The Market-Based Statistics of "Actual" Returns of Investors

Olkhov, Victor

Independent

2 April 2023

The Market-Based Statistics of "Actual" Returns of Investors

Victor Olkhov

Independent, Moscow, Russia

victor.olkhov@gmail.com

ORCID: 0000-0003-0944-5113

ABSTRACT

We describe three successive approximations of market-based statistical moments of "actual" return that investors gain within their market trade sales. We derive how the market-based statistical moments of "actual" return depend on the statistical moments of the sale values, the purchase values, and the trade volumes of stocks. That differs from the conventional assessments of returns' statistical moments based on the frequency analysis of the time series of investors' returns. We start with the statistical moments of return that investors gain from a single trade sale after making multiple purchases in the past. Then we describe the statistical moments of return that an investor gains from numerous trade sales during the "trading day." Finally, we introduce the trading day portfolio composed of stocks, which were sold by all investors during the "trading day". We derive the statistical moments of return of the trading day portfolio and the statistical moments of average returns of different investors. That describes the distribution of the "actual" profitability of market trade over all "selling" investors during the "trading day" and can serve as a benchmark for "purchasing" investors.

Keywords : asset pricing, stock returns, volatility, correlations, probability, market trades

JEL: C1, E4, F3, G1, G12

This research received no support, specific grants, or financial assistance from funding agencies in the public, commercial, or nonprofit sectors. We welcome valuable offers of grants, support, and positions.

1. Introduction

The literature describes the two kinds of stock returns. For convenience, we note them as "anticipated" and "actual" returns. The most studied "anticipated" return $r(t,\tau)$ is determined by the ratio $r(t,\tau)=p(t)/p(t-\tau)$ of the stock price $p(t)$ traded "today" at time t and the price $p(t-\tau)$ traded at $t-\tau$ in the past. The "anticipated" stock returns $r(t,\tau)$ describe the expected, anticipated gains or losses that investors could get if they bought stocks at time $t-\tau$ in the past and then sold them at time t "today." Modeling and predictions of the "anticipated" stock return at horizon T define the core issues of financial economics (Fisher and Lorie, 1964; Mandelbrot, Fisher, and Calvet, 1997; Campbell, 1985; Brown, 1989; Fama, 1990; Fama and French, 1992; Lettau and Ludvigson, 2003; Greenwood and Shleifer, 2013; van Binsbergen and Koijen, 2015; Martin and Wagner, 2019). The description of probabilistic properties of the "anticipated" stock return during any specific averaging time interval Δ "today" or, as we note, a "trading day", and predictions of the probability of return at horizon T "next day", deliver the most desired results for investors and traders. The frequency-based analysis of the return time series assesses the probability distributions of the "anticipated" return during a "trading day" (Amaral et al., 2000; Andersen et al., 2001; Knight and Satchell, 2001; Tsay, 2005; Andersen and Benzoni, 2009).

However, the "anticipated" stock return $r(t,\tau)=p(t)/p(t-\tau)$ with the time shift τ describes the gains or losses that may or may not match the real, "actual" return of investors. As "actual", we consider the return that a particular investor gains from selling the stocks at the time t, which he has purchased in 10 minutes, a day, a week, or any time in the past. Obviously, not all stocks that investors sell at time t "today" were purchased at time $t-\tau$ in the past. Some stocks were purchased earlier or later than $t-\tau$ and at a price that differs from $p(t-\tau)$. Thus, investors who sell stocks at time t gain returns that are different from "anticipated" returns, $r(t,\tau)=p(t)/p(t-\tau)$. So, "anticipated" returns $r(t,\tau)$ describe an option investors may gain, and "actual" returns describe the benefits investors realize. During a "trading day", traders and investors sell stocks that were initially purchased at different times in the past. Investors sell stocks during the "trading day" and gain returns on stocks they purchased in 10 minutes, a week, or any time in the past. To assess the average returns, or statistical moments of "actual" returns, that investors gain, one should take into account returns with different time shifts. That differs from the description of the statistical properties of "anticipated" and "actual" returns. Investigation of the "actual" returns of institutional, professional, or individual investors forms a separate problem. Different aspects of "actual" returns were studied by

(Schlarbaum, Lewellen and Lease, 1978; Stanley, Lewellen and Schlarbaum, 1980; Baker and Wurgler, 2004; Ivković, Sialm and Weisbenner, 2004; Gabaix, et al 2005; Daniel and Hirshleifer, 2016; Koijen, Richmond and Yogo, 2020; Hardouvelis, Karalas and Vayanos, 2021) and others.

However, the statistical properties of "anticipated" and "actual" returns are mostly studied in the same way. To assess the probability $P(r)$ of "anticipated" or "actual" returns during the time interval Δ, one studies the time series of returns and assesses the frequency m_r/N:

$$P(r) \sim \frac{m_r}{N} \tag{1.1}$$

In (1.1), m_r denotes the number of returns that equal r, and N is the total number of terms of the time series during Δ. That is the regular assessment of the probability of any event, and analyzing its time series during Δ follows the firm ground of probability theory (Shephard, 1991; Shiryaev, 1999; Shreve, 2004). For convenience, we further refer to such assessments (1.1) as the frequency-based probabilities of stock returns. It is regular and completely correct to assess the probability of return if the time series of return $r(t_i,\tau)$ during the averaging interval Δ are considered independent, self-reliant variables. However, return at time t_i is determined by stock price $p(t_i)$ at time t_i and price $p(t_i\text{-}\tau)$ at time $t_i\text{-}\tau$. Moreover, stock prices $p(t_i)$ themselves are determined by market trade values $C(t_i)$ and volumes $U(t_i)$ at time t_i:

$$C(t_i) = p(t_i)U(t_i) \tag{1.2}$$

For convenience, in this paper, all prices are adjusted to the present time. We consider market trade values, volumes, market prices, and returns as random variables during the selected time-averaging interval Δ. We believe that the consideration of market prices and stock returns as financial and economic matters should take into account the impact of the size of the trade values $C(t_i)$ and volumes $U(t_i)$ (1.2) on the average price, return, volatility, and statistics of returns. The well-known Volume Weighted Average Price (VWAP) (Berkowitz et al., 1988; Duffie and Dworczak, 2018), which differs from the frequency-based assessment of the mean price, demonstrates the impact of the size of trade volumes $U(t_i)$ on the average price. It is reasonable that the statistical properties of the market trade values $C(t_i)$ and volumes $U(t_i)$ should determine the statistics of the market price, and the price statistics should determine the statistics of stock return. The stochasticity of market trade should determine the probabilities of price and return, and that differs from the frequency-based probabilities of the price and return time series. We note the statistics of stock return determined by a random time series of the market trade values $C(t_i)$ and volumes $U(t_i)$ during the interval Δ as market-based probability of return. A description of the statistical properties of market prices and "anticipated" stock returns determined by the statistical moments of market trade values $C(t_i)$ and volumes $U(t_i)$ is given in Olkhov(2021-2023) and we use the main results to describe the statistical moments of "actual" return.

