An Extended Goodwin Model with Endogenous Technical Change: Theory and Simulation for the US Economy (1960-2019)

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Abstract

This paper extends the two-dimensional Goodwin model of distributive cycles by incorporating endogenous technical change, inspired on some insights originally formulated by Marx. We introduce a three-dimensional dynamical system, expanding the model to include wage share, employment rate, and capital-output ratio as state variables. Theoretical analysis demonstrates an economically meaningful and locally stable equilibrium point, and the Hopf bifurcation theorem reveals the emergence of stable limit cycles as the mechanization-productivity elasticity surpasses a critical value. Econometric estimation of model parameters using ARDL bounds cointegration tests is performed for the US economy from 1965 to 2019. Simulations show damped oscillations, limit cycles, and unstable oscillations, contributing to the understanding of complex capitalist dynamics.

Keywords: Goodwin model, endogenous technical change, Hopf bifurcation, ARDL, numerical simulations

JEL Classification: C61, E11, E32, O33, O41

1. Introduction

As mentioned by Barrales-Ruiz et al. (2022), the theory of distributive cycles posits that economic growth and its cyclical fluctuations result from the conflictual interaction between profit-seeking capital and workers employed on its behalf. The original impetus for this theory

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was formulated by Marx (1867/1975) in his discussion on capitalist accumulation and the reserve army of labor. One of the first mathematical formulations of this theory can be traced to the model of endogenous distributive cycles introduced by Goodwin (1967).[2] The Goodwin model has received attention from both theoretical and empirical fronts.[3] Particularly, a relevant line of empirical research emerged with the econometric estimations for the United Kingdom presented by Desai (1984) and subsequently extended for ten OECD economies by Harvie (2000). Grasselli and Maheshwari (2018) further enhanced these estimations using cointegration techniques for the same ten economies. While these works represent valuable contributions to the empirical understanding of distributive cycles, they are confined by the two-dimensional framework imposed by the original version of Goodwin's (1967) model. In this version, clockwise cyclic patterns are identified solely in the phase plane formed by the wage share and the employment rate. This limitation is relevant as the two-dimensional fails version of Goodwin's model fails to represent complex dynamics associated with how income distribution affects other economic processes such as technical change.

This paper attempts to expand the theoretical and empirical study of distributive cycles by econometrically estimating an extended version of Goodwin's (1967) model. This extension accounts for endogenous cycles in three dimensions: wage share, employment rate, and capital-output ratio.[4] Justifying the inclusion of a third dimension, the paper formulates a theoretical model that extends the original work of Goodwin (1967). This extension assumes the existence of endogenous technical change, driven by the capitalist inclination to enhance labor productivity

[2] For a literature review on the analytical representations of Marx's insights about capitalist accumulation and the reserve army of labor, see Cámara Izquierdo (2022).

[3] For a brief literature review of the theoretical extensions and the empirical studies based on Goodwins' (1967) model, see Azevedo Araujo et al. (2019).

[4] Other extensions of the Goodwin model where the capital-output ratio is a state variable alongside the employment rate and the wage share include Shah and Desai (1981), Foley (2003), and Julius (2005), although they consider specifications for technical change that differ to those employed in this paper.

through mechanization of the production process. Simultaneously, mechanization itself is viewed as a response by capitalists during their distributive struggle with the working class. These assumptions draw from insights originally formulated by Marx. To validate the relevance of the theoretical model, the paper analytically establishes that the model possesses an economically meaningful equilibrium point that is locally stable. Additionally, the paper employs the existence part of the Hopf bifurcation theorem for three-dimensional dynamical systems to demonstrate that the model tends to generate stable limit cycles near its equilibrium point when the mechanization-productivity elasticity surpasses a critical value.

Having formulated and validated the theoretical model, the paper proceeds to the econometric estimation of its parameters, following a procedure analogous to that applied by Grasselli and Maheshwari (2018). Specifically, we use the ARDL bounds cointegration test proposed by Pesaran et al. (2001) to estimate the parameters of the dynamic equations describing the dynamics of mechanization and the real wage, while other parameters are estimated using log-regressions as well as historical means. The methodology is applied to data from the AMECO[5] database for the United States spanning from 1965 to 2019. Using the estimated parameters, the paper presents numerical simulations of the theoretical three-dimensional model, yielding trajectories characterized by damped oscillations. These simulated trajectories are then compared with the actual time series of the wage share, the employment rate, and the capital-output ratio observed for the US economy. The paper also constructs simulations to illustrate the model's capacity to generate limit cycles and unstable oscillations. Consequently, the paper endeavors to contribute both to the theoretical study of high-dimensional dynamic systems and the empirical estimation of an extension of the Goodwin model of distributive cycles.

[5] Annual macro-economic database of the European Commission's Directorate General for Economic and Financial Affairs.

The rest of the paper has the following structure. Following this introduction, Section 2 briefly reviews the Goodwin model with general capital accumulation rate formulated by Grasselli and Maheshwari (2018), which serves as a baseline. Section 3 presents the extended version of Goodwin's model with endogenous technical change, resulting in a three-dimensional dynamical system. The paper analytically establishes the local stability of this theoretical model and its propensity to generate stable limit cycles around its equilibrium point (Appendix B). Section 4 delineates the main results of the simulation of the extended model calibrated using parameters estimated for the US economy from 1965 to 2019 (Appendix C), along with some descriptive statistics. Section 5 utilizes additional simulations to illustrate the capability of the model to generate limit cycles and discuss the role played by the mechanization-productivity elasticity. Finally, Section 6 concludes.

2. The Goodwin Model with a General Capital Accumulation Rate

In this section, we introduce the Goodwin model with a general capital accumulation rate, as formulated originally by Grasselli and Maheshwari (2018). This model serves as the baseline for the extended version we present in the following section. Building upon Goodwin (1967), we establish the following initial assumptions: Consider a closed economy without government that relies solely on labor and fixed capital as inputs for producing a single commodity, which can be allocated for either consumption or investment (there are no intermediate goods). Labor productivity and labor supply exhibit constant growth rates, while the capital needed per unit of output remains constant, as indicated by a fixed capital/output ratio. There are two social classes: workers, who earn wages and allocate their entire income toward consumption without saving, and capitalists, who earn profits and consistently save a fixed proportion of their income to finance investment. The economic system is characterized by the equality of investment and savings, with no presence of debt or inflation, and all economic variables are measured in real terms. The real wage experiences growth, particularly in proximity to full employment of labor, and all variables are measured in continuous time.[6]

Given these initial assumptions, let w represent the real wage, l denote the employed labor force, and q signify real output.[7] We introduce the wage share u as:

$$u = \frac{wl}{q} \quad (1)$$

Building upon the initial assumptions, $1 - u$ corresponds to the profit share, and $q(1 - u)$ represents total profits. Furthermore, we introduce s as the savings-accumulation rate ($0 < s \leq 1$), k as the total capital, and δ as the depreciation rate of capital. The real net investment $\dot{k}$ is then defined as follows:[8]

$$\dot{k} = sq(1 - u) - \delta k \quad (2)$$

Concerning the capital required for production, we define the capital/output ratio σ as:

$$\sigma = \frac{k}{q} \quad (3)$$

Upon applying logarithms and time differentiation to equation (3), we observe that the assumption of a constant capital/output ratio ($\dot{\sigma} = 0$) holds when the growth rate of capital equals the growth rate of output:

$$\frac{\dot{k}}{k} = \frac{\dot{q}}{q} \quad (4)$$

Now, if we define labor productivity a as:

[6] For a discrete-time version of Goodwin's (1967) model, see Grasetti et al. (2020).
[7] A summary of the notation employed in this paper is available in Appendix A. Furthermore, all mathematical derivations and numerical simulations featured in this study were conducted using a Mathematica notebook that is available as supplementary material. More details are available upon request to the author.
[8] For any variable x, $\dot{x} = dx/dt$ denotes its time derivative, while $\dot{x}/x$ represents its growth rate.

$$a = \frac{q}{l} \quad (5)$$

Then, by applying logarithms and time differentiation to equation (5), we get an expression for the growth rate of productivity:

$$\frac{\dot{a}}{a} = \frac{\dot{q}}{q} - \frac{\dot{l}}{l} \quad (6)$$

Given the initial assumption that productivity grows at a constant rate, we can express this concept as:

$$\frac{\dot{a}}{a} = \alpha \quad (7)$$

where α represents an exogenous constant. Regarding labor dynamics, if n represents the total labor supply, we can define the employment rate as:

$$v = \frac{l}{n} \quad (8)$$

Applying logarithms and time differentiation to equation (8) gives:

$$\frac{\dot{v}}{v} = \frac{\dot{l}}{l} - \frac{\dot{n}}{n} \quad (9)$$

Under the initial assumption that labor supply maintains a constant growth rate, we express this rate as:

$$\frac{\dot{n}}{n} = \beta \quad (10)$$

where β is an exogenous constant.[9]

By considering equations (1) to (10), we can identify the influence of the wage share u on the dynamics of capital accumulation. For instance, when the wage share decreases ($\downarrow u$), ceteris paribus, it triggers a chain of effects: it augments the profit share, elevates total profits, fosters

[9] For a model of capitalist dynamics that assumes an endogenous growth rate of labor supply, see Harris (1983).

increased savings, and stimulates investment, resulting in an acceleration of total capital ($\uparrow \dot{k}/k$). This sequence of effects is represented by dividing equation (2) by the total capital k, yielding:

$$\frac{\dot{k}}{k} = \frac{sq(1-u)}{k} - \delta \quad (11)$$

The acceleration of capital causes a higher growth rate in output, a consequence of the assumption of a constant capital/output ratio. This effect is observed by substituting equations (3) and (4) into (11):

$$\frac{\dot{q}}{q} = \frac{s(1-u)}{\sigma} - \delta \quad (12)$$

To achieve accelerated output growth, there must be a corresponding increase in the growth rate of the employed labor force, discounting the influence of productivity. This effect is noted by substituting equation (12) into (6) and solving for the growth rate of the employed labor force:

$$\frac{\dot{l}}{l} = \frac{s(1-u)}{\sigma} - \delta - \frac{\dot{a}}{a} \qquad (13)$$

A stronger growth rate of the labor force employed implies an acceleration of the employment rate, discounting the growth rate of labor supply. This consequence becomes apparent by substituting equation (13) into (9), resulting in:

$$\frac{\dot{v}}{v} = \frac{s(1-u)}{\sigma} - \delta - \frac{\dot{a}}{a} - \frac{\dot{n}}{n} \qquad (14)$$

Substituting equations (7) and (10) into (14) and rearranging terms yields:

$$\frac{\dot{v}}{v} = \frac{s}{\sigma} - (\alpha + \beta + \delta) - \frac{su}{\sigma} \qquad (15)$$

Equation (15) is equivalent to the first dynamic equation originally formulated by Goodwin (1967), although it introduces two additional components suggested by Grasselli and Maheshwari (2018): the savings-accumulation rate s and the depreciation rate δ. To simplify notation, we can express equation (15) as follows:

$$\frac{\dot{v}}{v} = A_0 - A_1 u, \qquad A_0 = \frac{s}{\sigma} - (\alpha + \beta + \delta), \qquad A_1 = \frac{s}{\sigma} \quad (16)$$

Here we note that A_1 is always positive. Assuming that s is sufficiently high and $(\alpha + \beta + \delta)$ is sufficiently low to ensure a positive value for A_0, equation (16) encapsulates the influence of the wage share on the dynamics of the employment rate. For instance, when the profit share falls ($\downarrow u$), it leads to an acceleration in the employment rate ($\uparrow \dot{v}/v$). Now, according to Marx, the dynamics of the employment rate v (associated with the reserve army of labor)[10] have consequences on the real wage w. In his own words:

> Taking them as a whole, the general movements of wages are exclusively regulated by the expansion and contraction of the industrial reserve army, and these again correspond to the periodic changes of the industrial cycle. They are, therefore, not determined by the variations of the absolute number of the working population, but by the varying proportions in which the working class is divided into active and reserve army, by the increase or diminution in the relative amount of the surplus population, by the extent to which it is now absorbed, now set free (Marx, 1867/1975, p. 631).

