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The foundation of economic growth theory

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Capital and Labor Augmented Technical Progress, Growth Model, Maximum Profit Condition, Production Functions, General Technological Progress, Capital-Labor-mix, Elasticity of Substitution, Normalized CES Functions, Total Factor Productivity, DSGE Model, ***Solow Model, Hicks, Harrod, Labor Saving***

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Abstract

In 2018 we adapted the implementation of technical growth to correct the Solow growth model. Within this article, we delve into some of the consequential aspects of this Modern Universal Growth Theory (MUGT) with respect to homogeneous degree 1 CES production functions. In particular, we demonstrate, that the well-known Cobb-Douglas and CES production functions can serve as the first and second order approximation of any arbitrary production function, respectively. Furthermore, contrary to what you can find in literature, we show that technical progress in the MUGT is always labor saving. Also interesting is the point that even a negative elasticity of substitution is allowed.

Keywords: Capital and Labor Augmented Technical Progress, Growth Model, Maximum Profit Condition, Production Functions, General Technological Progress, Capital-Labor-mix, Estimation of the Elasticity of Substitution, DSGE, Total Factor Productivity, Solow model, Hicks, Harrod, Labor Saving

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1 Introduction

The realm of economic theory is characterized by its dynamic nature, continually evolving to address the complex interplay of factors that drive economic growth and development. At the forefront of this evolution stands the Growth Theory, a pivotal framework that attempts to elucidate the mechanisms behind the expansion of economies over time. Rooted in historical context and enriched by the contributions of visionary economists, the Growth Theory has undergone significant transformations, with the works of Robert Solow, Hirofumi Uzawa, John Hicks, and Sir Roy Harrod serving as milestones in its development.

The Growth Theory, as envisaged by Robert Solow in the mid-20th century, emerged during a period of post-World War II recovery and reconstruction. Solow's pathbreaking research laid the foundation for understanding the drivers of economic growth by introducing the concept of technological progress as a central determinant. His seminal model, often referred to as the Solow-Swan model, highlighted the roles of capital accumulation and technological advancements in fostering sustained economic growth. By distinguishing between short-term fluctuations and long-term trends, Solow's work established a framework that would inspire subsequent economists to delve deeper into the intricate dynamics of growth.

Building upon Solow's work, Hirofumi Uzawa ventured into the realm of endogenous growth theory, which sought to explain the sources of technological progress itself. Uzawa's groundbreaking contributions illuminated the role of human capital and education in propelling economies forward. He postulated that investments in education and research could lead to self-sustaining growth, where the pursuit of knowledge fuels innovation and productivity enhancements. Uzawa's insights challenged the conventional wisdom of exogenous technological progress and spurred a new wave of research into the determinants of innovation-driven growth.

In parallel, John Hicks and Sir Roy Harrod enriched the Growth Theory by introducing concepts that delved into the nuances of economic instability and fluctuations. Hicks' theory of capital utilization and its dynamic adjustment in response to changes in demand provided a lens through which economists could understand the cyclical nature of growth. Harrod, on the other hand, delved into the intricacies of economic instability arising from the mismatch between savings and investment. His work highlighted the potential for instability even within a framework of long-term growth, emphasizing the need for policy interventions to mitigate economic fluctuations.

The amalgamation of these visionary contributions not only expanded the scope of the Growth Theory but also paved the way for a more comprehensive understanding of the intricate forces at play in the realm of economic growth. From Solow's fundamental insights into capital accumulation and technological progress to Uzawa's emphasis on human capital and endogenous innovation, and from Hicks' and Harrod's analysis of economic fluctuations to their implications for policy, these economists collectively wove a tapestry of theories that continues to shape modern discussions on economic development.

And it is here where our contribution to this theory starts. To be more specific, in order to describe economic growth in a mathematical way in production functions, factors for labor and capital improvement were introduced to represent technical progress.

In the 1960s, pioneering economists like Solow employed factors for capital and labor alongside Cobb-Douglas and CES functions to model economic growth. Despite wide adoption, a lingering sense of incongruence persisted. Even in those days there was a lot of discussion, realizing that something wasn't right. Despite the inconsistency up until now this theory is common knowledge and to be found in all textbooks about Modern Growth Theory. In 2018 Marcel R. de la Fonteijne solved this inconsistency by showing how the factor of technical progress had to be implemented in a simple two-factor homogeneous degree 1 CES production function to make a balanced growth path (BGP) with constant capital to income ratio possible.

We will use the term Modern Universal Growth Theory to refer to the corrected version of the growth theory originally introduced by Solow, among other, in the ’60’s of the XX century, in which the factors of technical progress have been adapted in order to solve the inconsistency. You can find detailed information about the proof of mentioned inconsistency in De la Fonteijne (2018).

