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Abstract

The core as a solution concept captures the state of allocations against which there exists no objection by any coalition of agents. Aumann and Maschler (1961) however emphasized the shortcomings of the objection mechanism and hence the core to further repercussions from agents. In that spirit, they introduced the bargaining set which later was adapted to the case of exchange economies by Mas-Colell (1989) and Vind (1992). In this paper, we consider a club economy where club goods are consumed parallel to private goods to capture the social aspects of consumption. We consider the framework proposed by Ellickson et al. (1999) in this regard and refer to the bargaining sets introduced in line with Mas-Colell and Vind as the local and global bargaining sets in our framework. We provide characterizations of the global bargaining set in terms of the size of the (counter-) objecting coalitions thereby extending the works of Schødt and Sloth (1994) and Hervés- Estvéz and Moreno-García (2015). We provide further interpretations of the global bargaining set in terms of several notions of robustly efficient states of a club economy.

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1 Introduction

The set of core allocations in an economy are allocations for which no subgroup of agents can re-contract among themselves to achieve a mutually beneficial outcome. This automatically implies that there exists some objection by a group of agents to any non-core allocation. However, such an objection is a one-time process, and the definition of the core has remained silent on the repercussions of an objection. Aumann and Maschler (1961) in their work thus argued for a mechanism based on "threats" and "counter-threats" and referred to the set of stable outcomes from such a mechanism as the "bargaining set". This two-step veto mechanism takes into consideration counter-objections to objections and hence reflects a more forward-looking behavior of agents in the economy. Since the end goal for any coalition of agents should be to reach a credible or stable outcome that reflects in some sense the market power of each player, we consider only objections that have no counter-objections as credible or justified.

Mas-Colell (1989) adopted the notion of bargaining set introduced by Aumann and Maschler (1961) for classical exchange economies and showed that under conditions similar to Aumann (1964) the bargaining set coincides with the set of Walrasian allocations. Further, the core equivalence theorem in Aumann (1964) implies that frivolous objections are not formed at all. However, such a definition of bargaining set for atomless economies is not universally accepted. Vind (1992) later proposed another notion of bargaining set where the objecting and counter-objecting allocation must be feasible for the entire set of agents and not just attainable for the respective objecting and counter-objecting coalitions. We follow the terminology introduced by Schødt and Sloth (1994) and later followed in Hervés- Estvéz and Moreno-García (2015) and refer to the bargaining set of Mas-Colell (1989) as the "local bargaining set"[1] and the one proposed by Vind (1992) as the "global bargaining set". We extend the works of Schødt and Sloth (1994) and Hervés- Estvéz and Moreno-García (2015) to the case of club economies where club goods are treated as articles of choice in a parallel fashion

[1]See Anderson et al. (1997), Liu and Zhang (2016), Liu (2017), Hervés-Estévez and Moreno-García (2018) and Beloso et al. (2018) for further works on local bargaining sets in exchange economies.

to private goods. Informally, one can think of club goods as an element belonging to the interior of a public-private spectrum. The main aspect of club economies involves the studying of social interaction alongside consumption. The initial literature on club goods (see Buchanan (1965), Tiebout (1956), Wiseman (1957)) has emphasized primarily finding "optimal sizes of clubs" and the "optimal number of clubs" based on individual decisions that involved comparing marginal benefits to congestion costs and externalities from other club members[2]. However, the analysis was restricted to the case of finitely many agents in the economy. This coupled with the indivisible nature of the club goods led to the absence of perfect competition from such economies. Ellickson et al. (1999) introduced a framework that dealt with both these issues. We thus resort to their framework for the purpose of. this paper.

Clubs in our framework are limited in size and can only allow for finitely many agents as members of the club. So any club type (gyms, swimming pools, libraries, etc.) is infinitesimal compared to the whole economy, although the number of clubs that can form in equilibrium can well be large enough. Each club endorses a non-Samuelson public project that is local to its members. Each agent is bestowed with an external characteristic (for example male or female) which is observable to other agents and inflicts some externality upon them. Each club is thus identified through the composition of its members and the public project it engages in. In the absence of the notion of money in our framework inputs required for the formation of such projects are contributed by agents from their initial endowments. Agents purchase memberships of clubs which basically grants them rights of entry to the club and use of the club project. Every membership embodies in itself a description of all relevant aspects such as the profile of characteristics of other members, the total number of members in question, the purpose of the club, and the resources necessary to form the club.

However, it is important to highlight some key differences between private goods and club goods. They are (i) prices for private goods can be always guaranteed to be strictly positive[3]. However, prices for club memberships can be positive, negative, or even zero[4]. (ii) A second key difference concerns the feasibility condition for an allocation (state). Compared to a classical exchange economy an economy with club

[2]See Scotchmer and Wooders (1987), Scotchmer (1996), Engl and Scotchmer (1996), Giles and Scotchmer (1997), and Scotchmer (1997) for further works on club economies.

[3]Due to the presence of monotonicity-like assumption.

[4]The prices for club memberships for an agent reflect the externality imposed by him or her on other members of the club. A strictly positive price indicates a negative externality imposed by an agent on other members of the club. On the other hand, a strictly negative price or subsidy indicates a positive externality exerted by the member.

goods requires that the consumption of private goods must additionally account for the inputs required for club projects, and (iii) A third obvious difference is the core indivisible nature of club goods vis-á-vis private goods. In such a setting we introduce restrictions on both objections and counter-objections (local and global) in the spirit of Schmeidler (1972) and obtain characterizations of both the local and global bargaining sets. We provide further characterizations of the global bargaining set in terms of the veto mechanism proposed by Vind (1972) and Hervés-Beloso and Moreno-García (2008).

The core of an economy with a continuum of agents is obtained by eliminating allocations that can be blocked by coalition(s) of agents. Underlying such coalition formation lies the requirement of communication between individual agents belonging to a coalition, which at times may be difficult. Particularly, a large coalition requires communication among a large number of agents which can be quite costly. Thus Schmeidler (1972) proposed that one should check for the robustness of the core to blocking by coalitions of small sizes. In his paper, he showed that given a coalition S which improves upon a non-core allocation and any positive ε less than the measure of S, one can always find a sub-coalition of measure ε to block a non-core allocation. Later, Vind (1972) extended Schmeidler's work and claimed that one can always find a coalition of measure ε, where ε is greater than zero and less than the measure of the grand coalition to block a non-core allocation. Thus, for a given ε close enough to the measure of the grand coalition, one can argue that core allocations are allocations that are agreed upon by the majority of a population (set of agents). In this paper, we show that the global bargaining set is robust to the size of both the objecting and counter-objecting coalition in the spirit of Schmeidler (1972). However, the same applies to the local bargaining set only when the objecting coalition is the grand coalition itself. We further show that the global bargaining set is robust to the scenario where the size of the counter-objecting coalitions is restricted ála Vind (1972).

The notions of core and bargainings set hinges on blocking by infinitely many coalitions for an atomless economy. A different notion of blocking was introduced by Hervés-Beloso and Moreno-García (2008). They considered blocking by the grand coalition only but exercised in a sequence of perturbed economies. These perturbed economies are constructed from the original economy by perturbing the initial endowment distribution of a coalition of individuals in the economy. Hervés-Beloso and Moreno-García (2008) referred to allocations that were non-dominated in such a sequence of perturbed economies as “robustly efficient allocations”. In the context of club economies, Bhowmik and Kaur (2023) provided a first-ever characterization of club equilibrium states in terms of approximately robustly efficient states. However, they showed that a similar characterization in terms of robustly efficient states is not possible. However, the reverse is not necessarily guaranteed. In this paper, we show robustly efficient states qualify as globally justified objections, however, the reverse fails to hold. One can attribute such a failure to analogous reasons for which the set of club equilibrium states fails to qualify as robustly efficient ones. In this paper, we posit a weaker notion of "sequentially robustly efficient states" and show that one can establish that the set of globally justified states is a strict subset of sequentially robustly efficient states. However, we remark that we are uncertain about the converse at this moment. Hence, we characterize the globally justified objections to a given state in our club economy in terms of approximately robustly efficient allocations introduced in Bhowmik and Kaur (2022) and vice versa thereby extending the work by Hervés-Estevez and Moreno-García (2015).

The paper is structured as follows. Section 2 lays out the economic model. Section 3 talks about the solution concepts in our framework thereby introducing the local and global bargaining set. Section 4 characterizes the objection and counter-objection mechanism in terms of the size of the (counter-) blocking coalition. Section 5 provides a characterization of the global bargaining set in terms of robustly efficient states.

2 Economic Model

We consider a pure exchange club economy where clubs are treated in a parallel fashion to private goods. Clubs in this framework are described through the profile of their members and the local activity they engage in. Agents purchase memberships of clubs, where memberships are dependent on the agent's characteristics and the club type. To begin with, we provide a simple example from Ellickson et al. (1999) to give a general idea about what do we mean by clubs in our setup.

Example 2.1. Consider an economy with a continuum of agents uniformly distributed over $[0, 10]$. There is only one private good in the economy and individual k is endowed with $e_k = k$ units of the private good. In addition, individuals have the opportunity to build and use a swimming pool either alone or as a member of a club. Constructing a swimming pool requires 6 units of private goods as input. The utility derived by a consumer from consuming x units of the private good is denoted by $u(x, 0) = x$ if no pool utility is enjoyed and $u(x, n) = \frac{4x}{n}$ if an individual shares a pool with n other members. No individual becomes a member of more than one club.

We normalize the price of private goods to 1, so consumer k has wealth k. Note that the preferences are that only the size of the club matters and not the composition. Hence, the price of a membership for a club of size n is the total inputs required for the pool split equally among the members, i.e. $\frac{6}{n}$. So if a consumer chooses no pool then he or she achieves a utility of k; while sharing a pool with n members yields a utility of

$$u(k - q_n; n) = \frac{4}{n}\left(k - \frac{6}{n}\right).$$

In equilibrium there is a stratification of the population based on wealth: the wealthiest consumers with wealth $(9, 10]$ have a pool of their own; consumers with wealth $(6, 9]$ share a pool with one other consumer (i.e. they are a member of a club of size 2) and the poorest consumers with wealth in $(0, 6]$ consume only the private good. Hence clubs of size greater than 2 don't form in equilibrium.