Goodwin (1967) represents this insight through a simplified real wage Phillips curve structured as follows:

$$\frac{\dot{w}}{w} = -\gamma + \rho v, \qquad \gamma, \rho > 0 \quad (17)$$

where γ represents an autonomous tendency of the real wage to fall and ρ is the effect of the employment rate on the real wage. Here, a decrease in γ or an increase in ρ may be interpreted as a strengthening of the working class's bargaining power, enabling them to negotiate the acceleration of the real wage.[11] If the real wage accelerates, a corresponding acceleration in the wage share may also be observed. This relationship becomes evident by applying logarithms and time derivatives to equation (1), yielding:

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[10] Implicitly, we assume that the reserve army of labor consists solely of unemployed workers.

[11] A similar interpretation of the parameters of the wage Phillips curve in terms of bargaining power can be found in the works of Mehrling (1986) and Cajas Guijarro and Vera (2022).

$$\frac{\dot{u}}{u} = \frac{\dot{w}}{w} + \frac{\dot{l}}{l} - \frac{\dot{q}}{q} \quad (18)$$

By substituting equation (6) into (18), we can include the effect of the growth rate of labor productivity on the dynamics of the wage share:

$$\frac{\dot{u}}{u} = \frac{\dot{w}}{w} - \frac{\dot{a}}{a} \quad (19)$$

Subsequently, incorporating equation (17) into (19) yields:

$$\frac{\dot{u}}{u} = -\gamma + \rho v - \frac{\dot{a}}{a} \quad (20)$$

Finally, substituting equation (7) into (20), we arrive at the second dynamic equation originally proposed by Goodwin (1967):

$$\frac{\dot{u}}{u} = -(\alpha + \gamma) + \rho v \quad (21)$$

To simplify notation, we can present this equation as:

$$\frac{\dot{u}}{u} = -B_0 + B_1 v, \qquad B_0 = \alpha + \gamma, \qquad B_1 = \rho \quad (22)$$

where B_0 and B_1 are always positive. Equation (22) captures how the employment rate shapes the evolution of the wage share. To illustrate, when the employment rate increases ($\uparrow v$), it propels an acceleration in the wage share ($\uparrow \dot{u}/u$). This, in turn, paves the way for potential future increases in the wage share ($\uparrow u$), which, as per equation (15), subsequently triggers a future deceleration in the employment rate ($\downarrow \dot{v}/v$). This cyclical interaction permanently reinforces the dynamic relationship between these variables. Consequently, as mentioned by Goodwin (1967), equations (15) and (22) jointly define a two-dimensional dynamical system that engenders closed clockwise cycles within the plane formed by the state variables u and v. In fact, when $A_0 > 0$, these equations resemble the structure of the predator-prey model, independently formulated by Lotka (1910) and Volterra (1927). In this analogy, the wage share u

corresponds to the predator, while the employment rate v takes on the role of the prey.[12] In this sense, Goodwin claimed that:

> It has long seemed to me that Volterra's problem of the symbiosis of two populations-partly complementary, partly hostile-is helpful in the understanding of the dynamical contradictions of capitalism, especially when stated in a more or less Marxian form (Goodwin, 1967, p. 55).

3. An Extended Goodwin Model with Endogenous Technical Change

The Goodwin model discussed in the preceding section incorporates Marx's insight regarding the influence of the reserve army of labor, indirectly represented by the employment rate, on the dynamics of the real wage. However, Marx's discussion about the role of the reserve army of labor in the dynamics of capitalist accumulation is notably intricate. An essential aspect in this regard is the interaction between the reserve army of labor and capitalist-driven technical change. For instance, consider the following intuitions expressed by Marx:

> Once given the general basis of the capitalistic system, then, in the course of accumulation, a point is reached at which the development of the productivity of social labour becomes the most powerful lever of accumulation (…)
>
> The degree of productivity of labour, in a given society, is expressed in the relative extent of the means of production that one labourer, during a given time, with the same tension of labour power, turns into products. The mass of the means of production which he thus transforms, increases with the productiveness of his labour. But those means of production play a double part. The increase of some is a consequence, that of the others a condition of the increasing productivity of labour. E. g., with the division of labour in manufacture, and with the use of machinery, more raw material is worked up in the same time, and, therefore, a greater mass of raw material and auxiliary substances enter into the labour process. That is the consequence of the increasing productivity of labour. On the other hand, the mass of machinery, beasts of burden, mineral manures, drain-pipes, etc., is a condition of the increasing productivity of labour. So also is it with the means of production concentrated in buildings, furnaces, means of transport, etc. But whether condition or consequence the growing extent of the means of production, as compared with the labour power incorporated with them, is an expression of the growing productiveness of labour. The increase of the latter appears, therefore, in the diminution of the mass of labour in proportion to the mass of means of production moved by it, or in the diminution of the subjective factor of the labour process as compared with the objective factor (Marx, 1867/1975, p. 617).

In the context of these insights, two critical aspects of technical change become apparent: increasing labor productivity and the mechanization of production processes. As an outcome of these concurrent processes, the bulk of means of production tends to accelerate at a more rapid

[12] Some comments about the predator-prey interpretation of the Goodwin model can be found in Solow (1990).

pace compared to the mass of labor employed. To formalize this concept, for the sake of simplicity, if we maintain the assumption that the economy exclusively relies on fixed capital and produces a single commodity applicable for both consumption and investment, we can introduce a mechanization ratio denoted as m:

$$m = \frac{k}{l} \quad (23)$$

Here, the total capital k represents the mass of the means of production, while l represents the mass of labor employed.

In accordance with the assumption of an absence of intermediate goods, we can infer, drawing from Marx's insights, that a higher mechanization ratio ($\uparrow m$) causes a heightened labor productivity ($\uparrow a$). Consequently, we extend the Goodwin model by replacing equation (7), which previously assumed a constant productivity growth rate, with a new dynamic equation. This equation assumes an endogenous productivity growth rate that depends on the dynamics of mechanization in the following way:

$$\frac{\dot{a}}{a} = \alpha_0 + \alpha_1 \left(\frac{\dot{m}}{m}\right), \qquad 0 < \alpha_0 < 1, \qquad 0 < \alpha_1 < 1 \quad (24)$$

In this expression, α_0 represents the inclination of productivity to increase autonomously, while α_1 denotes the elasticity of labor productivity with respect to mechanization, which we refer to as the mechanization-productivity elasticity. Here we assume that mechanization has positive but decreasing returns, implying that $0 < \alpha_1 < 1$.

If both productivity a and mechanization m undergo changes, it inevitably results in an alteration of the capital/output ratio. To illustrate this, we can substitute equations (5) and (23) into (3), yielding:

$$\sigma = \frac{m}{a} \quad (25)$$

By applying logarithms and time differentiation to equation (25), we derive:

$$\frac{\dot{\sigma}}{\sigma} = \frac{\dot{m}}{m} - \frac{\dot{a}}{a} \quad (26)$$

At this stage of the discussion, we require to explain the factors driving the dynamics of mechanization. Here, once again, we draw upon Marxian insights for inspiration. Specifically, we turn our attention to an observation articulated by Paul Sweezy:

> The reserve army is recruited primarily from those who have been displaced by machinery, whether this takes the more striking form of the repulsion of laborers already employed, or the less evident but not less real form of the more difficult absorption of the additional laboring population through the usual channels. That Marx thought of the introduction of labor-saving machinery as a more or less direct response on the part of capitalists to the rising tendency of wages is clearly indicated in the following passage:
>
> "Between 1849 and 1859, a rise of wages took place in the English agricultural districts. This was the result of an unusual exodus of the agricultural surplus population caused by the demands of war, the vast extension of railroads, factories, mines, etc. (…) What did the farmers do now? (…) They introduced more machinery and in a moment the laborers were redundant again in a proportion satisfactory even to the farmers. There was now 'more capital' laid out in agriculture than before, and in a more productive form. With this the demand for labor fell not only relatively but absolutely"
>
> So far as the individual capitalists are concerned, each takes the wage level for granted and attempts to do the best he can for himself. In introducing machinery he is therefore merely attempting to economize on his own wage bill. The net effect of all capitalists' behaving in this way, however, is to create unemployment which in turn acts upon the wage level itself. It follows that the stronger the tendency of wages to rise, the stronger also will be the counteracting pressure of the reserve army, and vice versa (Sweezy, 1964, p. 88).

