What determines the Direction of Technological Progress(2023.11.16)?

Li, Defu and Bental, Benjamin

School of Economics and Management, Tongji University

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What determines the Direction of Technological Progress? *

Defu Li[1]
School of Economics and Management, Tongji University
Benjamin Bental[2]
Department of Economics, University of Haifa.

Abstract: Technological progress relates not only to its rate but also to its direction and bias. The rate has been analyzed by the endogenous technical change models and the bias has been analyzed by the directed technical change model, but the determinants of the direction has not been uncovered yet. This paper tries to provide a framework where the equilibrium direction of technical change can be studied to reveal its determinants in steady state. The crucial introductions of the framework are the generalized factor accumulation processes and a generalized production function. The generalizations admit unrestricted factor supply elasticities and marginal transformation rates of production factors into effective factors. These turn out to be the key determinants of the steady-state direction of technological progress, whereby that direction tends towards the factor with the relatively smaller supply elasticity or marginal transformation rate. The neoclassical growth model as a special case cannot admit capital-augmenting technical change in steady state because of the assumptions of capital with infinite supply elasticity and constant marginal transformation rate. Similarly, labor-augmenting technical change cannot be part of a Malthusian steady state owing to labor with infinite supply elasticity. These results provide new insights for understanding the puzzle of Uzawa's (1961) steady-state theorem and indicates that the size and change of factor supply elasticities may be crucial elements in explaining the Malthusian trap before the industrial revolution, the Kaldor (1961) facts afterwards and the industrial revolution itself.

Key Words: Economic Growth, Direction of Technological Progress, Factor Supply Elasticities, Marginal transformation rates of Factors to Effective Factors, Uzawa Steady-State Theorem, Industrial Revolution, Adjustment Cost

JEL: E13; E25; O33; O41

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[1] School of Economics and Management, Tongji University, 1500 Siping Road, Shanghai, P.R. China, tjldf@tongji.edu.cn.

[2] Department of Economics, University of Haifa, 199 Aba Khoushy Ave., Mount Carmel, Haifa, Israel, bbental@econ.haifa.ac.il.

What determines the Direction of Technological Progress?

1 Introduction

Technological progress relates not only to its rate but also to its direction and bias. Over two decades ago, Acemoglu (2002) pointed out that the rate of technological progress has been deeply studied, but the direction and bias are not addressed,[1] and proposed a directed technical change model where the equilibrium bias can be studied, but the equilibrium direction still cannot be analyzed. The evidence is that, when Acemoglu (2009, ch15.6) used the directed technical change model to investigate why technical change might be purely labor-augmenting, he not only failed to resolve this issue but also arrived at the incorrect Proposition 15.12 (the proof see Li, 2016). Why Acemoglu's (2002) directed technical change model can analyze the bias but cannot analyze the direction of technical change, because the determinants of the direction are overlooked unintentionally and implicitly for analyzing the bias. Suppose that the aggregate production function is $Y = F(BK, AL)$, the bias of technical change refers to the impact of the change of relative technology on the relative marginal product of the two factors, namely $\frac{\partial(MP_K/MP_L)}{\partial(B/A)}$, while the direction of technical change refers to the ratio of the rates of factor-augmenting technical change, namely ($\frac{\dot{B}/B}{\dot{A}/A}$). Although they are closely related, their determinants in steady state are different. Therefore, after twenty years, we have to say that what determines the direction of technological progress is still an unresolved issue. However, uncovering the determinants of direction of technological progress is necessary to shed light on several facets of economic growth theory.

First is the Uzawa's (1961) steady-state theorem that Acemoglu (2009, ch15.6) investigated but came to an incorrect proposition 15.12. The theorem (elegantly and intuitively re-proven by Schlicht 2006) points out that unless the production function is Cobb-Douglas, the steady state of the neoclassical growth model requires technological progress to be purely labor-augmenting. Even the models with the endogenous choice of capital- and labor-augmenting technical change converge to a steady-state path with only labor-augmenting technical change (Acemoglu, 2003; Irmen,2017; Irmen and

[1] Acemoglu (2002) said in the beginning, "There is now a large and influential literature on the determinants of the aggregate technical progress (see, among others, Romer (1990), Segerstrom, Anant and Dinopoulos (1990), Grossman and Helpman (1991), Aghion and Howitt (1992), Young (1993)). This literature does not address questions related to the ***direction*** and ***bias*** of technical change."

Tabakovic, 2017). However, as has already been pointed by Acemoglu (2009, pp.59), "This result is very surprising and troubling, since there are no compelling reasons for why technological progress should take this form." Schlicht (2006) also said, "This is an extremely restrictive, and consequently extremely decisive, requirement, establishing that steady-state growth is a highly singular and therefore highly improbable case." However, what assumptions lead to the extreme result. Solow (1970, p. ix) remarks that growth theory "is set up to generate labor-augmenting technical change because that is the only kind that combines with the *other standard assumptions* to permit a steady state." In trying to identify these "other standard assumptions" Irmen (2018) points out that features like a micro-founded research sector that employs resources to generate new capital- or labor-augmenting technological knowledge, a micro-founded production sector possibly operating under imperfect competition, or the presence of knowledge spill-overs are certainly *not* among them. Yet, Irmen (2018) does not spell out what *does* belong to the set of these "standard assumptions" which are responsible for the inadmissibility of capital augmentation as a feature of the steady-state technological progress in the neoclassical growth model. Why do these seemingly "standard assumptions" lead to extreme results? Are these "standard assumptions" really sufficient "standard"? Whether these "standard assumptions" can be replaced by more general, realistic and reasonable assumptions to relax the extremely restrictive requirement of the Uzawa's steady-state theorem?

Second, although technological progress has been the crucial factor of economic growth throughout human history, empirical research on growth shows that humanity has stagnated in the Malthusian trap for a long time (Maddison, 2003; Ashraf and Galor, 2011). That is, technological progress resulted in increased population density and land productivity, but hardly in improved labor productivity. However, the stylized facts of modern economic growth show that after the industrial revolution, per capita income and per capita capital continued to grow, while the ratio of capital to output or capital productivity remained basically unchanged (Kaldor, 1961). What factors are responsible for the fact that technological progress had almost no effect on labor productivity before the industrial revolution, but afterwards improved labor productivity exclusively? What caused the transition from one regime to the other?

These questions require a framework where the equilibrium direction of technical change can be studied. We present a framework for this purpose by introducing the generalized factor accumulation processes and a generalized production function. The former does not limit the supply elasticities of factors, and the latter does not limit the marginal transformation rate of production factors to effective production factors. These generalizations avoid the implicit restriction on the direction of technological progress imposed by the "standard assumptions" related to the factor accumulation functions and the production function in the existing growth models. The paper starts from a reduced form growth model, directly applies the definitions of the steady-state equilibrium, direction of technological progress, and factor supply elasticities, to reveal the determinants of the direction of technological progress assuming the steady-state equilibrium exists. The determinants are identified for the first time in the literature, providing the core conclusion of this paper: ***the direction of technological progress depends on the relative size of factor supply elasticities and the marginal transformation rates of production factors to effective production factors, and tends to the factor with the smaller supply elasticity and marginal transformation rate. In particular, when the supply elasticity of a factor is infinite and its marginal transformation rate is constant, technological progress will not improve its productivity at all in the steady-state equilibrium.*** Since the results arise from a reduced form growth model, they are very general and independent from other settings of a specific growth model.

Following this general statement, the previous reduced form model is extended to include the micro foundation. This allows us to prove the existence of a steady-state equilibrium under the derived generalized factor accumulation processes and generalized production function. Just as in the reduced form model, the steady-state direction of technological progress in the micro-founded environment also depends on the relative size of supply elasticities and the marginal transformation rates of factors to effective factors and tends to the factor with the smaller ones. The neoclassical growth model as a special case cannot admit capital-augmenting technical change in steady state because of the assumptions of capital with infinite supply elasticity and constant marginal transformation rate. Similarly, labor-augmenting technical change cannot be part of a Malthusian steady state owing to labor with infinite supply elasticity.

At the same time, the dynamic analysis of the complete model reveals comprehensively the three factors that affect the direction of technological progress, namely the relative price proposed by Hicks (1932), the relative market size proposed by Acemoglu (2002), and the relative marginal productivity of scientists. While these three factors affect the direction of technological progress by affecting the relative wage rates of scientists and the sectoral distribution of scientists in different sectors, the direction of technological progress in turn also affects these three factors through changing the relative technologies. As a result of their interaction, finally, it is the determinants mentioned earlier that determine the direction of technological progress in steady state.

Based on the results of the framework the paper provides new insights for understanding the puzzle of the Uzawa's steady-state theorem, and indicates that the size and change of factor supply elasticities may be crucial elements in explaining the Malthusian trap before the industrial revolution, the Kaldor (1961) facts afterwards and the industrial revolution itself. In addition, it suggests an alternative explanation for the seeming contradiction between the continuous decline in investment good prices and the Kaldor facts that has attracted much attention in recent years (Grossman et al., 2017).

The plan of the paper is as follows. Section **2** discusses the related literature; Section **3** derives the determinants of technological change in a reduced form growth model with the generalized factor accumulation processes and a generalized production function; Section **4** develops a specific growth model which provides micro-foundations to the reduced form model. It is used to reveal the particular determinants of technological change and verifies the conclusions of the reduced form growth model; Section **5** focuses on some applications; Section **6** discusses the impact on the core conclusions of alternative formulations of the factor accumulation processes and the innovation possibilities frontier; Section **7** contains concluding remarks.

2 Related Literature

Although there exists more in-depth literature on the *rate* of technological progress than on its *direction and bias*, the literature on the latter preceded that on the former. Early in 1932, Hicks (1932) pointed out that changing in relative prices of factors may affect that direction and bias. Brozen (1953) too pointed out that the direction of technological progress was endogenously determined by economic forces. However, lacking a dynamic growth framework (to be developed by Solow a few years later), the early contributions only provided intuitive insight into the direction and bias of technological progress, and could not provide formal analyses on its micro mechanism in transitional dynamics and its determinants in steady state, and did not distinguish between the direction and bias.

The neoclassical growth models (Solow, 1956; Swan, 1956; Cass, 1965; Koopmans,1965) prove that technological progress is the key factor of economic growth in the long run. However, the rate of technological progress in the neoclassical growth models is exogenous and its direction turns out to be a cumbersome issue as manifested by the puzzle of Uzawa's steady-state theorem. The puzzle is why that seems to be very standard assumptions lead to extremely restrictive requirement about the direction of technical change in steady state. Understanding the puzzle of the Uzawa's steady-state theorem has been a main motivation in research on the direction of technological progress.

The first attempt to overcome the problem of the neoclassical growth model is the induced innovation literature (Samuelson, 1965; Drandakis and Phelps, 1966). These papers assume that firms can simultaneously adopt both capital- and labor-augmenting technological progress, but face a trade-off between the two kinds of technological progress summarized by an innovation possibilities frontier (von Weizsäcker, 1962; Kennedy, 1964). Assuming that the firms' goal is to reduce current total cost, these papers prove that the steady-state growth with constant income shares can be obtained only if firms choose pure labor-augmentation. Thereby that literature restates Hicks' (1932) intuitive analysis, that relative price changes encourage technological progress to be biased towards the relatively scarce factors. This seems to provide a reasonable economic rationale for the Uzawa's steady-state theorem and was even considered to have saved the neoclassical growth model at that time (Nordhaus, 1973). However, there are two serious deficiencies in the induced innovation literature. First, it lacked micro foundations: in a perfectly competitive market structure, it was not clear who undertook the R&D activities nor how they were financed and priced. Moreover, firms were assumed to be cost-minimizers rather than profit-maximizers (Nordhaus, 1973; Acemoglu, 2001, 2003). Second, the determinants of the direction of technological progress in steady state were not provided. Changing relative prices of factors of production indeed would induce invent to economizing the use of the relatively expensive factor, but it also would incentivize to increase the supply of the expensive factor. In equilibrium, it may be uncertain whether it promotes the technological progress directed to this factor or increases the supply of this factor.

Due to the defects of the induced innovation literature, the research on the direction and bias of technological progress has been silent for almost 30 years. It was reactivated with the emergence of the endogenous technological progress models (Romer, 1990; Grossman and Helpman, 1991; Aghion and Howitt, 1992). By extending the technological progress from one dimension to two, Acemoglu (2002) developed a framework in which the direction and bias could be endogenized. Within the extended framework, Acemoglu (2002) proposed a market size effect as another key factor affecting the direction and bias of technological progress besides the price effect of Hicks (1932). Unlike Acemoglu, some authors (Funk, 2002; Irmen, 2017; Irmen et al.,

2017) constructed models with endogenous capital- and labor-augmenting technological progress based on the framework of perfect competition, which also provided micro foundations for the direction and bias of technological progress. Although these contributions make up for the lack of micro mechanism in the early induced innovation literature, they still do not distinguish between the direction and the bias of technological progress, and do not recognize their difference in the determinants. Although their model's direction of technological progress in steady state is endogenous, they do not reveal their determinants. For example, for the aggregate production function $Y = F(BK, AL)$, in steady state, $\frac{\dot{B}/B}{\dot{A}/A} = 1$ in Acemoglu (2002), while $\frac{\dot{B}/B}{\dot{A}/A} = 0$ in Acemoglu (2003), Irmen (2017) and Irmen et al. (2017). However, they did not point out what led to this different outcome. [2] This shows that, while profit incentives can answer who undertakes innovation and how it is financed and priced, they cannot identify the factors determining the relative investments in different innovations.

In recent years, some authors have recognized that the standard assumptions concerning the factor accumulation functions and production function in the neoclassical growth model are reasons why the steady-state equilibrium requires that technological progress be purely labor-augmenting. Accordingly, some papers obtain capital-augmenting technological progress in the steady-state equilibrium by expanding the capital accumulation processes (Sato, 1996; Sato et al., 1999, 2000; Irmen, 2013), and others by expanding the production function (Grossman et al., 2017; Casey and Horii, 2022). However, the purpose of these papers is only to admit capital-augmenting technological progress in the steady state, rather than identify the determinants of the direction of technological progress.

Unlike Acemoglu's (2002) directed technical change model where the equilibrium bias of technical change can be studied, this article develops a framework for analyzing the equilibrium direction technological progress to reveal its determinants. This framework synthesized existing works to extend the Acemoglu's model mainly in the following aspects: ***First***, it introduces investment adjustment costs to generate generalized capital and labor accumulation processes which do not implicitly limit the elasticity of factor supplies. The Sato (1996) and Irmen (2013) specifications just correspond to different specific adjustment cost functions respectively. ***Second***, it

[2] In fact, it is precisely because of $\frac{\dot{B}/B}{\dot{A}/A} = 1$ that Acemoglu (2002) can give the determinants of the equilibrium relative technology (B/A).

provides a generalized production function which does not restrict the marginal transformation rates of production factors to effective production factors. The production functions used by Grossman et al. (2017) and Casey and Horii (2022) can be regarded as two special cases of the generalized formulation; ***Third*,** it replaces the Acemoglu (2002, 2003) perfect scientists mobility assumption by an adjustment process affecting the movement of scientists from one sector to another. This modifies and supplements the micro mechanism of underlying the direction of technological progress. It can clearly and simultaneously reveal three forces that affect technological progress in the transitional dynamic process: relative prices of factors, relative market size, and relative marginal productivity of scientist.

3 The Direction of Technological Progress in a Reduced Form Growth Model

3.1. The Environment

The economy that this article focuses on includes two material inputs and one final output. The output can be used for consumption or investment to increase future consumption. The output depends on the quantity of input factors and the factor-augmenting technologies (or the quality of factors). The quantity of each factor and each factor-augmenting technology both can be accumulated or improved by investing economic resources. These resources may be the final output (i.e. investment), or some specific resources (for example, scientists specializing in research and development). The core issue of this article is the determinants of the relative rates of the two factor-augmenting technological progress in steady-state, that is, the determinants of the direction of technological progress, especially in the decentralized decision-making of marketization. Like Acemoglu (2002) discussing the equilibrium bias of technological progress, the framework of this article also does not limit the two inputs to be capital and labor, but rather, according to the needs of problem, they can be either usual capital and labor, land and labor, or skilled labor and unskilled labor, etc.

In this section, we introduce a generalized production function and two generalized factor accumulation processes. In order to highlight that the key determinants of the direction of technological progress are implicit in these two generalized functions, which are ignored coincidentally by and existing literature using specific functions, this article first uses a reduced form growth model to analyze the determinants of the direction of technological progress. At this stage, the "reduced form" formulation abstracts from the behavior of households and firms, and assumes the steady state exist. In the next section, we will expand the reduced model to include both household and firm behavior to construct a complete growth model with micro foundation to prove that the steady state does exist in the decentralized decision-making of marketization.

(1) A generalized production function

We refer to factors that do not include technology as pure factors, while factors that include technologies are referred to as effective factors. Effective factors are a function of the quantity of factors and the factor-augmenting technologies. Specifically, let $\mathrm{K(t)}$ and $\mathrm{L(t)}$ be capital and labor respectively, that is, pure inputs, and $\widehat{K}(t)$ and $\widehat{L}(t)$ represent effective capital and effective labor respectively, that is, effect factors. If $\mathrm{Y(t)}$ represents output, then the usual neoclassical production function is:

$$\mathrm{Y(t)} = \mathrm{F}\big[\widehat{K}(t), \widehat{L}(t)\big] \tag{1}$$

We assume that equation (1) satisfies all the properties of the neoclassical production function such as, the constant return to scale, the diminishing marginal return, and the Inada (1963) conditions. Furthermore, we assume that the transformations of capital (labor) into effective capital (labor) are given by: $\widehat{K} = \mathrm{B}K^{\phi}$ and $\widehat{L} = A\mathrm{L}^{\varphi}$, where $0 < \phi \leq 1$, $0 < \varphi \leq 1$ determine the marginal transformation rates while B and A represent capital- and labor-augmenting elements, respectively. Substituting this into equation (1) we obtain the generalized production function (2) as follows:

$$\mathrm{Y(t)} = \mathrm{F}\big[\mathrm{B}(t)\mathrm{K}(t)^{\phi}, A(t)\mathrm{L(t)}^{\varphi}\big], 0 < \phi \leq 1, 0 < \varphi \leq 1 \tag{2}$$

The production function (2) introduces two parameters ϕ and φ that generalize the neoclassical functions in existing literature which is the special case ϕ=φ=1. Their proximate meaning is to describe the marginal transformation rates of pure factors into effective factors.

