total income should equal total expenditure at the equilibrium. Denoting $Y(r) \equiv (24 - br)w - ar$, the budget constraint implies the following:

$$Y(r) \equiv (168 - br)w - ar = wl + c + P(r)h. \tag{4}$$

Households in this model differ only in their hourly wages w. Limited by data availability and computational tractability, we categorize households into ten equal-sized groups based on hourly wages.[16] We refer to households whose hourly wages belong to the ith decile as Type i households and add subscript i to all relevant variables to differentiate among types. To ease the notation, we suppress the subscripts in most parts of this paper, and the readers should keep in mind that income levels differ across types of households.

3.1.2 Optimal Consumption Allocation and Indirect Utility Function

Taking equilibrium market rent $P^*(r)$ as given, a household living r miles from the CBD maximizes the utility function (Equation 1) subject to the budget constraint (Equation 4):

$$V(P^*(r), r) = \max_{(c,h,l)} U(c, h, l) \text{ s.t. } Y(r) = wl + c + P^*(r)h, \tag{5}$$

where $Y(r) \equiv (168 - br)w - ar$.

[15] A household can only live in one location. Hence, the consumption set is not convex. The First Welfare Theorem does not apply here. See Rogerson (1988) and Shell and Wright (1993) for more discussion.

[16] The income distribution of the U.S. is well-approximated by a log-normal distribution. Hence, each group's income range is different to generate equal-sized groups.

Solving this maximization problem yields the following optimal choices of consumption, housing, and leisure:

$$h(P^*(r), r) = \frac{\alpha\kappa(P^*(r))Y(r)}{P^*(r)}, \tag{6}$$

$$c(P^*(r), r) = \alpha[1 - \kappa(P^*(r))]Y(r), \tag{7}$$

$$l(P^*(r), r) = \frac{(1-\alpha)Y(r)}{w} = (1-\alpha)(168 - br - \frac{ar}{w}), \tag{8}$$

where $\kappa(x; \theta, \rho) = \frac{(\frac{1-\theta}{\theta})^{\frac{1}{\rho}} x^{\frac{\rho-1}{\rho}}}{1+(\frac{1-\theta}{\theta})^{\frac{1}{\rho}} x^{\frac{\rho-1}{\rho}}}$.

Finally, substituting the optimal choices shown in Equation (6)-(8) in the utility function shown in Equation (1) yields the indirect utility function $V(P^*(r), r)$:

$$V(P^*(r), r) = [1 - \kappa(P^*(r); \theta, \rho)]^{\frac{\alpha\rho}{\rho-1}} KY(r)(\frac{1}{w})^{1-\alpha}, \tag{9}$$

where $K = \alpha^\alpha(1-\alpha)^{1-\alpha}\theta^{\frac{\alpha}{1-\rho}}$.

3.1.3 Work Hours and Housing Expenditure Share

In our model, the weekly work hours are obtained by subtracting leisure time and time cost of commute, br, from 168 hours. Further, Equation (8) shows that the optimal leisure time is approximately a constant, 168α, if the time and monetary cost of commute, ar and br, are small. Hence, we conclude that all types of households have roughly the same work hours (Stylized Fact 1) when the commute cost is low. In the calibration section later, we show ar and br are indeed small relative to income.

The $\kappa(P^*(r); \theta, \rho)$ function defined in the previous subsection provides a measure of housing expenditure share in the model. Equation (6) and (7) together imply that $\kappa(P^*(r); \theta, \rho) = \frac{P^*(r)h}{c+P^*(r)h}$. Noting that the total expenditure is equal to $c + P^*(r)h + ar$, $\kappa(P^*(r); \theta, \rho)$ measures the ratio of housing expenditures to total after-commute-cost expenditures and is hereafter referred to as the ACC housing expenditure share. Since ar is relatively small compared to the total expenditure, $\kappa(P^*(r); \theta, \rho)$ is quite close to the housing expenditure share.

Note that $\kappa(P^*(r); \theta, \rho)$ does not depend on hourly wages. Since all households have the same preferences (θ and ρ), we simply write the $\kappa(P^*(r); \theta, \rho)$ function by suppressing the θ and ρ arguments. In Appendix A, we show that $\kappa(P^*(r))$ is increasing in equilibrium rent $P^*(r)$ if the elasticity of substitution between consumption and housing is smaller than one.

Proposition 1. *If $\rho > 1$, then $\frac{\partial\kappa(x)}{\partial x} > 0$.*

As we will explain in the following subsection, this proposition is vital for generating the negative relationship between income and housing expenditure share.

3.2 Basic Analysis of the Equilibrium

This subsection defines and characterizes the equilibrium of our model. We will show that our monocentric city model is broadly consistent with the empirical evidence documented in Section 2. Also, these characterizations assist our calibration in a later section.

3.2.1 Bid-rent Functions and Market Rent Curves

As with many spatial equilibrium models, all households bid for land on a featureless plane. The common practice is to solve the bid-rent function, which expresses a household's willingness to pay for the equilibrium utility level u_i^*. For a Type $i \in \{1, 2, ..., 10\}$ household, the maximization problem can be mathematically expressed as follows:

$$\psi_i(u_i^*, r) = \max_{(c,h,l)} \left\{ \frac{Y_i(r) - c - w_i l}{h} | U\,(c, h, l) = u_i^* \right\}, \tag{10}$$

where $Y_i(r) \equiv (168 - br)w_i - ar$.

Technically speaking, this bid-rent maximization problem is the dual problem to the utility maximization problem defined in Equation (5). Hence, we can obtain the following *bid-rent function* by inverting the indirect utility function $V_i(P^*(r), r)$:

$$\psi_i(u_i^*, r) = \left\{ \left[\frac{Y_i(r)K}{u_i^* w_i^{1-\alpha}} \right]^{\frac{\rho-1}{\alpha\rho}} - 1 \right\}^{\frac{\rho}{\rho-1}} (\frac{\theta}{1-\theta})^{\frac{1}{\rho-1}}, \tag{11}$$

where $K = \alpha^\alpha (1-\alpha)^{1-\alpha} \theta^{\frac{\alpha}{1-\rho}}$.

In the model, all of the lands are rented out via auctions. The ten types of households and agricultural workers can bid for any location indexed by its distance from the CBD, r.[17] For each location, the right of usage goes to the agent who offers the highest bid. Therefore, the equilibrium rent curve $P^*(r)$ is the upper envelope of the bid rent curves $\psi_i(u_i^*, r)$ of the ten types of households and the agricultural rent P_a. As the family moves away from the CBD, its bid rent declines and eventually hits 0. It means that beyond a certain distance $R_f^\star$, the agricultural rent P_a dominates the bids offered by *all* of the households in the economy. Hence, no one resides there. We introduce the function $t^\star(r)$ to indicate the type of the residents at distance r. Formally, within the fringe distance $R_f^\star$, $t^\star(r)$ is given by:

$$t^\star(r) = \arg\max_i \psi_i(u_i^*, r). \tag{12}$$

The equilibrium rent $P^*(r)$ is then given by:

$$P^*(r) = \max(\sum_{i \in \{1,2,...,10\}} \psi_i(u_i^*, r) I(t^\star(r) = i), P_a), \tag{13}$$

where $I(\cdot)$ is an indicator function that takes the value 1 when the condition in the bracket is satisfied and 0 otherwise.

[17] Following the urban economics literature, the agricultural workers are assumed to be self-sustained, except for the participation of the land auction. They would not affect any other aspect of the model economy.

To understand how different agents are distributed spatially, we consider the spatial order of two adjacent types of households. It is determined by their bid rent curves' relative steepness at the intersection point. The one with the steeper curve resides closer to the CBD. In other words, the condition for the equilibrium location of Household 1 being farther from the CBD than that of Household 2 is $\frac{\partial \psi_1(\cdot)/\partial r}{\partial \psi_2(\cdot)/\partial r} < 1$. In Appendix A, we show that $\frac{\partial \psi_i(\cdot)}{\partial r}$ is always negative and that $|\frac{\partial \psi_i(\cdot)}{\partial r}|$ is larger for households with lower hourly wage w_i. Hence, the following proposition holds.

Proposition 2. *Households with higher hourly wage w_i live farther from the CBD at the equilibrium.*

The equilibrium rent $P^*(r)$ decreases in the distance r. Combining Proposition 2 and 1, we conclude that households with higher hourly wages spend a smaller fraction of their total after-commute-cost expenditures on housing services.

Proposition 3. *Consider two arbitrary households. We denote their types as i and j, respectively. Let r_i^* and r_j^* denote their distance from the CBD at the equilibrium. If $\rho > 1$ and $w_i > w_j$, then $\kappa(P(r_i^*)) < \kappa(P(r_j^*))$ at the equilibrium.*

3.2.2 Population Density and Land Market Clearance

Proposition 3 shows how agents are allocated across the city given the equilibrium rent schedule. Now we explain how the equilibrium rent schedule is determined. Following the literature, the total number of households for each type i, $i \in \{1, 2, ..., 10\}$, is exogenously given at $\overline{N_i}$ in this model. Suppose that Type i households in equilibrium occupy the locations r miles from the CBD, and $L(r)$ represents the amount of land available per unit distance. Our model has $L(r) = 2\pi r$.[18]

The clearing of the land market means that within the fringe distance $R_f^\star$, the following equation holds, $L(r) = h_i(P^*(r), r)m_i(P^*(r), r)$, where $m_i(P^*(r), r)$ is the equilibrium number of households per unit distance, assuming that distance r is occupied by Type i household and $u_i^\star$ is the equilibrium utility of Type i household. All households find residence locations, implying the following *population constraint.*

$$\int_0^\infty m_i(P^*(r), r)I(t^\star(r) = i)dr = \overline{N}_i, \text{ for } i \in \{1, 2, ..., 10\}. \tag{14}$$

It is easy to verify that the household number distribution function $m(r)$ is given by:

$$m(r) = \sum_{i \in \{1,2,...,10\}} m_i(P^*(r), r)I[t^\star(r) = i]. \tag{15}$$

[18]In reality, one unit of land might correspond to more housing units at locations closer to the city's center. One way to incorporate this possibility is to let $L(r) = 2\pi r\mathcal{D}(r)$, while $\mathcal{D}(r)$ is decreasing in r. It is important to note that our main welfare result (Proposition 4) remains in this case because its validity does not depend on the choice of $L(r)$.

3.2.3 Stationary Equilibrium

We now define the general equilibrium of this model economy. In the stationary equilibrium, no household has an incentive to relocate. It can be formally defined as follows:

Definition 1. An equilibrium is a set of utility levels $\{u_1^*, u_2^*, ..., u_{10}^*\}$, market rent curve $P^*(r)$, and type function $t^*(r)$ that show the equilibrium occupant of the location at distance r that satisfy the following conditions.

• Households offer their bids according to Equation (11). The land is rented out through an auction. The household that offers the highest bid wins a particular location if the bid is higher than the agricultural rent. Otherwise, the land is used for agriculture. Type function $t^*(r)$ records the auction's winner.

• Market rent $P^*(r)$ is determined as the upper envelope of bids from all types of households and the agricultural rent according to Equation (13).

• All Type i households attain the same utility level, i.e., $u_i^* = V_i(P^*(r), r)$ for all r such that $t^*(r) = i$.

• Each household rents a certain amount of land according to Equation (6). The land market clears, and the population constraint (Equation 14) holds.

