Numerical Simulation of an Endogenously Growing Economy and Its Balanced Growth Path

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Abstract

In this paper, I simulate how an economy grows endogenously and reaches a balanced growth path supposing that households behave under the MDC (maximum degree of comfortability)-based procedure, where MDC indicates the state at which a household feels most comfortable with its combination of income and assets. Although it is not easy to numerically simulate the path to a steady state in dynamic economic growth models in which households behave generating rational expectations, it is easy if households are supposed to behave under the MDC-based procedure to reach a steady state. The simulation results indicate that an economy can indeed grow endogenously as predicted theoretically, although some small scale effects exist. If uncompensated knowledge spillovers are restrained, however, large scale effects are generated. A lower degree of risk aversion increases the growth rate. In addition, economies converge if productivities are identical, but they diverge if they are not.

JEL Classification: E17, E60, O11, O30, O40

Keywords: Convergence; Endogenous growth; Scale effects; Simulation; Uncompensated knowledge spillovers

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1 INTRODUCTION

In this paper, I numerically simulate the paths of economies that endogenously grow and reach a balanced growth path by using the novel simulation method presented by Harashima (2022c, 2023a, 2023b). In dynamic economic growth models in which households behave generating rational expectations, it is not easy to numerically simulate the path to a steady state and a balanced growth path because there is no closed-form solution to these models. However, Harashima (2022c) presented a completely different way to simulate it on the basis of the concept of the maximum degree of comfortability (MDC), where MDC indicates the state at which a household feels most comfortable with its combination of incomes and assets.

Usually, it is assumed that households behave by generating rational expectations to reach a steady state, but Harashima (2018[1]) showed an alternative procedure for households to reach a steady state. With this procedure, households maintain their capital-wage ratio (CWR) at the MDC. He showed that the behavior of households based on rational expectations (i.e., the behavior under the RTP [rate of time preference]-based procedure) is equivalent to that under the MDC-based procedure (Harashima 2018, 2021, 2022a[2]). Unlike the case of the RTP-based procedure, the path to a steady state will easily be simulated if we suppose that households behave under the MDC-based procedure because households are not required to do something equivalent to computing complex models.

Indeed, Harashima (2022c) numerically simulated the path to a steady state under the MDC-based procedure and showed that households can reach a stabilized (steady) state without generating any rational expectations, as predicted theoretically (Harashima, 2010[3], 2012a[4], 2014), and a government can achieve a stabilized (steady) state by appropriately intervening, although heterogeneous households cannot necessarily reach their intrinsic CWRs at MDC (this state is called "approximate sustainable heterogeneity"). Furthermore, Harashima (2023a) simulated the effect of economic rents obtained heterogeneously among households, and Harashima (2023b) examined the mechanism underlying why economic inequality can increase in democratic countries using the same simulation method. In these simulations, a household was set to increase or decrease its consumption according to simple formulae that are supposed to well capture and represent a household's behaviors under the MDC-based procedure.

However, in these simulations, technologies are exogenously given and therefore

[1] Harashima (2018) is also available in Japanese as Harashima (2019a).
[2] Harashima (2022a) is also available in Japanese as Harashima (2022b).
[3] Harashima (2010) is also available in Japanese as Harashima (2017).
[4] Harashima (2012a) is also available in Japanese as Harashima (2020a).

paths of endogenously growing economies are not simulated. The purpose of this paper is to numerically simulate and examine how an economy endogenously grows and reaches a balanced growth path by extending the method of numerical simulation shown in Harashima (2022c, 2023a, 2023b). To do so, a mechanism of endogenous growth, on which the simulations are based, has to be specified first. Many kinds of mechanisms of endogenous growth have been presented. In this paper, I use the asymptotically non-scale endogenous growth model presented in Harashima (2013) because it avoids the familiar and problematic "scale effects" (i.e., as the population increases, the growth rate increases) (see, e.g., Romer, 1990; Grossman and Helpman, 1991; Aghion and Howitt, 1992, 1998; Jones, 1995b; Kortum, 1997; Segerstrom, 1998; Eicher and Turnovsky, 1999; Young, 1998; Peretto, 1998; Dinopoulos and Thompson, 1998; Peretto and Smulders, 2002). In simulations based on the extended method, firms behave to always keep the marginal products of capital and technology equal, following the mechanism shown in Harashima (2013). This behavior makes an economy grow endogenously and reach a balanced growth path in simulations.

The results of the simulations indicate that, as predicted theoretically in Harashima (2013), an economy can indeed grow endogenously and reach a balanced growth path if firms behave keeping the marginal products of capital and technology equal. Scale effects exist but they are small; even if the population is very small, the growth path is almost the same as that when the population is very large. If uncompensated knowledge spillovers are restrained as the population increases, however, large scale effects emerge. Furthermore, although the degree of risk aversion (DRA) is not explicitly included in the simulation method, the effect of its surrogate variable (i.e., the adjustment speed of consumption) in simulations suggests that a smaller DRA increases the growth rate. Finally, I simulate whether the growth paths of different economies eventually converge. The results of the simulations indicate that, if productivities are identical, economies converge regardless of their past economic paths, but if productivities are heterogeneous, economies diverge even if they are initially identical.

2 ENDOGENOUS GROWTH

2.1 Non-scale endogenous growth

Scale effects have been a central issue in the study of endogenous growth. Early endogenous growth models (e.g. Romer, 1986, 1987; Lucas, 1988) commonly included scale effects. However, the existence of scale effects in present-day economies is not supported by empirical evidence (Jones, 1995a). The source of scale effects lies in the assumption of a linear relation between capital and technology. Many kinds of solutions to this problem have been presented, but most of them generate other problems in the attempt to solve this problem (Romer, 1990; Grossman and Helpman, 1991; Aghion and Howitt, 1992, 1998; Jones, 1995b; Kortum, 1997; Segerstrom, 1998; Eicher and Turnovsky, 1999; Young, 1998; Peretto, 1998; Dinopoulos and Thompson, 1998; Peretto and Smulders, 2002).

Harashima (2013[5]) showed a different kind of solution and presented an asymptotically non-scale endogenous growth model that does not have the problem of scale effects. A key mechanism in this model is substitution between investments in capital and technology. The simulation method used in this paper adopts this mechanism as the driving force of endogenous growth.

2.2 *Essence of endogenous growth*

The asymptotically non-scale endogenous growth model presented in Harashima (2013) is summarized in Appendix 4; its essence is briefly explained in this section.

2.2.1 Production of technologies

In the model, Y_t is output (production) and is the sum of consumption C_t, the increase in capital K_t, and the increase in technology A_t in period t such that

$$Y_t = C_t + \dot{K}_t + v\dot{A}_t \quad .$$

Thus,

$$\dot{k}_t = y_t - c_t - \frac{v\dot{A}_t}{L_t} - n_t k_t \quad ,$$

where $y_t = \frac{Y_t}{L_t}$, $c_t = \frac{C_t}{L_t}$, and $k_t = \frac{K_t}{L_t}$. L_t is labor input, $n_t \left(= \frac{\dot{L}_t}{L_t} \right)$ is the population growth rate in period t, $v(>0)$ is a constant, and a unit of K_t and v^{-1} of a unit of A_t are equivalent; that is, they are produced using the same quantities of inputs (capital, labor, and technology). This means that technologies are produced with capital, labor, and technology in the same way as consumer goods and services and capital. The production function is assumed to be $Y_t = K_t^{1-\alpha}(A_t L_t)^\alpha$; thus,

$$y_t = A_t^\alpha k_t^{1-\alpha} \quad .$$

[5] Harashima (2013) is also available in Japanese as Harashima (2019b).

It is assumed for simplicity that the population growth rate (n_t) is zero.

2.2.2 Substitution between investments in capital and technology

For any period,

$$m = \frac{M_t}{L_t} \ ,$$

where M_t is the number of firms (which are assumed to be identical) and $m(>0)$ is a constant. For any period,

$$\frac{\partial Y_t}{\partial K_t} = \frac{\varpi}{M_t^{1-\rho}} \frac{\partial Y_t}{\partial (v A_t)} \ ; \tag{1}$$

thus,

$$\frac{\partial y_t}{\partial k_t} = \frac{\varpi L_t^{\rho}}{m^{1-\rho} v} \frac{\partial y_t}{\partial A_t} \tag{2}$$

is always kept, where $\varpi(>1)$ and $\rho(0 \le \rho < 1)$ are constants. The parameter ρ describes the effect of uncompensated knowledge spillovers, and the parameter ϖ indicates the effect of patent protection. For simplicity, the patent period is assumed to be indefinite, and no capital depreciation is assumed. Equations (1) and (2) indicate that the marginal products of capital and technology are always kept equal through arbitrage in markets.

Because

$$\frac{\varpi L_t^{\rho}}{m^{1-\rho} v} \frac{\partial y_t}{\partial A_t} = \frac{\partial y_t}{\partial k_t} \Leftrightarrow \frac{\varpi L_t^{\rho} \alpha}{m^{1-\rho} v} A_t^{\alpha-1} k_t^{1-\alpha} = (1-\alpha) A_t^{\alpha} k_t^{-\alpha} \ ,$$

then

$$A_t = \frac{\varpi L_t^{\rho} \alpha}{m^{1-\rho} v (1-\alpha)} k_t$$

by equation (2), which indicates that $\frac{A_t}{k_t}$ = constant for L_t^{ρ} = constant, and the model can therefore show balanced endogenous growth.

2.2.3 Uncompensated knowledge spillovers

Equations (1) and (2) also indicate that the investing firm cannot obtain all of the returns on its investment in technology because knowledge spills over to other firms without compensation and other firms possess complementary technology. If the number of firms increases and uncompensated knowledge spillovers increase, the compensated fraction in $\frac{\partial Y_t}{\partial A_t}$ that the investing firm can obtain becomes smaller, as do its returns on the investment in technology. The parameter ρ describes the magnitude of this effect.

Because of the non-rivalness of technology, all firms can equally benefit from uncompensated knowledge spillovers, regardless of the number of firms. Hence, it is quite likely that the probability that a firm can utilize a unit of new technology developed by each of the other firms without compensation will be kept constant even if the population and the number of firms increase. As a result, uncompensated knowledge spillovers will eventually increase to the point that they increase at the same rate as the increase in the number of firms. The investing firm's fraction of $\frac{\partial Y_t}{\partial A_t}$ that it can obtain will thereby be reduced at the same rate as the increase in the number of firms, which means that ρ will naturally decrease to zero as a result of firms' profit-seeking behavior.

Based on $\rho = 0$,

$$\frac{\partial Y_t}{\partial K_t} = \frac{\varpi}{M_t} \frac{\partial Y_t}{\partial (vA_t)}$$

by equations (1) and (2); thus,

$$\frac{\partial y_t}{\partial k_t} = \frac{\varpi}{mv} \frac{\partial y_t}{\partial A_t} \tag{3}$$

is always maintained.

3 SIMULATION METHOD

Simulations in this paper are undertaken on the basis of the sustainable heterogeneity (SH) concepts presented in Harashima (2010, 2012a, 2014) and the MDC-based procedure developed in Harashima (2018, 2021, 2022a). These concepts are briefly summarized in Appendixes 1 and 2. The method of simulations is basically the same as that used in Harashima (2022c, 2023a, 2023b), which is explained in Appendix 3, but it is extended to simulate endogenous growth.

3.1 Basic simulation assumptions

No technological progress and capital depreciation are assumed, and all values are expressed in real and per capita terms. It is assumed that there are H economies in a country, the number of households in each of economy is identical, and households within each economy are identical. The production function of Economy i ($1 \leq i \leq H$) is

$$y_{i,t} = \omega_i A_t^{\alpha} k_{i,t}^{1-\alpha} \ , \tag{4}$$

where $y_{i,t}$ and $k_{i,t}$ are the production and capital of a household in Economy i in period t, respectively; ω_i is the productivity of a household in Economy i; A_t is technology in period t; and α ($0 < \alpha < 1$) is a constant and indicates the labor share. All variables are expressed in per capita terms. In simulations, I set $\alpha = 0.65$, $A_t = 1$, and $\omega_i = 1$ for any t and i. The initial capital a household owns is set at 1 for any household.

By equation (4), the production of a household in Economy i in period t ($y_{i,t}$) is calculated, for any i, by

$$y_{i,t} = k_{i,t}^{1-\alpha} \ .$$

The amount of capital used (not owned) by each household (i.e., $k_{i,t}$) is kept identical among households although the amount of capital owned (not used) by each household can be heterogeneous. For any i,

$$k_{i,t} = \frac{\sum_{i=1}^{H} \check{k}_{i,t}}{H} \ ,$$

where $\check{k}_{i,t}$ is the amount of capital a household in Economy i owns (not uses).

The capital income of a household in Economy i in period t ($x_{K,t}$) is calculated by

$$x_{K,i,t} = r_t \check{k}_{i,t} \ ,$$

where r_t is the real interest rate in period t and

$$r_t = \frac{\partial k_{i,t}}{\partial y_{i,t}} \ .$$

The labor income of a household in Economy i in period t ($x_{L,i,t}$) is calculated by extracting its capital income from its production such that

$$x_{L,i,t} = y_{i,t} - r_t k_{i,t} = y_{i,t} - r_t \frac{\sum_{i=1}^{H} \check{k}_{i,t}}{H} .$$

Household savings in Economy i in period t ($s_{i,t}$) are calculated by

$$s_{i,t} = x_{L,i,t} + x_{K,i,t} - c_{i,t} \;\; ,$$

where $c_{i,t}$ is the consumption of a household in Economy i in period t. In period t + 1, these savings ($s_{i,t}$) are added to the capital the household owns, and therefore,

$$\check{k}_{i,t+1} = \check{k}_{i,t} + s_{i,t} .$$

The following simple consumption formula is used.