Production function (2) is inspired by Grossman et al. (2017) and Casey and Horii (2022). They point out that the existing neoclassical production functions implicitly limit the direction of technological progress in the steady state, and extend it to include capital-augmenting technological progress in steady state. Equation (2) symmetrically introduces these two parameters to generalize their extensions,[3] which not only can

[3] Grossman et al. (2017) assume $\widehat{K} = \mathrm{B}D(s_t)^{a}K$, where s_t represents schooling, and $\frac{\partial \mathrm{D}(s_t)^{a}}{\partial s_t} < 0$, therefore $\frac{\partial \widehat{K}/\partial K}{\partial s_t} < 0$. Casey and Horii (2022) assume $\widehat{K} = \mathrm{BR}^{1-a}\mathrm{K}^{a}$, $0 < a < 1$, where R represents nonrenewable factors such as land or natural resources, therefore $\frac{\partial^2 \widehat{K}}{\partial \mathrm{K}^2} = (a-1)\mathrm{BR}^{1-a}\mathrm{K}^{a-2} < 0$.

obtain their steady-state equilibrium results, but also make the production function more suitable for analyzing the equilibrium direction of technological progress. Accordingly, if $\phi < 1$ or $\varphi < 1$, the production function (2) has diminishing returns to scale for K and L; if $\phi = \varphi = 1$, the production function has constant returns to scale for K and L.[4]

Output per effective labor is expressed by $y \equiv Y/A\mathrm{L}^{\varphi}$ and the ratio of effective capital to effective labor is expressed by $k \equiv \mathrm{BK}^{\phi}/A\mathrm{L}^{\varphi}$. Accordingly, the production function in the intensive form takes the form as $\mathrm{y(t)} = \mathrm{F}\left[\frac{\mathrm{B}(t)\mathrm{K}(t)^{\phi}}{A(t)\mathrm{L}(\mathrm{t})^{\varphi}}, 1\right] \equiv f\big(k(t)\big)$.

If the market is not completely competitive, then factor prices will be smaller than their marginal products. To simplify, we assume that factor prices are proportional to their marginal products. By equation (2) the corresponding prices of K and L can be written as:

$$\begin{cases} w(t) = \xi_L \varphi A(t) L(t)^{\varphi-1}\big[f\big(k(t)\big) - kf'\big(k(t)\big)\big] \\ \mathrm{r(t)} = \xi_K \phi B(t) K(t)^{\phi-1} f'\big(k(t)\big) \end{cases} \tag{3}$$

where $\xi_L \leq 1$ and $\xi_K \leq 1$, whereby $\xi_L = \xi_K = 1$ when the factor markets are completely competitive.

Equations (3) show that the parameters φ and ϕ have important influence on factor prices. Notice that when they are smaller than 1, even if k is constant, factor prices decrease with their quantities. Therefore, in this case, only factor-augmenting technological progress can keep factor prices constant or make them increase.

(2) Generalized factor accumulation processes

If the production function describes the process of converting effective inputs into final output, then the factor accumulation functions essentially describe the process of converting the final output into effective inputs. Any macroeconomic system is essentially a circular circulation process composed of these two processes.

In principle, each factor of production can be accumulated through investment, so investment is an important factor affecting the accumulation of each factor. Although the neoclassical growth models usually assume exogenous labor growth, denying that investment is a factor affecting labor growth is not in line with the facts. Whether the quantity or quality of population, investment is very important. Even land expansion is

[4] If $\phi > 1$ or $\varphi > 1$, there are increasing marginal returns in the transformation of K and L into $\widehat{K}$ and $\hat{L}$ which may lead to negative technological progress in steady state. Therefore, we do not consider these cases.

not in depended on investment since ancient times, but a result of investment and development by humankind. Of course, the marginal return of investment in accumulating different factors may vary in different periods and places. Therefore, we set two symmetric generalized factor accumulation functions as follows although their specific forms can vary.

$$\begin{cases} \dot{K}(t) = G[I_K(t), K(t), B(t)] \\ \dot{L}(t) = H[I_L(t), L(t), A(t)] \end{cases} \tag{4}$$ [5]

where $\dot{K}(t) \equiv \frac{dK(t)}{dt}$, where $\dot{L}(t) \equiv \frac{dL(t)}{dt}$. $I_K(t)$ and $I_L(t)$ represent investment in two factors accumulation, respectively.

These generalized functions indicate that factor accumulation is not only a function of investment, but also a function of factor stock and factor-augmenting technologies. The standard neoclassical capital accumulation function and the labor accumulation function of exogenous growth both can be seen as a special case of equation (4). Sato (1996) constructed a nonlinear capital accumulation process based on Leviathan and Samuelson (1969) whereby $\mathrm{K(t)}\Phi\left[\frac{I_K(t)}{B(t)K(t)}\right] - \delta_K \mathrm{K(t)}, \Phi' > 0, \Phi'' < 0$. Irmen (2013) proposed a capital accumulation function with decreasing marginal return on investment: $\dot{K}(t) = b_K I_K^{\alpha_K} - \delta_K K, 0 \le \alpha_K \le 1$. We prove that these two functions are the results of two specific investment adjustment cost functions, respectively.

On the one hand, when investment has adjustment costs, the marginal return of investment to factor accumulation decreases weakly. On the other hand, the marginal return of investment will increase with investment-specific technical change. Therefore, the effects of investment adjustment costs and investment-specific technical change on factor accumulation are opposite. As long as there is depreciation or death, the larger the stock of factors, the harder it is to accumulate them. The impact of factor-augmenting technology on factor accumulation is often overlooked, but Sato's (1996) capital accumulation function indicates that the higher the level of technology, the more difficult it is to accumulate the factor. To sum up, we assume, without losing generality, that the derivative properties of each variable in the functions (4) are as follows:

Assumption: $\frac{\partial \mathrm{G}}{\partial I_K(\mathrm{t})} > 0$ and $\frac{\partial^2 \mathrm{G}}{\partial I_K(\mathrm{t})^2} \le 0$; $\frac{\partial \mathrm{H}}{\partial I_L(\mathrm{t})} > 0$ and $\frac{\partial^2 \mathrm{H}}{\partial I_L(\mathrm{t})^2} \le 0$; $\frac{\partial \mathrm{G}}{\partial K(\mathrm{t})} \le 0$,

[5] This function can also be derived from the investment adjustment cost theory, see in Appendix A.

$\frac{\partial \mathrm{H}}{\partial L(\mathrm{t})} \leq 0; \frac{\partial \mathrm{G}}{\partial B(\mathrm{t})} \leq 0, \frac{\partial \mathrm{H}}{\partial A(\mathrm{t})} \leq 0.$

(3) Remarks on the generalizations

When people become accustomed to a particular setting, they may take it for granted as a general situation and forget that they are just in a special case of a generalized setting, and in turn, be amazed that the resulting results are so special! We argue that this is the problem with current growth theories. By reintroducing the generalized production function and factor accumulation functions, it can be clearly revealed which conclusions are only the results of special assumptions, and the specificity and the narrowness of analysis scope of existing growth models. As for what specific assumptions are more realistic, they are not the focus of our concern.

3.2. Definitions

(1) Supply Elasticity of Factor

Existing literature has noticed that Hicks (1932) proposed that the change in relative prices encourages innovation directed to save relatively expensive factor, but has overlooked that it also encourages to increase the supply of relatively expensive factor. After the relative supply of the expensive factor increases, the relative price may decrease, leading to the innovation that saves the factor with increase relative price in short-term may not be reasonable. When innovators expect this, the incentive effect of change in relative prices on innovation must be affected. Therefore, it is uncertain how the change in relative prices affect the direction of innovation, especially in the long term. Of course, if the relative supply of factors is not sensitive to change in relative prices, then the change in relative prices may have a significant impact on the direction of innovation. Therefore, whether the relative supply of factors is sensitive to change in relative prices, that is, the size of the supply elasticity of factor, is the key factor affecting the direction of technological progress. However, although supply elasticity is a widely used concept in microeconomics, there is no clear definition of factor supply elasticity in economic growth models and it is rarely used. Therefore, we define them in a dynamic environment as follows:

Definition 1: we define the *supply elasticity of factor* in a dynamic economy as follows:

$$\begin{cases} \varepsilon_K(t) \equiv \dfrac{\dot{\mathrm{K}}(\mathrm{t})/K(t)}{\dot{r}(\mathrm{t})/r(t)} \\ \varepsilon_L(t) \equiv \dfrac{\dot{\mathrm{L}}(\mathrm{t})/L(t)}{\dot{w}(\mathrm{t})/w(t)} \end{cases} \tag{5}$$

where $\varepsilon_K(t)$ and $\varepsilon_K(t)$ represent the supply elasticities of capital and labor, respectively. Equations (5) show that the factor accumulation functions (4) are the key determinants of the factor supply elasticity. Therefore, the specific form of factor accumulation function may implicitly limit the value of factor supply elasticity. Since the changes in factor supply and factor prices are both endogenous variables, in steady-state equilibrium, the supply elasticity of factor will be determined endogenously by the parameters of the model.

Equation (5) is similar in form to the supply elasticity of factor defined in microeconomics, but there are subtle differences in their meanings. First, in microeconomics, the supply elasticities of capital and labor are defined as follows: $\varepsilon_K \equiv \frac{\partial K}{\partial r}\frac{r}{K}$ and $\varepsilon_L \equiv \frac{\partial L}{\partial w}\frac{w}{L}$ which only considers the impact of price changes on factor supply changes, do not consider the impact of time. The length of time that factor price changes occur and the length of time that factor supply changes do not take into account. Price changes of 1% in a year or in a day are not important. However, equation (5) clearly refers to the change in prices and the change in supply of factors in the same period dt. Second, only when factor supply is a univariate function of price, i.e. K(t)=K(r(t)), and due to $\varepsilon_K(t) = \frac{\frac{dK(r(t))}{d(r(t))}\frac{d(r(t))}{dt}/K(t)}{(dr(t)/dt)/r(t)} = \frac{dK(r(t))}{d(r(t))}\frac{r(t)}{K(t)}$, the two definitions of supply elasticity of factor are mathematically equal. However, it is these two differences that make the traditional definition of factor supply elasticity unsuitable for economic growth models, and require us to redefine them using equation (5).

(2) the Direction of Technological Progress

Acemoglu (2002) defined the bias of technical change as the impact of the change of relative technologies on the relative marginal productivity of factors, and distinguished the two concepts of factor-biased and factor-augmenting. However, he did not provide a clear definition of the direction of technical change, nor clearly distinguish it from the bias of technical change. When it comes to the direction, existing literature usually only focuses on Hicks neutral, Harold neutral, and Solow neutral. If the direction of technological progress refers to the relative rates of different factor-augmenting technical changes, then the direction in theory should be an infinite set from zero to infinity, far from just three neutral technological progress. For this purpose, we provide a clear definition of the direction of technological progress as follows.

Definition 2: The *direction of technological progress, DTP,* is the ratio between the augmentation rates of capital and labor, i.e.

$$DTP(t) \equiv \frac{\dot{\mathrm{B}}(\mathrm{t})/\mathrm{B}(\mathrm{t})}{\dot{\mathrm{A}}(\mathrm{t})/A(t)} \quad (6)$$

DTP can take any value in $[0, \infty]$. When $\dot{B}/B = 0$ and $\dot{A}/A > 0$ then $DTP = 0$, and technological progress is purely labor-augmenting (i.e. Harrod-neutral); when $\dot{B}/B > 0$ and $\dot{A}/A = 0$ then $DTP \to +\infty$, and technological progress is purely capital-augmenting (i.e. Solow-neutral); when $\dot{B}/B = \dot{A}/A > 0$ then $DTP = 1$, and technological progress is Hicks-neutral. The three types of neutrality are just three special directions of technological progress.

Figure 1 shows different directions of technological progress.

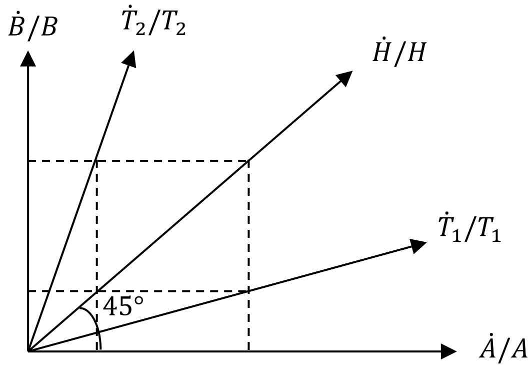

Figure 1: Direction of technological progress

Clearly, the axes represent Harrod-neutral (horizontal) and Solow-neutral (vertical) technological changes. The diagonal $\dot{H}/H$ represents the location of Hicks-neutral technological changes. The ray $\dot{T}_1/T_1$ indicates technological progress which tends to be more labor augmenting, while $\dot{T}_2/T_2$ is more capital augmenting.

3.3. The Determinants of DTP

Use $k \equiv \frac{\mathrm{BK}^{\phi}}{A\mathrm{L}^{\varphi}}$ and equations (5) and (6), at time t, the direction of technological progress in the economy given by equations (1), (2), (3) and (4) can be expressed as following:[6]

[6] Proof is given in Appendix B.

$$DTP(t)=\frac{\left[\dfrac{1+\varepsilon_L(t)-\varepsilon_L(t).\sigma_{\hat{L}}\dfrac{\dot{k}(\mathrm{t})/k(\mathrm{t})}{\dot{\mathrm{L}}(\mathrm{t})/L(t)}}{1+(1-\varphi)\varepsilon_L(t)-\varepsilon_L(t).\sigma_{\hat{L}}\dfrac{\dot{k}(\mathrm{t})/k(\mathrm{t})}{\dot{\mathrm{L}}(\mathrm{t})/L(t)}}\right]+\dfrac{\dot{\mathrm{k}}(\mathrm{t})}{\mathrm{k}(\mathrm{t})}\Big/\dfrac{\dot{\mathrm{A}}(\mathrm{t})}{A(t)}}{\left[\dfrac{1+\varepsilon_K(t)-\varepsilon_K(t).\sigma_{\hat{K}}\dfrac{\dot{k}(\mathrm{t})/k(\mathrm{t})}{\dot{\mathrm{K}}(\mathrm{t})/K(t)}}{1+(1-\phi)\varepsilon_K(t)-\varepsilon_K(t).\sigma_{\hat{K}}\dfrac{\dot{k}(\mathrm{t})/k(\mathrm{t})}{\dot{\mathrm{K}}(\mathrm{t})/K(t)}}\right]} \tag{7}$$

where $\sigma_{\hat{L}}\equiv\frac{-[k(t)]^2f''(k(t))}{f(k(t))-k(t)f'(k(t))}$, $\sigma_{\hat{K}}\equiv\frac{k(t)f''(k(t))}{f'(k(t))}$.

When $\frac{\dot{\mathrm{k}}(\mathrm{t})}{\mathrm{k}(\mathrm{t})}\neq 0$, the economy is in an unbalanced growth path. Equation (7) shows that the direction of technological progress is changing. On the one hand, owing to $\frac{\dot{\mathrm{k}}(\mathrm{t})}{\mathrm{k}(\mathrm{t})}\neq 0$, $\frac{\dot{\mathrm{k}}(\mathrm{t})}{\mathrm{k}(\mathrm{t})}$ will affect directly the direction of technological progress $DTP(t)$; On the other hand, $\frac{\dot{\mathrm{k}}(\mathrm{t})}{\mathrm{k}(\mathrm{t})}\neq 0$ will cause k(t) to rise or fall continuously, thus affecting the marginal output and price of factors, thus affecting the accumulation rate of factors and the supply elasticities of factor $\varepsilon_L(t)$ and $\varepsilon_K(t)$, further affect the direction of technological progress $DTP(t)$. However, if there is a steady state of the economy, in steady state there will be $\frac{\dot{\mathrm{k}}(\mathrm{t})}{\mathrm{k}(\mathrm{t})}=0$, $\varepsilon_L(t)$ and $\varepsilon_K(t)$ will also be ε_L and ε_K. Substitute them into equation (7) to obtain proposition 1:

Proposition 1: *If the production function takes the form of equation (2) and factor prices are proportional to their respective marginal products as shown in equations (3), then the steady-state direction of technological progress is given by:*

$$DTP=\frac{(1+\varepsilon_L)/[1+(1-\varphi)\varepsilon_L]}{(1+\varepsilon_K)/[1+(1-\phi)\varepsilon_K]} \tag{8}$$

Compared with equation (7) and (8) shows that the determinants of the direction of technological progress in the steady state are simpler and clearer than those in unbalanced growth path, the crucial determinants of the direction of technological progress are the elasticities of the factor supplies (ε_L and ε_K) and the parameters (ϕ and φ) which determine the marginal returns of transforming capital and labor into effective capital and labor which have so far been absent from the existing literature. The former reflects the factor accumulation processes, and the latter reflect the production function. Given ϕ and φ, technological progress tends towards the factor with the smaller supply elasticity. Similarly, given the supply elasticities, it tends to the factor with the smaller marginal transformation rate to effective factor.

Proposition 1 has an immediate corollary for the case where both marginal transformation rates are constant (i.e., $\phi = 1$ and $\varphi = 1$).

Corollary 1: when the marginal transformation rates of both factors are constant (i.e., $\phi = 1$ and $\varphi = 1$), then the direction of technological progress is given by*:*

$$DTP = \frac{1 + \varepsilon_L}{1 + \varepsilon_K} \tag{9}$$

Furthermore, from equation (9) we can obtain:[7]

(i) If $\varepsilon_L < \varepsilon_K$, then $DTP \equiv \frac{\dot{\mathrm{B}}(\mathrm{t})/\mathrm{B}(\mathrm{t})}{\dot{\mathrm{A}}(\mathrm{t})/A(t)} < 1$, i. e. $\frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})} < \frac{\dot{\mathrm{A}}(\mathrm{t})}{\mathrm{A}(\mathrm{t})}$, especially, if $0 \le \varepsilon_L < \infty$ and $\varepsilon_K = \infty$, then $DTP \equiv \frac{\dot{\mathrm{B}}(\mathrm{t})/\mathrm{B}(\mathrm{t})}{\dot{\mathrm{A}}(\mathrm{t})/A(t)} = 0$, i. e. $\frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})} = 0, \frac{\dot{\mathrm{A}}(\mathrm{t})}{\mathrm{A}(\mathrm{t})} > 0$;

(ii) If $\varepsilon_L > \varepsilon_K$, then $DTP \equiv \frac{\dot{\mathrm{B}}(\mathrm{t})/\mathrm{B}(\mathrm{t})}{\dot{\mathrm{A}}(\mathrm{t})/A(t)} > 1$, i. e. $\frac{\dot{\mathrm{A}}(\mathrm{t})}{\mathrm{A}(\mathrm{t})} < \frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})}$, especially, if $0 \le \varepsilon_K < \infty$ and $\varepsilon_L = \infty$, then $DTP \equiv \frac{\dot{\mathrm{B}}(\mathrm{t})/\mathrm{B}(\mathrm{t})}{\dot{\mathrm{A}}(\mathrm{t})/A(t)} = \infty$, i. e. $\frac{\dot{\mathrm{A}}(\mathrm{t})}{\mathrm{A}(\mathrm{t})} = 0, \frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})} > 0$;

(iii) If $\varepsilon_L = \varepsilon_K$, then $DTP \equiv \frac{\dot{\mathrm{B}}(\mathrm{t})/\mathrm{B}(\mathrm{t})}{\dot{\mathrm{A}}(\mathrm{t})/A(t)} = 1$, i. e. $\frac{\dot{\mathrm{A}}(\mathrm{t})}{\mathrm{A}(\mathrm{t})} = \frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})}$, especially, if $\varepsilon_K = \varepsilon_L = \infty$, then $DTP \equiv \frac{\dot{\mathrm{B}}(\mathrm{t})/\mathrm{B}(\mathrm{t})}{\dot{\mathrm{A}}(\mathrm{t})/A(t)} = 1$ and $\frac{\dot{\mathrm{A}}(\mathrm{t})}{\mathrm{A}(\mathrm{t})} = \frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})} = 0$.