3.3 Calibration

3.3.1 Parameter Set

This section shows that our baseline model can be calibrated to match the stylized facts of the U.S. circa 2017 documented in Section 2. The parameters of our model can be divided into three categories, which are (1) macroeconomic environment parameters P_a and $\overline{N}_i$, (2) budget constraint parameters w_i, a, and b, and (3) preference parameters α, θ, and ρ. Below we describe the calibration of each category of parameters. Table 3 summarizes our calibration exercise.

We start with the macroeconomic environment parameters, i.e., agricultural rent P_a and the number of households $\overline{N}_i$. In 2017, the average population and area of cities with more than 100,000 people were 303,322 and 94.08 square miles.[19] In our model, all households commute to the CBD for work. In the U.S., the employment-to-population rate is consistently around 60%. Hence, we fix the total number of households at $303,322 \times 0.6 = 181,993$. Each income group constitutes 10% of the total population implying that $\overline{N}_i = 18,199$ for all i. Given the total number of households (and other parameters), agricultural rent P_a determines the city's size. The lower P_a is, the larger the city is. We set agricultural rent to $P_a = \$505,340$ per square mile per week to match the endogenous calibration targets for the fringe distance R_f^*, which is $\sqrt{94.08/\pi}$ miles.

We then describe the calibration of the budget constraint parameters. Note that our model is static and does not feature income tax. Hence, instead of calibrating hourly wage w_i to match annual before-tax income (reported in the first row of Table 1), we

[19] Authors' calculation based on the Census data.

calibrate w_i to match total yearly expenditures (reported in the second row of Table 1). Recall that Table 2 shows that the hours worked on average are around 8 hours for all income groups. Hence, under a reasonable assumption that a typical worker works five days each week, the weekly and annual work times are around $5 \times 8 = 40$ hours and $40 \times 52 = 2080$ hours, respectively. Therefore, we obtain the hourly wages for each income group by dividing the average total annual expenditures reported in the second row of Table 1 by 2080.

The weekly per mile pecuniary cost a and time cost b are chosen to match average transportation expenditure and commute time. The average annual spending on transportation for each income group has been reported in Table 1. Dividing those numbers by 52 (weeks) yields the calibration targets for the group-specific averages of ar. In the U.S., the average one-way commute time is 26.1 minutes, according to 2017 Census data. We assume each household does a round-trip commute to the CBD for five days weekly. Hence, the calibration target for the average br is $26.1 \times 2 \times 5/60 = 4.35$ hours. The calibrated values of a and b are 17.46 and 1.56, respectively.

The last set of parameters to be determined are the preference parameters. Equation (8) shows that $\alpha = 1 - \frac{l}{168 - br - \frac{ar}{w}} = 1 - \frac{24 - br - n}{168 - br - \frac{ar}{w}}$, where n represents the weekly hours of work. As documented above, n is roughly 40 hours and does not vary much across households. The average commute time br is 4.35 hours. The average $\frac{ar}{w}$ can be obtained this way. We first calculate the ratio of the group-specific averages of ar to group-specific weekly wage w_i. We then take the average over income groups. The result is 1.81 hours. Hence, we set α to be $1 - \frac{168 - 4.35 - 40}{168 - 4.35 - 1.81} = 0.236$.

Finally, we calibrate parameters θ and ρ to match the observed pattern of the housing expenditure share. Intuitively speaking, ρ determines the speed at which this fraction decreases with total income; for any fixed ρ, θ pins down the overall level of this fraction. We search for the set of θ and ρ that minimizes the sum of the squared differences between model-implied shares for each income group and their data counterparts. We find that $\theta = 0.0068$ and $\rho = 2.85$.

3.3.2 Baseline Equilibrium

Table 4 summarizes the baseline equilibrium outcomes. Panel A of Table 4 shows that the calibrated model can closely match data on total expenditures, housing expenditures, housing expenditure share, work hours, commute time, and pecuniary commute cost. Figure 1 shows that our model can match the observed negative relationship between income, or equivalently total expenditures, and the housing expenditure share. The red dashed line is obtained by plotting the fifth row of Panel A of Table 4 against the first row of Panel A of Table 4. The housing expenditure share is roughly monotonically decreasing in total weekly spending. The solid blue line, obtained by plotting the sixth row of Panel A against Panel A's second row, represents the model-implied counterpart of the red line. Figure 1shows these two lines are very close.

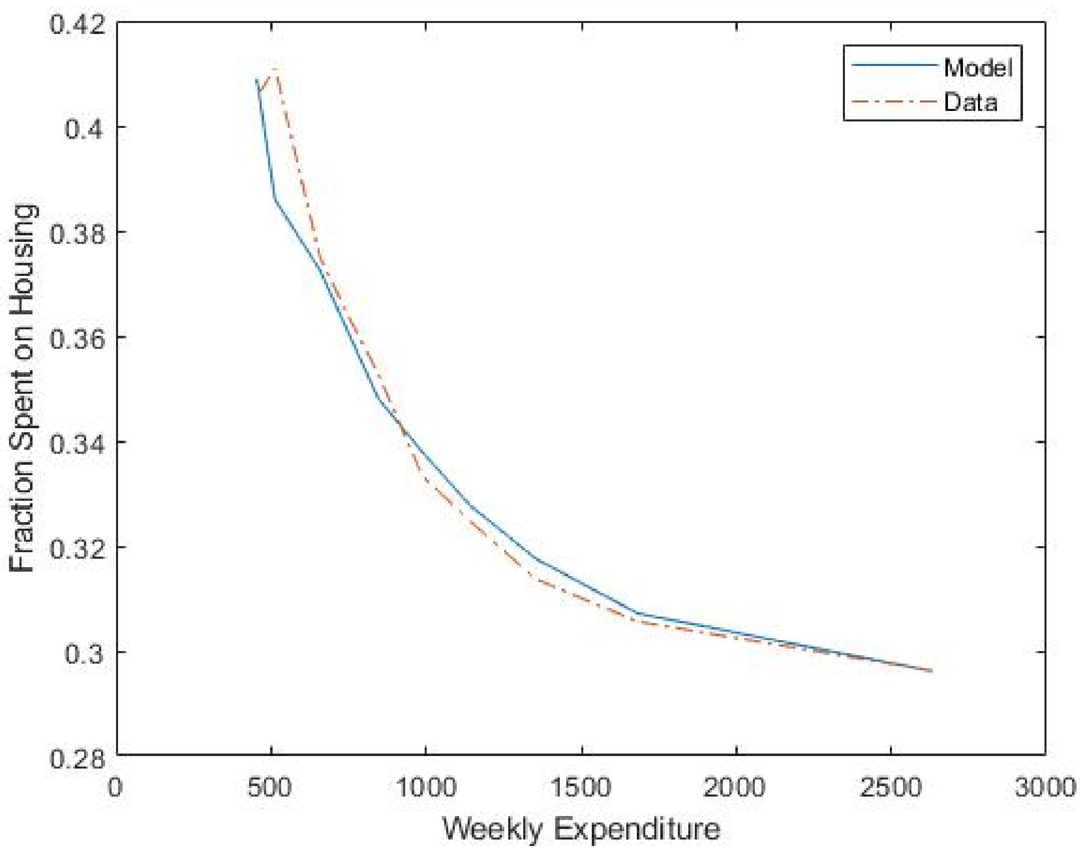
Table 3: Calibration Targets and Results

Parameter	Interpretation	Targeted Moment	Value: CES Spatial	Non-homothetic Spaceless	Non-homothetic Spatial
Macroeconomic Environment Parameters					
P_a	Weekly Agricultural Rent (dollars per square mile)	Average Size of U.S. Cities	505,340	703,070	644,910
$\overline{N}_i$	Population Size of Type i	Average Working Population of U.S. Cities		18,199	
Budget Constraint Parameters					
w_i	Hourly Wage Rate (dollars)	Total Expenditures by Income Type		Varies by Income Type	
a	Weekly Commute Cost per Mile (dollars)	Average Transportation Expenditure		17.46	
b	Weekly Commute Cost per Mile (hours)	Average Commute Time		1.56	
Preference Parameters					
α (CES)	Importance of Consumption relative to Leisure	Weekly Hours of Work	0.236	N.A.	
θ (CES)	Importance of Numerare Goods relative to Housing	Relationship between Housing Expenditure Share	0.0068	N.A.	
ρ (CES)	Inverse of the Elasticity of Substitution between Numerare Goods and Housing		2.85	N.A.	
β_C (Non-homothetic)	Importance of Numerare Goods relative to Housing	and Hourly Wage Rate w_i	N.A.	0.7191	
$\bar{c}$ (Non-homothetic)	"Free" Consumption of Numerare Goods (dollars)		N.A.	240	

Note: Here we report type-specific hourly wages: $w_1 = 11.34$, $w_2 = 12.82$, $w_3 = 16.45$, $w_4 = 18.90$, $w_5 = 21.14$, $w_6 = 24.69$, $w_7 = 28.56$, $w_8 = 33.85$, $w_9 = 42.03$, $w_{10} = 65.80$.

Panel B of Table 4 presents the additional results. Notably, the second row of Panel B shows that more affluent households reside farther away from the CBD, where the rent is lower. More affluent households can afford housing with larger lot sizes with higher disposable income, as shown in the first row of Panel B. Finally, the last row of Panel B reports the average value of κ for each income group. Comparing them to the sixth row of Panel A, we find that κ is systematically larger than but quite close to the housing expenditure share. This is consistent with our interpretation of κ in Section 3.1.2.

Figure 1: Total Annual Expenditure and the Fraction Spent on Housing



4 A Monocentric City Model: Housing Expenditure Share Cap

In this section, we examine the welfare implications of imposing a cap $\bar{\kappa}$ on the after-commute-cost (ACC) housing expenditure share, $\frac{P^*(r)h}{c+P^*(r)h}$, in the context of the monocentric city model described in Section 3. In Appendix B, we show that our results are robust to adding dynamics and unobserved preference shock to the baseline static deterministic model.