Economic Agents: The set of economic agents (A, Σ, λ) is a complete, finite, and atomless probability space, where A denotes the set of agents, the σ-algebra Σ denotes the set of allowable coalitions whose economic weight in the market is given by the measure λ.

Private Commodity Space: Let N denote the set of private commodities. We assume that the commodities are perfectly divisible[5]. Thus, the space of private goods is described as the N-dimensional Euclidean space $\mathbb{R}^N$. The consumption set of private commodities for each agent is encompassed by the non-negative orthant $\mathbb{R}^N_+$. Furthermore, let $\mathbb{R}^N_{++}$ denote the strictly positive elements of $\mathbb{R}^N$. For any two commodity bundles $x, y \in \mathbb{R}^N$, $x \geq y$ implies $x_i \geq y_i$ for all $i \in N$; $x > y$ implies that $x \geq y$, however $x \neq y$; and $x \gg y$ implies that $x_i > y_i$ for each $i \in N$. We denote $\|x\|_1 := \sum_{n=1}^{N} |x_n|$ for all $x \in \mathbb{R}^N$.

Club profile and project: Each agent is endowed with some external characteristics that are observable to other agents and inflict externality upon them. Examples of such characteristics can be sex, religion, appearance, etc. Let Ω denote the set of possible external characteristics that can be bestowed upon an agent. We assume that there exist only finitely many such characteristics. An element $\omega \in \Omega$ denotes one particular external characteristics. For each club, we identify the number of agents of each particular characteristics who are members of the club and call it the profile of the club. Each club endorses a public project local to the club. Let Γ denote the (finite) set of public projects available to a club. The set of such public projects constitutes an abstract set in the sense that there exists no common unifying order on these projects and each agent ranks them subjectively.[6] More formally, a **club type** as a pair (π, γ),

[5]Without loss of generality we assume that N also denotes the cardinality for the set commodities.

[6]See Mas-Colell (1980) for further discussion on such an abstract set of public projects. Examples of such projects can be a park, a medical center, and others.

where $\pi : \Omega \to \mathbb{Z}_+$ is a map[7] and $\gamma \in \Gamma$. We identify the composition of a club with such a map π and term it as **profile** of a club. Therefore, for a profile π of a club, $\pi(\omega)$ denotes the number of members of characteristic ω for every $\omega \in \Omega$. The total number of members in a club is denoted by $\|\pi\|_1 := \sum_{\omega\in\Omega} \pi(\omega)$. The set of possible club types in the economy is denoted by $Clubs = \{(\pi, \gamma)\}$, which is assumed to be finite. The input requirement of a club is denoted by $inp(\pi, \gamma)$. In the absence of money in our framework, club projects are financed jointly by members of the club by contribution from their initial endowments. This leads to $inp(\pi, \gamma) \in \mathbb{R}^N_+$.

Club memberships: We assume that agents are bestowed with external characteristics and they contribute towards club activities from their initial endowments. A **club membership** is a triplet given by $m = (\omega, \pi, \gamma)$. An agent with external characteristics ω becomes a member of a club only if $\pi(\omega) \geq 1$ i.e. the profile of the club allows memberships of that particular characteristic ω. Thus, club membership can be interpreted as an opportunity to become a part of a given club type for an individual of a given characteristic. We denote the set of all possible memberships by $\mathscr{M}$. Notice that $\mathscr{M}$ has finitely many elements. Individuals may choose to belong to any number of clubs or none. A map specifying the number of club memberships of each type is known as a list, where a **list** is a mapping $l : \mathscr{M} \to \{0, 1, 2, \dots\}$, where $l\,(\omega, \pi, \gamma)$ is the number of memberships of type $(\omega, \pi, \gamma) \in \mathscr{M}$. Further, the set of all possible *list* is defined as

$$Lists = \{l : l \text{ is a } list\}.$$

Letting $\mathbb{R}^{\mathscr{M}}$ be the set of all mappings from the set $\mathscr{M}$ to the real line, we can frequently view $Lists$ as a subset of $\mathbb{R}^{\mathscr{M}}$. Throughout the rest of the paper, we also assume that there is an exogenously given upper bound M on the number of memberships an individual may choose. We denote such a bounded set by

$$Lists_M = \{l \in Lists : \|l\|_1 \leq M\}$$

2.1 Club Economy

Definition 2.2. A **club economy** $\mathscr{E}$ is a **measurable mapping** $a \mapsto (\omega_a, X_a, e_a, u_a)$, satisfying the following conditions:

(1) Each agent a is associated with some characteristics $\omega_a \in \Omega$.

(2) The choice set of an agent a, denoted by X_a, specifies the set of all possible pairs of private goods and club membership consumption. Thus, $X_a \subset \mathbb{R}^N \times Lists$. For

[7] $\mathbb{Z}_+$ denotes the set of non-negative integers.

simplicity, we restrict our attention to only non-negative bundles of private goods. Moreover, club memberships embody a notion of excludability in themselves. Thus, the feasible consumption set for agent a is $X_a = \mathbb{R}^N_+ \times Lists_a$, where $Lists_a \subseteq Lists$ denotes the set of feasible lists for $a \in A$. We assume that an individual can only belong to a club type offering memberships with his/her external characteristics; formally, $l(\omega, \pi, \gamma) = 0$ if $l \in Lists$, $(\omega, \pi, \gamma) \in \mathscr{M}$ and $\omega \neq \omega_a$.

(3) The utility function of agent $a \in A$ is denoted by $u_a : X_a \to \mathbb{R}$. Further, the mapping $(a, x, l) \mapsto u_a(x, l)$ is jointly measurable with $u_a(\cdot, l)$ is continuous and strongly monotonic for all $a \in A$.

(4) The initial endowment of agent $a \in A$ is denoted by e_a. Additionally, the mapping $a \mapsto e_a$ is assumed to be integrable with $\int_A e_a d\lambda \in \mathbb{R}^N_{++}$. Endowments are said to be **desirable** in our framework if we have $u_a(e_a, 0) > u_a(0, l_a)$ for every agent $a \in A$ and $l_a \in Lists_a$. In simple words, an agent will prefer to stay put with his or her initial endowment compared to consuming only club memberships.

2.2 Club Consistency and Feasible States

In everyday life, club memberships are indivisible hence the need for a consistency requirement. Clubs in our framework are such that their sizes are limited and they have no market power. Therefore, juxtaposed to the continuum of agents in our model, the above requirement translates to finite club sizes. Since clubs are composed of members, individual memberships to clubs must be bounded and finite.[8] All these make clubs infinitesimal relative to the society. Also, external characteristics, as stated earlier, inflict externalities, but such externalities are confined within the clubs, thereby enabling the model to remain competitive.

A **state** of $\mathscr{E}$ is basically a measurable mapping $(f, l) : A \to \mathbb{R}^N_+ \times \mathbb{R}^{\mathscr{M}}_+$, which specifies for any agent $a \in A$ the amount of private good consumption f_a and the club membership vector l_a. It is said to be **individually feasible** if $(f_a, l_a) \in X_a$ λ-a.e. In a standard general equilibrium model, social feasibility requires market clearance for private goods. However, in this framework, an additional condition of consistency for the states is required. To this end, we introduce the concept of a consistent membership vector. Before that, we define a **coalition** as a measurable subset B of A whose measure

[8] All these restrictions on club sizes and individual memberships to be bounded along with a finite number of public goods makes the choices finite-dimensional as pointed out by Ellickson et al. (1999)

is positive. Furthermore, a **sub-coalition** of a coalition B is a coalition B' such that $B' \subseteq B$ For any coalition B, a choice function $\mu : B \to Lists$ and $j \in \mathbb{N}$, let

$$E^j_\mu(\omega, \pi, \gamma) = \{a \in B : \mu_a(\omega, \pi, \gamma) = j\}$$

denote the set of agents in the coalition B who choose j memberships of the type $(\omega, \pi, \gamma) \in \mathscr{M}$. It can be then duly observed that $j|E^j_\mu(\omega, \pi, \gamma)|$ denotes the number of 'j' memberships of type (ω, π, γ) bought by agents belonging to coaition B, where $|E^j_\mu(\omega, \pi, \gamma)|$ denotes the number of agents in B who subscribes to 'j' many memberships of type (ω, π, γ). Then the aggregate membership choice of the coalition B can be captured through the following sum

$$\overline{\mu}_B(\omega, \pi, \gamma) = \sum_{j=1}^{\infty} j|E^j_\mu(\omega, \pi, \gamma)|$$

The fact that any membership choice vector for a particular agent must belong to $Lists_M$ ensures that the above sum is defined and finite. We say that a choice function $\mu : B \to Lists$ is **integer consistent for the set *B*** if for each $(\pi, \gamma) \in \mathscr{M}$ there exists a non-negative integer $\alpha(\pi, \gamma)$ such that

$$\overline{\mu}_B(\omega, \pi, \gamma) = \alpha(\pi, \gamma) \cdot \pi(\omega)$$

for every $\omega \in \Omega$. Notice that $\alpha(\pi, \gamma)$ is independent of ω and it only means that $\alpha(\pi, \gamma)$ many clubs of type (π, γ) need to be formed to satisfy the demand for the coalition B. For example, consider a finite economy consisting of 200 males and 100 females. The set of external characteristics of this economy is $\Omega := \{male, female\}$. We assume that there exists only a single club type (π, γ), where $\pi(male) = 20$ and $\pi(female) = 10$, and γ is to construct and use a gym. Consider a coalition B of agents containing 100 males and 50 females. If each agent of B demands one membership of the gym then $\overline{\mu}_B(male, \pi, \gamma) = 100$ and $\overline{\mu}_B(female, \pi, \gamma) = 50$. Thus, $\alpha(\pi, \gamma) = 5$, which means 5 gyms need to be formed to satisfy the demand for the coalition B. However, if each male agent of B demands one membership of the gym and a female agent demands two memberships then $\overline{\mu}_B(male, \pi, \gamma) = 100$ and $\overline{\mu}_B(female, \pi, \gamma) = 100$. In this case, we cannot fulfill all individual demands by constructing any specific number of clubs. Now turning our attention to the case of a continuum economy where the set of agents denoted by $[0, 100]$ is endowed with the Lebesgue σ-algebra and Lebesgue measure λ. Considering any coalition B, $\lambda(E^j_\mu(\omega, \pi, \gamma))$ denotes the measure of agents in B who purchase j many memberships of type (ω, π, γ). Hence, $j\lambda(E^j_\mu(\omega, \pi, \gamma))$ is the measure of j memberships of type (ω, π, γ) bought by these agents. Then

$$\overline{\mu}_B(\omega, \pi, \gamma) = \int_B \mu_a(\omega, \pi, \gamma) d\lambda = \sum_{j=1}^{\infty} j\lambda(E^j_\mu(\omega, \pi, \gamma))$$

is the total number of memberships of type (ω, π, γ) being chosen by the agents in the coalition B and as before is well defined and finite. Note that the above-defined sum can well be a fraction as $0 \leq \lambda(E^j_\mu(\omega, \pi, \gamma)) \leq 1$ and thus $\alpha(\pi, \gamma)$ here is not necessarily an integer anymore. On this note, we provide the formal definition of a consistent membership vector.

