A linear model for freight transportation

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Abstract

We propose a new freight transportation problem to distribute the benefit of ore transportation throughout a linear route to compensate the communities located on it. Such distribution aims to avoid conflicts generated by the transportation that causes external costs such as air, water, and land pollution to communities that can block the route, forcing more expensive alternatives. We propose a solution based on stability and fairness principles. In particular, we present some reasonable properties to characterize a family of assignment rules for determining compensation to local communities.

Keywords: Cooperative game theory, core, mining, freight transportation.

1 Introduction

Mining makes an essential contribution to countries' economies, especially in developing countries, where mining projects are increasingly expected to deliver sustainable benefits to local, regional, and national stakeholders (Wall, 2011; Gustafsson and Scurrah, 2019). Sustainable development needs to guarantee

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the investment of the mining industry in the countries' markets and to maintain a peaceful atmosphere in the local communities. A key aspect is to set how to share compensation from the benefit, which is generated by the mining industry. Also, this situation can be found in different contexts like wood, fruit, oil, cement, and other industries.

As a motivational example, we mention the case of Peru, where the Minerals and Metals Group (MMG) *Las Bambas* transports ore through the Peruvian Southern mining corridor, which is defined as the national road from *MMG Las Bambas* exploitation place (Fuerabamaba, Apurimac) and the train station (Pillones Station, Arequipa). This National road goes through several peasant communities. Since *MMG Las Bambas* began with the exploitation, social conflicts with the communities have taken place many times in the Apurimac-Cusco region, resulting in serious results like life losses.

We focus on one key question: How can communities be compensated for the transportation of minerals through their land in a way that assures stability? To answer this question, our model considers a set of players. One of them (the mining firm) is asymmetric with respect to the rest of the players. This player requires the use of the main road. The rest of the players are the local communities. We assume that communities have either a) private rights on the road that prevent the mining firm from using it without their consent or b) communities can block the main road with the same results.

There exist studies based on linear routes. For instance, Ye et al. (2020) optimize transportation system, Lin et al. (2021) study a railway freight transport system. Other studies focus on networks. For instance, (Liu et al., 2020) minimize fuel consumption in balanced networks, and Ma et al. (2018) study urban road networks. Other studies are based on the transportation of freight on rivers (Alcalde-Unzu et al., 2021; van den Brink et al., 2018; Sun et al., 2019). Our study lies on the intersection between these two approaches since our model is linear and there are networks as alternative routes.

The asymmetric player (mining firm) is an essential player in the sense that no benefit is possible without it. Such a player is commonly known as a veto player (Bahel, 2016). In particular, a veto player is an agent that belongs to all coalitions of positive value. Games with a single veto player generalize other families of games, such as clan games (Potters et al., 1989) and big-boss games (Muto et al., 1988). The game that arises from a freight transportation problem is not a clan game nor a big-boss game, and hence no result can be derived from these previous works. As opposed, Bahel (2016) shows that games with a single veto player have a non-empty core, and it also coincides with the bargaining set (Aumann and Maschler, 1964; Davis and Maschler, 1967).

The paper is organized as follows. In Section 2, we present the model and define the freight transportation problem. In Section 3, we study the core. In Section 4, we define a family of core allocation rules. In Section 5, we present a brief discussion and the conclusions.

2 Model

Let $\mathbb{R}_+$ denote the set of non-negative real numbers and $\mathbb{R}_{++}$ denote the set of positive real numbers. For notational convenience, given a finite set S and $x \in \mathbb{R}^S$, we define

$$x(S) = \sum_{i \in S} x_i.$$

2.1 Cooperative games

Let $\mathcal{U} = [1, \infty)$ be the universe of (potential) players, and let $\mathcal{N}$ be the set of the nonempty, finite subsets of $\mathcal{U}$, with generic element $N = \{1, 2, \ldots, n\}$. A *cooperative game with transferable utility (TU-game)* is a pair (N, v) where $N \in \mathcal{N}$ is a set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function with $v(\emptyset) = 0$, where $v(S)$ is the worth of coalition $S \subseteq N$, which can be interpreted as the benefit that players in S can generate by themselves.

A TU-game (N, v) is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subset N$ with $S \cap T = \emptyset$. This means that two different coalitions can get at least as much benefit working together as separately. A TU-game (N, v) is *monotonic* if $v(S) \leq v(T)$ for all $S \subseteq T \subseteq N$.

An *imputation* of (N, v) is an allocation $x \in \mathbb{R}^N$, satisfying $x(N) = v(N)$ and $x_i \geq v(\{i\})$ for all $i \in N$. We denote the set of imputations as $I(N, v)$, i.e.,

$$I(N, v) = \left\{ x \in \mathbb{R}^N : x(N) = v(N), x_i \geq v(\{i\}) \; \forall i \in N \right\}.$$

The notion of imputation comprises the most basic requirements for a reasonable allocation. It requires that each player receives at least its own stand-alone value. It also requires that the worth of the grand coalition is fully shared, which makes sense under the reasonable condition of superadditivy. Under superadditivy, the set of imputations is always nonempty.

The *core* (Gillies, 1959) of (N, v) is the set of stable imputations, defined as:

$$Core(N, v) = \{x \in I(N, v) : x(S) \geq v(S) \ \ \forall S \subset N\}.$$

The core also has an intuitive interpretation. We are interested in payoff allocations where no coalition of players can improve by themselves. Also, the core implies the efficiency and individual rationality of players. The main problem with the core is that it may be empty even for superadditive games.

An *assigning rule* is a function that assigns to each TU-game (N, v) in a class of games a vector $\phi(N, v) \in \mathbb{R}^N$ such that $\phi_i(N, v)$ is interpreted as the payoff allocated to player $i \in N$.

2.2 Freight transportation problems

In this subsection, we introduce the freight transportation problem. This problem adds structure to the problems with constraints and claims (Bergantiños and Lorenzo, 2008; Lorenzo, 2010), which are generalizations of the well-known bankruptcy (or claims) problems first studied by O'Neill (1982).

Let $\mathcal{N}^1 = \{N \in \mathcal{N} : 1 \in N\}$. In our context, $N \in \mathcal{N}^1$ is the set of players where 1 represents a mining firm and

$$N' = N \setminus \{1\}$$

is the set of local communities with a fixed order.

Consider the network where all players are in a line. We assume that the order of the players in the line is given by its numerical value, i.e., $2, 3, \ldots, n$ when $N = \{1, 2, \ldots, n\}$. A coalition $S \subseteq N'$ of communities is *connected* if the sub-network restricted to agents in S has a single component. We denote the set of connected coalitions as $\mathcal{C}^N$. Formally,

$$\mathcal{C}^N = \{S \subseteq N' : i, k \in S, \ j \in N, \ i < j < k \Rightarrow j \in S\} \setminus \{\emptyset\}.$$

Notice that, by definition, $N', \{i\} \in \mathcal{C}^N$ for all $i \in N'$. We also assume $\emptyset \notin \mathcal{C}^N$ for notational convenience.

Since the communities are located on a linear route, we can work with the notion of consecutive communities, formally defined as follows:

Definition 2.1. *Coalitions* $S, T \in \mathcal{C}^N$ *are* consecutive *if* $S \cup T \in \mathcal{C}^N$ *and* $S \cap T = \emptyset$.

Let $\mathcal{A} \subseteq \mathcal{C}^N$ denote the set of consecutive communities that have an alternative route option. An example will clarify this notion:

Example 2.1. *Assume* $N' = \{2, 3, 4, 5\}$. *A set of consecutive communities is* $\mathcal{A} = \{\{2, 3\}, \{3\}, \{4\}, \{5\}, \{4, 5\}\}$. *This means an alternative road can be built to circumvent communities* 2 *and* 3 *altogether, and another alternative circumventing communities* 4 *and* 5 *altogether. Moreover, there is another alternative circumventing community* 3, *another one circumventing community* 4, *and another one circumventing community* 5, *so it is possible to avoid, say, only community* 4. *As opposed, there is no alternative to avoid community* 2 *alone. If community* 2 *does not cooperate, then cooperation from community* 3 *alone is not enough.*

Definition 2.2. *A* path compatible with $\mathcal{A}$ *is a pair* $(\mathcal{P}, f)$ *where*

- $\mathcal{P} = \left\{P^1, \ldots, P^k\right\} \subset \mathcal{C}^N$ *such that* $P^l \in \mathcal{A}$ *whenever* $\left|P^l\right| > 1$, P^{l-1} *and* P^l *are consecutive for all* $l = 2, \ldots, k$, *and* $\bigcup_{l=1}^k P^l = N'$.
- $f : \mathcal{P} \to \{col, alt\}$ *is a function that assigns either col (collaborate) or alt (alternative) to each* $P^l \in \mathcal{P}$ *and such that* $f\left(P^l\right) = alt$ *whenever* $\left|P^l\right| > 1$ *and* $f\left(P^l\right) = col$ *whenever* $P^l \notin \mathcal{A}$.

Let $\mathbb{P}(\mathcal{A})$ denote the set of paths compatible with $\mathcal{A}$. Notice that $\mathbb{P}(\mathcal{A})$ is always nonempty, as $(\mathcal{P}, f)$ with $\mathcal{P} = \{\{2\}, \{3\}, \ldots, \{n\}\}$ and $f(\{i\}) = col$ for all $i \in N'$ always belongs to it. Some compatible paths in the previous example are the following:

Example 2.2. *Assume* $N' = \{2, 3, 4, 5\}$. *A set of consecutive communities is* $\mathcal{A} = \{\{2, 3\}, \{3\}, \{4\}, \{5\}, \{4, 5\}\}$. *The following are paths compatible with* $\mathcal{A}$*:*

- $(\mathcal{P}^1, f^1)$ *with* $\mathcal{P}^1 = \{\{2\}, \{3\}, \{4, 5\}\}$, $f^1(\{2\}) = col$, $f^1(\{3\}) = alt$, $f^1(\{4, 5\}) = alt$. *In this path, the mining freight goes through community* 2, *and avoids communities* 3, 4, *and* 5.
- $(\mathcal{P}^2, f^2)$ *with* $\mathcal{P}^2 = \{\{2\}, \{3\}, \{4, 5\}\}$, $f^2(\{2\}) = col$, $f^2(\{3\}) = col$, $f^2(\{4, 5\}) = alt$. *In this path, the mining freight goes through communities* 2 *and* 3, *and avoids communities* 4 *and* 5.

- $(\mathcal{P}^3, f^3)$ *with* $\mathcal{P}^3 = \{\{2,3\},\{4\},\{5\}\}$, $f^3(\{2,3\}) = alt$, $f^3(\{4\}) = col$, $f^3(\{5\}) = alt$. *In this path, the mining freight goes through community* 4, *and avoids communities* 2, 3, *and* 5.

Definition 2.3. *A freight transport problem is a tupla* $\mathcal{F} = (N, E, c, \mathcal{A}, a)$ *where* $N \in \mathcal{N}^1$, $E > 0$, $c \in \mathbb{R}_+^{N'}$, $\mathcal{A} \subseteq \mathcal{C}^N$, *and* $a \in \mathbb{R}_{++}^{\mathcal{A}}$ *satisfies*

$$a_A > c(A) \tag{1}$$

for all $A \in \mathcal{A}$,

$$a_A + c(B \setminus A) \leq a_B \tag{2}$$

for all $A, B \in \mathcal{A}$ *with* $A \subset B$, *and*

$$a_A \leq a_{A_1} + a_{A_2} + \cdots + a_{A_k} \tag{3}$$

whenever $A, A_1, A_2, \ldots, A_k \in \mathcal{A}$ *and* $A = \bigcup_{l=1}^{k} A_l$.