This pure theoretical paper describes the market-based statistical moments of the "actual" stock returns of investors, which they gain during the averaging interval Δ that we call a "trading day". We call all stocks that are sold by all investors during the "trading day" a trading day portfolio. We derive the dependence of statistical moments of "actual" return of investors on the statistical moments of market trade values $C(t_i)$ and volumes $U(t_i)$. In turn, the statistical moments of market trade values and volumes are assessed by the regular frequency-based (1.1) probability (Shephard, 1991; Shiryaev, 1999; Shreve, 2004). We consider statistical moments of return, which all investors gain from their trade sale of the trading day portfolio, as benchmarks for investors who purchase these stocks during the same "trading day".

In Sec. 2, we briefly present the main relations between the market-based statistical moments of price and the "anticipated" stock return. In Sec. 3, we consider the market-based statistical moments of the "actual" return of a single trade sale. Sec. 4 presents statistical moments of "actual" returns, which an investor gains if he makes a lot of trade sales during the averaging interval Δ, which we call a "trading day". In Sec. 5, we introduce the trading day portfolio as the set of stocks that were sold by all investors during the "trading day". We derive the market-based statistical moments of "actual" return of the trading day portfolio and the statistical moments of returns of different investors. Sec.6 Conclusion. We assume that readers are familiar with the basics of probability theory, statistical moments, etc.

2. The market-based statistical moments of "anticipated" stock return

As "anticipated," we consider the stock return $r(t_i,\tau)=p(t_i)/p(t_i-\tau)$ determined as the ratio of market trade price $p(t_i)$ at time t_i with respect to the price $p(t_i-\tau)$ at time $t_i-\tau$. We consider the trade of identical stocks and adjust all prices to the present. Let us consider the time series of the trade values $C(t_i)$ and volumes $U(t_i)$ at time t_i and assume that the time series determines the initial time axis division with a constant shift ε:

$$t_{i+1} - t_i = \varepsilon \quad ; \qquad \varepsilon - const$$

Market trade time series present irregular and highly variable data. To forecast the stock returns, one should choose the time-averaging interval $\Delta >> \varepsilon$ and estimate the average variables. Average variables are much more useful for the forecasts than the initial irregular market time series. The choice of the interval Δ introduces a new time axis division multiple of Δ at times t_k, $k=0,1,\ldots$. We consider the present time $t=t_0$ as "today," and the time $t_k=t-k\Delta$, $k=1,2,..$ describes $k\Delta$ intervals in the past:

$$\Delta_k = \left[t_k - \frac{\Delta}{2}; t_k + \frac{\Delta}{2}\right] \quad ; \quad t_k = t - \Delta \cdot k \quad ; \quad k = 0, 1, 2, \ldots \tag{2.1}$$

We assume that each interval Δ_k, $k=0,1,..$ contains the same number N of terms t_i of the time series. The present time interval Δ contains N terms of the time series t_i:

$$t_i \in \Delta \quad ; \quad i = 1, 2, .. N \tag{2.2}$$

We consider the trade values $C(t_i)$, volumes $U(t_i)$, and prices $p(t_i)$ during each interval Δ_k (2.1) as the random variables. To describe a random variable, one can equally use the probability density function, the characteristic function, or the set of statistical moments of the random variable (Shephard, 1991; Shiryaev, 1999; Shreve, 2004). We take the set of statistical moments and assess the probabilistic properties of trade values, volumes, price, and return. We present the statistical moments of the trade value $C(t;n)$ and volume $U(t;n)$ averaged during Δ (2.1) "today" using frequency-based probability (1.1) as:

$$C(t;n) \equiv E[C^n(t_i)] \sim \frac{1}{N}\sum_{i=1}^{N} C^n(t_i) \tag{2.3}$$

$$U(t;n) \equiv E[U^n(t_i)] \sim \frac{1}{N}\sum_{i=1}^{N} U^n(t_i) \quad ; \quad n = 1,2,. \tag{2.4}$$

We use $E[\ldots]$ to denote mathematical expectation (2.3; 2.4) during Δ (2.1) and the symbol "~" to underline that the finite number N of trades during Δ determines assessments, the estimators of the statistical moments of market trade value $C(t;n,)$ and volume $U(t;n)$ at time $t=t_0$ "today" averaged during Δ. The sets of statistical moments (2.3; 2.4) for all $n=1,2,..$ describe the trade value $C(t_i)$ and volume $U(t_i)$ as the random variables during Δ "today." Actually, any reasonable averaging interval Δ contains only a finite number of terms of the market trade time series. Thus, relations (2.3; 2.4) can assess only a finite number of the statistical moments of market trade value $C(t;n,)$ and volume $U(t;n)$. The statistical moments (2.3; 2.4) for a finite n assess only approximations of the probability density functions and the characteristic functions of random variables $C(t_i)$ and $U(t_i)$. Thus, a finite number of statistical moments (2.3; 2.4) can describe only approximations of the probability and characteristic functions of price and return as random variables during Δ. We refer to Olkhov (2021; 2022a; 2023) for details.

To describe how the statistical moments (2.4; 2.5) define statistical moments $r(t,\tau;n)=E[r^n(t_i,\tau)]$ of "anticipated" return, we transform (1.2) as:

$$C(t_i) = p(t_i)U(t_i) = \frac{p(t_i)}{p(t_i-\tau)} p(t_i-\tau)U(t_i) = r(t_i,\tau)\, C_p(t_i,\tau)$$

We take the "anticipated" return $r(t_i,\tau)$ with time shift τ as the ratio of prices (2.5):

$$r(t_i, \tau) = \frac{p(t_i)}{p(t_i - \tau)} \tag{2.5}$$

Equations (2.6) link up the sale value $C(t_i)$, return $r(t_i,\tau)$, and "purchased" value $C_p(t_i,\tau)$:

$$C(t_i) = r(t_i, \tau)\, C_p(t_i, \tau) \quad ; \quad C_p(t_i, \tau) = p(t_i - \tau)U(t_i) \tag{2.6}$$

We mention that we consider all prices adjusted to the present time *t*. We introduce the "purchased" value $C_p(t_i,\tau)$ (2.6) of the trade volume $U(t_i)$ at the trade price $p(t_i\text{-}\tau)$ at time $t_i\text{-}\tau$ in the past. In other words, $C_p(t_i,\tau)$ (2.6) equals the purchased value of the volume $U(t_i)$ at the price $p(t_i\text{-}\tau)$ at time $t_i\text{-}\tau$ in the past. Equation (2.6) is similar to the trade equation (1.2), but the price $p(t_i\text{-}\tau)$ at $t_i\text{-}\tau$ defines the "purchased" value $C_p(t_i,\tau)$. Similar to (2.3), we assess the frequency-based statistical moments $C_p(t,\tau;n)$ of value $C_p(t_i,\tau)$:

$$C_p(t, \tau; n) \equiv E\left[C_p^n(t_i, \tau)\right] \sim \frac{1}{N} \sum_{i=1}^{N} C_p^n(t_i, \tau) \tag{2.7}$$