Table 4: Summary of Baseline Equilibrium Outcomes

Item	Lowest 10 percent	Second 10 percent	Third 10 percent	Fourth 10 percent	Fifth 10 percent	Sixth 10 percent	Seventh 10 percent	Eighth 10 percent	Ninth 10 percent	Tenth 10 percent
Panel A: Model-fit										
Average Weekly Expenditures										
Data	\$454	\$513	\$658	\$756	\$846	\$988	\$1,142	\$1,354	\$1,681	\$2,632
Model	\$456	\$519	\$664	\$763	\$852	\$991	\$1,142	\$1,345	\$1,657	\$2,554
Average Weekly Housing Expenditures										
Data	\$184	\$211	\$247	\$274	\$298	\$329	\$371	\$425	\$514	\$780
Model	\$186	\$200	\$247	\$274	\$297	\$335	\$374	\$427	\$509	\$756
Fraction of Expenditures Spent on Housing										
Data	0.4056	0.4109	0.3749	0.3631	0.3527	0.3334	0.3247	0.3137	0.3056	0.2962
Model	0.4090	0.3859	0.3725	0.3597	0.3481	0.3379	0.3277	0.3176	0.3072	0.2962
Average Weekly Work Hours										
Data	around 40 hours									
Model	40.17	40.46	40.38	40.36	40.31	40.15	39.98	39.74	39.42	38.81
Average Weekly Commute Time										
Data	around 4.35 hours									
Model	1.01	1.90	2.61	3.30	3.95	4.61	5.30	6.04	6.85	7.92
Average Weekly Pecuniary Commute Cost										
Data	\$20	\$23	\$29	\$36	\$41	\$47	\$53	\$62	\$73	\$102
Model	\$11	\$21	\$29	\$37	\$44	\$52	\$59	\$68	\$77	\$89
Panel B: Additional Outcomes										
Average Lot-size (unit: 10^{-4} miles^2)										
Model	1.6268	1.9479	2.6137	3.1254	3.6306	4.4020	5.2905	6.5291	8.4751	14.0459
Average Weekly Rent (unit: per miles^2)										
Model	\$1,145,300	\$1,028,100	\$946,900	\$877,900	\$817,100	\$760,900	\$707,100	\$654,400	\$600,600	\$538,400
κ										
Model	0.4364	0.4096	0.3960	0.3838	0.3726	0.3619	0.3512	0.3402	0.3286	0.3157

4.1 Optimal Consumption and Indirect Utility Function

When a cap of $\bar{\kappa}$ is imposed on $\frac{P*(r)h}{c+P*(r)h}$, we modify the utility maximization problem defined in Equation (5) accordingly to obtain the indirect utility function $\tilde{V}_i(P^*(r), r)$:

$$\tilde{V}_i(P^*(r), r) = \max_{(c,h,l)} U(c, h, l)$$

$$\text{s.t. } Y_i(r) = w_i l + c + P^*(r)h, \text{ and } \frac{P^*(r)h}{c + P*(r)h} \leq \bar{\kappa}, \tag{16}$$

where $Y_i(r) \equiv (168 - br)w_i - ar$.

There are two possibilities. First, the cap is not binding. Hence, the utility maximization problem is identical to the one defined in Equation (5). As shown in Section 3.1.2, in this case, the optimal value of $\frac{P^*(r)h}{c+P*(r)h}$ is equal to $\kappa(P^*(r)) = \frac{(\frac{1-\theta}{\theta})^{\frac{1}{\rho}} P^*(r)^{\frac{\rho-1}{\rho}}}{1+(\frac{1-\theta}{\theta})^{\frac{1}{\rho}} P^*(r)^{\frac{\rho-1}{\rho}}}$, which is monotonically increasing in the market rent $P^*(r)$. Hence, the cap binds when $P^*(r)$ is sufficiently high. It is easy to verify that the cutoff value for $P^*(r)$ is $(\frac{\kappa}{1-\kappa})^{\frac{\rho}{\rho-1}}(\frac{\theta}{1-\theta})^{\frac{1}{\rho-1}}$.

The second possibility is that the cap binds. The choices of consumption, housing, and leisure are given by:

$$\tilde{h}_i(P^*(r), r) = \frac{\alpha\bar{\kappa}Y_i(r)}{P^*(r)}, \tag{17}$$

$$\tilde{c}_i(P^*(r), r) = \alpha(1 - \bar{\kappa})Y_i(r), \tag{18}$$

$$\tilde{l}_i(P^*(r), r) = \frac{(1-\alpha)Y_i(r)}{w_i} = (1-\alpha)(168 - br - \frac{ar}{w_i}). \tag{19}$$

Comparing these choices to their counterparts in the unconstrained case (Equation 6-8), we find that under a binding housing expenditure cap, households spend less money on housing and more money on consumption, and enjoy the same amount of leisure time. Moreover, we can combine results for both the constrained and the unconstrained cases in the following expression:

$$\tilde{V}_i(P^*(r), r) = [(1-\tilde{\kappa})^{1-\rho} + \frac{1-\theta}{\theta}\frac{\tilde{\kappa}^{1-\rho}}{P^*(r)^{1-\rho}}]^{\frac{\alpha}{1-\rho}} K Y_i(r)(\frac{1}{w_i})^{1-\alpha}, \tag{20}$$

where $\tilde{\kappa} = \min(\kappa(P^*(r)), \bar{\kappa})$ represents the ACC housing expenditure share chosen by the household at the equilibrium.[20] We will see later whether the cap binds directly affect the welfare results.

4.2 Bid-rent Function and Spatial Order under Housing Expenditure Cap

This section studies how agents are distributed in the monocentric city under the housing expenditure cap. Recall that the spatial order of two adjacent types of households is determined by the relative steepness of the bid-rent function $\tilde{\psi}_i(u_i^*, r)$ at the intersection, and the household with a steeper bid-rent function lives closer to the CBD.

[20] We obtain the indirect utility function for this constrained case by substituting the choices shown in Equation (17)-(19) in the utility function shown in Equation (1).

When the cap is non-binding, this bid-rent function is identical to the one shown in Equation (11). Hence, as discussed in Section 3.2.1, among the unconstrained households, those with higher hourly wages w_i live farther away from the CBD.

When the cap binds, we obtain the bid-rent function $\tilde{\psi}_i(u_i^*, r)$ by inverting the indirect utility function $\tilde{V}_i(P^*(r), r)$ shown in Equation (20) with $\tilde{\kappa}$ taking the value of the cap $\bar{\kappa}$:

$$\tilde{\psi}_i(u_i^*, r) = \bar{\kappa} \left\{ \left[\frac{Y_i(r)K}{u_i^* w_i^{1-\alpha}} \right]^{\frac{\rho-1}{\alpha}} - (1-\bar{\kappa})^{1-\rho} \right\}^{\frac{1}{\rho-1}} (\frac{\theta}{1-\theta})^{\frac{1}{\rho-1}}. \tag{21}$$

As with the unconstrained case, we can show that 1) $\frac{\partial \tilde{\psi}_i(\cdot)}{\partial r}$ is always negative, and 2) $|\frac{\partial \tilde{\psi}_i(\cdot)}{\partial r}|$ is larger for households with lower hourly wage w_i in the constrained case as well. Hence, those with higher hourly wages w_i also live farther away from the CBD among the constrained households.

To sum up, we conclude that Proposition 2, which states that households with higher hourly wage w_i live farther from the CBD at the equilibrium, remains valid under the housing expenditure cap.[21]

4.3 Welfare Analysis

This subsection analyzes and quantifies the welfare implications of imposing a cap on the ACC housing expenditure share, especially on how the cap would affect the welfare/utility inequality. Since utility is unit-free, direct cross-sectional comparisons of individual utility do not convey meaningful information. Hence, we adopt a widely used consumption-equivalent measure (CE) to quantify the cap's effect on the utility level for each type of household (Cooley and Hansen, 1989). We focus on the cross-sectional relationship between CE and income level. For example, if low-income households tend to have smaller gains or more losses in CE under the cap, we would conclude that the cap policy leads to a more considerable welfare inequality. Formally, the consumption-equivalent measure ξ is defined as follows:

Definition 2. Consider a household whose numeraire goods consumption, housing consumption, and leisure time in the baseline equilibrium are c_{base}, h_{base}, and l_{base}, respectively. Further, denote u_{cap} as the utility level this household obtains under a cap on the ACC housing expenditure share. The consumption-equivalent measure ξ for this household satisfies the following equations:

$$U((1+\xi)c_{base}, (1+\xi)h_{base}, l_{base}) = u_{cap}.$$

Definition 2 states that ξ measures the faction of numeraire goods and housing consumption that a household needs to obtain in the baseline to achieve the same level of

[21]Noting that $\frac{\partial \tilde{\psi}_i(\cdot)}{\partial r}$ is negative for all households, we also conclude that the equilibrium rent $P^*(r)$ is decreasing in the distance from the CBD r. Hence, Proposition 1 ($\frac{\partial \kappa(x)}{\partial x} > 0$) implies that the housing expenditure share, $\tilde{\kappa} = \min(\kappa(P^*(r)), \bar{\kappa})$, is weakly monotonically decreasing in the distance r and in the hourly wage w_i at the equilibrium.

utility in the economy with the cap policy. A positive ξ represents a utility gain, while a negative ξ represents a utility loss.

It is easy to verify that, for any given cap $\bar{\kappa}$, households of the same type have the same ξ. We then denote the consumption-equivalent measure for type i household as ξ_i. The Appendix shows that when a commute's monetary and time costs are negligible, ξ_i is roughly the same for all households. Formally, the following proposition holds.

Proposition 4. *For any $\bar{\kappa} \in (0, 1]$ and any Type i and Type j households, we have*

$$\lim_{a->0^+, b->0^+} \xi_i - \xi_j = 0.$$

Proposition 4 implies that, in contrast to the single housing market, the welfare costs incurred by different types of households are similar under the policy cap inflicted in a monocentric city model when commuting costs are negligible. The intuition is as follows. A monocentric city model contains infinitely many housing submarkets indexed by the distance r. Because the equilibrium exhibits perfect sorting by income, constrained low-income households only need to compete with other deprived low-income families interested in the same submarket, resulting in a lower equilibrium rent for these locations. Hence, the utility loss due to constrained housing expenditure share is compensated by the utility gain from lower market rent for the constrained households.

Next, we quantify the effect of the cap on equilibrium utility based on our calibrated model. We consider five different values for $\bar{\kappa}$, 20%, 25%, 30%, 35%, and 40%. For each value, we solve for the equilibrium utility and the consumption-equivalent measure ξ_i for all types of households.

Panel A of Table 5 reports the results. As shown in Section 3.3, commute costs are quite small in our calibrated model. Hence, consistent with Proposition 4, we find very similar ξ_i across household types for all $\bar{\kappa}$ that we consider. If anything, we find that ξ_i is generally smaller or more negative for relatively higher-income households.[22] Both the theoretical and quantitative results suggest that restricting the fraction of total income a household can spend on housing consumption could lead to a smaller welfare inequality in a monocentric city environment.

5 Spaceless Model and Non-homothetic Preference

This section will further explore the interaction of non-homothetic preference and the assumption of space. In particular, we will show that adding a simple Stone-Geary type non-homothetic utility function to a spaceless model drastically alters the implications for welfare inequality (Geary, 1950; Stone, 1954). In comparison, we also consider a monocentric city model with non-homothetic preference and discuss the implications of these results for the value of explicitly modeling space.

[22] In many cases, we see ξ_i is small but positive for all types of households. Notice that this does *not* represent an economy-wide Pareto improvement. Equilibrium market rents in these cases are lower than in the baseline, meaning the absentee landlord bears the cost.

