A Note on the Euler Equation of the Growth Model

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Abstract: The neoclassical Euler equation provides the necessary conditions for households to maximize lifetime utility by allocating income between consumption and investment, and is the core equation for solving the steady-state of the neoclassical growth model. The existing textbooks (Barro and Sala-i-Martin, 2004, ch6.3; Acemoglu, 2009, ch13.2, ch15.6; Aghion and Howitt, 2009, ch3.2.2) ignore the premise of this equation and directly apply it to solve the steady state of other growth models, which not only leads to incorrect results but also limits the ability of growth models to analyze the steady-state technological progress direction. This note first points out and rigorously verifies the errors in existing textbooks; Then, by replacing the capital accumulation function with exogenous growth rate with the generalized capital accumulation function considering adjustment costs of investment in the Acemoglu (2009, ch15.6) model, the note put forward the generalized Euler equation and steady-state equilibrium including capital-augmenting technological progress, which reveals the necessary conditions for the neoclassical Euler equation and Uzawa's (1961) steady-state theorem; Finally, it is pointed out that the possible reasons for the misuse of the neoclassical Euler equation in existing textbooks maybe confuse the rental price of capital and the interest rate of investment.

Key words: Neoclassical Euler equation, Uzawa's steady-state theorem, Growth model, the direction of technical change, the rental price of capital, the interest rate of investment

JEL: E130, O30, 0400, O410

First draft. Any comment would be appreciated!

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0 Introduction

Uzawa's (1961) steady-state theorem says that a neoclassical growth model requires technological progress to be purely labor-augmenting in steady state, unless the production function is Cobb-Douglas. However, as Acemoglu (2009, ch2, p59) pointed out, there is no compelling reasons for why technological progress should take this form, leading to the Uzawa's steady-state theorem becoming a puzzle in economic growth theory (Jones and Scrimgeour, 2008). Acemoglu (2009, ch15.6) intended to provide a micro foundation for this theorem within the framework of directed technical change (Acemoglu,2002), but came up with an incorrect Proposition 15.12 (Li, 2016).[3] Peters and Simsek (2010) provided supplementary proof for this proposition, but did not find this simple error.

Why did Acemoglu come up with this simple and clearly incorrect proposition, and as a well-known textbook widely used worldwide, why has the error not been pointed out and corrected for so long time? This is indeed confusing. We think that the possible reason may be the proof process of Acemoglu (2009, ch15.6). Acemoglu provided a detailed derivation of the equation of capital return, but directly applied the neoclassical Euler equation $\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(r(t) - \rho)$ without any derivation based on the assumptions of Acemoglu's (2009, ch15.6) model. Perhaps the existing growth theory implicitly argues that the neoclassical Euler equation can be applied to all growth models, so there is no need to re-derive it. As a result, no one doubts the validness of Proposition 15.12. However, verifying that Proposition 15.12 is not valid is also very simple, which shows that the neoclassical Euler equation does not hold true in all environment, but has strict premises.

If only Acemoglu (2009, ch15.6) directly applies the neoclassical Euler equation to lead to incorrect results, it may be due to Acemoglu's negligence in order to simplify the derivation. However, it is not only Acemoglu (2009, ch15.6), existing famous textbooks (Barro and Sala-i-Martin, 2004, ch6.3; Acemoglu, 2009, ch13.2; Aghion and Howitt, 2009, ch3.2.2) also directly apply it to solve the steady-state of endogenous technological progress growth models with knowledge spillovers innovation function, which also leads to incorrect results. And as the best of our knowledge, there is still no literature that has pointed them out. Therefore, the first purpose of this note is to clearly identify and strictly verify these errors in order to push these textbooks to correct them.

If directly applying the neoclassical Euler equation only leads to mathematic solving errors, but it does not affect the core conclusions of the model, then it does not

[3] This article will prove that this proposition is not valid again.

seriously impact on the development of economic growth theory. However, the implicit error that neoclassical Euler equation holds true in any growth model, seriously hindering to resolve the puzzle of Uzawa's steady-state theorem and severely limiting the ability of growth model to analyze the determinants of the direction of technological progress. On the one hand, if the neoclassical Euler equation holds, for any growth model as long as with a neoclassical production, then the steady-state technological progress can only be purely labor-augmenting; However, on the other hand, from the concepts of technological progress and steady-state, there is no compelling reason to accept that the steady state is incompatible with other types of technological progress. If the Uzawa's steady-state theorem holds for any model, the economic growth model cannot actually analyze the determinants of the direction of technological progress, and there is no need for analysis, because other types of technological progress cannot exist in steady-state at all. However, this is not in line with the facts.

If income can be used for both consumption and investment, consumption generates current utility, while investment decreases current consumption but increases it in future. Therefore, in order to maximize household's lifetime utility, it is necessary to appropriately allocate his income between consumption and investment. However, this allocation depends not only on the utility function and time discount factor, but also on the form of investment function. The neoclassical Euler equation is a necessary condition for the optimal allocation of income between consumption and investment under the investment function assumed by the neoclassical growth model. Why cannot the neoclassical Euler equation be applied under Acemoglu (2009, ch15.6)? Because this model assumes that capital is not accumulated through investment, but rather through exogenous growth, household's income can only be used for consumption, and there is no problem of allocating income between consumption and investment. All income is consumed that is the optimal choice for household, and naturally there is no Euler equation. Why cannot the neoclassical Euler equation be applied to the endogenous technological progress model with a knowledge spillover innovation function? Because these models assume that production does not require capital, income can only be consumed, and consuming all income is the optimal choice for household. However, unlike Acemoglu's (2009,15.6) model, household need to allocate labor appropriately, as labor can be used not only to produce the final product consumed in current, but also for research and development to increase output and consumption in the future. Therefore, there exists a Euler equation for how household allocates labor, but it is different from the neoclassical Euler equation for optimal allocation of income between consumption and investment.

Why do existing textbooks implicitly argue that the neoclassical Euler equation holds true in any growth models? This is indeed a mystery! It is likely because existing textbooks implicitly treat the rental price of capital and the return on investment (i.e. interest rate) as the same variable. In a competitive market, the rental price of capital is equal to the market value of the marginal output of capital, and the interest rate of investment reflects the opportunity cost of income used for investment relative to consumption. Whether the two variables are equal depends on the specific environment of a model. When the model assumes that income cannot be used for investment but can only be used for consumption, the opportunity cost of consumption is zero, and the return on investment is also zero. At this point, if there is capital input in production, the value of marginal output of capital cannot be zero. If investment can accumulate capital, the return on investment should be the amount of capital converted from investment multiplied by the value of marginal output of capital, rather than directly equal to the value of marginal output of capital. When the adjustment cost of investment is marginal increasing, then capital accumulation for investment is marginal decreasing. At this point, the marginal output of capital must continue to rise to maintain the marginal return on investment to be constant. By clearly pointing out the difference between the interest rate of investment and the market rental price of capital, it can reveal that the neoclassical Euler equation does not hold true in all environment, and that ignoring its prerequisites and directly applying it will lead to incorrect results. Pointing out and verifying this error in existing textbooks is not only beneficial for correcting the errors in these textbooks, but more importantly, it is beneficial for understanding the Uzawa's steady-state theorem, and improving the analysis of the direction of technological progress in growth model.