Equations (1.2; 2.6; 2.5) allow us to derive the market-based *n-th* statistical moments of the price $p(t;n)$, purchased market price $p_p(t,\tau;n)$, and the n-th statistical moments of return $r(t,\tau;n)$. For $n=1,2,\ldots$ we take the *n-th* powers (2.8-2.10) of equations (1.2; 2.6; 2.5):

$$C^n(t_i) = p^n(t_i)U^n(t_i) \tag{2.8}$$

$$C_p^n(t_i, \tau) = p^n(t_i - \tau)U^n(t_i) \tag{2.9}$$

$$C^n(t_i) = r^n(t_i, \tau)\, C_p^n(t_i, \tau) \tag{2.10}$$

We use (2.3; 2.4; 2.7) and introduce (2.11; 2,12) the statistical moments $p(t;n)$ of price $p(t_i)$:

$$C(t; n) = p(t; n)U(t; n) \tag{2.11}$$

$$p(t; n) \equiv E[p^n(t_i)] = \frac{C(t;n)}{U(t;n)} = \frac{1}{\sum_{i=1}^{N} U^n(t_i)} \sum_{i=1}^{N} p^n(t_i)U^n(t_i) \tag{2.12}$$

The relations (2.13; 2.14) give the n-th statistical moments $p_p(t,\tau;n)$ of the "purchased" price:

$$C_p(t, \tau; n) = p_p(t, \tau; n)U(t; n) \tag{2.13}$$

$$p_p(t, \tau; n) \equiv E\left[p_p^n(t_i - \tau)\right] = \frac{C_p(t,\tau;n)}{U(t;n)} = \frac{1}{\sum_{i=1}^{N} U^n(t_i)} \sum_{i=1}^{N} p^n(t_i - \tau)U^n(t_i) \tag{2.14}$$

The relations (2.15; 2.16) define the n-th statistical moments $r(t,\tau;n)$ of the "anticipated" return with time shift τ:

$$C(t; n) = r(t, \tau; n)C_p(t, \tau; n) \tag{2.15}$$

$$r(t, \tau; n) \equiv E[r^n(t_i, \tau)] = \frac{C(t;n)}{C_p(t,\tau;n)} = \frac{1}{\sum_{i=1}^{N} C_p^n(t_i,\tau)} \sum_{i=1}^{N} r^n(t_i, \tau)C_p^n(t_i, \tau) \tag{2.16}$$

The relations (2.11-2.16) introduce the dependences of the statistical moments of price $p(t;n)$, of" purchased" price $p_p(t,\tau;n)$, and of "anticipated" return $r(t,\tau;n)$ on the statistical moments of trade values $C(t;n)$ (2.3), "purchased" values $C_p(t,\tau;n)$ (2.7), and volumes $U(t;n)$ (2.4), that are alike to the definition of VWAP (Berkowitz et al., 1988; Duffie and Dworczak, 2018). We define volume weighted average "purchased" price $p_p(t,\tau;1)$ (VWAPp) (2.14; 2.17):

$$p_p(t,\tau;1) = \frac{C_p(t,\tau;1)}{U(t;1)} = \frac{1}{\sum_{i=1}^{N} U(t_i)} \sum_{i=1}^{N} p(t_i,-\tau)U(t_i) \tag{2.17}$$

We call the "anticipated" average return $r(t,\tau;1)$ as "purchased" value weighted average returns (V_pWAR) (2.18):

$$r(t,\tau;1) = \frac{C(t;1)}{C_p(t,\tau;1)} = \frac{1}{\sum_{i=1}^{N} C_p(t_i,\tau)} \sum_{i=1}^{N} r(t_i,\tau)C_p(t_i,\tau) \tag{2.18}$$

The volatility $\sigma^2(t,\tau)$ of the "anticipated" stock returns takes the form:

$$\sigma^2(t,\tau) \equiv E\left[((r(t_i,\tau) - r(t,\tau;1))^2\right] = r(t,\tau;2) - r^2(t,\tau;1) \tag{2.19}$$

$$\sigma^2(t,\tau) = \frac{C(t;2)}{C_p(t,\tau;2)} - \frac{C^2(t;1)}{C_p^2(t,\tau;1)} \tag{2.20}$$

The relations (2.11-2.20) describe the market-based statistical moments of price and "anticipated" stock return in a similar way, and we refer to Olkhov (2021; 2022a; 2023) for further details. In Sections 3, 4, and 5, we use relations (2.8-2.20) to derive the market-based statistical moments of the "actual" return of investors.

3. The "actual" return of a single sale

In this section, we consider the statistical moments of "actual" return that an investor gains within a single sale of the volume $U(t_i)$ of stocks. We propose that the investor, at time t_i *"*today,*"* sells $U(t_i)$ stocks at a price $p(t_i)$. We assume that the investor purchased this amount of stocks $U(t_i)$ by small stakes of shares $U(t_j(i))$ at different times $t_j(i)$, $j=1,2, ..M(i)$ in the past at a price $p(t_j(i))$. We consider all prices $p(t_j(i))$ at times $t_j(i)$ in the past adjusted to the present at time t. The investor at time $t_j(i)$ purchases the value $C_p(t_j(i))$ of the volume $U(t_j(i))$ of stocks at price $p(t_j(i))$:

$$C_p(t_j(i) = p\left(t_j(i)\right) U\left(t_j(i)\right) \tag{3.1}$$

For each volume $U(t_j(i))$ of stocks purchased in the past at a price $p(t_j(i))$ we use equation (3.2) similar to (2.6):

$$C\left(t_i,t_j(i)\right) = p(t_i)U\left(t_j(i)\right) = \frac{p(t_i)}{p\left(t_j(i)\right)} p\left(t_j(i)\right) U\left(t_j(i)\right) = r\left(t_j(i)\right)\ C_p\left(t_j(i)\right)$$

$$r\left(t_i,t_j(i)\right) \equiv \frac{p(t_i)}{p\left(t_j(i)\right)} \quad ; \quad C_p\left(t_j(i)\right) \equiv p\left(t_j(i)\right) U\left(t_j(i)\right)$$

$$C(t_i) = \sum_{j=1}^{M(i)} C\left(t_i,t_j(i)\right) \quad ; \quad C\left(t_i,t_j(i)\right) = r\left(t_i,t_j(i)\right)\ C_p\left(t_j(i)\right) \tag{3.2}$$