Drawing from this observation, we propose the following simplified dynamic equation in which the growth rate of mechanization is treated as an endogenous variable that depends on the distribution process between wages and capitalist profits:

$$\frac{\dot{m}}{m} = -\psi_0 + \psi_1 u, \qquad \psi_0, \psi_1 > 0 \quad (27)$$

Here, ψ_0 represents an autonomous stabilizing effect, while ψ_1 denotes the impact of the wage share on the growth rate of mechanization. This effect is assumed to be positive, given the assumption that an increase in the wage share ($\uparrow u$) subsequently leads to a decline in the profit share ($\downarrow (1-u)$). This, in turn, incentivizes the capitalist class to adopt novel production techniques that enable them to 'substitute labor with machinery.' Consequently, this leads to an acceleration in mechanization ($\uparrow \dot{m}/m$). Thus, a higher value for ψ_1 implies that capitalists possess a stronger power to mechanize production as a response to its distributive struggle with the working class.[13]

Substituting equation (27) into (24) yields an expression that describes a positive influence of the wage share u on the growth rate of productivity:[14]

$$\frac{\dot{a}}{a} = \alpha_0 - \alpha_1\psi_0 + \alpha_1\psi_1 u \quad (28)$$

Further, by substituting equations (27) and (28) into (26), we derive a dynamic equation governing the growth rate of the capital/output ratio σ. As a result, this term becomes a new state variable in the extended version of the Goodwin model with endogenous technical change presented in this paper:

$$\frac{\dot{\sigma}}{\sigma} = -[\alpha_0 + (1 - \alpha_1)\psi_0] + (1 - \alpha_1)\psi_1 u \quad (29)$$

With σ now being an endogenous variable, other equations require adjustments to accurately extend the Goodwin model. To begin, since σ is no longer a constant term, we should substitute equation (4) with the following expression, which is derived by applying logarithms and time differentiation to equation (3):

$$\frac{\dot{\sigma}}{\sigma} = \frac{\dot{k}}{k} - \frac{\dot{q}}{q} \quad (30)$$

[13] The assumption underlying the term ψ_1 and its sign is not arbitrary but is rooted in the Marxian literature on cyclical models. In fact, a similar assumption can be found in Eagly's (1972) Marxian model of economic cycles. Specifically, Eagly posits that a higher employment rate translates into enhanced bargaining power for the working class, enabling them to negotiate higher wage rates. In response, the capitalist class accelerates mechanization to diminish the bargaining power of workers. In contrast to Eagly, equation (27) directly links the dynamics of mechanization with the distribution process through the inclusion of the wage share.

[14] For an exploration of various methods for endogenizing labor productivity, which includes the potential positive influence of the wage share, refer to Dutt (2013).

Incorporating equations (3) and (30) into (11) and rearranging terms yields a new expression for the growth rate of output, which should replace equation (12):

$$\frac{\dot{q}}{q} = \frac{s(1-u)}{\sigma} - \delta - \frac{\dot{\sigma}}{\sigma} \quad (31)$$

By substituting equation (31) into (6) and solving for the growth rate of labor force employed, we obtain a new expression that should substitute equation (13):

$$\frac{\dot{l}}{l} = \frac{s(1-u)}{\sigma} - \delta - \frac{\dot{a}}{a} - \frac{\dot{\sigma}}{\sigma} \quad (32)$$

Combining equation (9), (26) and (32) gives:

$$\frac{\dot{v}}{v} = \frac{s(1-u)}{\sigma} - \frac{\dot{n}}{n} - \delta - \frac{\dot{m}}{m} \quad (33)$$

Substituting equations (10) and (27) into (33) and rearranging terms yields:

$$\frac{\dot{v}}{v} = \frac{s}{\sigma} - (\beta + \delta - \psi_0) - \left(\frac{s}{\sigma} + \psi_1\right) u \quad (34)$$

This expression supplants equation (15) and represents an extended dynamical equation for the growth rate of the employment rate. It embraces the effects linked to endogenous mechanization and productivity, highlighting the altered nature of σ from a constant term to an additional endogenous state variable within the model. Importantly, this variable introduces a non-linear influence on the growth rate of v.

Finally, by combining equations (17), (19), and (28), we arrive to an extended dynamic equation for the growth rate of the wage share, supplanting equation (21):

$$\frac{\dot{u}}{u} = -(\alpha_0 + \gamma - \alpha_1\psi_0) - \alpha_1\psi_1 u + \rho v \quad (35)$$

Equations (29), (34), and (35) constitute a three-dimensional dynamical system that characterizes the fully extended version of the Goodwin model, incorporating endogenous technical change as proposed in this paper. The state variables within this system encompass the wage share (u), the employment rate (v), and the capital/output ratio (σ). In the steady state ($\dot{u} = \dot{v} = \dot{\sigma} = 0$), this dynamical system has a non-trivial equilibrium point denoted as (u^*, v^*, σ^*), which is given by:

$$u^* = \frac{Z_2}{\psi_1 Z_1}, \qquad v^* = \frac{Z_3}{\rho Z_1}, \quad \sigma^* = \frac{Z_5}{Z_4} \quad (36)$$

where:

$$Z_1 = 1 - \alpha_1, \qquad Z_2 = \alpha_0 + (1 - \alpha_1)\psi_0, \qquad Z_3 = \gamma(1 - \alpha_1) + \alpha_0$$

$$Z_4 = \psi_1[(1 - \alpha_1)(\beta + \delta) + \alpha_0], \qquad Z_5 = s[(1 - \alpha_1)(\psi_1 - \psi_0) - \alpha_0]$$

In Appendix B, we analytically prove that this equilibrium point is positive and stable under the following conditions:

$$0 < \alpha_0 < 1, \qquad \alpha_1^c < \alpha_1 < 1, \qquad \psi_0 < \psi_1, \qquad \alpha_0 < (1 - \alpha_1)(\psi_1 - \psi_0) \quad (37)$$

Here, the lower bound α_1^c is equal to:

$$\alpha_1^c = \frac{Z_6 - \sqrt{Z_7}}{2Z_8} \quad (38)$$

where:

$$Z_6 = (\alpha_0 + \beta + \delta)\psi_1 + (\alpha_0 + \beta + \delta + \psi_0)(\psi_1 - \psi_0) - \alpha_0\psi_0$$

$$Z_7 = (\beta + \delta + \psi_1 - \psi_0)[(\beta + \delta + \psi_1 - \psi_0)\psi_0^2 + 4\alpha_0\psi_1(\alpha_0 + \beta + \delta)]$$

$$Z_8 = (\beta + \delta)\psi_1 + \psi_0(\psi_1 - \psi_0)$$

Within the same Appendix, we employ the existence part of the Hopf bifurcation theorem for three-dimensional dynamical systems to establish that the model described by equations (29), (34), and (35) can generate limit cycles near its equilibrium point. These limit cycles represent closed periodic solutions revolving around the equilibrium point, gradually attracting nearby trajectories over time. They come into existence as the parameter α_1, representing the mechanization-productivity elasticity, approaches the critical value α_1^c as defined in equation

(38). Building upon these analytical findings, the subsequent section features numerical simulations of the extended Goodwin model proposed in this paper, specifically applied to the US economy. The objective is to illustrate the cyclic nature of the model's dynamics and discuss the role played by the term α_1.

4. Simulating the Extended Goodwin Model for the US Economy

To perform simulations of the extended Goodwin model with endogenous technical change, as discussed in the preceding section, it is necessary to assign numerical values to the model's parameters. In this regard, we utilized the parameter estimates presented in Table 1, which have been derived from empirical estimations using annual data for the US economy spanning from 1960 to 2019. The estimation procedure follows an approach analogous to that employed by Grasselli and Maheshwari (2018), and the details of the procedure are provided in Appendix C.

Table 1. Parameter Estimates for the Extended Goodwin Model (US Economy)

$\hat{\delta}$	$\hat{s}$	$\hat{\beta}$	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\psi}_0$	$\hat{\psi}_1$	$\hat{\gamma}$	$\hat{\rho}$
0.05198494	0.5626761	0.01399579	0.0120001	0.3334868	0.2678959	0.4210462	0.2677537	0.3065009

Note: Parameter estimates obtained based on the procedure detailed in Appendix C.

Using the parameter values provided in Table 1 and inserting them into equation (36), we obtained the following equilibrium values for the state variables:

$$u^* = 0.6790, \qquad v^* = 0.9323, \quad \sigma^* = 2.1504 \quad (39)$$

where:

$$Z_1 = 0.6665, \qquad Z_2 = 0.1905, \qquad Z_3 = 0.1904, \qquad Z_4 = 0.0235, \qquad Z_5 = 0.0506$$

Furthermore, it can be confirmed that the values presented in Table 1 satisfy the conditions outlined in expression (37), ensuring the stability of the positive equilibrium (u^*, v^*, σ^*):

$$0 < \alpha_0 = 0.012 < 1, \qquad \alpha_1^c = 0.1527 < \alpha_1 = 0.3334 < 1, \qquad \psi_0 = 0.2678 < \psi_1 = 0.421,$$

$$\alpha_0 = 0.012 < (1 - \alpha_1)(\psi_1 - \psi_0) = 0.102$$

where:

$$Z_6 = 0.0825, \qquad Z_7 = 0.0037, \qquad Z_8 = 0.0688$$

Having confirmed that the estimated parameter values for the US economy meet the theoretical conditions for a positive and stable equilibrium, the next step in constructing our numerical simulation is defining the initial conditions of the state variables $\{u_1, v_1, \sigma_1\}$. To select these initial conditions, we implemented the following procedure: First, we initiated the model trajectories using each historical triplet $\{u_t^H, v_t^H, \sigma_t^H\}$ observed for the US economy from $t = 1960$ to $t = 2019$ as a candidate initial condition. This resulted in $S = 1,2,3, \ldots, 60$ preliminary simulations. Next, for each simulation S, we compared the complete historical series of values $\{u_t^H, v_t^H, \sigma_t^H\}_{t=1960}^{2019}$ with the corresponding simulated series $\{u_t^S, v_t^S, \sigma_t^S\}_{t=1960}^{2019}$. This comparison was done by calculating an Average Euclidean Distance (AED) as follows:

$$AED^S = \frac{1}{60} \sum_{t=1960}^{2019} \sqrt{(u_t^H - u_t^S)^2 + (v_t^H - v_t^S)^2 + (\sigma_t^H - \sigma_t^S)^2}, \qquad S = 1,2,3, \ldots, 60 \quad (40)$$

Finally, we selected the definitive initial condition as the historical triplet $\{u_t^H, v_t^H, \sigma_t^H\}$ that corresponds to the simulation with the minimum Average Euclidean Distance. As a result of this procedure, we selected the following initial values for the state variables:

$$\{u_1, v_1, \sigma_1\} = \{u_{1962}^H, v_{1962}^H, \sigma_{1962}^H\} = \{0.6703, \;\; 0.9500, \;\; 3.1460\} \quad (41)$$

Using the parameter values from Table 1 and the initial conditions defined in expression (41), we conducted a definitive simulation to generate time series, two-dimensional, and three-dimensional trajectories of the model's state variables (u, v, σ). These simulated trajectories were then compared with historical data from the US economy spanning from 1960 to 2019. The comparison results are depicted in Figures 1, 2, and 3. These figures consistently illustrate that the extended Goodwin model effectively generates stable ‘cyclical’ trajectories, specifically damped oscillations, when calibrated for the US economy.