Definition 2.3. Given a membership vector $\overline{\mu} \in \mathbb{R}^{\mathscr{M}}$, if for each club type $(\pi, \gamma) \in Clubs$, there exists a number $\alpha(\pi, \gamma) \in \mathbb{R}$ such that

$$\overline{\mu}(\omega, \pi, \gamma) = \alpha(\pi, \gamma)\, \pi(\omega)$$

for all $\omega \in \Omega$, then we call such a membership vector $\overline{\mu}$ **consistent**. For any coalition B, a choice function $\mu : B \to Lists$ is **consistent for B** if the corresponding aggregate membership vector $\overline{\mu}_B = \int_B \mu_a d\lambda$ is consistent.

Define

$$\mathscr{C}ons := \left\{\overline{\mu} \in \mathbb{R}^{\mathscr{M}} : \overline{\mu} \text{ is consistent}\right\}.$$

Recognized that $\mathscr{C}ons$ is a vector subspace of $\mathbb{R}^{\mathscr{M}}$. Since membership choices are always non-negative integers for any agent $a \in A$, aggregate membership choices for any coalition of agents are therefore restricted to $\mathscr{C}ons_+$, the positive part of $\mathscr{C}ons$.

Next, we will define conditions under which a state is feasible to society as a whole. Private goods need to achieve clearance over and above the already defined conditions of consistency and individual feasibility. This is guaranteed by material balance. Material balance states for a coalition of agents require their aggregate consumption of private goods and aggregate contribution to inputs for club projects on their behalf to match their aggregate initial endowments. We define the allocation rule as in Ellickson et al. (1999) and it takes the form given by $\frac{1}{\|\pi\|_1} inp(\pi, \gamma)$. Thus, each member of the club contributes the average input requirement for the club project. It then follows that an individual 'a' with external characteristic 'ω' and $l_a(\omega, \pi, \gamma)$ many memberships of the club type (π, γ) contributes $\frac{1}{\|\pi\|_1} inp(\pi, \gamma) \cdot l_a(\omega, \pi, \gamma)$ amount of input in total for the club type (π, γ). Summing over all possible membership choices gives the total input contribution made by agent 'a' and is denoted as

$$\tau(l_a) := \sum_{(\omega,\pi,\gamma)} \frac{1}{\|\pi\|_1} inp(\pi, \gamma)\, l_a(\omega, \pi, \gamma).$$

Definition 2.4. A state (f, l) is **feasible for a coalition B** if it abides by the following conditions:

- **Individual Feasibility**: $(f_a, l_a) \in X_a$ λ-a.e. on B;

- **Material Balance**: $\int_B f_a d\lambda + \int_B \tau(l_a) d\lambda = \int_B e_a d\lambda$; and
- **Consistency**: $\bar{l}_B \in \mathscr{C}ons$.

For $B = A$ then we simply call it **feasible**.

2.3 Role of Externalities

Our club economy is governed by the existence of only finitely many club types where a club type as defined before is described through the profile of a club and its club project. Now given one particular project $\widetilde{\gamma} \in \Gamma$ the cardinality of the set $\{(\pi, \widetilde{\gamma})\}$ may well be greater than one, i.e. there may exist multiple clubs with different number and composition of members providing for the same project $\widetilde{\gamma}$ for its members. We denote such a collection by

$$Clubs_{\widetilde{\gamma}} = \{(\pi_1, \widetilde{\gamma}), (\pi_2, \widetilde{\gamma}), \cdots (\pi_k, \widetilde{\gamma})\}$$

A central discussion in the club literature has been the analysis of marginal benefits and marginal costs a typical member faces from other members. In our framework, an agent considers the external characteristics of other members in conjunction with the analysis of marginal benefit to cost while becoming a member of a club.[9] The higher the number of other members the greater the congestion cost. On the other hand, since inputs are divided among the members, an additional member reduces an agent's contribution and thus provides additional benefit. Moreover, for an agent $a \in A$ with external characteristic ω faced with the choice of two different clubs (π_1, γ) and (π_2, γ) where $\|\pi_1\|_1 = \|\pi_2\|_1$ will choose to be a member of the club with the more desired profile for him or her. In reference to Example 2.1 we have in equilibrium, the highest-income individual chooses to build a pool for themselves (i.e. a club with a swimming pool where each high-income agent is the only member), middle-income individuals choose to share a pool with one other agent (i.e. a club where two middle-income agents are a member of a club with swimming pool) and the rest decide to be part of no club. One should further note from the above discussion that although an entire pool to themselves would yield more utility to middle-income agents, this would come at the expense of forgone private goods consumption. Since utility depends on both private and club goods these would reduce their overall utilities.

[9] It also happens that sometimes individuals more care about the composition of individuals in the profile of a club rather than the total number of $\|\pi_i\|_1 = \|\pi_j\|_2$ for some

3 Solution Concepts

Core states are states for which no coalition of agents can trade among themselves to achieve an improved state or utility. The bargaining set as an evolution over the core is an outcome of the two-step veto mechanism "objection and counter-objection". Thus, we can repeat similar exercises of finding coalitions of fixed sizes whether arbitrarily small or large, and check whether the objection counter-objection process is robust to the size of the blocking coalitions. To begin with, we shall first introduce the bargaining sets proposed by both Mas-Colell (1989) and Vind (1992). We shall refer to the bargaining set in the line of Mas-Colell (1989), the **local bargaining set** and the one defined parallel to Vind (1992) as the **global bargaining set**. We then propose restricted objection and counter-objection mechanisms in line with Schmeidler (1972) and Vind (1972), introduced by Schødt and Sloth (1994).

3.1 Bargaining Sets

Definition 3.1. A **local objection** to the state (f, l) in the economy $\mathscr{E}$ is a pair $(S,(g,\mu))$, where (g,μ) is a state which is feasible for the coalition S, such that $u_a(g_a,\mu_a) \geq u_a(f_a,l_a)$ for all $a \in S$ and

$$\lambda(\{a \in S : u_a(g_a,\mu_a) > u_a(x_a,l_a)\}) > 0.$$

A state (f, l) is said to be in the **core** of the economy $\mathscr{E}$ if there does not exist a local objection to it. Equivalently, the core of the economy $\mathscr{E}$ could be defined as the set of states (f, l), such that there do not exist any global objection to (f, l), where a **global objection** to the state (f, l) in the economy $\mathscr{E}$ is a pair $(S,(g,\mu))$ if (g,μ) is a feasible state and $(S,(g,\mu))$ is also a local objection to (f, l).

Remark 3.2. In our economy $\mathscr{E}$ there exists finitely many possible club types which are listed as $\{(\pi_1,\gamma_1),(\pi_2,\gamma_2),\cdots,(\pi_m,\gamma_m)\}$. Now, given any agent 'a' of characteristic 'ω' and $l_a \in Lists_a$ one can say that he or she purchases $l_a(\omega,\pi_i,\gamma_i)$ many memberships of the club type (π_i,γ_i) where $\pi_i(\omega) \geq 1$. Now each club contains finitely many agents who act as members of the club and impose an externality effect on other members of the club as discussed earlier. Hence, one can note that since individual 'a' can purchase only finitely many memberships, he or she faces an externality effect from only finitely many members. This is closely related to the framework of externality proposed by Hammond et al. (1989). However, there are some major differences. (i) Firstly, the trade of private goods is not restricted within the clubs. Thus whenever a coalition S blocks a state (f, l) through a state (g,μ), the trade of private goods takes place within the overall coalition S. Since S is a positive measurable set we have that $(f\mid_{A\setminus S}, g\mid_S, l\mid_{A\setminus S}, \mu\mid_S)$ differs from (f, l) over a positive measurable coalition of agents. This basically implies that a redistribution of the initial endowment distribution within the Aumann coalition S affects the average state of the economy as a whole. This stands in sharp contrast to the blocking by coalitions containing finitely many agents where the average allocation or consumption of the economy is unchanged as in Hammond et al. (1989). (ii) Secondly, the dependence of the agent's decision on a club profile basically reflects the externalities imposed by the characteristics of other members (finite) in the club on his or her preference. This is clearly a case of local externality as opposed to the notion of widespread externality where agents are concerned with the aggregate consumption as a whole.

Definition 3.3. A **local counter-objection** to a local objection $(S,(g,\mu))$ of a state (f,l) is a pair $(T,(h,\nu))$, where (h,ν) is a state which is feasible for the coalition T, such that $u_a(h_a,\nu_a) > u_a(g_a,\mu_a)$ for all $a \in S \cap T$; and $u_a(h_a,\nu_a) > u_a(f_a,l_a)$ for all $a \in T \setminus S$. A local objection is said to be a **local justified objection** if there is no local counter-objection against it. The **local bargaining set** of the economy $\mathscr{E}$, denoted by $\mathscr{B}^\ell(\mathscr{E})$, is the set of feasible states of the economy $\mathscr{E}$ against which there does not exist any local justified objection.

Definition 3.4. A **global counter-objection** to a global objection $(S,(g,\mu))$ of a state (f,l) is a pair $(T,(h,\nu))$, where (h,ν) is a state which is feasible for the coalition T, such that $u_a(h_a,\nu_a) > u_a(g_a,\mu_a)$ for all $a \in T$. A global objection is said to be a **global justified objection** if there is no global counter-objection against it. The **global bargaining set** of the economy $\mathscr{E}$, denoted by $\mathscr{B}^g(\mathscr{E})$, is the set of feasible states of the economy $\mathscr{E}$ against which there does not exist any global justified objection.

Remark 3.5. There are three key differences between the local and global bargaining sets:

- The global objection mechanism juxtaposed to the local notion not only proposes a feasible state for the objecting coalition but proposes a feasible state for the overall economy as a whole. Thus, objecting to any feasible state is harder in this scenario.

