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Abstract

This paper highlights the links between the descriptions of macroeconomic variables and statistical moments of market trade, price, and return. We consider economic transactions during the averaging time interval Δ as the exclusive matter that determines the change of any economic variables. We regard the stochasticity of market trade values and volumes during Δ as the only root of the random properties of price and return. We describe how the market-based n-th statistical moments of price and return during Δ depend on the n-th statistical moments of trade values and volumes or equally on sums during Δ of the n-th power of market trade values and volumes. We introduce the secondary averaging procedure that defines statistical moments of trade, price, and return during the averaging interval $\Delta_2 >> \Delta$. As well, the secondary averaging during $\Delta_2 >> \Delta$ introduces statistical moments of macroeconomic variables, which were determined as sums of economic transactions during Δ. In the coming years, predictions of the market-based probabilities of price and return will be limited by Gaussian-type distributions determined by the first two statistical moments. We discuss the roots of the internal weakness of the conventional hedging tool, Value-at-Risk, that could not be solved and thus remain the source of additional risks and losses. One should consider economic theory as a set of successive approximations, each of which describes the next array of the n-th statistical moments of market transactions and macroeconomic variables, which are repeatedly averaged during the sequence of increasing time intervals.

Keywords: economic theory, price and return, statistical moments, market-based probabilities

JEL: C0, E4, F3, G1, G12

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1. Introduction

Studies of economic theory have no beginning and, probably, no end. Starting at least with the *Essay* by Cantillon (1730), published almost three centuries ago, numerous results in economic theory were presented by (Hicks, 1937; Schumpeter, 1939; Neumann, 1945; Solow, 1956; Leontief, 1973; Sargent, 1979; Blaug, 1985; Greenwald and Stiglitz, 1987; Romer, 1996; Krueger, 2002; Kurz and Salvadori, 2003; Wickens, 2008; Vines and Wills, 2018). Any review of the current state of economic theory should discuss the results of hundreds of papers.

To avoid that impossible task, we direct our efforts to the investigation of the general framework of economic theory. We consider the composition of macroeconomic variables and highlight the rules for their econometric assessments using variables of economic agents and market transactions. We believe that market transactions are, on the one hand, the only origin of economic evolution and development and, on the other hand, the only roots of economic uncertainty and stochasticity. The randomness of market trade can be explained by many factors, such as the uncertainty of agents' expectations, economic or political shocks, etc. A lot of hidden reasons cause the stochasticity of market trade. However, all these hidden factors result in the irregularities and randomness of the market trade time series. We don't study why the market time series are random but describe how the records of random market data determine the stochasticity of price, return, and other economic variables. An investigation of the records of random trade time series gives firm ground for the description of statistical moments and probabilities of market price and return of the traded assets, stocks, commodities, etc. Any change in the economic environment is the result of economic transactions, and these market transactions are the source of economic randomness. Market price and return are the conventional indicators of that stochasticity. We consider the randomness of market trade values and volumes as the source of price and return stochasticity and describe the dependence of statistical moments of price and return on statistical moments of market trade value and volume. We call it the market-based approach to describing the probability of price and return. The market-based approach highlights the ties between the trade statistical moments and the factors that describe the change in macroeconomic variables. In some sense, the uncertainty of macroeconomic variables is described by statistical moments of market trade. We show that the description of macroeconomic evolution is almost equal to the description of trade statistical moments. Economic theory, to a large extend, is the description of trade statistical moments. We present a pure theoretical consideration of the problems that should be developed to make economic models and forecasts more reliable and sustainable.

The rest of the paper is as follows: In Section 2, we discuss general issues with the approximations of real economic processes. In Section 3, we describe the composition of economic variables aggregated during a particular time interval. In Sections 4 and 5, we discuss the definitions and descriptions of the market-based statistical moments of price and return. After these introductory sections, in Section 6, we consider the relations between economic theory and market trade statistical moments, introduce the secondary averaging procedure of statistical moments, and discuss parallels between macroeconomic variables and trade statistical moments. In Section 7, we discuss some practical outcomes. Section 8 – Conclusion.

We believe that readers are familiar with conventional models of price and return probabilities, have skills in the use of statistical moments, etc., and know or can find on their own the definitions, notions, and terms that are not given in the text.

2. General considerations

To study economic theory, one should identify the main elements that compose economic relations. We don't consider particular economic matters such as production and consumption, credits and loans, demand and supply, etc. Instead, we take economic agents as the elementary bricks that establish the economic system as a whole. As agents, we consider international companies, large banks, hedge funds, small firms, shops, households, and all participants in economic, financial, and market relations. We assume that agents have many economic and financial variables like income and consumption, taxes and production, investment and profits, etc. We believe that all macroeconomic variables that define the evolution of the economic system as a whole are determined by the aggregations, by the sums of corresponding variables of economic agents, or depend on them (Fox et al., 2017). That is the conventional treatment of macroeconomic variables.