Equation (3.2) introduces the current $C(t_i,t_j(i))$ sale value of the small stakes of shares of stock $U(t_j(i))$ that was purchased at price $p(t_j(i))$ in the past. At time t_i, the investor gains the "actual" return $r(t_i,t_j(i))$ by selling the volume $U(t_j(i))$ at price $p(t_i)$. The $C_p(t_j(i))$ (3.2) is the value of a purchase in the past at a price $p(t_j(i))$ adjusted to the present. The total sale volume $U(t_i)$ (3.3) was purchased in the past by $M(i)$ small stakes of shares $U(t_j(i))$ at prices $p(t_j(i))$. The purchased value $C_p(t_i)$ (3.3) of the volume $U(t_i)$ (3.4) takes the form:

$$C_p(t_i) = \sum_{j=1}^{M(i)} C_p(t_j(i)) = \sum_{j=1}^{M(i)} p\left(t_j(i)\right) U\left(t_j(i)\right) \tag{3.3}$$

$$U(t_i) = \sum_{j=1}^{M(i)} U\left(t_j(i)\right) \tag{3.4}$$

Obviously, the purchases of stocks in the past at different prices $p(t_j(i))$ result in a different "actual" return $r(t_i,t_j(i))$. If the total number $M(i)$ of the purchases is sufficiently large, then equation (3.2) gives the statistical moments of the "actual" returns $r(t_i,t_j(i))$ of a single market sale at a time t_i. One should follow Sec. 2 and, similar to (2.10), take the *n-th* power of (3.2):

$$C^n(t_i, t_j(i)) = r^n\left(t_i, t_j(i)\right) C_p^n\left(t_j(i)\right) \tag{3.5}$$

We introduce the n-th statistical moments $C(t_i;n)$ (3.6) at t_i of the sale value $C(t_i,t_j(i))$ and the n-th statistical moments $C_p(t_i;n)$ (3.7) at t_i of the purchased value $C_p(t_j(i))$ as:

$$C(t_i; n) \equiv E\left[C^n(t_i, t_j(i))\right] \sim \frac{1}{M(i)} \sum_{j=1}^{M(i)} C^n(t_i, t_j(i)) \tag{3.6}$$

It is clear that the average sale value $C(t_i;1)$ (3.6) for $n=1$ multiplied by $M(i)$ equals the value $C(t_i)$ (3.2) of the total sale at time t_i:

$$C(t_i) = \sum_{j=1}^{M(i)} C(t_i, t_j(i)) = M(i) C(t_i; 1)$$

The statistical moments $C_p(t_i;n)$ (3.7) of the purchased value $C_p(t_j(i))$ take the form:

$$C_p(t_i; n) \equiv E\left[C_p^n\left(t_j(i)\right)\right] \sim \frac{1}{M(i)} \sum_{j=1}^{M(i)} C_p^n(t_j(i)) \tag{3.7}$$

Similar to (2.11-2.16), obtain the statistical moments $r(t_i;n)$ of the "actual" return that the investor gains with a single sale of the volume $U(t_i)$ at t_i and numerous purchases in the past:

$$r(t_i; n) \equiv E\left[r^n\left(t_i, t_j(i)\right)\right] \tag{3.8}$$

$$C(t_i; n) = r(t_i; n) C_p(t_i; n) \tag{3.9}$$

$$r(t_i; n) = \frac{C(t_i;n)}{C_p(t_i;n)} = \frac{1}{\sum_{j=1}^{M(i)} C_p^n(t_j(i))} \sum_{j=1}^{M(i)} r^n\left(t_i, t_j(i)\right) C_p^n\left(t_j(i)\right) \tag{3.10}$$

The average sale value $C(t_i;1)$ (3.6) and the average purchase value $C_p(t_i;1)$ determine the average return $r(t_i;1)$ (3.8; 3.10) of a single trade at t_i. The volatility $\sigma^2(t_i)$ of the return of a single trade at t_i takes the form:

$$\sigma^2(t_i) \equiv E\left[r^2\left(t_i, t_j(i)\right)\right] - E^2\left[r\left(t_i, t_j(i)\right)\right] = r(t_i; 2) - r^2(t_i; 1) \tag{3.11}$$

$$\sigma^2(t_i) = \frac{C(t_i;2)}{C_p(t_i;2)} - \frac{C^2(t_i;1)}{C_p^2(t_i;1)} \tag{3.12}$$

4. The "Actual" return of a single investor

Now assume that at present time t during the averaging interval Δ (4.1):

$$\Delta = \left[t - \frac{\Delta}{2}; t + \frac{\Delta}{2}\right] \quad (4.1)$$

an investor makes N trade sales at the times t_i, $i=1,...N$. Each sale value $C(t_i)$ was purchased in the past by pieces $C_p(t_j(i))$ at times $t_j(i)$, $j=1,2,..M(i)$ and (3.2-3.4) valid for all i. During Δ, the investor makes N trade sales, and the total number N_Δ of purchases in the past equals:

$$N_\Delta = \sum_{i=1}^{N} M(i) \quad (4.2)$$

One can use equation (3.5) to define the n-th statistical moments of "actual" return $r(t;n)$ the investor gains due to all trade sales during Δ (4.1). At first, we introduce the sums of the *n-th* power of trade values during Δ:

$$C_\Delta(t;n) = \sum_{i=1}^{N} \sum_{j=1}^{M(i)} C^n\left(t_i, t_j(i)\right) \quad (4.2)$$

$$C_{\Delta p}(t;n) = \sum_{i=1}^{N} \sum_{j=1}^{M(i)} C_p^n\left(t_j(i)\right) \quad (4.3)$$

The n-th statistical moments of the sale values $C(t;n)$ (4.4) and purchased values $C_p(t;n)$ (4.5) during Δ (4.1) take the form:

$$C(t;n) = \frac{1}{N_\Delta} C_\Delta(t;n) = \frac{1}{N_\Delta} \sum_{i=1}^{N} \sum_{j=1}^{M(i)} C^n\left(t_i, t_j(i)\right) \quad (4.4)$$

$$C_p(t;n) = \frac{1}{N_\Delta} C_{\Delta p}(t;n) = \frac{1}{N_\Delta} \sum_{i=1}^{N} \sum_{j=1}^{M(i)} C_p^n\left(t_j(i)\right) \quad (4.5)$$

Relations (4.2-4.5) introduce the *n-th* statistical moments of "actual" return $r(t;n)$ that the investor gains after N trade sales during the interval Δ (4.1):

$$r(t;n) \equiv E\left[r^n\left(t_i, t_j(i)\right)\right]$$

$$C(t;n) = r(t;n) C_p(t;n) \quad ; \quad C_\Delta(t;n) = r(t;n) C_{\Delta p}(t;n) \quad (4.6)$$

Equations (4.6) equally define the n-th statistical moments of the "actual" return $r(t;n)$ (4.7):

$$r(t;n) = \frac{C_\Delta(t;n)}{C_{\Delta p}(t;n)} = \frac{C(t;n)}{C_p(t;n)} = \frac{1}{\sum_{i=1}^{N} \sum_{j=1}^{M(i)} C_p^n\left(t_j(i)\right)} \sum_{i=1}^{N} \sum_{j=1}^{M(i)} r^n\left(t_i, t_j(i)\right) C_p^n\left(t_j(i)\right) \quad (4.7)$$

The statistical moments of the "actual" return $r(t;n)$ (4.7) define the properties of return as a random variable during the interval Δ (4.1).