Table 5: Consumption Equivalent Measure ξ_i (%) for Alternative Housing Expenditure Share Cap $\bar{\kappa}$

Item / Cap $\bar{\kappa}$	Lowest 10 percent	Second 10 percent	Third 10 percent	Fourth 10 percent	Fifth 10 percent	Sixth 10 percent	Seventh 10 percent	Eighth 10 percent	Ninth 10 percent	Tenth 10 percent
Panel A: Monocentric City Model: CES Preference										
40%	0.07	0.06	0.04	0.04	0.04	0.03	0.03	0.03	0.03	0.02
35%	0.56	0.54	0.48	0.44	0.41	0.38	0.35	0.33	0.31	0.28
30%	1.65	1.62	1.53	1.47	1.43	1.37	1.31	1.25	1.18	1.08
25%	1.56	1.52	1.40	1.33	1.27	1.18	1.10	1.00	0.89	0.70
20%	-2.91	-2.97	-3.14	-3.24	-3.33	-3.46	-3.58	-3.73	-3.90	-4.21
Panel B: Spaceless Model: Non-Homothetic Preference										
40%	-0.19	-0.06	0	0	0	0	0	0	0	0
35%	-1.22	-0.83	-0.27	-0.12	-0.04	0	0	0	0	0
30%	-3.31	-2.65	-1.55	-1.14	-0.88	-0.59	-0.40	-0.24	-0.12	-0.01
25%	-6.71	-5.77	-4.08	-3.39	-2.94	-2.40	-2.00	-1.63	-1.27	-0.81
20%	-11.81	-10.56	-8.24	-7.26	-6.59	-5.78	-5.16	-4.55	-3.94	-3.10
Panel C: Monocentric City Model: Non-Homothetic Preference										
40%	-0.13	0.02	0.05	0.04	0.04	0.03	0.03	0.02	0.01	0.01
35%	-0.91	-0.49	0	0.13	0.17	0.17	0.14	0.11	0.08	0.05
30%	-2.33	-1.64	-0.68	-0.34	-0.15	0.04	0.14	0.21	0.24	0.21
25%	-4.78	-3.84	-2.40	-1.84	-1.49	-1.11	-0.86	-0.64	-0.47	-0.30
20%	-8.86	-7.64	-5.68	-4.89	-4.36	-3.79	-3.38	-3.01	-2.68	-2.28

5.1 Spaceless Non-homothetic Model

Recall that the work hours are roughly constant across income groups. Hence, we consider a simpler version where all households work the same amount of time (40 hours per week), and a household's utility depends only on its consumption of numeraire good c and housing lot size h. Formally, we assume that its preference is *non-homothetic* and its utility is given by:

$$U(c, h) = (c + \bar{c})^{\beta_c} h^{1-\beta_c}, \tag{22}$$

where $\bar{c}$ is "free" consumption of numeraire goods, which, as shown below, can help generate a decreasing (with income) housing expenditure share.

In this spaceless model, all households reside at a location r^* miles away from their workplace, the CBD. The (weekly) budget constraint faced by a household with an hourly wage of w is given by

$$40w = c + P^*(r^*)h + ar^*, \tag{23}$$

where, as before, $P^*(r^*)$ is the equilibrium market rent and ar^* is the pecuniary cost of commuting.

Solving the household's problem yields the following equations for the optimal housing expenditure and the indirect utility function:

$$P^*(r^*)h(P^*(r^*), r^*) = (1 - \beta_c)(40w - ar^* + \bar{c}), \tag{24}$$

$$V(P^*(r^*), r^*) = \beta_c^{\beta_c}(1 - \beta_c)^{1-\beta_c}(40w - ar^* + \bar{c})[P^*(r^*)]^{\beta_c - 1}. \tag{25}$$

The optimal after-commute-cost (ACC) housing expenditure share in this model is then given by:

$$\frac{P^*(r^*)h(P^*(r^*), r^*)}{40w - ar^*} = (1 - \beta_c)(1 + \frac{\bar{c}}{40w - ar}). \tag{26}$$

It is trivial to see that this housing expenditure share is decreasing in hourly wage w, which is consistent with our second stylized fact.

Next we examine the implications of imposing a cap of $\bar{\kappa}$ on the ACC housing expenditure share, $\frac{P^*(r^*)h(P^*(r^*),r^*)}{40w-ar^*}$. Notice that because our model features an open city, the equilibrium rent equals the exogenous agricultural rent P_a with or without the housing expenditure cap. Hence, if the cap is not binding, the utility maximization problem faced by a particular household is identical to the baseline case. Consider a household with an hourly wage of w_i. If the optimal ACC housing expenditure share is lower than $\bar{\kappa}$, the cap is not binding for this household. In this case, this household will attain the same level of utility as in the baseline. On the other hand, if this share is greater than $\bar{\kappa}$, this household cannot allocate consumption expenditures optimally. As a result, it will attain utility that is strictly lower than the baseline level.

Because high-income (higher w) households have a lower optimal housing expenditure share, they are less likely to be constrained by the housing expenditure share cap $\bar{\kappa}$ compared to low-income households. Therefore, high-income households may not be affected for a moderate $\bar{\kappa}$, while low-income families may be constrained and attain lower utility (than in the baseline). Since high-income households have relatively higher utility than low-income households in the baseline ($V(P^*(r^*), r^*)$ is increasing in w), this policy cap on the ACC housing expenditure share would increase utility/welfare inequality.

We also quantitatively examine the welfare implications of the cap policy in this spaceless model. We calibrate the model to match the same targeted moments as in Section 3.3.1. The only three parameters calibrated to have different values from those for the monocentric city model are the agricultural rent P_a and preference parameters β_c and $\bar{c}$. P_a is the minimum market rent in the monocentric city model and the unique market rent in the spaceless model. Hence, to keep the city's size constant, P_a in the spaceless model needs to be larger than its counterpart in the monocentric city model. The calibrated weekly agricultural rent P_a is \$703,070 per mile^2. The preference parameters, β_c and $\bar{c}$, are chosen to match the relationship between housing expenditure share and total weekly expenditure/income (Figure 1) as closely as possible. Their calibrated values are $\beta_c = 0.7191$ and $\bar{c} = \$240$, respectively.[23]

Panel B of Table 5 reports the welfare results from this spaceless model. Consistent with the discussion above, a household experiences a drop in welfare (negative CE measure) only when its optimal ACC housing expenditure share is higher than $\bar{\kappa}$. Since low-income families have a higher optimal ACC housing expenditure share, they are more heavily (negatively) influenced by the cap. As a result, the consumption-equivalent measure ξ_i tends to be much more negative for low-income households. For example, a 20% cap imposes 3.10% welfare loss (in terms of ξ_i) for families in the top decile of the income distribution and 11.81% welfare loss for households in the bottom decile of the income distribution.

5.2 Spatial Non-homothetic Model

The previous sections have shown the welfare implications of the housing expenditure cap in a monocentric city model with CES preference and a spaceless model the non-homothetic preference (Panels A and B of Table 5). In this section, we will show in parallel the welfare implications of having non-homothetic preference in a monocentric city model (Panel C of Table 5). Expressly, we assume a household's problem is that described in Section 5.1 while keeping all the other model settings the same as those described in Section 3.

Appendix C.2 shows that this spatial non-homothetic model can generate the same spatial order as the spatial CES model - wealthier households live farther away from the CBD. We focus on quantitatively evaluating the welfare implications of imposing a housing expenditure share cap, $\bar{\kappa}$. Most calibrated parameters of the spatial non-homothetic model are the same as those of the spaceless non-homothetic model. The only exception is the weekly agricultural rent P_a, calibrated to be 644,910 per mile^2.

[23] See Appendix C.1 for the fit of the spaceless model.

We again measure welfare changes using the consumption-equivalent (ξ_i) defined in Definition 2. Panel C of Table 5 reports the welfare results. The general pattern is qualitatively similar to those of the spaceless model (Panel B). When a cap is imposed on the housing expenditure share, poorer households suffer a larger welfare drop than more affluent households. For example, a 20% cap imposes 2.28% welfare loss for households in the top decile of the income distribution and 8.86% welfare loss for households in the bottom decile of the income distribution, which results in a larger welfare inequality.

However, the quantitative differences are worth noting. Again using a 20% cap as an example, the welfare drop in the spatial model is smaller than that in the spaceless model (3.10% for the top decile and 11.81% for the bottom decile). We then turn our attention to the effect of the cap on welfare inequality. One measure of welfare inequality change is the difference between the top income decile's and bottom income decile's CE, $\xi_{10} - \xi_1$. This difference's positive (negative) value indicates an increase (decrease) in welfare inequality. For a 20% cap, this difference equals 6.58% (8.86% - 2.28%) in the spatial model and 8.71% (11.81% - 3.10%) in the spaceless model. In Figure 2, we plot this measure of welfare inequality change for a wide range of $\bar{\kappa}$ and for all three models we have considered. For all levels of $\bar{\kappa}$, Figure 2 shows that while, unlike the case in the spatial CES model, we still see aggravated welfare inequality due to the cap policy in the spatial non-homothetic model, the magnitude of this effect is smaller compared to that in the spaceless non-homothetic model.

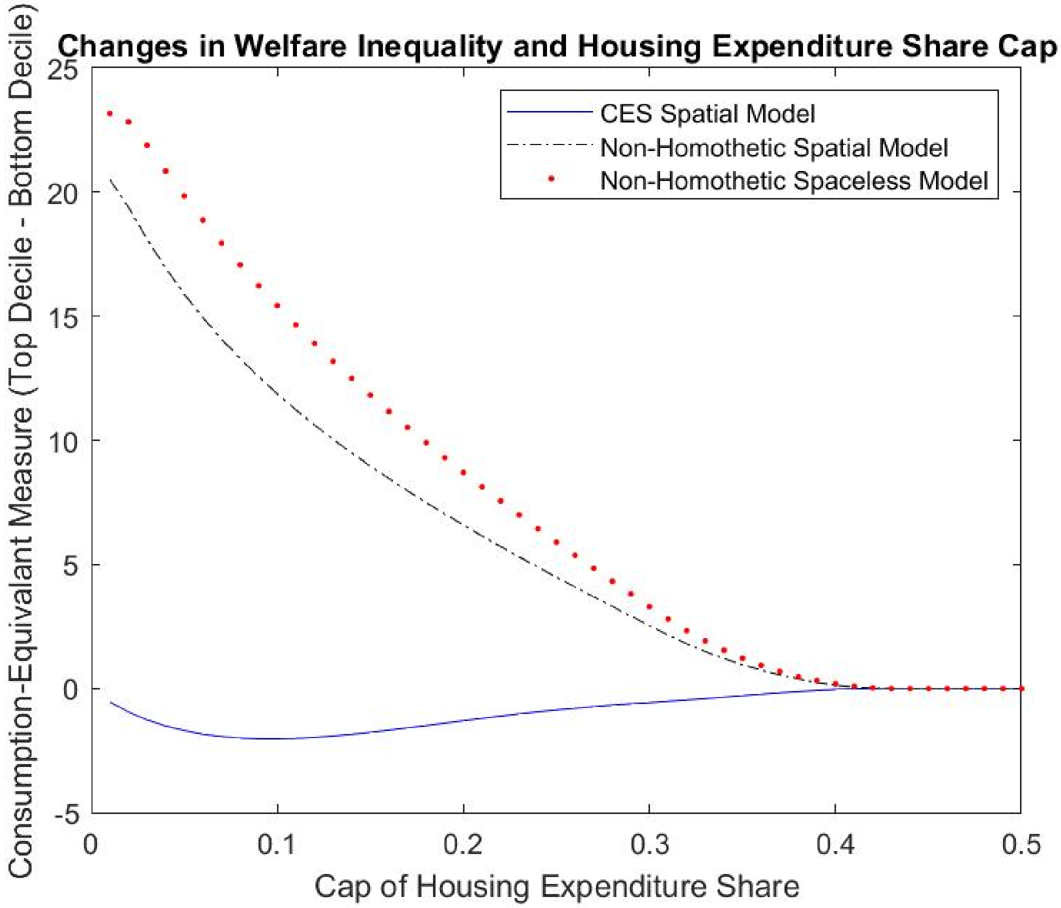
The quantitative difference between the welfare results in the spatial non-homothetic model and the spaceless non-homothetic model stems from the general equilibrium price adjustment and income-based spatial sorting discussed in Section 4.3. The welfare inequality still increases in the spatial non-homothetic model because the general equilibrium channel is not strong enough when households have (non-homothetic) Stone-Geary preference. In this case, market rent only affects welfare through its influence on housing affordability. In contrast, in the CES model, market rent influences housing affordability and determines the optimal housing expenditure share. This second additional effect strengthens the general equilibrium channel, allowing it to entirely offset the aggravating impact of the cap on welfare inequality in the spatial CES model.