- The local counter-objection mechanism proposes a state that improves upon the original state for agents who belong only to the counter-objecting coalition and not the objecting one. The global counter-objection mechanism on the other hand proposes that all agents in the counter-objecting state must improve upon the objecting state necessarily. However, this follows quite trivially from the definition of global objection.

- The local and global bargaining set both contain the core of an atomless club economy $\mathscr{E}$. Now, Mas-Colell (1989) showed that the local bargaining set coincides with the set of equilibrium allocation, which generalizes the core-Walras equivalence theorem for exchange economies. However, a similar result fails to hold for the global bargaining set, and in fact, it is larger than the set of equilibrium allocations as remarked by Vind (1992).

3.2 The $\varepsilon\delta$-Bargaining Sets

Schødt and Sloth (1994) introduced a notion of $\varepsilon\delta$-bargaining set by restricting the sizes of both objecting and counter-objecting coalitions in tandem with the core characterization proposed by Schmeidler (1972). In what follows, we adapt this notion of restricted objection and counter-objection in the spirit of Schødt and Sloth (1994) and introduce the $\varepsilon\delta$-bargaining set for our club economy.

Definition 3.6. Given a $\delta > 0$, we say that $(S, (g, \mu))$ is a **δ-feasible local objection** to (f, l) if $(S, (g, \mu))$ is a local objection to (f, l) and $\lambda(S) \leq \delta$. Given an $\varepsilon > 0$ we say that $(T, (h, \nu))$ is an **ε-feasible local counter-objection** to $(S, (g, \mu))$ if $(T, (h, \nu))$ is a local counter-objection to $(S, (g, \mu))$ and $\lambda(T) \leq \varepsilon$. Given $\varepsilon, \delta > 0$, a δ-objection $(S, (g, \mu))$ to (f, l) is said to be **ε-justified** if there does not a ε-counter-objection $(T, (h, \nu))$ to it. The **local $\varepsilon\delta$-bargaining set**, denoted by $\mathscr{B}^{\ell}_{\varepsilon\delta}(\mathscr{E})$, is the set of states against which there exists no ε-justified δ-objection. The **global $\varepsilon\delta$-bargaining set**, denoted by $\mathscr{B}^{g}_{\varepsilon\delta}(\mathscr{E})$, is defined analogously.

4 The Sizes of (Counter-) Objecting Coalitions

Lemma 4.1. *Let (f, l) be a feasible state in the economy $\mathscr{E}$ and $(S, (g, \mu))$ be a local (global) objection to (f, l). Let $(T, (h, \nu))$ be a local (global) counter-objection to $(S, (g, \mu))$. Then for every $\varepsilon \in (0, \lambda(T))$, there exists a coalition R such that $\lambda(R) = \varepsilon$ and $(R, (h, \nu))$ forms a local (global) counter-objection to it.*

Proof. Let $\varepsilon \in (0, \lambda(T))$. Since $(T, (h, \nu))$ is a counter-objection to $(S, (g, \mu))$, we have

$$\int_T h_a d\lambda + \int_T \tau(\nu_a)\, d\lambda = \int_T e_a d\lambda \text{ and } \int_T \nu_a d\lambda \in \mathscr{C}ons.$$

We now define a vector-valued measure κ on $\Sigma|_T$ as

$$\kappa\left(B\right):=\left\{\left(\lambda\left(B\right),\int_B\left(e_a-h_a+\tau\left(\nu_a\right)\right)d\lambda,\int_B\nu_a d\lambda\right):B\in\Sigma|_T\right\}.$$

Therefore, we have

$$\kappa\left(\emptyset\right)=\left(0,0,0\right)\ \text{and}\ \kappa\left(T\right)=\left(\lambda\left(T\right),0,\int_T\nu_a d\lambda\right).$$

It follows from Lyapunov's convexity theorem that $\{\kappa\left(B\right):B\in\Sigma|_T\}$ is a convex set. For $\alpha=\frac{\varepsilon}{\lambda(T)}$, the convexity of $\{\kappa\left(B\right):B\in\Sigma|_T\}$ guarantees existence of a coalition $R\subseteq T$ such that

$$\kappa\left(R\right)=\left(1-\alpha\right)\cdot\kappa(\emptyset)+\alpha\cdot\kappa(T)=\left(\varepsilon,\ 0,\ \alpha\cdot\int_T\nu_a d\lambda\right).$$

Therefore, we have that $\lambda\left(R\right)=\varepsilon$ and the following conditions are satisfied

(i) $\int_R h_a d\lambda+\int_R\tau\left(\nu_a\right)d\lambda=\int_R e_a d\lambda$; and

(ii) $\int_R\nu_a d\lambda\in\mathscr{C}ons$, as $\mathscr{C}ons$ is a subspace.

Thus, $(R,(h,\nu))$ forms a counter-objection to $(S,(g,\mu))$. This completes the proof. □

Lemma 4.2. *Let (f,l) be a state of the economy $\mathscr{E}$. Any local (global) objection $(S,(g,\mu))$ to the allocation (f,l) is justified if and only if it is ε-justified.*

Proof. The 'If' part follows from the previous lemma and the 'Only if' part follows from the fact that ε-feasible counter-objection is also a counter-objection. □

Theorem 4.3. *Let $\mathscr{E}$ be a club economy. For any $\varepsilon,\delta>0$ we have $\mathscr{B}^\ell\left(\mathscr{E}\right)\subseteq\mathscr{B}^\ell_{\varepsilon\delta}\left(\mathscr{E}\right)$ and $\mathscr{B}^g\left(\mathscr{E}\right)\subseteq\mathscr{B}^g_{\varepsilon\delta}\left(\mathscr{E}\right)$.*

Proof. By abuse of notation, let us denote by $\mathscr{B}(\mathscr{E})$ both the local and global bargaining sets of the economy $\mathscr{E}$. Likewise, the notation $\mathscr{B}_{\varepsilon\delta}\left(\mathscr{E}\right)$ is employed to denote both local and global $\varepsilon\delta$-bargaining sets of the economy $\mathscr{E}$. Suppose $(f,l)\in\mathscr{B}\left(\mathscr{E}\right)\setminus\mathscr{B}_{\varepsilon\delta}\left(\mathscr{E}\right)$. This means that there exists an ε-justified δ-objection to the feasible state (f,l). Let $(S,(g,\mu))$ be an ε-justified δ-objection to (f,l), where $\lambda(S)\leq\delta$. By Lemma 4.2, it follows that $(S,(g,\mu))$ itself constitutes a justified objection to (f,l). This is a contradiction as $(f,l)\in\mathscr{B}(\mathscr{E})$. So $\mathscr{B}\left(\mathscr{E}\right)\subseteq\mathscr{B}_{\varepsilon\delta}\left(\mathscr{E}\right)$. □

Remark 4.4. Consider $\delta \geq \lambda(A)$. Lemma 4.2 guarantees that any justified objection is ε-justified. Thus, for the local bargaining set $\mathscr{B}^{\ell}(\mathscr{E})$ of the economy it can be readily verified that $\mathscr{B}^{\ell}_{\varepsilon\delta}(\mathscr{E}) \subseteq \mathscr{B}^{\ell}(\mathscr{E})$. Combined with Theorem 4.3 it can be inferred that $\mathscr{B}^{\ell}_{\varepsilon\delta}(\mathscr{E}) = \mathscr{B}^{\ell}(\mathscr{E})$. However, as established in Theorem 3 of Schødt and Sloth (1994) the equivalence fails to hold if $\delta < \lambda(A)$ for a typical exchange economy. It can be observed that their argument for an atomless economy corresponds to a special case for our club economy. Hence, we can claim that if $\delta < \lambda(A)$, $\mathscr{B}^{\ell}_{\varepsilon\delta}(\mathscr{E}) \neq \mathscr{B}^{\ell}(\mathscr{E})$ for our club economy.

Although the equivalence fails for the local bargaining set, we show that the global bargaining set $\mathscr{B}^{g}(\mathscr{E})$ is robust to any $0 \leq \delta \leq \lambda(A)$. This was a key observation made by Schødt and Sloth (1994). We provide the proof for our case below.

Theorem 4.5. *Let $\mathscr{E}$ be a club economy. For any $\varepsilon, \delta > 0$, we have $\mathscr{B}^{g}(\mathscr{E}) = \mathscr{B}^{g}_{\varepsilon\delta}(\mathscr{E})$*

Proof. By Theorem 4.3, we have $\mathscr{B}^{g}(\mathscr{E}) \subseteq \mathscr{B}^{g}_{\varepsilon\delta}(\mathscr{E})$. We show that $\mathscr{B}^{g}_{\varepsilon\delta}(\mathscr{E}) \subseteq \mathscr{B}^{g}(\mathscr{E})$. Let $(f,l) \in \mathscr{B}^{g}_{\varepsilon\delta}(\mathscr{E})$. Then, either (f,l) has no objection, and if there exists a δ-objection to it, there exists an ε-counter-objection to it. If (f,l) has no objection, then $(f,l) \in \mathscr{C}(\mathscr{E}) \subseteq \mathscr{B}^{g}(\mathscr{E})$. So let, $(S,(g,\mu))$ be an objection to (f,l). Lyapnov's convexity theorem guarantees the existence of a sub-coalition $\widetilde{S}$ of S such that $\lambda(\widetilde{S}) \leq \delta$ and $(\widetilde{S},(g,\mu))$ constitutes a δ-feasible objection to (f,l). Since $(f,l) \in B^{g}_{\varepsilon\delta}(\mathscr{E})$, there exists an ε-feasible counter-objection $(T,(h,\nu))$ to $(\widetilde{S},(g,\mu))$. It can be easily observed that $(T,(h,\nu))$ constitutes an objection to $(S,(g,\mu))$, which means that $(f,l) \in \mathscr{B}^{g}(\mathscr{E})$. This completes the proof of the theorem. □

Later, the objection mechanism was looked at under the specs of a normative notion, and whether the set of core states in the economy can be thought of as outcomes that emerge out of a "majority voting" mechanism. Vind (1972) in his paper formalized the question and provided an affirmative answer. He claimed that given any $\varepsilon > 0$, lying between zero and the size of the grand coalition, one can always construct a blocking coalition to a non-core allocation whose measure is less or equal to ε. Thus, for a large enough $\varepsilon > 0$, one can think of core states as outcomes of a majority voting mechanism. Following along the lines of Vind (1972), given any $\alpha > 0$ we show that it is sufficient to consider coalitions of size α to counter-object. We begin with the extension of Lemma 3.2 in Hervés-Estévez and Moreno-García (2008) and extend Theorem 3.2 of their paper.