The current values of agents' economic and financial variables at time t describe the state and shape of the economic system. The only factor that results in the evolution of the economy as well as in the change of the values of agents' economic variables is a market. Agents make numerous market transactions with other agents, and these trades are the only origins of the change in agents' variables and of the change in macroeconomic variables. However, the frequency of trades in stock markets, FOREX, commodities markets, etc. is very high. A time interval ε between market trades can be equal to a second or even a fraction of a second. Such a high frequency of trades results in highly irregular trade time series, which result in irregular changes in the corresponding agents' variables. In addition, high-frequency time series are of little help for the description of long-term macroeconomic relations. The collisions between the high frequency of market trades and the description of long-term relations raise the important problem of the choice of the averaging time interval Δ that determines the time axis division of the macroeconomic model. Indeed, high-frequency time series of market trades at time t_i such as:

$$t_{i+1} - t_i = \varepsilon \quad ; \quad i = 1, 2.. \tag{2.1}$$

introduces the initial market time axis division multiple of ε. However, this initial market time axis division is too precise for macroeconomic modeling. To describe macroeconomic relations at a long horizon $T >> \varepsilon$, one should roughen the initial time axis division. To do that, one should choose the time interval Δ, such as $\varepsilon << \Delta < T$, and aggregate the market trade time series during Δ. The choice of the interval Δ and the aggregation of market trade time series during Δ smooths the irregularity of the initial trade time series and determines the collective impact of market trade during Δ on macroeconomic and agents' variables.

Any approximations of real economic processes or any economic model require the choice of a time-averaging interval Δ. The choice of duration of the interval Δ significantly determines the properties, reliability, and stability of the economic models and predictions that use economic variables aggregated during Δ. The sequence of several time intervals $\Delta < \Delta_2 < \Delta_3 <$.. determines the sequence of economic models with more and more smooth changes of variables.

3. Properties of aggregated variables

Different research goals result in the choice of different durations of intervals Δ. Investment decisions during days or weeks require the choice of the averaging interval Δ that equals days or weeks. If one models financial markets to take decisions during hours, then the interval Δ could not be longer than hours. Long-term economic forecasts on the horizon of months or years could require the interval Δ to equal at least a month. Different durations of Δ result in different models of economic variables.

Let us consider how the averaging interval Δ determines the properties of collective economic variables. We start with a simple model and consider market trading with a particular asset. As such, one can take stocks of a large company or commodities, such as oil, metals, gold, FOREX trading, etc. We assume that the frequency of market trades is determined by (2.1) and propose that Δ defines the intervals Δ_k:

$$\Delta_k = \left[t_k - \frac{\Delta}{2}; t_k + \frac{\Delta}{2}\right] \;\; ; \;\; t_k = t_0 + \Delta \cdot k \;\; ; \;\; k = 0, 1, 2, \dots \tag{3.1}$$

We renumber the time series t_i in such a way that t_{ik} belongs to the intervals Δ_k:

$$t_{ik} \in \Delta_k \;\; ; \;\; i = 1, 2, \dots N \tag{3.2}$$

The interval Δ substitutes the initial market time axis division t_i (2.1) that is a multiple of ε with a new one, t_k (3.1), that is a multiple of Δ. To express the change in economic variables as a result of market trades during Δ, let us define market trade value $C(t_{ik})$ and trade volume $U(t_{ik})$ at a time t_{ik}. During the interval Δ, the total trade value $C_\Delta(t_k)$ and volume $U_\Delta(t_k)$ take the form:

$$C_\Delta(t_k; 1) = \sum_{i=1}^{N} C(t_{ik}) \quad ; \quad U_\Delta(t_k; 1) = \sum_{i=1}^{N} U(t_{ik}) \tag{3.3}$$

Relations (3.3) define the collective change of market trade value $C_\Delta(t_k;1)$ and volume $U_\Delta(t_k;1)$ during Δ. Obviously, the time series of the trade value $C_\Delta(t_k)$ and volume $U_\Delta(t_k)$ at time t_k, $k=0,1,..$ demonstrate more smooth dynamics than the initial high frequency and irregular market time series of the trade value $C(t_{ik})$ and volume $U(t_{ik})$ during Δ_k (3.1; 3.2). We use index 1 in (3.3) to highlight that the sums (3.3) are taken over the first power of the variables on the right side. That index will play an important role in our further consideration, and we outline its importance now. The duration of the interval Δ can be equal to a day, a week, a month, or a quarter, and that introduces the change of macroeconomic variables during the corresponding time interval. One can consider any market transactions that are performed in the economy during the interval Δ alike to collective trade value and volume (3.3). The perfect methodology for aggregation of economic variables as sums of transactions that are made during the averaging interval Δ is presented by Fox et al. (2017). It describes the procedures that are in use for assessments of the official statistics of the National Accounts for aggregating additive economic variables and subsequent assessments of non-additive variables such as prices, returns, inflation, etc. The use of collective variables, which are alike to (3.3), opens the way for modeling the dynamics of monthly, quarterly, or annual investments and sales, consumption and production, profits or expenses, etc. Most macroeconomic theories (Leontief, 1955; Sargent, 1979; Blaug, 1985; Romer, 1996; Krueger, 2002) describe relations between economic variables, which are composed of corresponding economic transactions during the selected interval Δ. Obviously, macroeconomic theories are not limited to using only variables similar to (3.3). They use many variables that describe price, return, rates, indices, etc. Each market trade with a particular asset of the value $C(t_{ik})$ and volume $U(t_{ik})$ at time t_{ik} defines the trade price $p(t_{ik})$ due to a trivial equation:

$$C(t_{ik}) = p(t_{ik}) U(t_{ik}) \tag{3.4}$$

For convenience, throughout this paper, at all times t_{ik}, we consider all prices adjusted to the current value at time t_0. Models and predictions of price define the core problems of financial economics and generate an endless row of studies (Muth, 1961; Sharpe, 1964; Fama, 1965; Black and Scholes, 1973; Merton, 1973; Friedman, 1990; Cochrane and Hansen, 1992; Cochrane, 2001; Campbell, 2018). These references present only a millesimal part of the asset pricing studies. At least since Bachelier (1900), the description of price as a random variable has become the most conventional: "in fact, the first author to put forward the idea to use a random walk to describe the evolution of prices was Bachelier" (Shiryaev, 1999). The descriptions of price and return probabilities and the forecasts of their probabilities at horizon T are among the most studied problems of modern finance. However, the hidden economic barriers almost prohibit any exact predictions of price probabilities.