Corollary 1 shows that for the case where the marginal transformation rates are constant for both factors (i.e., $\phi = 1$ and $\varphi = 1$), the steady-state direction of technological progress is determined solely by the relative size of the factor supply elasticities and is biased towards the one with the relatively smaller elasticity. In particular, when the supply elasticity of a factor is infinite, technological progress will not improve its productivity at all in the steady-state equilibrium. These results are summarized in Table 1.

Table 1：the DTP under different relative elasticities of factor supplies in steady state

Labor \ Capital	$0 \le \varepsilon_K < \infty$			$\varepsilon_K = \infty$
$0 \le \varepsilon_L < \infty$	$\varepsilon_L < \varepsilon_K$	$\varepsilon_L = \varepsilon_K$	$\varepsilon_L > \varepsilon_K$	$\frac{\dot{\mathrm{A}}}{\mathrm{A}} > \frac{\dot{\mathrm{B}}}{\mathrm{B}} = 0$
	$\frac{\dot{\mathrm{A}}}{\mathrm{A}} > \frac{\dot{\mathrm{B}}}{\mathrm{B}} > 0$	$\frac{\dot{\mathrm{A}}}{\mathrm{A}} = \frac{\dot{\mathrm{B}}}{\mathrm{B}} > 0$	$0 < \frac{\dot{\mathrm{A}}}{\mathrm{A}} < \frac{\dot{\mathrm{B}}}{\mathrm{B}}$	
$\varepsilon_L = \infty$	$\frac{\dot{\mathrm{B}}}{\mathrm{B}} > \frac{\dot{\mathrm{A}}}{\mathrm{A}} = 0$			$\frac{\dot{\mathrm{A}}}{\mathrm{A}} = \frac{\dot{\mathrm{B}}}{\mathrm{B}} = 0$

[7] See proof in Appendix C.

Proposition 1 and its corollary are based on a reduced form growth model, which does not consider the behavior of households and firms, nor the market structure, but only the generalized production function, generalized factor accumulation processes, and the aforementioned definitions. There are three remarkable features of this proposition:

First of all, according to the definition of the direction of technological progress, $DTP \equiv \frac{\dot{B}(t)/B(t)}{\dot{A}(t)/A(t)}$ can be any value from zero to infinity, that is, technological progress can be any type from purely labor-augmenting to purely capital-augmenting. However, due to the Uzawa's (1961) steady state theorem, the existing literature generally argues that the steady state equilibrium of the economic growth model requires that technological progress must be purely labor-augmenting. Proposition 1 and its corollary show that, the steady state can be compatible with *any* direction of technological progress, dependent on the relative size of the factor supply elasticities which are determined by concrete factor accumulation functions.

Secondly, Proposition 1 and its corollary give many theoretically verifiable predictions about the direction of technological progress in steady state. The following three examples verify their predictions from both the positive and negative sides. First, if the factors supply are given exogenously, their supply elasticities are zero, namely $\varepsilon_L = 0$ and $\varepsilon_K = 0$, then the direction of technological progress must be $\frac{\dot{B}/B}{\dot{A}/A} = 1$, that is, Hicks neutral, which is exactly the setting of Acemoglu (2002) model and it does have $\frac{\dot{B}/B}{\dot{A}/A} = 1$ in steady state. Second, if the setting of a model makes the elasticities there are $\varepsilon_K = \infty$ and $\varepsilon_L < \infty$ in steady state, substituted them into equation (9), there is $\frac{\dot{B}/B}{\dot{A}/A} = 0$, that is, technological progress is purely labor-augmenting. As well known, the neoclassical growth model requires that technological progress must be purely labor-augmenting in steady state, so we can check that whether its factor accumulation functions does implicitly $\varepsilon_K = \infty$ and $\varepsilon_L < \infty$. In fact, it does. This not only verify the correctness of equation (9), but also help to understand Uzawa's (1961) steady-state theorem. Finally, if the factor accumulation functions in one model imply $\varepsilon_K < \infty$ and $\varepsilon_L = 0$, then it can be inferred from equation (9) that there is no $\frac{\dot{B}/B}{\dot{A}/A} = 0$, that is, technological progress in steady state cannot be purely labor-augmenting in this model. Acemoglu (2009, ch15.6) just gave such a model. The model assumes that the L is given exogenously, therefore, $\varepsilon_L = 0$, the capital accumulation rate is given as $\frac{\dot{K}(t)}{K(t)} = s_K$, where s_K is an exogenous constant, so there also must be $\varepsilon_K < \infty$.However, the Proposition 15.12 of Acemoglu (2009, ch15.6) argues that the technological progress in steady state must be purely labor-augmenting in this model. Therefore, we can infer that the Proposition cannot be established, and Li (2016) confirmed this point.

Finally, although the factor accumulation functions (4) are not used in the process of obtaining equation (7), they are crucial functions to the direction of technological progress because they determine the supply elasticities of factors. Setting a specific form for the factor accumulation function may unintentionally limit the value of the factor supply elasticity and then restrict the direction of technological progress. This may be the key reason why the existing literature cannot give the determinants of direction of technological progress, and also the key reason why Uzawa's (1961) steady-state theorem has not been reasonably explained for a long time.

4 The Direction of Technological Progress in a Specific Growth Model

Proposition 1 is derived without specifying the micro-structure of households and enterprises. Now, we provide a well-founded micro-based model to analyzes the adjustment mechanism of the direction of technological progress in transitional dynamics and its determinants in steady state. We verify the existence of steady state and the validity of proposition 1.

For this purpose, we extend the Acemoglu (2002, 2003) framework. That framework expands the Romer (1990) technology from one dimension to two, making it suitable for the analysis of potential directions of technological progress. However, as commented above, the marginal transformation rates of factors into effective factors in the Acemoglu framework were assumed to be one. and both supply elasticities of labor to be zero in Acemoglu (2002, 2003), capital supply elasticity to be zero in Acemoglu (2002) but infinite in Acemoglu (2003). Thereby, both papers ignore the two aforementioned aspects that determine the steady-state direction of technological progress. As a result, although Acemoglu's models provide the micro foundations for the direction of technological progress, they do not identify its determinants in the steady-state equilibrium. In addition, since Acemoglu assumes perfect mobility of scientists (responsible for R&D) from one sector to another, he only emphasizes the demand side factors of innovation (relative prices and market size) and ignores the supply side factors of innovation (relative marginal productivity reflecting the relative crowding effects) in the transitional dynamics.

Accordingly, in this section, we extend Acemoglu's framework in the following aspects: *first*, the factor accumulation functions and production function are replaced by the generalized functions proposed earlier. *Second*, a migration function of scientists is proposed, assuming that their transition between the innovation sectors is induced by wage differences during the process of transitional dynamics; *Third*, we distinguish between instantaneous and steady-state equilibrium, which enables us to reveal the difference between the factors affecting the direction of technological progress in the process of transitional dynamics and in the steady-state equilibrium.

4.1. The Environment

Following Acemoglu (2002, 2003), there are two material factors and three production sectors: final goods, intermediate goods, and research and development (R&D). The symbols K and L represent the two kinds of material production factors. S represent "scientists" who specialize in research and development of new intermediate products, respectively.

(1) The production functions

The final goods sector is competitive, using the following CRS production function:

$$Y = \left[\gamma {Y_L}^{(\varepsilon-1)/\varepsilon} + (1-\gamma){Y_K}^{(\varepsilon-1)/\varepsilon}\right]^{\varepsilon/(\varepsilon-1)}, \quad 0 \leq \varepsilon < \infty \tag{10}$$

where Y is output and Y_L and Y_K are the two inputs, with the factor-elasticity of substitution given by ε.

The inputs Y_L and Y_K are also produced competitively by constant elasticity of substitution (CES) production functions using a continuum of intermediate inputs, $X(i)$ and $Z(j)$:

$$Y_L = \left[\int_0^N X(i)^{\varphi\beta} di\right]^{1/\beta} \ and \ Y_K = \left[\int_0^M Z(j)^{\phi\beta} dj\right]^{1/\beta} \tag{11}$$

where the elasticity of substitution is given by $v = 1/(1-\beta)$ and N and M represent the measure of the two types of the intermediate inputs, respectively. The specification of the production functions extends that of Acemoglu's by introducing the parameters φ and ϕ which are assumed to satisfy $0 < \varphi \leq 1$ and $0 < \phi \leq 1$. Since $\left[\int_0^N [\lambda X(i)]^{\varphi\beta} di\right]^{1/\beta} = (\lambda)^{\varphi} Y_L$, φ determines the return to scale of $X(i)$, similarly, ϕ determines the return to scale of $Z(j)$.

The intermediate factors $X(i)$ are produced by labor, whereas $Z(j)$ are produced by capital, where the respective production functions are linear:

$$X(i) = L(i) \text{ and } Z(j) = K(j) \tag{12}$$

Accordingly, Y_L and Y_K represent labor-intensive and capital-intensive inputs respectively.

(2) Factor accumulation processes

In the previous reduced-form model, the factor accumulation functions are given generally by equations (4), implicitly reflecting investment adjustment cost. In order to make the model tractable, here we specify functional forms for these equations. Following Irmen (2013), we posit:[8]

$$\begin{cases} \dot{K} = b_K I_K^{\alpha_K} - \delta_K K, & b_K > 0, 0 \leq \alpha_K \leq 1, \delta_K > 0 \\ \dot{L} = b_L I_L^{\alpha_L} - \delta_L L, & b_L > 0, 0 \leq \alpha_L \leq 1, \delta_L > 0 \end{cases} \tag{13}$$

where I_K and I_L denote investment into capital and labor accumulation, and α_K and α_L reflect the impact of investment adjustment cost. The parameters b_K and b_L reflect investment-specific technologies. If they are given as constants, it indicates that the investment-specific technology remains unchanged. If $\dot{b_K}(t)/b_K(t) = g_q \geq 0$, there is an exogenously investment-specific technological progress in capital accumulation. While investment-specific technological progress is not focal in this paper, considering it shows that this model can admit a continuous decline of investment goods prices and obtain the results of Grossman et al. (2017).

(3) Innovation possibilities Frontier

New intermediates are developed by an R&D sector. The innovation possibilities frontier is implicitly specified as follows:[9]

$$\begin{cases} \dot{M} = d_M M\Omega(S_M) - \delta M = d_M M {S_M}^{\mu} - \delta M \\ \dot{N} = d_N N\Omega(S_N) - \delta N = d_N N {S_N}^{\mu} - \delta N \end{cases} \quad 0 < \mu \leq 1 \tag{14}$$

where $d_M > 0$ and $d_N > 0$ are innovation productivity parameters and δ is the deprecation (or obsolescence) rate affecting blueprints of new varieties of intermediate inputs.[10] The variables S_N and S_M represent respectively the number of scientists

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[8] As shown in appendix A, Irmen (2013)'s factor accumulation is a special case of equation (4). Later, we discuss the factor accumulation function corresponding to another adjustment cost function proposed by Sato (1996). The conclusion remains valid.

[9] This function is consistent with Acemoglu (2003), but the crowding effect of scientists is internalized. The frontier of innovation possibilities based on the extended lab equipment model (Rivera-Natiz and Romer, 1991) will be discussed Section VII. Both do not change the conclusion.

[10] In order to ensure the existence and uniqueness of the steady-state equilibrium of the model, we

engaged in innovation of the two kinds of intermediates, and $\Omega(.)$ with $\Omega'(.) > 0$, $\Omega''(.) < 0$, reflects a crowding effect among scientists, here specified as $\Omega(S) = S^{\mu}$, $0 < \mu < 1$. This captures the notion that scientists' marginal productivity declines when more of them are present in one innovation sector. We further assume $\Omega'(0) \to \infty$ to ensure that the economy has a unique steady-state equilibrium.[11] Finally, the exogenously set total number of scientists, S, is assumed to be sufficiently large to guarantee that the technology of the two sectors does not regress. In particular, letting $\bar{S}_N = \Omega^{-1}\left(\frac{\delta}{d_N}\right) = \left(\frac{\delta}{d_N}\right)^{1/\mu}$ and $\bar{S}_M = \Omega^{-1}\left(\frac{\delta}{d_M}\right) = \left(\frac{\delta}{d_M}\right)^{1/\mu}$, it is assumed that $\bar{S}_N + \bar{S}_M \leq S$ (see also Acemoglu 2003).

Equations (14) describe the input-output relationship of innovation as in endogenous technological progress models. Given S, $S_N + S_M \leq S$ equations (14) also describe the trade-off between the two kinds of innovation described by the innovation possibilities frontier proposed by the induced innovation literature (von Weizsäcker, 1962; Kennedy, 1964). The distribution of scientists in different innovation sectors determines the relative rates of technological progress in different sectors, and then determines the direction of technological progress.

Finally, once a new intermediate input is invented, the inventor obtains a permanent patent, as in Romer's (1990) model.

(4) Migration of scientists

Unlike Acemoglu (2002, 2003), we assume that although scientists are homogeneous, it takes time for them to move from one sector to another, say because scientists need time to finish existing research projects in the current innovation sector and adapt to the environment of the new sector. This means that at any time, the wage of all scientists within a given innovation sector is equal but may be different across sectors. However, a wage gap will induce scientists to move to the higher-paying sector, thereby changing the rates of technological progress in the different sectors and the direction of technological progress.

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need to impose $0 \leq g_q < \frac{1-\phi\beta}{\phi\beta}\delta + \frac{1-\varphi\beta}{\phi\beta}\frac{1-\phi\alpha_K}{1-\varphi\alpha_L}[d_N S^{\mu} - \delta]$, which means that the investment-specific technological progress should not be "too fast".

[11] The number of scientists may have both positive spillovers (stimulating more inspiration) and negative spillovers (crowding effect) on innovation, and which is dominant in reality is an empirical question. Some empirical researches (Borjas and Doran, 2012; Bloom et al., 2020) shows that crowding effect may be more important. Because the crowding effect makes the steady state of the model unique and more stable, this paper inherits the assumption of Acemoglu (2003) that the crowding effect is dominant.

We posit the following migration function of scientists:

$$\frac{\dot{S_M}}{S_M} = \psi\left[\frac{w_M}{w_N}\right], \psi[1] = 0, \psi'[.] > 0 \tag{15}$$

where w_M and w_N are the wages of scientists in the two sectors, respectively. Because w_M and w_N are assumed to be determined by the marginal productivity of scientists, $\frac{\dot{S_M}}{S_M}$ is implicitly a function of S_M (implicitly assuming that S_N=S-S_M). Assuming that the respective marginal productivity of scientists is very large when S_M=0 or S_N=0, according to (15), these points are not dynamically stable. In the first case scientists migrate from the N-sector to the M-sector while the opposite is true in the second case.[12]

(5) Household's preference and budget constraint

The household's goal is to maximize the discounted flow of utility, given by:

$$\mathrm{U} = \int_0^\infty \frac{C(t)^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt \tag{16}$$

where $C(t)$ is consumption at time t, $\rho > 0$ is the discount rate, and $\theta > 0$ is a utility curvature coefficient of the household.

The household's periodic budget constraint is given by:

$$C + I_K + I_L \leq Y = wL + rK + \mathrm{w}_N S_N + \mathrm{w}_M S_M + \Pi \tag{17}$$

where the LHS stands for expenditures consisting of consumption and investments, I_K and I_L, into capital and labor, and the RHS is income, obtained from renting out labor at the rate w, capital at the rate r, scientists at the wages w_N and w_M. Π stands for total profits, which are positive when the returns to scale of the final good production function are decreasing.

The equilibrium consists of two stages: the first stage is the *instantaneous equilibrium* (or short run equilibrium). Specifically, given K, L, M, N, S_M and S_N and setting the final good as the numeraire, the prices r, w, w_M and w_N clear all markets (the product market, factor markets and scientist markets), and underly the households'

[12] It turns out that the innovation possibilities frontier (14) with a crowding effect coupled with $\Omega'(0) \to \infty$, imply that the model in this paper has only one steady-state equilibrium $0 < S_M^* < S$, which is saddle-point stable. Moreover, Bental et al. (2022) prove that if innovation has a crowding effect and scientists are imperfectly mobile between sectors, Acemoglu's (2003) model also has a unique locally saddle-point stable steady-state equilibrium even the substitute elasticity of capital and labor greater than 1.

intertemporal utility maximization as well as the enterprises' profit maximization.

The second stage is the *steady-state equilibrium* (or long run equilibrium), comprised of the instantaneous equilibrium with the additional condition that the scientists' wage rate across sectors is equalized and the dynamics of factor accumulation and technological progress is adjusted accordingly.

We start by formally analyzing the instantaneous equilibrium, then move to the steady-state equilibrium of the model.

4.2. The Instantaneous equilibrium

(1) Enterprise profit maximization and goods market equilibrium

When enterprise maximize profits, the goods market clears, the production function takes the form of a CES special case of function (2), as summarized in Proposition 2.

Proposition 2: In the instantaneous equilibrium, the final output production function takes the form:

$$Y = \left[\gamma(AL^{\varphi})^{(\varepsilon-1)/\varepsilon} + (1-\gamma)(BK^{\phi})^{(\varepsilon-1)/\varepsilon}\right]^{\varepsilon/(\varepsilon-1)} \tag{18}$$

where $A \equiv N^{(1-\varphi\beta)/\beta}$ *and* $B \equiv M^{(1-\phi\beta)/\beta}$.

Proof: See Appendix D.

The CES production function shown in equation (18), which is *derived* from the extended Acemoglu (2002, 2003) model, is characterized by constant returns to scale to AL^{φ} and BK^{ϕ}. With respect to L and K, it has constant returns to scale if $\varphi = 1$ and $\phi = 1$ and diminishing returns to scale when $\varphi < 1$ or $\phi < 1$. This shows that the formulation of the production function (2) can be supported under specific micro foundations.