In particular, a is a vector whose coordinates are the costs of using alternative routes avoiding each coalition in $\mathcal{A}$. Condition (1) assures that it is sub-optimal to use any alternative route. The reason for (2) and (3) is that communities in $A \in \mathcal{A}$ would be irrelevant if we allow a_A to be too high so that they could then be avoided at no additional cost by the mining firm, either by using a supra-alternative (2) or several intra-alternatives (3).

We analyze in depth this and the rest of the components of a freight transportation problem:

1. N is the set of players with $1 \in N$ the mining firm and $N' = N \setminus \{1\}$ the ordered communities.

2. E represents the total benefit generated by freight transportation, i.e., the benefit the mining industry gets from the ore once arrives at its destination.

3. The vector cost $c \in \mathbb{R}_+^{N'}$ represents the costs that the transportation of minerals affects the respective local community. Hence, a community $i \in N'$ suffers a cost c_i only if the mineral goes through its land.

4. $\mathcal{A}$ represents the set of communities that have an alternative route avoiding the respective community.

5. Finally, $a \in \mathbb{R}^{\mathcal{A}}_{++}$, such that a_A with $A \in \mathcal{A}$, is the cost of building/using the respective alternative route that avoids going thought players in A.

Example 2.3. *Let $\mathcal{F} = (N, E, c, \mathcal{A}, a)$ with $N = \{1, 2, 3, 4, 5\}$, $E = 11$, $c_i = 0$ for all $i \in N'$, $\mathcal{A} = \{\{2, 3\}, \{3\}, \{4\}, \{5\}, \{4, 5\}\}$, $a_{\{2,3\}} = 4$, $a_{\{3\}} = a_{\{4\}} = 2$, $a_{\{5\}} = 4$, and $a_{\{4,5\}} = 5$. See Figure 1. It is straightforward to check that (1), (2) and (3) are satisfied.*

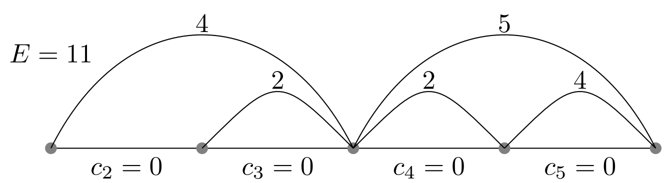

Figure 1: Example of a freight transportation problem.

Example 2.4. *Let $\mathcal{F} = (N, E, c, \mathcal{A}, a)$ with $N = \{1, 2, 3, 4, 5\}$, $E = 8$, $c_2 = c_3 = c_4 = 1$, $c_5 = 0$, $\mathcal{A} = \{\{2\}, \{3\}, \{4\}, \{4, 5\}, \{3, 4, 5\}\}$, $a_{\{2\}} = a_{\{3\}} = 2$, $a_{\{4\}} = 4$, $a_{\{4,5\}} = 4$ and $a_{\{3,4,5\}} = 6$. See Figure 2. It is straightforward to check that (1), (2) and (3) are satisfied.*

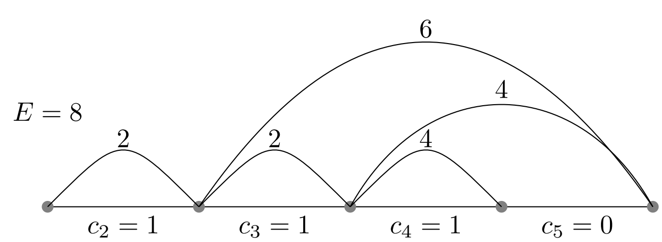

Figure 2: Example of a freight transportation problem.

We also assume the next conditions:

Assumption 1 There is enough benefit of cooperation even without collaboration from the local communities, i.e., there exists a partition of N', $\mathcal{A}' = \{A^1_N, \ldots, A^k_N\} \subseteq \mathcal{A}$ such that $E \geq \sum_{A \in \mathcal{A}'} a_A$.

In other words, there exists at least one alternative that avoids all communities. Let denote as $\mathcal{A}^{(a)} \subset \mathcal{A}$ a partition with minimum $\sum_{A \in \mathcal{A}^{(a)}} a_A$.

Assumption 2. The alternative options can not properly overlap, i.e., given $i < j \leq k < l \in N'$ and $\{i, i+1, ..., k\} \in \mathcal{A}$, then $\{j, j+1, ..., l\} \notin \mathcal{A}$.

In other words, if an alternative exists between i and k, then there is no alternative option between j and l.

Notice that both freight transportation problems given in Example 2.3 and Example 2.4, respectively, satisfy Assumption 1 and Assumption 2.

Proposition 2.1. *Assumption 2 holds if and only if $S \cap T \in \{S, T, \emptyset\}$ for all $S, T \in \mathcal{A}$.*

Proof. ($\Rightarrow$) Let $S, T \in \mathcal{A}$. Hence, there exist i, j, k, l with $i \leq k$ and $j \leq l$ such that $S = \{i, \ldots, k\}$ and $T = \{j, \ldots, l\}$. Assume w.l.o.g. $i \leq j$. In case $i = j$, then either $S \cap T = S$ or $S \cap T = T$. Hence, we assume $i < j$. We have three cases:

- If $l \leq k$, then $T \subseteq S$ and hence $S \cap T = T$.
- If $l > k$ and $j > k$, then $S \cap T = \emptyset$.
- If $l > k$ and $j \leq k$, under Assumption 2, $T \notin \mathcal{A}$, which is a contradiction.

($\Leftarrow$) Let $i < j \leq k < l \in N$ and $S = \{i, \ldots, k\} \in \mathcal{A}$. We have to prove that $T = \{j, \ldots, l\} \notin \mathcal{A}$. Assume, on the contrary, that $T \in \mathcal{A}$. Under our hypothesis, we have three cases:

- $S \cap T = S$. Since $S \cap T = \{j, \ldots, k\}$, we deduce $i = j$, which is a contradiction because $i < j$.
- $S \cap T = T$. Since $S \cap T = \{j, \ldots, k\}$, we deduce $k = l$, which is a contradiction because $k < l$.
- $S \cap T = \emptyset$. Since $S \cap T = \{j, \ldots, k\}$, we deduce $k < j$, which is a contradiction because $j \leq k$.

■

Proposition 2.2. *Under Assumption 1 and Assumption 2, there exists a unique coarsest partition $\mathcal{A}'$ of N' formed by elements in $\mathcal{A}$.*

Proof. Existence is guaranteed by Assumption 1. Assume there exist two different partitions of N', named $\mathcal{A}^1, \mathcal{A}^2 \subset \mathcal{A}$ such that no other such partition is coarser than any of them. For each $i \in N'$, let $\mathcal{A}_i^1 \in \mathcal{A}^1$ such that $i \in \mathcal{A}_i^1$ and let $\mathcal{A}_i^2 \in \mathcal{A}^2$ such that $i \in \mathcal{A}_i^2$. Since $i \in \mathcal{A}^1 \cap \mathcal{A}^2 \neq \emptyset$, under Proposition 2.1 either $\mathcal{A}_i^1 \subseteq \mathcal{A}_i^2$ or $\mathcal{A}_i^2 \subseteq \mathcal{A}_i^1$. Hence $\mathcal{A}_i^0 = \mathcal{A}_i^1 \cup \mathcal{A}_i^2 \in \mathcal{A}$ for all $i \in N'$. Then, $\mathcal{A}^0 = \bigcup_{i \in N'} \{\mathcal{A}_i^0\}$ is partition of N' formed by elements in $\mathcal{A}$ that is coarser than both $\mathcal{A}^1$ and $\mathcal{A}^2$, which is a contradiction. □

Corollary 2.1. *For each $N \in \mathcal{N}^1$ and $\mathcal{A} \in \mathcal{C}^N$,*

$$E' = \sum_{A \in \mathcal{A}^{(a)}} a_A = \sum_{A \in \mathcal{A}'} a_A.$$

for all $\mathcal{F} = (N, E, c, \mathcal{A}, a)$ satisfying Assumption 1 and Assumption 2, where $\mathcal{A}'$ is the coarsest partition given in Proposition 2.2.

Proof. Let $\mathcal{P} \subset \mathcal{A}$ be any partition of N'. Then, $\mathcal{A}'$ is coarser than $\mathcal{P}$, and hence, under (3), $\sum_{A \in \mathcal{A}'} a_A \leq \sum_{A \in \mathcal{P}} a_A$ and hence $E' = \sum_{A \in \mathcal{A}'} a_A = \sum_{A \in \mathcal{A}^{(a)}} a_A$. □

Under Corollary 2.1, we can assume $\mathcal{A}^{(a)} = \mathcal{A}'$ independent of a.

Notice that Assumption 2 is satisfied in the alternative sets given in Example 2.1 and Example 2.2.

Let $\mathfrak{F}$ be the class of all freight transportation problems satisfying Assumption 1 and Assumption 2.

Given $\mathcal{F} = (N, E, c, \mathcal{A}, a) \in \mathfrak{F}$, we associate a TU-game $\left(N, v^{\mathcal{F}}\right)$, where N is the set of players including the mining firm and local communities and $v^{\mathcal{F}}$ is a characteristic function defined as follows. Given $S \subseteq N$ with $1 \notin S$, $v^{\mathcal{F}}(S) = 0$. Assume now $1 \in S$. Then,

$$v^{\mathcal{F}}(S) = \max_{(\mathcal{P}, f) \in \mathbb{P}^S(\mathcal{A})} \left\{ E - \sum_{P^l \in \mathcal{P}: f(P^l) = alt} a_{P^l} - \sum_{P^l \in \mathcal{P}: f(P^l) = col} c\left(P^l\right) \right\}$$

where

$$\mathbb{P}^S(\mathcal{A}) = \left\{ (\mathcal{P}, f) \in \mathbb{P}(\mathcal{A}) : P^l \in \mathcal{P}, P^l \nsubseteq S \Rightarrow f(P^l) = alt \right\}.$$

Clearly, $S \subset T$ implies $v^{\mathcal{F}}(S) \leq v^{\mathcal{F}}(T)$, i.e., $v^{\mathcal{F}}$ is a monotonic TU-game.

Example 2.5. *The following Tables 1 and 2 show the worth of some coalitions in the cooperative games associated with the freight transportation problems given in Example 2.3 and Example 2.4, respectively:*

Table 1: The worth of some coalitions in Example 2.3 and their respective optimal paths, with $f(P) = col$ unless stated otherwise.

S	$v^{\mathcal{F}}(S)$	Optimal $\mathcal{P}$	Optimal f
$\{1\}$	2	$\{\{2,3\},\{4,5\}\}$	$f(\{2,3\}) = f(\{4,5\}) = alt$
$\{1,2\}$	4	$\{\{2\},\{3\},\{4,5\}\}$	$f(\{3\}) = f(\{4,5\}) = alt$
$\{1,2,3\}$	6	$\{\{2\},\{3\},\{4,5\}\}$	$f(\{4,5\}) = alt$
$\{1,2,4,5\}$	9	$\{\{2\},\{3\},\{4\},\{5\}\}$	$f(\{3\}) = alt$
N	11	$\{\{2\},\{3\},\{4\},\{5\}\}$	$f(\{i\}) = col \;\; \forall i \in N'$

Table 2: The worth of some coalitions in Example 2.4 and their respective optimal paths, with $f(P) = col$ unless stated otherwise.