The notion of price probability itself has multiple treatments. Below, we consider the market-based probabilities of price and return determined by the randomness of the market trade.

4. The market-based statistical moments of price

It is conventional to assume that during the averaging time interval Δ all N trades have equal probabilities and the probability $P(p)$ of price p is proportional to the frequency m_p/N of trades at price p:

$$P(p) \sim m_p/N \tag{4.1}$$

For convenience, we note (4.1) as the frequency-based approach to price probability that, during the last century, has been studied in a great number of papers. Researchers check up almost all standard random distributions (Forbes et al., 2011; Walck, 2011) to verify their adequacy to different assessments of price probability. The frequency-based treatment (4.1) of price probability completely follows contemporary probability theory (Shiryaev, 1999; Shreve, 2004) and serves as a perfect example of the use of modern methods of probability theory in financial economics.

However, the conventional frequency-based approach (4.1) to price probability almost 35 years ago was supplemented by a different treatment of the average price that takes into account the size of volumes of market trades and that is well known now as volume weighted average price (VWAP) (Berkowitz et al., 1988; Duffie and Dworczak, 2018). Using relations (3.1-3.4), VWAP $p(t_k;1)$ during the interval Δ_k takes the form:

$$p(t_k;1) = \frac{1}{\sum_{i=1}^{N} U(t_{ik})} \sum_{i=1}^{N} p(t_{ik})U(t_{ik}) = \frac{C_\Delta(t_k;1)}{U_\Delta(t_k;1)} \tag{4.2}$$

Relation (4.2) determines the average price $p(t_k;1)$ at time t_k, and the averaging is made during the interval Δ_k (3.1; 3.2). The average price $p(t_k;1)$, or as it is called, the 1-st statistical moment of price, is not enough to determine the price probability. If one considers price irregular time series during Δ_k (3.1; 3.2) as a random variable, then, to identify the price probability or characteristic function that equally describes the properties of a random variable (Shiryaev, 1999; Shreve, 2004; Klyatskin, 2005), one should determine all statistical moments $p(t_k;n)$ of price:

$$p(t_k;n) = E[p^n(t_{ik})] \quad ; \quad n = 1,2,\dots \tag{4.3}$$

We use the notation $E[..]$ to note mathematical expectation during Δ_k. For any n, the finite set of price statistical moments $p(t_k;m)$, $m=1,..n$ defines an approximation of the price probability measure or characteristic function. The frequency-based approach to price probability (4.1) is valid if one considers a random price time-series $p(t_{ik})$ during Δ_k as an independent random variable. If one studies only irregular price time series during Δ_k, then the conventional frequency-based probability (4.1) is a completely correct approach to the description of the random price (Shiryaev, 1999; Shreve, 2004).

However, one should keep in mind that the market price $p(t_{ik})$ is a result of market trade (3.4). To define the properties of price as a random variable that is determined by (3.4), one should take into account the random properties of the time series of the trade value $C(t_{ik})$ and volume $U(t_{ik})$ (Olkhov, 2021a; 2021b; 2022a; 2023a; 2023b). Indeed, equation (3.4) states that the given properties of the random trade value $C(t_{ik})$ and volume $U(t_{ik})$ determine the properties of the random price $p(t_{ik})$. We consider the price (3.4) as a result of market trade and describe the dependence of the market-based statistical moments of price on the statistical moments of market trade value $C(t_{ik})$ and volume $U(t_{ik})$. We call that the market-based approach to price probability. To support that simple proposal, we refer to Fox et al. (2017), who present the methodology for the assessment of aggregate price and other non-additive macroeconomic variables as a result of the aggregation of additive economic variables. The time series of the trade value $C(t_{ik})$ and volume $U(t_{ik})$ are examples of additive economic variables, and the sums of the trade value $C_\Delta(t_k)$ and volume $U_\Delta(t_k)$ (3.3) during the interval Δ_k determine the VWAP $p(t_k;1)$ (4.2).

In total, there is no single solution, no single rule, or law that determines a single definition of the probability of price as a non-additive economic variable. The votaries of econometric data and the adherents of empirical evidence supporting any theoretical results and conclusions in economics and finance should be disappointed and discouraged. No econometric data exists that can provide any empirical evidence in favor of the frequency-based or market-based approach to the definition of price probability. These two different considerations of price probability exist simultaneously.

In this paper, we describe the market-based probabilities of price and return and show how that approach highlights mutual relations with general problems of economic and financial theory. We show that on the one hand, the market-based approach to price and return probabilities is determined by the stochasticity of market trade, and on the other hand, it highlights the origin of the successive approximations of the economic and financial theories.