Proposition 4.6. *Let (f,l) be a state in $\mathscr{E}$ and $(S,(g,\mu))$ a global objection to (f,l). Let $(T,(h,\nu))$ be a counter-objection to $(S,(g,\mu))$. Then for all $\alpha \in (0,1)$ there exists a global counter-objection $(E,(y,\kappa))$ to $(S,(g,\mu))$ such that $\lambda(E) = \alpha$.*

Proof. Since $(T,(h,\nu))$ is a counter-objection to $(S,(g,\mu))$ then it must follow that

(i) $(h_a,\nu_a) \in X_a$ λ-a.e. on T;

(ii) $u_a(h_a,\nu_a) > u_a(g_a,\mu_a)$ λ-a.e. in T;

(iii) $\int_T h_a d\lambda + \int_T \tau(\nu_a) d\lambda = \int_T e_a d\lambda$; and

(iv) $\int_T \nu_a d\lambda \in \mathscr{C}ons$.

By Lemma 4.1, without loss of generality, we assume that $\lambda(T) < \alpha < 1$. Define δ such that

$$\delta := 1 - \frac{\alpha - \lambda(T)}{\lambda(A \backslash T)}.$$

By the continuity of monotonicity of preferences, we can find a function $\zeta : T \to \mathbb{R}^N_+$ such that $u_a(\zeta_a,\nu_a) > u_a(g_a,\mu_a)$ λ-a.e. on T and

$$\int_T \zeta_a \, d\lambda = \int_T h_a \, d\lambda - z.$$

By Lemma 3.1 in Bhowmik and Kaur [4], there exists an allocation (ξ,ν') such that

(i) $u_a(\xi_a,\nu'_a) > u_a(g_a,\mu_a)$, λ-a.e. on T;

(ii) $\int_T \xi_a \, d\lambda = \int_T (\delta \cdot \zeta_a + (1-\delta)\cdot g_a) \, d\lambda$; and

(iii) $\int_T \nu'_a \, d\lambda = \int_T (\delta \cdot \nu_a + (1-\delta)\cdot \mu_a) \, d\lambda$.

Furthermore, by the Lyapunov convexity theorem, there exists a coalition $R \subseteq A \backslash T$ such that

(1) $\lambda(R) = (1-\delta)\lambda(A \backslash T)$;

(2) $\int_R \mu_a d\lambda = (1-\delta)\int_{A\backslash T} \mu_a d\lambda$; and

(3) $\int_R (g_a + \tau(\mu_a) - e_a) d\lambda = (1-\delta)\int_{A\backslash T}(g_a + \tau(\mu_a) - e_a) d\lambda$.

Lastly, let us define $E := T \cup R$ and an allocation $(y,\kappa) : A \to \mathbb{R}^N_+ \times \mathscr{R}^{\mathscr{M}}$ such that

$$(y_a,\kappa_a) = \begin{cases} (\xi_a, \, \nu'_a) & \text{for } a \in T, \\ \left(g_a + \frac{z\delta}{\lambda(R)}, \, \mu_a\right) & \text{for } a \notin T. \end{cases}$$

It can be readily verified that

$$\int_E \kappa_a \, d\lambda = \int_T \nu'_a \, d\lambda + \int_R \mu_a \, d\lambda.$$

Since $\int_T \nu'_a \in \mathscr{C}ons$ and $\int_A \mu_a d\lambda \in \mathscr{C}ons$, we have $\int_E \kappa_a d\lambda \in \mathscr{C}ons$. Therefore, we have

$$\int_E \tau(\kappa_a) \, d\lambda = \int_T \tau(\nu'_a) \, d\lambda + \int_R \tau(\mu_a) \, d\lambda.$$

Using the above equality, we derive that

$$\begin{aligned} \int_E [y_a + \tau(\kappa_a) - e_a] \, d\lambda &= \alpha \int_S [h_a + \tau(\nu'_a) - e_a] \, d\lambda + (1-\alpha) \int_A [g_a + \tau(\mu'_a) - e_a] \, d\lambda \\ &= 0 \end{aligned}$$

Thus, $(E, (y, \kappa))$ forms a global counter-objection to $(S, (g, \mu))$ and $\lambda(E) = \alpha$. □

In view of the above proposition, we characterize the bargaining set in terms of the restriction imposed on the counter-objection process. Let α-$\mathscr{B}^g(\mathscr{E})$ denote the set of feasible states in $\mathscr{B}^g(\mathscr{E})$ such that if they have global objection the they also are counter-objected by coalitions in $\mathscr{C}_\alpha = \{S \in \Sigma : \lambda(S) = \alpha\}$.

Theorem 4.7. *α-$\mathscr{B}^g(\mathscr{E}) = \mathscr{B}^g(\mathscr{E})$ for every $\alpha \in (0, 1)$.*

Proof. We just need to versify that $\mathscr{B}^g(\mathscr{E}) \subseteq \alpha$-$\mathscr{B}^g(\mathscr{E})$. To this end, let $(f, l) \in B^g(\mathscr{E})$. If (f, l) has no objection, then $(f, l) \in \mathscr{C}(\mathscr{E}) \subseteq \mathscr{B}^g(\mathscr{E})$. Thus, we assume that (f, l) has a global objection $(S, (g, \mu))$, which is globally counter-objected by $(T, (h, \nu))$. Then, by Proposition 4.6, we can guarantee the existence of a global counter-objection $(E, (y, \kappa))$ such that it forms a global counter-objection to $(S, (g, \mu))$ and $\lambda(E) = \alpha$. Hence, $(f, l) \in \alpha$-$\mathscr{B}^g(\mathscr{E})$. □

5 Robust Efficiency

Hervés-Beloso and Moreno-García (2008) proposed an alternative veto mechanism where one only allows for blocking by the grand coalition in a collection of infinitely perturbed economies formed by perturbing the initial endowments of a certain coalition of agents in the economy. In their paper, they show that competitive equilibrium allocations can be characterized as allocations that are not dominated in any of these infinite perturbed economies. Later, Hervés-Estevéz and Moreno-Garcia (2015) in their paper provided a characterization of justified objections through robustly efficient allocations in a typical exchange economy. We extend their characterization to the case of club economy and show that the global bargaining set consists of states that cannot be objected to by "approximately" robustly efficient states. To this end we define

For any $\mu : A \to \mathbb{R}^{\mathscr{M}}$, coalition S and real number $\alpha \in (0, 1]$, define

$$\mathcal{A}(\mu, S, \alpha) := \left\{ B \in \Sigma_S : \lambda(B) = \alpha\lambda(S) \text{ and } \int_B \mu_a \, d\lambda = \alpha \int_S \mu_a \, d\lambda \right\}.$$

By the Lyapunov convexity theorem, we have $\mathcal{A}(l, S, \alpha) \neq \emptyset$. For any feasible state (g, μ), coalitions S, B with $B \in \Sigma_S$ and real number $\alpha \in (0, 1]$, we define an economy $\mathscr{E}(S, B, g, \mu, \alpha)$ whose initial endowment state of private goods and club memberships are given below:

$$e_a(S, g, \alpha) := \begin{cases} e_a, & \text{if } a \in A \setminus S; \\ (1-\alpha)e_a + \alpha g_a, & \text{if } a \in S \ , \end{cases}$$

and

$$\beta_a(B, \mu) := \begin{cases} \mu_a, & \text{if } a \in B; \\ 0, & \text{if } a \in A \setminus B. \end{cases}$$

Definition 5.1. A state (g, μ) is said to be **dominated** in an economy $\mathscr{E}(S, B, g, \mu, \alpha)$ if there exists a state (h, ν) such that

(i) $u_a(h_a, \nu_a) > u_a(g_a, \mu_a)$ λ-a.e. on A;

(ii) $\int_A h_a d\lambda + \int_A \tau(\nu_a) \, d\lambda = \int_A e(S, g, \alpha) \, d\lambda + \int_A \tau(\beta_a(B, \mu)) \, d\lambda$; and

(iii) $\int_A \nu_a d\lambda, \ \int_B \mu_a d\lambda \in \mathscr{C}ons.$

A state (g, μ) is said to be **robustly efficient** if it is not dominated in any perturbed economy $\mathscr{E}(S, B, g, \mu, \alpha)$.

Proposition 5.2. *Let $\mathscr{E}$ be a club economy in which endowments are desirable and uniformly bounded from above. Further, let $(S, (g, \mu))$ be a global objection to the state (f, l) in the economy $\mathscr{E}$. If (g, μ) is a robustly efficient state of $\mathscr{E}$ then $(S, (g, \mu))$ is justified.*

Proof. Let (g, μ) be a robustly efficient state of $\mathscr{E}$. Suppose, by way of contradiction, we assume that $(S, (g, \mu))$ is not a justified objection. By Proposition 4.6, there exists a coalition T with $\lambda(T) < \lambda(A)$ and a state (h, ν) such that

(i) $u_a(h_a, \nu_a) > u_a(g_a, \mu_a)$, λ-a.e. on T;

(ii) $\int_T g_a d\lambda + \int_T \tau(\nu_a) d\lambda = \int_T e_a d\lambda$; and

(iii) $\int_T \nu_a d\lambda \in \mathscr{C}ons$.

By (A.4), there exists a function $\zeta : T \to \mathbb{R}^N_+$ and some $z \in \mathbb{R}^N_+ \setminus \{0\}$ such that $u_a(\zeta_a, \nu_a) > u_a(g_a, \mu_a)$, λ-a.e. on T, and

$$\int_T \zeta_a d\lambda = \int_T h_a d\lambda - z.$$

Choose an element $\alpha \in (0, 1]$. In view of Lemma 3.1 of Bhowmik and Kaur [4], we can find some state (φ, κ) such that

(A) $u_a(\varphi_a, \kappa_a) > u_a(g_a, \mu_a)$, λ-a.e. on T;

(B) $\int_T \varphi_a d\lambda = \int_T (\alpha\zeta_a + (1 - \alpha)g_a) d\lambda$; and

(C) $\int_T \kappa_a d\lambda = \int_T (\alpha\nu_a + (1 - \alpha)\mu_a) d\lambda$.