Figure 1. Observed and Simulated Time Series (US Economy)

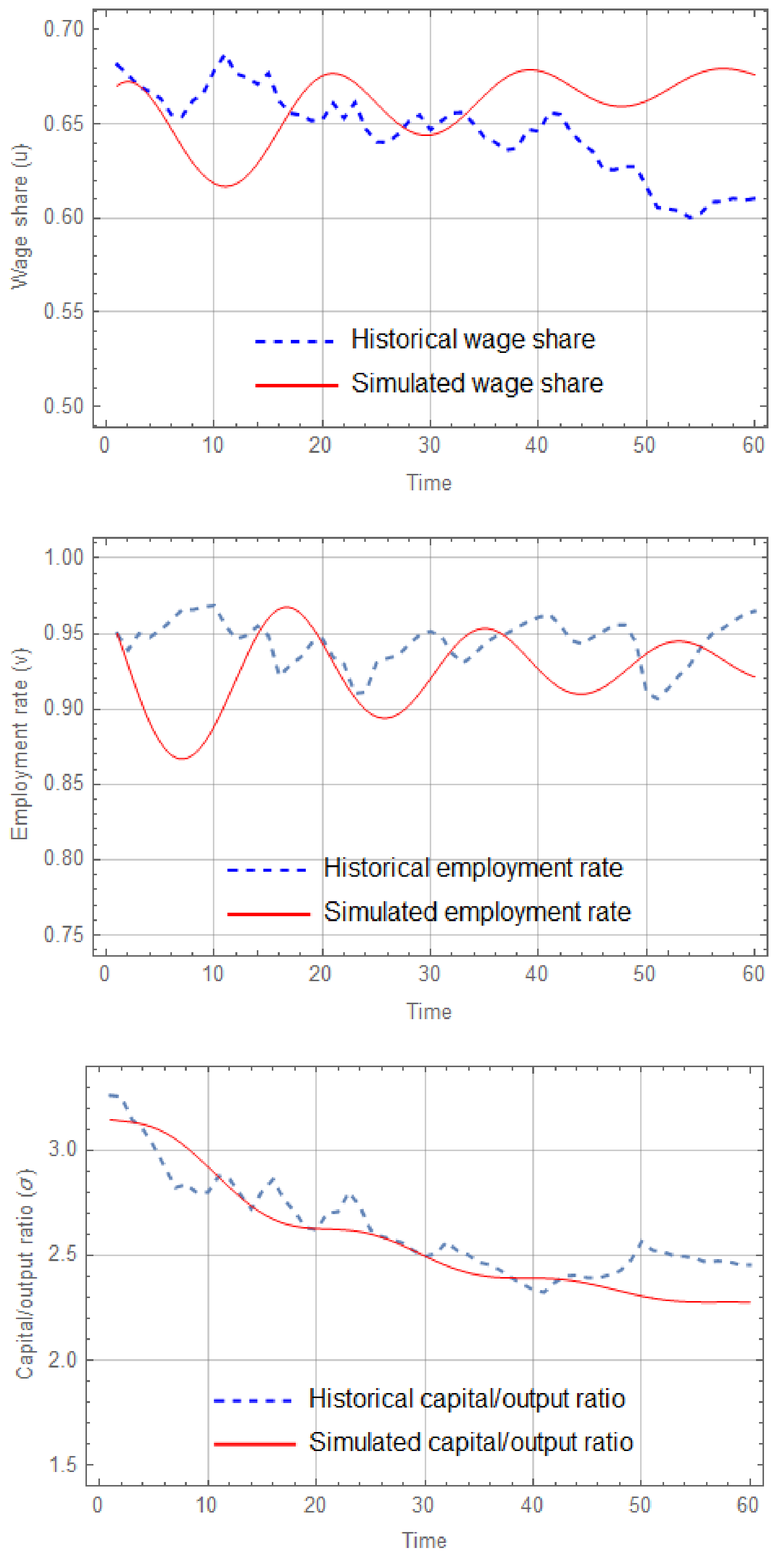


Figure 2. Observed and Simulated 2D Trajectories (US Economy)

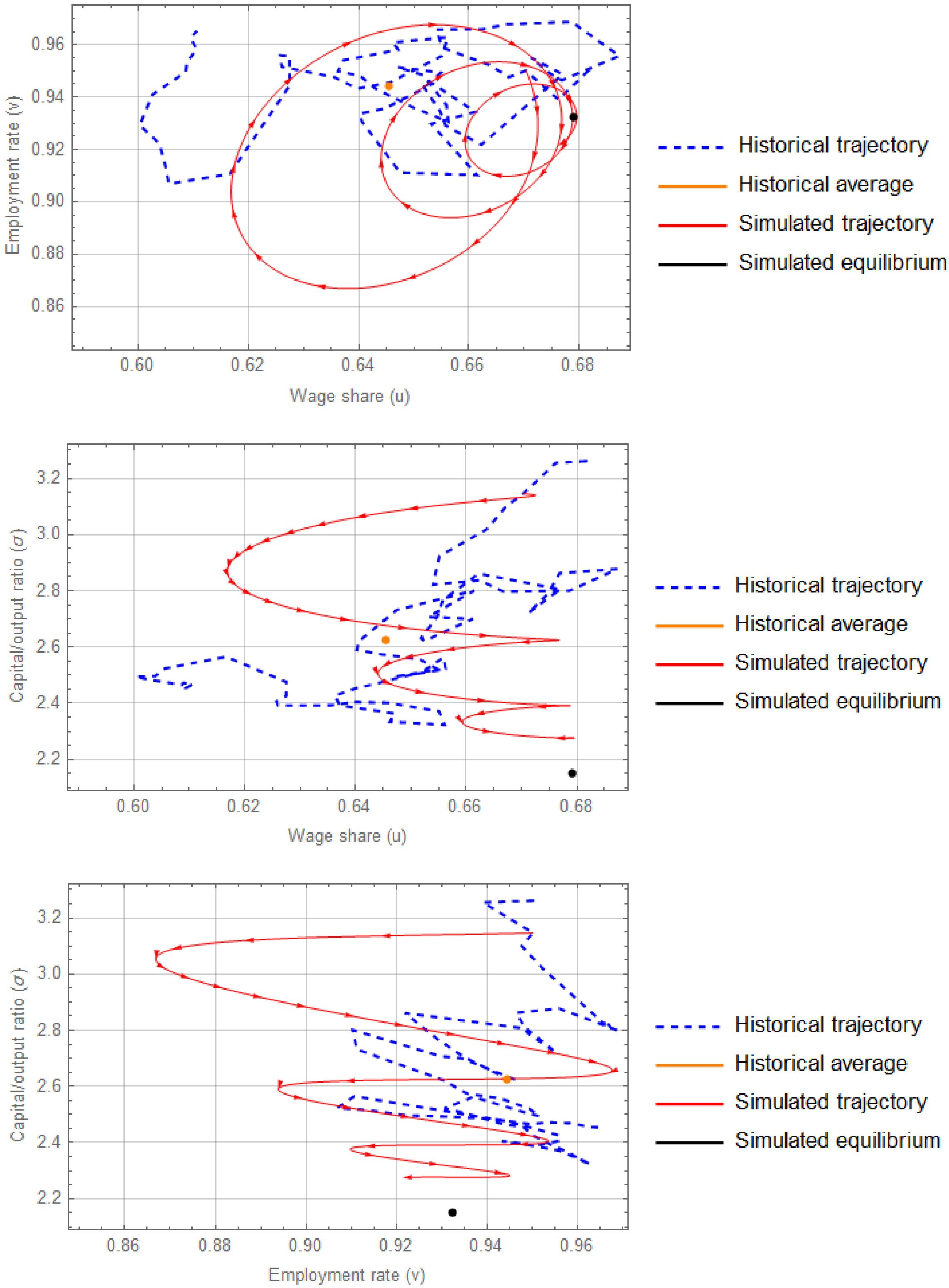
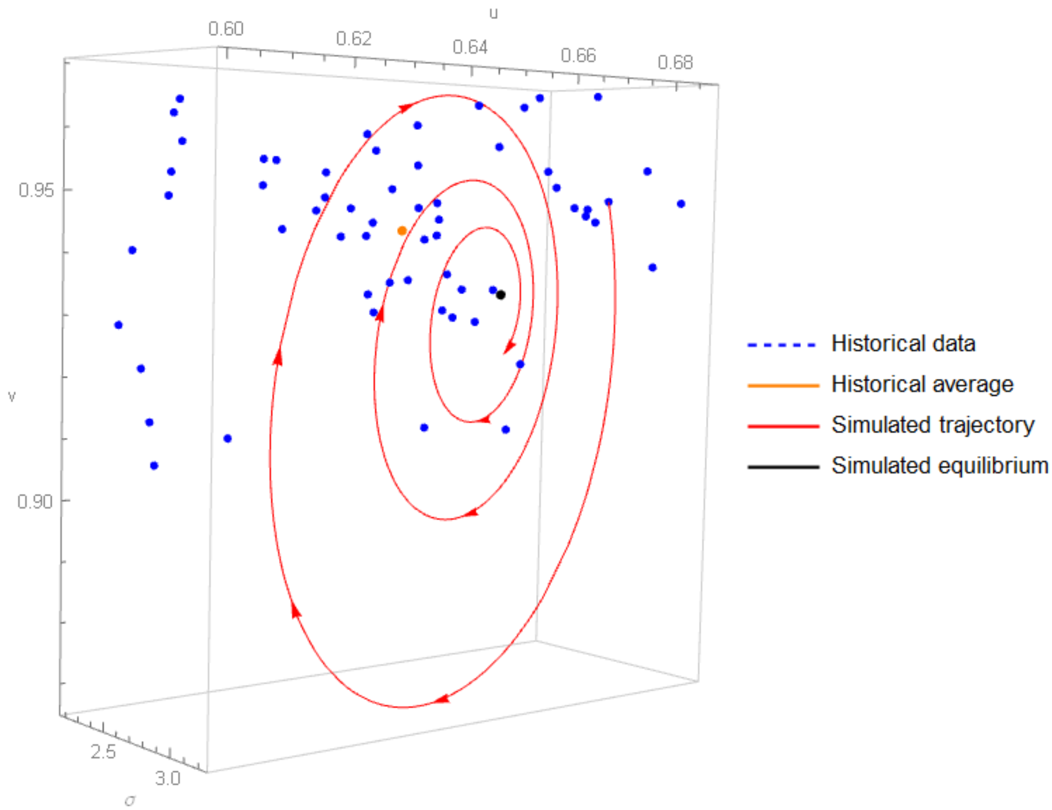


Figure 3. Observed Data and Simulated 3D Trajectory (US Economy)



In terms of comparing the simulated trajectories to historical data, we followed a similar approach to Harvie (2000) and Grasselli and Maheshwari (2018). We contrasted the estimated equilibrium values, as presented in equation (39), with the historical means of the state variables. The results of this comparison are summarized in Table 2, where we observe differences between the estimated equilibrium values and the historical means for the wage share (u), employment rate (v), and capital/output ratio (σ). Specifically, the estimated equilibrium values are 5.2% higher, 1.28% lower, and 18.04% lower than the historical means, respectively.