We consider the irregular time series of trade value $C(t_{ik})$ and volume $U(t_{ik})$ during the averaging interval Δ_k as the only source of the random properties of price and return. We consider the trade value and volume as random variables during Δ_k and assess their statistical moments by conventional frequency-based probability (4.1). To describe the properties of a random variable, one can equally use a probability measure, a characteristic function, or a set of statistical moments (Shiryaev, 1999; Shreve, 2004). A finite number N of trades during Δ_k means that one can assess only a finite number of statistical moments of a random variable. Hence, the records of the market trade time series during Δ_k allow estimate only approximations of the trade value and volume probabilities, determined by a finite number of statistical moments. We denote the n-th statistical moments of trade value $C(t_k;n)$ and volume $U(t_k;n)$ using frequency-based probability:

$$C(t_k;n) \equiv E[C^n(t_{ik})] \sim \frac{1}{N}\sum_{i=1}^{N} C^n(t_{ik})\,;\; U(t_k;n) \equiv E[U^n(t_{ik})] \sim \frac{1}{N}\sum_{i=1}^{N} U^n(t_{ik}) \quad (4.4)$$

We use the symbol "~" to mention that (4.4) are the estimates of the n-th statistical moments during Δ_k by a finite number N of terms of time series.

Let us take the expressions of the total trade value $C_\Delta(t_k;1)$ and volume $U_\Delta(t_k;1)$ (3.3) during Δ_k (3.1; 3.2) and introduce the similar variables as sums of the n-th power of trade value $C_\Delta(t_k;n)$ and volume $U_\Delta(t_k;n)$ during Δ_k:

$$C_\Delta(t_k;n) = \sum_{i=0}^{N} C^n(t_{ik}) \quad ; \quad U_\Delta(t_k;n) = \sum_{i=0}^{N} U^n(t_{ik}) \quad ; \quad n = 1,2,\dots \quad (4.5)$$

The n-th statistical moments (4.4) and the sums of the n-th power of the trade values and volumes (4.5) are linked by trivial relations (4.6):

$$C_\Delta(t_k;n) = N \cdot C(t_k;n) \quad ; \quad U_\Delta(t_k;n) = N \cdot U(t_k;n) \quad (4.6)$$

As we show below, the relations (4.6) result in the equal dependence of the market-based n-th statistical moments of price and return on the sums of the n-th power of market trade and on the n-th statistical moments of market trade. To derive the dependence of the n-th statistical moments of price $p(t_k;n)$ (4.3) on random properties of the trade value and volume (4.4-4.6) take the n-th power of equation (3.4):

$$C^n(t_{ik}) = p^n(t_{ik})U^n(t_{ik}) \quad ; \quad n = 1,2,\dots \quad (4.7)$$

From (3.4; 4.7), we define the n-th statistical moment of price $p(t_k;n)$ (4.3) in a form that is similar to the form of VWAP (4.2) as follows:

$$p(t_k;n) = \frac{1}{\sum_{i=1}^{N} U^n(t_{ik})} \sum_{i=1}^{N} p^n(t_{ik}) U^n(t_{ik}) = \frac{C_\Delta(t_k;n)}{U_\Delta(t_k;n)} = \frac{C(t_k;n)}{U(t_k;n)} \quad (4.8)$$

The definition (4.8) is similar to (4.2) and introduces the n-th statistical moment of price or, equally, the average n-th power of price $p(t_k;n)$ (4.3) as a ratio of the sum of the n-th power of trade value $C_\Delta(t_k;n)$ during the interval Δ_k (3.1; 3.2) to the sum of the n-th power of trade volume $U_\Delta(t_k;n)$ during Δ_k. That is completely equal to the n-th power price $p^n(t_{ik})$ weighted by the n-th power volume $U^n(t_{ik})$, or equal to the ratio of the n-th statistical moment of the trade value $C(t_k;n)$ to the n-th statistical moment of the trade volume $U(t_k;n)$. The relations (4.8) define the n-th statistical moment of price $p(t_k;n)$ as the result of equation (4.7) in the same sense as the definition of VWAP $p(t_k;1)$ (4.2) is the result of the trade equation (3.4). The relations (4.8) define the dependence of the market-based price statistical moments on the statistical moments of the trade value and volume.

The simple consequences of the definitions of the market-based statistical moments (4.2; 4.8) result in zero correlations between the time series of the n-th power of price $p^n(t_{ik})$ and trade volumes $U^n(t_{ik})$. Indeed, from (4.4; 4.8), obtain:

$$C(t_k;n) = E[C^n(t_{ik})] = E[p^n(t_{ik})U^n(t_{ik})] = p(t_k;n)U(t_k;n) \quad (4.9)$$

From (4.9), we obtain that for all $n=1,2,..$ the correlation $corr\{p^n,U^n;t_k\}$ (4.10) between the n-th power of price $p^n(t_{ik})$ and the n-th power of trade volume $U^n(t_{ik})$ equals zero:

$$corr\{p^n, U^n; t_k\} = E[p^n(t_{ik})U^n(t_{ik})] - p(t_k;n)U(t_k;n) = 0 \quad (4.10)$$

However, time series of price $p(t_{ik})$ and trade volume $U(t_{ik})$ during averaging interval Δ_k (3.1; 3.2) are not statistically independent, and, for example, one can derive a non-zero correlation between price $p(t_{ik})$ and squares of trade volume $U^2(t_{ik})$ (Olkhov, 2021a; 2022a). Actually, numerous researchers investigate correlation $corr\{p,U;t_k\}$ between time series of price $p(t_{ik})$ and trade volume $U(t_{ik})$ (Tauchen and Pitts, 1983; Karpoff, 1987; Gallant et al., 1992; Campbell et al., 1993; Llorente et al., 2001; DeFusco et al., 2017). However, these researchers assess the correlation determined by the frequency-based approach to price probability (4.1). The differences between their results and our consideration of the market-based correlations (4.10) highlight the fact that different treatments of price probability result in different properties of the price-volume correlations.