In De la Fonteijne (2018) we showed that in a growing economy with constant factors as capital and labor technical progress as conceptualized by influential economists like Solow, Hicks or Harrod did, **never** will lead to a Balanced Growth Path (BGP) with constant capital to income ratio if we use a CES homogeneous degree 1 production function. Furthermore, these economic factors might not entirely encapsulate the essence of Total Factor Productivity in its purest form, as perceived from our analytical perspective.

Lemma: In the old theory with only constant growth parameters, it is **never** possible to achieve a Balances Growth Path (BGP) with constant capital to income ratio when using a CES production function.

In the Modern Universal Growth Theory technical progress causes changes in a mutual progress term for capital and labor ξ_{TFP}, in a capital-labor-mix term α_{CES}, expressing the balance between capital and labor, and in the elasticity of substitution σ_{CES}.

This report refrains from reiterating the analysis of the "why." Instead, we commence with the application of the corrected homogeneous degree 1 CES production function within the framework of the MUGT.

In Section 2, we elucidate that the Cobb-Douglas and the CES production functions are the first and second order approximation of an arbitrary production function.

Moving to section 3, we show how to determine the parameters of the model.

Contrary to conventional literature, section 4 of this report presents a novel perspective by demonstrating that technical progress in the MUGT framework is always labor saving, avoiding the explanation of the complex difference between augmented capital or labor technical progress needed in the old theory.

In Section 5, we present an argument that elucidates how a production function inherently exhibits CES-like characteristics around its base point. Additionally, we provide a concise and insightful definition of a production function.

Section 6 delves into a particularly intriguing aspect, where we explore the notion that the elasticity of substitution within certain industries can span a spectrum from negative to positive infinity. This observation sheds new light on the economic dynamics within these sectors.

Within Section 7, we advocate for the utilization of multiple sectors, underlining the substantial variations in parameter values across different economic sectors, thereby enhancing the precision of economic modeling.

Concluding our discussion in Section 8, we offer a summary of key insights and remarks drawn from our analysis.

2 First and second order approximation of an arbitrary growth path in its base point with Cobb-Douglas and CES production functions.

In this chapter we will show that any arbitrary growth path Y can be approximated by a CES production function in a small surrounding near its base point.

We start with the basic equation of the Modern Universal Growth Theory (MUGT)

$$Y = Y_0 \xi_{TFP} \left[\alpha_1 \left(\frac{K}{K_0}\right)^{\gamma} + (1-\alpha_1)\left(\frac{L}{L_0}\right)^{\gamma}\right]^{1/\gamma} \tag{1}$$

with

$$\alpha_1 = \alpha_2 \left(\frac{1}{\xi_{TFP}}\right)^{\gamma} \tag{2}$$

where α_2 is the initial capital-labor-mix α_0 in the base point including an extra change due to the technical progress process.

We calculate the first order derivatives

$$\frac{\partial Y}{\partial \xi_{TFP}} = \frac{Y}{\xi_{TFP}} + \frac{Y}{\alpha_1\left(\frac{K}{K_0}\right)^{\gamma} + (1-\alpha_1)\left(\frac{L}{L_0}\right)^{\gamma}} \alpha_0 \left(-\xi_{TFP}{}^{-\gamma-1}\right) \left(\left(\frac{K}{K_0}\right)^{\gamma} - \left(\frac{L}{L_0}\right)^{\gamma}\right) \tag{3}$$

$$\frac{\partial Y}{\partial \alpha_2} = \frac{Y}{\alpha_1\left(\frac{K}{K_0}\right)^{\gamma} + (1-\alpha_1)\left(\frac{L}{L_0}\right)^{\gamma}} \xi_{TFP}{}^{-\gamma} \left(\left(\frac{K}{K_0}\right)^{\gamma} - \left(\frac{L}{L_0}\right)^{\gamma}\right) \tag{4}$$

$$\frac{\partial Y}{\partial K} = \frac{\alpha_0 Y}{\alpha_1\left(\frac{K}{K_0}\right)^{\gamma} + (1-\alpha_1)\left(\frac{L}{L_0}\right)^{\gamma}} \left(\frac{K}{K_0}\right)^{\gamma-1} \frac{1}{K_0} \tag{5}$$

$$\frac{\partial Y}{\partial L} = \frac{(1-\alpha_0) Y}{\alpha_1\left(\frac{K}{K_0}\right)^{\gamma} + (1-\alpha_1)\left(\frac{L}{L_0}\right)^{\gamma}} \left(\frac{L}{L_0}\right)^{\gamma-1} \frac{1}{L_0} \tag{6}$$