Letting $k \equiv \frac{BK^{\phi}}{AL^{\varphi}} = \frac{M^{(1-\phi\beta)/\beta}K^{\phi}}{N^{(1-\varphi\beta)/\beta}L^{\varphi}}$, from equation (18), the intensive production function is:

$$f(k) \equiv Y/AL^{\varphi} = \left[\gamma + (1-\gamma)k^{(\varepsilon-1)/\varepsilon}\right]^{\varepsilon/(\varepsilon-1)} \tag{19}$$

(2) Scientist market equilibrium

The instantaneous equilibrium of scientist market refers to the balance between scientists supply and demand equilibrium in each sector, for given M, N, S_M and S_N. The demand for scientists depends on the marginal patent output of scientists and the market value of each invention patent. Substituting the wages w_M and w_N into the migration function (15), we obtain (see Appendix E):

$$\frac{\dot{S_M}}{S_M} = \psi\left[\frac{d_M {S_M}^{\mu-1}}{d_N(S-S_M)^{\mu-1}} \cdot \frac{\varphi(1-\phi\beta)}{\phi(1-\beta\varphi)} \cdot \frac{r}{w} \cdot \frac{K}{L}\right] \tag{20}$$

Because the moving of scientists affects on their distribution of scientist in different innovation sectors, which in turn affects the direction of technological progress, therefore, from equation (20) we derive Proposition 3:

Proposition 3： In the instantaneous equilibrium, the forces affecting the direction of technological progress are the relative factor price $\frac{r}{w}$, the relative market size $\frac{K}{L}$ and the relative marginal productivity of scientists in the different sectors $\frac{d_M {S_M}^{\mu-1} M}{d_N(S-S_M)^{\mu-1} N}$.

In the equation (20), the relative wages, $\frac{w_M}{w_N} = \frac{d_M {S_M}^{\mu-1}}{d_N(S-S_M)^{\mu-1}} \cdot \left(\frac{\varphi(1-\phi\beta)}{\phi(1-\beta\varphi)} \cdot \frac{r}{w} \cdot \frac{K}{L}\right) = \frac{\partial\dot{M}/\partial S_M}{\partial\dot{N}/\partial S_N} \cdot \frac{V_Z}{V_X}$, which determines the movement of scientists, comprehensively include three factors that affect the direction of technological progress: namely the relative price ($\frac{r}{w}$) proposed by Hicks (1932), the relative market size ($\frac{K}{L}$) proposed by Acemoglu (2002), and the relative marginal productivity of scientists ($\frac{d_M {S_M}^{\mu-1} M}{d_N(S-S_M)^{\mu-1} N}$). While these three factors affect the direction of technological progress by affecting the relative wage rate of scientists and the sectoral distribution of scientists in different sectors, the direction of technological progress in turn also affects these three factors through changing the relative technologies (M/N). As long as $\frac{\dot{S_M}}{S_M} \neq 0$ the direction of technological progress will change.

If there is a crowding effect, a greater number of scientists in a sector reduces their marginal output. This will induce scientists to eventually be evenly distributed in the two sectors, which ensures the uniqueness of the steady state. [13]

[13] While Acemoglu (2003) proposed a crowding effect, he did not analyze its impact on the direction

(3) Maximization of household utility

Households are price takers. Given the factor accumulation functions, the representative household maximizes the intertemporal utility by allocating income between consumption and the two different kinds of investment. The resulting Euler equations are as follows (see Appendix F):

$$\begin{cases} \dfrac{\dot{C}}{C} = \dfrac{1}{\theta}\left\{r\alpha_K b_K I_K^{\alpha_K - 1} - (\alpha_K - 1)\dfrac{\dot{I}_K}{I_K} - g_q - \rho - \delta_K\right\} \\ \dfrac{\dot{C}}{C} = \dfrac{1}{\theta}\left\{w\alpha_L b_L I_L^{\alpha_L - 1} - (\alpha_L - 1)\dfrac{\dot{I}_L}{I_L} - \rho - \delta_L\right\} \end{cases} \tag{21}$$

The Euler equations indicate that the allocation of household income should equalize returns of the current consumption C and investment in capital accumulation I_K and investment in labor accumulation I_L. It is worth noting that when $\alpha_K = 1$ and $g_q = 0$ ($b_K = 1$), the first Euler equation in equations (21) degenerates to $\dot{C}/C = (r - \rho - \delta_K)/\theta$. This is the famous Euler equation of the usual neoclassical growth model. Therefore, the Euler equation of the neoclassical growth model is only a special condition under the specific capital accumulation function and cannot be used for other capital accumulation functions.

(4) The dynamics of the instantaneous equilibrium

The instantaneous equilibrium delivers the optimal behavior of households and firms which underly the economy's dynamics. To obtain the equations describing that dynamics, we need to use the state variable $k(t)$. In addition we define the investment rate of capital and labor $\mathrm{s_K(t)} \equiv \mathrm{I_K}(t)/\mathrm{Y(t)}$ and $\mathrm{s_L(t)} \equiv \mathrm{I_L}(t)/\mathrm{Y(t)}$ as well as the following growth rates: $g(t) \equiv \dot{Y}(t)/Y(t)$, $\mathrm{g_K(t)} \equiv \dot{\mathrm{K}}(t)/\mathrm{K(t)}$, $\mathrm{g_L(t)} \equiv \dot{\mathrm{L}}(t)/\mathrm{L(t)}$, $i_K(t) \equiv \dot{I}_K(t)/I_K(t)$, $i_L(t) \equiv \dot{I}_L(t)/I_L(t)$.

Given the above definitions, the equations describing the dynamic evolution of the economy take the following form (see Appendix G):

of technological progress.

$$\begin{cases} \frac{\dot{k}}{k} = \frac{1-\phi\beta}{\beta}[d_M {S_M}^{\mu} - \delta] + \phi g_K - \frac{1-\varphi\beta}{\beta}[d_N(S-S_M)^{\mu} - \delta] - \varphi g_L \\ \frac{\dot{S_M}}{S_M} = \psi\left[\frac{d_M {S_M}^{\mu-1}}{d_N(S-S_M)^{\mu-1}} . \frac{\phi(1-\phi\beta)}{\varphi(1-\varphi\beta)} . \frac{1-\gamma}{\gamma} k^{\frac{\varepsilon-1}{\varepsilon}}\right] \\ \frac{\dot{s}_K}{s_K} = i_K - g \\ \frac{\dot{s}_L}{s_L} = i_L - g \\ \frac{\dot{g}_K}{g_K} = \frac{g_K+\delta_K}{g_K}\left(g_q + \alpha_K i_K - g_K\right) \\ \frac{\dot{g}_L}{g_L} = \frac{g_L+\delta_L}{g_L}(\alpha_L i_L - g_L) \end{cases} \tag{22}$$

whereby (g, i_K, i_L) are represented by $(S_M$, k, s_K, s_L, g_K, $g_L)$ as follows:

$$\begin{cases} g = \left\{\frac{1-\varphi\beta}{\beta}[d_N(S-S_M)^{\mu} - \delta] + \varphi g_L\right\} - \frac{\dot{k}}{k}\frac{kf'(k)}{f(k)} \\ i_K = \left[\frac{\theta(g - s_K i_K - s_L i_L)}{(1-\alpha_K)(1-s_K-s_L)} + \frac{g_q+\rho+\delta_K}{(1-\alpha_K)} - \frac{\alpha_K\beta\phi^2\left(g_K+\delta_K\right)}{(1-\alpha_K)s_K}\frac{kf'(k)}{f(k)}\right] \\ i_L = \left[\frac{\theta(g - s_K i_K - s_L i_L)}{(1-\alpha_L)(1-s_K-s_L)} + \frac{\rho+\delta_L}{(1-\alpha_L)} - \frac{\alpha_L\beta\varphi^2\left(g_L+\delta_L\right)}{(\alpha_L-1)s_L}\frac{[f(k)-kf'(k)]}{f(k)}\right] \end{cases} \tag{23}$$

Equations (22) consist of a set of differential equations, including six independent equations and six variables other than time t, which describe the dynamic behavior of economy in instantaneous general equilibrium.

4.3. The Steady State equilibrium (or the long run equilibrium)

According to definition 2, the steady-state equilibrium of the model is the zero solution of the dynamic equations (22), that is, when each of the rates of change in (22) equals 0.

(1) Existence and uniqueness of steady state

For the existence and uniqueness of the steady state of the specific model described in this section, we provide proposition 4:

Proposition 4: An economy characterized by equations (22) possesses a unique steady-state growth path described by equations (24).

$$
\begin{cases}
S_M{}^* = S_M(S, \beta, \phi, \varphi, d_M, d_N, \alpha_K, \alpha_L, g_q, \delta, \mu) \\
k^* = \left[\dfrac{d_N(S - S_M{}^*)^{\mu-1}}{d_M S_M{}^{*\mu-1}} \cdot \dfrac{\varphi(1-\varphi\beta)}{\phi(1-\phi\beta)} \cdot \dfrac{\gamma}{1-\gamma}\right]^{\frac{\varepsilon}{\varepsilon-1}} \\
s_K{}^* = \dfrac{\alpha_K \beta \phi^2 (g_q + \alpha_K g^* + \delta_K)}{(1-\alpha_K+\theta)g^* + \rho + \delta_K} \cdot \dfrac{(1-\gamma)k^{*\frac{\varepsilon-1}{\varepsilon}}}{\gamma + (1-\gamma)k^{*(\varepsilon-1)/\varepsilon}} \\
s_L{}^* = \dfrac{\alpha_L \beta \varphi^2 (\alpha_L g^* + \delta_L)}{(1-\alpha_L+\theta)g^* + \rho + \delta_L} \cdot \dfrac{\gamma}{\gamma + (1-\gamma)k^{*(\varepsilon-1)/\varepsilon}} \\
g_M{}^* = \dfrac{\beta}{1-\phi\beta}\left[1 - \phi\left(\alpha_K + \dfrac{g_q}{g^*}\right)\right] g^* \\
g_N{}^* = \dfrac{\beta}{1-\varphi\beta}(1 - \varphi\alpha_L) g^* \\
g_L{}^* = \alpha_L g^* \\
g_K{}^* = g_q + \alpha_K g^* \\
g^* = g_C{}^* = i_K{}^* = i_L{}^* = \dfrac{1-\varphi\beta}{\beta} \dfrac{[d_N(S - S_M{}^*)^{\mu} - \delta]}{(1-\varphi\alpha_L)}
\end{cases}
\tag{24}
$$

In Appendix H, we formally prove that the equation group (22) lead to a unique steady-state solution. Here we briefly discuss the economic intuition of the proposition.

First, the **uniqueness** of steady state. Remeber that the innovation functions are assumed to be characterized by a crowding effect, that is, $\Omega'(S_M) > 0, \Omega''(S_M) < 0$ and $\Omega'(0) \to \infty$. Accordingly, when scientists are moving from one sector to another, the wages in the absorbing sector decline and the wages of the contributing sector increase. The assumption $\Omega'(0) \to \infty$ serves as an "Inada condition" which guarantees that scientists will be present in both innovation sectors, and that the scientist market has only one equilibrium interior point, that is, the equation $\frac{\dot{S_M}}{S_M} = 0$ possesses only one solution $0 < S_M{}^* < S$.

Second, the **existence** of the steady state of the model. Once $S_M{}^*$ is obtained, k^* is given. Then, recursively the rates of intermediates invention $g_M{}^*$ and $g_N{}^*$, the rates of output growth g^*, consumption growth $g_C{}^*$, investment growth $i_K{}^*$ and $i_L{}^*$, and factors accumulation $g_L{}^*$ and $g_K{}^*$ in the steady state can be obtained. Therefore, the ratio of investment $s_K{}^*$ and $s_L{}^*$ and consumption $(1 - s_K{}^* - s_L{}^*)$ can also be solved. This shows that as long as $S_M{}^*$ exists, equations (24) define the steady state.

Third, about the **stability** of the steady state of the model. Unfortunately, we are currently unable to provide rigorously mathematical proof of the stability of the steady-state of the model, but we attempt to analyze the convergence of each state variable near the steady-state which indicates that the steady-state of the model is likely to be locally saddle point stable (see Appendix H).

(2) Determinants of the direction of technological progress in steady state

In this complete specific growth model, the two material factors of production and two factor-augmenting technological progress are endogenous. By applying $A \equiv N^{(1-\varphi\beta)/\beta}$ and $B \equiv M^{(1-\phi\beta)/\beta}$, from equations (24), the steady-state capital- and labor-augmenting technological progress are:

$$\begin{cases} (\dot{B}/B)^* = \left[1 - \phi\left(\alpha_K + g_q/g^*\right)\right]g^* \\ (\dot{A}/A)^* = (1 - \varphi\alpha_L)g^* \end{cases} \tag{25}$$

Substituting equation (25) into the definition of the direction of technological progress (6), we obtain another core proposition of this paper:

Proposition 5: In the steady state of the growth model described by equations (10)-(17), depending on the values of the exogenous parameters the direction of technological progress can range from purely labor-augmenting to purely capital-augmenting, and is given by:

$$DTP = \frac{1 - \phi\left(\alpha_K + g_q/g^*\right)}{1 - \varphi\alpha_L} \tag{26}$$

In general, equation (26) indicates that the determinants of the direction of technological progress are given by the set $\left(S, \mu, \beta, \phi, \varphi, d_M, d_N, \alpha_K, \alpha_L, g_q, \delta\right)$. However, if investment-specific technological progress is excluded, i.e., $g_q = 0$, then equation (26) degenerates to $DTP = \frac{1-\phi\alpha_K}{1-\varphi\alpha_L}$, depending only on the tuple $(\phi, \varphi, \alpha_K, \alpha_L)$. If the investment adjustment costs are not considered and the marginal transformation rate of capital into effective capital is constant, that is, $\phi = \alpha_K = 1$, then $DTP = 0$ (or purely labor-augmenting). Alternatively, assuming that both factors are supplied inelastically, that is, $\alpha_K = \alpha_L = 0$, then $DTP = 1$ (or Hicks neutral). Both cases are independent of any other parameters of the model. In fact, the former represents the standard assumptions of the neoclassical growth model on the production function and capital accumulation functions, and the latter resembles Acemoglu (2002). If the values of these parameters are unintentionally and implicitly set, the modelers may be surprised as to why the direction of technological progress in the steady state is of a particular type and not another.

In addition, in order to check whether equation (26) is consistent with equation (8), we substitute the steady-state accumulation rates of capital and labor and the growth rates of the capital and labor prices into the definition of the supply elasticities (equations (5)) to obtain the supply elasticities of both factors (see Appendix I):

$$\begin{cases} \varepsilon_K = \dfrac{\alpha_K + g_q/g^*}{1-\left(\alpha_K + g_q/g^*\right)} \\ \varepsilon_L = \dfrac{\alpha_L}{1-\alpha_L} \end{cases} \tag{27}$$

From equation (27) we can extract α_K *and* α_L , substitute them into equation (26) and obtain:

$$DTP = \frac{1-\phi\left(\alpha_K + g_q/g^*\right)}{(1-\varphi\alpha_L)} = \frac{(1+\varepsilon_L)/[1+(1-\varphi)\varepsilon_L]}{(1+\varepsilon_K)/[1+(1-\phi)\varepsilon_K]} \tag{28}$$

Equation (28) shows that although the determinants of the direction of technological progress include many exogenous parameters, they can be divided into two categories: the ones pertain to the marginal transformation rates of production factors into effective production factors (φ and ϕ), the others are associated with the factor supply elasticities (ε_L and ε_K) in steady state. Therefore, the direction of technological progress still depends on the relative size of marginal transformation rates and factor supply elasticities.

(3) Relationship between income shares and the direction of technological progress

Define $\sigma_K \equiv \frac{rK}{Y}$ and $\sigma_L \equiv \frac{wL}{Y}$ as the income shares of capital and labor respectively. Then the relative factor income share $\frac{\sigma_K}{\sigma_L} = \frac{rK}{wL}$ becomes:

$$\frac{\sigma_K}{\sigma_L} = \frac{d_N\varphi(1-\varphi\beta)}{d_M\phi(1-\phi\beta)}\left(\frac{{S_M}^*}{S-{S_M}^*}\right)^{1-\mu} \tag{29}$$

Because $\mu < 1$, $\frac{\partial(\sigma_K/\sigma_L)}{\partial {S_M}^*} > 0$. Since the capital-augmenting technological progress is positively related to ${S_M}^*$, equation (29) shows that, given other factors, when technological progress is more capital-augmenting, the income share of capital becomes relatively higher, which verifies the conclusion of the induced innovation literature. However, as has been proven before, the direction of technological progress in the steady state of this specific model can be of any type, and it is not required to be purely labor-augmenting. This shows once again that although the innovation possibilities frontier proposed by the induced innovation literature is very important for the micro mechanism of the direction of technological progress, just introducing the innovation possibility frontier into the neoclassical growth model cannot provide the determinants of the direction of technological progress, if the key parameters of the production function and factor accumulation processes are implicitly ignored.

5 Applications

In this section, we discuss some of the applications of the above findings. First, we address the puzzle of the Uzawa's steady-state theorem, next we explain the characteristics of economic growth before and after the industrial revolution, and finally we discuss the contradiction between the decline of investment prices and Kaldor's facts.

5.1. The puzzle of Uzawa's (1961) steady-state theorem

Although the direction of technological progress itself is of great importance to economic growth, the interest in it first arose from the puzzle of the Uzawa's steady-state theorem. The theorem states that if the neoclassical growth model possesses a steady-state growth path, then technological progress must be purely labor-augmenting at least in that steady state, unless the production function is Cobb-Douglas. This theorem has plagued the economic growth theory ever since it was published and attracted much attention of many authors. Based on the previous results of the model on the determinants of technological progress direction, we will first provide our explanation of the puzzle of Uzawa's steady-state theorem, and then provide our comments on the explanations of existing literature, pointing out their shortcomings.

(1) Why is Uzawa's steady-state theorem a puzzle?

Why does existing literature argue that the Uzawa's steady-state theorem is a puzzle? Because on the one hand, existing literature argues that the requirement of the theorem is an extremely restrictive and troubling requirement (Aghion and Howitt, 1998, 16n; Schlicht, 2006; Acemoglu,2009, p59), but on the other hand, they argue that the assumptions of neoclassical growth model seem very standard. The puzzle is that why the "standard assumptions" lead to extremely result.

However, we have pointed out that the capital accumulation function of neoclassical growth models implicitly infinite supply elasticity of capital which is a very special case and not "standard" at all. Corollary 1 previously indicates that in this situation, technological progress is inevitably purely labor-augmenting, and it is not surprising at all.