4.1 Statistics of different trade sales

Actually, different trade sales of the investor at times t_i, $i=1,2,..N$ result in different returns $r(t_i;n)$ (3.10). One can consider different average returns $r(t_i;n)$ (3.10), which the investor gains during interval Δ (4.1), as a random variable. To describe the random properties of average returns $r(t_i;n)$ (3.10), which the investor gains at time t_i, $i=1,2,...N$ during Δ, one should perform the secondary averaging procedure. Similar to (4.2; 4.3), we define sums of trade values:

$$C_\Delta(t_i;n) = \sum_{j=1}^{M(i)} C^n\left(t_i, t_j(i)\right) \quad ; \quad C_{\Delta p}(t_i;n) = \sum_{j=1}^{M(i)} C_p^n\left(t_j(i)\right) \tag{4.8}$$

From (3.6; 3.7) obtain that (3.9) can take the equal form:

$$C_\Delta(t_i;n) = r(t_i;n)C_{\Delta p}(t_i;n) \tag{4.9}$$

If the investor makes numerous sales $N>>1$ during Δ, then one can consider (4.9) similar to equation (3.5). Let us take the m-th power of (4.9):

$$C_\Delta^m(t_i;n) = r^m(t_i;n)\; C_{\Delta p}^m(t_i;n)$$

Let us define the sums $C_\Delta(t;n|m)$ and $C_{\Delta p}(t;n|m)$ during the interval Δ:

$$C_\Delta(t;n|m) = \sum_{i=1}^{N} C_\Delta^m(t_i;n) \quad ; \quad C_{\Delta p}(t;n|m) = \sum_{i=1}^{N} C_{\Delta p}^m(t_i;n) \tag{4.10}$$

It is obvious that for $n=1$, $m=1$ it is valid that $C_\Delta(t;1|1)$ (4.10) equals $C(t;1)$ (4.2) – the total sale value during the interval Δ. Respectively, $C_{\Delta p}(t;1|1)$ (4.10) equals $C_p(t;1)$ (4.3) – total purchased value in the past. One can define the secondary *m-th* statistical moments of the primary *n-th* statistical moments of sale trade values $C(t;n|m)$ and purchased value in the past $C_p(t;n|m)$ in a way similar to (4.4; 4.5) as follows:

$$C(t;n|m) = \frac{1}{N}C_\Delta(t;n|m) \quad ; \quad C_p(t;n|m) = \frac{1}{N}C_{\Delta p}(t;n|m) \tag{4.11}$$

We define the secondary *m-th* statistical moments of the primary *n-th* statistical moments of the "actual" returns $r(t;n|m)$:

$$C_\Delta(t;n|m) = r(t;n|m)C_{\Delta p}(t;n|m) \tag{4.12}$$

$$r(t;n|m) \equiv E[r^m(t_i;n)] \quad ; \quad r(t;n|m) = \frac{C_\Delta(t;n|m)}{C_{\Delta p}(t;n|m)} = \frac{C(t;n|m)}{C_p(t;n|m)} \tag{4.13}$$

$$r(t;n|m) = \frac{1}{\sum_{i=1}^{N} C_{\Delta p}^m(t_i;n)} \sum_{i=1}^{N} r^m(t_i;n)\, C_{\Delta p}^m(t_i;n) \tag{4.14}$$

We indicate that all definitions of statistical moments of price (2.12) and return (2.16; 3.10; 4.7) have the same form VWAP as (2.12) for the case n=1.

It is obvious that, as $C_\Delta(t;1|1)$ (4.10) equals $C(t;1)$ (4.2) and $C_{\Delta p}(t;1|1)$ (4.10) equals $C_p(t;1)$ (4.3), hence for $n=m=1$, relations (4.12-4.14) define average "actual" return $r(t;1|1)$ that is equal to average "actual" return $r(t;1)$ (4.7). However, for n>1 or m>1, relevant statistical moments are different. In particular, volatility $\sigma^2(t;1)$ (4.15) of "actual" return $r(t;1)$ (4.7) and volatility $\sigma^2(t;1|1)$ (4.16) of average return $r(t;1|1)$ (4.14) for $n=m=1$ are different:

$$\sigma^2(t;1) \equiv r(t;2) - r^2(t;1) = \frac{C(t;2)}{C_p(t;2)} - \frac{C^2(t;1)}{C_p^2(t;1)} \tag{4.15}$$

$$\sigma^2(t;1|1) \equiv r(t;1|2) - r^2(t;1|1) = \frac{C(t;1|2)}{C_p(t;1|2)} - \frac{C^2(t;1|1)}{C_p^2(t;1|1)} \tag{4.16}$$

The investor can analyze the effectiveness of sales of his assets, taking into account the volatility $\sigma^2(t_i;1)$ (3.12) of a single sale at time t_i as the result of numerous purchases in the past at different prices. As well, the investor can take into account the volatility $\sigma^2(t;1)$ (4.15) as the result of numerous sales during the interval Δ (4.1) of his assets that were purchased in the past. Finally, the investor can assess the volatility $\sigma^2(t;1|1)$ (4.16) of the average return $r(t_i;1)$ of each of his trade sales during Δ (4.1). The variety of ways to assess return and volatility uncovers the complexity of financial markets and their statistical properties.

4.2 Zero correlations of return and purchased values

Let us underline that the market-based assessments of statistical moments of "actual" return (3.10; 4.7; 4.14) cause that there are no correlations between the time series of the n-th power of return $r(t_i,t_j(i))$ (3.5) and the n-th power of the purchased value $C_p^n(t_j(i))$ (3.1). That is valid for each trade sale at time t_i and for time series during the interval for Δ (4.1) for t_i, $i=1,2,..N$.
Moreover, zero correlations are valid for the time series of the n-th statistical moments of "actual" return $r(t_i;n)$ (3.10) and the time series of the n-th statistical moments of purchased values $C_p(t_i;n)$ (3.7). We highlight that the assessments of correlations between time series completely depend on the choice of probability and the definition of statistical moments that describe the given time series.