Although the neoclassical Euler equation is a fundamental equation that almost everyone who has learned economic growth theory or macroeconomics is familiar with it, so far, only Li (2016) has explicitly pointed out and verified the error of Proposition 15.12 in Acemoglu (2009, Ch15.6), but he has not pointed out that this incorrect proposition is the result of directly applying the neoclassical Euler equation. Li and Bental (2022) pointed out that the neoclassical Euler equation is only a special result under a special capital accumulation function, but they did not indicate whether there were errors caused by directly applying the neoclassical Euler equation in existing literature.

The remainder of this note is arranged as follows: Section 1 introduces the derivation process and necessary conditions of the neoclassical Euler equation; Section 2 points out and verifies the error of Proposition 15.12 in Acemoglu (2009, Ch15.6); The section 3 pointed out and verified the errors in the steady-state solution of the growth model with knowledge spillovers innovation function in the existing textbooks (Barro and Sala-i-Martin, 2004, ch6.3; Acemoglu, 2009, ch13.2; Aghion and Howitt, 2009, ch3.2.2); In Section 4, the Acemoglu (2009, ch15.6) model, which replaces the capital accumulation function of exogenous growth with a capital accumulation function that includes investment adjustment costs, provides a generalized Euler equation and a steady-state equilibrium that includes capital-augmenting technological progress, revealing the conditions under which the neoclassical Euler equation and Uzawa's (1961) steady-state theorem are applicable; Section 5 points out that if the neoclassical Euler equation is valid for any growth model, it will inevitably lead to a dilemma of Uzawa's steady-state theorem; Section 6 speculates the possible reasons for the widespread misuse of Euler equations in existing literature. Section 7 is concluding remarks.

1. The Euler Equation of the Neoclassical Growth Model

The neoclassical growth model (Ramsey, 1928; Cass, 1965; Koopmans, 1965) is the foundation of modern economic growth theory. Compared to the Solow (1956) model, it points out the necessary conditions for households to allocate their income between consumption and investment to maximize their lifetime utility, that is, the Euler equation, later referred to as the neoclassical Euler equation. This section first provides the derivation process of the equation and points out its prerequisite conditions.

1.1 Setup of the Neoclassical Growth Model[4]

Population within each household grows at the rate *n*, starting with $L(0) = 1$, so that total population in the economy is

$$L(t) = L(0)\exp(nt) \tag{1.1}$$

All members of the household supply their one unit of labor inelastically.

Each household wishes to maximize overall utility, U, as given by

$$U = \int_0^\infty \exp(-(\rho - n)t)\, u\big(c(t)\big)dt \tag{1.2}$$

where c(t) is consumption per capita at time t; ρ is the subjective discount rate; and the effective discount rate is $\rho - n > 0$. The function $u\big(c(t)\big)$ is the felicity

[4] The derivation of the neoclassical Euler equation is the basic content in economic growth theory, and the derivation process in this note is based on Acemoglu (2009, ch8).

function which is increasing in $c(t)$ and concave, that is, $u'(c(t)) > 0$ and $u''(c(t)) < 0$.

In existing literature, it is usually assumed that the per capita asset of a household is $a(t)$, and each member has one unit of labor. The return on assets $r(t)$ and wage rate $w(t)$ are both determined by market competition. The household income is the sum of asset income and labor income. Income can be used for consumption or to accumulate assets, so the dynamic equation for per capita household assets is

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t) \tag{1.3}$$

To rule out chain-letter possibilities, the appropriate restriction is that the present value of assets must be asymptotically nonnegative, that is,

$$\lim_{t\to\infty}\left[a(t)exp\left(-\int_0^t (r(s) - n)ds\right)\right] \geq 0 \tag{1.4}$$

1.2 Household Optimization and Euler Equation

The optimization problem is to solve the following maximization problem under the constraints of equation (1.3),

$$\max_{c(t)} \int_0^\infty \exp(-(\rho - n)t)\, u(c(t))dt \tag{1.5}$$

Using optimal control technology to solve equation (1.5) and construct the present value Hamilton equation as following

$$H(t, a, c, \mu) = \exp(-(\rho - n)t)\, u(c(t)) + \mu(t)[w(t) + (r(t) - n)a(t) - c(t)] \tag{1.6}$$

First-Order Conditions are as following

$$\begin{cases} H_c(t, a, c, \mu) = \exp(-(\rho - n)t)\, u'(c(t)) - \mu(t) = 0 \\ H_a(t, a, c, \mu) = \mu(t)(r(t) - n) = -\dot{\mu}(t) \end{cases} \tag{1.7}$$

The Euler equation obtained from (1.7) is as following

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}(r(t) - \rho) \tag{1.8}$$

Where $\varepsilon_u(c(t))$ is the elasticity of the marginal utility $u'(c(t))$ as follows,

$$\varepsilon_u(c(t)) \equiv -\frac{u''(c(t))c(t)}{u'(c(t))} \tag{1.9}$$

When the felicity function is as following

$$u(c(t)) = \frac{c(t)^{1-\theta} - 1}{1 - \theta}, 0 < \theta < 1 \tag{1.10}$$

then $\varepsilon_u\big(c(t)\big) = \theta$, substituting it into the equation (1.8) to obtain the familiar neoclassical Euler equation as following

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(r(t) - \rho) \tag{1.11}$$

An important characteristic of equations (1.8) and (1.11) is that when the consumption growth rate remains unchanged, the interest rate also remains unchanged. This is a core result of the neoclassical growth model, which is consistent with the fact that interest rate is no trend in reality, and is consistent with the famous Kaldor (1961) stylized facts in economic growth theory.

1.3 Necessary Conditions for the Neoclassical Euler Equation

The process of solving the Euler equation above seems to be only related to household preferences and budget constraints, and is not related to the model's factor accumulation and production function. Therefore, it seems to be applicable for any growth model with the same household preference. Perhaps it is precisely for this reason that existing textbooks (Barro and Sala-i Martin, 2004, ch6.3; Acemoglu, 2009, ch13.2, ch15.6; Aghion and Howitt, 2009, ch3.2.2) directly apply the neoclassical Euler equation in almost all growth models. However, this led to a clearly incorrect Proposition 15.12 in Acemoglu (2009, ch15.6), indicating that the neoclassical Euler equation depends not only on household preference but also on other prerequisites and cannot be used in all growth models. One crucial prerequisite is that household income, in addition to consumption, must also be used to accumulate assets (capital or patented technology) to increase future consumption. Otherwise, the household's income can only be used for consumption, and consuming all income is the household's optimal choice, as a result there is no neoclassical Euler equation. For the neoclassical growth model, the assets that households can accumulate are the capital K(t) used for final product production, and the per capita assets are the per capita capital $a(t) = k(t) \equiv \frac{K(t)}{L(t)}$. The accumulation equation of per capita capital is also the accumulation equation of per capita assets as following

$$\dot{k}(t) = \dot{a}(t) = (r(t) - n)k(t) + w(t) - c(t) \tag{1.12}$$

However, the later models indicate that not all growth models allow households' income to accumulate assets.