Consumption formula 1: The consumption of a household in Economy i in period t is

$$c_{i,t} = \left(x_{L,i,t} + x_{K,i,t}\right)\left(\frac{\Gamma(\tilde{s}_i)}{\Gamma_{i,t}}\right)^{\gamma} \;\; ,$$

and equivalently

$$c_{i,t} = \left(x_{L,i,t} + x_{K,i,t}\right)\left(\frac{\theta_i}{\Gamma_{i,t}\dfrac{1-\alpha}{\alpha}}\right)^{\gamma} \;\; ,$$

where $\Gamma_{i,t}$ is the capital-wage ratio (CWR) of a household in Economy i in period t, $\Gamma(\tilde{s}_i)$ is $\Gamma_{i,t}$ of a household in Economy i in period t when the household is at its MDC, and γ is a parameter. In this paper, I set the value of γ to be 0.5. It is assumed that the intrinsic $\Gamma(\tilde{s}_i)$ (i.e., CWR at MDC) of a household is identical across households and economies, and I set this common $\Gamma(\tilde{s}_i)$ to be $0.04 \times 0.65/(1-0.65) = 0.0743$, which corresponds to an RTP of 0.04.

In a heterogeneous population, Consumption formula 1 should be modified to Consumption formula 2. Let $\Gamma_{R,i,t}$ be the adjusted value of $\Gamma_{i,t}$ of a household in Economy i in period t in a heterogeneous population, and $\Gamma(S_t)$ be the CWR of the country (i.e., the aggregate CWR).

Consumption formula 2: In a heterogeneous population, the consumption of a household in Economy i in period t is

$$c_{i,t} = \left(x_{L,i,t} + x_{K,i,t}\right)\left(\frac{\Gamma(\tilde{s}_i)}{\Gamma_{R,i,t}}\right)^{\gamma}$$
$$= \left(x_{L,i,t} + x_{K,i,t}\right)\left(\frac{\Gamma(\tilde{s}_i)}{r_t \dfrac{\alpha}{1-\alpha}}\right)^{\gamma} = \left(x_{L,i,t} + x_{K,i,t}\right)\left(\frac{\Gamma(\tilde{s}_i)\dfrac{1-\alpha}{\alpha}}{r_t}\right)^{\gamma} ,$$

and equivalently,

$$c_{i,t} = \left(x_{L,i,t} + x_{K,i,t}\right)\left(\frac{\theta_i}{r_t}\right)^{\gamma} .$$

Let κ_i be the $\check{k}_{i,t}$ that a government aims for to force a household in Economy i to own capital at a stabilized (steady) state (i.e., κ_i is the target value set by the government). Under these conditions, the bang-bang (two-step) control of government transfers is set as follows.

Transfer rule: The amount of government transfers from a household in Economy i to a household in Economy i + 1 in period t is T_{low} if $\check{k}_{i,t}$ is lower than κ_i, and T_{high} if $\check{k}_{i,t}$ is higher than κ_i, where T_{low} and T_{high} are constant amounts of capital predetermined by the government, and if $i = H$, i + 1 is replaced with 1.

In the simulations, I set T_{low} to be -0.1 and T_{high} to be 0.5. The value of κ_i is varied in each simulation depending on what stabilized (steady) state the government is aiming to achieve.

3.2 *Extension to simulate endogenous growth*

In the basic simulation assumptions shown in Section 3.1, no technological progress is assumed. In this section, the simulation is extended to allow technology to progress endogenously.

3.2.1 Investments in capital and technology

Let the population (i.e., the number of households) in an economy be L and be constant. The total amount of investments in technology in the economy in period t is described as

vdA_t, and to finance these investments, $\frac{vdA_t}{L}$ is allocated from the savings of each household (s_t) indirectly through financial institutions in period t. Hence, for a household,

$$s_t = dk_t + \frac{vdA_t}{L} \ . \tag{5}$$

Equation (5) means that a household's savings are allocated for either investments in capital,

$$dk_t = \frac{dK_t}{L} \ ,$$

or those in technology,

$$\frac{vdA_t}{L} \ .$$

On the other hand, as shown in Section 2.2, firms behave so as to keep equation (3); therefore,

$$dA_t = \frac{\varpi\alpha}{mv(1-\alpha)} dk_t \tag{6}$$

holds (see Appendix 4). Here, by equations (5) and (6),

$$dk_t = \frac{L}{\left[\frac{\varpi\alpha}{m(1-\alpha)} + L\right]} s_t \ . \tag{7}$$

By equations (5) and (7),

$$dA_t = \frac{v^{-1}}{\frac{1}{L} + \frac{m(1-\alpha)}{\varpi\alpha}} s_t \ . \tag{8}$$

As indicated by equation (5), a household's savings are indirectly allocated for either investments in capital or technology. In simulations, a household's indirect investments in capital are calculated by equation (7) and those in technology are calculated by equation (8).

3.2.2 Multiple economies

Suppose that there are H economies that are identical and each of them has the same population L. These economies are open to each other, although labor is immobilized within each economy; therefore, technologies generated by an economy can be used by any other economy. Because equation (6) holds in any economy, for any i (=1, 2,…, H),

$$dA_t = \frac{\varpi\alpha}{mv(1-\alpha)} dk_{i,t} \ . \tag{9}$$

Let $A_{i,t}$ be technologies that are newly generated in Economy i in period t. Because $k_{i,t}$ is identical for any i, then $dA_{i,t}$ is also identical for any i, and therefore,

$$dA_t = \sum_{i=1}^{H} dA_{i,t} = HdA_{i,t} \ . \tag{10}$$

On the other hand, a household's savings are indirectly allocated for investments in capital or technology or for transactions with other economies. Hence, for any i,

$$s_{i,t} = dk_{i,t} + \tau_{i,t} + \frac{vdA_{i,t}}{L} = dk_{i,t} + \tau_{i,t} + \frac{v\frac{dA_t}{H}}{L} = d\check{k}_{i,t} + \frac{v\frac{dA_t}{H}}{L} \tag{11}$$

by equation (10), where $\tau_{i,t}$ is the current account balance of Economy i in transactions with the other economies and

$$\sum_{i=1}^{H} \tau_{i,t} = 0 \ . \tag{12}$$

By equations (11) and (12),

$$S_t = \sum_{i=1}^{H} s_{i,t} = \sum_{i=1}^{H} dk_{i,,t} + \frac{vdA_t}{L} = Hdk_{1,t} + \frac{vdA_t}{L} \ ;$$

thus,

$$S_t - Hdk_{1,t} = \frac{vdA_t}{L} \ . \tag{13}$$

Hence, by equations (9) and (13),

$$dA_t = \frac{v^{-1}}{\frac{1}{L} + H\frac{m(1-\alpha)}{\varpi\alpha}} S_t \ . \tag{14}$$

Here, by equation (11),

$$d\check{k}_{i,t} = s_{1,t} - \frac{v}{HL} dA_t \ . \tag{15}$$

By equations (14) and (15),

$$d\check{k}_{i,t} = s_{i,t} - \frac{1}{\frac{1}{L} + H\frac{m(1-\alpha)}{\varpi\alpha}} \frac{S_t}{HL} \ . \tag{16}$$

In simulations with multiple economies, the total increase in technologies in period t (dA_t) is calculated by equation (14), and a household's increase in capital owned in Economy i in period t ($d\check{k}_{i,t}$) is calculated by equation (16).

4 SIMULATION RESULTS

In simulations, I set $v = 1$, $\frac{\varpi\alpha}{m} = 0.3$, and the population at 100,000,000, unless otherwise noted.

4.1 Base case

I first simulate the growth path of an economy simply by applying the three parameter values given above. The simulated paths of capital owned and consumption are shown in Figure 1, and their growth rates are shown in Figure 2. In addition, the simulated path of technology and its growth rate are shown in Figures 3 and 4, respectively. These figures clearly indicate that the economy endogenously grows and reaches a balanced growth path. After reaching a balanced growth path, the economy grows at a constant rate of 2.08% every period.

Note that on this balanced growth path, the real interest rate is 0.0536, which is greater than the assumed RTP of 0.04. This occurs because the economy is not at a static steady state; rather, it is growing endogenously.

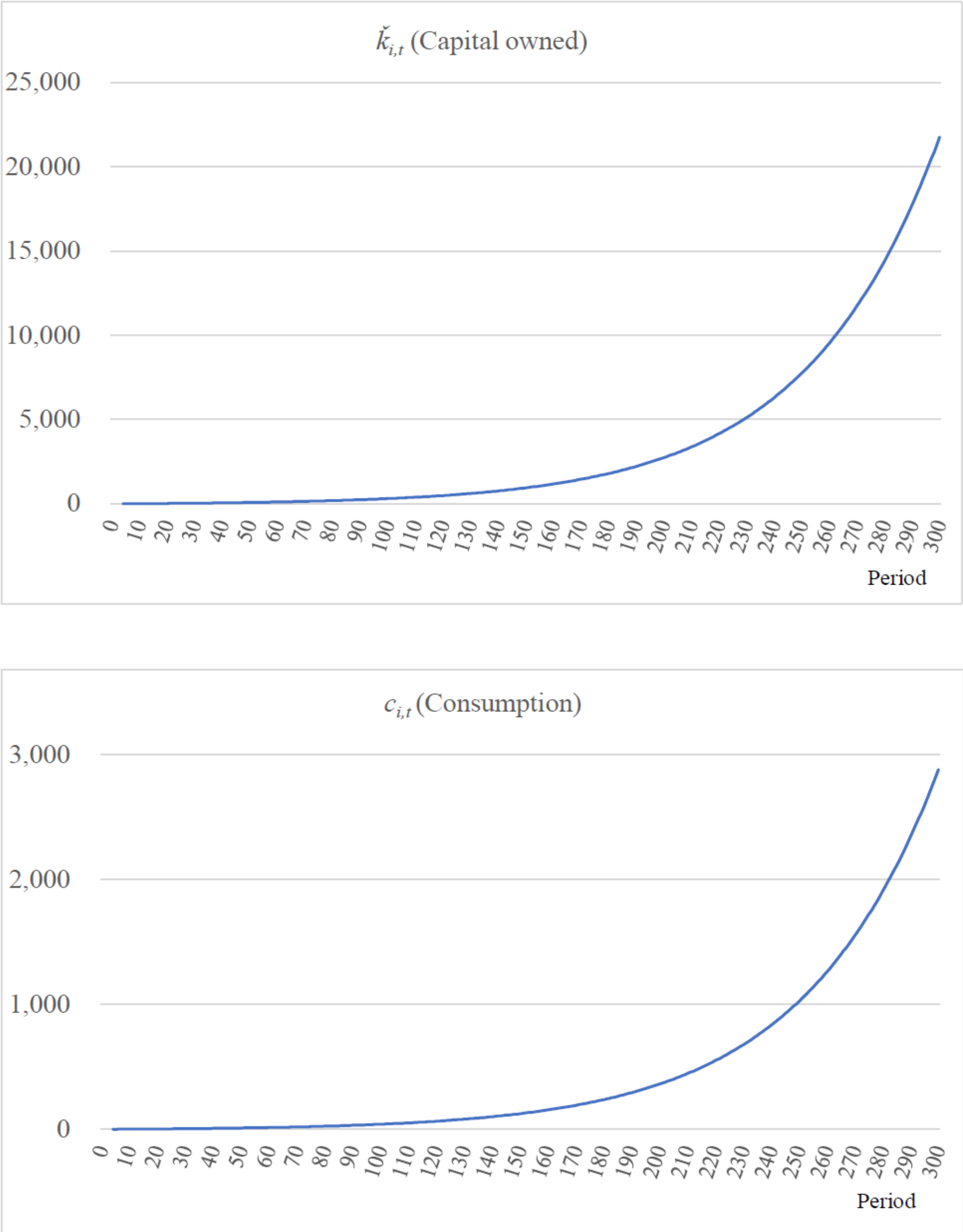
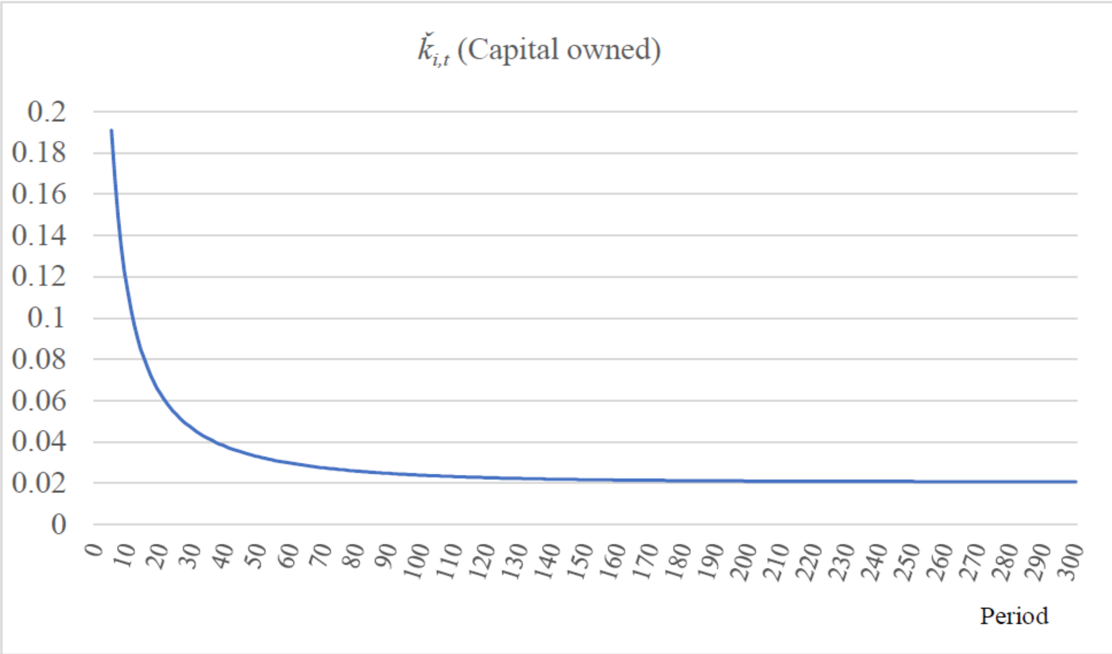