5.3 Discussion: The Value of Modeling Space

In examining the effect of a housing expenditure share cap, the value of modeling space is three-fold. First, it enables our model to match the empirical relationship between commute distance and household income. In contrast, by definition, a spaceless model is silent on this empirical finding.

Second, because of the general equilibrium channel discussed above, keeping household preference unchanged, the aggravating effect of the cap on welfare inequality is always more potent in a model without space than in a model with space. This is the case even when the preference is non-homothetic (Stone-Geary).

Figure 2: Changes in Welfare Inequality and Housing Expenditure Share Cap



Third, perhaps more importantly, by explicitly modeling space, we can generate the negative relationship between income and housing expenditure share using homothetic preference (CES). This presents an important *unidentification* issue because both the spatial CES model and the spatial Stone-Geary model can match the three stylized facts documented in Section 2 but have drastically different policy predictions. Intuitively, these two models represent two extremes. The optimal housing expenditure share is determined entirely by equilibrium market rents in the former and entirely by hourly wages in the latter. In reality, this share likely depends on both. That means the true effect of the cap policy on welfare inequality is somewhere between the blue solid line and the black dashed line in Figure 2. To point-identify this effect, one needs first to bring in more data to recover households' preferences precisely.

6 Conclusion

This paper reinforces the intuitive and essential point that spatial consideration matters in economic analysis. We frame the statement in the context of the housing expenditure share cap. Variants of this policy have been explicitly and implicitly adopted as some components of macroprudential policies worldwide. We document three cross-sectional, stylized facts: the housing expenditure share decreases with income, the apparent constancy of work hours across income groups, and the positive relationship between income and the commute distance from the CBD. While non-homothetic preference is needed for a spaceless model to match the first two facts, a simple monocentric city model can match all three facts with both homothetic and non-homothetic preferences.

Moreover, because all households face the same housing price, the spaceless model would predict that the cap policy leads to a disproportionately high welfare loss for low-income families constrained by the cap, resulting in substantially larger welfare inequality. The monocentric city model has a different prediction. Because of income-based spatial sorting and geographically differentiated housing price adjustments, constrained low-income families experience a disproportionally more considerable drop in endogenous equilibrium rents than unconstrained high-income families. Depending on households' preferences, the welfare gain from this mechanism can partially (non-homothetic preference) or entirely (homothetic preference) offset the welfare loss due to the cap policy, resulting in a smaller increase (non-homothetic preference) or even a mild reduction (homothetic preference) in welfare inequality after the cap is imposed.

Our results suggest that policymakers should incorporate spatial considerations in the design of macroprudential policies. Moreover, our finding that the exact welfare results depend on households' preferences highlights the importance for future research to precisely estimate a flexible utility function that allows the housing expenditure share to depend on both household income and housing rents/prices.

We have considered several extensions in this paper, including introducing idiosyncratic location preference shock and extending to a more realistic, dynamic setting. Future research can explore other extensions, including collateral constraints, dynamic location choice, endogenous housing supply, endogenous tenure choice, multi-community within a city, and portfolio choices.[24] These new features can facilitate a more comprehensive and in-depth analysis of macroprudential policies.

[24] See Beraja et al. (2019), Combes, Duranton, and Gobillon (2021), Hanushek and Yilmaz (2007, 2013), Leung and Teo (2011), Piazzesi, Schneider, and Stroebel (2020), and Yao (2023), among others.

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Appendices

A Proofs

Proposition 1. *If* $\rho > 1$*, then* $\frac{\partial \kappa(x)}{\partial x} > 0$.

Proof. Recall that $\kappa(x;\theta,\rho) = \frac{(\frac{1-\theta}{\theta})^{\frac{1}{\rho}} x^{\frac{\rho-1}{\rho}}}{1+(\frac{1-\theta}{\theta})^{\frac{1}{\rho}} x^{\frac{\rho-1}{\rho}}}$. Taking the derivative of this function with respect to x, we have:

$$\begin{aligned}
\frac{\partial \kappa(x;\theta,\rho)}{\partial x} &= \frac{[1+(\frac{1-\theta}{\theta})^{\frac{1}{\rho}} x^{\frac{\rho-1}{\rho}}](\frac{1-\theta}{\theta})^{\frac{1}{\rho}} x^{\frac{\rho-1}{\rho}-1}\frac{\rho-1}{\rho}}{[1+(\frac{1-\theta}{\theta})^{\frac{1}{\rho}} x^{\frac{\rho-1}{\rho}}]^2} \\
&- \frac{(\frac{1-\theta}{\theta})^{\frac{1}{\rho}} x^{\frac{\rho-1}{\rho}} (\frac{1-\theta}{\theta})^{\frac{1}{\rho}} x^{\frac{\rho-1}{\rho}-1}\frac{\rho-1}{\rho}}{[1+(\frac{1-\theta}{\theta})^{\frac{1}{\rho}} x^{\frac{\rho-1}{\rho}}]^2} \\
&= \frac{\rho-1}{\rho}\frac{(\frac{1-\theta}{\theta})^{\frac{1}{\rho}} x^{\frac{-1}{\rho}}}{[1+(\frac{1-\theta}{\theta})^{\frac{1}{\rho}} x^{\frac{\rho-1}{\rho}}]^2}. \qquad (27)
\end{aligned}$$

This derivative has the same sign as $\frac{\rho-1}{\rho}$. Hence, if $\rho > 1$, then $\frac{\rho-1}{\rho} > 0$, which implies that $\frac{\partial \kappa(x;\theta,\rho)}{\partial x} > 0$.

□

Proposition 2. *Households with higher hourly wage* w_i *live farther from the CBD at the equilibrium.*

Proof. Recall that the bid-rent function for Type i household, $\psi_i(u_i^*, r)$, is given by:

$$\psi_i(u_i^*, r) = \left\{\left[\frac{Y_i(r)K}{u_i^* w_i^{1-\alpha}}\right]^{\frac{\rho-1}{\alpha\rho}} - 1\right\}^{\frac{\rho}{\rho-1}} (\frac{\theta}{1-\theta})^{\frac{1}{\rho-1}}, \qquad (28)$$

where $Y_i(r) \equiv (168 - br)w_i - ar$.

The first order derivative of this function with respect to the distance r is given by:

$$\begin{aligned}
\frac{\partial \psi_i(\cdot)}{\partial r} &= \psi_i(u_i^*, r)\frac{\rho}{\rho-1}\left\{\left[\frac{Y_i(r)K}{u_i^* w_i^{1-\alpha}}\right]^{\frac{\rho-1}{\alpha\rho}} - 1\right\}^{-1} \frac{\rho-1}{\alpha\rho}\left[\frac{Y_i(r)K}{u_i^* w_i^{1-\alpha}}\right]^{\frac{\rho-1}{\alpha\rho}} \frac{Y_i'(r)}{Y_i(r)} \\
&= \frac{\psi_i(u_i^*, r)}{\alpha}\frac{\left[\frac{Y_i(r)K}{u_i^* w_i^{1-\alpha}}\right]^{\frac{\rho-1}{\alpha\rho}}}{\left[\frac{Y_i(r)K}{u_i^* w_i^{1-\alpha}}\right]^{\frac{\rho-1}{\alpha\rho}} - 1}\frac{-(a+bw_i)}{(168-br)w_i - ar}. \qquad (29)
\end{aligned}$$

The fact that $\psi_i(u_i^*, r) > 0$ implies that $\left[\frac{Y_i(r)K}{u_i^* w_i^{1-\alpha}}\right]^{\frac{\rho-1}{\alpha\rho}} - 1 > 0$. Hence, we conclude that $\frac{\partial \psi_i(\cdot)}{\partial r} < 0$ from Equation (29). It means all bid-rent functions are decreasing in the distance from the CBD, r.

Let $r_{B,i}^*$ denote the distance from the CBD at which Type i households and Type $i+1$ households intersect in the baseline equilibrium. Note that, at the equilibrium, we have $\psi_i(u_i^*, r_{B,i}^*) = \psi_{i+1}(u_{i+1}^*, r_{B,i}^*)$ (same rental rate for both types of households at the intersection). Together with Equation (28), it implies that:

$$\frac{\psi_i(u_i^*, r_{B,i}^*)}{\alpha} \frac{\left[\frac{Y_i(r_{B,i}^*)K}{u_i^* w_i^{1-\alpha}}\right]^{\frac{\rho-1}{\alpha\rho}}}{\left[\frac{Y_i(r_{B,i}^*)K}{u_i^* w_i^{1-\alpha}}\right]^{\frac{\rho-1}{\alpha\rho}} - 1} = \frac{\psi_{i+1}(u_{i+1}^*, r_{B,i}^*)}{\alpha} \frac{\left[\frac{Y_{i+1}(r_{B,i}^*)K}{u_{i+1}^* w_{i+1}^{1-\alpha}}\right]^{\frac{\rho-1}{\alpha\rho}}}{\left[\frac{Y_{i+1}(r_{B,i}^*)K}{u_{i+1}^* w_{i+1}^{1-\alpha}}\right]^{\frac{\rho-1}{\alpha\rho}} - 1}. \tag{30}$$

Denoting the value of this term as $Cons_{\psi,i}^*$, we have:

$$\begin{aligned} |\frac{\partial \psi_i(u_i^*, r_{B,i}^*)}{\partial r}| &= Cons_{\psi,i}^* \frac{a + bw_i}{(168 - br)w_i - ar}, \\ |\frac{\partial \psi_{i+1}(u_{i+1}^*, r_{B,i}^*)}{\partial r}| &= Cons_{\psi,i}^* \frac{a + bw_{i+1}}{(168 - br)w_{i+1} - ar}. \end{aligned} \tag{31}$$

Then, the difference between the two slopes is given by:

$$\begin{aligned} |\frac{\partial \psi_i(\cdot)}{\partial r}| - |\frac{\partial \psi_{i+1}(\cdot)}{\partial r}| &= Cons_{\psi,i}^* [\frac{a + bw_i}{(168 - br)w_i - ar} - \frac{a + bw_{i+1}}{(168 - br)w_{i+1} - ar}] \\ &= \frac{Cons_{\psi,i}^* \{(a + bw_i)[(168 - br)w_{i+1} - ar] - (a + bw_{i+1})[(168 - br)w_i - ar]\}}{[(168 - br)w_i - ar][(168 - br)w_{i+1} - ar]} \\ &= \frac{168 Cons_{\psi,i}^* a(w_{i+1} - w_i)}{[(168 - br)w_i - ar][(168 - br)w_{i+1} - ar]} \\ &> 0, \end{aligned} \tag{32}$$

where the last line follows from the fact that Type i households has lower hourly wages than Type $i+1$ households ($w_i < w_{i+1}$).