2 Proposition 15.12 of Acemoglu (2009, Ch15.6)

Acemoglu (2009, Ch15.6) proposed a growth model with endogenous direction of technological progress, and the core result of this model is the Proposition 15.12, as follows:

"Proposition 15.12: *Consider the baseline model of directed technological change with the two factors corresponding to labor and capital. Suppose that the innovation possibilities frontier is given by the knowledge spillovers specification with extreme state dependence, that is, δ=1, and that capital accumulates according to (15.45) (i.e., $\dot{K}(t)/K(t) = s_K$). Then there exists a unique BGP allocation in which there is only labor-augmenting technological change, the interest rate is constant and consumption and output grow at constant rates."* (Acemoglu, 2009, Ch15.6, p521)

This proposition suggests that the model exists a steady-state equilibrium where interest rates remain constant and technological progress is purely labor-augmenting. We first prove that the proposition is incorrect based on Li (2016), and then point out how Acemoglu (2009, Ch15.6) applied the neoclassical Euler equation to obtain this erroneous conclusion.

2.1 Setup of the Model

The aggregate production function combining the outputs of two intermediate sectors with a constant elasticity of substitution:

$$Y(t) = \left[\gamma_L Y_L(t)^{\frac{\varepsilon-1}{\varepsilon}} + \gamma_K Y_K(t)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \tag{2.1}$$

Where $Y(t)$ is the final output. Where $Y_L(t)$ and $Y_K(t)$ denote the outputs of two intermediate goods. As the indices indicate, the first is L-intensive, while the second is K-intensive. The parameter $\varepsilon \in [0,\infty)$ is the elasticity of substitution between these two intermediate goods.

The two intermediate goods are produced competitively with the following production functions::

$$\begin{cases} Y_L(t) = \dfrac{1}{1-\beta}\left(\displaystyle\int_0^{N_L(t)} x_L(v,t)^{1-\beta} dv\right) L^{\beta} \\ Y_K(t) = \dfrac{1}{1-\beta}\left(\displaystyle\int_0^{N_K(t)} x_K(v,t)^{1-\beta} dv\right) K^{\beta} \end{cases} \tag{2.2}$$

where $x_L(v,t)$ and $x_K(v,t)$ denote the quantities of the different machine varieties (used in the production of one or the other intermediate good) and $\beta \in (0, 1)$. These machines are again assumed to depreciate after use. The range of machines complementing labor, L, is [0, $N_L(t)$], while the range of machines complementing factor K is [0, $N_K(t)$]. Once invented, each machine can be produced at the fixed marginal cost ψ>0 in terms of the final good, which is normalized to $\psi \equiv 1-\beta$. Thus, total resources devoted to machine production at time t are

$$X(t) = (1-\beta)\left(\int_0^{N_L(t)} x_L(v,t)dv + \int_0^{N_K(t)} x_K(v,t)dv\right) \quad (2.3)$$

The innovation possibilities frontier (written in terms of labor- and capital-augmenting technologies) takes the form

$$\begin{cases} \dot{N}_L(t) = \eta_L N_L(t) S_L(t) \\ \dot{N}_K(t) = \eta_K N_K(t) S_K(t) \\ S_L + S_K = S \end{cases} \quad (2.4)$$

The total amount of scientists S is given exogenously.

Labor supply L is given exogenously.

Capital accumulates at an exogenous rate, that is,

$$\frac{\dot{K}(t)}{K(t)} = s_K \quad (2.5)$$

s_K is given exogenously. We will prove that equation (2.5) is the crucial assumption that led to the failure of Proposition 15.12. However, Acemoglu (2009, ch15.6) obviously did not recognize the importance of the capital accumulation function for the steady-state direction of technological progress in growth model. Instead, he emphasized the importance of state dependent parameters in the innovation function. Therefore, he argued that for the sake of simplifying the analysis, the capital accumulation function could be set in this form without affecting his desired conclusion.[5]

2.2. Proposal 15.12 is incorrect

The following is the proof process for the proposition to be invalid.

Let the final good as the numeraire, maximizing the profits of the final good enterprise to obtain

$$\begin{cases} p_K(t) = \dfrac{\partial Y(t)}{\partial Y_K(t)} = \left[\gamma_L \left(\dfrac{Y_L(t)}{Y_K(t)}\right)^{\frac{\varepsilon-1}{\varepsilon}} + \gamma_K\right]^{\frac{1}{\varepsilon-1}} \gamma_K \\ p_L(t) = \dfrac{\partial Y(t)}{\partial Y_L(t)} = \left[\gamma_L + \gamma_K \left(\dfrac{Y_K(t)}{Y_L(t)}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{1}{\varepsilon-1}} \gamma_L \end{cases} \quad (2.6)$$

$p_L(t)$ and $p_K(t)$ denote the prices of $Y_L(t)$ and $Y_K(t)$ respectively.

Since $Y_L(t)$ and $Y_K(t)$ are produced by firms in competitive market, which employ labor at wage $w(t)$ and capital at the rental price $r(t)$, and use the machine

[5] As he say "To state this result in the simplest possible way and to facilitate the analysis in the rest of this section, let us simplify the analysis and suppose that capital accumulates at an exogenous rate,…" (Acemoglu (2009, ch15.6,p520).

as intermediates input which prices are $p_{x_K}(v,t)$ and $p_{x_L}(v,t)$, respectively. The profit maximization of enterprises producing $Y_K(t)$ and $Y_L(t)$ yields $r(t)$, $w(t)$, $p_{x_L}(v,t)$ and $p_{x_K}(v,t)$, respectively, as follows

$$\begin{cases} r(t) = p_K(t)\dfrac{\partial Y_K(t)}{\partial K(t)} = p_K(t)\dfrac{\beta}{1-\beta}\left(\displaystyle\int_0^{N_K(t)} x_K(v,t)^{1-\beta}dv\right)K^{\beta-1} \\ w(t) = p_L(t)\dfrac{\partial Y_L(t)}{\partial L} = p_L(t)\dfrac{\beta}{1-\beta}\left(\displaystyle\int_0^{N_L(t)} x_L(v,t)^{1-\beta}dv\right)L^{\beta-1} \end{cases} \tag{2.7}$$

$$\begin{cases} p_{x_K}(v,t) = p_K(t)x_K(v,t)^{-\beta}K^{\beta} \\ p_{x_L}(v,t) = p_L(t)x_L(v,t)^{-\beta}L^{\beta} \end{cases} \tag{2.8}$$