To calculate $\frac{\partial Y}{\partial \sigma}$ we write equation 1 as

$$Y = Y_0 \left[\alpha_2 \left(\frac{K}{K_0}\right)^{\gamma} + ({\xi_{TFP}}^{\gamma} - \alpha_2) \left(\frac{L}{L_0}\right)^{\gamma}\right]^{1/\gamma} \qquad (7)$$

$$\frac{\partial Y}{\partial \sigma} = \frac{\partial Y}{\partial \gamma}\frac{\partial \gamma}{\partial \sigma} = Y \log \left[\alpha_2 \left(\frac{K}{K_0}\right)^{\gamma} + ({\xi_{TFP}}^{\gamma} - \alpha_2) \left(\frac{L}{L_0}\right)^{\gamma}\right] \frac{\partial Y}{\partial \gamma}\frac{\partial \gamma}{\partial \sigma} +$$

$$+ \frac{1}{\gamma} \frac{Y}{\alpha_2\left(\frac{K}{K_0}\right)^{\gamma} + ({\xi_{TFP}}^{\gamma} - \alpha_2)\left(\frac{L}{L_0}\right)^{\gamma}} \left[\alpha_2 \left(\frac{K}{K_0}\right)^{\gamma} \log\left(\frac{K}{K_0}\right) + {\xi_{TFP}}^{\gamma} \left(\frac{L}{L_0}\right)^{\gamma} \log(\xi_{TFP}) +\right.$$

$$\left. + ({\xi_{TFP}}^{\gamma} - \alpha_2) \left(\frac{L}{L_0}\right)^{\gamma} \log\left(\frac{L}{L_0}\right)\right] \frac{\partial \gamma}{\partial \sigma} \qquad (8)$$

which results in the first order derivatives in the base point

$$\left(\frac{\partial Y}{\partial \xi_{TFP}}\right)_0 = \frac{Y_0}{(\xi_{TFP})_0} = Y_0 \qquad (9)$$

$$\left(\frac{\partial Y}{\partial \alpha_2}\right)_0 = 0 \qquad (10)$$

$$\left(\frac{\partial Y}{\partial K}\right)_0 = \frac{\alpha_0 Y_0}{K_0} \qquad (11)$$

$$\left(\frac{\partial Y}{\partial L}\right)_0 = \frac{(1-\alpha_0) Y_0}{L_0} \qquad (12)$$

$$\left(\frac{\partial Y}{\partial \sigma}\right)_0 = 0 \qquad (13)$$

Notice that by using a homogeneous degree 1 CES production function you introduce a relationship between $\frac{\partial Y}{\partial K}$ and $\frac{\partial Y}{\partial L}$.

In total by first order approximation income Y is

$$Y = Y_0 + \left(\frac{\partial Y}{\partial \xi_{TFP}}\right)_0 (\xi_{TFP} - (\xi_{TFP})_0) + \left(\frac{\partial Y}{\partial \alpha_2}\right)_0 (\alpha_2 - (\alpha_2)_0) + \left(\frac{\partial Y}{\partial K}\right)_0 (K - K_0) + \left(\frac{\partial Y}{\partial L}\right)_0 (L - L_0) +$$

$$+ \left(\frac{\partial Y}{\partial \sigma}\right)_0 (\sigma_0 - (\sigma_0)_0) =$$

$$= Y_0 \xi_{TFP} + \frac{\alpha_0 Y_0}{K_0}(K - K_0) + \frac{(1-\alpha_0) Y_0}{L_0}(L - L_0) \qquad (14)$$

As illustrated by equation 14, it becomes apparent that this equation lacks the necessary degrees of freedom to generate an arbitrary GDP growth trajectory along with its precise derivatives within a narrow proximity of the base point. This limitation stems from the inherent dependency on α_{CES} in equation 11 and 12, which is a direct consequence of our decision to employ a homogeneous degree 1 CES production function.

We can introduce an extra parameter ν in the model, as is common in literature, although in that case the model is no longer homogeneous degree 1.