S	$v^{\mathcal{F}}(S)$	Optimal $\mathcal{P}$	Optimal f
$\{1\}$	0	$\{\{2\},\{3,4,5\}\}$	$f(\{2\}) = f(\{3,4,5\}) = alt$
$\{1,2\}$	1	$\{\{2\},\{3,4,5\}\}$	$f(\{3,4,5\}) = alt$
$\{1,2,3\}$	2	$\{\{2\},\{3\},\{4,5\}\}$	$f(\{2\}) = f(\{4,5\}) = alt$
$\{1,2,4,5\}$	4	$\{\{2\},\{3\},\{4\},\{5\}\}$	$f(\{3\}) = alt$
N	5	$\{\{2\},\{3\},\{4\},\{5\}\}$	$f(\{i\}) = col \;\; \forall i \in N'$

We also define the core of $\mathcal{F} \in \mathfrak{F}$ as follows:

$$Core(\mathcal{F}) = Core\left(N, v^{\mathcal{F}}\right).$$

Our aim is to study whether the core of a freight transportation problem is nonempty and, if so, how to find a core allocation.

2.3 Essential problems

To study the core in freight transportation games, we use the concept of *essential freight transportation problems*, defined as follows:

Definition 2.4. *Given $\mathcal{F} = (N, E, c, \mathcal{A}, a) \in \mathfrak{F}$, we say that $\mathcal{F}$ is* essential *if*

1. $c_i = 0$ *for all* $i \in N'$,

2. $a_A < a_B$ *for all* $A, B \in \mathcal{A}$ *with* $A \subset B$, *and*

3. $a_{A_0} < \sum_{l=1}^{k} a_{A_l}$ *for all* $A_0, A_1, \ldots, A_k \in \mathcal{A}$ *with* $A_0 = \bigcup_{l=1}^{k} A_l$.

It is not difficult to check that the freight transportation problem defined in Example 2.3 is essential. By contrast, the freight transportation problem presented in Example 2.4 is not essential because it does not satisfy any of the three conditions: $c_i \neq 0$ for all $i \in \{2, 3, 4\}$, $a_{\{4\}} = 4 = a_{\{4,5\}}$, and $a_{\{3,4,5\}} = 6 = a_{\{3\}} + a_{\{4,5\}}$. However, there exists an essential problem $\mathcal{F}^*$ that generates the same cooperative game as the problem defined in Example 2.4. It is given by $\mathcal{F}^* = (N, E^*, c^*, \mathcal{A}^*, a^*)$ with $E^* = 5$, $c_i^* = 0$ for all $i \in N'$, $\mathcal{A}^* = \{\{2\}, \{3\}, \{4, 5\}\}$, $a^*_{\{2\}} = a^*_{\{3\}} = 1$, and $a^*_{\{4,5\}} = 3$ (see Figure 3).

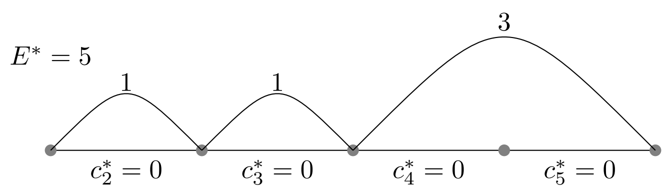

Figure 3: Essential freight transportation problem with the same characteristic function as those given in Example 2.4.

This result is general, as shown in Proposition 2.4. Another advantage of essential problems is that the optimal paths are easily determined, as the next result shows.

Proposition 2.3. *If* $\mathcal{F} = (N, E, c, \mathcal{A}, a) \in \mathfrak{F}$ *is essential, then* $v^{\mathcal{F}}(N) = E$ *and* $a_A = E - v^{\mathcal{F}}(N \setminus A)$ *for all* $A \in \mathcal{A}$. *In particular, for any* $A \in \mathcal{A}$, *the (unique) optimal path for* $v^{\mathcal{F}}(N \setminus A)$ *is* $(\mathcal{P}, f) \in \mathbb{P}^{N \setminus A}(\mathcal{A})$ *given by* $\mathcal{P} = \{A\} \cup \{\{i\}\}_{i \in N' \setminus A}$, $f(A) = alt$, *and* $f(\{i\}) = col$ *for all* $i \in N' \setminus A$.

Proof. $v^{\mathcal{F}}(N) = E$ follows from $c_i = 0$ for all $i \in N$. Fix $A \in \mathcal{A}$. There exists an optimal path $(\mathcal{P}, f) \in \mathbb{P}^{N \setminus A}(\mathcal{A})$ such that

$$v^{\mathcal{F}}(N \setminus A) = E - \sum_{P \in \mathcal{P}: f(P) = alt} a_P.$$

We need to prove that $\mathcal{P} = \{A\} \cup \{\{i\}\}_{i \in N' \setminus A}$, $f(A) = alt$, and $f(\{i\}) = col$ otherwise. Assume, on the contrary, that $A \notin \mathcal{P}$. Then, under Assumption 2, either there exists $B \in \mathcal{P}$ such that $A \subset B$ or there exist $A_1, , \ldots, A_k \in \mathcal{P}$ such that $A = A_1 \cup \cdots \cup A_k$.

- If there exists $B \in \mathcal{P}$ such that $A \subset B$, then we define $(\mathcal{P}', f')$ as follows: $\mathcal{P}' = (\mathcal{P} \setminus \{B\}) \cup \{A\} \cup \{\{i\}\}_{i \in B \setminus A}$, f' defined as $f'(A) = alt$, $f'(\{i\}) = col$ for all $i \in B \setminus A$, and $f'(A') = f(A')$ otherwise. Since $\mathcal{F}$ is essential and $A \subset B$, we deduce $a_A < a_B$. Then,

$$\sum_{P \in \mathcal{P}' : f'(P) = alt} a_P < \sum_{P \in \mathcal{P} : f(P) = alt} a_P$$

  and hence

$$v^{\mathcal{F}}(N \setminus A) = E - \sum_{P \in \mathcal{P} : f(P) = alt} a_P < E - \sum_{P \in \mathcal{P}' : f'(P) = alt} a_P \leq v^{\mathcal{F}}(N \setminus A)$$

  which is a contradiction.

- If there exist $A_1, \ldots, A_k \in \mathcal{P}$ such that $A = A_1 \cup \cdots \cup A_k$, then we define $(\mathcal{P}', f')$ as follows: $\mathcal{P}' = (\mathcal{P} \setminus \{A_1, \ldots, A_k\}) \cup \{A\}$, f' defined as $f'(A) = alt$, $f'(A') = f(A')$ otherwise. Since $\mathcal{F}$ is essential and $A = \bigcup_{l=1}^{k} A_l$, we deduce $a_A < \sum_{l=1}^{k} a_{A_l}$. Then,

$$\sum_{P \in \mathcal{P}' : f'(P) = alt} a_P < \sum_{P \in \mathcal{P} : f(P) = alt} a_P$$

  and hence

$$v^{\mathcal{F}}(N \setminus A) = E - \sum_{P \in \mathcal{P} : f(P) = alt} a_P < E - \sum_{P \in \mathcal{P}' : f'(P) = alt} a_P \leq v^{\mathcal{F}}(N \setminus A)$$

  which is a contradiction.

Assume now $A \in \mathcal{P}$ and there exists $i' \in N' \setminus A$ such that $\{i'\} \notin \mathcal{P}$. Since $\mathcal{P}$ is a partition of N', there exists $A' \in \mathcal{P}$, $A' \neq \{i'\}$, such that $i' \in A'$. We define $(\mathcal{P}', f')$ as follows: $\mathcal{P}' = (\mathcal{P} \setminus \{A'\}) \cup \{\{i\}\}_{i \in A'}$, f' defined as $f'(\{i\}) = col$ for all $i \in A'$, $f'(B) = f(B)$ otherwise. Since $\mathcal{F}$ is essential, we deduce $c(A') = 0 < a_{A'}$. Then,

$$\sum_{P \in \mathcal{P}' : f'(P) = alt} a_P < \sum_{P \in \mathcal{P} : f(P) = alt} a_P$$

and hence

$$v^{\mathcal{F}}(N \setminus A) = E - \sum_{P \in \mathcal{P} : f(P) = alt} a_P < E - \sum_{P \in \mathcal{P}' : f'(P) = alt} a_P \leq v^{\mathcal{F}}(N \setminus A)$$

which is a contradiction. Hence, $\mathcal{P} = \{A\} \cup \{\{i\}\}_{i \in N' \setminus A}$. We still need to prove that $f(A) = alt$ and $f(\{i\}) = col$ otherwise.

- $f(A) = alt$ follows from the fact that $(P, f) \in \mathbb{P}^{N \setminus A}(\mathcal{A})$.
- $f(\{i\}) = col$ follows from the fact that $c_i = 0 < a_{\{i\}}$ for all $i \in N'$.

□

Proposition 2.4. *For all $\mathcal{F} \in \mathfrak{F}$, there exists a unique $\mathcal{F}^* \in \mathfrak{F}$ essential such that $v^{\mathcal{F}} = v^{\mathcal{F}^*}$.*

Proof. Given $\mathcal{F} = (N, E, c, \mathcal{A}, a) \in \mathfrak{F}$, we define $\mathcal{F}^* = (N, E^*, c^*, \mathcal{A}^*, a^*)$ where $E^* = E - c(N')$, $c_i^* = 0$ for all $i \in N'$, and $\mathcal{A}^*$ and a^* are defined as follows. Let

$$\mathcal{A}^1 = \mathcal{A} \setminus \{A \in \mathcal{A} : \exists B \in \mathcal{A}, \text{ such that } A \subset B, a_A + c(B \setminus A) = a_B\} \tag{4}$$

$$\mathcal{A}^* = \mathcal{A}^1 \setminus \left\{ A \in \mathcal{A}^1 : \exists A_1, \dots, A_k \in \mathcal{A}^1 \text{ such that } A = \bigcup_{l=1}^{k} A_l \text{ and } \sum_{l=1}^{k} a_{A_l} = a_A \right\} \tag{5}$$

and a^* is defined as $a^*_A = a_A - c(A)$ for all $A \in \mathcal{A}^*$. It is straightforward to check that $\mathcal{F}^* \in \mathfrak{F}$ and it is essential. Moreover, $v^{\mathcal{F}}(S) = v^{\mathcal{F}^*}(S)$ for all $S \subseteq N$. To see why, notice that we remove alternatives A that are redundant because it is equivalent to using either

- the longer alternative B that also avoids communities in $B \setminus A$ (equation (4)), or
- the partition of alternatives $A_1, \dots, A_k$ that also avoids A (equation (5)).