Each machine $x_L(v,t)$ and $x_K(v,t)$ are exclusively produced by a monopolist with patent rights. Profit maximization implies that each monopolist sets the quantity of each machine and obtains the monopoly profit from each machine as follows,

$$\begin{cases} x_K = p_K(t)^{\frac{1}{\beta}}K \\ x_L = p_L(t)^{\frac{1}{\beta}}L \end{cases} \tag{2.9}$$

$$\begin{cases} \pi_{x_K} = p_K(t)^{\frac{1}{\beta}}K(1-\psi) = \beta p_K(t)^{\frac{1}{\beta}}K \\ \pi_{x_L} = p_L(t)^{\frac{1}{\beta}}L(1-\psi) = \beta p_L(t)^{\frac{1}{\beta}}L \end{cases} \tag{2.10}$$

Substituting equation (2.9) into equation (2.2) yields

$$\begin{cases} Y_L(t) = \dfrac{1}{1-\beta}p_L(t)^{\frac{1-\beta}{\beta}}N_L(t)L \\ Y_K(t) = \dfrac{1}{1-\beta}p_K(t)^{\frac{1-\beta}{\beta}}N_K(t)K(t) \end{cases} \tag{2.11}$$

Substituting equation (2.9) into equation (2.7) yields

$$\begin{cases} r(t) = \dfrac{\beta}{1-\beta}p_K(t)^{\frac{1}{\beta}}N_K(t) \\ w(t) = \dfrac{\beta}{1-\beta}p_L(t)^{\frac{1}{\beta}}N_L(t) \end{cases} \tag{2.12}$$

Substituting equation (2.9) into equation (2.6) yields

$$\begin{cases} p_K(t) = \left[\gamma_L \left(\dfrac{p_L(t)^{\frac{1-\beta}{\beta}} N_L(t)L}{p_K(t)^{\frac{1-\beta}{\beta}} N_K(t)K(t)}\right)^{\frac{\varepsilon-1}{\varepsilon}} + \gamma_K\right]^{\frac{1}{\varepsilon-1}} \gamma_K \\ p_L(t) = \left[\gamma_L + \gamma_K \left(\dfrac{p_K(t)^{\frac{1-\beta}{\beta}} N_K(t)K(t)}{p_L(t)^{\frac{1-\beta}{\beta}} N_L(t)L}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{1}{\varepsilon-1}} \gamma_L \end{cases} \tag{2.13}$$

From equation (2.12), the following equation (2.14) can be obtained

$$\frac{r(t)}{w(t)} = \left(\frac{p_K(t)}{p_L(t)}\right)^{\frac{1}{\beta}} \frac{N_K(t)}{N_L(t)} = p(t)^{\frac{1}{\beta}} \frac{N_K(t)}{N_L(t)} \tag{2.14}$$

The relative prices $p(t)$ of $Y_K(t)$ and $Y_L(t)$ can be obtained from equation (2.13) as following

$$p(t) \equiv \frac{p_K(t)}{p_L(t)} = \frac{\gamma_K}{\gamma_L}\left(\frac{Y_K(t)}{Y_L(t)}\right)^{\frac{-1}{\varepsilon}} = \left(\frac{\gamma_K}{\gamma_L}\right)^{\frac{\varepsilon\beta}{\sigma}} \left(\frac{N_K(t)K}{N_L(t)L}\right)^{\frac{-\beta}{\sigma}} \tag{2.15}$$

The salary of scientist is equal to the market value of their marginal output of innovation. It can be obtained from the function of innovation possibilities frontier and the monopoly profit of each machine patent as follows,

$$\begin{cases} w_{S_N}(t) = \eta_L N_L(t)\pi_{x_L} = \beta\eta_L p_L(t)^{\frac{1}{\beta}} N_L(t)L \\ w_{S_M}(t) = \eta_K N_K(t)\pi_{x_K} = \beta\eta_K p_K(t)^{\frac{1}{\beta}} N_K(t)K \end{cases} \tag{2.16}$$

The following discussion will be divided into two situations:

The first case, assuming that the wage rates of scientists $w_{S_N}(t)$ and $w_{S_M}(t)$ for innovation in both sectors are equal in the steady-state, it can be obtained that

$$1 = \frac{\eta_K}{\eta_L}\left(\frac{p_K(t)}{p_L(t)}\right)^{\frac{1}{\beta}} \frac{N_K(t)K}{N_L(t)L} \tag{2.17}$$

Substituting equation (2.15) into equation (2.17) yields

$$\frac{N_K(t)K(t)}{N_L(t)L} = \left[\left(\frac{\eta_K}{\eta_L}\right)^{-1} \left(\frac{\gamma_K}{\gamma_L}\right)^{-\frac{\varepsilon}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \tag{2.18}$$

From equation (2.18) the following equation can be obtained.

$$\frac{\dot{N}_L(t)}{N_L(t)} + \frac{\dot{L}}{L} = \frac{\dot{N}_K(t)}{N_K(t)} + \frac{\dot{K}}{K} \tag{2.19}$$

Given L a constant and substituting equations (2.4) and (2.5) into equation (2.19) yields

$$\eta_L[S - S_K(t)] - s_K = \eta_K S_K(t) \tag{2.20}$$

According to equation (2.20), the allocation of scientist at steady state is as follows,

$$\begin{cases} S_K^* = \dfrac{\eta_L S - s_K}{\eta_L + \eta_K} \\ S_L^* = \dfrac{\eta_K S + s_K}{\eta_L + \eta_K} \end{cases} \tag{2.21}$$

Substituting equation (2.21) into the functions of innovation possibility frontier (2.4), the technological progress in steady state is obtained as follows,

$$\begin{cases} \dfrac{\dot{N}_K}{N_K} = \eta_K S_K^* = \eta_K \dfrac{\eta_L S - s_K}{\eta_L + \eta_K} \\ \dfrac{\dot{N}_L}{N_L} = \eta_L S_L^* = \eta_L \dfrac{\eta_K S + s_K}{\eta_L + \eta_K} \end{cases} \tag{2.22}$$

Equations (2.22) provide the steady-state technical progress in the Acemoglu (2009, ch15.6) model. Obviously, under the reasonable assumptions about of parameters η_K, η_L, s_K and S, the steady-state technological progress of this model can include capital augmentation rather than pure labor-augmentation. Therefore, Proposition 15.12 does not hold.