Figure 1: Simulation of the base case of endogenous growth: capital owned by each household ($\check{k}_{i,t}$) and consumption ($c_{i,t}$)



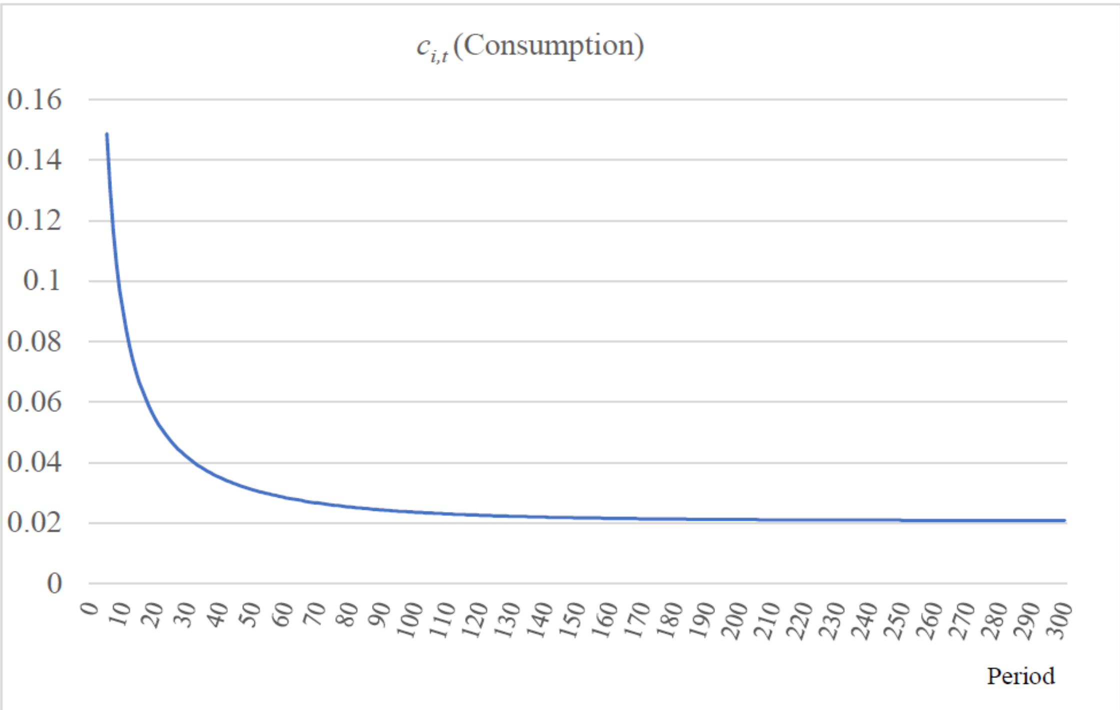


Figure 2: Simulation of the base case of endogenous growth: growth rates of capital owned by each household ($\check{k}_{i,t}$) and consumption ($c_{i,t}$)

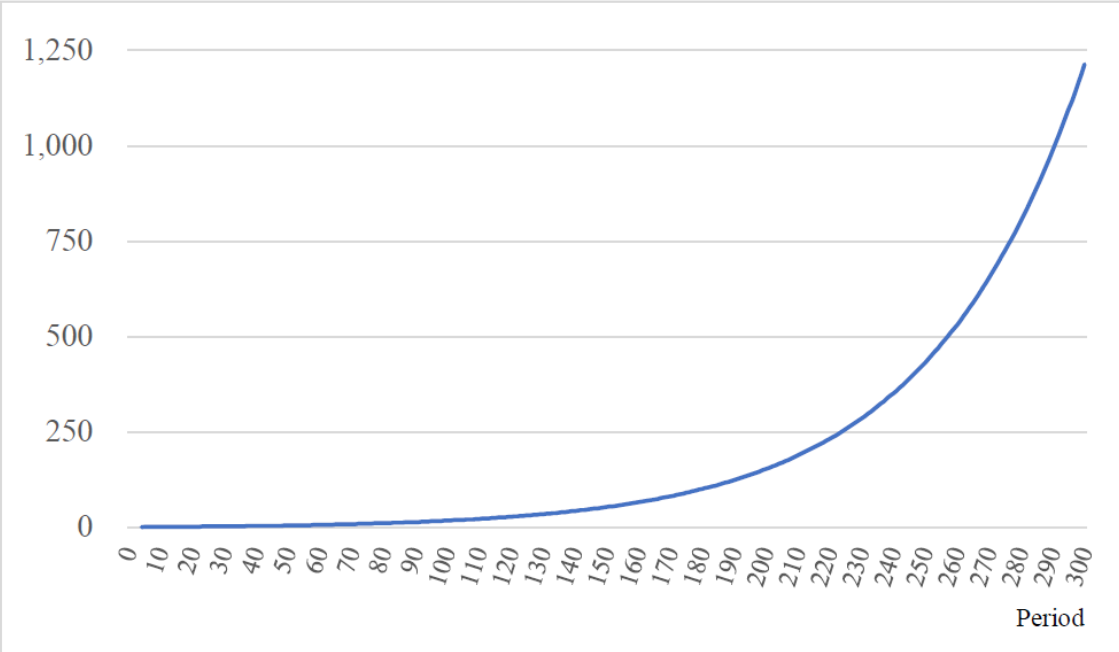


Figure 3: Simulation of the base case of endogenous growth: technology (A_t)

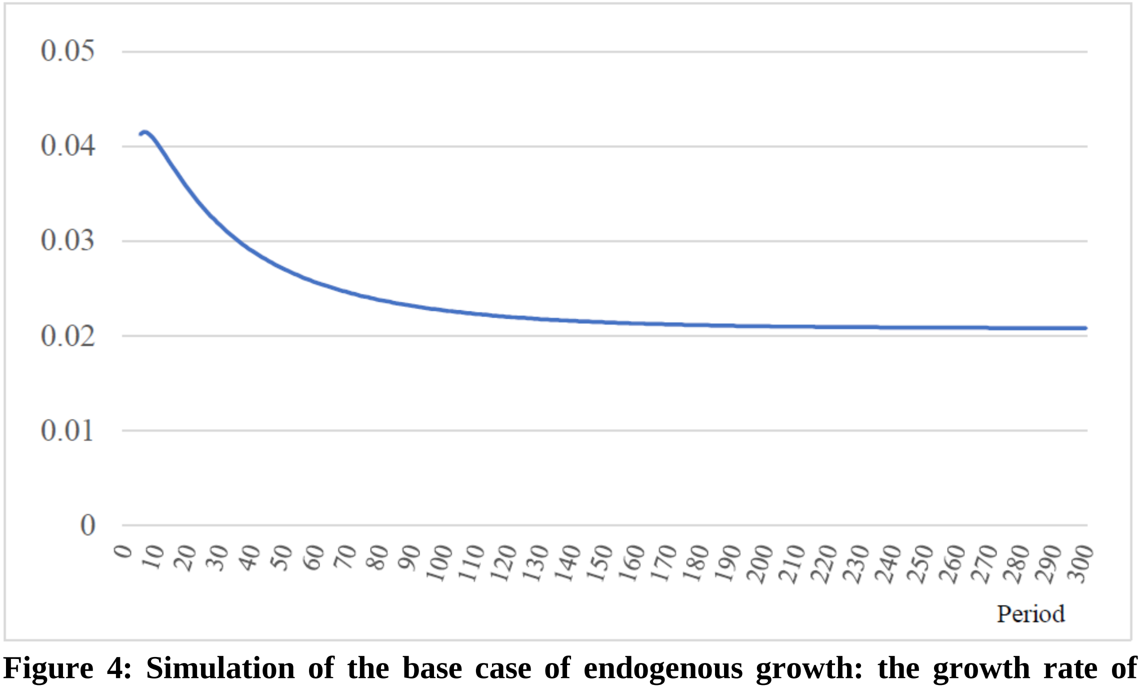


Figure 4: Simulation of the base case of endogenous growth: the growth rate of technology (A_t)

4.2 Scale effects

Next, I examine scale effects by simulating three economies with different populations (100, 100,000, and 100,000,000), while the other parameter values are the same as the base case in Section 4.1. The simulated paths of consumption of the three economies are shown in Figure 5, and their consumption growth rates are shown in Figure 6.

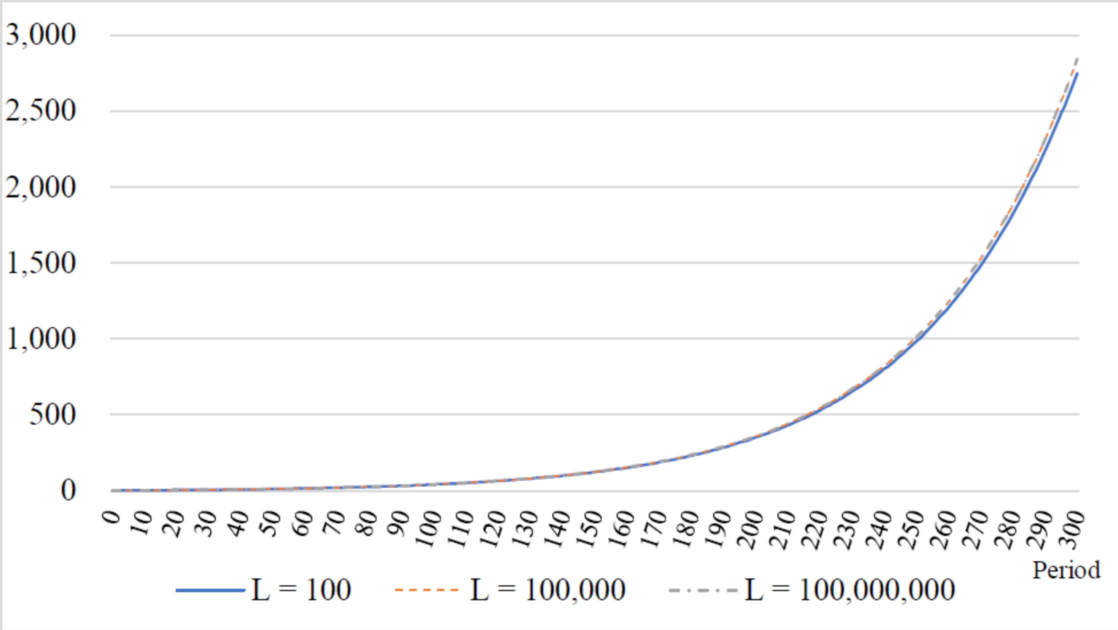


Figure 5: Simulation of scale effects: consumptions ($c_{i,t}$) of economies with populations of 100, 100,000, and 100,000,000

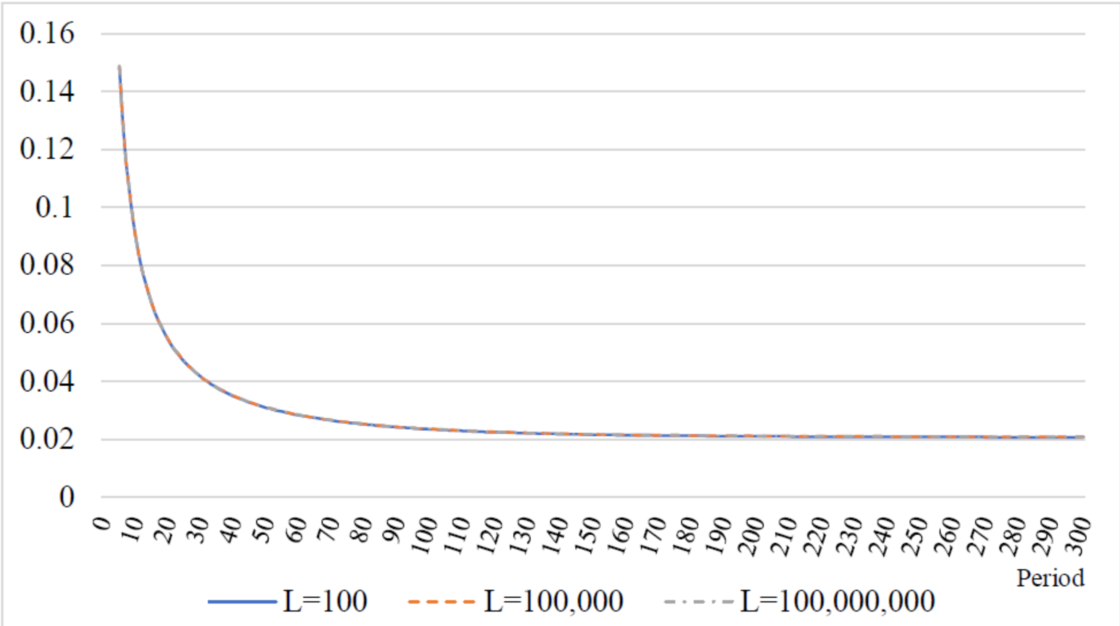


Figure 6: Simulation of scale effects: growth rates of consumption ($c_{i,t}$) of economies with populations of 100, 100,000, and 100,000,000

Although the populations are very different, the simulated paths of the three economies are very similar, which means that scale effects are very small. For example, the growth rate of the economy with the smallest population in period 300 is lower than that those with the larger populations, but not by much (2,750, 2,844, and 2,850, respectively). The reason for this very small effect is that knowledge can sufficiently spillover without compensation even if the population is small. Sufficient uncompensated knowledge spillovers are possible because ρ = 0 is kept through firms' fierce competition in markets, regardless of population.