The production function then is

$$Y = Y_0 \xi_{TFP} \left[\alpha_1 \left(\frac{K}{K_0}\right)^\gamma + (1-\alpha_1)\left(\frac{L}{L_0}\right)^\gamma \right]^{\nu/\gamma} \tag{15}$$

To calculate the derivative with respect to ν we write the production function as

$$\log(Y) = \log(Y_0) + \log(\xi_{TFP}) + {}^{\nu}/_{\gamma} \log\left(\alpha_1 \left(\frac{K}{K_0}\right)^\gamma + (1-\alpha_1)\left(\frac{L}{L_0}\right)^\gamma \right) \tag{16}$$

$$\frac{\partial Y}{\partial \nu} = {}^{1}/_{\gamma} \log\left(\alpha_1 \left(\frac{K}{K_0}\right)^\gamma + (1-\alpha_1)\left(\frac{L}{L_0}\right)^\gamma \right) \tag{17}$$

$$\left(\frac{\partial Y}{\partial \nu}\right) \nu_0 = 0 \tag{18}$$

Formula 14 649 changes to

$$Y = Y_0 \xi_{TFP} + \frac{\nu \alpha_0 Y_0}{K_0}(K - K_0) + \frac{\nu(1-\alpha_0)Y_0}{L_0}(L - L_0) \tag{19}$$

ξ_{TFP}, ν and α_0 generate enough freedom to make any growth path possible in a small surrounding round the base point. Of course, the model is changed by using ν and because we don't have enough data points, we have to know ν as a priori information in the same way as holds for the elasticity of substitution σ_{CES}.

In this report, we continue using $\nu = 1$, which will result in adapted values for ξ_{TFP} and α_2. In this case we are also able to approximate any arbitrary growth path. Keep in mind that by doing so this is a simplification of the model, in order to escape from the fact that we have not enough data point to independently determine the derivatives to labor L and capital K.

Lemma: In the Modern Universal Growth Theory technical progress causes changes in a mutual progress term for capital and labor ξ_{TFP}, in a capital-labor-mix term α_2 expressing the balance between capital and labor and in the elasticity of substitution σ_{CES}.

If an arbitrary production function f (intensive form) with $\xi_{TFP} = 1$ is known, then we can calculate the parameters of a CES function to approximate f in the base point.

As is well-known for homogeneous degree 1 production function we can write α_{CES} and σ_{CES} in term of the derivatives of the arbitrary production function f in the base point.

$$\alpha_{CES} = f' \beta_0 \tag{20}$$

$$\sigma_{CES} = -\frac{f'(f - kf')}{kff''} = \frac{f'\left(\frac{1}{k_0} - \frac{1}{y_0} f'\right)}{f''} \tag{21}$$

In total this results in a second order approximation of any arbitrary production function in its base point.

For the first order approximation we only deal with the first derivative f' and σ_{CES} is no longer in the equation. This means that we can choose σ_{CES} as we please, e.g., $\sigma_{CES} = 1$, with which equation 1 will convert into a Cobb-Douglas production function.

$$y = y_0 {\xi_g}^{1-\alpha_{CES}} \left(\frac{k}{k_0}\right)^{\alpha_{CES}} = y_0 \xi_{TFP} \left(\frac{k}{k_0}\right)^{\alpha_{CES}} \quad (22)$$

$$\alpha_{CES} = f' \beta_0 \quad (23)$$

In other words, any arbitrary production function can be approximated by a Cobb-Douglas function in its base point. Of course, in case of a second order approximation we still cannot estimate the parameter σ_{CES} out of real data.

For more generality we introduce the term General Balanced Growth Path (GBGP), which is a growth path on which the maximum profit principle is maintained and also c_1 may vary in time.

Definition: A general balanced growth path (GBGP) is a growth path on which the maximum profit principle is maintained and e.g., c_1 may vary in time.

The formulas, however, still are valid even on a General Balanced Growth Path (GBGP). The parameter α_{CES} expresses capital share and will not change as a result of changing capital use by using a Cobb-Douglas function. Any discrepancy with measured data will be attributed to an exogenous change due to the technical growth process.

Realize that equation 23 can be used by assuming that producers are acting in accordance with the maximum profit principle.

In case of the first order approximation, using equation 22 and 23 the parameters can be calculated from real data. It is also possible to use the extensive form, even with unemployment. Realize that it is a pointwise approximation and results in time dependent functions $\xi_{TFP} = \xi_{TFP}(t)$ and $\alpha_{CES} = \alpha_{CES}(t)$. We even can calculate how much of total growth can be attributed to exogenous TFP growth due to the technical growth process and how much can be attributed to endogenous growth due to increasing capital. It also makes clear that this first order approximation is exactly what is done in traditional growth accounting, although it seems reasonable to work with ξ_{TFP} and not with ξ_g to estimate the exogenous determinants of TFP.

This approach also explains why the use of a Cobb-Douglas production function is quite acceptable to analyze the commonly used set on American data. It simply is its first order approximation.