5. The market-based statistical moments of return

To derive the market-based statistical moments of return, follow (Olkhov, 2023a; 2023b). Let us choose a time shift τ and consider return $r(t_{ik},\tau)$ as a ratio of price $p(t_{ik})$ at time t_{ik} to price $p(t_{ik}-\tau)$ at a time $t_{ik}-\tau$ in the past:

$$r(t_{ik},\tau) = \frac{p(t_{ik})}{p(t_{ik}-\tau)} \tag{5.1}$$

We take the time shift τ to be a multiple of ε, and hence, $t_{ik}-\tau$ belongs to the time series t_{ik}. Now we transform equation (3.4) as follows:

$$C(t_{ik}) = \frac{p(t_{ik})}{p(t_{ik}-\tau)}\, p(t_{ik}-\tau)U(t_{ik}) = r(t_{ik},\tau)C_a(t_{ik},\tau) \tag{5.2}$$

$$C_a(t_{ik},\tau) = p(t_{ik}-\tau)U(t_{ik}) \tag{5.3}$$

The relations (5.3) define the value $C_a(t_{ik}-\tau)$ of the trade volume $U(t_{ik})$ at time t_{ik} at a price $p(t_{ik}-\tau)$ in the past. The n-th power of (5.2) defines the equation (5.4) on the n-th power of return $r^n(t_{ik},\tau)$ in a form that is similar to the form of the equation on price $p(t_{ik})$ (4.7).

$$C^n(t_{ik}) = r^n(t_{ik},\tau)C_a^n(t_{ik},\tau) \tag{5.4}$$

Using (5.4), we define the n-th statistical moments of return $r(t_k,\tau;n)$, similar to the definition of the n-th statistical moments of price $p(t_k;n)$ (4.8) as:

$$r(t_k,\tau;n) = E[r^n(t_{ik},\tau)] = \frac{1}{\sum_{i=1}^{N} C_a^n(t_{ik},\tau)} \sum_{i=1}^{N} r^n(t_{ik},\tau)C_a^n(t_{ik},\tau) \tag{5.5}$$

$$r(t_k,\tau;n) = \frac{C_\Delta(t_k;n)}{C_{a\Delta}(t_k,\tau;n)} = \frac{C(t_k;n)}{C_a(t_k,\tau;n)} \tag{5.6}$$

$$C_{a\Delta}(t_k,\tau;n) = \sum_{i=1}^{N} C_a^n(t_{ik},\tau) \quad ; \quad C_a(t_k,\tau;n) = \frac{1}{N}\sum_{i=1}^{N} C_a^n(t_{ik},\tau) \tag{5.7}$$

$$C_a(t_k,\tau;n) = p_a(t_k,\tau;n)U(t_k;n) \tag{5.8}$$

$$p_a(t_k,\tau;n) = \frac{1}{\sum_{i=1}^{N} U^n(t_{ik})} \sum_{i=1}^{N} p^n(t_{ik}-\tau)U^n(t_{ik}) = \frac{C_{a\Delta}(t_k,\tau;n)}{U_\Delta(t_k;n)} = \frac{C_a(t_k,\tau;n)}{U(t_k;n)} \tag{5.9}$$

The relation (5.8; 5.9) introduces the n-th statistical moment $p_a(t_k,\tau;n)$ of the past price $p(t_{ik}-\tau)$ determined by the n-th statistical moment of the trade volume $U(t_k;n)$ (4.4; 4.5) and the n-th statistical moment of the past value $C_a(t_{ik}-\tau)$ (5.7). The relations (4.8) and (5.5-5.7) establish simple relations between the *n*-th statistical moments of price $p(t_k;n)$ and return $r(t_k,\tau;n)$:

$$p(t_k;n) = r(t_k,\tau;n)\, p_a(t_k,\tau;n) \tag{5.10}$$

The relations (5.10) have the form similar to (4.8; 4.9; 5.6; 5.8; 5.9) and show the mutual dependence between the statistical moments of price and return. For details and consequences of relations (4.7; 4.8) and (5.1-5.7), we refer to Olkhov (2021a; 2022a; 2023a; 2023b).

6. Economic theory and statistical moments

Now we discuss the main matter of this paper: the relations between economic theory and the description of the statistical moments of economic variables.

Conventional economic models describe the change in macroeconomic variables. The change in each macroeconomic variable is determined by the changes in corresponding variable of agents during a particular time interval Δ. For example, the change in macroeconomic investments, credits, taxes, etc. is determined by the change in the corresponding variables of economic agents during Δ. In its turn, agents' variables change because of numerous market transactions during Δ. In total, the change in macroeconomic variables during Δ is determined by market transactions during Δ. We highlight that economic and financial transactions of economic agents are the only processes that change agents' variables. Any change in macroeconomic investments, credits, GDP, etc., during the interval Δ occurs only as a result of agents' transactions. Almost all macroeconomic variables are composed of sums of the 1st power of market transactions. For example, the change in macroeconomic investment is determined by the sum of the investment deals of agents during Δ. For convenience, we call them the 1-degree economic variables. We call conventional economic models, which describe relations between the 1-degree variables as the 1-degree economic theories or the 1-degree approximations.