Historically, before the industrial revolution that began in the 18th century, both the population and economic growth rate were far smaller than those in the present day, which suggests that scale effects actually existed in the past. However, the low growth rate before the industrial revolution probably was caused not by a small population but by immature capitalism and strict restriction on market activities, which would significantly deter generation of new technologies. On the other hand, in these simulations, it is implicitly assumed that markets function sufficiently well and are not restricted, even if the population is very small.

4.3 *Restrained uncompensated knowledge spillovers*

4.3.1 Positive ρ

If ρ in equations (1) and (2) is positive, scale effects can clearly emerge because a positive ρ means that uncompensated knowledge spillovers do not increase at the same rate as the population increases. If ρ is positive,

$$dk_{i,t} = \frac{L}{\left[\frac{\varpi \alpha L^{\rho}}{m^{1-\rho}(1-\alpha)} + L\right]} s_{i,t}$$

and

$$dA_t = \frac{v^{-1}}{\frac{1}{L} + \frac{1}{\left[\frac{\varpi \alpha L^{\rho}}{m^{1-\rho}(1-\alpha)}\right]}} s_{i,t}$$

by the same procedure shown in Section 3.2.

4.3.2 Simulation results

Here, I simulate the case where ρ = 0.01 (i.e., ρ only slightly deviates from ρ = 0) for three different economies (populations of 100, 100,000, and 100,000,000); the other parameter values are the same as in the base case in Section 4.1. The simulated paths of consumption of the three economies are shown in Figure 7, and their consumption growth rates are shown in Figure 8.

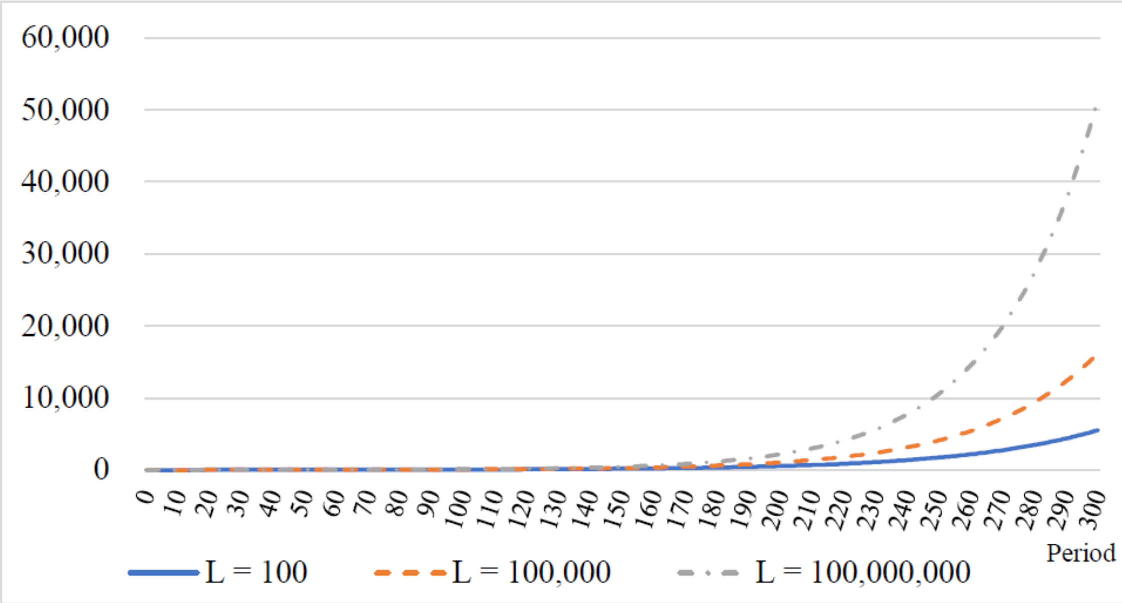


Figure 7: Simulation of restrained uncompensated knowledge spillovers (ρ = 0.01): consumption ($c_{i,t}$) of economies with populations of 100, 100,000, and 100,000,000

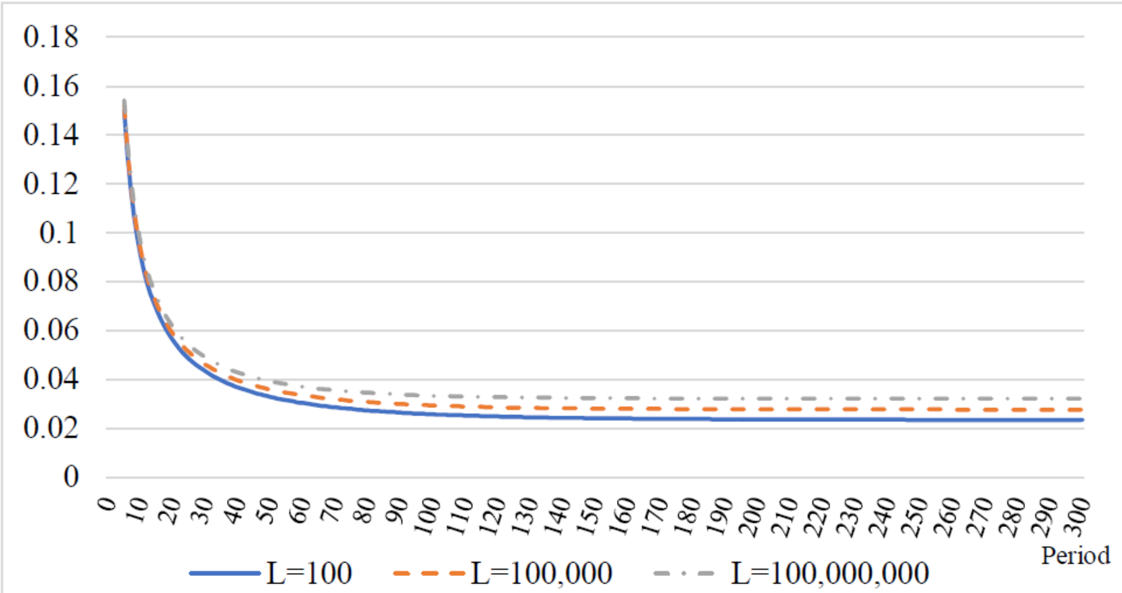


Figure 8: Simulation of restrained uncompensated knowledge spillovers (ρ = 0.01): growth rates of consumption ($c_{i,t}$) of economies with populations of 100, 100,000, and 100,000,000

Unlike the case where ρ = 0 (Section 4.2), large scale effects are observed. The economy with a population of 100,000,000 grows far more rapidly than an economy with a population of 100. The growth rates of consumption in economies with populations of 100, 100,000, and 100,000,000 after reaching a balanced growth path are 2.34%, 2.77%, and 3.21%, respectively.

Note that in all three economies, the growth rate of consumption after reaching a balanced growth path is higher than that in the case with $\rho = 0$ (Figure 6). This means that, if uncompensated knowledge spillovers are restrained and as a result the returns of firms that invest in technology increase as a population increases as indicated by equation (2), a greater amount of investments in technology will be undertaken in the economy as the population increases.

Nevertheless, firms will actually never restrain themselves from utilizing uncompensated knowledge spillovers to the greatest extent possible merely because the population and consequently the number of firms increase. Therefore, the returns of firms that invest in technology will be certainly reduced at the same rate as the increase in the number of firms (i.e., $\rho = 0$ will be always kept) (see Harashima, 2013).

4.4 Effects of degree of risk aversion (DRA)

4.4.1 Degree of risk aversion

As shown in Harashima (2018), under the MDC-based procedure, a household responds to technological progress such that:

(a) If a new version (variety) of a product is introduced into markets that performs better at the same price as the old version (variety), a household will buy the new version (variety) instead of the old one without changing its capital-wage ratio (CWR).

(b) If a household feels that its income has unexpectedly and permanently increased and that its current CWR is deviating from (particularly, is higher than) its most comfortable CWR, it will begin to adjust its consumption such that its CWR returns to its most comfortable CWR according to the consumption formulae. Because of the permanent increase in income, the household will accumulate more capital to make its CWR return to its most comfortable CWR.

How sensitively a household responds to new versions (varieties) in Channel (a) and to increases in income in Channel (b) will differ depending on its DRA (ε), and ε will eventually influence firms' plans to invest in technology.

In simulations, the sensitivity of a household's response (i.e., consumption adjustment) is represented by γ, the parameter that reflects consumption adjustment (see Section 3.1). In this sense, it is highly likely that ε and γ are correlated. Furthermore, if a household's ε is smaller, the household will tolerate larger fluctuations in consumption; the same is true if its γ is larger. Therefore, ε is negatively correlated with γ.

4.4.2 Simulation result

I simulate the paths of three economies with different values of γ (0.4, 0.5, and 0.6); the other parameter values are the same as in the base case in Section 4.1. $\gamma = 0.4$ is the highest level of risk aversion, and γ = 0.6 is the lowest. The simulated paths of consumption in the three economies are shown in Figure 9, and the consumption growth rates are shown in Figure 10.

After the economies have reached balanced growth paths, the growth rates of economies with γ = 0.4, 0.5, and 0.6 are 1.69%, 2.08%, and 2.46%, respectively. That is, the economy with γ = 0.6 grows most rapidly, and the economy with γ = 0.4 grows most slowly. This relation corresponds to the theoretical prediction. In this sense, the results of simulations are generally consistent with the standard growth theory based on the Ramsey growth model.

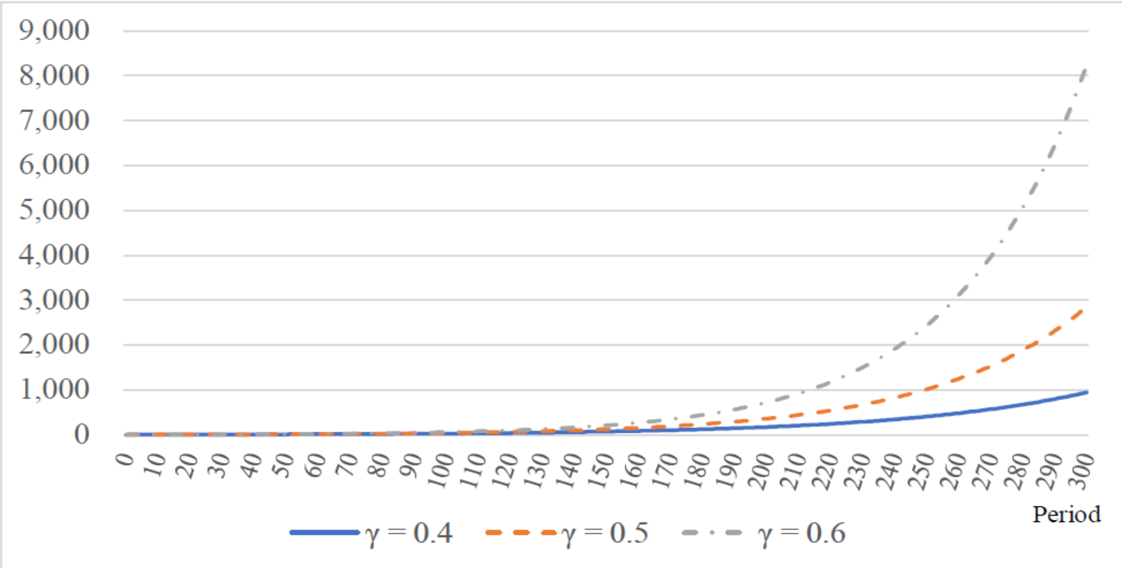


Figure 9: Simulation of effects of γ: consumptions ($c_{i,t}$) of economies with γ = 0.4, 0.5, and 0.6

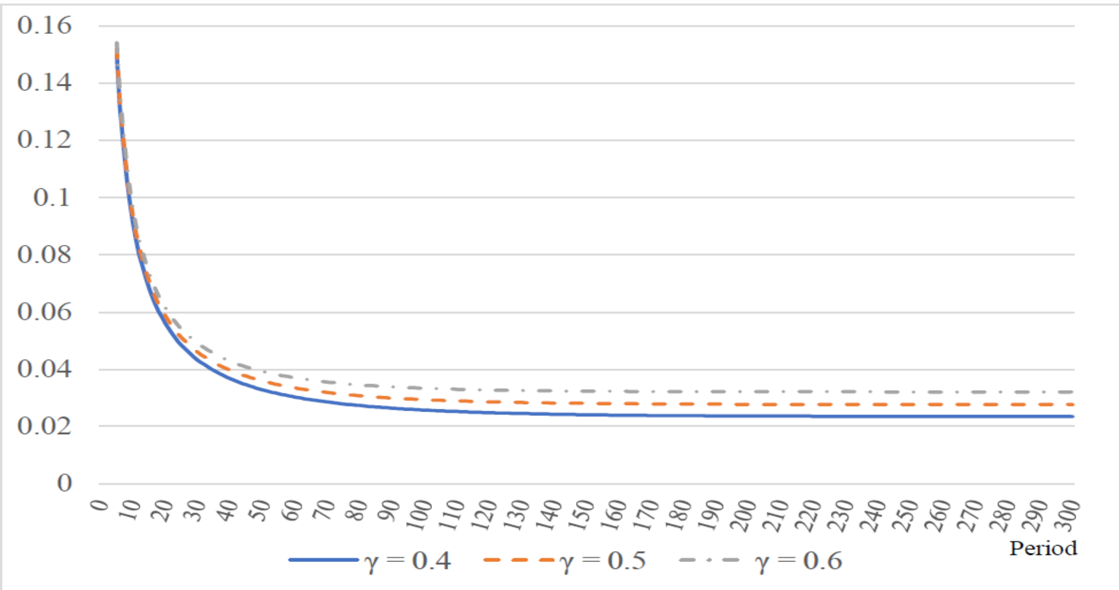


Figure 10: Simulation of effects of γ: growth rates of consumption ($c_{i,t}$) of economies with γ = 0.4, 0.5, and 0.6

Note that I preliminarily simulated several cases with different values of γ, and the results indicate that they indeed seem to be negatively correlated, but this correlation is not simple. It will be non-linear and complicated, and γ will probably be correlated not only with ε but also with other related elements. Because the correlation is not simple, for a given constant value of γ, the corresponding value of ε may change with time before reaching a balanced growth path in simulations. However, after reaching it, the value of ε is kept constant because the economy is on a balanced growth path.