Although existing literature (Schlicht, 2006; Jones and Scrimgeour, 2008; Acemoglu, 2009, ch2) recognizes that the capital accumulation function $\dot{K} = I - \delta_K K$ is the crucial assumption leading to the theorem, they are not aware that this function implies capital with infinite supply elasticity ($\varepsilon_K = \infty$) and it is not a "standard assumption". If realizing that capital accumulation has infinite supply elasticity, then it would be an intuitive conclusion that technological progress does not improve capital productivity, that is, excluding capital-augmentation in steady state.

(2) Explanations in existing literature on the Uzawa's steady-state theorem

Existing literature (Acemoglu,2003; Schlicht, 2006; Jones and Scrimgeour, 2008) argue that the intuition for result of Uzawa's steady-state theorem is the asymmetry that capital, K, can be accumulated, while labor, L, cannot. More specifically, capital is produced from output while labor follows an independent accumulation equation, producing the need for labor augmenting technical progress to balance the growth rates of effective capital and labor along a balanced growth path. This may be the mainstream viewpoint in existing literature. However, it not only fails to explain the puzzle of Uzawa's steady state theorem, but also is incorrect and may lead to incorrect conclusions.

First, even if this explanation is correct, it does not answer why the requirement of the theorem is an extremely restrictive, very surprising and troubling result. Existing literature is surprised by Uzawa's steady-state theorem is not that it requires technological progress to include labor-augmenting, but rather not to include any capital-augmenting, that is, to be purely labor-augmenting. Obviously, only by recognizing that the capital accumulation function of neoclassical growth model implies infinite supply elasticity of capital, rather than a "standard assumption", can we realize that this requirement is in line with intuition.

Moreover, this explanation is not correct. In fact, as previously proved in this article, if the capital accumulation function is $\dot{K}(t) = I_K^{\alpha_K} - \delta_K K$, $0 < \alpha_K < 1$ as proposed by Irmen (2013), while labor growth exogenously as $\dot{L}(t) = nL(t)$, then the technological change is not purely labor-augmenting, include capital-augmenting, and the rate of capital-augmentation can even greater than the rate of labor-augmentation. However, in this case the capital is produced from output while labor follows an independent accumulation equation.

Furthermore, this explanation that the core is the asymmetry between accumulation of capital and labor, may lead to incorrect results. Acemoglu (2009, ch15.6) may follow this explanation precisely, allowing capital to accumulate infinitely but with finite supply elasticity ($\dot{K}/K = s_K$, where s_K is given exogenously), while labor is given exogenously, so capital and labor accumulation are asymmetric. Therefore, he argues that technological progress in this model should still be purely labor-augmenting in steady state, but on the contrary, technological progress can include capital-augmenting in this model. As a result, he came up with the incorrect Proposition 15.12 (see the proof by Li, 2016).

(3) Defects of Irmen's Generalized Uzawa's steady-state Theorem Irmen (2018) proposed a generalized Uzawa's steady-state theorem emphasizing two conditions (regarding capital accumulation and the CRS property of the production function) but neither of them is accurate. First, it is not accurately pointed out that to obtain a purely labor-augmenting technological change the capital accumulation function must imply an infinite capital supply elasticity; Second, it is not clearly pointed out that the production function requires only capital to have a constant marginal transformation rate into effective capital, but not labor. In other words, it is unnecessary to impose constant returns to scale for both capital and labor. As a result, one may accept false consequences and reject true one. As pointed by Irmen himself, the model of Acemoglu (2009, chapter 15.6) satisfies the conditions of the generalized Uzawa's steady-state theorem. Nevertheless, as argued above, in that model the steady-state technological progress of that model cannot be purely labor-augmenting. In contrast, if the production function is specified as $\mathrm{Y} = \mathrm{F}[\mathrm{BK}, AL^{\varphi}]$, and $\varphi < 1$, but the capital accumulation function $\dot{K} = sY - \delta_K K$ remains unchanged, despite the fact that the model does not meet the requirements of Irmen's generalized Uzawa's steady-state theorem, the technological progress must be purely labor-augmenting in steady state.

5.2. Why is modern economic growth characterized by the Kaldor facts?

Kaldor's (1961) stylized facts characterize principal features of modern economic growth. They have been recently verified by Jones (2016). Why does modern economic growth display these characteristics? The neoclassical growth model yields steady-state growth which is consistent with the Kaldor facts by *assuming* that technological progress is purely labor-augmenting. However, the neoclassical growth model cannot, on the one hand, be surprised that the Uzawa's steady-state theorem requires technological progress to be purely labor-augmenting, and on the other hand, argue that it has successfully explained the Kaldor facts. However, according to corollary 1 of this article, when capital has infinite supply elasticity and labor has finite supply elasticity, that is, $\varepsilon_K = \infty$ and $\varepsilon_L < \infty$, technological progress in the steady state will be purely labor-augmenting, and the characteristics of the steady state are consistent with the Kaldor facts. Therefore, although the infinite supply elasticity of capital is only a special case, it does not mean that it cannot exist in reality. In fact, it may be basically in line with the historical circumstances that enabled western European countries to start the process of modern economic growth. Actually, for quite a long time after the industrial revolution, only few western European countries entered the phase of capital-based industrialization. The resources needed to produce capital and accumulate it were drawn from the entire globe, making the capital supply elasticity quite large. Perhaps it is precisely because the capital accumulation function of neoclassical growth model has long been in line with the history of modern economic growth in western European countries lead that existing literature regards this capital accumulation function as a "standard assumption", completely unaware that it is a special case implicit capital with infinite supply elasticity. In addition, due to the demographic change, the higher per capita income no longer increased the birth rate but rather reduced it. The elasticity of the labor supply became finite. Such changes are consistent with the emergence of a purely labor-augmenting economic growth path and hence also with the Kaldor facts.

5.3. Why did technological progress not increase per capita income before the industrial revolution?

Maddison (2003) and Ashraf and Galor (2011) argue that before the industrial revolution technological progress only increased land productivity and population density but had little impact on labor productivity. According to corollary 1, if the production function has constant returns to scale and if labor has an infinite supply elasticity, there can be no labor-augmenting technological progress in the steady-state. The growth model of Section IV also shows that when the $\alpha_L = 1$, the labor supply elasticity is infinite and per capita income remains unchanged in the steady state. In that steady state, technological progress and land growth can only lead to population growth and increased population density, which is consistent with the empirical findings (Maddison, 2003; Ashraf and Galor, 2011). Therefore, the stagnation of technology and the shortage of land are not the crucial causes of the Malthusian trap. Rather, it is the infinite labor supply elasticity that is fundamental to the result.

5.4. What led to the industrial revolution?

While this paper does not build a Unified Growth Model in the spirit of Galor (2011), it may shed some light on the transformation from the Malthusian trap to modern growth. From the perspective of the direction of technological progress, industrial revolution amounts to a transition from a path that completely excludes labor augmentation to one that includes it, that is, from $\frac{\dot{A}(t)}{A(t)} = 0$ to $\frac{\dot{A}(t)}{A(t)} > 0$. From corollary 1 or Table 1, we can see that the fundamental reason for such a transition is that the labor supply elasticity changes from infinite to finite (if the production function has constant returns to scale with respect to the factors). While this paper provides no mechanism that may cause such an elasticity change, this is one of the core contents of Galor's (2011) Unified Growth Model.

5.5. Can investment-specific technological progress and purely labor-augmenting technological progress be compatible in steady state?

Although purely labor-augmenting technological progress can explain the Kaldor facts, empirical data also find another important feature characterizing modern economic growth; The relative price of capital equipment, adjusted for quality, has been falling steadily for decades, as shown in figure 2 (Gordon, 1990; Greenwood et al., 1997; Grossman et al., 2017; Jones, 2016).

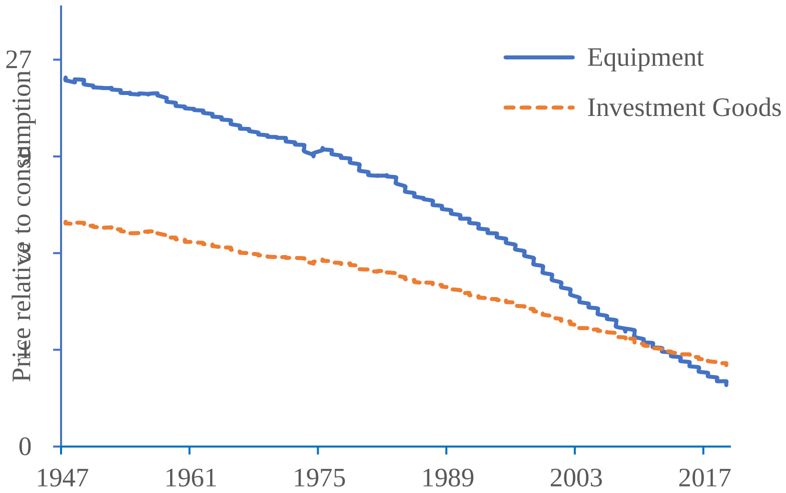

Figure 2 US Relative Price of Equipment and Investment goods, 1947–2019. Source: Federal Reserve Bank Economic Data (FRED), series PIRIC and PERIC

This finding indicates the presence of investment-specific technological progress. Unless the production function is Cobb-Douglas, with such technological progress the steady-state growth path of the existing neoclassical growth models will no longer be purely labor-augmenting or even fail to exist. This poses a new difficult problem for the neoclassical growth model.

At present, there are two solutions for the problem in the literature: one is to argue that the production function is indeed Cobb-Douglas, at least in the steady state (León-Ledesma and Satchi, 2018); the other is to introduce capital-augmenting technological progress into the production function (Grossman et al., 2017; Casey and Horii, 2022). The empirical evidence showing that the substitution elasticity of capital and labor is not unitary, indicates that the Cobb-Douglas production function specification may be empirically invalid.[14] While the steady-state equilibrium can admit capital-augmenting technological progress, in that case the capital/output ratio will continue to decline in steady-state, which is inconsistent with the Kaldor facts.

We have shown above in section IV, that with constant return to scale to the factors, i.e., $\phi = \varphi = 1$, both investment-specific and purely labor-augmenting technological progress can concurrently be present at the steady state, provided the investment adjustment costs increase at a rate that just offsets the investment-specific technological progress, (that is, $\alpha_K = 1 - g_q/g^*$). The CES specification of the production function implies that the elasticity of factor substitution is not required to be 1. Indeed, there is a steady state in the model which is consistent with the long-term stability of factor income shares for any elasticity of substitution between capital and labor. At the same time, technological progress in the final good production is purely labor-augmenting which is consistent with the Kaldor facts. Of course, whether this resolution of the neoclassical model's conflict with the empirical findings is more plausible than those suggested by the existing literature is a matter of further empirical study.

6 Discussion on Alternative Assumptions

In the specific model of Section III, the factor accumulation processes take the form proposed by Irmen (2013) and the innovation possibilities frontier is drawn from the knowledge spillover model. In this section, we discuss alternative factor accumulation processes proposed by Sato (1996), and the alternative innovation possibilities frontier implied by the lab equipment model proposed by Rivera-Batiz and Romer (1991). In the discussion, where we replace only one assumption at a time, we do not consider investment-specific technological progress and keep the other assumptions unchanged. The results show that these alternative assumptions do not affect the core conclusions on the determinants of direction of technological progress in steady state but do require a slight extension.

[14] See, e.g., Karabarbounis and Neiman, 2014; Oberfield and Raval, 2014; Chirinko and Mallick, 2014; Lawrence, 2015; Knoblach, Roessler and Zwerschke, 2020.

6.1. Different form of factor accumulation processes [15]

Sato (1996) constructed a nonlinear capital accumulation process based on Leviathan and Samuelson (1969) whereby $\frac{\dot{\mathrm{K}}(\mathrm{t})}{\mathrm{K}(\mathrm{t})}=\frac{I_K(t)}{B(t)\mathrm{K}(\mathrm{t})}-\delta_K$, but only considered the constant marginal transformation rate of capital to effective capital. We extend this specification to consider the diminishing marginal transformation rates and also include labor as follows:

$$\begin{cases}\dfrac{\dot{\mathrm{K}}(\mathrm{t})}{\mathrm{K}(\mathrm{t})}=\dfrac{I_K(t)}{B(t)\mathrm{K}(\mathrm{t})^{\phi}}-\delta_K\\ \dfrac{\dot{\mathrm{L}}(\mathrm{t})}{L(t)}=\dfrac{I_L(t)}{A(t)L(t)^{\varphi}}-\delta_L\end{cases} \tag{30}$$

From equations (30), the dynamic capital and labor growth rate are modified as follows:

$$\begin{cases}\dfrac{\dot{g}_K}{g_K}=\dfrac{g_K+\delta_K}{g_K}\left(i_K-\dfrac{1-\phi\beta}{\beta}[d_M S_M{}^{\mu}-\delta]-\phi g_K\right)\\ \dfrac{\dot{g}_L}{g_L}=\dfrac{g_L+\delta_L}{g_L}\left(i_L-\dfrac{1-\varphi\beta}{\beta}[d_N(S-S_M)^{\mu}-\delta]-\varphi g_L\right)\end{cases} \tag{31}$$

Analogously, the Euler equations of the household are become:

$$\begin{cases}\theta\dfrac{\dot{C}}{C}=\beta\phi^2 f'(k)-\phi\delta_K+\dfrac{1-\varphi\beta}{\beta}[d_N(S-S_M)^{\mu}-\delta]-\rho\\ \theta\dfrac{\dot{C}}{C}=\beta\varphi^2[f(k)-kf'(k)]-\varphi\delta_L+\dfrac{1-\varphi\beta}{\beta}[d_N(S-S_M)^{\mu}-\delta]-\rho\end{cases} \tag{32}$$

Here too the steady-state direction of technological progress depends on the parameters of the model. These can also be divided into two categories: one that contains the parameters that determine the marginal transformation rates of production factors into effective production factors (ϕ and φ), and the other that relates to the parameters which determine the factor supply elasticities. The direction of technological progress can admit all types from pure labor augmentation to pure capital augmentation, and Proposition 1 of the reduced form model still holds.

6.2. Different Forms of Technical Progress[16]

The lab equipment model was suggested by Rivera-Natiz and Romer (1991), and is used in Acemoglu (2002, 2003, 2009). In those models, the main input of the R&D

[15] The detailed solution process of the steady state equilibrium is available from the authors upon request.

[16] The detailed solution process of the steady state equilibrium is available from the authors upon request.

sectors is final output. As a result, the accumulation processes and the R&D sectors compete for resources. We expand the lab equipment model to consider the nonlinear characteristics of the return to innovation investment which yields the existence of the steady state growth. Specifically, the innovation functions take the form:

$$\begin{cases} \dot{M} = b_M I_M^{\alpha_M} - \delta_M M, \ \ b_M > 0, \ \ 0 \leq \alpha_M \leq \dfrac{\beta}{1-\phi\beta}, \delta_M > 0 \\ \dot{N} = b_N I_N^{\alpha_N} - \delta_N N, \ \ b_N > 0, \ \ 0 \leq \alpha_N \leq \dfrac{\beta}{1-\varphi\beta}, \delta_N > 0 \end{cases} \tag{33}$$

where I_M and I_N are final-good investments needed to develop new varieties M and N of the respective intermediate inputs, and δ_M and δ_N are respectively deprecation (or obsolescence) rates affecting blueprints of new varieties of capital- and labor-intensive intermediate inputs.

Different from the knowledge spillover model, the following knife-edge conditions must be met for a steady state equilibrium in the lab equipment model to exist:

$$\begin{cases} \dfrac{1-\phi\beta}{\beta}\alpha_M + \phi\alpha_K = 1 \\ \dfrac{1-\varphi\beta}{\beta}\alpha_N + \varphi\alpha_L = 1 \end{cases} \tag{34}$$

Accordingly, when $\phi = \alpha_K = 1$, then α_M must be set to 0. However, Acemoglu (2003) assumes $\alpha_M = 1$, which violates the knife-edge condition. As a result, there is no steady-state path in his setting. When this knife-edge condition is satisfied, the steady-state equilibrium exists, and Proposition 1 also holds. The direction of technological progress can be of any type.

7 Concluding Remarks

Technological progress may be the fundamental factor in all economic problems, with the three core elements such as speed, direction, and bias. The determinants of its equilibrium speed can be studied by endogenous technological progress models (Romer,1990; Grossman and Helpman,1991; Aghion and Howitt,1992), while the determinants of its equilibrium bias can be studied by the directed technological progress model (Acemoglu, 2002). Based on these models, this article provides a framework for studying the determinants of its equilibrium direction. Although profit incentives are the same core forces that simultaneously affect the speed, bias, and direction of technological progress, the determinants of the direction of technological progress in steady state are the factor supply elasticities implied by the factor accumulation functions and the marginal transformation rate of factors into effective factors in the production function. The change in relative prices of factors not only incentives invention to economize the use of the relatively expensive factor, but also promotes its supply. However, the equilibrium direction of technological innovation does not depend on the relative supply or the relative price of factors, but rather on the relative size of the supply elasticities of factors, and biased to the factor with smaller elasticity. The results of this framework provide new insights for understanding the puzzle of Uzawa's steady-state theorem and other economic growth problems.

The puzzle of Uzawa's steady-state theorem is a reflection of the deficiencies of existing growth models in analyzing the equilibrium direction of technological progress, reflecting the neglect of the factor supply elasticities implied by factor accumulation functions. The neoclassical growth models, on the one hand, argue that steady-state technological progress must be purely labor-augmenting is an extreme requirement, and on the other hand, think its capital accumulation function is a "standard assumption". Therefore, it leads to a puzzle: "standard assumptions" lead to the extreme result of Uzawa's steady-state theorem. On the contrary, if it is recognized that the capital accumulation function of the neoclassical growth model implies that capital has infinite supply elasticity, then in the steady state, technological progress does not include capital-augmentation but pure labor-augmentation, is an intuitive conclusion. Therefore, the problem is not why technological progress in steady state is purely labor-augmenting, but whether the capital accumulation function with infinite supply elasticity is in line with the reality.

While *theoretically* factor supply elasticities may range from zero to infinity, their *actual* values are subject to an empirical investigation. In view of their key impact on the direction of technological progress, estimating these parameters should be an important task for future empirical research. This is particularly true where investment adjustment costs are concerned because of their impact on factor supply elasticities and thereby on the direction of technological progress. Moreover, the presence of investment adjustment costs affects the standard perpetual inventory method used to calculate capital stocks, as the contribution of current investment may be no longer linear. Potentially, the commonly used assessments of the capital stock may be systematically upwards biased. This, in turn, may affect the estimated aggregate production functions and the ensuing consequences such as the elasticity of substitution between labor and capital and the like. It is in this sense that we believe that the theoretical challenges presented in this paper are also a call to empirically reassess the neoclassical growth model.