We now prove the uniqueness. Assume there exist two essential problems $\mathcal{F}^1 = (N, E^1, c^1, A^1, a^1)$ and $\mathcal{F}^2 = (N, E^2, c^2, A^2, a^2)$ with $v^{\mathcal{F}^1} = v^{\mathcal{F}^2} = v^{\mathcal{F}}$. We check that $\mathcal{F}^1 = \mathcal{F}^2$. First,

$$E^1 = v^{\mathcal{F}^1}(N) = v^{\mathcal{F}^2}(N) = E^2$$

and $c_i^1 = 0 = c_i^2$ for all $i \in N'$ by definition of essential problems. To check $\mathcal{A}^1 = \mathcal{A}^2$ and $a^1 = a^2$, we proceed by an induction argument on the size of coalitions. It is trivially true for $|A| = 0$, because $\{A \in \mathcal{A}^1 : |A| = 0\} = \emptyset = \{A \in \mathcal{A}^2 : |A| = 0\}$. Given $s \in \{1, 2, \dots\}$, assume $\{A \in \mathcal{A}^1 : |A| < s\} =$

$\{A \in \mathcal{A}^2 : |A| < s\}$. We have to prove that $\{A \in \mathcal{A}^1 : |A| = s\} = \{A \in \mathcal{A}^2 : |A| = s\}$. In particular, we prove $\{A \in \mathcal{A}^1 : |A| = s\} \subseteq \{A \in \mathcal{A}^2 : |A| = s\}$ because the other way around is equivalent. We proceed by contradiction. Let $A \in \mathcal{A}^1$ with $|A| = s$ and assume $A \notin \mathcal{A}^2$. Let $(\mathcal{P}^2, f^2) \in \mathbb{P}^{N\setminus A}(\mathcal{A}^2)$ such that $v^{\mathcal{F}^2}(N\setminus A) = E^2 - \sum_{P\in\mathcal{P}^2 : f(P)=alt} a^2_P$. Since $A \notin \mathcal{A}^2$, under Assumption 2 either there exists $B \in \mathcal{A}^2$ such that $A \subset B$ and $v^{\mathcal{F}^2}(N \setminus A) = E^2 - a^2_B$, or there exist $A_1, \ldots, A_k \in \mathcal{A}^2$, $k > 1$, such that $A = \bigcup_{l=1}^k A_l$ and $v^{\mathcal{F}^2}(N\setminus A) = E^2 - \sum_{l=1}^k a^2_{A_l}$. We study both cases:

- There exists $B \in \mathcal{A}^2$ such that $A \subset B$ and $v^{\mathcal{F}^2}(N \setminus A) = E^2 - a^2_B$. Under Proposition 2.3, $a^1_A = E^1 - v^{\mathcal{F}^1}(N \setminus A)$ and $a^2_B = E^2 - v^{\mathcal{F}^2}(N \setminus B)$. Hence,

$$E^1 - a^1_A = v^{\mathcal{F}^1}(N \setminus A) = v^{\mathcal{F}^2}(N \setminus A) = E^2 - a^2_B.$$

  Since $E^1 = E^2$, we deduce $a^1_A = a^2_B$. We have two cases:

  - If $B \in \mathcal{A}^1$, then $a^1_B = E^1 - v^{\mathcal{F}^1}(N\setminus B) = E^2 - v^{\mathcal{F}^2}(N\setminus B) = a^2_B = a^1_A$. But $A \subset B$ implies $a^1_A < a^1_B$, which is a contradiction.
  - If $B \notin \mathcal{A}^1$, then we have two subcases:
    * There exists $B' \in \mathcal{A}^1$ such that $B \subset B'$ and $v^{\mathcal{F}^1}(N \setminus B) = E^1 - a^1_{B'}$. In this case, $v^{\mathcal{F}^1}(N \setminus B) = v^{\mathcal{F}^1}(N \setminus B')$. We also know that $v^{\mathcal{F}^1}(N \setminus B) = v^{\mathcal{F}^2}(N \setminus B) = v^{\mathcal{F}^1}(N \setminus A)$. Hence, $E^1 - a^1_{B'} = v^{\mathcal{F}^1}(N \setminus B') = v^{\mathcal{F}^1}(N \setminus A) = E_1 - a^1_A$. Hence, $a^1_{B'} = a^1_A$. But $A \subset B'$ implies $a^1_A < a^1_{B'}$, which is a contradiction.
    * There exist $B_1 = A, B_2, \ldots, B_k \in \mathcal{A}^1$, $k > 1$, such that $B = \bigcup_{l=1}^k B_l$ and $v^{\mathcal{F}^1}(N \setminus B) = E^1 - \sum_{l=1}^k a^1_{B_l}$. Moreover,

$$v^{\mathcal{F}^1}(N \setminus B) = v^{\mathcal{F}^2}(N \setminus B) = E^2 - a^2_B = E^1 - a^1_A.$$

      Hence, $\sum_{l=1}^k a^1_{B_l} = a^1_A$. Since $B_1 = A$, we deduce $\sum_{l=2}^k a^1_{B_l} = 0$, which is a contradiction because $a^1_{B_l} > c^1(B_l) = 0$ for all l.

- There exist $A_1, \ldots, A_k \in \mathcal{A}^2$, $k > 1$, such that $A = \bigcup_{l=1}^k A_l$ and

$$v^{\mathcal{F}^2}(N \setminus A) = E^2 - \sum_{l=1}^{k} a^2_{A_l}.$$

  Under the induction hypothesis, $A_1, \ldots, A_k \in \mathcal{A}^1$ and $a^1_{A_l} = a^2_{A_l}$. Moreover, $v^{\mathcal{F}^1}(N \setminus A) = E^1 - a^1_A$. Since $E^1 = E^2$, we deduce that $a^1_A = \sum_{l=1}^k a^1_{A_l}$. This is a contradiction because essentially of $\mathcal{F}^1$ implies $a^1_A < \sum_{l=1}^k a^1_{A_l}$.

Finally, we prove that $a^1 = a^2$. Given $A \in \mathcal{A}^1 = \mathcal{A}^2$, under Proposition 2.3,

$$a_A^1 = E^1 - v^{\mathcal{F}^1}(N \setminus A) = E^2 - v^{\mathcal{F}^2}(N \setminus A) = a_A^2.$$

■

3 The core

In this section, we study the core of freight transportation problems. In particular, we study whether it is always possible to find a core allocation, i.e., an agreement that no coalition of players has incentives to reject.

The following results, which can also be derived from Bahel (2016), show that the answer is affirmative. The core is always nonempty in a freight transportation problem.

Theorem 3.1. *Given $\mathcal{F} = (N, E, c, \mathcal{A}, a) \in \mathfrak{F}$, the core associated to $\mathcal{F}$ is:*

$$Core(\mathcal{F}) = \left\{ x \in \mathbb{R}_+^N : x(N) = E - c(N'),\ x(A) \leq a_A - c(A)\ \forall A \in \mathcal{A} \right\}.$$

Proof. Let $\mathcal{F} \in \mathfrak{F}$. The result is trivially true for $N' = \emptyset$. Assume $N' \neq \emptyset$.

"$\subseteq$" Let $x \in Core(\mathcal{F})$. Hence, $x_1 \geq v^{\mathcal{F}}(\{1\}) = E - \sum_{A \in \mathcal{A}'} a_A$, which is non-negative by Assumption 1. Moreover, for each $i \in N'$, $x_i \geq v^{\mathcal{F}}(\{i\}) = 0$. Thus, $x \in \mathbb{R}_+^N$. For the second condition:

$$x(N) = v^{\mathcal{F}}(N) = E - c(N').$$

And for the third condition, let $A \in \mathcal{A}$. Then,

$$x(N \setminus A) \geq v^{\mathcal{F}}(N \setminus A) = E - c(N' \setminus A) - a_A$$

and hence

$$\begin{aligned} x(A) = x(N) - x(N \setminus A) &= v^{\mathcal{F}}(N) - x(N \setminus A) \\ &\leq v^{\mathcal{F}}(N) - E + c(N' \setminus A) + a_A \\ &= E - c(N') - E + c(N' \setminus A) + a_A = a_A - c(A). \end{aligned}$$

"$\supseteq$" Let $x \in \mathbb{R}_+^N$ such that $x(N) = E - c(N')$ and $x(A) \leq a_A - c(A)$ for all $A \in \mathcal{A}$. We first prove that x is an imputation, i.e., $x \in I(N, v^{\mathcal{F}})$:

$$x(N) = E - c(N') = v^{\mathcal{F}}(N).$$

Moreover, given $i \in N$, it is straightforward to check that $x_i \geq 0 = v^{\mathcal{F}}(\{i\})$. We now prove that $x(S) \geq v^{\mathcal{F}}(S)$ for all $S \subset N$. Let $S \subset N$. Assume first $1 \notin S$. Since $x \in \mathbb{R}^N_+$, $v^{\mathcal{F}}(S) = 0 \leq x(S)$. Assume then $1 \in S$. There exists $(\mathcal{P}, f) \in \mathbb{P}^S(\mathcal{A})$ such that

$$v^{\mathcal{F}}(S) = E - \sum_{P \in \mathcal{P}: f(P) = alt} a_P - \sum_{P \in \mathcal{P}: f(P) = col} c(P)$$

and such that $P \in \mathcal{P}$ and $P \nsubseteq S$ imply $f(P) = alt$. Equivalently, $f(P) = col$ implies $P \subseteq S$. Hence, $\{1\} \cup S' \subseteq S$ where

$$S' = \{j \in N' : j \in P \text{ for some } P \in \mathcal{P}, f(P) = col\}.$$

Thus,

$$\begin{aligned}
v^{\mathcal{F}}(S) &= E - \sum_{P \in \mathcal{P}: f(P) = alt} a_P - \sum_{P \in \mathcal{P}: f(P) = col} c(P) \\
&\leq E - \sum_{P \in \mathcal{P}: f(P) = alt} (x(P) + c(P)) - \sum_{P \in \mathcal{P}: f(P) = col} c(P) \\
&= E - \sum_{P \in \mathcal{P}: f(P) = alt} x(P) - \sum_{P \in \mathcal{P}: f(P) = alt} c(P) - \sum_{P \in \mathcal{P}: f(P) = col} c(P) \\
&= E - \sum_{P \in \mathcal{P}: f(P) = alt} x(P) - c(N') \\
&= x(N) - \sum_{P \in \mathcal{P}: f(P) = alt} x(P) \\
&= x_1 + \sum_{P \in \mathcal{P}: f(P) = col} x(P) \\
&= x(\{1\} \cup S') \overset{\{1\} \cup S' \subseteq S}{=} x(S) - x(S \setminus (\{1\} \cup S')) \overset{x \in \mathbb{R}^N_+}{\leq} x(S).
\end{aligned}$$

■

Corollary 3.1. *The core of any freight transport problem satisfying Assumption 1 and Assumption 2 is always nonempty.*

Proof. Fix $\mathcal{F} \in \mathfrak{F}$ and let $x \in \mathbb{R}^N$ given by $x_1 = E - c(N')$ and $x_i = 0$ for all $i \in N'$. Under (1) and Assumption 1, $x_1 = E - c(N') > E - \sum_{A \in \mathcal{A}'} a_A \geq 0$. Hence, $x \in \mathbb{R}^N_+$. Moreover, $x(N) = E - c(N')$. Finally, given $A \in \mathcal{A}$,

$$a_A - c(A) \overset{(2)}{>} 0 = x(A).$$

Under Theorem 3.1, $x \in Core(\mathcal{F})$. Hence, $Core(\mathcal{F}) \neq \emptyset$. □

Corollary 3.1 also follows from Lemma 2 in Bahel (2016). However, our proof is constructive and provides a particular and relevant core allocation that will be denoted as $\phi^0(\mathcal{F})$ in the next Section.

4 A family of core allocation rules

In this section, we propose a family of core allocation rules. As the first step, we study the concept of levels structure, a concept first formalized in the context of cooperative game theory by Winter (1989).