4.5 Heterogeneous households and sustainable heterogeneity

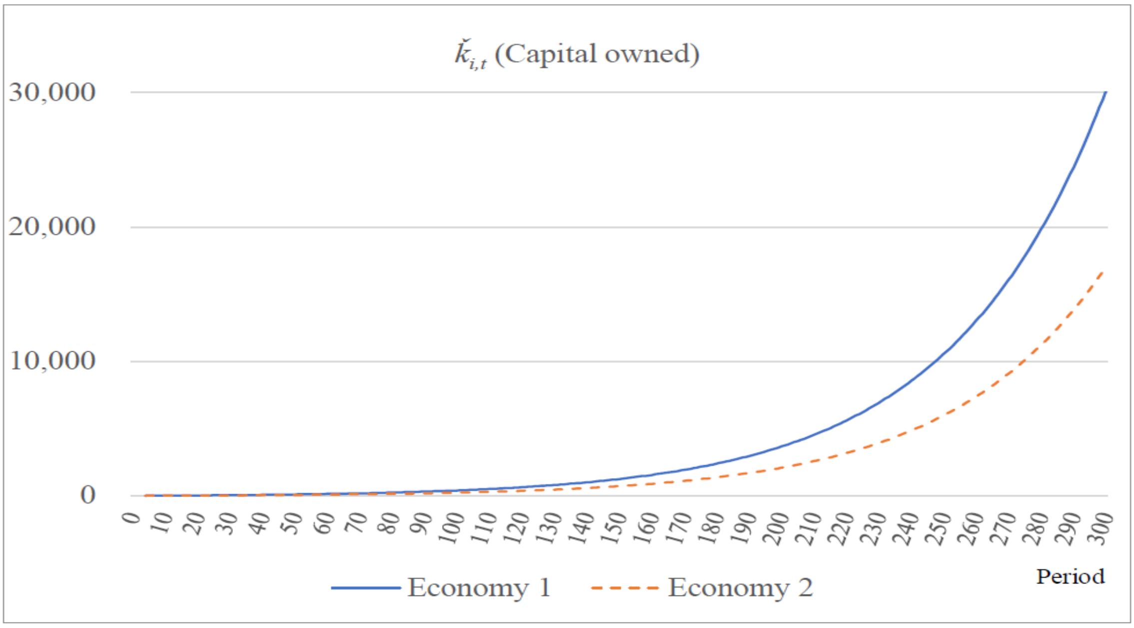
4.5.1 Sustainable heterogeneity

As shown in Harashima (2012a), under the RTP-based procedure, if the DRA (ε) is heterogeneous among households, it is difficult to achieve a balanced growth path without government intervention. Harashima (2018) showed that the same is true under the MDC-based procedure. However, if a government appropriately intervenes, sustainable heterogeneity (SH) (i.e., all optimality conditions of all heterogeneous households are satisfied) can be achieved under the RTP-based procedure (see Appendix 1). On the other hand, Harashima (2018) showed that under the MDC-based procedure, only approximate SH can be achieved with government intervention (see Appendix 2), but at the approximate SH, a balanced growth path can be realized even if the parameter that corresponds to ε under the RTP-based procedure is heterogeneous among households.

4.5.2 Simulation results

Suppose that there are two economies (Economies 1 and 2) in a country, and they are identical except for their values of γ. In Economy 1, γ is set at 0.6, and in Economy 2, γ is 0.4. First, I simulate the case that the government of the country does not intervene to achieve an approximate SH in the country. The simulated paths of capital owned and consumption are shown in Figure 11, and their growth rates are shown in Figure 12. Figures 11 and 12 show that, even without government intervention, the two economies eventually reach a balanced growth path. Nevertheless, the levels of capital owned and consumption of Economy 2, whose γ is smaller, are smaller than those of Economy 1.

As discussed above, theoretically, if the values of ε are heterogeneous, a balanced growth path is difficult to achieve without government intervention, but Figure 12 indicates that a balanced growth path can be achieved even if the values of γ are heterogeneous. This result means that the correlation between ε and of γ is not simple, as discussed in Section 4.4. The balanced growth path achieved in Figure 11 implicitly indicates that the values of ε eventually become identical in Economies 1 and 2 even though their values of γ are different. This means that a constant ε does not necessarily guarantee a constant γ in simulations. Hence, if the value of ε is assumed to be constant in simulations, the value of γ may have to be reset every period in the simulations. In actuality, for example, if a household with a constant ε feels that the current speed of increase in technology is lower than expected, it may increase γ to pursue more stimulation from unexpectedly smaller numbers of enjoyable new technologies.



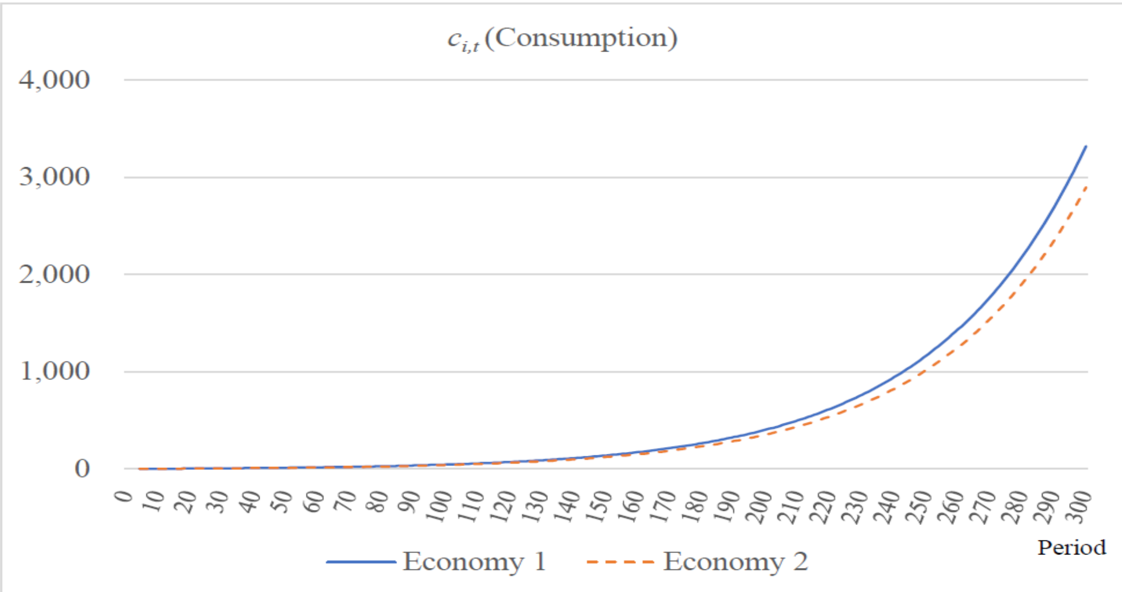
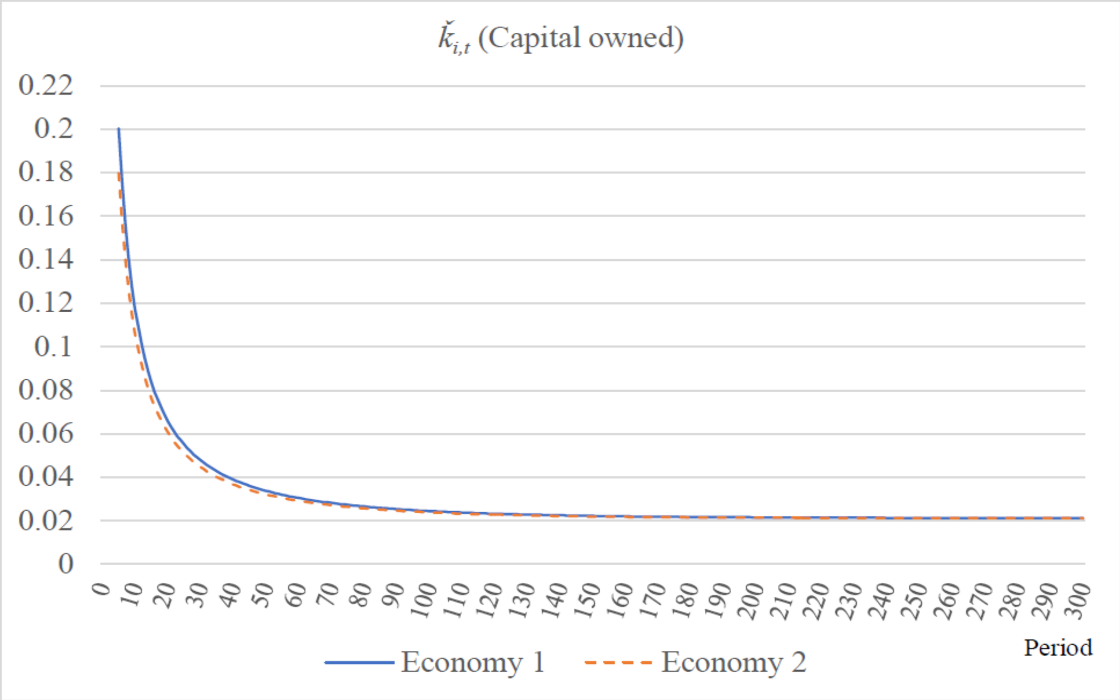


Figure 11: Simulation of heterogeneous economies (γ = 0.6 and 0.4) without government intervention: capital owned by each household ($\check{k}_{i,t}$) and consumption ($c_{i,t}$)



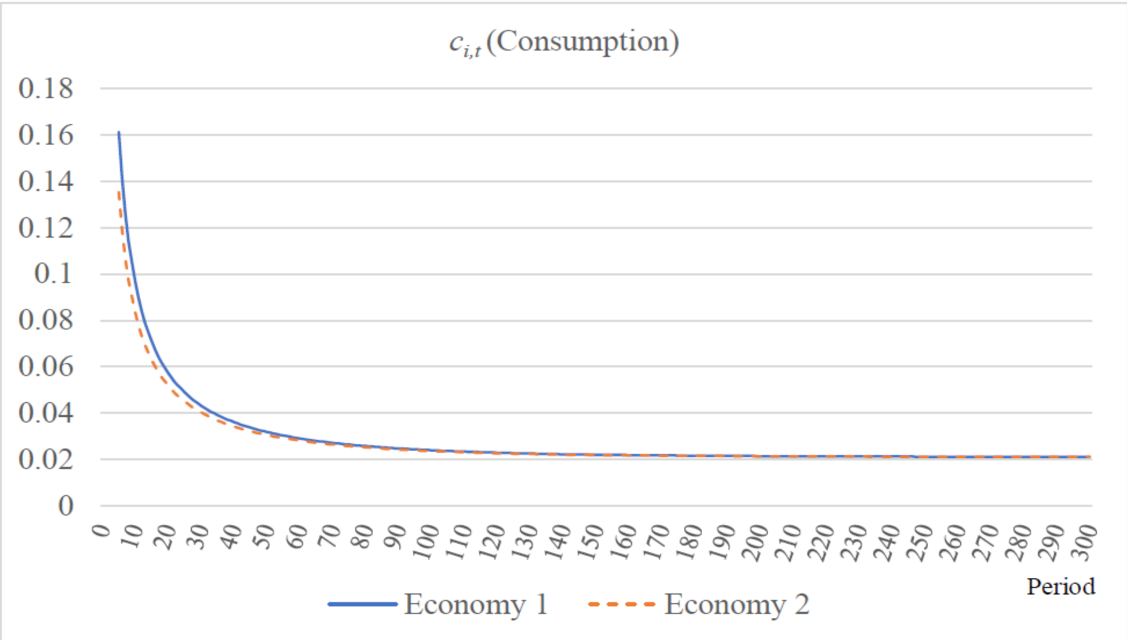
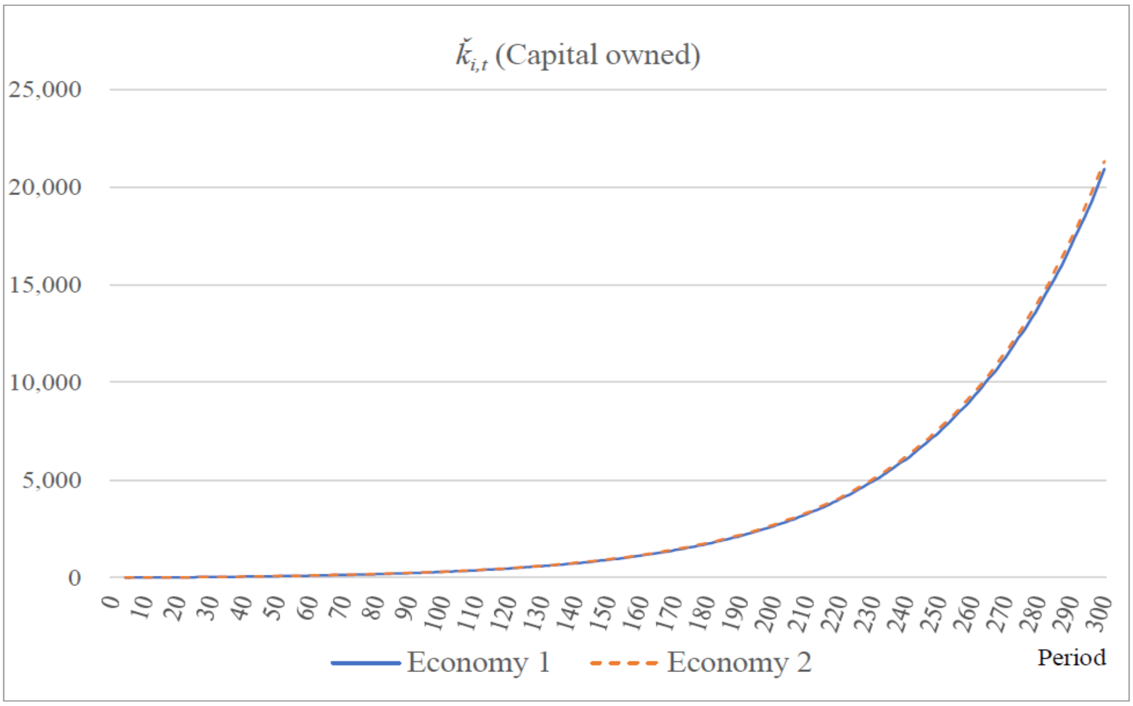


Figure 12: Simulation of heterogeneous economies (γ = 0.6 and 0.4) without government intervention: growth rates of capital owned by each household ($\check{k}_{i,t}$) and consumption ($c_{i,t}$)

Figures 11 and 12 also indicate that the levels of capital owned and consumption are clearly different in the two economies. A larger value of γ results in higher levels of capital owned and consumption, which indicates that ε and γ are generally negatively correlated, although this correlation is non-linear and complex.