Altogether, this means, that we can approximate any arbitrary real growth data by a Cobb-Douglas production function, where the parameters $\xi_{TFP} = \xi_{TFP}(t)$ and $\alpha_{CES} = \alpha_{CES}(t)$ are functions of time.

Lemma: Utilizing the framework of the Modern Universal Growth Theory (MUGT) the following holds true:

Any arbitrary real data of growing per capita income can be approximated by a Cobb-Douglas function, for which the parameters $\xi_{TFP} = \xi_{TFP}(t)$ and $\alpha_{CES} = \alpha_{CES}(t)$ are functions of time and can be determined. The Cobb-Douglas production function serves as a first order approximation and is more accurate if the real coefficient of substitution σ_{CES} is closer to one. Without losing generality you can use the formula in intensive form with $\sigma_{CES} = 1$, resulting in a Cobb-Douglas production function

$$y = y_0 \xi_{TFP} \left(\frac{k}{k_0}\right)^{\alpha_{CES}}$$

Keep in mind that by using a homogeneous degree one production function you introduce a relationship between $\frac{\partial Y}{\partial K}$ and $\frac{\partial Y}{\partial L}$.

If we use the second order approximation using a CES function then the approximation will be more accurate. However, as we discussed before, we need to have a priori information to determine σ_{CES} direct or indirect. If the elasticity of substitution σ_{CES} is not constant in time we will use the term variable elasticity of substitution production function (VES).

$$y = y_0 \xi_{TFP} \left[\alpha_1 \left(\frac{k}{k_0}\right)^{\gamma} + (1 - \alpha_1)\right]^{1/\gamma} \tag{24}$$

Lemma: Utilizing the framework of the Modern Universal Growth Theory (MUGT) the following holds true:

Any arbitrary real growth per capita data can be approximated by a VES function, for which the parameters $\xi_{TFP} = \xi_{TFP}(t)$, $\alpha_{CES} = \alpha_{CES}(t)$ and $\sigma_{CES} = \sigma_{CES}(t)$ are functions of time. We need a priori information to determine σ_{CES}. The VES function serves as a second order approximation and is more accurate if the used elasticity of substitution σ_{CES} is closer to the real one. Without losing generality you can use the formula in intensive

$$y = y_0 \xi_{TFP} \left[\alpha_1 \left(\frac{k}{k_0}\right)^{\gamma} + (1 - \alpha_1)\right]^{1/\gamma}$$

Notice that y is describing labor productivity and k capital deepening.

In fact, also the knowledge about α_{CES} is based on information over a time interval, because we assume the producer are capable of maintaining the maximum profit principle, causing the capital to income share ks to be equal to α_{CES} in a considered base point.

3 Determination of the parameters from measured data using the intensive form and the definition of a production function

From 2 data points on a general balanced growth path (GBGP) we can determine total factor productivity (TFP) by using a first order approximation if we don't know the elasticity of substitution σ. As explained, we can choose $\sigma = 1$ and use the Cobb-Douglas production function with constant α_{CES}. Alternatively, we use a second order approximation or CES production function if the elasticity of substitution σ is known or can be estimated by using some kind of a priori information.