By the Lyapunov convexity theorem, we can choose some coalition $C \in \Sigma_{A \setminus T}$ such that $\lambda(C) = \alpha\lambda(A \setminus T)$ and

$$\int_C \mu_a d\lambda = \alpha \int_{A\setminus T} \mu_a d\lambda.$$

Consider a coalition B such that[10] $C \subseteq B$ and $\int_B \mu_a d\lambda \in \mathscr{C}ons$. Let us define two functions $\xi : A \to \mathbb{R}^N_+$ and $\eta : A \to \mathbb{R}^{\mathscr{M}}$ by letting

$$\xi_a := \begin{cases} \varphi_a, & \text{if } a \in T; \\ g_a + \frac{\alpha z}{\lambda(A\setminus T)}, & \text{if } a \in A \setminus T; \end{cases}$$

and

$$\eta_a := \begin{cases} \kappa_a + \mu_a, & \text{if } a \in T \cap (B \setminus C); \\ \kappa_a, & \text{if } a \in T \setminus (B \setminus C); \\ 2\mu_a, & \text{if } a \in (A \setminus T) \cap (B \setminus C); \\ \mu_a, & \text{if } a \in (A \setminus T) \setminus (B \setminus C). \end{cases}$$

In view of (A.4), it follows that $u_a(\xi_a, \eta_a) > u_a(g_a, \mu_a)$, λ-a.e. on A. Next, we now show that (g, μ) is dominated in the economy $\mathscr{E}(A \setminus T, B, g, \mu, \alpha)$. To see this, we note that

[10] One such B is A. Note also that if $\lambda(C)$ is sufficiently small, then $\lambda(B)$ can be chosen to be small.

$$
\begin{aligned}
\int_A (\eta_a - \beta_a(B,\mu))d\lambda &= \int_T \kappa_a d\lambda + \int_{B\setminus C} \mu_a d\lambda + \int_{A\setminus T} \mu_a d\lambda - \int_B \mu_a d\lambda \\
&= \alpha \int_T \nu_a d\lambda + (1-\alpha) \int_T \mu_a d\lambda + \int_{A\setminus T} \mu_a d\lambda - \int_C \mu_a d\lambda \\
&= \alpha \int_T \nu_a d\lambda + (1-\alpha) \int_T \mu_a d\lambda + \int_{A\setminus T} \mu_a d\lambda - \alpha \int_{A\setminus T} l\mu_a d\lambda \\
&= \alpha \int_T \nu_a d\lambda + (1-\alpha) \int_T \mu_a d\lambda + (1-\alpha) \int_{A\setminus T} \mu_a d\lambda \\
&= \alpha \int_T \nu_a d\lambda + (1-\alpha) \int_A \mu_a d\lambda.
\end{aligned}
$$

Since $\int_T \nu_a d\lambda$, and $\int_A \mu_a d\lambda$ both are in $\mathscr{C}ons$, we have

$$
\int_A (\eta_a - \beta_a(B,\mu))d\lambda \in \mathscr{C}ons.
$$

It follows that

$$
\int_A [\tau(\eta_a) - \tau(\beta_a(B,\mu))]d\lambda = \alpha \int_T \tau(\nu_a) d\lambda + (1-\alpha) \int_A \tau(\mu_a) d\lambda.
$$

Consequently,

$$
\begin{aligned}
&\int_A \xi_a d\lambda + \int_A [\tau(\eta_a) - \tau(\beta_a(B,l))]d\lambda - \int_A e_a(A\setminus T, g, \alpha) d\lambda \\
&= \int_T (\alpha\zeta_a + (1-\alpha)g_a) d\lambda + \int_{A\setminus T} g_a d\lambda + \alpha z + \alpha \int_T \tau(\nu_a) d\lambda + (1-\alpha) \int_A \tau(\mu_a) d\lambda \\
&- \int_T e_a d\lambda - \int_{A\setminus T} ((1-\alpha)e_a + \alpha g_a) d\lambda \\
&= \alpha \int_T [h_a + \tau(\nu_a) - e_a] d\lambda + (1-\alpha) \int_A [g_a + \tau(\mu_a) - e_a] d\lambda \\
&= 0
\end{aligned}
$$

This completes the proof. □

Remark 5.3. We consider Example 2.1 to show that the converse of the above result is not true. Define the state $(g,\mu) : A \to \mathbb{R}^N_+ \times \mathbb{R}^{\mathscr{M}}$ as

$$
g_a := \begin{cases} k & \text{if } a \in [0,6]; \\ k-3 & \text{if } a \in (6,9]; \\ k-6 & \text{if } a \in (9,10]. \end{cases}
$$

and

$$\mu_a := \begin{cases} (0,0) & \text{if } a \in [0,6]; \\ (0,1) & \text{if } a \in (6,9]; \\ (1,0) & \text{if } a \in (9,10]. \end{cases}$$

where the first coordinate in μ denotes the demand for membership of the club type "own pool" and the second coordinate denotes the demand for membership of the club type "shared pool with one other consumer". Recall that (g, μ) constitutes an equilibrium of the economy. Note that by Theorem (5.1) of Ellickson et al. (1999) (g, μ) constitutes a core state of the economy and has no objection against it. Now consider the coalition $S = (9, 10]$. It can be observed that $(S, (g, \mu))$ constitutes a global objection to the initial endowment state, and since it has no further objection to it one can claim that $(S, (g, \mu))$ constitutes a global justified objection to the initial endowment state. In what follows, we can proceed in similar lines of Example 3.7 of Bhowmik and Kaur (2023) to show that (g, μ) fails to qualify as a robustly efficient state of the economy.

The above non-equivalence highlights the fact that globally justified objections of an economy $\mathscr{E}$ are a much larger class than the class of robustly efficient states. This is primarily attributed to the weaker nature of the objection mechanism embodied in the definition of robustly efficient states. Thus, we propose a stronger notion of objection in the form of sequentially robust efficiency where any state that fails to qualify as a sequentially robust efficient state needs to be dominated in a sequence of economies and not just one. It then follows that such a class of states is a much larger class than robustly efficient states. We begin by establishing that any globally justified objection is sequentially robustly efficient. To this end, we start with the definition of a sequentially robustly efficient state. We know that the cardinality of the set $Clubs = \{(\pi, \gamma)\}$ is finite in our setup. Moreover, the maximum number of memberships that can be bought by one particular agent is bounded above by some exogenously given finite number M. So, denote by $\mathbb{Y} = \{y_1, y_2, \cdots, y_k\}$ the set of feasible membership choice vectors lying between the vectors $0\mathbf{1}$ and $M\mathbf{1}$ in $\mathbb{Z}_+^{\mathscr{M}}$, where $\mathbf{1} = (1, 1, \cdots, 1) \in \mathbb{Z}_+^{\mathscr{M}}$. Fix any feasible state (g, μ), we note that $\mu_a \in \mathbb{Y}$ for all $a \in A$. Thus, given a coalition S and $y_i \in \mathbb{Y}$, we define $S_i^{\mu} := \{a \in S : \mu_a = y_i\}$ as the set of agents whose membership vector is y_i. Further, define $\mathbb{I}_S^{\mu} := \{i : \lambda(S_i) > 0\}$.

Definition 5.4. A state (g, μ) is said to be **sequentially ε-dominated** if there exist a sequences $\{\mathscr{E}(S_n, B_n, g, \mu, \alpha_n) : n \geq 1\}$ of economies and a sequence $\{(h^n, \nu^n) : n \geq 1\}$ of states such that (g, μ) is dominated by (h^n, ν^n) in $\mathscr{E}(S_n, B_n, g, \mu, \alpha_n)$ and the following conditions are satisfied:

(i) there is a coalition R such that $u_a(\psi_a^n, \nu_a^n) > u_a(g_a, \mu_a)$ for all $\psi_a^n \in h_a^n + \mathbb{B}(0, \varepsilon)$ with $a \in R$ and $n \geq 1$; and

(ii) $\mathbb{I}_{B_n}^{\mu} = \mathbb{I}_{S_n}^{\mu}$ and $\lambda(B_n^i) \geq \alpha_n.\lambda(S_n^i)$ for all $n \geq 1$ and $i \in \mathbb{I}_{S_n}^{\mu}$; and

(iii) $\{(\alpha_n, \lambda(B_n)) : n \geq 1\}$ converges to $(0, 0)$.

A state (g, μ) is called **ε-sequentially robustly efficient** if it is not sequentially ε-dominated. Furthermore, a state (g, μ) is said to be **sequentially robustly efficient** if it is ε-sequentially robustly efficient for all $\varepsilon > 0$.

If we denote by $\mathrm{RE}_\varepsilon(\mathscr{E})$ the set of ε-sequentially robustly efficient states and $\widetilde{\mathrm{RE}}(\mathscr{E})$ the set of sequentially robustly efficient states then $\{\mathrm{RE}_\varepsilon(\mathscr{E}) : \varepsilon > 0\}$ is an accending sequence satisfying

$$\widetilde{\mathrm{RE}}(\mathscr{E}) = \bigcap \{\mathrm{RE}_\varepsilon(\mathscr{E}) : \varepsilon > 0\}.$$

Theorem 5.5. *Let $\mathscr{E}$ be a club economy and (f, l) be a feasible state of $\mathscr{E}$. Further, let $(S, (g, \mu))$ be a justified objection to (f, l). Then (g, μ) is a sequentially robustly efficient state of $\mathscr{E}$.*

Proof. Assume that $(S, (g, \mu))$ is a justified objection to the state (f, l). Thus, $(g, \mu) \in \mathscr{C}(\mathscr{E})$ and by Theorem 5.1 of Ellickson et al. (1999), we can claim that (g, μ) is a club equilibrium state of the economy $\mathscr{E}$. Let (p, q) be an equilibrium price. Suppose, by way of contradiction, that (g, μ) is not an ε- sequentially robustly efficient state for some $\varepsilon > 0$. This implies that there exists a sequence $\{\mathscr{E}(S_n, B_n, g, \mu, \alpha_n) : n \geq 1\}$ of economies and a sequence $\{(h^n, \nu^n) : n \geq 1\}$ of states such that (g, μ) is dominated by (h^n, ν^n) in $\mathscr{E}(S_n, B_n, g, \mu, \alpha_n)$, which means

(i) $u_a(h_a^n, \nu_a^n) > u_a(g_a, \mu_a)$, λ-a.e. on A;

(ii) $\int_A h_a^n \, d\lambda + \int_A \tau(\nu_a^n) \, d\lambda = \int_A e(S_n, g, \alpha_n) \, d\lambda + \int_A \tau(\beta_a(B_n, \mu)) \, d\lambda$; and

(iii) $\int_A \nu_a^n \, d\lambda, \ \int_A \beta_a(B_n, \mu) \, d\lambda \in \mathscr{C}ons$.