However, economic and market transactions significantly depend on agents' expectations of future prices, returns, volatilities, etc. As we show above, the market-based n-th statistical moments of price and return depend on the corresponding n-th statistical moments of market trade values and volumes. For example, the volatilities of price $\sigma^2_p(t_k)$ and return $\sigma^2_r(t_k,\tau)$ (Olkhov, 2021a; 2021b; 2022a; 2023a) are determined by the 1-st and 2-d statistical moments of the trade value and volume:

$$\sigma_p^2(t_k) = E\left[\left(p(t_{ik}) - p(t_k;1)\right)^2\right] = p(t_k;2) - p^2(t_k;1) \tag{6.1}$$

$$\sigma_p^2(t_k) = \frac{C(t_k;2)}{U(t_k;2)} - \frac{C^2(t_k;1)}{U^2(t_k;1)} = \frac{C_\Delta(t_k;2)}{U_\Delta(t_k;2)} - \frac{C_\Delta^2(t_k;1)}{U_\Delta^2(t_k;1)} \tag{6.2}$$

$$\sigma_r^2(t_k,\tau) = E\left[\left(r(t_{ik},\tau) - r(t_k,\tau;1)\right)^2\right] = r(t_k,\tau;2) - r^2(t_k,\tau;1) \tag{6.3}$$

$$\sigma_r^2(t_k,\tau) = \frac{C(t_k;2)}{C_a(t_k,\tau;2)} - \frac{C^2(t_k;1)}{C_a^2(t_k,\tau;1)} = \frac{C_\Delta(t_k;2)}{C_{a\Delta}(t_k,\tau;2)} - \frac{C_\Delta^2(t_k;1)}{C_{a\Delta}^2(t_k,\tau;1)} \tag{6.4}$$

Thus, agents' expectations of price and return volatilities (6.1-6.4), which impact agents' decisions on market transactions, depend on modeling the 2-d statistical moments of the trade values and volumes. In simple words, to develop the 1-degree economic theories, which describe macroeconomic variables that are composed of sums of market transactions, one should also model the relations between variables of the 2-degree. The attempts to approximate the market-based price or return probabilities by the first n statistical moments imply the requirement to develop economic theories that describe the relations between variables composed of sums of the m-th power of market transactions for $m=1,\ldots n$. We believe that the general economic theory should be considered as a sequence of successive n-th approximations starting with the description of the 1-degree economic variables, which are described by conventional economic models. Each next n-th approximation adds an additional layer of economic approximation that is formed of economic variables composed of the sums of the n-th power of market transactions during the selected time averaging interval Δ (Olkhov, 2021b; 2021c; 2022b; 2023c).

The choice of the averaging time interval adds complexity to economic approximations. Indeed, one can consider the sequence of the averaging intervals $\Delta < \Delta_2 < \Delta_3 < \ldots$. The transition from economic approximation, which is determined by the averaging interval Δ, to economic approximation, which is determined by the averaging interval $\Delta_2 >> \Delta$, induces two possible approximations. The first one simply uses the averaging interval Δ_2 instead of the Δ and describes the market-based price statistical moments $p(t_k;n)$ (4.7; 4.8) that are determined by trade statistical moments (4.4), which are averaged during Δ_2.

The lack of huge amounts of the initial market data that is required for averaging transactions during Δ_2 could make the first way impossible. However, one can use the recurrent, repeated averaging approximations of economic processes. Actually, the statistical moments, which are averaged during Δ, can demonstrate irregular or random behavior during a long interval $\Delta_2 >> \Delta$. Due to (4.6), the sums of the n-th power of the trade value $C_\Delta(t_k;n)$ and volume $U_\Delta(t_k;n)$ would also behave randomly. One can use the definitions of the market-based statistical moments (4.4; 4.8) as a starting point for the secondary averaging procedure. Assume that:

$$\Delta_2 = M \cdot \Delta \qquad ; \qquad M \gg 1 \tag{6.5}$$

$$t_k \in \left[t - \frac{\Delta_2}{2} ; t + \frac{\Delta_2}{2}\right] \quad ; \quad k = 0, 1, 2, \ldots M \tag{6.6}$$

Let us consider the sums of the n-th power of the trade value $C_\Delta(t_k;n)$ and volume $U_\Delta(t_k;n)$, as a finite time series t_k, $k=1,2,..M$ in the interval Δ_2 (6.6). Let us take the equation that determines the relations between the sums of the n-th power of the trade value $C_\Delta(t_k;n)$ and volume $U_\Delta(t_k;n)$ (4.5) and the n-th statistical moment of price $p(t_k;n)$ (4.8):

$$C_\Delta(t_k;n) = p(t_k;n) U_\Delta(t_k;n) \qquad ; \quad n = 1,2, \ldots \tag{6.7}$$

Sums of the trade value $C_\Delta(t_k;n)$ and volume $U_\Delta(t_k;n)$ (4.5) are additive, and hence, one can consider (6.7) as a starting equation similar to price equations (3.4) and replicate the same averaging procedure. One can take the m-th power of (6.7) as:

$$C_\Delta^m(t_k;n) = p^m(t_k;n)\, U_\Delta^m(t_k;n) \tag{6.8}$$

Now one can introduce the sums of secondary *m*-th power of the trade value *C(t;n,m)* and volume *U(t;n,m)* during the interval Δ_2 (6.5; 6.6):

$$C_{\Delta_2}(t;n,m) = \sum_{k=1}^{M} C_{\Delta}^{m}(t_k;n) \quad ; \quad U_{\Delta_2}(t;n,m) = \sum_{k=1}^{M} U_{\Delta}^{m}(t_k;n) \qquad (6.9)$$