At first glance, these results might suggest that the extended Goodwin model discussed in this paper struggles to accurately match the historical means of the wage share and, especially, the capital/output ratio. However, a closer examination of the historical and simulated time series for these variables in Figure 1 reveals that averages may not necessarily provide the best estimators of equilibrium values. For instance, the historical wage share in the US exhibits a significant decreasing trend, particularly since 1970, a pattern the model struggles to replicate, a challenge also encountered in simulations by Grasselli and Maheshwari (2018). In contrast, the historical capital/output ratio demonstrates a notable decreasing trend from 1960 to 2000. In this case, the extended Goodwin model seems to satisfactorily replicate this trend. However, it fails to completely capture the subsequent recovery in the capital/output ratio during the following years. When considering the employment rate, both historical and simulated time series appear to fluctuate around similar values.

To assess the model's fit more quantitatively, Table 2 also provides the root-mean-square error (RMSE) of the simulated trajectories of each state variable as a proportion of its historical mean. The RMSE accounts for 5.93% of the historical mean for the wage share, 4.13% for the employment rate, and 4.63% for the capital/output ratio. It's worth noting that, in comparison, Grasselli and Maheshwari (2018) reported lower RMSE values of 5.4% for the wage share and 1.4% for the employment rate in their simulations for the US economy. While our model exhibits slightly lower goodness-of-fit in the case of the wage share and employment rate, this difference appears to be modest. Furthermore, this discrepancy may be offset by the increased theoretical and analytical complexity introduced by the extended Goodwin model with endogenous technical change presented in this paper, particularly due to the inclusion of the capital/output ratio as a third state variable. Importantly, this variable seems to be simulated with acceptable accuracy by the model.

Table 2. Mean, Equilibrium, and RMSE of Simulated Trajectories (US Economy)

State variables	Historical mean (A)	Estimated equilibrium (B)	Difference (A-B) as a proportion of (A)	RMSE of simulated trajectories as a proportion of (A)
Wage share (u)	0.6454	0.6790	5.20 %	5.93 %
Employment rate (v)	0.9444	0.9323	-1.28 %	4.13 %
Capital/output ratio (σ)	2.6237	2.1504	-18.04 %	4.63 %

5. Limit Cycles and the Mechanization-Productivity Relationship

Referring back to Appendix B, our utilization of the existence part of the Hopf bifurcation theorem for three-dimensional dynamical systems suggests that the extended Goodwin model under study possesses the capacity to generate limit cycles as the mechanization-productivity elasticity α_1 approaches the critical value α_1^c defined in equation (38). Building upon this analytical outcome and recognizing that substituting the values provided in Table 1 into equation (38) yields an estimated critical value of $\alpha_1^c = 0.1527$, we can proceed to create a new numerical simulation of the extended Goodwin model capable to generate limit cycles. This simulation involves the replacement of the estimated value of α_1 with the critical value α_1^c, as outlined in Table 3.

Table 3. Parameter Estimates for the Extended Goodwin Model (Limit Cycle)

$\hat{\delta}$	$\hat{s}$	$\hat{\beta}$	$\hat{\alpha}_0$	$\hat{\alpha}_1 = \hat{\alpha}_1^c$	$\hat{\psi}_0$	$\hat{\psi}_1$	$\hat{\gamma}$	$\hat{\rho}$
0.05198494	0.5626761	0.01399579	0.0120001	0.15269969	0.2678959	0.4210462	0.2677537	0.3065009

Note: The parameter values used are consistent with those in Table 1, except for substituting the estimated value of α_1 with the critical value α_1^c. For details, see Appendix C.

Substituting the values from Table 3 into equation (37) yields a revised equilibrium point, which slightly differs from the equilibrium point in equation (39):

$$u^* = 0.6699, \qquad v^* = 0.91979, \quad \sigma^* = 2.3175 \quad (42)$$

Applying the Average Euclidean Distance minimization method described in the preceding section, we get new initial conditions:

$$\{u_1, v_1, \sigma_1\} = \{u_{1963}^H, v_{1963}^H, \sigma_{1963}^H\} = \{0.6672,\ \ 0.9475,\ \ 3.1006\} \quad (43)$$

Using the parameter values from Table 3 and the initial conditions specified in equation (43), we conducted a new simulation of the model to derive two-dimensional and three-dimensional trajectories of the state variables (u, v, σ). We then compared the results with the historical data for the US. The comparison is illustrated in Figures 4 and 5, showcasing the extended Goodwin model's capability to generate limit cycles as α_1 approaches α_1^c.

Regarding the model's fit, Table 4 indicates that the trajectories simulated in the case with limit cycles revolve around equilibrium values that closely align with the historical means of the state variables, representing an improvement over the simulation presented in the previous section. However, the RMSE is slightly higher for the trajectories associated with both the wage share and the employment rate, while being lower for the trajectory linked to the capital/output ratio. This reaffirms the notion that the extended Goodwin model may serve as an effective analytical tool for understanding the dynamics of the latter variable even in the case of limit cycles.

Table 4. Mean, Equilibrium, and RMSE of Simulated Trajectories (Limit Cycles)

State variables	**Historical mean (A)**	**Estimated equilibrium (B)**	**Difference (A-B) as a proportion of (A)**	**RMSE of simulated trajectories as a proportion of (A)**
Wage share (u)	0.6454	0.6699	3.78 %	6.63 %
Employment rate (v)	0.9444	0.9197	-2.61 %	5.82 %
Capital/output ratio (σ)	2.6237	2.3175	-11.67 %	4.31 %

Figure 4. Observed and Simulated 2D Trajectories (Limit Cycle)

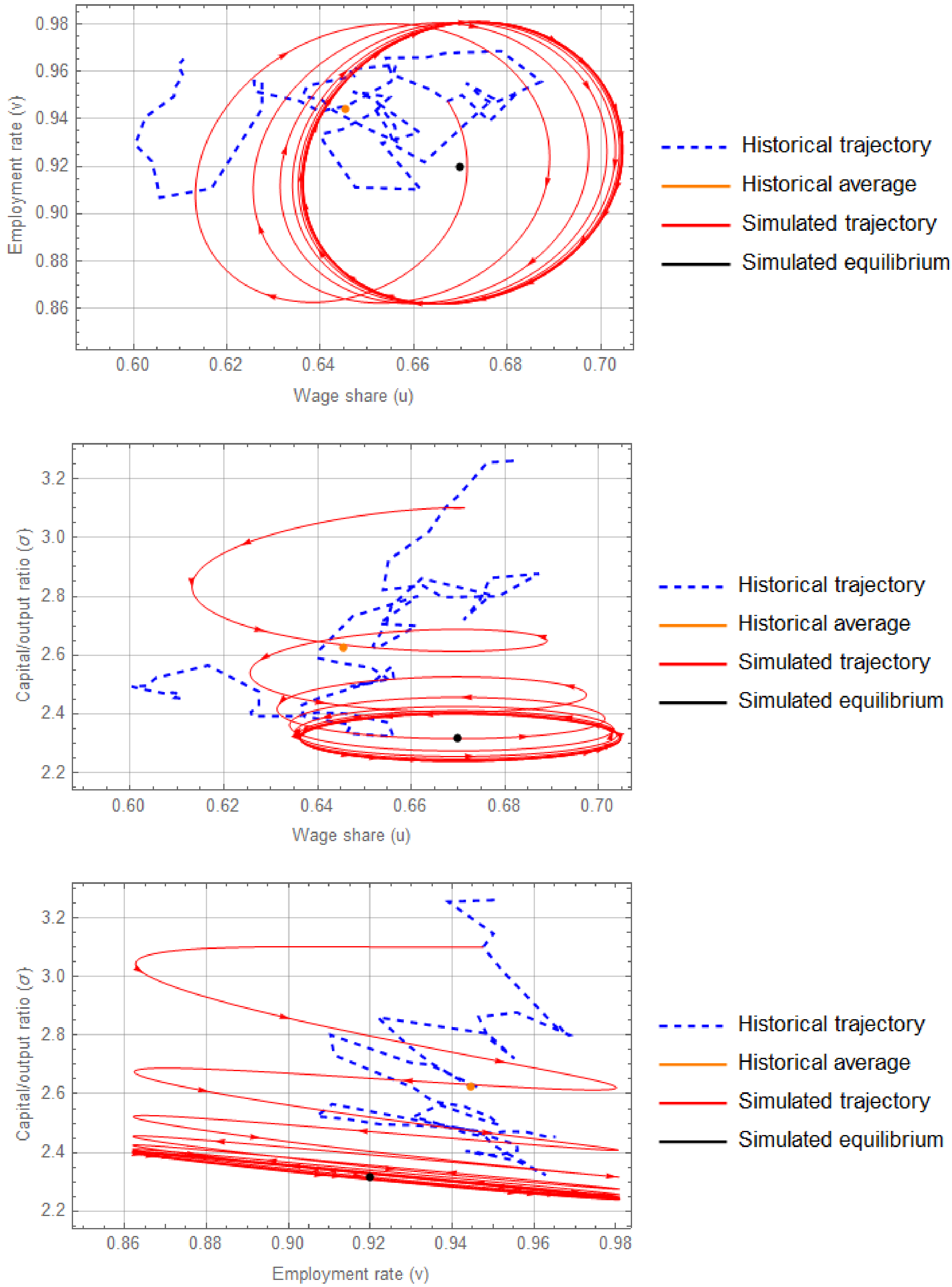
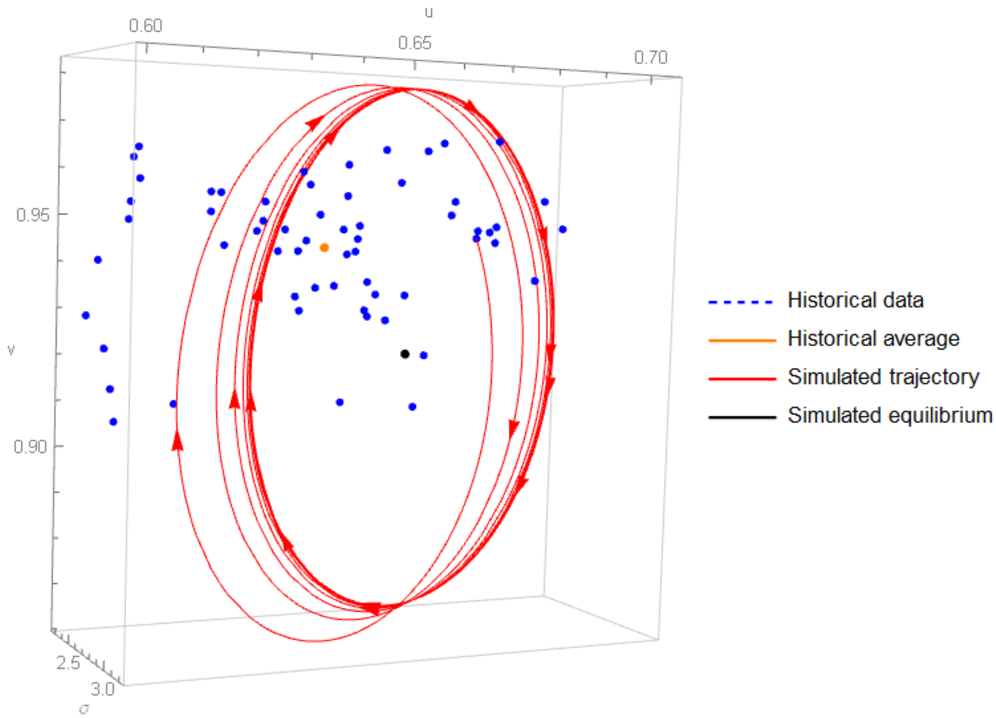
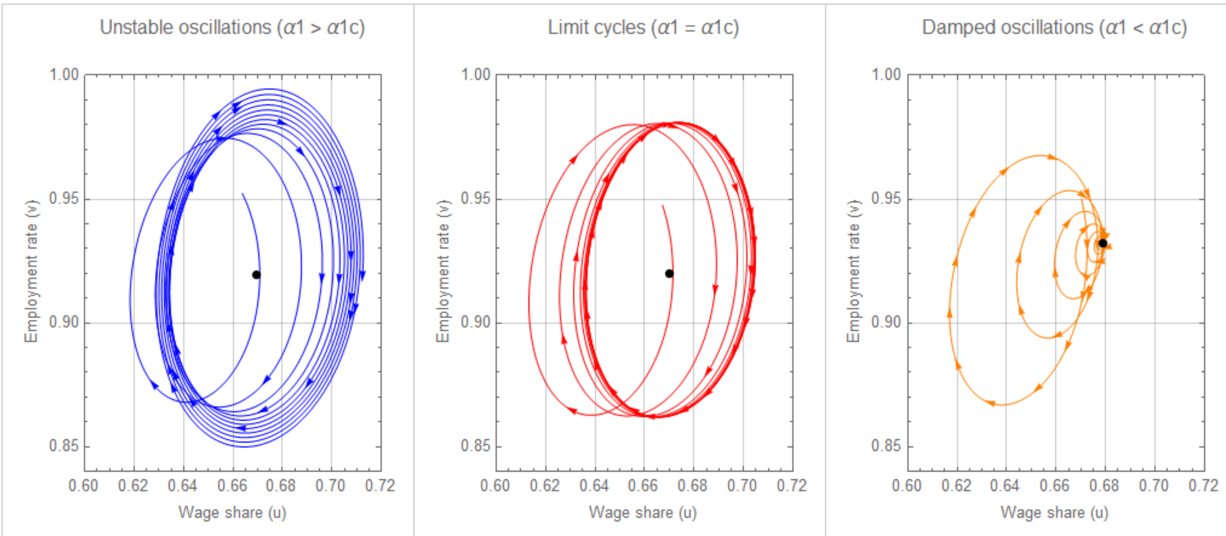