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Appendix A: The implications of investment adjustment costs

Consider the following standard specification of the factor accumulation process:

$$\dot{K}(t) = I_{KN}(t) - \delta_K K(t) \tag{A1}$$

where $\dot{K}(t) \equiv \frac{dK(t)}{dt}$, represents the newly increased capital at time t, $I_{KN}(t)$ is the investment to increase capital, δ_K is the rate of deprecation.

The presence of the investment adjustment costs is specified as follows:

$$I_{KN}(t)\{1 + \mathrm{h}[I_{KN}(t), K(t), B(t)]\} = I_K(t) \tag{A2}$$

That is, investing a total of $I_K(\mathrm{t})$ units in factor K enhances the amount of capital only by $I_{KN}(t)$ units, whereby a proportional addition of $\mathrm{h}[I_{KN}(t), K(t), B(t)]$ units is spent as adjustment costs.[17] Different from the existing literature, equation (A2) assumes that the adjustment cost is not only a function of the investment quantity $I_{KN}(t)$ and the capital stock $K(t)$, but also a function of the capital-augmenting technology $B(t)$ whereby it is assumed that higher levels of capital-augmenting technology imply higher equipment installation costs, that is, $\frac{\partial \mathrm{h}()}{\partial B(t)} > 0$. We also assume $\mathrm{h}[0, K(t), B(t)] = 0$, $\frac{\partial \mathrm{h}()}{\partial I_{KN}(t)} > 0$, $\frac{\partial^2 \mathrm{h}()}{\partial I_{KN}(t)^2} \geq 0$, that is, the adjustment costs are non-decreasing also at the margin. Monotonicity allows us to obtain the inverse function $\mathrm{I_{KN}(t)} = \mathrm{G}[I_K(\mathrm{t}), K(t), B(t)]$. Substituting it into equation (A1) yields the generalized factor accumulation process that incorporates the investment adjustment costs:

$$\dot{K}(t) = G[I_K(t), K(t), B(t)] - \delta_K K(t) \tag{A3}$$

with $\frac{\partial \mathrm{G}}{\partial I_K(\mathrm{t})} > 0$ and $\frac{\partial^2 \mathrm{G}}{\partial I_K(\mathrm{t})^2} \leq 0$. Equation (A3) is equation (4) in the text which shows that with adjustment costs, capital accumulation is a nonlinear function of investment, and the marginal efficiency of turning investment into a production factor is (weakly) decreasing.

(1) the case of Sato (1996)

Let the investment adjustment cost function be:

$$\mathrm{h}[I_{KN}(t), K(t), B(t)] = B(t)K(t)^{\phi-1}\psi\left[\frac{I_{KN}(t)}{K(t)}\right] - 1, \psi' > 0, \psi'' > 0 \quad (A4)$$

Substituting (A4) into equation (A2) we obtain:

[17] Equations (A1) and (A2) are equations (3.25) and (3.26) in Barro and Sala-i-Martin (2004), Chapter 3, page 152, except that their adjustment cost function is specified as $\mathrm{h}(I_K/K)$.

$$I_{KN}(t)B(t)K(t)^{\phi-1}\psi\left[\frac{I_{KN}(t)}{K(t)}\right]=I_K(t) \tag{A5}$$

Rearrange this equation to get:

$$\frac{I_{KN}(t)}{K(t)}\psi\left[\frac{I_{KN}(t)}{K(t)}\right]=\frac{I_K(t)}{B(t)K(t)^{\phi}} \tag{A6}$$

The inverse function of equation (A6) is:

$$\frac{I_{KN}(t)}{K(t)}=\Phi\left[\frac{I_K(t)}{B(t)K(t)^{\phi}}\right],\Phi'>0,\Phi''<0 \tag{A7}$$

Substitute (A7) into equation (A1) to obtain:

$$\dot{\mathrm{K}}(\mathrm{t})=\mathrm{K}(\mathrm{t})\Phi\left[\frac{I_K(t)}{B(t)K(t)^{\phi}}\right]-\delta_K\mathrm{K}(\mathrm{t}),\Phi'>0,\Phi''<0 \tag{A8}$$

When $\phi=1$, equation (A8) is the Sato (1996) capital accumulation function. However, he did not point out that this function is related to investment adjustment costs.

(2) the case of Irmen (2013)

Let the investment adjustment cost function be:

$$\mathrm{h}[I_{KN}(t),K(t),B(t)]=I_{KN}(t)^{(1-\alpha)/\alpha}-1,\quad 0<\alpha\le 1 \tag{A9}$$

Substitute it into equation (A2) to obtain:

$$I_{KN}(t)^{1/\alpha}=I_K(t) \tag{A10}$$

Substitute (A10) into equation (A1) to obtain Irmen's (2013) capital accumulation function:

$$\dot{K}(t)=I_K(t)^{\alpha}-\delta_K K(t),\qquad \alpha<0<1 \tag{A11}$$

Irmen pointed out that this function reflected investment adjustment cost but did not show the specific adjustment cost function.

Due to its simplicity, we use equation (A11) as the capital accumulation function in the specific model in Section 4 but discuss the results when the capital accumulation function is described by equation (A8) in Section 5.

Appendix B：The derivation of equation (6)

From $k(\mathrm{t})\equiv\frac{\mathrm{B}(t)\mathrm{K}(t)^{\phi}}{A(t)\mathrm{L}(\mathrm{t})^{\varphi}}$ we can obtain

$$\phi\frac{\dot{\mathrm{K}}(\mathrm{t})}{\mathrm{K}(\mathrm{t})}+\frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})}=\varphi\frac{\dot{\mathrm{L}}(\mathrm{t})}{\mathrm{L}(\mathrm{t})}+\frac{\dot{\mathrm{A}}(\mathrm{t})}{\mathrm{A}(\mathrm{t})}+\frac{\dot{\mathrm{k}}(\mathrm{t})}{\mathrm{k}(\mathrm{t})} \tag{B1}$$

Dividing the denominator of equation (7) by the two sides of equation (B1) yields:

$$DTP \equiv \frac{\dot{\mathrm{B}}(\mathrm{t})/\mathrm{B}(\mathrm{t})}{\dot{\mathrm{A}}(\mathrm{t})/A(t)} = \frac{1+\varphi \frac{\dot{\mathrm{L}}(\mathrm{t})}{L(t)} \Big/ \frac{\dot{\mathrm{A}}(\mathrm{t})}{A(t)} + \frac{\dot{\mathrm{k}}(\mathrm{t})}{\mathrm{k}(\mathrm{t})} \Big/ \frac{\dot{\mathrm{A}}(\mathrm{t})}{A(t)}}{1+\phi \frac{\dot{\mathrm{K}}(\mathrm{t})}{\mathrm{K}(\mathrm{t})} \Big/ \frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})}} \tag{B2}$$

From the equations (3) the growth rates of $r(t)$ and $w(t)$ can be can obtained as following:

$$\begin{cases} \frac{\dot{w}(\mathrm{t})}{w(t)} = \frac{\dot{\mathrm{A}}(\mathrm{t})}{A(t)} + (\varphi - 1)\frac{\dot{\mathrm{L}}(\mathrm{t})}{L(t)} + \frac{-[k(t)]^2 f''(k(t))}{f(k(t)) - k(t)f'(k(t))}\frac{\dot{k}(t)}{k(t)} \\ \frac{\dot{r}(\mathrm{t})}{r(t)} = \frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})} + (\phi - 1)\frac{\dot{\mathrm{K}}(\mathrm{t})}{\mathrm{K}(\mathrm{t})} + \frac{k(t)f''(k(t))}{f'(k(t))}\frac{\dot{k}(t)}{k(t)} \end{cases} \tag{B3}$$

Let $\sigma_{\hat{L}} \equiv \frac{-[k(t)]^2 f''(k(t))}{f(k(t)) - k(t)f'(k(t))}$, $\sigma_{\hat{K}} \equiv \frac{k(t)f''(k(t))}{f'(k(t))}$, substitute them into equation (B3) then yields:

$$\begin{cases} \frac{\dot{w}(\mathrm{t})}{w(t)} = \frac{\dot{\mathrm{A}}(\mathrm{t})}{A(t)} + (\varphi - 1)\frac{\dot{\mathrm{L}}(\mathrm{t})}{L(t)} + \sigma_{\hat{L}}\frac{\dot{k}(t)}{k(t)} \\ \frac{\dot{r}(\mathrm{t})}{r(t)} = \frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})} + (\phi - 1)\frac{\dot{\mathrm{K}}(\mathrm{t})}{\mathrm{K}(\mathrm{t})} + \sigma_{\hat{K}}\frac{\dot{k}(t)}{k(t)} \end{cases} \tag{B4}$$

Substituting equations (B4) into equations (4) then yields:

$$\begin{cases} \varepsilon_L(t) = \frac{\dot{\mathrm{L}}(\mathrm{t})/L(t)}{\frac{\dot{\mathrm{A}}(\mathrm{t})}{A(t)} + (\varphi - 1)\frac{\dot{\mathrm{L}}(\mathrm{t})}{L(t)} + \sigma_{\hat{L}}\frac{\dot{k}(t)}{k(t)}} \\ \varepsilon_K(t) = \frac{\dot{\mathrm{K}}(\mathrm{t})/K(t)}{\frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})} + (\phi - 1)\frac{\dot{\mathrm{K}}(\mathrm{t})}{K(t)} + \sigma_{\hat{K}}\frac{\dot{k}(t)}{k(t)}} \end{cases} \tag{B5}$$

Rearrange equation (B5) to obtain:

$$\begin{cases} \frac{1}{\varepsilon_L(t)} = \frac{\dot{\mathrm{A}}(\mathrm{t})/A(t)}{\dot{\mathrm{L}}(\mathrm{t})/L(t)} + (\varphi - 1) + \sigma_{\hat{L}}\frac{\dot{k}(\mathrm{t})/k(\mathrm{t})}{\dot{\mathrm{L}}(\mathrm{t})/L(t)} \\ \frac{1}{\varepsilon_K(t)} = \frac{\dot{\mathrm{B}}(\mathrm{t})/B(t)}{\dot{\mathrm{K}}(\mathrm{t})/K(t)} + (\phi - 1) + \sigma_{\hat{K}}\frac{\dot{k}(\mathrm{t})/k(\mathrm{t})}{\dot{\mathrm{K}}(\mathrm{t})/K(t)} \end{cases} \tag{B6}$$

Rearrange equation (B6) to obtain:

$$\begin{cases} \frac{\dot{\mathrm{L}}(\mathrm{t})}{L(t)} \Big/ \frac{\dot{\mathrm{A}}(\mathrm{t})}{A(t)} = \frac{\varepsilon_L(t)}{1 + (1-\varphi)\varepsilon_L(t) - \varepsilon_L(t).\sigma_{\hat{L}}\frac{\dot{k}(\mathrm{t})/k(\mathrm{t})}{\dot{\mathrm{L}}(\mathrm{t})/L(t)}} \\ \frac{\dot{\mathrm{K}}(\mathrm{t})}{\mathrm{K}(\mathrm{t})} \Big/ \frac{\dot{\mathrm{B}}(\mathrm{t})}{\mathrm{B}(\mathrm{t})} = \frac{\varepsilon_K(t)}{1 + (1-\phi)\varepsilon_K(t) - \varepsilon_K(t).\sigma_{\hat{K}}\frac{\dot{k}(\mathrm{t})/k(\mathrm{t})}{\dot{\mathrm{K}}(\mathrm{t})/K(t)}} \end{cases} \tag{B7}$$

Substitute equations (B7) into equation (B2) and rearrange to get

$$DTP(t) = \frac{\left[\dfrac{1+\varepsilon_L(t)-\varepsilon_L(t).\sigma_{\hat{L}}\dfrac{\dot{k}(\mathrm{t})/k(\mathrm{t})}{\dot{\mathrm{L}}(\mathrm{t})/L(t)}}{1+(1-\varphi)\varepsilon_L(t)-\varepsilon_L(t).\sigma_{\hat{L}}\dfrac{\dot{k}(\mathrm{t})/k(\mathrm{t})}{\dot{\mathrm{L}}(\mathrm{t})/L(t)}}\right]+\dfrac{\dot{\mathrm{k}}(\mathrm{t})}{\mathrm{k}(\mathrm{t})}\Big/\dfrac{\dot{\mathrm{A}}(\mathrm{t})}{A(t)}}{\left[\dfrac{1+\varepsilon_K(t)-\varepsilon_K(t).\sigma_{\hat{K}}\dfrac{\dot{k}(\mathrm{t})/k(\mathrm{t})}{\dot{\mathrm{K}}(\mathrm{t})/K(t)}}{1+(1-\phi)\varepsilon_K(t)-\varepsilon_K(t).\sigma_{\hat{K}}\dfrac{\dot{k}(\mathrm{t})/k(\mathrm{t})}{\dot{\mathrm{K}}(\mathrm{t})/K(t)}}\right]} \tag{B8}$$

Appendix C：Proof of Corollary 1

Substituting $\varphi = 1$ into (B4), we obtain:

$$\varepsilon_L = \frac{\dot{\mathrm{L}}(\mathrm{t})/L(t)}{\dot{\mathrm{A}}(\mathrm{t})/A(t)} \tag{$C1$}$$

Since $\frac{\dot{\mathrm{L}}(\mathrm{t})}{L(t)} < \infty$, if $0 \leq \varepsilon_L < \infty$, then $\frac{\dot{\mathrm{A}}(\mathrm{t})}{A(t)} > 0$, if $\varepsilon_L = \infty$, then $\frac{\dot{\mathrm{A}}(\mathrm{t})}{A(t)} = 0$.

An analogous proof applies for $\phi = 1$.

Appendix D: Proof of Proposition 2.

Letting the final good serve as numeraire, the representative competitive final good producer faces the input prices p_{L} and p_{K} and selects the respective Y_K and Y_L to maximize

$$\pi_Y = Y - p_{\mathrm{L}}Y_L - p_{\mathrm{K}}Y_K \tag{$D1$}$$

subject to the production function (10), yielding the demand functions:

$$\begin{cases} p_K = (1-\gamma)[\gamma + (1-\gamma)(Y_K/Y_L)^{(\varepsilon-1)/\varepsilon}]^{1/(\varepsilon-1)}(Y_K/Y_L)^{-1/\varepsilon} \\ p_L = \gamma[\gamma + (1-\gamma)(Y_K/Y_L)^{(\varepsilon-1)/\varepsilon}]^{1/(\varepsilon-1)} \end{cases}. \tag{$D2$}$$

The reperesentative producer of Y_K and Y_L maximizes profits by choosing $Z(j)$ and $X(i)$, given the intermediate input prices $p_{\mathrm{Z}}(j)$ and $p_{\mathrm{X}}(i)$:

$$\begin{cases} \pi_K = p_K Y_K - \int_0^M p_Z(j)Z(j)dj \\ \pi_L = p_L Y_L - \int_0^N p_X(i)X(i)di \end{cases} \tag{$D3$}$$

subject to their respective production functions (11). This generates the demand functions

$$\begin{cases} Z(j) = (Y_K)^{\frac{1-\beta}{1-\phi\beta}}(\phi p_K/p_Z(j))^{1/(1-\phi\beta)} \\ X(i) = (Y_L)^{\frac{1-\beta}{1-\varphi\beta}}(\varphi p_L/p_X(i))^{1/(1-\varphi\beta)} \end{cases} \quad (D4)$$

The intermediate input producers, who hold the exclusive right to produce their particular type of input, face the prices of the primary inputs and choose, respectively, $(p_Z(j), K(j))$ and $(p_X(i), L(i))$ to maximize

$$\begin{cases} \pi_Z(j) = p_Z(j)Z(j) - rK(j) \\ \pi_X(i) = p_X(i)X(i) - wL(i) \end{cases} \quad (D5)$$

subject to their technologies (12) and the demand functions (D4).

From the maximization (D5) we obtain:

$$\begin{cases} p_Z(j) = p_Z = r/\phi\beta \\ p_X(i) = p_X = w/\varphi\beta \end{cases} \quad (D6)$$

which imply that all intermediate inputs have the same mark-up over marginal cost. Substituting equations (D6) into (D4), we find that all capital-intensive and all labor-intensive intermediate goods are produced in equal (respective) quantities.

$$\begin{cases} Z(j) = (Y_K)^{\frac{1-\beta}{1-\phi\beta}}(\beta\phi^2 p_K/r)^{1/(1-\phi\beta)} \\ X(i) = (Y_L)^{\frac{1-\beta}{1-\varphi\beta}}(\beta\varphi^2 p_L/w)^{1/(1-\varphi\beta)} \end{cases} \quad (D7)$$

By the production functions of the intermediate inputs (12), all monopolists have the same respective demand for labor and capital.

The material factor market clearing conditions imply:

$$\begin{cases} Z(j) = K/M \\ X(i) = L/N \end{cases} \quad (D8)$$

Substituting equations (D8) into (11), we obtain the equilibrium quantities of the labor-intensive and capital-intensive inputs:

$$\begin{cases} Y_L = \left[\int_0^N X(i)^{\varphi\beta} di\right]^{1/\beta} = N^{(1-\varphi\beta)/\beta} L^{\varphi} \\ Y_K = \left[\int_0^M Z(j)^{\phi\beta} dj\right]^{1/\beta} = M^{(1-\phi\beta)/\beta} K^{\phi} \end{cases} \quad (D9)$$

Substituting equations (D9) into equation (10), we obtain:

$$Y = \left[\gamma(N^{(1-\varphi\beta)/\beta}L^{\varphi})^{(\varepsilon-1)/\varepsilon} + (1-\gamma)(M^{(1-\phi\beta)/\beta}K^{\phi})^{(\varepsilon-1)/\varepsilon}\right]^{\varepsilon/(\varepsilon-1)} \quad (D10)$$

Define $A \equiv N^{(1-\varphi\beta)/\beta}$ and $B \equiv M^{(1-\phi\beta)/\beta}$, then equation (D10) replicates equation (18) in the text.

Appendix E： Scientists' wages at the instantaneous equilibrium.

An innovating firm obtains invention patents by employing scientists for research and development. Its revenue is the market value of new patents paid by intermediate product manufacturers, and its cost is the salary of scientists. Therefore, the profit function of innovating firms is:

$$\begin{cases} \pi_M = V_Z\dot{M} - w_M S_M \\ \pi_X = V_X\dot{N} - w_N S_N \end{cases} \tag{E1}$$

where π_M and π_X are profits of the two kinds of innovating firms, $\dot{M}$ and $\dot{N}$ are the respective invention patents, V_Z and V_X are the market values of the two kinds of patents, and S_M and S_N are the respective scientist employment levels.