Next, I simulate the case in which the government intervenes to achieve an approximate SH within the country. The government intervenes according to the Transfer rule shown in Section 3.1 to make the amount of capital owned by Economies 1 and 2 identical. The simulated paths of capital owned and consumption are shown in Figure 13, and their growth rates are shown in Figure 14.



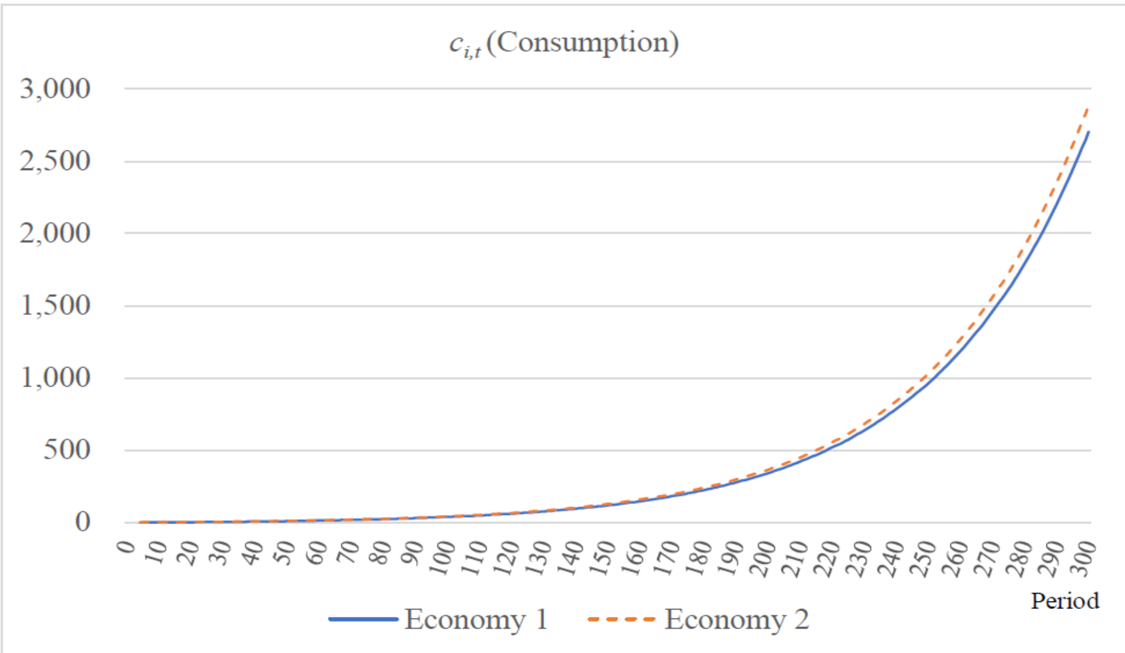
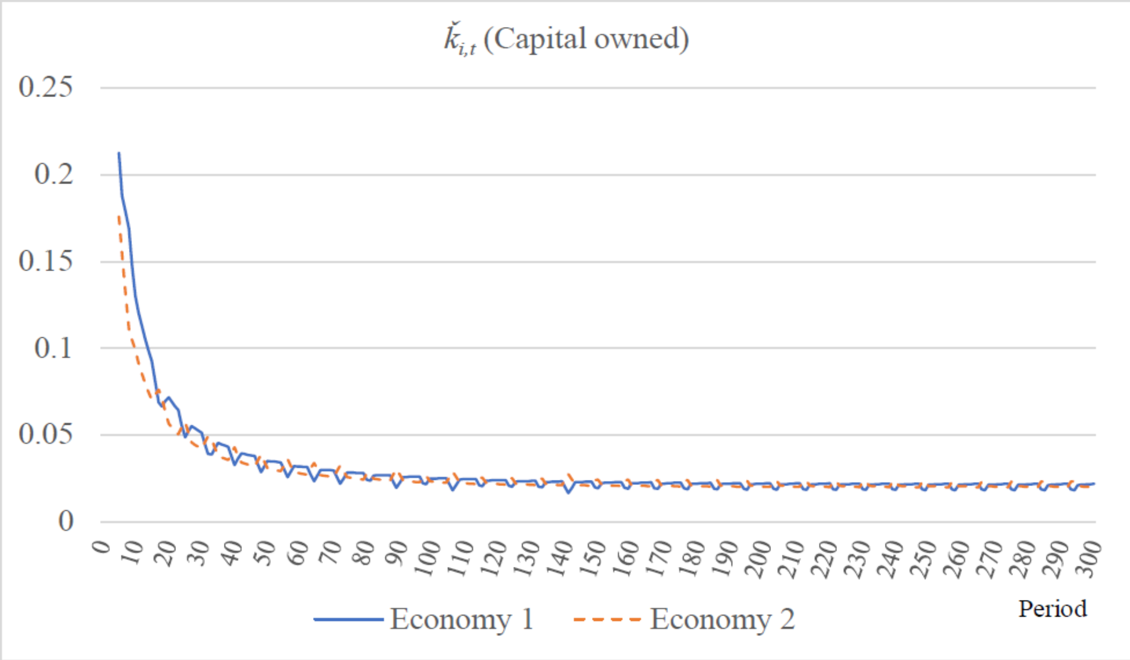


Figure 13: Simulation of heterogeneous economies (γ = 0.6 and 0.4) with government intervention (SH): capital owned by each household ($\check{k}_{i,t}$) and consumption ($c_{i,t}$)



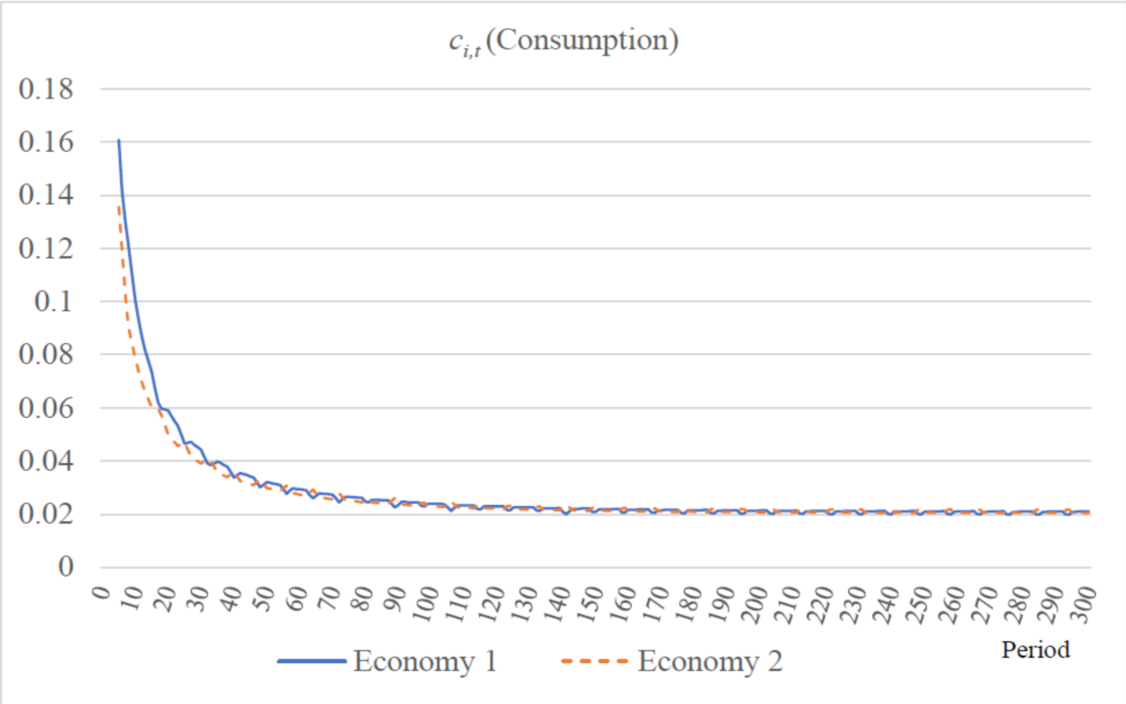


Figure 14: Simulation of heterogeneous economies (γ = 0.6 and 0.4) with government intervention (SH): growth rates of capital owned by each household ($\breve{k}_{i,t}$) and consumption ($c_{i,t}$)

The results indicate that an approximate SH is established between the two heterogeneous economies, as predicted theoretically in Harashima (2018). The amounts of capital owned on the balanced growth path are between those on the paths of Economies 1 and 2 when the government does not intervene (Figure 11). Note that the curves in Figure 14 are not smooth because the Transfer rule shown in Section 3.1 is based on a very simple bang-bang (two-step) control.

4.6 Convergence

4.6.1 Heterogeneity in productivities

Estimates of total factor productivity (TFP) vary substantially among countries, particularly those of developed and developing countries. Neo-classical Ramsey growth models naturally predict that these currently diverse estimates of TFP will eventually converge. On the other hand, many endogenous growth models do not support the convergence hypothesis (e.g., Romer, 1986, 1987). Prescott (1998) has concluded that a theory of TFP is needed to answer this question.

Harashima (2012b[6]) presented a new model of TFP, in which TFP reflects the fruits of human intelligence. This model indicates that TFP is an increasing function of ordinary workers' intelligence. In the production function (i.e., equation (4)), the element of ordinary workers' intelligence is represented by ω_i (i.e., the productivity of economy i). If ω_i is heterogeneous among economies, TFP ($\omega_i A_t^{\alpha}$ in equation (4)) is also heterogeneous even though A_t is common to all economies.

4.6.2 Simulation results

I first simulate the case of two economies (Economies 1 and 2) that are identical (i.e., their productivities are also identical), but Economy 2 starts to be a "market economy" 100 periods after Economy 1 did so. Here, "market economy" means a modern industrialized and capitalist economy. For simplicity, capital, production, consumption, and other related variables of Economy 2 are set as zero during periods 0–99, and in period 100, Economy 2 begins to produce goods and services and accumulate capital by initially borrowing capital from Economy 1. Economy 1 starts the market economy in period 0, and its values are the same as in the base case in Section 4.1.

The simulated paths of capital owned and consumption are shown in Figure 15, and the ratios of capital owned and consumption of Economy 2 to those of Economy 1 are shown in Figure 16. Figures 15 and 16 indicate that even if Economy 2 starts a market economy 100 periods after Economy 1 did, Economy 2 rapidly catches up to Economy 1, and the ratios of Economy 2 to Economy 1 eventually approach unity. That is, the two economies eventually converge. Therefore, if productivities are identical, economies eventually converge.

[6] Harashima (2012b) is also available in Japanese as Harashima (2020b).