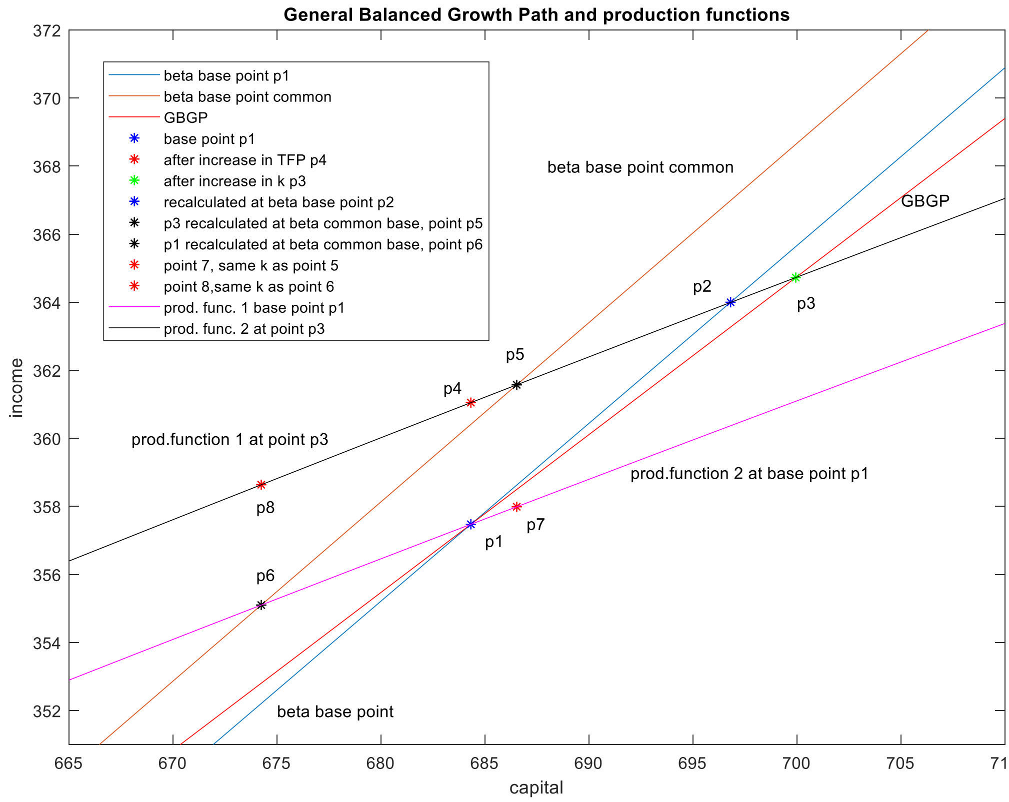

Fig. 1 General Balanced Growth Path with data point p_1 and p_3. The point p_1 and p_3 are real points, the others points are virtual point on a virtual GBGP and are used to split the growth process into a part with influence of technical process only (e.g., p_1 to p_4) and a part with growth of capital only (e.g., p_4 to p_3), to ease the understanding of the growth process. The production function is the collection of points what might have been the end result of this one step growth process if only capital would have changed differently. All points of a production function are referring to the same point t in time, so on a production function the values of ξ_{TFP}, α and γ are not changing due to technical progress. Notice that the blue line is a balanced growth path (BGP) as well.

We start with 2 data points (p_1 and p_3, see fig. 1) located on a GBGP:

$$y_i, k_i, ks_i, \alpha_i, \sigma_i, \beta_i, \gamma_i \quad \text{is representing point } p_i \tag{25}$$

We choose y_1, k_1 as our base point. Because we are on GBGP $\alpha_1 = ks_1$. This characterizes production function 2. We will describe production function 1 in terms of base point p_2. Because $\beta_2 = \beta_1$, a change in α is due to the technical progress process only. Moving along a production function will change ks_3 into ks_2 and because we are on GBGP, in point p_2 we describe production function 1 with $\alpha_2 = ks_2$.

$$\alpha_2 = ks_2 = ks_3 \left(\frac{\beta_2}{\beta_3}\right)^{\gamma_2} \tag{26}$$

The resulting change in α due to the technical progress process is

$$\Delta\alpha_{1-4} = \alpha_4 - \alpha_1 = (\alpha_2 - \alpha_1)\left(\frac{1}{\xi_{TFP}}\right)^{\gamma_2} \tag{27}$$

The reason why we can use the properties of a production function is that we assume the production function the collection of points what might have been the end result of this one step growth process if only capital would have changed differently. All points of a production function are referring to a point t in time, so on a production function the values of ξ_{TFP}, α and γ are not changing due to technical progress.

Lemma: Within the framework of the Modern Universal Growth Theory (MUGT), we describe a homogeneous degree 1 CES production function in its base point k_0, y_0, α_0 in its intensive form as follows:

$$y = y_0 \left[\alpha_0 \left(\frac{k}{k_0}\right)^{\gamma} + (1 - \alpha_0)\right]^{1/\gamma}$$

Notice that y is describing labor productivity and k capital deepening.

The production function is the collection of points what **might** have been the end result of this one step growth progress if only capital **would** have changed differently. All points of the production function are referring to same point t in time.

Finally, we calculate ξ_{TFP} from the inverse equation of equation 15

$$\xi_{TFP} = \left[\left(\frac{y_3}{y_1}\right)^{\gamma_2} - \alpha_2 \left(\frac{k_3}{k_1}\right)^{\gamma_2} + \alpha_2\right]^{1/\gamma_2} \tag{28}$$

If we look in fig. 1 at the triangle p_1, p_4, p_3 notice that p_1 to p_3 is the real process (GBGP) and point p_4 is a virtual point just to help understand the process. On p_1 to p_4 there are only changes due to technical progress and on p_4 to p_3 there is only change due to a change in capital k. As a result, we also can calculate ξ_{TFP} as

$$\xi_{TFP} = \frac{y_4}{y_1} \tag{29}$$

where $k_4 = k_1$.