Figure 5. Observed Data and Simulated 3D Trajectory (Limit Cycle)



The theoretical and simulation results obtained thus far highlight the significant sensitivity of the extended Goodwin model's 'cyclical' behavior to the magnitude of the mechanization-productivity elasticity α_1. In fact, running an additional simulation of the model with $\alpha_1 > \alpha_1^c$, *ceteris paribus*, yields trajectories characterized by unstable oscillations. This finding complements the previous observations of limit cycles for $\alpha_1 \approx \alpha_1^c$ and damped oscillations for $\alpha_1 < \alpha_1^c$. The relationship between these 'cyclical' trajectories and the elasticity α_1 is summarized in Figure 6. Therefore, a thorough discussion regarding the determinants of the behavior of the mechanization α_1 becomes crucial for a deeper understanding of the dynamics of capitalism, particularly in the context of endogenous technical change and distributive cycles generated by class-struggle.

Figure 6. Relationship between 'Cycles' and Mechanization-Productivity Elasticity



Note: Unstable oscillations simulated with $\alpha_1 = 0.1427$, while other parameters match those presented in Table 1.

Damped oscillations and limit cycles correspond to simulations presented in Figures 2 and 4, respectively.

While a comprehensive discussion of the determinants of α_1 extends beyond the scope of this paper, it is worthwhile to propose some initial insights. Unlike interpreting α_1 solely as an elasticity term influenced by technological factors, which is the conventional perspective in orthodox economics, a Marxian perspective may suggest that the relationship between mechanization and labor productivity, represented by α_1, is also shaped by the broader context of the class struggle and its impact on variables such as the average labor intensity. In line with Marx's observations (1867/1975), heightened mechanization of production and a larger reserve army of labor can potentially boost productivity. This boost may occur as these factors contribute to increased labor intensity through intensified work processes prompted by the threat of dismissal or similar mechanisms. Consider, for instance, the following passages:

> In proportion as the use of machinery spreads, and the experience of a special class of workmen habituated to machinery accumulates, the rapidity and intensity of labor increases (Marx, 1867/1975, p. 412).

> The development of the capitalist mode of production (…) enables the capitalist, with the same outlay of variable capital, to ser in action more labor by greater exploitation (extensive or intensive) of each

> individual labor power (…) The overwork of the employed part of the working class swells the ranks of the reserve, whilst conversely the greater pressure that the latter by its competition exerts on the former, forces these to submit to overwork and to subjugation under the dictates of capital. The condemnation of one part of the working class to enforced idleness by the overwork of the other part, and the converse, becomes a means of enriching the individual capitalists, and accelerates at the same time the production of the industrial reserve army on a scale corresponding with the advance of social accumulation (Marx, 1867/1975, pp. 629–630).

We could therefore postulate that the impact of mechanization on productivity is closely linked to the employment rate due to the influence of the reserve army of labor on labor intensity. As the employment rate decreases, the relative weight of the unemployed reserve army in the labor supply rises, diminishing the bargaining power of the working class. In this scenario, capitalists can leverage their position to demand higher productivity from workers, for a given mechanization growth rate, potentially through intensified work processes driven by the fear of job loss or similar factors. Consequently, within the context of the extended Goodwin model presented in this paper, we might posit an inverse relationship between the mechanization-productivity elasticity and the employment rate, denoted as $\alpha_1 = f(v)$ with $f' < 0$. However, this assumption may significantly increase the complexity of the three-dimensional dynamical system originally defined by equations (29), (34), and (35) depending on the specific form of f derived from the econometric analysis of historical data from the US or other capitalist economy. Therefore, we propose this additional extension of the model as a subject for future theoretical and empirical discussion.[15]

6. Conclusions

This paper has extended the Goodwin model to delve into the dynamic nature of capitalism, encompassing the influence of endogenous technical change, as inspired by Marx's intuitions regarding the interaction between mechanization and labor productivity within the framework of

[15] For a more comprehensive theoretical exploration of the interplay between class struggle, labor intensity, productivity, and endogenous cycles, see Cajas Guijarro and Vera (2022) and Cajas Guijarro (2023).

distributive cycles resulting from the class struggle between workers and capitalists. The analytical study of this extended, coupled with numerical simulations using parameter values estimated for the US economy from 1960 to 2019, has yielded significant insights from both theoretical and empirical standpoints.

From a theoretical perspective, the application of the existence part of the Hopf bifurcation theorem for three-dimensional dynamical systems has identified the mechanization-productivity elasticity (α_1) as a pivotal determinant of the cyclical behavior of the trajectories generated by the extended Goodwin model proposed here. These findings, reinforced through the numerical simulation of the model for the US economy, reveal the existence of damped oscillations when α_1 exceeds a critical value α_1^c, which is equal to 0.1526. Additional simulations confirm the presence of limit cycles as α_1 approaches this critical value and unstable oscillations when α_1 falls below it, underscoring the decisive role of α_1 in shaping capitalist 'cyclical' dynamics within a context of distributive cycles and endogenous technical change.

However, our theoretical discussion, again guided by Marx's insights, posits that α_1 is not merely a technological factor. Instead, it operates within a broader context influenced by the dynamics of the class struggle. In accordance with this Marxian perspective, the connection between mechanization and labor productivity is intricately tied to variables such as labor intensity. Elevated labor intensity, driven by intensified work processes due to the threat of job insecurity, may be considered as a critical component of the cyclical dynamics within a context of capitalist endogenous technical change, warranting future exploration through more intricate versions of the extended Goodwin model proposed herein. To initiate this exploration, it may be useful to consider the possibility of an inverse relationship between the mechanization-productivity elasticity and the employment rate, indirectly representing the influence of the reserve army of unemployed on labor intensity. Another avenue for future theoretical exploration involves expanding the extended Goodwin model to incorporate additional insights from Marxian economics like, for instance, the potential implications of a falling rate of profit. In this context, the model of long cycles developed by Nikolaos et al. (2022) could serve as a useful reference.

From an empirical standpoint, the numerical simulation of the extended Goodwin model, calibrated for the US economy, provides evidence of the model's capability to generate stable 'cyclical' trajectories that can be compared to historical data for the state variables, namely, the wage share (u), the unemployment rate (v), and the capital/output ratio (σ). The adjustment between simulated trajectories and historical data reveals promising results. While the estimated equilibrium values for these variables exhibit modest deviations from their historical means, a detailed examination of historical and simulated time series offers a more comprehensive understanding. The historical wage share in the US has shown a pronounced declining trend since the 1970s, a pattern that the model struggles to replicate, challenge also encountered in simulations by Grasselli and Maheshwari (2018). In contrast, the historical capital/output ratio portrays a decreasing trend up to 2000, a trend the model accurately mirrors, albeit not entirely capturing the subsequent recovery observed in historical data. Notably, the employment rate displays relatively consistent trajectories in both historical and simulated data. A quantitative perspective, incorporating the root-mean-square error (RMSE) of simulated trajectories as a proportion of their historical means, provides further insights. While the RMSE for the wage share and the employment rate is slightly higher compared with Grasselli and Maheshwari (2018), these discrepancies remain within a reasonable range. On the contrary, the RMSE for the capital/output ratio is relatively lower, reinforcing the notion that the extended Goodwin model is a suitable analytical tool for comprehending the dynamics of this variable. Even in the case of limit cycles, the simulated trajectories of the state variables exhibit an acceptable fit when compared to historical data.