For simplicity and brevity, we derive zero correlations between the time series of the n-th statistical moments of "actual" return $r(t_i;n)$ (3.10) at time t_i, and the time series of the n-th statistical moments of purchased values $C_p(t_i;n)$ (3.7). The market-based correlations $corr_{rCp}(t,n|t,n)$ between time series of the *n-th* statistical moments of "actual" return $r(t_i;n)$ (3.9; 3.10) and n-th statistical moments of purchased values $C_p(t_i;n)$(3.7) take the form:

$$corr_{r\,C_p}(t;n|t;n) \equiv E\big[r(t_i;n)C_p(t_i;n)\big] - E[r(t_i;n)]\,E\big[C_p(t_i;n)\big]$$

From (3.9; 4.10; 4.11; 4.13) obtain:

$$E\big[r(t_i;n)C_p(t_i;n)\big] = E[C(t_i;n)] = C(t;n|1)$$

$$E\big[C_p(t_i;n)\big] = C_p(t;n|1) = \frac{1}{N}C_{\Delta p}(t;n|1) \quad ; \qquad E[r(t_i;n)] = r(t;n|1)$$

$$C(t;n|1) = r(t;n|1)C_p(t;n|1)$$

Hence, correlations $corr_{rCp}(t,n|t,n)$ equal zero for all n=1,2,.. :

$$corr_{r\,C_p}(t;n|t;n) = C(t;n|1) - r(t;n|1)C_p(t;n|1) = 0 \tag{4.17}$$

For the particular case $n=1$, relations (4.17) give zero correlations between the time series of the "actual" return $r(t_i;1)$ (3.10) and purchased values $C_p(t_i;1)$ (3.7):

$$corr_{r\,C_p}(t;1|t;1) \equiv E\big[r(t_i;1)C_p(t_i;1)\big] - E[r(t_i;1)]\,E\big[C_p(t_i;1)\big] = 0$$

This result is similar to zero correlations between the time series of trade volumes and prices in the case of VWAP (Olkhov, 2021; 2022a). However, some researchers may assess the correlations between two random variables without taking into account their market-based statistical moments but using the frequency-based analysis of two given time series only. Such assessments differ from analyses based on the market origin of the stochasticity of price and return, which is determined by the randomness of market trade values and volumes. Those who study econometrics of the market time series should make a choice between the investigation of the market nature of the stochasticity of prices and returns described above and the frequency-based assessments of time series. These are two different approaches to the investigation of economic and financial stochasticity.

5. Statistics of the trading day portfolio and of different investors

In this section, we assess the statistical moments of "actual" return of the trading day portfolio as a whole and the statistical moments of average return that different investors gain during the averaging interval Δ, which we note as a "trading day". We define the trading day portfolio as a set of stocks that all investors sold during the "trading day". It is obvious that the trading day portfolio differs from the conventional notion of a market portfolio. Statistical moments of the trading day portfolio depend on the duration of the "trading day" and on factors that impact the financial markets. The statistical moments of return of the trading day portfolio describe random properties of the portfolio as a whole. Besides that, one can derive statistical moments of average return that different investors gain during the "trading day". Indeed, the average return that different investors gain from selling their stocks during the "trading day" varies a lot. One can consider the time series of the average returns of different investors as a random variable. Statistical moments of average returns describe the distribution of trade results of different investors after their sales during the "trading day". The statistical moments of return of the trading day portfolio, which was sold by investors, and the statistical moments of average return of different investors can serve as financial benchmarks for those investors who purchased stocks during the "trading day".

The statistical moments of "actual" return $r(t;n)$ (4.7) during the "trading day", the secondary statistical moments of "actual" return $r(t;n|m)$ (4.14), volatility $\sigma^2(t;1)$ (4.15) and volatility $\sigma^2(t;1|1)$ (4.16) can vary significantly for different investors. If the number Q of investors $Q>>1$, then one can consider the trading day portfolio as the total trade sale of all investors and assess the statistical moments of average return of different investors during the "trading day". We highlight that the "trading day" or the averaging interval Δ can be equal to an hour, a day, a week, or whatever. Assume that the investor q, $q=1,..Q$ makes $N(q)$ trade sales, and N is the total number of sales of all Q investors during the "trading day":

$$N = \sum_{q=1}^{Q} N(q) \tag{5.1}$$

Relations (4.2) define the total number of all purchases by a single investor q in the past. During the "trading day", the total number N_t of all purchases in the past takes the form (5.2):

$$N_t = \sum_{q=1}^{Q} \sum_{i=1}^{N(q)} M(i) \tag{5.2}$$

5.1 Statistical moments of a single sale returns

Each trade sale $C(t_i(q),t_j(i,q))$ of the investor q at time $t_i(q)$ of a stake of shares that was purchased in the past at time $t_j(i,q)$ at value $C_p(t_j(i,q))$ results in a return $r(t_i(q),t_j(i,q))$:

$$r\left(t_i(q), t_j(i,q)\right) = \frac{C\left(t_i(q), t_j(i,q)\right)}{C_p\left(t_j(i,q)\right)} \tag{5.3}$$

We use notations similar to section 3 and denote the time of sale $t_i(q)$ and the times $t_j(i,q)$, $j=1,2, ..M(i,q)$ of purchases in the past of investor q, $q=1,2,\ldots Q$. During the averaging interval Δ, which we call "trading day", there were N_t (5.2) of different purchases, each resulting in a return (5.3). To derive statistical moments of this set of returns (5.3) we use the results of the previous section and define the sum $C_{t\Delta}(t;n)$ of the *n-th* power of sales $C^n(t_i(q),t_j(i,q))$ (similar to 4.2) and the sum $C_{t\Delta p}(t;n)$ of the n-th power of purchases $C^n_p(t_j(i,q))$ (similar to 4.3) of all investors during the "trading day":

$$C_{t\Delta}(t;n) = \sum_{q=1}^{Q} \sum_{i=1}^{N(q)} \sum_{j=1}^{M(i,q)} C^n\left(t_i(q), t_j(i,q)\right) \tag{5.4}$$

$$C_{t\Delta p}(t;n) = \sum_{q=1}^{Q} \sum_{i=1}^{N(q)} \sum_{j=1}^{M(i,q)} C_p^n\left(t_j(i,q)\right) \tag{5.5}$$

Similar to (4.4; 4.5), define the *n-th* statistical moments of the trade sale values $C_t(t;n)$ and purchased values $C_{tp}(t;n)$:

$$C_t(t;n) = \frac{1}{N_t} C_{t\Delta}(t;n) \quad ; \quad C_{tp}(t;n) = \frac{1}{N_t} C_{t\Delta p}(t;n) \tag{5.6}$$

Similar to (4.6; 4.7) define the n-th statistical moments of "actual" return $r_t(t;n)$:

$$C_t(t;n) = r_t(t;n) C_{tp}(t;n) \quad ; \quad C_{t\Delta}(t;n) = r_t(t;n) C_{t\Delta p}(t;n) \tag{5.7}$$

$$r_t(t;n) = \frac{C_t(t;n)}{C_{tp}(t;n)} = \frac{C_{t\Delta}(t;n)}{C_{t\Delta p}(t;n)} \tag{5.8}$$

$$r_t(t;n) = \frac{1}{\sum_{q=1}^{Q} \sum_{i=1}^{N(q)} \sum_{j=1}^{M(i,q)} C_p^n\left(t_j(i,q)\right)} \sum_{q=1}^{Q} \sum_{i=1}^{N(q)} \sum_{j=1}^{M(i,q)} r^n\left(t_i(q), t_j(i,q)\right) C_p^n\left(t_j(i,q)\right) \tag{5.9}$$

Relations (5.7-5.9) define the *n-th* statistical moments of "actual" return $r_t(t;n)$ of the trading day portfolio, which is composed of all stocks that were sold by all investors during Δ.