Definition 4.1. *We define a levels structure over N' as a finite sequence $\mathfrak{A} = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m)$ of partitions of N' such that, for each $A \in \mathcal{A}_l$, $l > 1$, there exists $\mathcal{B} \subseteq \mathcal{A}_{l-1}$ with $A = \bigcup_{B \in \mathcal{B}} B$.*

Lemma 4.1. *If $\mathcal{F} \in \mathfrak{F}$ is essential, then there exists a unique levels structure $\mathfrak{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_m)$ over N' such that:*

1. *All partitions A_l are different.*
2. *$A \in \mathcal{A} \cup \{N'\} \cup \{\{i\}\}_{i \in N'}$ if and only if there exists l such that $A \in \mathcal{A}_l$.*
3. *If $A \in \mathcal{A}_l$, $l > 1$, and $\mathcal{B} \subseteq \mathcal{A}_{l-1}$ is such that $A = \bigcup_{B \in \mathcal{B}} B$, then either $|\mathcal{B}| > 1$ or $|A| = 1$.*

In particular, $\mathcal{A}_1 = \{\{i\}\}_{i \in N'}$ and $\mathcal{A}_m = \{N'\}$.

Proof. The proof is constructive. The first partition is defined as $\mathcal{A}_m = \{N'\}$. Assume we have defined $\mathcal{A}_{l+1}, \ldots, \mathcal{A}_m$. We define

$$\widetilde{\mathcal{A}}_l = \{A \in \mathcal{A} : A \subset B \in \mathcal{A} \implies B \in \mathcal{A}_{l'} \text{ for some } l' > l\}$$

and

$$\mathcal{A}_l = \widetilde{\mathcal{A}}_l \cup \{\{i\}\}_{i \in N' \setminus \bigcup_{A \in \tilde{\mathcal{A}}_l} A}.$$

The process continues until all $A \in \mathcal{A}$ belong to some $\mathcal{A}_l$, so that condition 1 is satisfied. In this case, $\widetilde{\mathcal{A}}_l = \widetilde{\mathcal{A}}_1 = \emptyset$, so that $\mathcal{A}_1 = \{\{i\}\}_{i \in N'}$. □

Example 4.1. *Assume $\mathcal{A} = \{\{2\}, \{3\}, \{4\}, \{2,3\}, \{2,3,4\}\}$. Then, $\mathfrak{A} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ where $\mathcal{A}_1 = \{\{2\}, \{3\}, \{4\}\}$, $\mathcal{A}_2 = \{\{2,3\}, \{4\}\}$ and $\mathcal{A}_3 = \{N'\}$.*

A *rule* ϕ is a function that assigns to each problem $\mathcal{F} = (N, E, c, \mathcal{A}, a) \in \mathfrak{F}$ a payoff allocation $\phi(\mathcal{F}) \in \mathbb{R}^N$.

We define a family of rules as follows: Let $\mathcal{F} = (N, E, c, \mathcal{A}, a) \in \mathfrak{F}$ and let $\mathcal{F}^* = (N, E^*, c^*, \mathcal{A}^*, a^*)$ be the (unique) essential problem associated with $\mathcal{F}$ as given by Proposition 2.4. Let $\mathfrak{A} = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m)$ be the unique level structure associated to $\mathcal{F}^*$ as given in Lemma 4.1. Given $i \in N'$ and $l \in \{1, \ldots, m\}$, let $A_l^i \in \mathcal{A}_l^*$ be the (unique) coalition in level l that contains agent i. Obviously, $i \in A_1^i \subseteq A_2^i \subseteq \cdots \subseteq A_m^i$. Let $\mathcal{A}' \subseteq \mathcal{A}$ be the partition of N' so that $v^{\mathcal{F}}(\{1\}) = E - \sum_{A \in \mathcal{A}^{(a)}} a_A$ (as given by Assumption 1). Given $\theta \in \mathbb{R}$, we define the rule ϕ^θ as follows:

$$\phi_1^\theta(\mathcal{F}) = E - \theta E' - (1 - \theta)\, c(N')$$

and

$$\phi_i^\theta(\mathcal{F}) = \theta\, (E' - c(N')) \prod_{l=1}^{m-1} \frac{\widetilde{a}_{A_l^i}}{\sum_{A \in \mathcal{A}_l : A \subseteq \mathcal{A}_{l+1}^i} (\widetilde{a}_A)} \tag{6}$$

for each $i \in N'$, where A_l^i is the (only) coalition in $\mathcal{A}_l$ that contains player i, and

$$\widetilde{a}_A = \min_{B \in \mathcal{A}^* : A \subseteq B} a_B^*.$$

Notice that $\widetilde{a}_A = a_A$ whenever $a \in \mathcal{A}^*$, and otherwise $A = \{i\}$ for some $i \in N'$, so that $\widetilde{a}_{\{i\}} = \min_{B \in \mathcal{A}^* : i \in B} a_B^*$. Moreover, for each $i \in N'$, there exists some $l^i \in \{1, \ldots, m\}$ such that $A_{l^i}^i \in \mathcal{A}^*$ and $A_l^i = \{i\}$ for all $l < l^i$, so that $\widetilde{a}_{\{i\}} = a^*_{A_{l^i}^i}$. Under Proposition 2.3, we deduce that

$$\widetilde{a}_{\{i\}} = v^{\mathcal{F}}(N) - v^{\mathcal{F}}\left(N \setminus A_{l^i}^i\right).$$

We now define some reasonable properties as follows:

Core selection (CS) Given $\mathcal{F}$, $\phi(\mathcal{F}) \in Core(\mathcal{F})$.

Core selection is a property that concerns the stability of a solution.

For the next property, we consider the problem that arises when a group of consecutive communities merge. This property prevents communities from manipulating their outcome by merging or splitting. It is a very relevant property in situations where the identity of the communities is unclear, as they can, for example, associate at different local levels or administrative levels (village, town hall, region, etc.).

Definition 4.2. *Two consecutive coalitions A, $A' \in \mathcal{A}$ are mergeable if they belong to the same supra-coalitions, i.e., for all $B \in \mathcal{A}$, $A \subsetneq B$ if and only if $A' \subsetneq B$.*

Independence of Merging of Mergeable Alternatives (IMMA) Given k pairwise disjoint consecutive and mergeable coalitions $A_1, \ldots, A_k \in \mathcal{A}$ with $i \in A = \bigcup_{l=1}^{k} A_l$, then, $\phi_j(\mathcal{F}) = \phi_j(\mathcal{F}^{A,i})$ for all $j \in N' \setminus A$, where

$$\mathcal{F}^{A,i} = \left((N \setminus A) \cup \{i\}, E, c^{A,i}, \mathcal{A}^{A,i}, a^{A,i}\right)$$

is given by

- $c_i^{A,i} = \sum_{j \in A} c_j$, $c_j^{A,i} = c_j$ otherwise,
- $\mathcal{A}^{A,i} = \{B \in \mathcal{A} : A \cap B = \emptyset\} \cup \{(B \setminus A) \cup \{i\} : A \subseteq B \in \mathcal{A}, \} \cup \{\{i\}\}$,
- $a_B^{A,i} = a_B$ if $A \cap B = \emptyset$, $a_{(B \setminus A) \cup \{i\}}^{A,i} = a_B$ if $A \subsetneq B$, and

$$a_{\{i\}}^{A,i} = \begin{cases} a_A & \text{if } A \in \mathcal{A} \\ \sum_{l=1}^{k} a_{A_l} & \text{if } A \notin \mathcal{A}. \end{cases}$$

IMMA states that no other agent should be affected if a group of communities with an essential alternative merge. Alternatively, if a community with an essential alternative splits into new consecutive communities, having the new group an essential alternative, no other agent should be affected.

Equivalence (EQUI). Given $\mathcal{F} = (N, E, c, \mathcal{A}, a) \in \mathfrak{F}$, $\phi(\mathcal{F}) = \phi\left(\widetilde{\mathcal{F}}\right)$, where $\widetilde{\mathcal{F}} = \left(N, E, c, \widetilde{\mathcal{A}}, \widetilde{a}\right)$ is defined as $\widetilde{\mathcal{A}} = \mathcal{A} \cup \{\{i\}\}_{i \in N'}$, $\widetilde{a}_A = a_A$ for all $A \in \mathcal{A}$, and

$$\widetilde{a}_{\{i\}} \geq \min_{A \in \mathcal{A} : i \in A} \{a_A - c(A \setminus \{i\})\}$$

for all $\{i\} \notin \mathcal{A}$.

EQUI states that the payoffs do not change if we assume that communities with no feasible alternative (i.e., those $i \in N'$ such that $\{i\} \notin \mathcal{A}$) do have an alternative, but it is so expensive that it is not worthy of using it.

Two communities $i, j \in N'$ are *symmetric* if they have the same cost, they belong to the same supra-coalitions, and their alternative costs (if any) are equal, i.e.,

- $c_i = c_j$,
- if $\{i\} \subsetneq A \in \mathcal{A}$, then $j \in A$; analogously, if $\{j\} \subsetneq A \in \mathcal{A}$, then $i \in A$, and
- if $\{i\} \in \mathcal{A}$, then $\{j\} \in \mathcal{A}$ and $a_{\{i\}} = a_{\{j\}}$; analogously, if $\{j\} \in \mathcal{A}$, then $\{i\} \in \mathcal{A}$ and $a_{\{i\}} = a_{\{j\}}$.

Symmetry (SYM): If $i, j \in N'$ are symmetric, then $\phi_i(\mathcal{F}) = \phi_j(\mathcal{F})$.

Additivity (ADD). Given $\mathcal{F}^1 = \left(N, E^1, c^1, \mathcal{A}, a^1\right), \mathcal{F}^2 = \left(N, E^2, c^2, \mathcal{A}, a^2\right) \in \mathfrak{F}$, $\phi(\mathcal{F}^1 + \mathcal{F}^2) = \phi(\mathcal{F}^1) + \phi(\mathcal{F}^2)$, where

$$\mathcal{F}^1 + \mathcal{F}^2 = \left(N, E^1 + E^2, c^1 + c^2, \mathcal{A}, a^1 + a^2\right).$$

In order to illustrate ADD, assume that a mining firm transports copper and gold. We consider that E is the benefit generated by copper transportation and E' by gold transportation in the mining transportation context, c and c' represent costs from different sources like air and land pollution, analogously for a and a'. Then, when computing the compensations that the mining firm should provide to the communities, ADD states that there is no difference between considering both problems separately or in aggregate.

A weaker version of ADD applies when a fixed amount x_i is added to each player $i \in N$. A natural requirement is that this change does not affect the final allocation. In particular, a change of x_i in the cost of a community $i \in N'$ that affects its alternatives and the estate on the same amount should not change player i's allocation. On the other hand, an additional change of x_1 in the estate only affects the mining firm.