Thus, we have shown that $|\frac{\partial \tilde{\psi}_i(\cdot)}{\partial r}|$ is larger for households with lower hourly wage w_i, which implies that households with higher hourly wage w_i lives farther from the CBD at the equilibrium. □

Proposition 3. *Consider two arbitrary households. We denote their types as i and j, respectively. Let r_i^* and r_j^* denote their distance from the CBD at the equilibrium. If $\rho > 1$ and $w_i > w_j$, then $\kappa(P(r_i^*)) < \kappa(P(r_j^*))$ at the equilibrium.*

Proof. If $w_i > w_j$, then Proposition 2 implies that $r_i^* > r_j^*$. Because the equilibrium rent function $P(r)$ is decreasing in the distance r, we have $P(r_i^*) < P(r_j^*)$. Hence, Proposition 1 implies that $\kappa(P(r_i^*)) < \kappa(P(r_j^*))$. □

Proposition 4. *For any $\bar{\kappa} \in (0, 1]$ and any Type i and Type j households, we have*

$$\lim_{a->0^+, b->0^+} \xi_i - \xi_j = 0.$$

Proof. Consider two equilibria, the baseline equilibrium and the equilibrium with the cap $\bar{\kappa}$. Let $r_{B,i}^*$ and $r_{\bar{\kappa},i}^*$ denote the distances from the CBD at which Type i households and Type $i+1$ households intersect in the baseline equilibrium and in the equilibrium with the cap, respectively. Let $P_B^*(r)$ and $P_{\bar{\kappa}}^*(r)$ denote the rent functions in the baseline equilibrium and in the equilibrium with the cap, respectively. Further, noting that $\tilde{\kappa} = \min(\kappa(P^*(r)), \bar{\kappa})$ only depends on the equilibrium rent $P^*(r)$ (and the cap $\bar{\kappa}$), we let $\tilde{\kappa}^*_{B,i}$ and $\tilde{\kappa}^*_{\bar{\kappa},i}$ denote the value of $\tilde{\kappa}$ at the distances from the CBD at which Type i households and Type $i+1$ households intersect in the baseline equilibrium and in the equilibrium with the cap, respectively.[25]

The utility that Type i household obtains in the baseline, $\tilde{V}^*_{B,i}$, is given by:

$$\tilde{V}^*_{B,i} = [(1-\tilde{\kappa}^*_{B,i})^{1-\rho} + \frac{1-\theta}{\theta}\frac{(\tilde{\kappa}^*_{B,i})^{1-\rho}}{P^*_B(r^*_{B,i})^{1-\rho}}]^{\frac{\alpha}{1-\rho}} KY_i(r^*_{B,i})(\frac{1}{w_i})^{1-\alpha}. \tag{33}$$

, where $K = \alpha^\alpha(1-\alpha)^{1-\alpha}\theta^{\frac{\alpha}{1-\rho}}$.

When this household is given ξ fraction more consumption (both c and h), it is trivial to see that the utility $\tilde{V}^*_{B,i}(\xi)$ is given by:

$$\begin{aligned}\tilde{V}^*_{B,i}(\xi) &= (1+\xi)^{\frac{\alpha}{1-\rho}}\tilde{V}^*_{B,i}(\xi) \\ &= (1+\xi)^{\frac{\alpha}{1-\rho}}[(1-\tilde{\kappa}^*_{B,i})^{1-\rho} + \frac{1-\theta}{\theta}\frac{(\tilde{\kappa}^*_{B,i})^{1-\rho}}{P^*_B(r^*_{B,i})^{1-\rho}}]^{\frac{\alpha}{1-\rho}} KY_i(r^*_{B,i})(\frac{1}{w_i})^{1-\alpha}. \end{aligned}\tag{34}$$

The utility that Type i household obtains in the equilibrium with the cap, $\tilde{V}^*_{\bar{\kappa},i}$, is given by:

$$\tilde{V}^*_{\bar{\kappa},i} = [(1-\tilde{\kappa}^*_{\bar{\kappa},i})^{1-\rho} + \frac{1-\theta}{\theta}\frac{(\tilde{\kappa}^*_{\bar{\kappa},i})^{1-\rho}}{P^*_{\bar{\kappa}}(r^*_{\bar{\kappa},i})^{1-\rho}}]^{\frac{\alpha}{1-\rho}} KY_i(r^*_{\bar{\kappa},i})(\frac{1}{w_i})^{1-\alpha}. \tag{35}$$

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The consumption-equivalent measure for Type i household, ξ_i, is defined as the solution to the equation $\tilde{V}^*_{B,i}(\xi_i) = \tilde{V}^*_{\bar{\kappa},i}$. Solving this equation, we have:

$$\xi_i = [\frac{Y_i(r^*_{\bar{\kappa},i})}{Y_i(r^*_{B,i})}]^{\frac{1-\rho}{\alpha}} \frac{(1-\tilde{\kappa}^*_{\bar{\kappa},i})^{1-\rho} + \frac{1-\theta}{\theta}\frac{(\tilde{\kappa}^*_{\bar{\kappa},i})^{1-\rho}}{P^*_{\bar{\kappa}}(r^*_{\bar{\kappa},i})^{1-\rho}}}{(1-\tilde{\kappa}^*_{B,i})^{1-\rho} + \frac{1-\theta}{\theta}\frac{(\tilde{\kappa}^*_{B,i})^{1-\rho}}{P^*_B(r^*_{B,i})^{1-\rho}}} - 1. \tag{36}$$

Following the same steps, we can show that the consumption-equivalent measure for Type $i+1$ household, ξ_{i+1}, is given by:

$$\xi_{i+1} = [\frac{Y_{i+1}(r^*_{\bar{\kappa},i})}{Y_{i+1}(r^*_{B,i})}]^{\frac{1-\rho}{\alpha}} \frac{(1-\tilde{\kappa}^*_{\bar{\kappa},i})^{1-\rho} + \frac{1-\theta}{\theta}\frac{(\tilde{\kappa}^*_{\bar{\kappa},i})^{1-\rho}}{P^*_{\bar{\kappa}}(r^*_{\bar{\kappa},i})^{1-\rho}}}{(1-\tilde{\kappa}^*_{B,i})^{1-\rho} + \frac{1-\theta}{\theta}\frac{(\tilde{\kappa}^*_{B,i})^{1-\rho}}{P^*_B(r^*_{B,i})^{1-\rho}}} - 1. \tag{37}$$

The key here is to notice that Type i and Type $i+1$ households share the same rent and housing expenditure share at the intersection of the two types.

Thus, we conclude that:

$$\xi_{i+1} - \xi_i = \{[\frac{Y_{i+1}(r^*_{\bar{\kappa},i})}{Y_{i+1}(r^*_{B,i})}]^{\frac{1-\rho}{\alpha}} - [\frac{Y_i(r^*_{\bar{\kappa},i})}{Y_i(r^*_{B,i})}]^{\frac{1-\rho}{\alpha}}\}Cons^*_i, \tag{38}$$

where $Cons^*_i \equiv \frac{(1-\tilde{\kappa}^*_{\bar{\kappa},i})^{1-\rho} + \frac{1-\theta}{\theta}\frac{(\tilde{\kappa}^*_{\bar{\kappa},i})^{1-\rho}}{P^*_{\bar{\kappa}}(r^*_{\bar{\kappa},i})^{1-\rho}}}{(1-\tilde{\kappa}^*_{B,i})^{1-\rho} + \frac{1-\theta}{\theta}\frac{(\tilde{\kappa}^*_{B,i})^{1-\rho}}{P^*_B(r^*_{B,i})^{1-\rho}}}$.

[25]In the baseline equilibrium, $\bar{\kappa} = 1$.

Recall that $Y_i(r) \equiv (168 - br)w_i - ar$. Hence, we have:

$$\begin{aligned}\frac{Y_i(r^*_{\bar{\kappa},i})}{Y_i(r^*_{B,i})} &= \frac{(168 - br^*_{\bar{\kappa},i})w_i - ar^*_{\bar{\kappa},i}}{(168 - br^*_{B,i})w_i - ar^*_{B,i}} \\ &= \frac{(168 - br^*_{B,i})w_i - ar^*_{B,i}}{(168 - br^*_{B,i})w_i - ar^*_{B,i}} + \frac{b(r^*_{B,i} - r^*_{\bar{\kappa},i})w_i + a(r^*_{B,i} - r^*_{\bar{\kappa},i})}{(168 - br^*_{B,i})w_i - ar^*_{B,i}} \\ &= 1 + \frac{(a + w_i b)(r^*_{B,i} - r^*_{\bar{\kappa},i})}{(168 - br^*_{B,i})w_i - ar^*_{B,i}} \\ &\xrightarrow[a \to 0^+,\ b \to 0^+]{} 1. \end{aligned} \tag{39}$$

This proves that $\xi_{i+1} - \xi_i \xrightarrow[a \to 0^+,\ b \to 0^+]{} 0$ for all $i = 1, 2, ..., 9$. As a result, we show that $\xi_j - \xi_i \xrightarrow[a \to 0^+,\ b \to 0^+]{} 0$ for all $i = 1, 2, ..., 10$ and $j = 1, 2, ..., 10$.

□

B Extensions

Our main analysis is based on a static model. While it is compatible with the previous literature (e.g, Hanushek and Yilmaz, 2022), there is room for further improvement. For one thing, a static model does not distinguish between the for-sale and rental markets. In practice, buyers typically do not pay in full but instead, make a mortgage loan. Therefore, many macroprudential measures target the debt-to-income ratio (IMF, 2014, 2018). In addition, our model features perfect sorting of income. In reality, while robust income-based sorting exists, such a sorting pattern is not perfect, i.e., households with different income levels reside in the same neighborhood (Davidoff, 2005; Gaignea et al., 2022). Therefore, we consider two extensions of the baseline model in this section. First, we extend our model to a two-period setting, which enables us to consider the for-sale market and mortgage loans. Hence, we can explicitly investigate the effect of a cap on the mortgage debt-to-income ratio, a commonly adopted macroprudential policy. We introduce unobserved preference shock to the baseline model in the second extension. This feature allows us to examine how our conclusions would change under imperfect income sorting. Our main finding that the cap policy does not lead to immense welfare inequality remains in both extensions.

B.1 A Two-Period Model

In this section, we extend our model to a two-period setting and then analyze the welfare effect of the cap on the debt-to-income ratio.

B.1.1 Model

A household's lifecycle is divided into two periods, with period t corresponding to q_t weeks. We normalize a household's lifespan to one week so that $q_1 + q_2 = 1$. To facilitate the comparison with the earlier sections, we assume that the periodic utility function in each period is the same as the one in the baseline static model (Equation 1) and that households do not discount future utility. Households only make residential decisions (location and housing lot size) in the first period. The total utility in this two-period model is given by:

$$U(c_1, c_2, h, l_1, l_2) = \sum_{t=1,2} q_t l_t^{1-\alpha}[\theta c_t^{1-\rho} + (1-\theta)h^{1-\rho}]^{\frac{\alpha}{1-\rho}}, \tag{40}$$

where h denotes the housing lot size and, for $t = 1, 2$, c_t denotes the consumption per week on numeraire goods and l_t denotes leisure time per week.

As before, in period t, the total income of a household living r miles away from the CBD with an hourly wage w is $q_t(168-l_t-br)w$, where b is the time cost per mile of weekly round-trip commute. In the first period, the household pays for the consumption goods and pecuniary commute cost ($q_1(c_1+ar)$) and purchases a housing unit by combining their period-1 income and D amount of mortgage debt. In the second period, the household pays back the mortgage and spends the remaining period-2 income on consumption goods and commute ($q_2(c_2 + ar)$). Since households do not discount future utility, we set the mortgage interest rate to zero, as alternative values for the interest rate would not change our substantive conclusions. Finally, we assume that the maximum amount of mortgage debt $\bar{D}$ that a household can take is proportional to the household's period-2 after-commute-cost (ACC) income, i.e., $\bar{D} = \bar{\kappa}_D q_2[(168 - l_2 - br)w - ar]$.[26] We refer to $\bar{\kappa}_D$ as the debt-to-income ratio. To summarize, the household's budget constraints are given as follows:

$$\begin{aligned} q_1(c_1 + ar) + P(r)h &= q_1[168 - l_1 - br]w + D, \\ q_2(c_2 + ar) + D &= q_2[168 - l_2 - br]w, \\ D &\leq \bar{\kappa}_D q_2[(168 - l_2 - br)w - ar]. \end{aligned} \tag{41}$$

Other aspects of the model, including the land allocation mechanism and market clearing conditions, remain the same as the baseline model.