Definitions of the secondary m-th statistical moments of trade value *C(t;n,m)* and volume *U(t;n,m)* (6.10) reproduce relations (4.6):

$$C(t;n,m) = \frac{1}{M} C_{\Delta_2}(t;n,m) \quad ; \quad U(t;n,m) = \frac{1}{M} U_{\Delta_2}(t;n,m) \qquad (6.10)$$

That result in the equations (6.11), which introduce the secondary *m*-th statistical moments of price *p(t;n,m)* in the form that is similar to (4.8):

$$p(t;n,m) = \frac{1}{\sum_{k=1}^{M} U_{\Delta}^{m}(t_k;n)} \sum_{i=1}^{N} p^{m}(t_k;n) U_{\Delta}^{m}(t_k;n) = \frac{C_{\Delta_2}(t;n,m)}{U_{\Delta_2}(t;n,m)} = \frac{C(t;n,m)}{U(t;n,m)} \qquad (6.11)$$

$$C_{\Delta_2}(t;n,m) = p(t;n,m)\, U_{\Delta_2}(t;n,m) \qquad (6.12)$$

$$C(t;n,m) = p(t;n,m)\, C(t;n,m) \qquad (6.13)$$

Thus, the successive time intervals $\Delta < \Delta_2 < \Delta_3 < \ldots$. introduce recurrent, repeated averaging procedures of statistical moments of trade value, volume, and price. Similar considerations result in the secondary averaging of the statistical moment of return.

Thus, one obtains the sequences of macroeconomic approximations generated by the corresponding approximations of the statistical moments of market trade value, volume, and price. The first sequence of approximations describes the *n*-th statistical moments of trade value *C(t;n)*, volume *U(t;n)* (4.4), and price *p(t;n)* (4.8) averaged during the selected time interval Δ for n=1,2,.. . The second averaging during the interval $\Delta_2 = M\ \Delta$ of statistical moments introduces the sequence of the secondary averaging approximations of statistical moments of trade value *C(t;n,m)*, volume *U(t;n,m)* (6.10), and price *p(t;n,m)* (6.11-6.13) for n=1,2,.. and m=1,2…. In the case of the third averaging interval $\Delta_3 = Q\ \Delta_2$, similar to (6.7-6.13), one can derive the third averaging approximations of statistical moments of the trade value *C(t;n,m,k)*, volume *U(t;n,m,k),* and price *p(t;n,m,k)* for n=1,2,.., m=1,2… , and k=1,2…. Each of these approximations of statistical moments matches the corresponding approximation of macroeconomic evolution. Indeed, the market-based statistical moments of price *p(t;n)* (4.8) and *p(t;n,m)* (6.11) depend on the sums of market trade values $C_{\Delta}(t;n)$ and volume $U_{\Delta}(t;n)$ (4.5), and on the sums $C_{\Delta 2}(t;n,m)$ and $U_{\Delta 2}(t;n,m)$ (6.9). These factors determine the change in macroeconomic variables for the corresponding level of approximation during the selected averaging intervals $\Delta, \Delta_2, \ldots$. For example, the sum of the values of investment deals made by agents during Δ determines the change in total investments in economics during Δ. The sum of squares of the values of investment deals during Δ determines the change of the 2-investments in economics during Δ, etc. These sums describe the change in macroeconomic variables and the evolution of the economy as a whole. Different averaging intervals, sums of different powers of market transactions and deals, recurrent averaging procedures, etc. generate a great array of economic approximations strongly linked to approximations of statistical moments of trade, price, and return.

Economics is a complex system with strong forward and backward links and constraints. Obviously, randomness cannot be the exclusive property of market trade, price, and return. As one agrees that random transactions play an exceptional role in the random change of any agents' variables, one should accept that all agents' variables are random, and their randomness in varying degrees is governed by the randomness of market trade. The duration of the averaging interval is the key factor for describing the stochasticity of the change in macroeconomic variables. The reasons to support that conclusion are completely the same as those above that determine the secondary averaging procedure (6.5-6.13). Indeed, the sums, during interval Δ, of the n-th power of transactions become irregular or random during the intervals $\Delta_2 >> \Delta$. To describe the random properties of change of macroeconomic variables, which are determined during Δ, one should assess the statistical moments of these variables during interval $\Delta_2 >> \Delta$ using their time series t_k (6.5; 6.6). That completely intertangles the models, which describe the dynamics of macroeconomic variables, and the models, which describe the statistical moments of economic transactions and macroeconomic variables.

The successive approximations of macroeconomic models generated by the sequence of the averaging intervals of different durations and by the hierarchy of the n-th statistical moments of market trade establish a rather complex picture of economic theory.

However, up until now, the models that describe economic evolution taking into account the variables of the first and second degrees are absent. Moreover, there is no econometric data and no methodology that could help assess the current values of most 2-degree macroeconomic variables. We believe that Douglas Fox, Stephanie McCulla, and coauthors (Fox et al., 2017) can probably develop the "NIPA Handbook-2", which would describe the Concepts and Methods of the 2-degree assessments of the National Accounts determined by the sums of squares of corresponding economic transactions during a particular interval Δ. That would open the doors for the development of macroeconomic models of the 2-dergee variables.