The second case. Assuming that the wage rate of scientists in the innovation sector with labor complementary machine is always higher than the potential wage that can be provided by the innovation sector with capital complementary machine, i.e. $w_{S_N}(t) > w_{S_M}(t)$. At this time, all scientists are concentrated in the innovation sector with labor complementary machine, namely, there are $S_L(t) = S$, $S_K = 0$, but scientists will no longer move. At this point, technological progress is indeed purely labor-augmenting, but equation (2.18) is no longer valid under reasonable assumptions about parameters. Only when there happens to be the total of scientists $S = \frac{s_K}{\eta_L}$, then $\frac{N_K(t)K(t)}{N_L(t)L}$ be constant. That is to say, although technological progress at this time is purely labor-augmenting, the model has not arrived at a steady state under reasonable assumptions about parameters. Therefore, at this point, Proposition 15.12 also does not hold.

In summary, under the assumptions given by Acemoglu (2009, ch15.6), the Proposal 15.12 is incorrect.

2.3. Proposal 15.12 is the result of incorrect application of the neoclassical Euler equation.

How did Acemoglu come up with the incorrect Proposition 15.12? The following is an introduction to Acemoglu's proof process to prove that Acemoglu (2009, ch15.6) caused this result owing to directly apply the neoclassical Euler equation.

Substituting (2.13) into equation (2.12) yields

$$r(t) = \frac{\beta}{1-\beta}(\gamma_K)^{\frac{\varepsilon}{\sigma}} N_K(t) \left[\left((\gamma_L)^{\frac{\varepsilon}{\sigma}} \left(\frac{N_L(t)L}{N_K(t)K(t)} \right)^{\frac{\sigma-1}{\sigma}} + (\gamma_K)^{\frac{\varepsilon}{\sigma}} \right) \right]^{\frac{1}{\sigma-1}} \quad (2.23)$$

Substituting (2.18) into equation (2.23) yields

$$r(t) = \left[(\gamma_K)^{\varepsilon} \left(\frac{\eta_K}{\eta_L} + 1 \right) \right]^{\frac{1}{\sigma-1}} N_K(t) \quad (2.24)$$

Equation (2.24) indicates that the rental price of capital in the Acemoglu model is a function of $N_K(t)$. Until this step, Acemoglu's proof process was correct. Next, however, he did not provide any derivation process and directly applied the neoclassical Euler equation $\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(r(t) - \rho)$, and from this equation obtained that $r(t)$ must be a constant in steady state, and then from equation (2.24) to obtain $\frac{\dot{N}_K(t)}{N_K(t)} = 0$, finally resulting in Proposition 15.12. On the contrary, it can be inferred from equations (2.22) and (2.24) that $r(t)$ cannot be a constant in the steady state of the Acemoglu model, indicating that the neoclassical Euler equation does not hold under the assumptions of this model.

Why is the neoclassical Euler equation no longer valid in Acemoglu's (2009, ch15.6) model? This is because in this model, households' income can only be used for consumption, cannot accumulate assets (material capital or technological patents) to increase future consumption. Therefore, for a given time discount rate ρ, consuming all income is the only option for households to maximize lifetime utility. As a result, not only does the neoclassical Euler equation not hold, but also the model does not have the usual Euler equation at all. In fact, in this model, households do not need to make intertemporal choices at all!

3. Solving the Steady State of Knowledge Spillover Models

This section we prove that it will lead incorrect result in the knowledge spillover model to solve the steady state equilibrium if directly applying the neoclassical Euler equation. If only Acemoglu (2009, ch15.6) directly applies the neoclassical Euler equation and leads to incorrect results, it may be just an Acemoglu's oversight in order to simplify the derivation process. However, existing textbooks (Barro and Sala-i

Martin, 2004, ch6.3; Acemoglu, 2009, ch13.2; Aghion and Howitt, 2009, ch3.2.2) also directly apply neoclassical Euler equation to solve the steady-state equilibrium in the knowledge spillover models and also leads incorrect results, indicating that it may be an implicit common error in existing literature that believing the neoclassical Euler equation is valid in all growth models. Therefore, it is necessary to clearly point out this error and correct it.[6]

In the endogenous technological progress model with knowledge spillover innovation function, households do need to tradeoff between higher current consumption and higher future consumption. However, what affects the current and future consumption of households is not the allocation of income between consumption and investment, is the allocation of labor between the production of final output and research and development. Therefore, the neoclassical Euler equation still does not hold. We first give the correct process for solving the steady-state equilibrium of the model, followed by the solving process in Acemoglu (2009, ch13.2), and verify its results are not optimal.

3.1 Setup of the Model

$$Y(t) = \frac{1}{1-\beta}\left(\int_0^{N(t)} x(v,t)^{1-\beta} dv\right) L_E(t)^{\beta} \tag{3.1}$$

Innovation function

$$\dot{N}(t) = \eta N(t) L_R(t) \tag{3.2}$$

Total labor is given exogenously, therefore

$$L_E(t) + L_R(t) = L \tag{3.3}$$

Utility function of representative household is

$$u\big(c(t)\big) = \frac{c(t)^{1-\theta} - 1}{1-\theta}, 0 < \theta < 1 \tag{3.4}$$

The goal of a household is to maximize lifetime utility

$$\max_{c(t)} \int_0^{\infty} \exp(-(\rho - n)t)\, u\big(c(t)\big) dt \tag{3.5}$$

Labor can produce the final good to current consumption, as well as research and development to improve production technology and increase future consumption. The more labor used for the production of the final good, the higher the current consumption, but the less labor used for research and development, and the lower the future

[6] Due to the subtle differences in the assumptions of the final product production function in these models, there are slight differences in the results of the same model in these three textbooks. This section of the note is based on Acemoglu (2009, ch13.2). In Barro and Sala-i-Martin (2004, ch6.3), the parameter α is the $(1-\beta)$, λL *is the* L_E, $(1-\lambda)L$ is the L_R in this note, in Aghion and Howitt (2009, ch3.2.2), the parameter α is the $(1-\beta)$, ε is the θ in this note.

consumption. Households need to allocate labor to maximize lifetime utility. This is the crucial issue in knowledge spillover models.

The households' constraints are

$$w(t)[L - L_R(t)] + p(t)N(t) = Lc(t) \tag{3.6}$$

$p(t)$ is the licensing fee for each patent in each period. The wage rates $w(t)$ and $p(t)$ are determined by the market.