Translation Invariance (TI) Given the problems $\mathcal{F}^1 = (N, E^1, c^1, \mathcal{A}, a^1)$, $\mathcal{F}^2 = (N, E^2, c^2, \mathcal{A}, a^2) \in \mathfrak{F}$, and $x \in \mathbb{R}^N$ such that $E^2 = E^1 + x(N)$, $c_i^2 = c_i^1 + x_i$ for all $i \in N'$, and $a_A^2 = a_A^1 + x(A)$ for all $A \in \mathcal{A}$, then

$$\phi\left(\mathcal{F}^2\right) = \phi(\mathcal{F}^1) + (x_1, 0, \ldots, 0).$$

Lemma 4.2. *Let ϕ be a rule that satisfies CS, IMMA, and SYM. Then, for each $\mathcal{F} = (N, E, c, \mathcal{A}, a)$ with $c_i = 0$ and $\{i\} \in \mathcal{A}$ for all $i \in N'$:*

$$\phi_i(\mathcal{F}) = (E - \phi_1(\mathcal{F})) \prod_{l=1}^{m-1} \frac{a_{A_l^i}}{\sum_{A \in \mathcal{A}_l : A \subseteq A_{l+1}^i} (a_A)} \qquad (7)$$

for all $i \in N'$, where $\mathfrak{A} = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m)$ is the unique level structure associated to $\mathcal{F}^*$ as given in Lemma 4.1 and $A_l^i \in \mathcal{A}_l$ is the (unique) coalition in level l that contains agent i.

We prove Lemma 4.2 in the Appendix. We illustrate a sketch of the proof in the following example.

Example 4.2. *Let $\mathcal{F} = (N, E, c, \mathcal{A}, a)$ with $N = \{1, 2, 3, 4\}$, $E = 10$, $c_i = 0$ for all $i \in N'$, $\mathcal{A} = \{\{2\}, \{3\}, \{4\}, \{2, 3\}, \{2, 3, 4\}\}$, $a_{\{2\}} = 3$, $a_{\{3\}} = 4$, $a_{\{4\}} = 3$, $a_{\{2,3\}} = 5$, and $a_{\{2,3,4\}} = 6$. See Figure 4.*

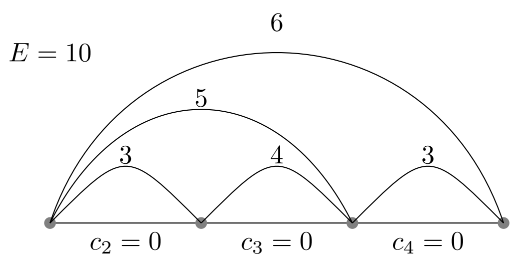

Figure 4: Example of a freight transportation problem, where communities 2 and 3 are mergeable, and so are coalitions $\{2, 3\}$ and $\{4\}$.

In this example, $m = 3$ and $\mathfrak{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ where

$$\mathcal{A}_1 = \{\{2\}, \{3\}, \{4\}\}$$
$$\mathcal{A}_2 = \{\{2, 3\}, \{4\}\}$$
$$\mathcal{A}_3 = \{\{2, 3, 4\}\}.$$

Assume ϕ satisfies CS, IMMA, and SYM. We need to prove (7), which in this example means $\phi_2(\mathcal{F}) = (E - \phi_1(\mathcal{F}))\frac{3}{7}\frac{5}{8}$, $\phi_3(\mathcal{F}) = (E - \phi_1(\mathcal{F}))\frac{4}{7}\frac{5}{8}$, and $\phi_4(\mathcal{F}) = (E - \phi_1(\mathcal{F}))\frac{3}{3}\frac{3}{8}$.

Under CS, $\phi_2(\mathcal{F}) + \phi_3(\mathcal{F}) + \phi_4(\mathcal{F}) = E - \phi_1(\mathcal{F})$. Communities 2 and 3 are mergeable. Hence, under IMMA, we can merge them and work with the reduced problem $\mathcal{F}' = \mathcal{F}^{\{2,3\},2}$ with communities $\{2, 4\}$ and $a'_{\{2\}} = 5$, $a'_{\{4\}} = 3$, and $a'_{\{2,4\}} = 6$. We apply IMMA again to split these new communities and work with the extended problem with eight symmetric communities, each of them with $a''_{\{i\}} = 1$. Under SYM, each split community receives $\frac{E - \phi_1(\mathcal{F})}{8}$. Under IMMA, we can remerge the eight communities into 2 and 4, so that $\phi_2(\mathcal{F}') =$

$5\frac{E-\phi_1(\mathcal{F})}{8}$ *and* $\phi_4(\mathcal{F}) = \phi_4(\mathcal{F}') = 3\frac{E-\phi_1(\mathcal{F})}{8}$. *We repeat the same reasoning with IMMA and SYM applied to coalition* $\{2\}$ *alone so that we conclude that* $\phi_2(\mathcal{F}) = 5\frac{E-\phi_1(\mathcal{F})}{8}\frac{3}{7}$ *and* $\phi_3(\mathcal{F}) = 5\frac{E-\phi_1(\mathcal{F})}{8}\frac{4}{7}$*, as desired.*

Theorem 4.1. *A rule* ϕ *satisfies CS, IMMA, EQUI, SYM, and TI if and only if there exists* $\theta \in [0,1]$ *such that* $\phi = \phi^\theta$.

Proof. It is straightforward to check that any ϕ^θ satisfies CS, IMMA, EQUI, and SYM. Let $\mathcal{F}^1 = (N, E^1, c^1, \mathcal{A}, a^1)$, $\mathcal{F}^2 = (N, E^2, c^2, \mathcal{A}, a^2) \in \mathfrak{F}$ and $x_i \in \mathbb{R}^N$ be as in the definition of TI, i.e., $E^2 = E^1 + x(N)$, $c_i^2 = c_i^1 + x_i$ for all $i \in N'$, and $a_A^2 = a_A^1 + x(A)$ for all $A \in \mathcal{A}$. We need to prove $\phi(\mathcal{F}^2) = \phi(\mathcal{F}^1) + (x_1, 0, \ldots, 0)$. The common $\mathcal{A}$ implies a common $\mathcal{A}'$ and a common levels structure $\mathcal{U} = (A_1, \ldots, A_m)$ for both problems. Hence,

$$\begin{aligned}
\phi_1^\theta\left(\mathcal{F}^2\right) &= E^2 - \theta \sum_{A\in\mathcal{A}'} a_A^2 - (1-\theta)c^2(N') \\
&= E^1 + x(N) - \theta \sum_{A\in\mathcal{A}'} a_A^1 - \theta \sum_{A\in\mathcal{A}'} x(A) - (1-\theta)c^1(N') - (1-\theta)x(N') \\
&= E^1 + x(N) - \theta \sum_{A\in\mathcal{A}'} a_A^1 - \theta x(N') - (1-\theta)c^1(N') - (1-\theta)x(N') \\
&= E^1 - \theta \sum_{A\in\mathcal{A}'} a_A^1 - (1-\theta)c^1(N') + x_1 = \phi_1(\mathcal{F}^1) + x_1.
\end{aligned}$$

Moreover, for each $A \in \mathcal{A}^*$,

$$\begin{aligned}
\left(a^2\right)_A^* &= a_A^2 - c^2(A) \\
&= a_A^1 + x(A) - c^1(A) - x(A) \\
&= a_A^1 - c^1(A)
\end{aligned}$$

and hence $\widetilde{a^2} = \widetilde{a^1}$. Thus, for each $i \in N'$:

$$\begin{aligned}
\phi_i^\theta(\mathcal{F}^2) &= \theta\left(\sum_{A\in\mathcal{A}'} a_A^2 - c^2(N')\right) \prod_{l=1}^{m-1} \frac{\widetilde{a^2}_{A_l^i}}{\sum_{A\in\mathcal{A}_l : A\subseteq\mathcal{A}_{l+1}^i}\left(\widetilde{a^2}_A\right)} \\
&= \theta\left(\sum_{A\in\mathcal{A}'} a_A^1 - c^1(N')\right) \prod_{l=1}^{m-1} \frac{\widetilde{a^1}_{A_l^i}}{\sum_{A\in\mathcal{A}_l : A\subseteq\mathcal{A}_{l+1}^i}\left(\widetilde{a^1}_A\right)} = \phi_i^\theta(\mathcal{F}^1).
\end{aligned}$$

Let ϕ be a rule satisfying CS, IMMA, EQUI, SYM, and TI. Under CS and TI, we can assume $E = E'$ and $c_i = 0$ for all $i \in N'$. We can also assume $E > 0$. Otherwise, the result is trivial under CS. Partition $\mathcal{A}'$ given in Assumption 1 is a pairwise disjoint consecutive and mergeable coalitions set. Fix a community in N'. We can assume w.l.o.g. it is community 2. Under IMMA, $\phi_1(\mathcal{F}) = \phi_1(\mathcal{F}^{N',2})$ and

$$\phi_2\left(\mathcal{F}^{N',2}\right) = \sum_{i\in N'} \phi_i\left(\mathcal{F}^{N',2}\right).$$

Under Theorem 3.1, it follows that $Core(\mathcal{F}^{N',2}) = \{(x_1, x_2) : x_1 + x_2 = E, x_1, x_2 \geq 0\}$. By CS, $\phi_2(\mathcal{F}^{N',2}) = \phi_2(\mathcal{F}) \in [0, E]$. We define $\theta = \frac{1}{E}\phi_2(\mathcal{F}) \in [0, 1]$. Under IMMA, $\phi_1(\mathcal{F}) = (1 - \theta)E$. Under CS, communities in N' share θE, i.e., $\sum_{i\in N'} \phi_i(\mathcal{F}) = \theta E$. Under EQUI, we can assume that $\{i\} \in \mathcal{A}$ for all $i \in N'$. We can then apply Lemma 4.2 to conclude the uniqueness part with

$$\begin{aligned}\phi_i(\mathcal{F}) &= (E - \phi_1(\mathcal{F})) \prod_{l=1}^{m-1} \frac{a_{A_l^i}}{\sum_{A\in\mathcal{A}_l : A\subset\mathcal{A}_{l+1}^i}(a_A)} \\ &= \theta E \prod_{l=1}^{m-1} \frac{a_{A_l^i}}{\sum_{A\in\mathcal{A}_l : A\subset\mathcal{A}_{l+1}^i}(a_A)} = \phi_i^\theta(\mathcal{F}).\end{aligned}$$

□

These properties are independent. We define rules that satisfy all the properties but exactly one.

- Rule ϕ^θ with $\theta \notin [0, 1]$ satisfies all the properties but CS.
- Let $\overline{\phi}$ defined as
$$\overline{\phi}(\mathcal{F}) = \begin{cases}\phi^0(\mathcal{F}) \text{ if } |N'| > 1\\ \phi^1(\mathcal{F}) \text{ if } |N'| = 1\end{cases} \tag{8}$$
for all $\mathcal{F} \in \mathfrak{F}$ with N as player set. This rule satisfies all the properties but IMMA.
- Let $\overline{\phi}^1$ defined, for each $\mathcal{F} \in \mathfrak{F}$ with N as player set, as
$$\overline{\phi}_1^1(\mathcal{F}) = E - E'$$
and, for each $i \in N'$,
$$\overline{\phi}_i^1(\mathcal{F}) = (E - c(N')) \prod_{l=1}^{m-1} \frac{\overline{a}_{A_l^i}}{\sum_{A\in\mathcal{A}_l : A\subset\mathcal{A}_{l+1}^i}(\overline{a}_A)}$$
where $\overline{a}_A = a_A$ for all $A \in \mathcal{A}^*$ and $\overline{a}_i = 0$ for all $\{i\} \in \mathcal{A}^* \setminus \mathcal{A}$. This rule satisfies all the properties but EQUI.