We can calculate ξ_g from (De la Fonteijne, 2018)

$$\xi_g = \frac{y_2}{y_1} \tag{30}$$

where the following equation still holds

$$\xi_g = \left(\frac{{\xi_{TFP}}^{\gamma_2} - \alpha_2}{1-\alpha_2}\right)^{1/\gamma_2} \tag{31}$$

If we execute this process for each adjacent pair of data points, we are able to calculate the change of α and ξ_{TFP} along the growth path.

Alternatively, we can calculate with respect to another path, e.g., with respect to a line with capital to income ratio β_c, in which case all the differences will be calculated with respect to a virtual growth path with capital to income ratio equal to a common β_c. Be aware that α and ξ_{TFP} will change, and it won't ease the economic interpretation.

We prefer the first method, because along the growth path (GBGP) you always refer to the real economic process and situation (data point).

As an example, we use a Cobb-Douglas approximation to calculate ξ_{TFP} from a known general balanced growth path (GBGP). Putting the elasticity of substitution to $\sigma = 1$, so $\gamma = 0$, which is the Cobb-Douglas case, will change equation 28 into

$$\xi_{TFP} = \frac{\frac{y_3}{{k_3}^{\alpha_2}}}{\frac{y_1}{{k_1}^{\alpha_2}}} \tag{32}$$

with indices as in fig. 1.

Then the model is (equation 22)

$$y = y_1 {\xi_g}^{1-\alpha_{CES}} \left(\frac{k}{k_1}\right)^{\alpha_2} = y_1 \xi_{TFP} \left(\frac{k}{k_1}\right)^{\alpha_2} \tag{33}$$

So far, we assumed that a change in α due to technical progress takes place simultaneously with a change in ξ_{TFP}. We can assume otherwise, but as we do not have information between point 1 and point 3 on GBGP, any more sophisticated process is assuming a priori information. Also notice that this first order approximation looks very similar to what is done in literature so far, i.e., dividing by capital and labor to eliminate their influence. Sometimes α is assumed not to change over the time interval considered, i.e., α is constant for all data points.

4 Capital and Labor Technical Progress is always Labor Saving.

Technical progress in the Modern Universal Growth Theory (MUGT) is determined by the factor ξ_{TFP}, the capital-labor-mix α_{CES} and σ_{CES}. Improvements in capital and/or labor, are labor saving, because an improvement will increment income per capita with the factor ξ_{TFP}. In other words, with less labor you can generate the same output as before.

If the factor time is involved then total growth is not only depending on the technical progress parameters ξ_{TFP}, the capital-labor-mix α_{CES} and σ_{CES} but also on the rest of the economic system, in our case the ratio of consumption to income ratio c_1, introducing a dynamic effect and change in capital use.

> Lemma: Utilizing the framework of the Modern Universal Growth Theory (MUGT) the following holds true:
>
> Technical progress in the Modern Universal Growth Theory (MUGT) is represented by the factor ξ_{TFP}, the change in the capital-labor-mix α_{CES} and the change in the elasticity of substitution σ_{CES}.

> Lemma: Utilizing the framework of the Modern Universal Growth Theory (MUGT) the following holds true:
>
> Technical progress in the Modern Universal Growth Theory (MUGT) is determined by the factor ξ_{TFP}, the capital-labor-mix α_{CES} and σ_{CES}. Improvements in capital and/or labor, are labor saving, because an improvement will increment income per capita with the factor ξ_{TFP}. In other words, with less labor you can generate the same output as before.

5 The character of the production function is always CES-like.

We remind you of the fact that a homogeneous degree 1 CES production function is a first or second order approximation in its base point and shows all possible alternative outcome that might have occurred if investment of capital would have been different.

> Lemma: Utilizing the framework of the Modern Universal Growth Theory (MUGT) the following holds true:
>
> Any arbitrary per capita production function can be approximated by a homogeneous degree 1 CES function in its base point (y_0, k_0), for which the parameters α_{CES} and σ_{CES} are constant. We need a priori information to determine σ_{CES}. This CES function serves as a second order approximation and is more accurate if the used elasticity of substitution σ_{CES} is closer to the real one and it shows all possible alternative outcome that might have occurred if investment of capital would have been different.
>
> $$y = y_0 \left[\alpha_{CES} \left(\frac{k}{k_0}\right)^{\gamma} + (1 - \alpha_{CES})\right]^{1/\gamma}$$
>
> Notice that y is describing labor productivity and k capital deepening.

Certain articles in the literature suggest that, in the short run, a production function exhibits characteristics resembling CES, while in the long run, it tends to converge towards a Cobb-Douglas form. From practical and theoretical point of view there is no evidence for this phenomenon. Interesting to see that there is no need to assume such a thing in the MUGT. Even if there is continuous growth along a balanced growth path (BGP) in equilibrium, i.e., the capital to income ratio is constant, the character of the production function is always CES in general and might be Cobb-Douglas but not necessarily, in which case you cannot even notice the influence of the elasticity of substitution σ_{CES} with respect to α_{CES}.