3.2. Correct processes for solving steady state

The Hamilton equation is

$$H(N, c, L_R, \nu, \lambda) = \exp(-\rho t)\frac{c(t)^{1-\theta} - 1}{1-\theta} + \nu(t)\eta N(t)L_R(t)$$
$$+\lambda(t)\{w(t)[L - L_R(t)] + pN(t) - c(t)\} \tag{3.7}$$

The first order conditions are

$$\begin{cases} \dfrac{\partial H}{\partial c} = \exp(-\rho t)\, c(t)^{-\theta} - \lambda(t) = 0 \\ \dfrac{\partial H}{\partial L_R} = \nu(t)\eta N(t) - \lambda(t)w(t) = 0 \\ \dfrac{\partial H}{\partial N} = \nu(t)\eta L_R(t) + \lambda(t)p(t) = -\dot{\nu}(t) \end{cases} \tag{3.8}$$

From equations (3.8) to yield

$$\begin{cases} -\rho - \theta\dfrac{\dot{c}(t)}{c(t)} = \dfrac{\dot{\lambda}(t)}{\lambda(t)} \\ \nu(t)\eta N(t) = \lambda(t)w(t) \\ \eta L_R(t) + \dfrac{\lambda(t)}{\nu(t)}p(t) = -\dfrac{\dot{\nu}(t)}{\nu(t)} \end{cases} \tag{3.9}$$

Substitute equation (3.2) into equation (3.9), and by simple calculations to obtain

$$\theta\frac{\dot{c}(t)}{c(t)} = \frac{\dot{w}(t)}{w(t)} + \frac{\eta N(t)p(t)}{w(t)} - \rho \tag{3.10}$$

From the profit maximization of the final goods enterprise to obtain following equation

$$\mathrm{w(t)} = \frac{\beta}{1-\beta}N(t) \tag{3.11}$$

From equation (3.11) to yield

$$\frac{\dot{w}(t)}{w(t)} = \frac{\dot{N}(t)}{N(t)} \tag{3.12}$$

Substituting equation (3.11) and (3.12) into equation (3.10) and using equation (3.2) to obtain

$$\theta\frac{\dot{c}(t)}{c(t)} = \frac{(1-\beta)\eta p(t)}{\beta} + \eta L_R(t) - \rho \tag{3.13}$$

In the case of free competition, the licensing fee $p(t)$ of a patent should be exactly equal to the monopoly profit of for producing a machine.

$$p(t) = \beta[L - L_R(t)] \tag{3.14}$$

Substituting equation (3.14) into equation (3.13) yields

$$\theta \frac{\dot{c}(t)}{c(t)} = (1-\beta)\eta L + \beta\eta L_R(t) - \rho \tag{3.15}$$

Equation (3.15) is the Euler equation for allocating households' labor in the knowledge spillover model.

Substituting $w(t)$ and $p(t)$ into the household constraint equation (3.6) yields

$$\beta\left(\frac{2-\beta}{1-\beta}\right)[L - L_R(t)]N(t) = Lc(t) \tag{3.16}$$

Since $L_R(t)$ is a constant in steady state, from equation (3.16) to yield

$$\frac{\dot{N}(t)}{N(t)} = \frac{\dot{c}(t)}{c(t)} = \eta L_R(t) \tag{3.17}$$

By combining equations (3.15) and (3.17), the optimal labor allocation in steady state can be obtained as following

$$\begin{cases} L_R^* = \dfrac{(1-\beta)\eta L - \rho}{\eta(\theta - \beta)} \\ L_E^* = \dfrac{\theta\eta L + \rho - \eta L}{(\theta - \beta)\eta} \end{cases} \tag{3.18}$$

Substituting (3.18) into equation (3.15), the growth rate of consumption in the steady-state is

$$\frac{\dot{c}(t)}{c(t)} = \frac{(1-\beta)\eta L - \rho}{\theta - \beta} \tag{3.19}$$

Equation (3.19) is the consumption growth rate of the knowledge spillover model solved by normal optimal control techniques in steady-state.

3.3. Process of Solving Steady State in Textbooks

The process of solving steady-state equilibrium in the textbook includes two steps. The first step applies that market equilibrium require wage rates for labor engaged in research and development and production of final good to be equal; The second step directly applies the neoclassical Euler equation. The first step is correct, but the latter step is incorrect.

The output of labor engaged in research and development is new patents, and their wage rates are equal to the market value of the patents of marginal output of innovation, i.e

$$w_R(t) = \frac{p(t)\eta N(t)}{r(t)} \tag{3.20}$$

$\eta N(t)$ is the marginal output of innovation，$p(t)$ is the licensing fees of patent in competition market. The present value of $p(t)\eta N(t)$ by discounting in interest rate $r(t)$ is the market value of the marginal output of R&D labor, which should be equal to the wage rate. This is the economic intuition of equation (3.20).

Since homogeneous labor can also produce the final good, the wage rates for different labor should be equal in equilibrium, i.e

$$w_R(t) = \frac{p(t)\eta N(t)}{r(t)} = w(t) = \frac{\beta}{1-\beta} N(t) \tag{3.21}$$

The licensing fee for a patent should be exactly equal to the monopolistic profit that can be obtained from monopolizing production of a machine by owning the patent. Substituting (3.14) into equation (3.21) yields

$$r(t) = (1-\beta)\eta L_E(t) \tag{3.22}$$

Until now, the derivation process is correct. However, in the following, the textbook directly applies the neoclassical Euler equation $\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(r(t) - \rho)$without any derivation process, and substituting equation (3.22) into it to obtain

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}\big((1-\beta)\eta L_E(t) - \rho\big) \tag{3.23}$$

Then substituting equation (3.17) into equation (3.23) yields

$$(1-\beta)\eta L - \rho = (1+\theta-\beta)\eta L_R(t) \tag{3.24}$$

Solved L_R^{**} from equation (3.24) as following

$$L_R^{**} = \frac{(1-\beta)\eta L - \rho}{(1+\theta-\beta)\eta} \tag{3.25}$$

Substituting equation (3.25) into $\frac{\dot{c}(t)}{c(t)} = \eta L_R(t)$ yields

$$\frac{\dot{c}(t)}{c(t)} = \frac{(1-\beta)\eta L - \rho}{1+\theta-\beta} \tag{3.26}$$

Equations (3.25) and (3.26) are the steady-state results given in the textbook.

By comparing equations (3.18) and (3.24), as well as equations (3.19) and (3.26), the following conclusions can be drawn. Since $(1+\theta-\beta) > (\theta-\beta)$, there is $L_R^{**} < L_R^{*}$. Therefore, the consumption growth rate given by equation (3.26) is also smaller than the rate given by equation (3.19). Under the same time discount rate ρ, a higher consumption growth rate represents a greater lifetime utility, so the consumption growth rate in equation (3.26) does not maximize lifetime utility. This proves that the results obtained in these textbooks are not optimal and incorrect. This is because under the assumption of the knowledge spillover model, household income can only be consumed and cannot be used for asset accumulation, the neoclassical Euler equation is also not valid.

4. Adjustment Costs of Investment and Generalized Euler Equation

The preceding two sections demonstrate through specific examples that directly applying the neoclassical Euler equation in existing textbooks lead to incorrect results. However, more importantly, this error leads to existing economic growth theory not being able to correctly understand Uzawa's (1961) steady-state theorem and cannot reveal the crucial determinants of the direction of technological progress in steady state. In fact, why technological progress after the Industrial Revolution is purely labor-augmenting in long-run is one of the main questions that Acemoglu (2002) plans to answer when establishing a framework with endogenous direction of technological progress. Acemoglu (2003) successfully obtained the steady state that technological progress is purely labor-augmenting under this framework. However, the Proposition 15.12 in Acemoglu (2009, ch15.6) indicates that he does not reveal the key determinants of direction of technological progress, and does not point out the prerequisite for purely labor-augmenting technical change in steady state.