5.2 Statistical moments of returns of different investors

The statistics of the trading day portfolio can be described in a different manner via the statistical moments of return of different investors. Indeed, during the "trading day," each investor q makes numerous trade sales $N(q)$ (5.1) that result in the n-th statistical moments of return $r(t;n|q)$ (4.7) of the investor q. Different investors have different statistical moments of return. One can assess the statistics of random results that different investors gain during the "trading day". Let us consider the statistical moments of return $r(t;n|q)$ (4.7) for $q=1,..Q$ as random variables and assess their statistical moments. We start with equation (4.6) that links the sums of the n-th power of trade value $C_\Delta(t;n|q)$ (4.2), sums of the n-th power of purchased value $C_{\Delta p}(t;n|q)$ (4.3), and return $r(t;n|q)$ (4.7) of investor q, $q=1,..Q$. We follow the above models and introduce the sums $C_\Delta(t;n,m)$ (5.10) by q of the m-th power of the n-th power of trade value $C^m_\Delta(t;n|q)$ (4.2), and the sums $C_{\Delta p}(t;n|m)$ (5.11) by q of the m-th power of the n-th power of purchase value $C^m_{\Delta p}(t;n,q)$ (4.3):

$$C_\Delta(t;n,m) = \sum_{q=1}^{Q} C_\Delta^m(t;n|q) \tag{5.10}$$

$$C_{\Delta p}(t;n,m) = \sum_{q=1}^{Q} C_{\Delta p}^m(t;n|q) \tag{5.11}$$

Similar to (4.4), relations (5.12) define the average m-th power $C(t;n,m)$ of $C^m_\Delta(t;n|q)$ (4.2), or the m-th statistical moment of the sum of n-th powers of trade values $C_\Delta(t;n|q)$:

$$C(t;n,m) = \frac{1}{Q}\sum_{q=1}^{Q} C_\Delta^m(t;n|q) \tag{5.12}$$

$$C_p(t;n,m) = \frac{1}{Q}\sum_{q=1}^{Q} C_{\Delta p}^m(t;n|q) \tag{5.13}$$

Relations (5.13) define the average m-th power $C_p(t;n,m)$ of $C^m_{\Delta p}(t;n|q)$ (4.3) or the m-th statistical moment of the sum of n-th powers of purchased values $C_{\Delta p}(t;n|q)$. Similar to (4.6; 4.7), the m-th statistical moments $r(t;n,m)$ (5.14; 5.15) of the n-th statistical moments $r(t;n|q)$ (4.7) of the return of investors q, $q=1,..Q$ take the form:

$$C(t;n,m) = r(t;n,m)C_p(t;n,m) \quad ; \quad C_\Delta(t;n,m) = r(t;n,m)C_{\Delta p}(t;n,m) \tag{5.14}$$

$$r(t;n,m) = \frac{1}{\sum_{q=1}^{Q} C_{\Delta p}^m(t;n|q)} \sum_{q=1}^{Q} r^m(t;n|q) C_{\Delta p}^m(t;n|q) \tag{5.15}$$

In particular, relations (5.10-5.15) define the average return $r(t;1,1)$ (5.16) that different investors gain during the "trading day". It is obvious that the average return $r(t;1,1)$ (5.16) coincides with the average return $r_t(t;1)$ (5.7; 5.8). To show that one can present (5.4; 5.5) using (4.8; 4.10):

$$C_{t\Delta}(t;n) = \sum_{q=1}^{Q}\sum_{i=1}^{N(q)}\sum_{j=1}^{M(i,q)} C^n\left(t_i(q), t_j(i,q)\right) = \sum_{q=1}^{Q} C_\Delta(t;n|q) = C_\Delta(t;n,1)$$

$$C_{t\Delta}(t;n|q) = \sum_{i=1}^{N(q)} C_\Delta(t_i;n|q) = \sum_{i=1}^{N(q)}\sum_{j=1}^{M(i,q)} C^n\left(t_i(q), t_j(i,q)\right)$$

The same relations are valid for:

$$C_{t\Delta p}(t;n) = C_{\Delta p}(t;n,1)$$

Hence, equations (5.7) on the n-th statistical moments of return $r_t(t;n)$ and (5.14) for m=1 on the 1-st statistical moment *r(t;n,1)* of return define the equal functions and:

$$r_t(t;n) = r(t;n,1) = \frac{C(t;1,1)}{C_p(t;1,1)} \tag{5.16}$$

Relations (5.14; 5.15) define the volatility $\sigma_Q^2(t;1)$ (5.19) of the average return *r(t;1,1)* (5.17) generated by all *Q* investors:

$$r(t;1,1) = \frac{C(t;1,1)}{C_p(t;1,1)} = \frac{1}{\sum_{q=1}^{Q} C_{\Delta p}(t;1|q)} \sum_{q=1}^{Q} r(t;1|q) C_{\Delta p}(t;1|q) \tag{5.17}$$

$$r(t;1,2) = \frac{C(t;1,2)}{C_p(t;1,2)} = \frac{1}{\sum_{q=1}^{Q} C_{\Delta p}^2(t;1|q)} \sum_{q=1}^{Q} r^2(t;1|q)\, C_{\Delta p}^2(t;1|q) \tag{5.18}$$

$$\sigma_Q^2(t;1) = r(t;1,2) - r^2(t;1,1) = \frac{C(t;1,2)}{C_p(t;1,2)} - \frac{C^2(t;1,1)}{C_p^2(t;1,1)} \tag{5.19}$$

The finite number *M* of the *m-th* statistical moments of return *r(t;1;m), m=1,2,…M* (5.15) defines approximations of characteristic function and probability measure (Olkhov, 2021; 2022a; 2023), which describe distributions of average return by *Q* different investors during the "trading day". Respectively, the finite number of the *m-th* statistical moments of return *r(t;2;m)* defines approximations of probability distributions of average squares or the 2-nd statistical moments of return by *Q* different investors during the "trading day". These approximations describe the distribution of the gains and losses as the results of trade sales by *Q* different investors. An analysis of the evolution in time for successive "trading days" of these approximations of probability distributions will give a description of variations in stock market profitability for different investors.

6. Conclusion

This paper describes three successive approximations of the n-th statistical moments of the "actual" return, which the investors gain within their market sales. We describe statistical moments of return generated by a single trade sale, by numerous trade sales of a single investor during the "trading day," and statistical moments of the trading day portfolio composed by sales of all investors.

The main issue of our model is as follows: the assessments of the statistical moments of the stock prices, the "anticipated" and "actual" stock returns, depend on statistical moments of market trade values and volumes. The statistics of return depend on market trade statistics. That differs our model from conventional frequency-based analysis of the return time series.

Let us highlight some problems that seem important for the further understanding and modeling of financial markets.

1. We introduce the notion of a trading day portfolio composed of stocks sold by all investors during the averaging time interval Δ, which we note as a "trading day". The share of stocks of company A in the trading day portfolio is proportional to the share of the sale value of these stocks in the total sale value of all stocks during the "trading day". Such a share varies for different durations of the "trading day" and varies in time. That differs the trading day portfolio from the conventional market portfolio, which defines shares of company A in the market portfolio as the share of capitalization of company A in the total capitalization of all stocks traded at the market.