The first-order condition for profit maximization is that the marginal value of scientists is equal to their wages:

$$\begin{cases} w_M = V_Z\dfrac{\partial\dot{M}}{\partial S_M} \\ w_N = V_X\dfrac{\partial\dot{N}}{\partial S_N} \end{cases} \tag{E2}$$

From the innovation function, equations (14), we obtain

$$\begin{cases} \dfrac{\partial\dot{M}}{\partial S_M} = d_M\Omega'(S_M)M = \mu d_M {S_M}^{\mu-1}M \\ \dfrac{\partial\dot{N}}{\partial S_N} = d_N\Omega'(S_N)N = \mu d_N (S - S_M)^{\mu-1}N \end{cases} \tag{E3}$$

The market value of a patent is the present value of the respective profit streams V_Z and V_X:

$$\begin{cases} V_Z(t) = \displaystyle\int_t^\infty exp[-[\bar{r}(t,v)+\delta](v-t)]\pi_Z(v)dv \\ V_X(t) = \displaystyle\int_t^\infty exp[-[\bar{r}(t,v)+\delta](v-t)]\pi_X(v)dv \end{cases} \tag{E4}$$

where π_Z and π_X are the monopoly profits of capital- and labor-intensive intermediate enterprises, $\bar{r}(t,v) = \frac{1}{v-t}\int_t^v r(\omega)d\omega$ is the interest rate at date t, and δ is the depreciation (obsolescence) rate of existing intermediate inputs.

Substituting equations (12) and (D6) into (D5), we obtain:

$$\begin{cases}\pi_Z(j) = (r/\phi\beta - r)Z(j)\\ \pi_X(i) = (w/\varphi\beta - w)X(i)\end{cases} \tag{E5}$$

Substituting (D8) into (E5) yields:

$$\begin{cases}\pi_Z = \left(\dfrac{1-\phi\beta}{\phi\beta}\right).\dfrac{rK}{M}\\ \pi_X = \left(\dfrac{1-\beta\varphi}{\beta\varphi}\right).\dfrac{wL}{N}\end{cases} \tag{E6}$$

Substitute (E6) into (E4) to obtain:

$$\begin{cases}V_Z(t) = \left(\dfrac{1-\phi\beta}{\phi\beta}\right).\dfrac{rK}{M}\displaystyle\int_t^\infty exp[-[\bar{r}(t,v)+\delta](s-t)]dv\\ V_X(t) = \left(\dfrac{1-\beta\varphi}{\beta\varphi}\right).\dfrac{wL}{N}\displaystyle\int_t^\infty exp[-[\bar{r}(t,v)+\delta](s-t)]dv\end{cases} \tag{E7}$$

From equation (E7) we can obtain

$$\frac{V_Z(t)}{V_X(t)} = \frac{\varphi(1-\phi\beta)}{\phi(1-\beta\varphi)}.\frac{r}{w}.\frac{K}{L}.\frac{N}{M} \tag{E8}$$

Substituting (E3) and (E8) into the wage equations (E2) yields:

$$\frac{w_M}{w_N} = \frac{\mu d_M {S_M}^{\mu-1} M}{\mu d_N (S-S_M)^{\mu-1} N}\left(\frac{\varphi(1-\phi\beta)}{\phi(1-\beta\varphi)}.\frac{r}{w}.\frac{K}{L}.\frac{N}{M}\right) \tag{E9}$$

Finally, substituting equations (E8) into the migration equation of scientists, equation (15), we get:

$$\frac{\dot{S_M}}{S_M} = \psi\left[\frac{d_M {S_M}^{\mu-1}}{d_N(S-S_M)^{\mu-1}}.\frac{\varphi(1-\phi\beta)}{\phi(1-\beta\varphi)}.\frac{r}{w}.\frac{K}{L}\right] \tag{E10}$$

Which replicates equation (20) in the text.

Appendix F: The household Euler equations

Let the Hamiltonian associated with the household optimization problem be:

$$H = U(C)e^{-\rho t} + \lambda_K\left(b_K I_K^{\alpha_K} - \delta_K K\right) + \lambda_L\left(b_L I_L^{\alpha_L} - \delta_L L\right)$$

$$+\mu[wL + rK + w_M S_M + w_N S_N + \Pi - C - (I_K + I_L)] \tag{F1}$$

The first-order conditions are:

$$\begin{cases} C^{-\theta}e^{-\rho t} = \lambda_K \alpha_K b_K I_K^{\alpha_K - 1} \\ C^{-\theta}e^{-\rho t} = \lambda_L \alpha_L b_L I_L^{\alpha_L - 1} \\ C^{-\theta}e^{-\rho t} = \mu \end{cases} \tag{F2}$$

Taking log-derivatives of both sides of (F2) over time, we obtain

$$\begin{cases} -\theta \dfrac{\dot{C}}{C} - \rho = \dfrac{\dot{\lambda}_K}{\lambda_K} + (\alpha_K - 1)\dfrac{\dot{I}_K}{I_K} + g_q \\ -\theta \dfrac{\dot{C}}{C} - \rho = \dfrac{\dot{\lambda}_L}{\lambda_L} + (\alpha_L - 1)\dfrac{\dot{I}_L}{I_L} \\ -\theta \dfrac{\dot{C}}{C} - \rho = \dfrac{\dot{\mu}}{\mu} \end{cases} \tag{F3}$$

The motion equations of λ_K and λ_L are:

$$\begin{cases} \dot{\lambda}_K = -\partial H/\partial K = \lambda_K \delta_K - \mu r \\ \dot{\lambda}_L = -\partial H/\partial L = \lambda_L \delta_L - \mu w \end{cases} \tag{F4}$$

Based on (F2) and (F4), we obtain

$$\begin{cases} \dot{\lambda}_K/\lambda_K = \delta_K - r\alpha_K b_K I_K^{\alpha_K - 1} \\ \dot{\lambda}_L/\lambda_L = \delta_L - w\alpha_L b_L I_L^{\alpha_L - 1} \end{cases} \tag{F5}$$

Using (F5) in (F3), we obtain the Euler equations (F6).

$$\begin{cases} \dfrac{\dot{C}}{C} = \dfrac{1}{\theta}\left\{ r\alpha_K b_K I_K^{\alpha_K - 1} - (\alpha_K - 1)\dfrac{\dot{I}_K}{I_K} - g_q - \rho - \delta_K \right\} \\ \dfrac{\dot{C}}{C} = \dfrac{1}{\theta}\left\{ w\alpha_L b_L I_L^{\alpha_L - 1} - (\alpha_L - 1)\dfrac{\dot{I}_L}{I_L} - \rho - \delta_L \right\} \end{cases} \tag{F6}$$

These replicate the Euler equations (21) in the text.

Appendix G: The dynamic equations (22) and (23)

Using equation (D11) and equations (D9), we transform the market prices of the capital-intensive and labor-intensive inputs (D2) into the following forms:

$$\begin{cases} p_K = \dfrac{\partial Y}{\partial Y_K} = f'(k) \\ p_L = \dfrac{\partial Y}{\partial Y_L} = f(k) - kf'(k) \end{cases} \tag{G1}$$

Substituting (D8) and (D9) into (D7), we obtain

$$\begin{cases} K/M = \left(M^{(1-\phi\beta)/\beta} K^{\phi}\right)^{\frac{1-\beta}{1-\phi\beta}} (\beta\phi^2 p_K/r)^{1/(1-\phi\beta)} \\ L/N = \left(N^{(1-\varphi\beta)/\beta} L^{\varphi}\right)^{\frac{1-\beta}{1-\varphi\beta}} (\beta\varphi^2 p_L/w)^{1/(1-\varphi\beta)} \end{cases} \tag{G2}$$

Substituting (G1) into (G2) and rearranging, we obtain the market prices of capital and labor:

$$\begin{cases} r = \beta\phi^2 K^{\phi-1} M^{(1-\phi\beta)/\beta} f'(k) \\ w = \beta\varphi^2 L^{\varphi-1} N^{(1-\varphi\beta)/\beta}[f(k) - kf'(k)] \end{cases} \qquad (G3)$$

From (G3) we obtain:

$$\frac{rK}{wL} = \frac{\beta\phi^2 M^{(1-\phi\beta)/\beta} K^{\phi} f'(k)}{\beta\varphi^2 N^{(1-\varphi\beta)/\beta} L^{\varphi}[f(k) - kf'(k)]} \qquad (G4)$$

Using $A \equiv N^{(1-\varphi\beta)/\beta}$, $B \equiv M^{(1-\phi\beta)/\beta}$ and $\mathrm{k} \equiv \frac{\mathrm{BK}^{\phi}}{\mathrm{AL}^{\varphi}}$, (G4) is rewritten as:

$$\frac{rK}{wL} = \frac{\beta\phi^2 kf'(k)}{\beta\varphi^2[f(k) - kf'(k)]} \qquad (G5)$$

Substitute $f(k) \equiv \frac{Y}{AL^{\varphi}} = \left[\gamma + (1-\gamma)k^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}$ into (G5) to obtain:

$$\frac{rK}{wL} = \frac{\beta\phi^2}{\beta\varphi^2} . \frac{1-\gamma}{\gamma} k^{\frac{\varepsilon-1}{\varepsilon}} \qquad (G6)$$

Substitute (G6) into equation (20) to obtain:

$$\frac{\dot{S_M}}{S_M} = \psi\left[\frac{d_M {S_M}^{\mu-1}}{d_N(S - S_M)^{\mu-1}} . \frac{\phi(1-\phi\beta)}{\varphi(1-\beta\varphi)} . \frac{1-\gamma}{\gamma} k^{\frac{\varepsilon-1}{\varepsilon}}\right] \qquad (G7)$$

Use $A \equiv N^{(1-\varphi\beta)/\beta}$and $B \equiv M^{(1-\phi\beta)/\beta}$ in $\mathrm{k} \equiv \frac{\mathrm{BK}^{\Phi}}{\mathrm{AL}^{\varphi}}$ to obtain:

$$\frac{\dot{k}}{k} = \frac{1-\phi\beta}{\beta}\frac{\dot{M}}{M} + \phi g_K - \frac{1-\varphi\beta}{\beta}\frac{\dot{N}}{N} - \varphi g_L \qquad (G8)$$

Substitute the innovation functions (14) into (G8) to obtain:

$$\frac{\dot{k}}{k} = \frac{1-\phi\beta}{\beta}[\mathrm{d_M S_M}^{\mu} - \delta] + \phi \mathrm{g_K} - \frac{1-\varphi\beta}{\beta}[\mathrm{d_N}(S - \mathrm{S_M})^{\mu} - \delta] - \varphi g_L \qquad (G9)$$

Using $i_K(t) \equiv \frac{\dot{I}_K(t)}{I_K(t)}$, $i_L(t) \equiv \frac{\dot{I}_L(t)}{I_L(t)}$ and $g(t) \equiv \frac{\dot{Y(t)}}{Y(t)}$ in the growth rates of $\mathrm{s_K(t)} \equiv \frac{\mathrm{I_K}(t)}{\mathrm{Y(t)}}$ and $\mathrm{s_L(t)} \equiv \frac{\mathrm{I_L}(t)}{\mathrm{Y(t)}}$, we can write:

$$\begin{cases} \frac{\dot{s}_K}{s_K} = i_K(t) - g(t) \\ \frac{\dot{s}_L}{s_L} = i_L(t) - g(t) \end{cases} \qquad (G10)$$

From the factor accumulation functions (13) we obtain:

$$\begin{cases} \frac{\dot{K}}{K} + \delta_K = \frac{b_K I_K^{\alpha_K}}{K}, & b_K > 0, 0 \le \alpha_K \le 1, \delta_K > 0 \\ \frac{\dot{L}}{L} + \delta_L = \frac{b_L I_L^{\alpha_L}}{L}, & b_L > 0, 0 \le \alpha_L \le 1, \delta_L > 0 \end{cases} \quad \text{(G11)}$$

Using $g_K(t) \equiv \frac{\dot{K}(t)}{K(t)}$ and $g_L(t) \equiv \frac{\dot{L}(t)}{L(t)}$, we rewrite (G11) as:

$$\begin{cases} \frac{\dot{g}_K}{g_K} = \frac{g_K + \delta_K}{g_K}\left(g_q + \alpha_K i_K - g_K\right) \\ \frac{\dot{g}_L}{g_L} = \frac{g_L + \delta_L}{g_L}\left(\alpha_L i_L - g_L\right) \end{cases} \quad (G12)$$

Collecting (G7), (G9), (G10) and (G12) replicates equation (22) in the text. The system describes the evolution of six variables (S_M, k, s_K, s_L, g_K, g_L), which depends on the six-tuple vector of the respective (endogenous) state variables but also on (g, i_K, i_L). Therefore, in order to solve the system, the latter must also be expressed in terms of the six-tuple (S_M, k, s_K, s_L, g_K, g_L).

From the Euler equations (21) we obtain:

$$\begin{cases} \theta\frac{\dot{C}}{C} = \left\{\alpha_K \frac{b_K(t) I_K^{\alpha_K}}{K}\frac{Y}{I_K}\frac{rK}{Y} - (\alpha_K - 1)\frac{\dot{I}_K}{I_K} - g_q - \rho - \delta_K\right\} \\ \theta\frac{\dot{C}}{C} = \left\{\alpha_L \frac{b_L I_L^{\alpha_L}}{L}\frac{Y}{I_L}\frac{wL}{Y} - (\alpha_L - 1)\frac{\dot{I}_L}{I_L} - \rho - \delta_L\right\} \end{cases} \quad (G14)$$

Using equation (G11), $s_K(t) \equiv \frac{I_K(t)}{Y(t)}$ and $s_L(t) \equiv \frac{I_L(t)}{Y(t)}$, (G14) is rewritten as:

$$\begin{cases} \theta\frac{\dot{C}}{C} = \left\{\alpha_K (g_K + \delta_K)\frac{1}{s_K}\frac{rK}{Y} - (\alpha_K - 1)\frac{\dot{I}_K}{I_K} - g_q - \rho - \delta_K\right\} \\ \theta\frac{\dot{C}}{C} = \left\{\alpha_L (g_L + \delta_L)\frac{1}{s_L}\frac{wL}{Y} - (\alpha_L - 1)\frac{\dot{I}_L}{I_L} - \rho - \delta_L\right\} \end{cases} \quad (G15)$$

The factor price equations (G3) can be written as:

$$\begin{cases} \frac{rK}{Y} = \frac{\beta\phi^2 K^{\phi} M^{(1-\phi\beta)/\beta} f'(k)}{Y} \\ \frac{wL}{Y} = \frac{\beta\varphi^2 L^{\varphi} N^{(1-\varphi\beta)/\beta}[f(k) - kf'(k)]}{Y} \end{cases} \quad (G16)$$

Using $k = \frac{K^{\phi} M^{(1-\phi\beta)/\beta}}{L^{\varphi} N^{(1-\varphi\beta)/\beta}}$ and $f(k) = \frac{Y}{L^{\varphi} N^{(1-\varphi\beta)/\beta}}$ in (G16), it takes the form:

$$\begin{cases} \frac{rK}{Y} = \beta\phi^2 \frac{kf'(k)}{f(k)} \\ \frac{wL}{Y} = \beta\varphi^2 \frac{[f(k) - kf'(k)]}{f(k)} \end{cases} \quad (G17)$$

Substitute (G17) into (G15) to obtain:

$$\begin{cases} \theta\frac{\dot{C}}{C} = \left\{\alpha_K(g_K+\delta_K)\frac{1}{s_K}\beta\phi^2\frac{kf'(k)}{f(k)} - (\alpha_K-1)\frac{\dot{I}_K}{I_K} - g_q - \rho - \delta_K\right\} \\ \theta\frac{\dot{C}}{C} = \left\{\alpha_L(g_L+\delta_L)\frac{1}{s_L}\frac{[f(k)-kf'(k)]}{f(k)}\beta\varphi^2 - (\alpha_L-1)\frac{\dot{I}_L}{I_L} - \rho - \delta_L\right\} \end{cases} \quad (G18)$$

Use $i_K(t) \equiv \frac{\dot{I}_K(t)}{I_K(t)}$, $i_L(t) \equiv \frac{\dot{I}_L(t)}{I_L(t)}$ and $g_C(t) \equiv \frac{\dot{C}(t)}{C(t)}$ in (G18) to obtain:

$$\begin{cases} i_K = \frac{1}{(1-\alpha_K)}\left[\theta g_C + g_q + \rho + \delta_K - \frac{kf'(k)}{s_K f(k)}\alpha_K\beta\phi^2(g_K+\delta_K)\right] \\ i_L = \frac{1}{(1-\alpha_L)}\left[\theta g_C + \rho + \delta_L - \frac{[f(k)-kf'(k)]}{s_L f(k)}\alpha_L\beta\varphi^2(g_L+\delta_L)\right] \end{cases} \quad (G19)$$

Calculating the growth rate on both sides of $Y = AL^{\varphi}f(k)$, yields:

$$g = \left\{\frac{1-\varphi\beta}{\beta}[d_N(S-S_M)^{\mu} - \delta] + \varphi g_L\right\} - \frac{\dot{k}}{k}\frac{kf'(k)}{f(k)} \quad (G20)$$

Calculating the growth rate on both sides of $C = Y(1-s_L-s_K)$, yields:

$$g_C = g - \frac{s_K}{(1-s_K-s_L)}\frac{\dot{s}_K}{s_K} - \frac{s_L}{(1-s_K-s_L)}\frac{\dot{s}_L}{s_L} \quad (G21)$$

Substitute (G10) into (G21) to obtain:

$$g_C = \frac{g - s_K i_K - s_L i_L}{1-s_K-s_L} \quad (G22)$$

Finally, substitute (G22) into (G19) to obtain:

$$\begin{cases} i_K = \left[\frac{\theta(g-s_K i_K - s_L i_L)}{(1-\alpha_K)(1-s_K-s_L)} + \frac{g_q+\rho+\delta_K}{(1-\alpha_K)} - \frac{\alpha_K\beta\phi^2(g_K+\delta_K)}{(1-\alpha_K)s_K}\frac{kf'(k)}{f(k)}\right] \\ i_L = \left[\frac{\theta(g-s_K i_K - s_L i_L)}{(1-\alpha_L)(1-s_K-s_L)} + \frac{\rho+\delta_L}{(1-\alpha_L)} - \frac{\alpha_L\beta\varphi^2(g_L+\delta_L)}{(\alpha_L-1)s_L}\frac{[f(k)-kf'(k)]}{f(k)}\right] \end{cases} \quad (G23)$$

whereby (G20) and (G23) are equations (23) in the text.

Appendix H: Proof of Proposition 4.