In summary, this extended Goodwin model with endogenous technical change seems to contribute to the understanding of the intricate and potentially cyclical dynamics of capitalism, from both theoretical and empirical perspectives. Particularly, the theoretical validation of the stability of the three-dimensional model and its capacity to generate limit cycles establishes a connection with other studies focused on high-dimensional dynamical systems, as mentioned by Azevedo Araujo et al. (2019). Moreover, the calibration of the three-dimensional system, encompassing endogenous distributive cycles in the context of the US economy, attempts to connect the theoretical examination of high-dimensional models and the empirical investigations originally formulated by authors like Desai (1984), Harvie (2000), and Grasselli and Maheshwari (2018), focused on econometrically estimate parameters associated with the Goodwin model. Perhaps, by integrating these diverse perspectives, we can gain better and deeper insights into the complexity and vulnerabilities of capitalism.

Declaration of Generative AI and AI-assisted technologies in the writing process

During the preparation of this work the author used ChatGPT to enhance the writing quality. After using this tool/service, the author reviewed and edited the content as needed and takes full responsibility for the content of the publication.

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Appendix A: Notation

Symbol	Description
q	Output
k	Total (fixed) capital
$\dot{k} = \frac{dk}{dt}$	Net investment
w	Real wage
l	Total labor force employed
a	Labor productivity
α	Growth rate of labor productivity
σ	Capital/output ratio
n	Labor supply
β	Growth rate of labor supply
u	Wage share
v	Employment rate
δ	Depreciation rate
s	Savings-accumulation rate
γ	Autonomous tendency of the real wage to fall
ρ	Effect of the employment rate on the real wage
m	Mechanization
α_0	Autonomous tendency of productivity to grow
α_1	Effect of mechanization on productivity
ψ_0	Autonomous tendency of mechanization to stabilize
ψ_1	Effect of the wage share on mechanization

Appendix B: Dynamic Analysis of the Extended Goodwin Model

Consider the extended Goodwin model with endogenous technical change, originally defined by equations (35), (34), and (29), which are restated here as equations (A1), (A2), and (A3) respectively:

$$\frac{\dot{u}}{u} = -(\alpha_0 + \gamma - \alpha_1\psi_0) - \alpha_1\psi_1 u + \rho v \quad (A1)$$

$$\frac{\dot{v}}{v} = \frac{s}{\sigma} - (\beta + \delta - \psi_0) - \left(\frac{s}{\sigma} + \psi_1\right) u \quad (A2)$$

$$\frac{\dot{\sigma}}{\sigma} = -[\alpha_0 + (1 - \alpha_1)\psi_0] + (1 - \alpha_1)\psi_1 u \quad (A3)$$

In the steady state ($\dot{u} = \dot{v} = \dot{\sigma} = 0$), this three-dimensional dynamical system has a non-trivial equilibrium point (u^*, v^*, σ^*), which is given by:

$$u^* = \frac{Z_2}{\psi_1 Z_1}, \qquad v^* = \frac{Z_3}{\rho Z_1}, \quad \sigma^* = \frac{Z_5}{Z_4} \quad (A4)$$

where:

$$Z_1 = 1 - \alpha_1, \qquad Z_2 = \alpha_0 + (1 - \alpha_1)\psi_0, \qquad Z_3 = \gamma(1 - \alpha_1) + \alpha_0$$

$$Z_4 = \psi_1[(1 - \alpha_1)(\beta + \delta) + \alpha_0], \qquad Z_5 = s[(1 - \alpha_1)(\psi_1 - \psi_0) - \alpha_0]$$

To guarantee the existence of a positive equilibrium point ($u^* > 0$, $v^* > 0$, $\sigma^* > 0$), we make the following assumptions:

$$0 < \alpha_0, \alpha_1 < 1, \qquad \psi_0 < \psi_1, \qquad \alpha_0 < (1 - \alpha_1)(\psi_1 - \psi_0) \quad (A5)$$

These assumptions guarantee that all terms Z_1 to Z_5 are positive. Now, linearizing equations (A1), (A2), and (A3) around the equilibrium point (u^*, v^*, σ^*) yields the following system of equations in matrix form:

$$\begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{v}\\ \Delta\dot{\sigma}\end{bmatrix} = \begin{bmatrix} -\frac{\alpha_1 Z_2}{Z_1} & \frac{s\rho Z_2}{\psi_1 Z_1} & 0 \\ -\frac{Z_6(sZ_4+\psi_1 Z_5)}{\rho Z_1 Z_4 Z_5} & 0 & -\frac{sZ_3 Z_4^2}{\rho\psi_1 Z_1^2 Z_5} \\ \frac{\psi_1 Z_1 Z_5}{sZ_4} & 0 & 0 \end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta v\\ \Delta\sigma\end{bmatrix}$$

The characteristic polynomial of the Jacobian matrix of this linearized system is equal to:

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$$

where:

$$b_1 = \frac{\alpha_1 Z_2}{Z_1}, \qquad b_2 = \frac{Z_2 Z_3(sZ_4 + \psi_1 Z_5)}{\psi_1 Z_1^2 Z_5}, \qquad b_3 = \frac{sZ_2 Z_3 Z_4}{\psi_1 Z_1^2}$$

According to the Routh-Hurwitz criteria, when b_1, b_2, b_3, are all positive and $b_1 b_2 - b_3 > 0$, all eigenvalues λ have negative real components, ensuring local stability of the model around its equilibrium point. In this sense, the assumptions represented in expression (A5) guarantee that b_1, b_2, and b_3 are positive. Concerning $y = b_1 b_2 - b_3$, firstly we note that:

$$y = b_1 b_2 - b_3 = \frac{Z_2 Z_3[Z_2(Z_4 + \alpha_1 Z_1 \psi_1^2) - \psi_1(\alpha_1 Z_2^2 - Z_1^2 Z_4)]}{\psi_1 Z_1^3 Z_5}$$

Given this result and previous assumptions, it can be proved that $b_1 b_2 - b_3$ is positive when:[16]

$$\alpha_1^c < \alpha_1 < 1 \quad (A6)$$

Here, the lower bound α_1^c is given by:

$$\alpha_1^c = \frac{Z_6 - \sqrt{Z_7}}{2Z_8} \quad (A7)$$

where:

[16] For the sake of simplicity, we focus on the critical value α_1^c that aligns with the assumptions presented in expression (A5) and holds economic significance when considering the parameter values estimated for the US economy from 1960 to 2019.

$$Z_6 = (\alpha_0 + \beta + \delta)\psi_1 + (\alpha_0 + \beta + \delta + \psi_0)(\psi_1 - \psi_0) - \alpha_0\psi_0$$

$$Z_7 = (\beta + \delta + \psi_1 - \psi_0)[(\beta + \delta + \psi_1 - \psi_0)\psi_0^2 + 4\alpha_0\psi_1(\alpha_0 + \beta + \delta)]$$

$$Z_8 = (\beta + \delta)\psi_1 + \psi_0(\psi_1 - \psi_0)$$

In addition to stability, it is possible to prove that the equilibrium point (u^*, v^*, σ^*) can transition from stable to unstable while the solutions oscillate around it. In other words, the model exhibits a Hopf bifurcation, indicating the potential to generate limit cycles when a bifurcation parameter approaches a critical value. Following Liu (1994), we can establish the existence of limit cycles by employing the existence component of the Hopf bifurcation theorem for three-dimensional dynamical systems. This process involves verifying the existence of a bifurcation parameter x, which has a critical value x^c that satisfies the following condition:

$$b_1(x^c), b_2(x^c), b_3(x^c) > 0, \qquad y(x^c) = 0, \qquad \left.\frac{dy}{dx}\right|_{x=x^c} \neq 0 \quad (A8)$$

By designating $x = \alpha_1$ as the bifurcation parameter and $x^c = \alpha_1^c$, as defined in equation (A7), as its critical value, we can demonstrate that α_1^c satisfies the conditions represented in expression (A8). On one hand, b_1, b_2, and b_3 remain positive when $\alpha_1 = \alpha_1^c$, as α_1^c aligns with the assumptions presented in (A5). On the other hand, it can be proved that the derivative $\left.\frac{dy}{d\alpha_1}\right|_{\alpha_1=\alpha_1^c}$ is positive, at least for relevant values of the model's parameters.[17]

In summary, the extended Goodwin model presented in this paper exhibits a stable positive equilibrium point (u^*, v^*, σ^*) under conditions (A5) and (A6). Furthermore, as α_1 approaches α_1^c, the simplified model can produce limit cycles near its equilibrium point.

[17] In the case of the parameters estimated for the US economy and detailed in Appendix C, the derivative is positive and equal to 0.0415782.

Appendix C: Parameter Estimation for Simulating the Extended Model

To develop numerical simulations for the extended Goodwin model with endogenous mechanization, we perform parameter estimation using annual data sourced from the AMECO database for the US economy from 1960 to 2019. Our estimation approach closely mirrors the procedure employed by Grasselli and Maheshwari (2018). In terms of data compilation and processing, our initial step involves extracting the variables outlined in Table A1 from the AMECO database. Subsequently, we compute initial time series necessary for the analysis of the extended Goodwin model, adhering to the definitions provided by Grasselli and Maheshwari (2018), as elaborated in Table A2.