2. Statistical moments of return of the trading day portfolio that all investors "actually" gain as a result of their sales during the "trading day" could serve as benchmarks and impact the decisions of "purchasing" investors in financial markets. "Purchasing" investors can assess their forecast of the expected returns at horizon T in comparison with the "actual" returns that "selling" investors already gain. Analysis of relations between the statistical moments of return that "selling" investors already gain and predictions of the statistical moments of return of "purchasing" investors can help further develop asset pricing models and portfolio theory. The statistical moments of return (5.15) describe the "actual" probability distribution of profitability over different investors in the stock market.

3. Probably, it is difficult to collect and study the market data records that assess statistical moments of "actual" return, which investors gain, and statistical moments of "actual" return of the trading day portfolio. The market data records that define "anticipated" statistical moments of return (see Sec. 2) are much more available. It is important to study relations between statistical moments of "anticipated" and "actual" return and derive possible dependence between these factors.

4. The fluctuations of the "anticipated" return due to the variations of the time shift τ can impact the duration of stock holding by the investors. That, in turn, can change the scales and fluctuations of "anticipated" returns determined by the time shift τ. Investigation of the hidden mutual dependence between the market-based statistics of the "actual" return of investors and the statistics of the "anticipated" return can help increase the efficiency of portfolio performance and asset pricing models.

References

Amaral, L.I., Plerou, V., Gopikrishnan, P., Meyer, M. and INVESTORS. E. Stanley, (2000). The Distribution of Returns of Stock Prices, Int.J.Theoretical and Applied Finance, 3(3), 365-369

Andersen, T., Bollerslev, T., Diebold, F.X, Ebens, INVESTORS. (2001). The Distribution of Realized Stock Return Volatility, Journal of Financial Economics, 61, 43-76

Andersen, T. and L. Benzoni, (2009). Realized Volatility, 555-570, in Andersen, T., Davis, R., Kreiß, J-P. and T. Mikosch, Handbook of Financial Time Series, Springer-Verlag Berlin Heidelberg, 1-1031.

Baker, M. and J. Wurgler, (2004). investor Sentiment And The Cross-Section Of Stock Returns, NBER, Cambridge, WP 10449, 1-47

Berkowitz, S.A., Dennis, E., Logue, D.E., Noser, E.A. Jr. (1988). The Total Cost of Transactions on the NYSE, *The Journal of Finance*, 43, (1), 97-112

van Binsbergen, J.INVESTORS. and R. Koijen, (2015). The Term Structure Of Returns: Facts And Theory, NBER WP 21234, Cambridge, 1-38

Brown, S.J. (1989). The Number of Factors in Security Returns, J. Finance, 44(5), 1247-1262

Campbell, J. (1985). Stock Returns And The Term Structure, NBER WP1626, 1-53

Daniel, K. and D. Hirshleifer, (2016). Overconfident investors, Predictable Returns, And Excessive Trading, NBER, Cambridge, WP 21945, 1-36

Duffie, D. and P. Dworczak, (2018). Robust Benchmark Design, NBER WP 20540, 1-56

Greenwood, R. and A. Shleifer, (2013). Expectations of Returns and Expected Returns, WP18686, NBER, Cambridge, 1-52

Fama, E.F. (1990). Stock Returns, Expected Returns, and Real Activity, J. Finance, 45(4), 1089-1108

Fama, E.F. and K. R. French, (1992). The Cross-Section of Expected Stock Returns, J.Finance, 47 (2), 427-465

Fisher, L. and J. INVESTORS. Lorie, (1964). Rates Of Return On Investments In Common Stocks, J. Business, 37(1), 1-21

Gabaix, X., Gopikrishnan, P., Plerou, V. and INVESTORS. E. Stanley, (2005). Institutional investors And Stock Market Volatility, NBER, Cambridge, WP 11722, 1-50

Greenwood, R. and A. Shleifer, (2013). Expectations Of Returns And Expected Returns, NBER, Cambridge, WP 18686, 1-51

Hardouvelis, G., Karalas, G. and D. Vayanos, (2021). The Distribution of investor Beliefs, Stock Ownership and Stock Returns, NBER, Cambridge, WP 28697, 1-47

Ivković, Z., Sialm, C. and S. Weisbenner, (2004). Portfolio Concentration and The Performance of Individual investors, NBER, Cambridge, WP 10675, 1-52

Knight, J. and S. Satchell, (Ed). (2001). Return Distributions In Finance, Butterworth-Heinemann, Oxford, 1-328

Koijen, R.S., Richmond, R.J. and M. Yogo (2020). Which investors Matter For Equity Valuations And Expected Returns?, NBER, Cambridge, WP 27402, 1-53

Lettau, M. and S. C. Ludvigson, (2003). Expected Returns And Expected Dividend Growth, WP 9605, NBER, Cambridge, 1-48

Mandelbrot, B., Fisher, A. and L. Calvet, (1997). A Multifractal Model of Asset Returns, Yale University, Cowles Foundation Discussion WP1164, 1-39

Markowitz, INVESTORS. (1952). Portfolio Selection, J. Finance, 7(1), 77-91

Martin, I. and C. Wagner (2019). What Is the Expected Return on a Stock?, J. Finance, 74(4), 1887-1929

Olkhov, V. (2021). Three Remarks On Asset Pricing, SSRN WP3852261, 1-24, https://ssrn.com/abstract=3852261

Olkhov, V. (2022a). The Market-Based Asset Price Probability, MPRA WP115382, 1-21, https://mpra.ub.uni-muenchen.de/115382/

Olkhov, V. (2022b). Price and Payoff Autocorrelations in the Consumption-Based Asset Pricing Model, SSRN, WP 4050652, 1- 18

Olkhov, V. (2023). The Market-Based Probability of Stock Returns, SSRN, WP 4350975, 1-25, https://ssrn.com/abstract=4350975

Schlarbaum, G.G., Lewellen, INVESTORS. G. and R. C. Lease, (1978). Realized Returns on Common Stock Investments: The Experience of Individual investors, J. of Business, 51(2) 299-325

Shephard, N.G. (1991). From Characteristic Function to Distribution Function: A Simple Framework for the Theory. *Econometric Theory*, 7 (4), 519-529

Shiryaev, A.N. (1999). Essentials Of Stochastic Finance: Facts, Models, Theory. World Sc. Pub., Singapore. 1-852

Shreve, S. E. (2004). Stochastic calculus for finance, Springer finance series, NY, USA

Stanley, K.L., Lewellen, INVESTORS.G. and G.G. Schlarbaum, (1980). Further Evidence of on the Value of Professional Investment Research, NBER, Cambridge, WP 536, 1-19

Tsay, R.S. (2005). Analysis of Financial Time Series, J.Wiley&Sons, Inc., New Jersey, 1-638