This section plans to replace the capital accumulation function of exogenous growth in Acemoglu (2009, ch15.6) with a capital accumulation function that considers adjustment cost of investment. This in order to not only further reveals the close relationship between the neoclassical Euler equation and the capital accumulation function, but also clearly reveals under what circumstances technological progress will be purely labor-augmenting, providing a clear explanation for Uzawa's (1961) steady-state theorem, answering the question that Acemoglu (2002, 2003, 2009) attempted to answer but did not successfully do.

Keep other assumptions of the Acemoglu (2009, ch15.6) model unchanged, but replace the capital accumulation function with the form proposed by Irmen (2013) that includes adjustment cost of investment, as follows:

$$\dot{K}(t) = I(t)^{\alpha} \tag{4.1}$$

$I(t)$ denotes total investment, $0 < \alpha \leq 1$. When $\alpha < 1$, it indicates that the adjustment cost of investment is marginal increasing, when $\alpha = 1$, it indicates no adjustment cost of investment.

Households are the owners of corporate capital. The budget constraints of households are

$$I(t) + C(t) = w(t)L(t) + r(t)K(t) \tag{4.2}$$

Equation (4.2) indicates that household income includes wages of labor and interest of capital, while household expenses include consumption and investment.

The optimization problem of household is to maximize the lifetime utility under the constraints of equations (4.1) and (4.2). The Hamilton equation for this problem is

$$H(t,K,C,I,\mu)=\exp(-\rho t)\frac{C(t)^{1-\theta}-1}{1-\theta}+\mu(t)I(t)^{\alpha}$$
$$+\lambda(t)[w(t)L(t)+r(t)K(t)-I(t)-C(t)] \qquad (4.3)$$

From equation (4.3) to obtain the Euler equation as following

$$\theta\frac{\dot{C}(t)}{C(t)}=\alpha I(t)^{\alpha-1}r(t)+(1-\alpha)\frac{\dot{I}(t)}{I(t)}-\rho \qquad (4.4)$$

Equation (4.4) is a generalized Euler equation based on the capital accumulation function (4.1). When $\alpha=1$, equation (4.1) indicates that the investment has no adjustment cost. At this point, the Euler equation degenerates into the usual neoclassical Euler equation

$$\theta\frac{\dot{C}(t)}{C(t)}=r(t)-\rho \qquad (4.5)$$

Equations (4.5) and (4.4) indicate that the neoclassical Euler equation (4.5) is only a special case of equation (4.4), as the capital accumulation function in the neoclassical growth model is a special case of equation (4.1). Equation (4.4) indicates that the Euler equation is closely related to the capital accumulation function. When the capital accumulation function changes, the form of the Euler equation must be also changed. Ignoring this change and still directly applying the neoclassical Euler equation inevitably leads to incorrect results.

The following is to solve the steady-state technical progress of the Acemoglu (2009, ch15.6) model when the capital accumulation equation is equation (4.1) instead of $\dot{K}(t)=s_K K(t)$ in original textbook, in order to answer the question that Acemoglu (2002, 2003, 2009, ch15) attempted to answer.

Substituting the market equilibrium $w(t)$ and $r(t)$ in Section 2, i.e. equation (2.12), into the household budget constraint equation (4.2), we can obtain

$$I(t)+C(t)=\frac{\beta}{1-\beta}N_L(t)Lp_L(t)^{\frac{1}{\beta}}\left(1+\left(\frac{p_K(t)}{p_L(t)}\right)^{\frac{1}{\beta}}\frac{N_K(t)K}{N_L(t)L}\right) \qquad (4.6)$$

Since $\frac{N_K(t)K}{N_L(t)L}$, $p_L(t)$ and $\frac{p_K(t)}{p_L(t)}$ all are constants in steady state, from equation (4.6) to yield the results in steady state as follows,

$$\frac{\dot{N}_L(t)}{N_L(t)}=\frac{\dot{I}(t)}{I(t)}=\frac{\dot{C}(t)}{C(t)} \qquad (4.7)$$

According to equation (4.1) of the capital accumulation function, it can be obtained as following

$$\frac{\dot{K}(t)}{K(t)} = \alpha \frac{\dot{I}(t)}{I(t)} \tag{4.8}$$

Substituting equation (4.7) into equation (4.8) yields

$$\frac{\dot{K}(t)}{K(t)} = \alpha \frac{\dot{N}_L(t)}{N_L(t)} \tag{4.9}$$

Substituting equation (4.9) into equation (2.19) yields

$$\frac{\dot{N}_L(t)}{N_L(t)} = \frac{\dot{N}_K(t)}{N_K(t)} + \alpha \frac{\dot{N}_L(t)}{N_L(t)} \tag{4.10}$$

Substituting the innovation functions (2.4) into equation (4.10) yields

$$(1-\alpha)\eta_L[S - S_K(t)] = \eta_K S_K(t) \tag{4.11}$$

From equation (4.11) the steady-state allocation of scientists can be obtained as follows,

$$\begin{cases} S_K{}^* = \dfrac{(1-\alpha)\eta_L S}{\eta_K + (1-\alpha)\eta_L} \\ S_L{}^* = S - S_K{}^* = \dfrac{\eta_K S}{\eta_K + (1-\alpha)\eta_L} \end{cases} \tag{4.12}$$

Substituting equations (4.12) into the innovation functions (2.4) yields the steady-state technological progress of this model are as follows,

$$\begin{cases} \dfrac{\dot{N}_K}{N_K} = \dfrac{(1-\alpha)\eta_L \eta_K S}{\eta_K + (1-\alpha)\eta_L} \\ \dfrac{\dot{N}_L}{N_L} = \dfrac{\eta_L \eta_K S}{\eta_K + (1-\alpha)\eta_L} \end{cases} \tag{4.13}$$

Equations (4.13) provide that if the capital accumulation function is equation (4.1) but not $\dot{K}(t) = s_K K(t)$ the technological progress of Acemoglu (2009, ch15.6) model in steady-state. When $\alpha < 1$, for the reasonable parameter assumptions, capital-augmenting technological progress $\frac{\dot{N}_K}{N_K} > 0$, which indicates that technological progress is not purely labor-augmenting under all circumstances. However, when $\alpha = 1$, the capital accumulation function is $\dot{K}(t) = I(t)$, which is the capital accumulation function in the Acemoglu (2003) model, where $\frac{\dot{N}_K}{N_K} = 0$, which means technological progress is purely labor-augmenting. Therefore, the capital accumulation function