Table A1. Original Variables obtained from AMECO Database

Variable from AMECO Database	Units	Symbol
Gross domestic product at current factor cost	Mrd ECU/EUR	GDP_fc
Price deflator gross domestic product	ECU/EUR: 2015 = 100	def_GDP
Compensation of employees: total economy	Mrd ECU/EUR	comp_emp
Number of self-employed: total economy (National accounts)	Number of people	self_emp
Employees, persons: total economy (National accounts)	Number of people	tot_emp
Total unemployment. Member States: definition EUROSTAT. Nonmember States: OECD	Number of people	tot_unemp
Net capital stock at 2015 prices: total economy	Mrd ECU/EUR	k_stock
Consumption of fixed capital at current prices: total economy	Mrd ECU/EUR	cons_fixk
Price deflator gross fixed capital formation: total economy	ECU/EUR: 2015 = 100	def_fixk
Gross fixed capital formation at current prices: total economy	Mrd ECU/EUR	fixkn
Price deflator gross fixed capital formation: total economy	ECU/EUR: 2015 = 100	fixkp

Table A2. Estimation of Initial Time Series for the Extended Goodwin Model

Symbol	Description	Estimation
q_t	Output	GDP_fc / def_GDP
k_t	Total (fixed) capital	k_stock
l_t	Total labor force employed	self_emp + tot_emp
n_t	Labor supply	self_emp + tot_emp + tot_unemp
v_t	Employment rate	l_t / n_t
W_t	Wage bill	(1 + self_emp / tot_emp) * (comp_emp / def_GDP)
w_t	Real wage	l_t / W_t
u_t	Wage share	W_t / q_t
δ_t	Depreciation rate	cons_fixk / (def_fixk * k_stock)
Π_t	Gross real profits	$q_t - W_t$
s_t	Savings-Accumulation rate	(fixkn / fixkp) / Π_t
σ_t	Capital/output ratio	k_t / q_t
a_t	Labor productivity	q_t / l_t
m_t	Mechanization	k_t / l_t

Note: Estimation using variables from Table A1 and definitions provided by Grasselli and Maheshwari (2018)

After obtaining the initial time series data as detailed in Table A2, we proceed to estimate the primary parameters of the extended Goodwin model. To this end, we follow the approach of Grasselli and Maheshwari (2018), utilizing historical means as parameter values for the depreciation rate ($\hat{\delta}$) and the average savings-accumulation rate ($\hat{s}$). This leads to the following parameter estimates:

$$\hat{\delta} = 0.05198, \qquad \hat{s} = 0.56267$$

For the growth rate of labor supply ($\hat{n}$) and the growth rate of labor productivity ($\hat{a}$), we estimate the following log-regressions:

$$\ln n_t = \underset{(0.01595)}{11.31} + \underset{(0.0004)}{0.013995}\, t + \hat{e}_{nt}, \qquad R^2 = 0.9423$$

$$\ln a_t = \underset{(0.7847)}{2.1011} + \underset{(0.0008)}{0.012001}\, t + \underset{(0.0671)}{0.3334868}\, m_t + \hat{e}_{at}, \qquad R^2 = 0.9912$$

where $\hat{e}_{nt}$ and $\hat{e}_{at}$ represent the estimated residuals. By applying time derivatives to these regressions, we obtain the following estimates:

$$\widehat{\frac{\dot{n}_t}{n_t}} = \hat{\beta} = 0.013995, \qquad \widehat{\frac{\dot{a}_t}{a_t}} = \hat{\alpha}_0 + \hat{\alpha}_1 m_t = 0.012001 + 0.3334868 m_t$$

Now, to estimate the remaining parameters of the model, we introduce the discrete-time versions of equations (17) and (27), respectively:

$$\Delta \ln w_t = -\hat{\gamma} + \hat{\rho} v_t + \hat{e}_{1t} \qquad (A9)$$

$$\Delta \ln m_t = \hat{\psi}_0 + \hat{\psi}_1 u_t + \hat{e}_{2t} \qquad (A10)$$

Here, for any variable X_t, $\Delta \ln X_t = \ln X_t - \ln X_{t-1}$ represents a discrete approximation of its growth rate, while $\hat{e}_{1t}$, and $\hat{e}_{2t}$ denote estimated residuals.

To estimate the values of $\hat{\gamma}$, $\hat{\psi}_0$, $\hat{\psi}_1$, and $\hat{\rho}$ we employ the long-run multipliers derived from the Autoregressive Distributive Lag (ARDL) estimator, following the bounds-testing procedure proposed by Pesaran et al. (2001). This approach requires that the time series used in the 'level equations' (A9), and (A10) are either stationary ($I(0)$) or, at most, integrated of order one ($I(1)$). We ascertain this requirement through the Augmented Dickey Fuller (ADF) test, which suggests that all the time series considered within these equations appear to be $I(1)$ with a 99% confidence level, as indicated in Table A3.

Table A3. Unit Root Tests

Variables	ADF test (levels)		ADF test (first difference)	
	Statistic	p-value	Statistic	p-value
$\Delta \ln w_t$	-2.6853	0.2979	-4.9251	<0.01
v_t	-2.5778	0.3413	-6.1129	<0.01
$\Delta \ln m_t$	-3.0684	0.1431	-5.7054	<0.01
u_t	-3.0771	0.1396	-5.3948	<0.01

Note: ADF tests conducted including 3 lags.

After confirming that all variables employed in equations (A9) and (A10) are $I(1)$, we proceed to estimate two Unrestricted Error Correction Models (UECM) characterized by the following equations:

$$\Delta(\Delta \ln w_t) = b_{1,0} + b_{1,1}(\Delta \ln w_{t-1}) + b_{1,2} v_{t-1} + \sum_{j=1}^{p_1 - 1} \psi_{1,j}\, \Delta\left(\Delta \ln w_{t-j}\right) + \sum_{j=0}^{q_1 - 1} \gamma_{1,j} \Delta v_{t-j} + \varepsilon_{1t} \quad (A11)$$

$$\Delta(\Delta \ln m_t) = b_{2,0} + b_{2,1}(\Delta \ln m_{t-1}) + b_{2,2} u_{t-1} + \sum_{j=1}^{p_2 - 1} \psi_{2,j}\, \Delta\left(\Delta \ln m_{t-j}\right) + \sum_{j=0}^{q_2 - 1} \gamma_{2,j} \Delta u_{t-j} + \varepsilon_{2t} \quad (A12)$$

These models can be estimated using Ordinary Least Squares (OLS). In addition, we incorporate dummy variables and select the lag lengths p_i and q_i to obtain satisfactory results in terms of the estimated residuals. To carry out these estimations for the US economy over the period from 1965 to 2019, we utilized the R package ARDL developed by Natsiopoulos and Tzeremes (2022). The results are presented in Tables A4 and A5, indicating that nearly all the regressors included in the two models exhibit statistical significance at the 95% confidence level. Concerning residuals diagnostics, Table A6 suggests that, at a 95% confidence level, we do not find evidence to reject the absence of serial correlation (Breusch-Godfrey test), homoskedasticity (Breusch-Pagan test), the absence of ARCH effects, conformity with normality (Jarque Bera test), no functional form misspecification (RESET test), and model stability (CUSUM OLS and recursive residuals).

Table A4. Estimation of Equation (A11)

Regressor	Coefficient	Standard Error	t-value	p-value
Intercept	-0.193238	0.098968	-1.953	0.056485
$\Delta \ln w_{t-1}$	-0.721699	0.128696	-5.608	8.85E-07
v_{t-1}	0.221201	0.104548	2.116	0.039365
Δv_t	0.570543	0.154254	3.699	0.000539
Δv_{t-1}	-0.411427	0.149772	-2.747	0.008339
D_{1970}	-0.010049	0.004569	-2.2	0.032482
D_{1980}	0.007243	0.003516	2.06	0.044628
D_{2000}	-0.004972	0.002817	-1.765	0.083624

Note: Each dummy variable D_τ assumes a value of 1 for years $t \geq \tau$.

Table A5. Estimation of Equation (A12)

Regressor	Coefficient	Standard Error	t-value	p-value
Intercept	-0.179973	0.065248	-2.758	0.00791
$\Delta \ln m_{t-1}$	-0.671803	0.118572	-5.666	5.82E-07
u_t	0.28286	0.099764	2.835	0.00643
D_{2000}	0.015157	0.004845	3.128	0.00283

Note: The dummy variable D_{2000} assumes a value of 1 for years $t \geq 2000$.

Table A6. Tests for Residuals

Tests		**Residuals of equation (A11)** ($\Delta \ln w_t$)		**Residuals of equation (A12)** ($\Delta \ln m_t$)	
		Statistic	p-value	Statistic	p-value
Breusch-Godfrey	1 lag	0.91394	0.3391	0.38171	0.5367
	2 lags	1.3114	0.5191	0.53716	0.7645
	3 lags	2.8452	0.4161	0.78497	0.8531
	4 lags	5.1545	0.2718	1.1656	0.8837
	5 lags	7.1279	0.2113	1.6035	0.9008
Breusch-Pagan		7.702	0.3596	2.8936	0.4083
ARCH LM		6.189	0.9063	2.9291	0.996
Jarque Bera		4.2707	0.1182	3.5303	0.1712
RESET		0.01268	0.9874	0.94783	0.3942
CUSUM OLS residuals		0.44539	0.9888	1.0361	0.2333
CUSUM recursive residuals		0.55577	0.503	0.64118	0.339

After checking the estimated residuals, we conduct an F test to examine the null hypothesis of no long-run relationship ($H_0: b_{i,1} = b_{i,2} = 0$), as well as a t test to evaluate the null hypothesis of the existence of a degenerate case ($H_0: b_{i,1} = 0$), as required by the bounds-testing procedure proposed by Pesaran et al. (2001). The results of these tests are presented in Table A7, providing evidence to reject the null hypotheses at the 99% confidence level. In other words, we can assert that equations (A11) and (A12) effectively establish the presence of long-run relationships among the considered variables. Given this outcome, we proceeded to estimate the long-run coefficients $\hat{\gamma}$, $\hat{\rho}$, $\hat{\psi}_0$, and $\hat{\psi}_1$ as indicated in Table A8. Notable, all these long-run coefficients are statistically significant at the 95% confidence level. Consequently, we consider these estimations to complete the parameter values necessary for simulating the extended Goodwin model.

Table A7. Long-Run Relationship Tests (statistics and p-value)

	Equation (A11) ($\Delta \ln w_t$)	**Equation (A12)** ($\Delta \ln m_t$)
F test	15.72732 (<0.01)	16.98549 (<0.01)
t test	-5.607783 (<0.01)	-5.665764 (<0.01)

Note: Tests obtained for the case of unrestricted intercept and no trend.

Table A8. Long-run Estimates for Equations (A9) and (A10)

	Dependent variable	Coefficient	Estimate	t-value	p-value
Equation (A1)	$\Delta \ln w_t$	$-\hat{\gamma}$	-0.2677537	-2.071292	0.04351168
		$\hat{\rho}$	0.3065009	2.27039	0.02752601
Equation (A2)	$\Delta \ln m_t$	$-\hat{\psi}_0$	-0.2678959	-2.771119	0.007646434
		$\hat{\psi}_1$	0.4210462	2.860701	0.005999657

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