- Let $\phi^{(last)}$ defined as the rule that gives, among all core allocations of the associated essential problem, the one that assigns the maximum possible to the last community, then the maximum possible to the second to last one, and so on, up to E'. Formally, for each $x, y \in \mathbb{R}_+^{N'}$, we say that $x \precsim_{lex} y$ if either $x = y$ or there exists $k \in N'$ such that $x_k = y_k$ and $x_i < y_i$ for all $i < k$. Moreover, given $X \subseteq \mathbb{R}_+^{N'}$, let $L(X) \in X$ such that $x \precsim_{lex} L(X)$ for all $x \in X$. We then define

$$\phi^{(last)}(\mathcal{F}) = L(Core(\mathcal{F})) \tag{9}$$

  for all $\mathcal{F} \in \mathfrak{F}$. This rule satisfies all the properties but SYM.

- The rule defined as $\phi(\mathcal{F}) = \phi^{\frac{E'}{E}}(\mathcal{F})$ for all $\mathcal{F} \in \mathfrak{F}$ satisfies all the properties but TI.

Moreover, ϕ^0 (i.e., ϕ^θ with $\theta = 0$) also satisfies ADD:

Proposition 4.1. *ϕ^0 satisfies ADD.*

Proof. Let $\mathcal{F}^1 = (N, E^1, c^1, \mathcal{A}, a^1)$ and $\mathcal{F}^2 = (N, E^2, c^2, \mathcal{A}, a^2)$ given as in the definition of ADD. Then,

$$\begin{aligned} \phi_1^0(\mathcal{F}^1 + \mathcal{F}^2) &= (E^1 + E^2) - \left(c^1(N') + c^2(N')\right) \\ &= E^1 - c^1(N') + E^2 - c^2(N') \\ &= \phi_1^0(\mathcal{F}^1) + \phi_1^0(\mathcal{F}^2) \end{aligned}$$

and $\phi_i^0(\mathcal{F}^1 + \mathcal{F}^2) = 0 = \phi_2^0(\mathcal{F}^1) + \phi_2^0(\mathcal{F}^2)$ for each $i \in N'$. □

For $\theta > 0$, ϕ^θ does not satisfy ADD, as the next example shows.

Example 4.3. *Let $\mathcal{F} = (N, E, c, \mathcal{A}, a) \in \mathfrak{F}$ given by $N = \{1, 2, 3\}$, $E = 4$, $c_2 = c_3 = 0$, $\mathcal{A} = \{\{2\}, \{3\}, \{2, 3\}\}$, $a_{\{2\}} = 2$, $a_{\{3\}} = 3$, and $a_{\{2,3\}} = 4$. Let $\mathcal{F}^1 = (N, E^1, c, \mathcal{A}, a^1) \in \mathfrak{F}$ given by $E^1 = 2$, $a^1_{\{2\}} = 1$, and $a^1_{\{3\}} = a^1_{\{2,3\}} = 2$. Let $\mathcal{F}^2 = (N, E^2, c, \mathcal{A}, a^2) \in \mathfrak{F}$ given by $E^2 = 2$, $a^2_{\{2\}} = a^2_{\{3\}} = 1$, and $a^2_{\{2,3\}} = 2$. Clearly, $\mathcal{F} = \mathcal{F}^1 + \mathcal{F}^2$. Moreover,*

$$\phi^\theta(\mathcal{F}) = \left(4 - 4\theta, \frac{8}{5}\theta, \frac{12}{5}\theta\right).$$

On the other hand,

$$\phi^\theta(\mathcal{F}^1) = \left(2 - 2\theta, \frac{2}{3}\theta, \frac{4}{3}\theta\right)$$

and

$$\phi^\theta(\mathcal{F}^2) = (2-2\theta, \theta, \theta)\,.$$

Hence, $\phi^\theta(\mathcal{F}) = \phi^\theta(\mathcal{F}^1 + \mathcal{F}^1) \neq \phi^\theta(\mathcal{F}^1) + \phi^\theta(\mathcal{F}^2)$ *whenever* $\theta > 0$.

Theorem 4.2. *A rule* ϕ *satisfies CS, IMMA, SYM, and ADD if and only if* $\phi = \phi^0$.

Proof. We know (Theorem 4.1 and Proposition 4.1) that ϕ^0 satisfies the four properties. Let ϕ be a rule satisfying CS, IMMA, SYM, and ADD. Fix $\mathcal{F} = (N, E, c, \mathcal{A}, a) \in \mathfrak{F}$. Under CS and TI, we can assume $E = E'$ and $c_i = 0$ for all $i \in N'$. We can assume $E > 0$. Otherwise, the result is trivial under CS. Partition A' given in Assumption 1 is a pairwise disjoint consecutive and mergeable coalitions set. Fix a community in N'. We can assume w.l.o.g. it is community 2. Let $\mathcal{F}^1 = \mathcal{F}^{N',2}$. Under IMMA,

$$\phi_1(\mathcal{F}^1) = \phi_1(\mathcal{F}). \tag{10}$$

Let $\mathcal{F}^2 = (N^2, E^2, c^2, \mathcal{A}^2, a^2)$ defined as $N^2 = \{1,2,3\}$, $E^2 = E$, $c^2_2 = c^2_3 = 0$, $\mathcal{A}^2 = \{\{2\},\{3\},\{2,3\}\}$, $a^2_{\{2\}} = E$, $a^2_{\{3\}} = \frac{E}{2}$, and $a^2_{\{2,3\}} = E$. It is straightforward to check that $\mathcal{F}^1 = (\mathcal{F}^2)^{\{2,3\},2}$. Let $\mathcal{F}^3 = (N^3, E^3, c^3, \mathcal{A}^3, a^3) \in \mathfrak{F}$ and $\mathcal{F}^4 = (N^4, E^4, c^4, \mathcal{A}^4, a^4) \in \mathfrak{F}$ defined as $N^3 = N^4 = \{1,2,3\}$, $E^3 = E^4 = \frac{E}{2}$, $c^3_2 = c^3_3 = c^4_2 = c^4_3 = 0$, $\mathcal{A}^3 = \mathcal{A}^4 = \{\{2\},\{3\},\{2,3\}\}$, $a^3_{\{2\}} = a^3_{\{2,3\}} = a^4_{\{2\}} = a^4_{\{3\}} = a^4_{\{2,3\}} = \frac{E}{2}$, and $a^3_{\{3\}} = 0$. It is straightforward to check that $\mathcal{F}^2 = \mathcal{F}^3 + \mathcal{F}^4$. Moreover, both $\mathcal{F}^3$ and $\mathcal{F}^4$ satisfy the conditions of Lemma 4.2. Hence,

$$\begin{aligned}
\phi_i(\mathcal{F}^3) &= \frac{E}{2} - \phi_1(\mathcal{F}^3) = \frac{E}{2} - \frac{\phi_1(\mathcal{F}^1)}{2} \\
\phi_i(\mathcal{F}^4) &= \frac{E}{4} - \frac{\phi_1(\mathcal{F}^3)}{2} = \frac{E}{4} - \frac{\phi_1(\mathcal{F}^1)}{4}.
\end{aligned}$$

and thus

$$\phi_i(\mathcal{F}^3 + \mathcal{F}^4) = \phi_i(\mathcal{F}^3) + \phi_i(\mathcal{F}^4) = \frac{3}{4}E - \frac{3}{4}\phi_1(\mathcal{F}^1).$$

On the other hand

$$\phi_i(\mathcal{F}^3 + \mathcal{F}^4) = \phi_i(\mathcal{F}^2) = \frac{2}{3}(E - \phi_1(\mathcal{F}^2)).$$

Hence,

$$\frac{3}{4}E - \frac{3}{4}\phi_1(\mathcal{F}^1) = \frac{2}{3}(E - \phi_1(\mathcal{F}^2)).$$

Which has, as a unique solution,

$$\phi_1(\mathcal{F}^2) = E.$$

Hence,

$$\phi_1(\mathcal{F}) = \phi_1(\mathcal{F}^1) = \phi_1(\mathcal{F}^2) = E = \phi_1^0(\mathcal{F}).$$

Under CS, we deduce $\phi_i(\mathcal{F}) = 0 = \phi_i^0(\mathcal{F})$ for all $i \in N'$. Hence, $\phi(\mathcal{F}) = \phi^0(\mathcal{F})$. □

Properties in Theorem 4.2 are independent:

- Rule $\phi^{(alt)}$ defined, for each $\mathcal{F} \in \mathfrak{F}$ with player set N, as $\phi_1^{(alt)}(\mathcal{F}) = E - E'$ and $\phi_i^{(alt)}(\mathcal{F}) = 0$ for all $i \in N'$, satisfies all the properties but CS.
- Rule $\overline{\phi}$ defined as in (8) satisfies all the properties but IMMA.
- Rule $\phi^{(last)}$ defined as in (9) satisfies all the properties but SYM.
- Rule ϕ^θ with $\theta \in (0, 1]$ satisfies all properties but ADD.

5 Conclusion

In this paper, we assess the potential of game theory through cooperative games applied to the freight transportation of minerals. In particular, this study shows a framework for allocating compensations to communities based on cooperative game theory, considering the principle of stability. We assess that benefits/costs would result from considering the bargaining power of communities to avoid the use of their land. It is a methodological contribution that analyses road use management for freight transportation.

In conclusion, we show that it is possible to establish compensation rules that assure stability for local communities. Specifically, we define several reasonable properties in the context of freight transport problems and propose a parametric family of solutions that satisfy each property. One of the properties is core selection, which allows us to assign stable solutions.

Appendix: Proof of Lemma 4.2

Proof. Let $\mathcal{F} = (N, E, c, \mathcal{A}, a)$ with $c_i = 0$ for all $i \in N'$. We can assume w.l.o.g. $N = \{1, ..., n\}$. Assume ϕ satisfies CS, IMMA, and SYM. We need to prove (7).

Under CS, $\sum_{i\in N'} \phi_i(\mathcal{F}) = E - \phi_1(\mathcal{F})$. Fix $i^* \in N'$. Let $\mathfrak{A} = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m)$ be the unique level structure associated to $\mathcal{F}^*$ as given in Lemma 4.1. We proceed by induction on $|\mathcal{A}|$, the number of alternatives. If $\mathcal{A} = \{\{i^*\}\}$, then Assumption 1 implies $N' = \{i^*\}$ and hence $m = 1$ with $\mathfrak{A} = (\mathcal{A}_1) = (\{N'\})$, from where (7) reduces to $\phi_{i^*}(\mathcal{F}) = E - \phi_1(\mathcal{F})$, which holds from CS. Assume (7) holds for problems with less than $|\mathcal{A}|$ alternatives and assume $|\mathcal{A}| > 1$. From Lemma (4.1), $m \geq 2$. Under condition 1 in Lemma 4.1, there is at least some coalition $A' \in \mathcal{A}_2$ with $|A'| > 1$, because otherwise $\mathcal{A}_1 = \mathcal{A}_2$. Coalitions in A' are mergeable. If $i^* \notin A'$, under IMMA, then i^* receives the same as in the problem where coalitions in A' merge. Since such merging reduces the size of $\mathcal{A}$, but it does not affect the structure nor alternatives of those coalitions community i^* belongs to, the result (7) arises from the induction hypothesis. We can then assume that $\mathcal{A}_2$ has all the coalitions singletons but $A_2^{i^*}$, i.e., there exist α, β such that $2 \leq \alpha \leq i^* \leq \beta \leq n$ and

$$\mathcal{A}_2 = \Big\{\{2\}, \ldots, \{\alpha - 1\}, A_2^{i^*}, \{\beta + 1\}, \ldots, \{n\}\Big\}.$$

Notice that $\alpha = 2$ covers the case where $A_2^{i^*}$ is at the beginning of the route (i.e., $2 \in A_2^{i^*}$), so that $\mathcal{A}_2 = \big\{A_2^{i^*}, \{\beta+1\}, \ldots, \{n\}\big\}$. Analogously, $\beta = n$ covers the case where $A_2^{i^*}$ is at the end of the route (i.e., $n \in A_2^{i^*}$), so that $\mathcal{A}_2 = \big\{\{2\}, \ldots, A_2^{i^*}\big\}$. When $2, n \in A_2^{i^*}$, then $\mathcal{A}_2 = \big\{A_2^{i^*}\big\}$, which is equivalent to $m = 2$ and it is also a possibility.