> Lemma: Utilizing the framework of the Modern Universal Growth Theory (MUGT) the following holds true:
>
> Any arbitrary per capita production function can be approximated by a CES function in its base point (y_0, k_0), for which the parameters α_{CES} and σ_{CES} are constant. The character of the production is **always** CES in general and **might** be Cobb-Douglas but not necessarily.
>
> $$y = y_0 \left[\alpha_{CES}\left(\frac{k}{k_0}\right)^{\gamma} + (1 - \alpha_{CES})\right]^{1/\gamma}$$
>
> Notice that y is describing labor productivity and k capital deepening.

6 The elasticity of substitution can vary between minus and plus infinity.

Any arbitrary per capita production function can be approximated by a CES function in its base point (y_0, k_0), for which the parameters α_{CES} and σ_{CES} are constant. If maximum profit conditions are applied then in the base point $\alpha_{CES} = ks$ and for economic reasons is restricted to

$0 \leq \alpha_{CES} \leq 1$

The parameter σ_{CES}, contrary to what is common in literature, is not restricted

$-\infty < \sigma_{CES} < \infty$

Whether we have stable and unique solutions depend not only on the parameters of the production function but also on the rest of the economic system. Under maximum profit conditions with fixed c_1 solutions are always stable and unique. However, optimizing profit with respect to c_1 will not always result in stable and unique solution. It also depends on the character of c_1.

Lemma: Utilizing the framework of the Modern Universal Growth Theory (MUGT) the following holds true:

Any arbitrary per capita production function can be approximated by a CES function in its base point (y_0, k_0), for which the parameters α_{CES} and σ_{CES} are constant. If maximum profit conditions are applied then in the base point $\alpha_{CES} = ks$ and for economic reasons restricted to

$0 \leq \alpha_{CES} \leq 1$

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Whether we have stable and unique solutions depend not only on the parameters of the production function but also on the rest of the economic system. Under maximum profit conditions with constant c_1, solutions are always stable and unique. However, optimizing profit with respect to c_1, in general, will not always result in stable and unique solution, because it depends on the nature of c_1.

Notice that y is describing labor productivity and k capital deepening.

7 The importance of using different production functions for different sectors.

Differences in the parameters describing the production between sectors can be extremely large. Additionally, the influence of technical progress process can be different across sectors. It seems to be a good idea to consider total income as the superposition of all those sectors in order to examine the influence of each individual sector.

Lemma: Utilizing the framework of the Modern Universal Growth Theory (MUGT) the following holds true:

Due to the big differences in parameter value over the several production sectors is seems necessary to calculate income as the sum of the several sectors. It is important that the capital-labor-mix α_{CES} and the elasticity of substitution σ_{CES} are not restricted and can be adjusted to match reality, the latter even negative, contrary to what is common in literature.

8 Some final remarks on growth regarding homogeneous degree 1 CES productions functions

In the MUGT framework we employ the technical growth parameters ξ_{TFP}, the capital-labor-mix α_{CES} and the elasticity of substitution σ_{CES}. When dealing with growth dynamics, it becomes imperative to adjust the capital-labor-mix α with the factor $\left(\frac{1}{\xi_{TFP}}\right)^{\gamma}$ in $\alpha_1 = \alpha_0 \left(\frac{1}{\xi_{TFP}}\right)^{\gamma}$ to make a Balanced Growth Path (BGP) with constant capital to income ratio possible. The decision to opt for a final solution that maintains a constant capital-to-income ratio might seem somewhat arbitrary, and in a certain way it is. On the other hand, it is the only viable choice possible if you require a BGP and a constant capital to income ratio.

If you like to divert from that idea then there are a lot of possibilities to end up with a feasible economy, albeit that the capital to income ratio is changing to a new value. E.g., use the term $\alpha_1 = \alpha_2 \left(\frac{1}{\xi_{TFP}}\right)^{\gamma}$ to express total change due to technical progress, where the difference between α_2 and α_0 is varying around zero, mirroring real-world scenarios. Moreover, should it be feasible to adjust the elasticity of substitution from above 1 to below 1 and vice versa, you can essentially attain a fluctuating capital-labor mix α_{CES}. Alternatively, if you possess predictive (a priory information) or influential capabilities regarding changes in the capital-to-labor-mix or the elasticity of substitution, you can leverage these insights to shape the economic growth trajectory accordingly.

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Literature

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