B.1.2 Model Solution and Welfare Analysis

As before, this dynamic model can be solved using a bid-rent function approach. We first note that, when the borrowing does not reach the limit, a household (optimally) chooses to have the same consumption and leisure time per week in the two periods, i.e., $c_1 = c_2$ and $l_1 = l_2$. In this case, the utility function and the budget constraints of the two-period model become identical to the static model. As a result, the equilibrium allocation and housing consumption of households in the two-period model are the same as those in the static model. The optimal (ACC) housing expenditure share given equilibrium housing price $P^*(r^*)$, $\frac{P^*(r^*)h}{q_1c_1+q_2c_2+P^*(r^*)h}$, is still given by $\kappa(P^*(r^*); \theta, \rho)$, where $\kappa(x; \theta, \rho) = \frac{(\frac{1-\theta}{\theta})^{\frac{1}{\rho}} x^{\frac{\rho-1}{\rho}}}{1+(\frac{1-\theta}{\theta})^{\frac{1}{\rho}} x^{\frac{\rho-1}{\rho}}}$. This equivalence result implies that we can assign the calibrated parameter

[26] Alternatively, we can assume that $\bar{D}$ is proportional to the household's period-1 ACC income, i.e., $\bar{D} = \bar{\kappa}_D q_1[(168 - l_1 - br)w - ar]$. This does not change our qualitative results.

values from the baseline static model to the two-period model. The only parameter unspecified in the static model is the length of the first period, q_1 (and $q_2 = 1 - q_1$). While this parameter does not influence the equilibrium outcomes when all households are unconstrained, it can affect our counterfactual welfare analysis. Assuming that the first period corresponds to ages 20-40 and the second period corresponds to ages 40-80, we let $q_1 = 1/3$ and $q_2 = 2/3$. Alternative parameterization of q_1 and q_2 does not lead to substantially different results.

When the borrowing constraint is binding, it restricts the amount of debt the household can take to finance their home purchase, effectively imposing a cap on the housing expenditure share. One can show that a household is borrowing constrained if and only if the unconstrained optimal housing expenditure share $\kappa(P^*(r^*); \theta, \rho)$ is larger than the cap on the debt-to-income ratio, $\bar{\kappa}_D$. Thus, if no one is constrained by the housing expenditure share cap in the static model ($\bar{\kappa}$), then no one will be constrained by a debt-to-income ratio cap ($\bar{\kappa}_D$) of the same value in the two-period model. However, it is essential to note that this result does not imply full equivalence regarding welfare implications between the housing expenditure share cap and the debt-to-income ratio cap. To examine the welfare implications of the debt-to-income ratio cap, we still need to numerically solve the two-period model with the cap and compute the consumption equivalence measures, ξ.

Recall that ξ is defined as the faction of numeraire consumption goods (c_1 and c_2) and housing consumption (h) that a household needs to obtain in the economy without debt limits to achieve the same level of utility in the economy with a debt limit. A higher value of ξ is associated with more utility gain in percentage terms. We solve for the equilibrium utility and the consumption-equivalent measure ξ_i for all types of households for $\bar{\kappa}_D = 20\%$, 25%, 30%, 35%, and 40%. Panel A of Table 6 reports the results. Comparing with the results in the baseline static model (Panel A of Table 5), we find that the welfare impact of the cap on the debt-to-income ratio is much smaller than that of the cap on the housing expenditure share. The intuition behind this result is simple. Even in the extreme case where the household cannot borrow against the future (i.e., $\bar{\kappa}_D = 0$), the household can still have a considerable amount of housing expenditures using their income from the first period. Despite these differences in the magnitude of the welfare impact, our qualitative conclusions about welfare inequality based on the static model remain valid in the two-period model. For all five levels of caps that we considered, ξ_i is generally smaller for relatively higher-income households. Thus, a cap on the debt-to-income ratio does not magnify the welfare inequality in a monocentric city model.

B.2 A Model with Unobserved Preference Shock

Notice that our principal result regarding the welfare consequences of the housing expenditure cap relates to income sorting in the housing market. Recall that low-income

Table 6: Consumption Equivalent Measure ξ_i (%) for Alternative Debt-to-Income Ratio Cap $\bar{\kappa}_D$ and Housing Expenditure Share Cap $\bar{\kappa}$

Item / Cap $\bar{\kappa}_D/\bar{\kappa}$	Lowest 10 percent	Second 10 percent	Third 10 percent	Fourth 10 percent	Fifth 10 percent	Sixth 10 percent	Seventh 10 percent	Eighth 10 percent	Ninth 10 percent	Tenth 10 percent
Panel A: Two-Period Model, Debt-to-Income Ratio Cap $\bar{\kappa}_D$										
40%	0.00	0	0	0	0	0	0	0	0	0
35%	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
30%	0.02	0.02	0.01	0.01	0.01	0.01	0.01	0.00	0.00	0.00
25%	0.04	0.03	0.03	0.03	0.03	0.03	0.03	0.02	0.02	0.02
20%	0.03	0.03	0.03	0.03	0.02	0.02	0.02	0.02	0.02	0.02
Panel B: Model with Unobserved Preference Shock, Housing Expenditure Share Cap $\bar{\kappa}$										
40%	0.04	0.03	0.02	0.02	0.02	0.02	0.02	0.02	0.01	0.01
35%	0.32	0.31	0.26	0.24	0.22	0.20	0.19	0.18	0.17	0.15
30%	0.92	0.89	0.83	0.79	0.76	0.72	0.68	0.64	0.60	0.54
25%	1.69	1.66	1.59	1.55	1.51	1.45	1.40	1.34	1.27	1.16
20%	-3.13	-3.17	-3.27	-3.34	-3.39	-3.48	-3.57	-3.66	-3.78	-3.99

households self-select into neighborhoods near the CBD in our static model. Thus, under the housing expenditure cap, they tend to compete with other low-income households, who are also constrained. Therefore, the equilibrium rent decreases and low-income households would not suffer more than their high-income counterparts. While we believe this is a reasonable first-order approximation, there are issues to be considered. Theoretically, the conditions needed for perfect income sorting may be strong and not be satisfied in all situations (Gravel and Oddoub, 2014). In practice, sorting is often imperfect, and households with different income levels live near each other. In addition, amenities may also affect the geographic sorting (Gaignea et al., 2022). Hence, it would be worthwhile to re-examine our main welfare conclusion under imperfect sorting. To proceed, we first build a model where imperfect sorting arises in equilibrium, which is the focus of the next section. We then calibrate the model to match the data and perform counterfactual policy analysis.

B.2.1 Model

Income sorting across neighborhoods is imperfect (Davidoff, 2005). One possibility is the amenities differ across neighborhoods (Carlino and Saiz, 2019).[27] In this section, we extend our baseline model to allow the utility that a household derives from a location to depend on an additive unobserved location-specific preference shock. Formally, when a housing expenditure share cap of $\bar{\kappa}$ is imposed, the total utility $\tilde{V}_i^T$ associated with a location r miles away from the CBD is given by:

$$\tilde{V}_i^T(P^*(r), r) = \tilde{V}_i(P^*(r), r) + \sigma\epsilon_{ir}, \tag{42}$$

where $\tilde{V}_i(P^*(r), r)$ is the indirect utility that household i obtains given location r and equilibrium rent $P^*(r)$ (Equation 16), ϵ_{ir} has an i.i.d. Type-I extreme value distribution, and σ measures the magnitude of the magnitude of the unobserved preference shock.[28]

As is standard in the literature, we assume a household chooses the location associated with the highest total utility $\tilde{V}_i^T$ to reside. This decision is made based on both a constant component $\tilde{V}_i(P^*(r), r)$ and an unobserved random component $\sigma\epsilon_{ir}$. As a result, the equilibrium location decision for a particular type of household is described by a household density distribution over all possible locations. In our model, each location corresponds to one dot on the two-dimensional plane within the fringe distance R_f^*, which is the distance at which the market rent $P^*(r)$ equals the agricultural rent P_a.

Then, under the assumption of Type-I extreme value distribution for ϵ_{ir}, the equilibrium density of Type i household at all locations r miles away from the CBD is given

[27] There are alternative explanations. For instance, De Bartolome and Ross (2003) explains the imperfect income sorting by fiscal competition between the inner city and suburb governments. Here we abstract from the fiscal competition and leave such possibility to future research.

[28] Note that, when $\bar{\kappa} = 1$, $\tilde{V}_i(P^*(r), r)$ equals the unconstrained indirect utility $V(P^*(r), r)$ given by Equation (5).

by:

$$m_i(P^*(r), r) = \frac{e^{\frac{\tilde{V}_i(P^*(r),r)}{\sigma}} L(r)}{\int_0^{R_f^*} e^{\frac{\tilde{V}_i(P^*(r),r)}{\sigma}} L(r)dr} \overline{N}_i, \tag{43}$$

where $L(r) = 2\pi r$ is the amount of locations available per unit distance, at a distance r, and $\overline{N}_i$ is the total number of Type i households.

To close the model, we impose the market clearing condition at all locations at the equilibrium. It means that for all distance $r \leq R_f^*$, the following equation is satisfied.

$$\sum_i^{10} m_i(P^*(r), r)\tilde{h}_i(P^*(r), r) = L(r), \tag{44}$$

where $\tilde{h}_i(P^*(r), r)$ (Equation 17) is the house size demanded by Type i household at distance r given equilibrium rent $P^*(r)$.

B.2.2 Re-calibration and Welfare Analysis

Compared to the baseline model, this extended model has one additional parameter, σ, which captures the magnitude of the unobserved preference shock. Intuitively, this parameter determines the degree of income-based sorting at the equilibrium. In one extreme case where $\sigma \to \infty$, we have $m_i(P^*(r), r) \to \frac{L(r)}{\int_0^{R_f^*} L(r)dr} \overline{N}_i$, implying that, for any Type i, households at all income levels are uniformly distributed across all locations. As a result, we do not observe income-based sorting at the equilibrium. In the other extreme case where $\sigma \to 0$, all equilibrium objects converge to their values in the case without the unobserved preference shock, and we observe perfect income-based sorting. Hence, it is essential to find a plausible value of σ for our policy analysis.

In the current context, the stronger the income-based sorting is, the more the average household income co-move with the commute distance. Hence, following the discussion above, we calibrate the value of σ to match our empirical finding reported in Section 2 that a 1,000 dollars increase in annual wage income is associated with a 0.0140-mile increase in commute distance.[29] More specifically, for each possible value of σ, we randomly sample households from the equilibrium and regress a household's annual gross income on the household's distance from the CBD.[30] We find that $\sigma = 0.4524$ is the value that produces a regression coefficient of 0.0140.[31] As a comparison, in the baseline model where the unobserved preference shock is not present, the regression coefficient is 0.0175.