(1) Existence and uniqueness of a steady-state equilibrium

According to its definition, at the steady state all dynamic equations (22) of the instantaneous equilibrium are set to zero:

$$
\begin{cases}
\frac{\dot{S_M}}{S_M} = \psi\left[\frac{d_M {S_M}^{\mu-1}}{d_N(S-S_M)^{\mu-1}}.\frac{\phi(1-\phi\beta)}{\varphi(1-\varphi\beta)}.\frac{1-\gamma}{\gamma}k^{\frac{\varepsilon-1}{\varepsilon}}\right] = 0 \\
\frac{\dot{k}}{k} = \frac{1-\phi\beta}{\beta}[d_M {S_M}^{\mu} - \delta] + \phi g_K - \frac{1-\varphi\beta}{\beta}[d_N(S-S_M)^{\mu} - \delta] - \varphi g_L = 0 \\
\frac{\dot{g}_K}{g_K} = g_q + \alpha_K i_K - g_K = 0 \\
\frac{\dot{g}_L}{g_L} = \alpha_L i_L - g_L = 0 \\
\frac{\dot{g}_L}{g_L} = i_K - g = 0 \\
\frac{\dot{g}_L}{g_L} = i_L - g = 0
\end{cases}
\tag{H1}
$$

From the system (H1) we obtain equations (H2) and (H3)

$$
\begin{cases}
g_K = g_q + \alpha_K g \\
g_L = \alpha_L g \\
i_K = i_L = g
\end{cases}
\tag{H2}
$$

$$
\frac{1-\phi\beta}{\beta}[d_M {S_M}^{\mu} - \delta] + \phi g_q = \frac{1-\varphi\beta}{\beta}[d_N(S-S_M)^{\mu} - \delta] - (\phi\alpha_K - \varphi\alpha_L)g \tag{H3}
$$

Substitute $\frac{\dot{k}}{k} = 0$ and (H2) into equation (23) to obtain:

$$
g = \frac{1-\varphi\beta}{\beta(1-\varphi\alpha_L)}[d_N(S-S_M)^{\mu} - \delta] \tag{H4}
$$

Subtitute equation (H4) into (H3) to obtain:

$$
\frac{1-\phi\beta}{\beta}[d_M {S_M}^{\mu} - \delta] + \phi g_q = \frac{1-\varphi\beta}{\beta}\frac{1-\phi\alpha_K}{1-\varphi\alpha_L}[d_N(S-S_M)^{\mu} - \delta] \tag{H5}
$$

Equation (H5) is a univariate equation of S_M. If it has a unique non-zero solution ${S_M}^*$, then the system of equations (H1) has a unique non-zero solution. Therefore, we first prove that (H5) has a unique non-zero solution, and then give the solution of (H1).

Construct the function $Q(S_M)$ as follows:

$$
Q(S_M) = \frac{1-\phi\beta}{\beta}[d_M {S_M}^{\mu} - \delta] + \phi g_q - \frac{1-\varphi\beta}{\beta}\frac{1-\phi\alpha_K}{1-\varphi\alpha_L}[d_N(S-S_M)^{\mu} - \delta] \tag{H6}
$$

where $S_M \in [0, S]$.

When $S_M = 0$, we get:

$$
Q(0) = -\delta\frac{1-\phi\beta}{\beta} + \phi g_q - \frac{1-\varphi\beta}{\beta}\frac{1-\phi\alpha_K}{1-\varphi\alpha_L}[d_N S^{\mu} - \delta] \tag{H7}
$$

By assumption, $0 \le g_q < \frac{1-\phi\beta}{\phi\beta}\delta + \frac{1-\varphi\beta}{\phi\beta}\frac{1-\phi\alpha_K}{1-\varphi\alpha_L}[d_N S^{\mu} - \delta]$, implying

$$
Q(0) = -\delta\frac{1-\phi\beta}{\beta} - \frac{1-\varphi\beta}{\beta}\frac{1-\phi\alpha_K}{1-\varphi\alpha_L}[d_N S^{\mu} - \delta] + \phi g_q < 0 \tag{H8}
$$

Equation (H8) means that as long as the exogenously given rate of the investment-specific technological progress g_q is not too large, $Q(0)$ must be negative.

When $S_M = S$, we obtain:

$$Q(S) = \frac{1-\phi\beta}{\beta}[d_M S^\mu - \delta] + \frac{1-\varphi\beta}{\beta}\frac{1-\phi\alpha_K}{1-\varphi\alpha_L}\delta + \phi g_q > 0 \quad \text{(H9)}$$

Equations (H8) and (H9) show that equation $Q(S_M) = 0$ has at least one solution.

The derivative of S_M for $Q(S_M)$ is:

$$\frac{dQ(S_M)}{dS_M} = \frac{1-\phi\beta}{\beta} d_M \mu {S_M}^{\mu-1} + \frac{1-\varphi\beta}{\beta} d_N \mu (S - S_M)^{\mu-1} > 0 \quad \text{(H10)}$$

Equation (H10) shows that $Q(S_M)$ increases monotonically, so there is a unique ${S_M}^*,\ 0 < {S_M}^* < S$ such that $Q({S_M}^*) = 0$. We denote the solution by:

$${S_M}^* = S_M\big(S, \mu, \beta, \phi, \varphi, d_M, d_N, \alpha_K, \alpha_L, g_q, \delta\big) \quad \text{(H11)}$$

With $0 < {S_M}^* < S, \frac{\dot{S_M}}{S_M} = 0$, we have:

$$\frac{d_M {S_M}^{\mu-1}}{d_N (S - S_M)^{\mu-1}} . \frac{\phi(1-\phi\beta)}{\varphi(1-\varphi\beta)} . \frac{1-\gamma}{\gamma} k^{\frac{\varepsilon-1}{\varepsilon}} = 1 \quad \text{(H12)}$$

Substitute (H11) into (H12) to obtatin

$$k^* = \left[\frac{d_N (S - {S_M}^*)^{\mu-1}}{d_M {S_M}^{*\mu-1}} . \frac{\varphi(1-\varphi\beta)}{\phi(1-\phi\beta)} . \frac{\gamma}{1-\gamma}\right]^{\frac{\varepsilon}{\varepsilon-1}} \quad \text{(H13)}$$

Substitute (H11) into (H4) and the innovation functions (14) to obtain

$$\begin{cases} {g_M}^* = d_M {S_M}^{*\mu} - \delta = \dfrac{\beta}{1-\phi\beta}\left[1 - \phi\left(\alpha_K + \dfrac{g_q}{g^*}\right)\right] g^* \\ {g_N}^* = d_N (S - {S_M}^{*\mu}) - \delta = \dfrac{\beta}{1-\varphi\beta}(1-\varphi\alpha_L) g^* \\ {g_L}^* = \alpha_L \dfrac{1-\varphi\beta}{\beta} \dfrac{[d_N (S - {S_M}^*)^\mu - \delta]}{(1-\varphi\alpha_L)} \\ {g_K}^* = g_q + \alpha_K \dfrac{1-\varphi\beta}{\beta} \dfrac{[d_N (S - {S_M}^*)^\mu - \delta]}{(1-\varphi\alpha_L)} \\ g^* = {g_C}^* = {i_K}^* = {i_L}^* = \dfrac{1-\varphi\beta}{\beta} \dfrac{[d_N (S - {S_M}^*)^\mu - \delta]}{(1-\varphi\alpha_L)} \end{cases} \quad \text{(H14)}$$

Substitute (H13) and (H14) into (G19) to obtain

$$\begin{cases} {s_K}^* = \dfrac{\alpha_K \beta \phi^2 \big(g_q + \alpha_K g^* + \delta_K\big)}{(1-\alpha_K+\theta) g^* + \rho + \delta_K} . \dfrac{(1-\gamma) k^{*\frac{\varepsilon-1}{\varepsilon}}}{\gamma + (1-\gamma) k^{*(\varepsilon-1)/\varepsilon}} \\ {s_L}^* = \dfrac{\alpha_L \beta \varphi^2 \big(\alpha_L g^* + \delta_L\big)}{(1-\alpha_L+\theta) g^* + \rho + \delta_L} . \dfrac{\gamma}{\gamma + (1-\gamma) k^{*(\varepsilon-1)/\varepsilon}} \end{cases} \quad (H15)$$

Equations (H11), (H13), (h14) and (H15) are the steady-state equilibrium solutions of the model, which together form equation (24) in the text.

(2) Stability of the steady-state equilibrium

A rigorous proof of stability of the steady state requires to use the properties of the eigenvalues of the coefficient matrix of the dynamic equation system (22) near the steady state. If all eigenvalues are positive, the steady state is divergent, if all 6 roots are negative, it is globally stable, and if the number of negative roots is less than 6 and greater than 0, it is saddle point stable. However, due to the fact that the characteristic equation of the coefficient matrix is a 6th degree equation, there is generally no real number solution. Therefore, we are currently unable to provide a rigorous generalized proof of stability. Another method is to use numerical calculations for given specific values of exogenous variables and parameters, calculate approximate steady-state equilibrium, and simultaneously provide steady-state convergence characteristics. This calculation program is not complex, but can only provide the stability characteristics of the specific steady-state equilibrium.

As following, we analyze whether the state variable near the steady state will converge to the steady state, that is, given that other state variables are in their steady state and one state variable deviates from the steady state, whether it will converge to the steady state.

First, k near its steady state will converge to k^* if $\varepsilon < 1$.

Owing to in the steady state, there two equations as follow.

$$\begin{cases} \dfrac{\dot{k}}{k} = \dfrac{1-\phi\beta}{\beta}\left[d_M {S_M^*}^{\mu} - \delta\right] + \phi g_K^* - \dfrac{1-\varphi\beta}{\beta}\left[d_N (S - S_M^*)^{\mu} - \delta\right] - \varphi g_L^* = 0 \\ \dfrac{\dot{S_M}}{S_M} = \psi\left[\dfrac{d_M {S_M^*}^{\mu-1}}{d_N (S - S_M^*)^{\mu-1}} \cdot \dfrac{\phi(1-\phi\beta)}{\varphi(1-\varphi\beta)} \cdot \dfrac{1-\gamma}{\gamma} {k^*}^{\frac{\varepsilon-1}{\varepsilon}}\right] = 0 \end{cases} \quad (H16)$$

When $k_0 > k^*$, if $\varepsilon < 1$, $(k_0)^{\frac{\varepsilon-1}{\varepsilon}} < (k^*)^{\frac{\varepsilon-1}{\varepsilon}}$, from (H16) we can obtain the following eqaution

$$\frac{\dot{S_M}}{S_M} = \psi\left[\frac{d_M {S_M^*}^{\mu-1}}{d_N (S - S_M^*)^{\mu-1}} \cdot \frac{\phi(1-\phi\beta)}{\varphi(1-\varphi\beta)} \cdot \frac{1-\gamma}{\gamma} {k^*}^{\frac{\varepsilon-1}{\varepsilon}}\right] < 0 \quad (H17)$$

As $\frac{\dot{S_M}}{S_M} > 0$, it will lead to $S_M > S_M^*$, and $S_N = S - S_M < S_N^*$, then from the equation (H16) there will be the following

$$\frac{\dot{k}}{k} = \frac{1-\phi\beta}{\beta}[d_M {S_M}^{\mu} - \delta] + \phi g_K^* - \frac{1-\varphi\beta}{\beta}[d_N (S - S_M)^{\mu} - \delta] - \varphi g_L^* < 0 \quad (H18)$$

Owing to $\frac{\dot{k}}{k} < 0$, the k will decrease to k^*.

On the contrary, when $k_0 < k^*$, the same reasoning suggests that there will be $\frac{\dot{k}}{k} > 0$, and it will lead to k increase to k^*.

Therefore, if $\varepsilon < 1$, k near to the steady state will converge to k^*.

However, if $\varepsilon > 1$, only from equation (H16), k maybe diverge from k^*.

Second, S_M near steady state will converge to S_M^*.

When $S_M^0 > S_M^*$, owing to $\mu < 1$, $\frac{{S_M^0}^{\mu-1}}{(S-S_M^0)^{\mu-1}} < \frac{{S_M^*}^{\mu-1}}{(S-S_M^*)^{\mu-1}}$, from (H16) we can obtain the following eqation

$$\frac{\dot{S_M}}{S_M} = \psi\left[\frac{d_M {S_M^0}^{\mu-1}}{d_N(S-S_M^0)^{\mu-1}}.\frac{\phi(1-\phi\beta)}{\varphi(1-\varphi\beta)}.\frac{1-\gamma}{\gamma}k^{*\frac{\varepsilon-1}{\varepsilon}}\right] < 0 \qquad (H19)$$

As $\frac{\dot{S_M}}{S_M} < 0$, the S_M will decrease to S_M^*.

On the contrary, when $S_M^0 < S_M^*$, the same reasoning suggests that there will be $\frac{\dot{S_M}}{S_M} > 0$, and it will lead to S_M increase to S_M^*.

Therefore, S_M near to the steady state will converge to S_M^*.

Third, g_K near $\boldsymbol{g_K^*}$ is convergent.

In steady state, there will be following

$$\frac{\dot{g}_K}{g_K} = \frac{g_K^* + \delta_K}{g_K^*}\left(g_q + \alpha_K i_K^* - g_K^*\right) = 0 \qquad (H20)$$

When $g_K^0 < g_K^*$, equation (H20) shows $\frac{\dot{g}_K}{g_K} > 0$, then g_K will increase to g_K^*. In contrast, when $g_K^0 > g_K^*$, the same reasoning suggests that g_K will decrease to g_K^*. Therefore, g_K will converges to g_K^* near it.

Fourth, g_L near $\boldsymbol{g_L^*}$ is convergent.

In steady state, there are following

$$\frac{\dot{g}_L}{g_L} = \frac{g_L^* + \delta_L}{g_L^*}(\alpha_L i_L^* - g_L^*) = 0 \qquad (H21)$$

When $g_L^0 < g_L^*$, equation (H21) shows $\frac{\dot{g}_L}{g_L} > 0$, then g_L will increase to g_L^*. In contrast, when $g_L^0 > g_L^*$, the same reasoning suggests that g_L will decrease to g_L^*. Therefore, g_L will converges to g_L^* near it.

Fifth, saving rate s_K and s_L is not **convergent to s_K^* and s_L^*** even **near the steady state.**

In steady state, there are following

$$\begin{cases} \dfrac{\dot{s}_K}{s_K} = \dfrac{\theta(g^* - s_K^* i_K^* - s_L^* i_L^*)}{(1-\alpha_K)(1-s_K^* - s_L^*)} - \dfrac{\alpha_K \beta \phi^2 (g_K^* + \delta_K)}{(1-\alpha_K) s_K^*} \dfrac{k^* f'(k^*)}{f(k^*)} + \dfrac{g_q + \rho + \delta_K}{(1-\alpha_K)} - g^* = 0 \\ \dfrac{\dot{s}_L}{s_L} = \dfrac{\theta(g^* - s_K^* i_K^* - s_L^* i_L^*)}{(1-\alpha_L)(1-s_K^* - s_L^*)} - \dfrac{\alpha_L \beta \varphi^2 (g_L^* + \delta_L)}{(\alpha_L - 1) s_L^*} \dfrac{[f(k^*) - k^* f'(k^*)]}{f(k^*)} + \dfrac{\rho + \delta_L}{(1-\alpha_L)} - g^* = 0 \end{cases} \quad (H22)$$

When $s_K^0 < s_K^*$ and $s_L^0 < s_L^*$, there are following equations

$$\begin{cases} \left(\dfrac{\partial(\dot{s}_K/s_K)}{\partial s_K} \middle| s_K = s_K^* \right) = \dfrac{\alpha_K \beta \phi^2 (g_K^* + \delta_K)}{(1-\alpha_K)} \dfrac{k^* f'(k^*)}{f(k^*)} \dfrac{1}{(s_K^*)^2} > 0 \\ \left(\dfrac{\partial(\dot{s}_L/s_L)}{\partial s_L} \middle| s_L = s_L^* \right) = \dfrac{\alpha_L \beta \varphi^2 (g_L^* + \delta_L)}{(\alpha_L - 1) s_L^*} \dfrac{k^* f'(k^*)}{f(k^*)} \dfrac{1}{(s_L^*)^2} > 0 \end{cases} \quad (H23)$$

Therefore, s_K and s_L is not convergent to s_K^* and s_L^* even near their steady state.

All in all, the above analysis indicates that if from a single variable alone, among the six state variables, S_M, g_K and g_L are convergent near their steady state, if $\varepsilon < 1$, then k is also convergent, but s_K and s_L are not convergent even near to their steady state. Therefore, the steady state of this model is likely to be the saddle point stable.

Appendix I: Proof of equation (25)

From equation (G3), we obtain that in steady state:

$$\begin{cases} \dfrac{\dot{w}}{w} = (\varphi - 1)\dfrac{\dot{L}}{L} + \dfrac{1-\varphi\beta}{\beta}\dfrac{\dot{N}}{N} \\ \dfrac{\dot{r}}{r} = (\phi - 1)\dfrac{\dot{K}}{K} + \dfrac{1-\phi\beta}{\beta}\dfrac{\dot{M}}{M} \end{cases} \quad (I1)$$

From $f(k) \equiv \frac{Y}{AL^{\varphi}} = \frac{Y}{BK^{\phi}} k$, in steady state we can obtain

$$\begin{cases} \dfrac{1-\varphi\beta}{\beta}\dfrac{\dot{N}}{N} = g^* - \varphi\dfrac{\dot{L}}{L} \\ \dfrac{1-\phi\beta}{\beta}\dfrac{\dot{M}}{M} = g^* - \phi\dfrac{\dot{K}}{K} \end{cases} \quad (I2)$$

Using equations (I2) in (I1) results in:

$$\begin{cases} \dfrac{\dot{w}}{w} = g^* - \dfrac{\dot{L}}{L} \\ \dfrac{\dot{r}}{r} = g^* - \dfrac{\dot{K}}{K} \end{cases} \quad (I3)$$

Substituting equations (H2) into (I3) results in

$$\begin{cases} \dfrac{\dot{w}}{w} = (1-\alpha_L) g^* \\ \dfrac{\dot{r}}{r} = (1-\alpha_K) g^* - g_q \end{cases} \quad (I4)$$

Using (I4) and equations (H2) in the factor supply elasticities equation (6) yields equation (27), that is:

$$\begin{cases} \varepsilon_K \equiv \dfrac{\dot{\mathrm{K}}/K}{\dot{r}/r} = \dfrac{\alpha_K + g_q/g}{1 - \left(\alpha_K + g_q/g\right)} \\ \varepsilon_L \equiv \dfrac{\dot{\mathrm{L}}/L}{\dot{w}/w} = \dfrac{\alpha_L}{1 - \alpha_L} \end{cases}$$