In addition, the following conditions are satisfied:

(iv) there is a coalition R such that $u_a(\psi_a^n, \nu_a^n) > u_a(g_a, \mu_a)$ for all $\psi_a^n \in h_a^n + \mathbb{B}(0, \varepsilon)$ with $a \in R$ and $n \geq 1$; and

(v) $\mathbb{I}_{B_n}^{\mu} = \mathbb{I}_{S_n}^{\mu}$ and $\lambda(B_n^i) \geq \alpha_n \lambda(S_n^i)$ for all $n \geq 1$ and $i \in \mathbb{I}_{S_n}$; and

(vi) $\{(\alpha_n, \lambda(B_n)) : n \geq 1\}$ converges to $(0, 0)$.

For each $n \geq 1$, there is a sub-coalition C_n of B_n such that $\lambda(C_n^i) = \alpha_n \lambda(S_n^i)$ for all $i \in \mathbb{I}_S$. Thus, we have

$$\int_{B_n} \mu_a \, d\lambda - \alpha_n \int_{S_n} \mu_a \, d\lambda = \int_{B_n \setminus C_n} \mu_a \, d\lambda.$$

Since $\{\lambda(B_n) : n \geq 1\}$ converges to 0, we have $\{q \cdot \int_{B_n \setminus C_n} \mu_a \, d\lambda : n \geq 1\}$ converges to 0. Let $n_0 \geq 1$ be an integer such that

$$q \cdot \int_{B_{n_0} \setminus C_{n_0}} \mu_a \, d\lambda < \frac{\varepsilon \lambda(R)}{2N}.$$

Letting

$$\delta := \frac{q}{\lambda(R)} \int_{B_{n_0} \setminus C_{n_0}} \mu_a \, d\lambda,$$

we note that $\delta < \frac{\varepsilon}{2N}$. It follows that $z_0 := (\delta, \cdots, \delta) \in \mathbb{B}(0, \varepsilon)$. Thus we consider $\widetilde{h} : A \to \mathbb{R}_+^N$ such that

$$\widetilde{h}_a = \begin{cases} h_a^{n_0} - z_0, & \text{if } a \in R; \\ \\ h_a^{n_0}, & \text{otherwise} . \end{cases}$$

As a consequence, we have

$$\int_A \widetilde{h}_a d\lambda = \int_A h_a^{n_0} d\lambda - \lambda(R) z_0.$$

It follows from (iv) that

$$p \cdot \widetilde{h}_a + q \cdot \nu_a^{n_0} > p \cdot e_a \geq p \cdot g_a + q \cdot \mu_a,$$

λ-a.e. on A. Thus,

$$(1 - \alpha_{n_0})(p \cdot \widetilde{h}_a + q \cdot \widetilde{\nu}_a) > (1 - \alpha_{n_0}) p \cdot e_a \text{ and } \alpha_{n_0}(p \cdot \widetilde{h}_a + q \cdot \widetilde{\mu}_a) > \alpha_{n_0}(p \cdot g_a + q \cdot \mu_a).$$

Consequently,

$$p \cdot \widetilde{h}_a + q \cdot \nu_a^{n_0} > p \cdot e(S_{n_0}, g, \alpha_{n_0}) + \alpha_{n_0} q \cdot \mu_a,$$

λ-a.e. on S_{n_0}, and hence

$$\int_{S_{n_0}} \left(p \cdot \widetilde{h}_a + q \cdot \nu_a^{n_0}\right) d\lambda > \int_{S_{n_0}} p \cdot e(S_{n_0}, g, \alpha_{n_0}) \, d\lambda + \alpha_{n_0} \int_{S_{n_0}} q \cdot \mu_a \, d\lambda.$$

This further implies that

$$\int_A \left(p \cdot \widetilde{h}_a + q \cdot \nu_a^{n_0}\right) d\lambda > \int_A p \cdot e\left(S_{n_0}, g, \alpha_{n_0}\right) d\lambda + \alpha_{n_0} \int_{S_{n_0}} q \cdot \mu_a \, d\lambda,$$

which immediately yields that[11]

$$\int_A \left(p \cdot h_a^{n_0} + q \cdot \nu_a^{n_0}\right) d\lambda - \lambda(R)\delta > \int_A p \cdot e\left(S_{n_0}, g, \alpha_{n_0}\right) d\lambda + \alpha_{n_0} \int_{S_{n_0}} q \cdot \mu_a \, d\lambda.$$

This is equivalent to

$$\int_A \left(p \cdot h_a^{n_0} + q \cdot \nu_a^{n_0}\right) d\lambda > \int_A p \cdot e\left(S_{n_0}, g, \alpha_{n_0}\right) d\lambda + \alpha_{n_0} \int_{S_{n_0}} q \cdot \mu_a \, d\lambda + \lambda(R)\delta.$$

Thus, we have that

$$\int_A \left(p \cdot h_a^{n_0} + q \cdot \nu_a^{n_0}\right) d\lambda > \int_A p \cdot e\left(S_{n_0}, g, \alpha_{n_0}\right) d\lambda + \int_{B_{n_0}} q \cdot \mu_a \, d\lambda. \qquad (5.1)$$

Now it can be observed that

$$\int_A p \cdot [\tau(\nu_a^{n_0}) - \tau(\beta_a(B_{n_0}, \mu))] \, d\lambda = p. \int_A \sum_{(\omega,\pi,\gamma)\in\mathscr{M}} \frac{1}{\|\pi\|_1} inp(\pi,\gamma)[\nu_a^{n_0}(\omega,\pi,\gamma) - \beta_a(B_{n_0},\mu)(\omega,\pi,\gamma)] \, d\lambda.$$

Now from (iii) we have that $\int_A (\nu_a^{n_0} - \beta_a(B_{n_0}, \mu)) \, d\lambda \in \mathscr{C}ons$ and thus there exists a real number $\delta(\pi, \gamma)$ such that

$$\int_A (\nu_a^{n_0}(\omega,\pi,\gamma) - \beta_a(B_{n_0},\mu)(\omega,\pi,\gamma)) \, d\lambda = \delta(\pi,\gamma)\pi(\omega).$$

The above equation along with the fact that $\sum_{\omega\in\Omega} \pi(\omega) q(\omega,\pi,\gamma) = p \cdot inp(\pi,\gamma)$[12] implies

$$\int_A p \cdot [\tau(\nu_a^{n_0}) - \tau(\beta_a(B_{n_0}, \mu))] \, d\lambda = \sum_{(\omega,\pi,\gamma)\in\mathscr{M}} \frac{1}{\|\pi\|_1} \left[\sum_{\omega\in\Omega} \pi(\omega) q(\omega,\pi,\gamma)\right] \delta(\pi,\gamma)\pi(\omega).$$

A simple algebraic manipulation yields

$$\int_A p \cdot [\tau(\nu_a^{n_0}) - \tau(\beta_a(B_{n_0}, \mu))] \, d\lambda = \sum_{(\omega,\pi,\gamma)\in\mathscr{M}} \frac{\pi(\omega)}{\|\pi\|_1} \sum_{\omega\in\Omega} \delta(\pi,\gamma)\pi(\omega) q(\omega,\pi,\gamma).$$

[11]Since $\|p\|_1 = 1$, we have $p \cdot z_0 = \delta$

[12]Refer to the definition of a club equilibrium.

We further observe that

$$\int_A p \cdot [\tau(\nu_a^{n_0}) - \tau(\beta_a(B_{n_0}, \mu))]\, d\lambda = \sum_{\omega\in\Omega} \frac{\pi(\omega)}{\|\pi\|_1} \sum_{(\omega,\pi,\gamma)\in\mathscr{M}} \delta(\pi,\gamma)\pi(\omega)q(\omega,\pi,\gamma),$$

which is equivalent to

$$\int_A p \cdot [\tau(\nu_a^{n_0}) - \tau(\beta_a(B_{n_0}, \mu))]\, d\lambda = \sum_{(\omega,\pi,\gamma)\in\mathscr{M}} \delta(\pi,\gamma)\pi(\omega)q(\omega,\pi,\gamma).$$

This further yields that

$$\int_A p\cdot[\tau(\nu_a^{n_0}) - \tau(\beta_a(B_{n_0}, \mu))]\, d\lambda = \sum_{(\omega,\pi,\gamma)\in\mathscr{M}} \int_A [\nu_a^{n_0}(\omega,\pi,\gamma) - \beta_a(B_{n_0},\mu)(\omega,\pi,\gamma)] q(\omega,\pi,\gamma)\, d\lambda.$$

which means

$$\int_A p\cdot[\tau(\nu_a^{n_0}) - \tau(\beta_a(B_{n_0}, \mu))]\, d\lambda = \int_A \sum_{(\omega,\pi,\gamma)\in\mathscr{M}} q(\omega,\pi,\gamma)[\nu_a^{n_0}(\omega,\pi,\gamma) - \beta_a(B_{n_0},\mu)(\omega,\pi,\gamma)]\, d\lambda,$$

equivalently,

$$\int_A p \cdot [\tau(\nu_a^{n_0}) - \tau(\beta_a(B_{n_0}, \mu))]\, d\lambda = \int_A q \cdot [\nu_a^{n_0} - \beta_a(B_{n_0}, \mu)] d\lambda.$$

It can be observed from the definition of the perturbed economy that

$$\int_A \beta_a(B_{n_0}, \mu)\, d\lambda = \int_{B_{n_0}} \mu_a\, d\lambda.$$

Thus, it follows from (ii) that

$$\int_A (p \cdot h_a^{n_0} + q \cdot \nu_a^{n_0})\, d\lambda = \int_A p \cdot e\left(S_{n_0}, g, \alpha_{n_0}\right)\, d\lambda + \int_{B_{n_0}} q \cdot \mu_a\, d\lambda.$$

This contradicts (5.1). □

Remark 5.6. Hervés-Estevez and Moreno-García (2015) characterized globally justified objections in terms of robustly efficient states. As already pointed out earlier dominating a state requires it to be dominated in a sequence of economies compared to only one in Hervés-Beloso and Moreno-García (2008). Therefore, the notion of blocking is much stronger in our case compared to the original definition of robust efficiency, thus, yielding that our class of sequentially robustly efficient states is larger than the set of robustly efficient states. Hence, it is unclear to us whether the converse of this result holds true or not.