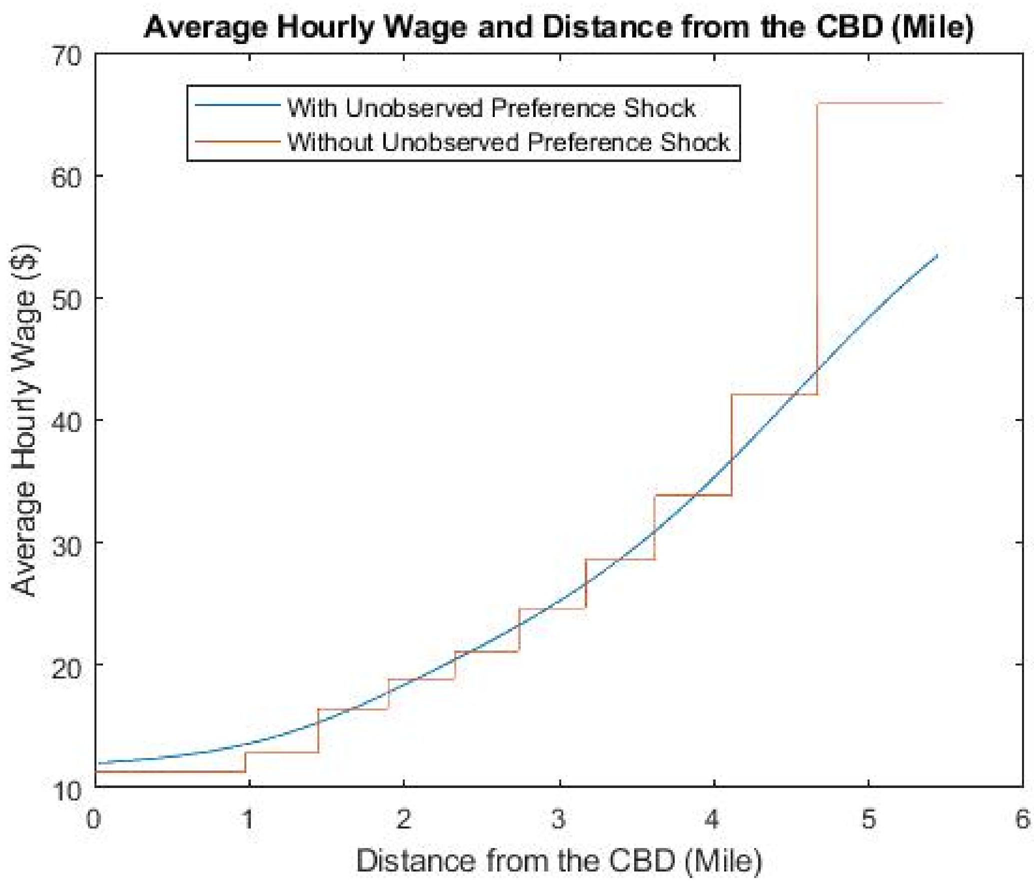
[29] We also need to re-calibrate the preference parameters to match the type-specific housing expenditure shares in the data, θ, and ρ. This is because the relationship between the housing expenditure share and income changes with the degree of income-based sorting.

[30] Note that, for each type of household, the wage rate w_i is calibrated to match annual expenditures instead of annual gross income. We multiply the model-implied annual expenditures by the ratio of annual pre-tax income (Item 1 in Table 1) to annual expenditures (Item 2 in Table 1) to obtain model-implied annual gross income. Hence, we ensure the consistency between the model and the data for each type of household.

[31] The re-calibrated values of ρ and θ are 5.47 and 3.1139×10^{-6}, respectively.

To visualize how unobserved preference shocks influence the degree of spatial sorting, we plot the relationship between the average wage rate and the distance from the CBD for both the baseline model and the current extended model in Figure 3. This figure shows that, as one moves away from the CBD, the average wage rate increases slower in the extended model, which has the unobserved preference shocks, than in the baseline model. The difference is particularly salient at locations closer to the city's boundary.

Figure 3: Average Hourly Wage and Distance from the CBD (Mile)



We then conduct welfare analysis based on the re-calibrated model. Note that the average utility of all Type i households is the same as the expected utility of a Type i household, which is given by:

$$\tilde{V}_i^E(P^*(r), r) = E[\tilde{V}_i^T(P^*(r), r)] = \sigma\gamma + \sigma \log[\int_0^{R_f^*} e^{\frac{\tilde{V}_i(P^*(r),r)}{\sigma}} L(r)dr], \tag{45}$$

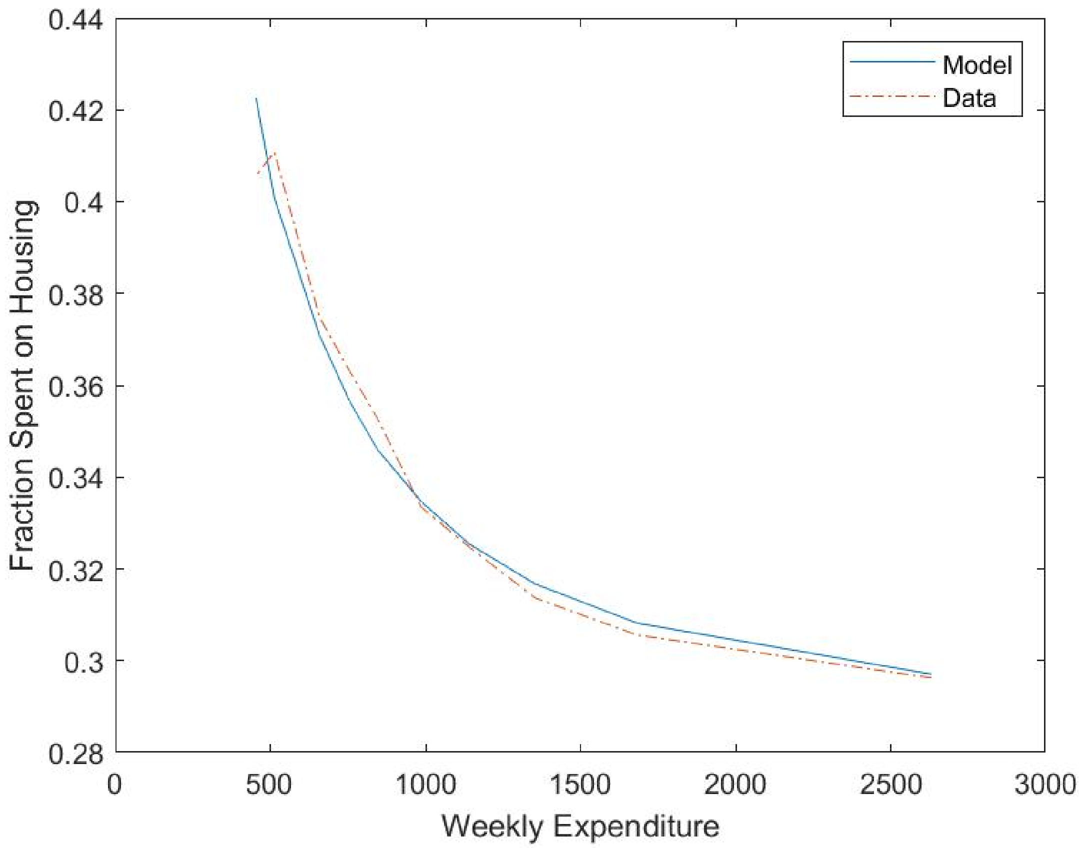
where $\gamma = 0.5772$ is the Euler's constant and $L(r) = 2\pi r$ is the amount of locations available per unit distance, at a distance r.

As before, we define the consumption equivalence measure, ξ_i, as the faction of numeraire goods and housing consumption that Type i household needs to obtain in the economy without the cap policy to achieve the same level of average utility $\tilde{V}_i^E(P^*(r), r)$ in the economy with the cap policy. We solve for the equilibrium utility and the consumption-equivalent measure ξ_i for all types of households for $\bar{\kappa} = 20\%$, 25%, 30%, 35%, and 40%. The results reported in Panel B of Table 6 are quantitatively similar to what we found earlier based on the baseline model (Panel A of Table 5). For all five levels of caps that we considered, ξ_i is generally less positive or more negative for relatively higher-income households. Thus, the housing expenditure ratio cap does not exacerbate welfare inequality under realistic income sorting.

C Non-homothetic Model

C.1 Fit of the Spaceless Model

Figure 4: Total Annual Expenditure and the Fraction Spent on Housing (Non-homothetic Model)



C.2 Spatial Order in the Monocentric City Model

The bid-rent function for Type i household in the spatial non-homothetic model can be obtained by inverting the indirect utility function shown in Equation (25):

$$\psi_i(u_i^*, r) = [\frac{\beta_c^{\beta_c}(1-\beta_c)^{1-\beta_c}(40w - ar + \bar{c})}{u_i^*}]^{\frac{1}{1-\beta_c}}. \quad (46)$$

The first order derivative of this function with respect to the distance r is given by:

$$\begin{aligned}\frac{\partial\psi_i(\cdot)}{\partial r} &= \psi_i(u_i^*, r)\frac{1}{1-\beta_c}[\frac{\beta_c^{\beta_c}(1-\beta_c)^{1-\beta_c}(40w_i - ar + \bar{c})}{u_i^*}]^{-1}\frac{\beta_c^{\beta_c}(1-\beta_c)^{1-\beta_c}(-a))}{u_i^*} \\ &= \psi_i(u_i^*, r)\frac{-a}{(1-\beta_c)(40w_i - ar + \bar{c})} \end{aligned} \tag{47}$$

It is easy to see that $\frac{\partial\psi_i(\cdot)}{\partial r} < 0$ from Equation (47). It means all bid-rent functions are decreasing in the distance from the CBD, r. Let $r_{B,i}^*$ denote the distance from the CBD at which Type i households and Type $i+1$ households intersect in the baseline equilibrium. Note that, at the equilibrium, we have $\psi_i(u_i^*, r_{B,i}^*) = \psi_{i+1}(u_{i+1}^*, r_{B,i}^*)$ (same rental rate for both types of households at the intersection). It implies that:

$$\begin{aligned}|\frac{\partial\psi_i(\cdot)}{\partial r}| - |\frac{\partial\psi_{i+1}(\cdot)}{\partial r}| &= \psi_i(u_i^*, r)[\frac{a}{(1-\beta_c)(40w_i - ar^* + \bar{c})} - \frac{a}{(1-\beta_c)(40w_{i+1} - ar^* + \bar{c})}] \\ &= \frac{\psi_i(u_i^*, r)a[(40w_{i+1} - ar^* + \bar{c}) - (40w_i - ar^* + \bar{c})]}{(1-\beta_c)(40w_i - ar^* + \bar{c})(40w_{i+1} - ar^* + \bar{c})} \\ &= \frac{40a\psi_i(u_i^*, r)(w_{i+1} - w_i)}{(1-\beta_c)(40w_i - ar^* + \bar{c})(40w_{i+1} - ar^* + \bar{c})} \\ &> 0, \end{aligned} \tag{48}$$

where the last line follows from the fact that Type i households has lower hourly wages than Type $i+1$ households ($w_i < w_{i+1}$).

Thus, we have shown that $|\frac{\partial\tilde{\psi}_i(\cdot)}{\partial r}|$ is larger for households with lower hourly wage w_i, which implies that households with higher hourly wage w_i lives farther from the CBD at the equilibrium in the spatial non-homothetic model.