One can say that, in some sense, economic and financial theories describe macroeconomic variables, or one can say that theories describe the statistical moments of economic transactions and macroeconomic variables. Economic randomness is a problem of a particular averaging interval. The sequence of averaging intervals generates the sequence of economic theories as a set of successive approximations of statistical moments. Each next level of the approximation models the next statistical moments of the economic variables. To a large extent, one should consider studies in economic theory as a description of market statistical moments. The recognition of the fact that the description of macroeconomic variables depends on modeling market statistical moments highlights the impact of probability approximations on economic modeling.

7. Practical outcomes

Nevertheless, our look at economic theory as a set of successive approximations, each of which describes the next n-th degree array of economic variables and statistical moments, could seem too abstract, but it has important practical applications. We briefly consider some. In the coming years, in the best case, the predictions of the price and return market-based probabilities are strictly limited by two statistical moments. Indeed, current records of high-frequency trades could supply a lot of data to assess "today" many statistical moments of trade value, volume, price, and return during any reasonable interval Δ that can be equal to hours, days, or weeks. High-frequency market transactions deliver sufficient data to assess 10, 20, or more statistical moments during Δ and derive rather precise approximations of the current probabilities of price and return "today". However, the attempts to predict the price or return probabilities at any reasonable horizon T that can be equal to weeks, months, or quarters meet tough problems. As we argued above, the predictions of the second statistical moments of price or return at a horizon T require forecasting the 2-d statistical moments of market trade. Such problems require the development of a 2-degree economic model that describes relations between the sums of the squares of economic and market transactions during the averaging interval Δ. The 2-degree economic model should describe the uncertainty of the macroeconomic variables as well as the uncertainty of market transactions at horizon T. However, such a model is absent now, and even the methodology for assessing the current values of the 2-degree economic variables is absent. Thus, any predictions of the market-based 2-d statistical moments of price and return, which neglect the 2-degree economic model, have low economic justification and are highly uncertain (Olkhov, 2023c). The development of 2-degree economic models may take years, but it is the only way to improve the predictability and reliability of economic theory. Predictions of the higher statistical moments require the development of economic models of the 3-degree, 4-degree, etc. Until then, the forecasts of price and return probabilities are limited at best by the predictions of the first two statistical moments. Thus, Gaussian-type probabilities of price and return are the only possible predictions for many years to come.

The second problem linked with the general look at economic theory concerns the reliability of the conventional risk-hedging tool, Value-at-Risk (VaR) (Olkhov, 2021a; 2022a). Indeed, the use of VaR (Longerstaey and Spencer, 1996; Duffie and Pan, 1997; Tobias and Brunnermeier, 2016) utilizes predictions of the price or return probabilities determined by the conventional frequency-based approach (4.1) at a horizon *T*. The heart of VaR matter is the assessment of the integrals of the left tails of the forecasts of probability densities at a horizon *T* as the benchmarks of possible losses. The origin of the troubles lies in the fact that basically the users of VaR are large banks, investment funds, financial institutions, etc., which manage hundreds of billions of USD of assets, stocks, and funds. To protect the value of the assets from a decline in price or return, they estimate possible losses through assessments of the left-tail integrals of the probability densities. The main trouble concerns the origin of these probability densities. Banks, funds, and financial companies that manage hundreds of billions of USD should assess the market-based price distributions that are determined by the randomness of the traded values and volumes, but not frequency-based price probabilities, which mostly describe the prices of trades with small values. Any market transactions with a huge volume of assets require knowledge and predictions of the market-based probabilities of price and return that take into account the impact of large values and volumes of market transactions. Conventional, frequency-based assessments of price probability ignore the dependence of price probability on the size of trade values and volumes, which could result in unexpected and excess losses. However, the current VaR hedge models use conventional frequency-based price and return probabilities. The differences between market-based and conventional frequency-based approaches to price and return probabilities are the origin of the additional risks and losses.

However, even the use of market-based probabilities of price and return, which are determined by the randomness of the trade values and volumes, carries hidden complexities. As we discussed above, the current state of economic theory at best limits the accuracy of any predictions of price and return probabilities by the 2-d market-based statistical moments. Thus, any predictions of the possible losses via the VaR model using market-based price or return probabilities are limited by Gaussian-type distributions. In total, the concept of VaR confirms the elementary thesis: no methods exist that can overcome internal economic obstacles using surrogates, like VaR, that don't solve but simply neglect the essence of the economic barriers.

8. Conclusion

This paper illuminates the similarity between the conventional development of economic theory and the description of the n-th statistical moments of market trade, price, and return. The change in macroeconomic variables during the averaging interval Δ is determined by the sums of the n-th power of economic transactions during Δ. We show that the description of the n-th statistical moments of price and return depends on the modeling of sums of the n-th power of economic transactions. We call such models n-degree economic theories. One should consider economic theory as a set of successive approximations of the statistical moments of economic variables. The tight links between economic theory and the successive descriptions of statistical moments of trade, price, return, and other macroeconomic variables highlight the unexpected features of economic theory. This paper doesn't consider agents' expectations, their influence on agents' trade decisions, and many other factors that, for sure, impact the evolution and randomness of market trade and economics as a whole. That would greatly complicate the economic theory, and we leave it for further studies. That unattainable goal will support economic studies for years.

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