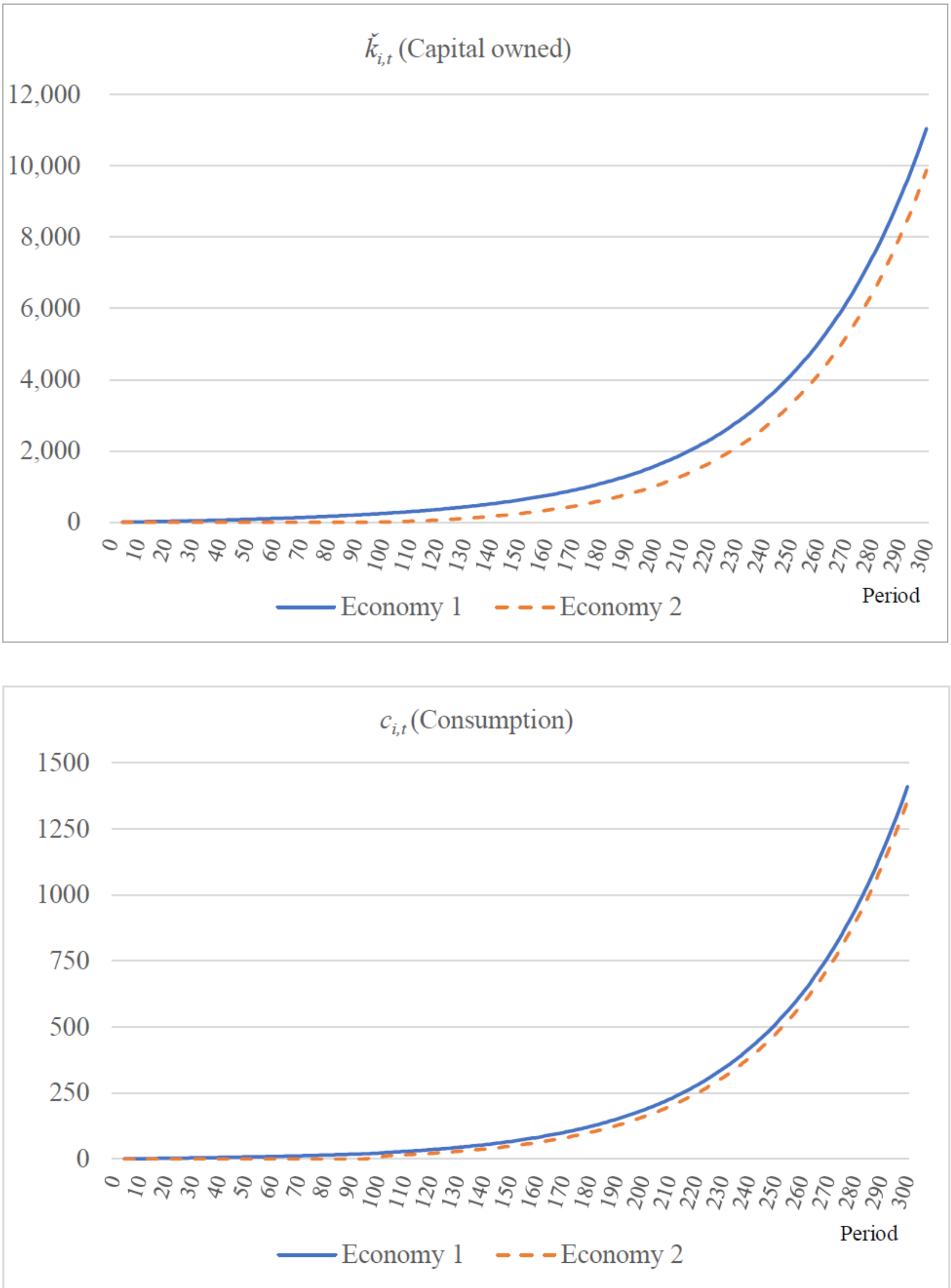


Figure 15: Simulation of convergence (homogenous productivity but different start periods): capital owned by each household ($\check{k}_{i,t}$) and consumption ($c_{i,t}$)

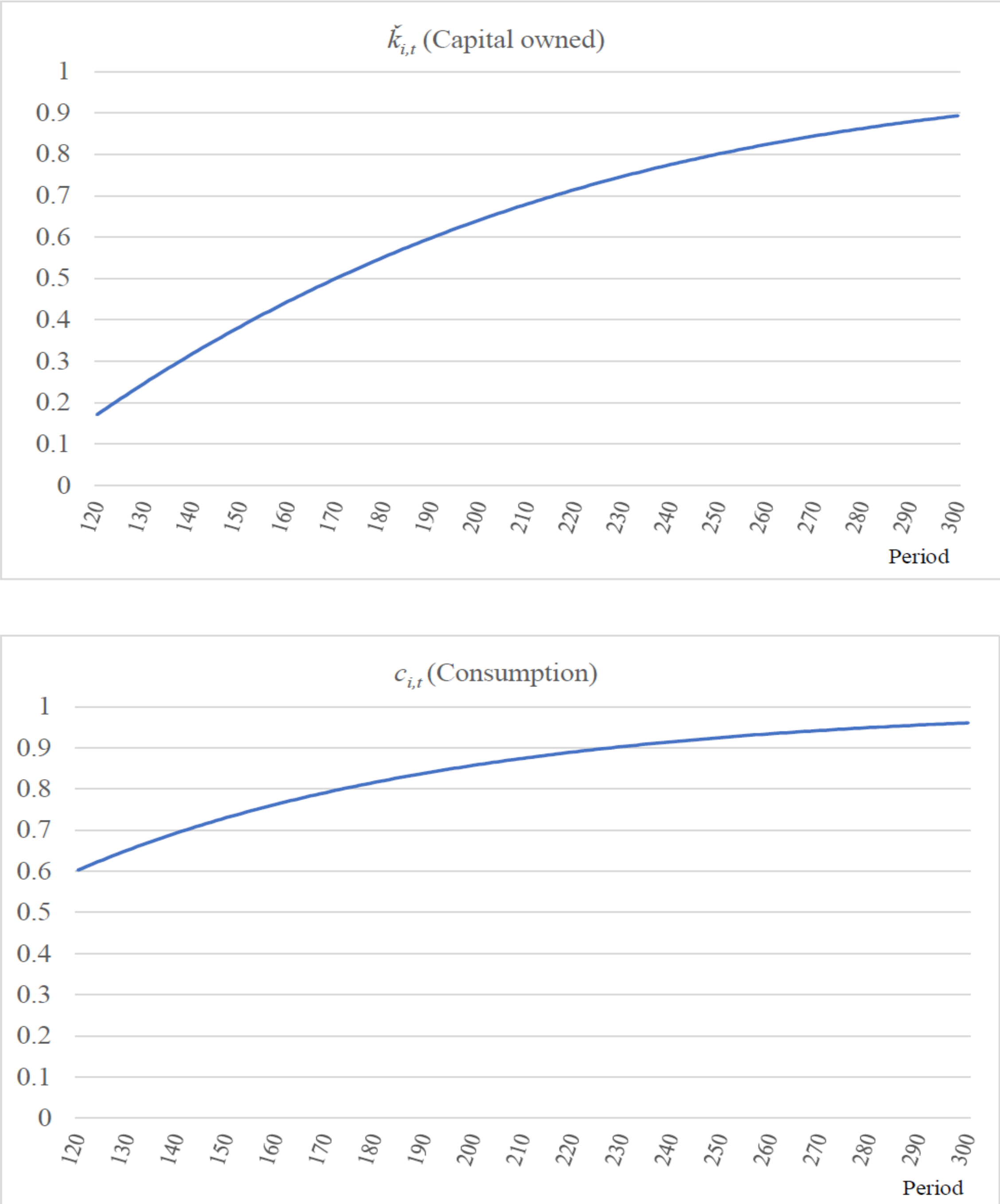
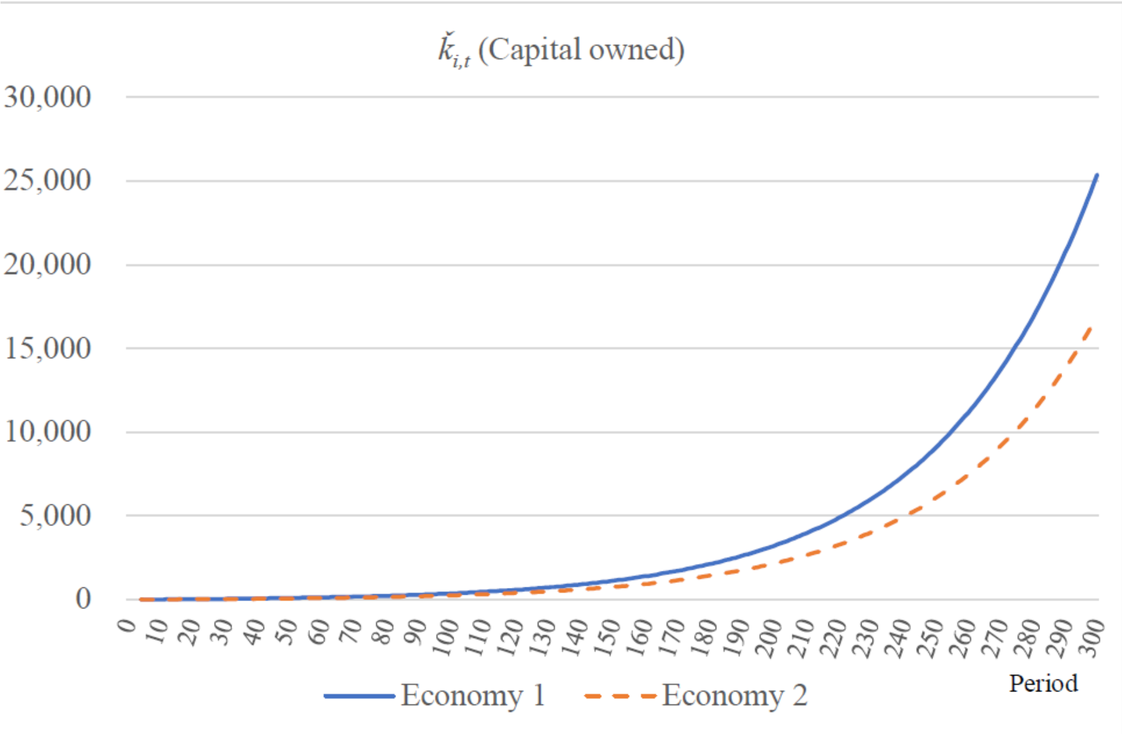


Figure 16: Simulation of convergence (homogenous productivity but different start periods): ratios of capital owned by each household ($\check{k}_{i,t}$) and consumption ($c_{i,t}$) of Economy 2 to those of Economy 1

Next, I simulate the case in which the two economies start the market economy simultaneously in period 0, but their productivities are heterogeneous. In this case, I set the productivity of Economy 1 higher ($\omega_1 = 1.2$) than that of Economy 2 ($\omega_2 = 0.8$). The simulated paths of capital owned and consumption are shown in Figure 17, and the ratios of capital owned and consumption of Economy 2 to those of Economy 1 are shown in Figure 18. Even if the two economies start the market economy at the same time, they proceed on different paths (i.e., they do not converge), and the ratios of Economy 2 to Economy 1 are eventually stabilized at almost 0.8/1.2 = 0.667.



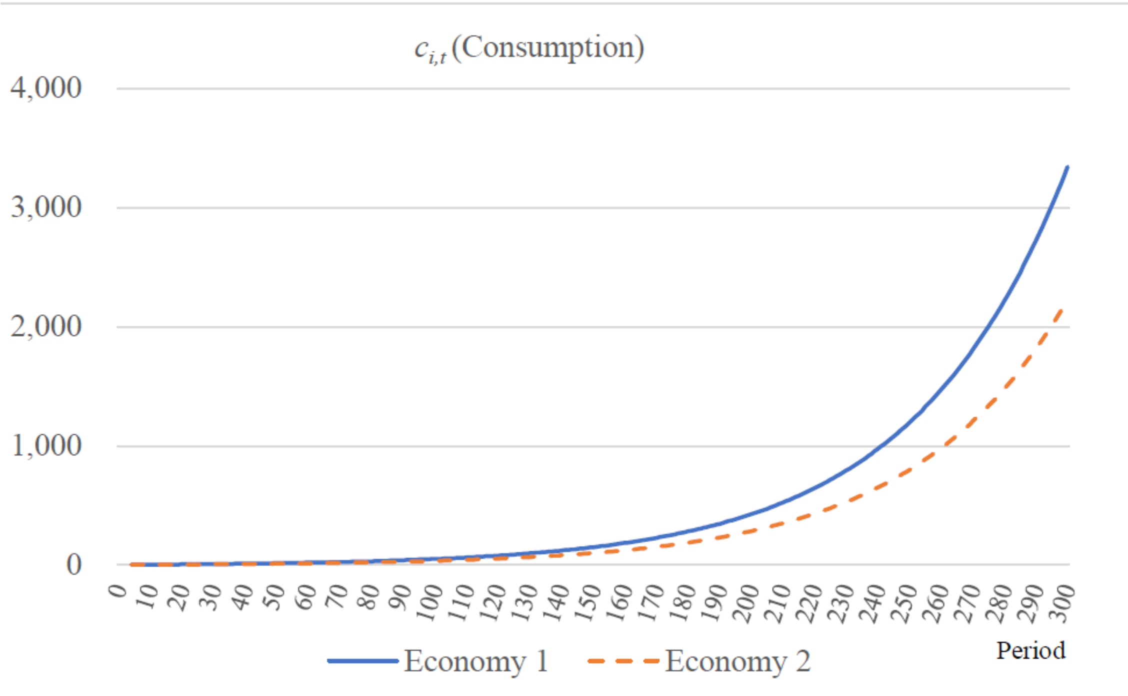
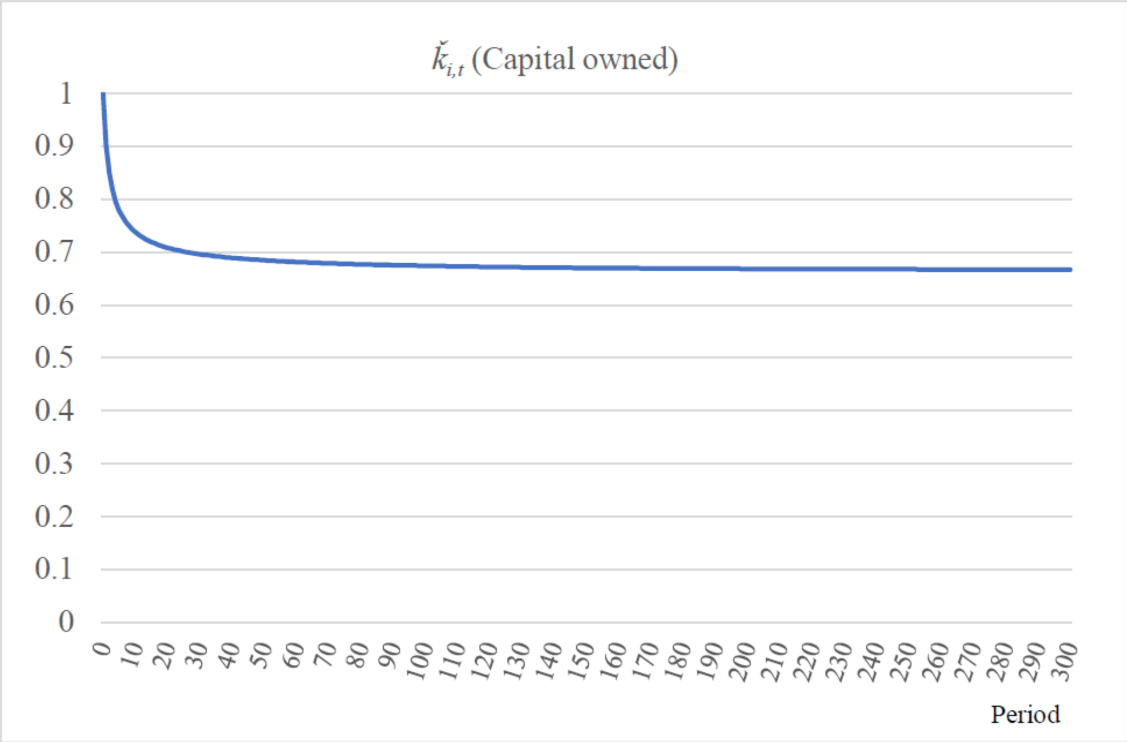