In what follows, we adopt the notion of "approximate domination" and "approximately robustly efficient" states from Bhowmik and Kaur (2023)[13]. To this end, consider an arbitrary economy $\widetilde{\mathscr{E}}$ which is the same as $\mathscr{E}$ except for initial endowment being an arbitrary state $(\widetilde{e}, \widetilde{l}) : A \to \mathbb{R}^L_+ \times \mathbb{R}^{\mathscr{M}}$. We say that a state (g, μ) is **approximately dominated** by (h, ν) in $\widetilde{\mathscr{E}}$ if the following conditions are satisfied:

(i) $u_a(h_a, \nu_a) > u_a(g_a, \mu_a)$, μ-a.e. on A;

(ii) $\int_A h_a \, d\lambda + \int_A \tau(\nu_a) \, d\lambda = \int_A \widetilde{e}_a \, d\lambda + \int_A \tau(\widetilde{l}_a) \, d\lambda$; and

(iii) $\int_A (\nu_a - \widetilde{l}_a) \, d\lambda \in \mathscr{C}ons$.

In the next lemma, we shall observe that a global objection $(S, (g, \mu))$ is justified if and only if the state (g, μ) is an approximate robustly efficient allocation. To this end, we first define a perturbed economy and an approximately robustly efficient state.

Definition 5.7. A state (g, μ) in $\mathscr{E}$ is said to be **approximately robustly efficient** if it is not approximately dominated in $\mathscr{E}(S, B, g, \mu, \alpha)$ for every $0 < \alpha \leq 1$, and coalitions B, S with $B \in \mathcal{A}(\mu, S, \alpha)$.

Theorem 5.8. *Let $(S, (g, \mu))$ be a global objection to the state (f, l) in the economy $\mathscr{E}$. Then $(S, (g, \mu)$ is justified if and only if (g, μ) is approximately robustly efficient.*

Proof. Let $(S, (g, \mu))$ be a justified global objection to the state (f, l) in the economy $\mathscr{E}$. Thus, (g, μ) belongs to the core of the economy $\mathscr{E}$. Hence, from Theorem 5.1 of Ellickson et al. (1999), we can claim that (g, μ) is a club equilibrium state of the economy $\mathscr{E}$. Suppose, by way of contradiction, let us assume that (g, μ) is not approximately robustly efficient. This means that there exists some $\alpha \in (0, 1]$, coalition S and sub-coalition $B \in \mathcal{A}(l, S, \alpha)$ such that (g, μ) is dominated in $\mathscr{E}(S, B, g, \mu, \alpha)$. Thus, there exist a state (h, ν) such that

(i) $u_a(h_a, \nu_a) > u_a(g_a, \mu_a)$, λ-a.e. on A;

(ii) $\int_A h_a \, d\lambda + \int_A \tau(\nu_a) \, d\lambda = \int_A e\,(S, g, \alpha) \; d\lambda + \int_A \tau(\beta_a(B, \mu)) \, d\lambda$; and

(iii) $\int_A (\nu_a - \beta_a(B, \mu)) d\lambda \in \mathscr{C}ons$.

[13]It is important to highlight the need for an approximation of the notion of robust efficiency. Ellickson et al. (1999) commented that due to the non-convex nature of the club goods, the second welfare theorem fails to hold for a club goods economy. This in conjunction with the contrapositive of Remark 3.2 in Hervés-Beloso and Moreno-García (2008) establishes that one cannot define robust efficiency for such economies. Thus, Bhowmik and Kaur (2023) proposed such an approximate notion.

Let $(p, q) \in \mathbb{R}^L_+ \times \mathbb{R}^{\mathscr{M}}$ be an equilibrium price corresponding to the state (g, μ). From (i), we have

$$p \cdot h_a + q \cdot \nu_a > p \cdot e_a \geq p \cdot g_a + q \cdot \mu_a, \lambda\text{-a.e. on } A.$$

Thus,

$$(1-\alpha)(p \cdot h_a + q \cdot \nu_a) > (1-\alpha)p \cdot e_a \text{ and } \alpha(p \cdot h_a + q \cdot \nu_a) > \alpha(p \cdot g_a + q \cdot \mu_a).$$

It follows that

$$(p \cdot h_a + q \cdot \nu_a) > (1-\alpha)p \cdot e_a + \alpha p \cdot g_a + \alpha q \cdot \mu_a, \lambda\text{-a.e. on } S.$$

Hence,

$$\int_A (p \cdot h_a + q \cdot \nu_a)\, d\lambda > \int_A p \cdot e_a(S, g, \alpha)\, d\lambda + \int_A q \cdot \beta_a(B, \mu)\, d\lambda. \tag{5.2}$$

From (iii), we have

$$\int_A [\nu_a - \beta_a(B, \mu)]\, d\lambda \in \mathscr{C}ons.$$

Then, invoking analogous arguments of the proof of Theorem 5.5, we obtain $\int_A p \cdot [\tau(\nu_a) - \tau(\beta_a(B, \mu))d\lambda = \int_A q \cdot (\nu_a - \beta_a(B, \mu))$. Thus, from (ii) we have

$$\int_A (p \cdot h_a + q \cdot \nu_a)\, d\lambda = \int_A (p \cdot e_a(S, g, \alpha) + q \cdot \beta_a(B, \mu))\, d\lambda,$$

which contradicts Equation (5.2). Hence, (g, μ) is an approximately robustly efficient state.

Conversely, let $(S, (g, \mu))$ be a global objection to the state (f, l) such that (g, μ) is an approximately robustly efficient state. Suppose, by way of contradiction, that $(S, (g, \mu))$ is not justified. Hence, by Proposition 4.6, there exists a coalition T with $\lambda(T) < \lambda(A)$ and a state (h, ν) such that

(i) $u_a(h_a, \nu_a) > u_a(g_a, \mu_a), \lambda$-a.e. on T;

(ii) $\int_T h_a\, d\lambda + \int_T \tau(\nu_a)\, d\lambda = \int_T e_a\, d\lambda$; and

(iii) $\int_T \nu_a\, d\lambda \in \mathscr{C}ons$.

Now, by Assumption (A.4), there exists a function $\zeta : T \to \mathbb{R}^N_+$ and some $z \in \mathbb{R}^N_+ \setminus \{0\}$ such that $u_a(\zeta_a, \nu_a) > u_a(g_a, \mu_a)$, λ-a.e. on T, and

$$\int_T \zeta_a d\lambda = \int_T h_a d\lambda - z.$$

Let $\alpha \in (0, 1]$. Applying Lemma 3.1 of Bhowmik and Kaur (2023), we can find some state (φ, κ) such that

(I) $u_a(\varphi_a, \kappa_a) > u_a(g_a, \mu_a)$, λ-a.e. on T;

(II) $\int_T \varphi_a \, d\lambda = \int_T (\alpha\zeta_a + (1-\alpha)g_a) \, d\lambda$; and

(III) $\int_T \kappa_a \, d\lambda = \int_T (\alpha\nu_a + (1-\alpha)\mu_a) \, d\lambda$.

Choose a coalition $B \in \mathcal{A}(\mu, S, \alpha)$. We define two functions $\xi : A \to \mathbb{R}^N_+$ and $\eta : A \to \mathbb{R}^{\mathscr{M}}$ by letting

$$\xi_a := \begin{cases} \varphi_a, & \text{if } a \in T; \\ g_a + \frac{\alpha z}{\lambda(A \setminus T)}, & \text{if } a \in A \setminus T; \end{cases}$$

and

$$\eta_a := \begin{cases} \kappa_a, & \text{if } a \in T; \\ \mu_a, & \text{if } a \in A \setminus T. \end{cases}$$

We note that

$$\begin{aligned} \int_A (\eta_a - \beta_a(B,\mu)) \, d\lambda &= \int_T \kappa_a \, d\lambda + \int_{A \setminus T} \mu_a \, d\lambda - \int_B \mu_a \, d\lambda \\ &= \alpha \int_T \nu_a \, d\lambda + (1-\alpha) \int_T \mu_a \, d\lambda + \int_{A\setminus T} \mu_a \, d\lambda - \alpha \int_{A \setminus T} \mu_a \, d\lambda \\ &= \alpha \int_T \nu_a d\lambda + (1-\alpha) \int_A \mu_a d\lambda \end{aligned}$$

Since $\int_T \nu_a \, d\lambda, \int_A \mu_a \, d\lambda \in \mathscr{C}ons$ and $\mathscr{C}ons$ is a linear space, we have

$$\int_A (\eta_a - \beta_a(B,\mu)) \, d\lambda \in \mathscr{C}ons.$$

Furthermore,

$$\int_A [\tau(\eta_a) - \tau(\beta_a(B,\mu))] \, d\lambda = \alpha \int_T \tau(\nu_a) \, d\lambda + (1-\alpha) \int_A \tau(\mu_a) \, d\lambda.$$

Finally, note that the following equations guarnatees material balance for the state (ξ, η)

$$\begin{aligned} &\int_A \xi_a \, d\lambda + \int_A [\tau(\xi_a) - \tau(\eta_a(B,\mu))] \, d\lambda - \int_A e_a(A \setminus T, g, \alpha) \, d\lambda \\ &= \int_T (\alpha\zeta_a + (1-\alpha)g_a) \, d\lambda + \int_{A \setminus T} g_a \, d\lambda + \alpha z + \alpha \int_T \tau(\nu_a) \, d\lambda + (1-\alpha) \int_A \tau(\mu_a) \, d\lambda \\ &- \int_T e_a d\lambda - \int_{A \setminus T} ((1-\alpha)e_a + \alpha g_a) \, d\lambda \\ &= \alpha \int_T [h_a + \tau(\nu_a) - e_a] \, d\lambda + (1-\alpha) \int_A [g_a + \tau(\mu_a) - e_a] \, d\lambda \\ &= 0 \end{aligned}$$

Hence, (g, μ) is not approximately robustly efficient, which is a contradiction. This completes the proof. □

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