Communities in $A_2^{i^*}$ are mergeable. Hence, under IMMA, we can merge them and work with the reduced problem

$$\mathcal{F}^1 = \mathcal{F}^{A_2^{i^*}, i^*} = (N^1, E, c^1, \mathcal{A}^1, a^1).$$

In particular, $N^1 = \{1, 2, \ldots, \alpha - 1, i^*, \beta + 1, \ldots, n\}$ and $a^1_{\{i^*\}} = \sum_{i \in A_N^{i^*}} a_{\{i\}}$. Since $|A_2^{i^*}| > 1$, we conclude that $|\mathcal{A}^1| < |\mathcal{A}|$. Let $\mathfrak{A}^1 = (\mathcal{A}_1^1, \mathcal{A}_2^1, \ldots, \mathcal{A}_{m^1}^1)$ be the unique level structure associated to $(\mathcal{F}^1)^*$ as given in Lemma 4.1. It is straightforward to check that $m^1 = m - 1$ and each $\mathcal{A}_l^1$ coincides with $\mathcal{A}_{l+1}$ after replacing coalition $A_2^{i^*}$ with i^*, i.e.,

$$\mathcal{A}_l^1 = \Big\{A \in \mathcal{A}_{l+1} : A_2^{i^*} \nsubseteq A\Big\} \cup \Big\{\Big(A \setminus A_2^{i^*}\Big) \cup \{i^*\} : A_2^{i^*} \subseteq A \in \mathcal{A}_{l+1}\Big\}$$

for each $l = 1, \ldots, m-1$. Let s^* be defined as

$$s^* = \sum_{i \in A_2^{i^*}} \phi_i\left(\mathcal{F}\right).$$

Under IMMA and the induction hypothesis:

$$\begin{aligned} s^* = \phi_{i^*}\left(\mathcal{F}^1\right) = \left(E - \phi_1\left(\mathcal{F}^1\right)\right) \prod_{l=1}^{m^1-1} \frac{a^1_{A_l^{1i^*}}}{\sum_{A \in \mathcal{A}_l^1 : A \subseteq A_{l+1}^{1i^*}} (a_A^1)} \\ = (E - \phi_1(\mathcal{F})) \prod_{l=1}^{m-2} \frac{a_{A_{l+1}^{i^*}}}{\sum_{A \in \mathcal{A}_{l+1} : A \subseteq A_{l+2}^{i^*}} (a_A)} \\ = (E - \phi_1(\mathcal{F})) \prod_{l=2}^{m-1} \frac{a_{A_l^{i^*}}}{\sum_{A \in \mathcal{A}_l : A \subseteq A_{l+1}^{i^*}} (a_A)}. \end{aligned}$$

Hence, it is enough to prove

$$\phi_{i^*}(\mathcal{F}) = \frac{a_{A_1^{i^*}}}{\sum_{A \in \mathcal{A}_1 : A \subseteq A_2^{i^*}} (a_A)} s^*$$

or, equivalently,

$$\phi_{i^*}(\mathcal{F}) = \frac{a_{\{i^*\}}}{\sum_{i \in A_2^{i^*}} (a_{\{i\}})} s^*. \tag{11}$$

Recall that $A_2^{i^*} = \{\alpha, \ldots, i^*, \ldots, \beta\}$ with $\alpha \leq i^* \leq \beta$. In particular, $\alpha = i^*$ covers the case where i^* is the first community in $A_2^{i^*}$ on the route, so that $A_2^{i^*} = \{i^*, \ldots, \beta\}$. Analogously, $\beta = i^*$ covers the case where i^* is the last community in $A_2^{i^*}$ on the route, so that $A_2^{i^*} = \{\alpha, \ldots, i^*\}$. Communities $\{\alpha, \ldots, i^* - 1\}$, if any, are mergeable, and so are communities $\{i^* + 1, \ldots, \beta\}$. Case $\alpha = \beta = i^*$ is not possible because $A_2^{i^*}$ has more than one element. Under IMMA, we can work with the reduced problem

$$\mathcal{F}^2 = \left(\mathcal{F}^{\{\alpha, \ldots, i^*-1\}, \alpha}\right)^{\{i^*+1, \ldots, \beta\}, \beta} = (N^2, E, c^2, \mathcal{A}^2, a^2)$$

obtained from $\mathcal{F}$ by merging these coalitions into α and β, respectively. Let $\epsilon_1, \epsilon_2, \ldots$ be a sequence of strictly positive scalars defined as

$$\epsilon_t = \frac{\sum_{i=\alpha}^{\beta} a_{\{i\}}}{t}$$

for each t, so that the sequence converges to zero as t increases. For each $x \in \mathbb{R}$, let $\lceil x \rceil$ denote the lowest integer higher or equal than x. In particular, when $x \in \mathbb{N}$, we have $x = \lceil x \rceil$. Under IMMA, we can work with a new problem $\mathcal{F}^3$ obtained from $\mathcal{F}^2$ by splitting community α into

$$n^\alpha = \left\lceil \frac{\sum_{i=\alpha}^{i^*-1} a_{\{i\}}}{\epsilon_t} \right\rceil$$

communities, the last one (if any, i.e., if $\alpha < i^*$), also denoted as α, with alternative cost $a^3_{\{\alpha\}} \in [0, \epsilon_t]$ bounded above by ϵ_t, and the rest (if any) with alternative cost ϵ_t each. Analogously, we split community β into

$$n^\beta = \left\lceil \frac{\sum_{i=i^*+1}^{\beta} a_{\{i\}}}{\epsilon_t} \right\rceil$$

communities, the first one (if any, i.e., if $i^* < \beta$), denoted also as β, with alternative cost $a^3_{\{\beta\}} \in [0, \epsilon_t]$ bounded above by ϵ_t, and the rest (if any) with alternative cost ϵ_t each. We complete the creation of $\mathcal{F}^3$ by splitting community i^* as follows:

Case I If $\alpha < i^* < \beta$, we split community i^* into $n^{i^*} = t - n^\alpha - n^\beta + 2$ communities: The first one, denoted as α', with alternative cost $a^3_{\{\alpha'\}} = \epsilon_t - a^3_{\{\alpha\}}$; the last one, denoted as β', with alternative cost $a^3_{\{\beta'\}} = \epsilon_t - a^3_{\{\beta\}}$; and the middle ones, with alternative cost ϵ_t each.

Case II If $\alpha \leq i^* < \beta$, we split community i^* into $n^{i^*} = t - n^\beta + 1$ communities: The last one, denoted as β', with alternative cost $a^3_{\{\beta'\}} = \epsilon_t - a^3_{\{\beta\}}$; and the other ones, with alternative cost ϵ_t each.

Case III If $\alpha < i^* \leq \beta$, we split community i^* into $n^{i^*} = t - n^\alpha + 1$ communities: The first one, denoted as α', with alternative cost $a^3_{\{\alpha'\}} = \epsilon_t - a^3_{\{\alpha\}}$; and the other ones, with alternative cost ϵ_t each.

When $\alpha < i^*$, communities α and α' are mergeable. When $i^* < \beta$, communities β and β' are mergeable. Hence, under IMMA, we can work with the reduced problem

$$\mathcal{F}^4 = \begin{cases} \left(\left(\mathcal{F}^3\right)^{\{\alpha,\alpha'\},\alpha}\right)^{\{\beta,\beta'\},\beta} & \text{if } \alpha < i^* < \beta \text{ \textit{(Case I)}} \\ \left(\mathcal{F}^3\right)^{\{\beta,\beta'\},\beta} & \text{if } \alpha \leq i^* < \beta \text{ \textit{(Case II)}} \\ \left(\mathcal{F}^3\right)^{\{\alpha,\alpha'\},\alpha} & \text{if } \alpha < i^* \leq \beta \text{ \textit{(Case III)}} \end{cases}$$

obtained from $\mathcal{F}^3$ by merging these communities. It is straightforward to check that $\mathcal{F}^4$ contains t symmetric communities, each of them with alternative cost ϵ_t and with aggregate value s^*. Under SYM, each of these communities receives

$$\phi_i(\mathcal{F}^4) = \frac{s^*}{t}$$

which, under IMMA and CS, implies that communities α' and β', when exist, receive in $\mathcal{F}^3$ something between 0 and $\frac{s^*}{t}$ each:

$$\phi_{\alpha'}(\mathcal{F}^3), \phi_{\beta'}(\mathcal{F}^3) \in \left[0, \frac{s^*}{t}\right]$$

which converge to 0 as t increases. Let x^t be defined as $x^t = \phi_{\alpha'}(\mathcal{F}^3)$ when α' exists, and $x^t = 0$ otherwise. Analogously, let y^t be defined as $y^t = \phi_{\beta'}(\mathcal{F}^3)$ when β' exists, and $y^t = 0$ otherwise. Under IMMA, community i^* receives in $\mathcal{F}^2$

$$\begin{aligned}
\phi_{i^*}(\mathcal{F}^2) &= (t - n^\alpha - n^\beta)\frac{s^*}{t} + x^t + y^t \\
&= \left(t - \left\lceil \frac{\sum_{i=\alpha}^{i^*-1} a_{\{i\}}}{\epsilon_t} \right\rceil - \left\lceil \frac{\sum_{i=i^*+1}^{\beta} a_{\{i\}}}{\epsilon_t} \right\rceil \right) \frac{s^*}{t} + x^t + y^t \\
&= \left(t - \left\lceil \frac{\sum_{i=\alpha}^{i^*-1} a_{\{i\}}}{\sum_{i=\alpha}^{\beta} a_{\{i\}}} t \right\rceil - \left\lceil \frac{\sum_{i=i^*+1}^{\beta} a_{\{i\}}}{\sum_{i=\alpha}^{\beta} a_{\{i\}}} t \right\rceil \right) \frac{s^*}{t} + x^t + y^t.
\end{aligned}$$

Since $\phi(\mathcal{F}^2)$ is independent of t, we deduce

$$\phi_{i^*}(\mathcal{F}^2) = \left(1 - \frac{\sum_{i=\alpha}^{i^*-1} a_{\{i\}}}{\sum_{i=\alpha}^{\beta} a_{\{i\}}} - \frac{\sum_{i=i^*+1}^{\beta} a_{\{i\}}}{\sum_{i=\alpha}^{\beta} a_{\{i\}}}\right) s^* = \frac{a_{\{i^*\}}}{\sum_{i=\alpha}^{\beta} a_{\{i\}}} s^*.$$

Under IMMA and CS, $\phi_{i^*}(\mathcal{F}^2) = \phi_{i^*}(\mathcal{F})$, and hence we get (11). □

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