$\dot{K}(t) = I(t)$ or $\alpha = 1$ in equation (4.1), is a crucial condition for pure labor-augmenting technological progress in steady state. Although Acemoglu (2003) obtained the result that steady-state equilibrium with purely labor-augmenting technological progress, he did not recognize $\dot{K}(t) = I(t)$ is the key condition that leads to this result. On the contrary, Acemoglu (2003) argues that the asymmetry between capital and labor accumulation is the key condition. Due to L is constant, $\frac{\dot{K}(t)}{K(t)} = s_K > 0$ keep the asymmetry between labor and capital accumulation. So, to simplify the analysis process, he replaced $\dot{K}(t) = I(t)$ with $\dot{K}(t) = s_K K(t)$ in Acemoglu (2009, ch15.6) and thought the core result that technological progress is pure labor augmentation in steady state could still be obtained. However, unfortunately, this replacement changes the key assumptions and leads to the incorrect Proposition 15.12. This not only indicates that the neoclassical Euler equation cannot be directly applied to other situations, but also indicates that the asymmetry of capital and labor accumulation is not a sufficient condition for steady state technological progress to be purely labor-augmenting.

Why is $\dot{K}(t) = I(t)$ a key condition for purely labor-augmenting technological progress in steady-state? According to Li and Bental (2022), for a neoclassical production function with constant returns to scale, the direction of technological progress in steady state depends on the relative size of factor supply elasticities. When capital has infinite supply elasticity, steady-state technological progress is pure labor augmentation. The key function determining the elasticity of capital supply is the capital accumulation function, and the key parameter determining the magnitude of supply elasticity is α. When $\alpha < 1$, capital accumulation has only limited elasticity, so there can be capital augmentation in steady state. When $\alpha = 1$, capital accumulation has infinite elasticity, and steady-state technological progress cannot include capital augmentation. This is also the condition for the Uzawa's steady-state theorem to hold. Although the capital accumulation function $\dot{K}(t) = s_K K(t)$ indicates that capital can accumulate indefinitely, the supply elasticity is not infinite.

5. The Neoclassical Euler Equation and the Puzzle of Uzawa's Theorem

This section proves that the neoclassical Euler equation cannot be unconditionally true by contradiction. If the neoclassical Euler equation is valid for any growth model, it will inevitably lead to a dilemma of Uzawa's steady-state theorem.

Firstly, assume a neoclassical production function as follows

$$Y(t) = F[B(t)K(t), A(t)L(t)] \tag{5.1}$$

$B(t)$ and $A(t)$ represent capital-augmenting and labor-augmenting technologies, respectively. Define $k \equiv \frac{B(t)K(t)}{A(t)L(t)}$, from the properties of the neoclassical production function it can be obtained as follow,

$$Y(t) = A(t)L(t)F[k, 1] \tag{5.2}$$

In a competitive market, the market rental price $r(t)$ of capital is equal to the marginal return of capital, which can be obtained from equation (5.2) as follows,

$$r(t) = \frac{\partial Y(t)}{\partial K(t)} = B(t)\frac{\partial F}{\partial k} \tag{5.3}$$

Since $k \equiv \frac{B(t)K(t)}{A(t)L(t)}$ must be a constant, as $\frac{\partial F}{\partial k}$ also be unchanged in steady-state, from equation (5.3) we can obtain the following equation

$$\frac{\dot{r}(t)}{r(t)} = \frac{\dot{B}(t)}{B(t)} \tag{5.4}$$

It is worth noting that equation (5.4) is only derived from the neoclassical properties of the production function and the competitive market condition where the rental price of capital equals the value of marginal output of capital, and is not related to the accumulation function of capital and labor. If the neoclassical Euler equation $\theta\frac{\dot{C}(t)}{C(t)} = r(t) - \rho$ is valid for all growth models, then according to it, there must be $\frac{\dot{r}(t)}{r(t)} = \frac{\dot{B}(t)}{B(t)} = 0$, that is, technological progress cannot include capital augmentation, but only be pure labor augmentation. However, as Acemoglu (2009, ch2, p59) pointed out, there is no compelling reasons that technological progress can only be purely labor-augmenting, and it also has been proven that Acemoglu (2009, ch15.6)'s model can include capital augmentation in the steady state. This is the dilemma of Uzawa's steady-state theorem. However, the analysis process above shows that this dilemma is led from the incorrect opinion that the neoclassical Euler equation is the unconditional valid for any model.

6. Rental prices of capital and interest rates of investments

The previous analysis indicates that some famous textbooks directly apply the neoclassical Euler equation ignoring the specific environment of the model to lead to incorrect results. Why do these famous textbooks make such errors that are not difficult to be verified and have not been pointed out and corrected for a long time? This is indeed confusing. This note argues that the possible reason is that existing literature overlooks the difference between market interest rates for investments and market rental prices for capital. If income can be used for both current consumption and investment to increase future consumption, then investment must have sufficient returns to attract households to sacrifice current consumption. In a competitive market, this return is the market interest rate of the investment. As a factor of production, the rental price of capital in a competitive market must be equal to the market value of the marginal output of capital. This indicates that the interest rate of investment and the rental price of capital are not exactly the same. Since $r(t)$ represents the market rental price of capital, to distinguish them, we define $i(t)$ as the market interest rate of the investment.

When capital is accumulated by household's investment, the market rental price of capital is closely related to the market interest rate of investment. If one unit of investment is converted into one unit of capital, then the market interest rate of the investment must be equal to the marginal output value of capital in equilibrium, so the market interest rate is equal to the rental price of capital. If the capital accumulation through investment decreases marginally due to adjustment cost, in order to keep interest rate to be stable, the value of the marginal output of capital must continue to rise, and as a result the rental price of capital must also continue to rise. For a specific model such as Acemoglu (2009, ch15.6), owing to capital accumulation rate is given exogenously, income cannot be invested to increase future consumption, which means that consuming all income is the optical choice to maximize their lifetime utility, then investment no return, that is, $i(t) = 0$. However, $r(t) = \left[(\gamma_K)^{\varepsilon}\left(\frac{\eta_K}{\eta_L} + 1\right)\right]^{\frac{1}{\sigma-1}} N_K(t)$, as the capital-augmenting technological progress, $r(t)$ will continue to rise.

7 Concluding Remarks

The Euler equation is one of the core equations of economic growth models, but the neoclassical Euler equation is only a special result of the special capital accumulation function in the neoclassical growth model and cannot directly be applied to other growth models. This note points out and verifies that the existing well-known textbooks all make the common mistakes that directly apply the neoclassical Euler equation in other models and lead wrong results. This note further proves that if the neoclassical Euler equation holds unconditionally for any growth model will inevitably lead to the dilemma that any economic growth model exhibits a steady state to require technological progress to be purely labor-augmenting, even in fact the model can include capital labor-augmentation.

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