Figure 17: Simulation of convergence (heterogeneous productivities but the same start period): capital owned by each household ($\check{k}_{i,t}$) and consumption ($c_{i,t}$)



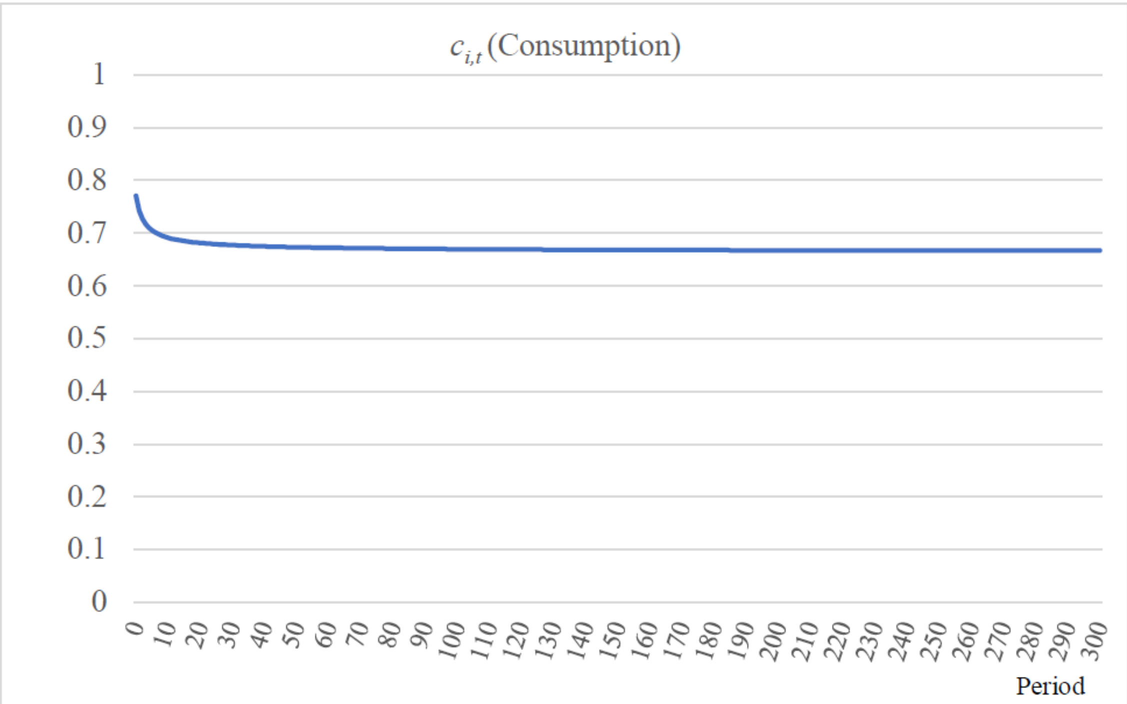


Figure 18: Simulation of convergence (heterogeneous productivities but the same start period): ratios of capital owned by each household ($\check{k}_{i,t}$) and consumption ($c_{i,t}$) of Economy 2 to those of Economy 1

These simulations indicate that if the productivities of the economies are almost the same, the economies will eventually almost converge regardless of the past economic paths. However, if their productivities are clearly different, they will not converge even if they were initially on identical economic paths.

5 CONCLUDING REMARKS

In this paper, I numerically simulate the paths of economies that endogenously grow and reach a balanced growth path on the basis of the simulation method presented by Harashima (2022c, 2023a, 2023b). In dynamic economic growth models in which households behave generating rational expectations, it is not easy to numerically simulate the path to a steady state and a balanced growth path because there is no closed-form solution to these models. However, the path to a steady state can easily be simulated if we suppose that households behave under the MDC-based procedure proposed by Harashima (2018, 2021, 2022a). Indeed, Harashima (2022c, 2023a, 2023b) numerically simulated the path to a steady state under the MDC-based procedure.

However, in the previous simulations, technologies were exogenously given and therefore endogenous growth was not simulated. In this paper, I simulate and examine how an economy grows endogenously and reaches a balanced growth path by extending the method of numerical simulation of reaching a steady state. The results of the simulations indicate that, as predicted theoretically in Harashima (2013), an economy grows endogenously if firms behave to always keep the marginal products of capital and technology equal. Scale effects exist, but they are small. However, if uncompensated knowledge spillovers are restrained as the population increases, large scale effects emerge. The effect of DRA’s surrogate variable (i.e., the adjustment speed of consumption) suggests that a smaller DRA increases the growth rate. Finally, if productivities are identical, economies converge regardless of the past economic paths, but if productivities are heterogeneous, economies diverge even if they are initially identical.

APPENDIX 1: Sustainable heterogeneity

A1.1 SH

Here, three heterogeneities—RTP, degree of risk aversion (DRA), and productivity—are considered. Suppose that there are two economies (Economy 1 and Economy 2) that are identical except for RTP, DRA, and productivity. Each economy is interpreted as representing a group of identical households, and the population in each economy is constant and sufficiently large. The economies are fully open to each other, and goods, services, and capital are freely transacted between them, but labor is immobilized in each economy. Households also provide laborers whose abilities are one of the factors that determine the productivity of each economy. Each economy can be interpreted as representing either a country or a group of identical households in a country. Usually, the concept of the balance of payments is used only for international transactions, but in this paper, this concept and the associated terminology are used even if each economy represents a group of identical households in a country.

The production function of Economy i (= 1, 2) is

$$y_{i,t} = A_t^{\alpha} k_{i,t}^{1-\alpha} \ ,$$

where $y_{i,t}$ and $k_{i,t}$ are the production and capital of Economy i in period t, respectively; A_t is technology in period t; and α ($0 < \alpha < 1$) is a constant and indicates the labor share. All variables are expressed in per capita terms. The current account balance in Economy 1 is τ_t and that in Economy 2 is $-\tau_t$. The accumulated current account balance

$$\int_0^t \tau_s ds$$

mirrors capital flows between the two economies. The economy with current account surpluses invests them in the other economy. Since $\frac{\partial y_{1,t}}{\partial k_{1,t}} \left(= \frac{\partial y_{2,t}}{\partial k_{2,t}} \right)$ is returns on investments,

$$\frac{\partial y_{1,t}}{\partial k_{1,t}} \int_0^t \tau_s ds \ \text{ and } \ \frac{\partial y_{2,t}}{\partial k_{2,t}} \int_0^t \tau_s ds$$

represent income receipts or payments on the assets that an economy owns in the other economy. Hence,

$$\tau_t - \frac{\partial y_{2,t}}{\partial k_{2,t}} \int_0^t \tau_s ds$$

is the balance on goods and services of Economy 1, and

$$\frac{\partial y_{1,t}}{\partial k_{1,t}} \int_0^t \tau_s ds - \tau_t$$

is that of Economy 2. Because the current account balance mirrors capital flows between the economies, the balance is a function of capital in both economies such that

$$\tau_t = \kappa(k_{1,t}, k_{2,t}) \, .$$

This two-economy model can be easily extended to a multi-economy model. Suppose that a country consists of H economies that are identical except for RTP, DRA, and productivity (Economy 1, Economy 2, … , Economy H). Households within each economy are identical. $c_{i,t}$, $k_{i,t}$, and $y_{i,t}$ are the per capita consumption, capital, and output of Economy i in period t, respectively; and θ_i, $\varepsilon_q = -\frac{c_{1,t} u_i''}{u_i'}$, ω_i, and u_i are the RTP, DRA, productivity, and utility function of a household in Economy i, respectively ($i = 1, 2, \ldots, H$). The production function of Economy i is

$$y_{i,t} = \omega_i A_t^{\alpha} k_{i,t}^{1-\alpha} \; .$$

In addition, $\tau_{i,j,t}$ is the current account balance of Economy i with Economy j, where i, $j = 1, 2, \ldots, H$ and $i \neq j$.

Harashima (2010) showed that if, and only if,

$$\lim_{t\to\infty} \frac{\dot{c}_{i,t}}{c_{i,t}} = \left(\frac{\sum_{q=1}^{H} \varepsilon_q \omega_q}{\sum_{q=1}^{H} \omega_q} \right)^{-1} \left\{ \left[\frac{\varpi \alpha \sum_{q=1}^{H} \omega_q}{Hmv(1-\alpha)} \right]^{\alpha} - \frac{\sum_{q=1}^{H} \theta_q \omega_q}{\sum_{q=1}^{H} \omega_q} \right\} \qquad \text{(A1.1)}$$

for any i (= 1, 2, … , H), all the optimality conditions of all heterogeneous economies are satisfied, where m, v, and ϖ are positive constants. Furthermore, if, and only if, equation (A1.1) holds,

$$\lim_{t\to\infty}\frac{\dot{c}_{i,t}}{c_{i,t}} = \lim_{t\to\infty}\frac{\dot{k}_{i,t}}{k_{i,t}} = \lim_{t\to\infty}\frac{\dot{y}_{i,t}}{y_{i,t}} = \lim_{t\to\infty}\frac{\dot{A}_t}{A_t} = \lim_{t\to\infty}\frac{\dot{\tau}_{i,j,t}}{\tau_{i,j,t}} = \lim_{t\to\infty}\frac{\frac{d\int_0^t \tau_{i,j,s}ds}{dt}}{\int_0^t \tau_{i,j,s}ds}$$

is satisfied for any i and j ($i \neq j$). Because all the optimality conditions of all heterogeneous economies are satisfied, the state at which equation (A1.1) holds is SH by definition.

A1.2 SH with government intervention

As shown above, SH is not necessarily naturally achieved, but if the government properly transfers money or other types of economic resources from some economies to other economies, SH is achieved.

Let Economy 1+2+…+ (H – 1) be the combined economy consisting of Economies 1, 2, …, and (H – 1). The population of Economy 1+2+… + (H – 1) is therefore (H – 1) times that of Economy i (= 1, 2, 3, …, H). $k_{1+2+\cdots+(H-1),t}$ indicates the capital of a household in Economy 1+2+…+ (H – 1) in period t. Let g_t be the amount of government transfers from a household in Economy 1+2+…+ (H – 1) to households in Economy H, and $\bar{g}_t$ be the ratio of g_t to $k_{1+2+\cdots+(H-1),t}$ in period t to achieve SH. That is,

$$g_t = \bar{g}_t k_{1+2+\cdots,+(H-1),t} \ .$$

$\bar{g}_t$ is solely determined by the government and therefore is an exogenous variable for households.

Harashima (2010) showed that if

$$\lim_{t\to\infty}\bar{g}_t = \left(\frac{\sum_{q=1}^{H}\varepsilon_q\omega_q}{\omega_H}\right)^{-1}\left\{\frac{\varepsilon_H\sum_{q=1}^{H}\omega_q - \sum_{q=1}^{H}\varepsilon_q\omega_q}{\sum_{q=1}^{H-1}\omega_q}\left[\frac{\varpi\alpha\sum_{q=1}^{H}\omega_q}{Hmv(1-\alpha)}\right]^{\alpha} - \frac{\varepsilon_H\sum_{q=1}^{H}\theta_q\omega_q - \theta_H\sum_{q=1}^{H}\varepsilon_q\omega_q}{\sum_{q=1}^{H-1}\omega_q}\right\}$$

is satisfied for any i (= 1, 2, …, H) in the case that Economy H is replaced with Economy i, then equation (A1.1) is satisfied (i.e., SH is achieved by government interventions even if households behave unilaterally). Because SH indicates a steady state, $\lim_{t\to\infty}\bar{g}_t$ = constant.

Note that the amount of government transfers from households in Economy 1+2+ … + (H – 1) to a household in Economy H at SH is

$$(H-1)g_t = (H-1)\ k_{1+2+\cdots+(H-1),t}\lim_{t\to\infty}\bar{g}_t \ .$$

Note also that a negative value of g_t indicates that a positive amount of money or other type of economic resource is transferred from Economy H to Economy $1+2+\cdots+(H-1)$ and vice versa.

APPENDIX 2: The MDC-based procedure

A2.1 “Comfortability” of CWR

Let k_t and w_t be per capita capital and wage (labor income), respectively, in period t. Under the MDC-based procedure, a household should first subjectively evaluate the value of $\frac{\tilde{w}_t}{\tilde{k}_t}$ where $\tilde{k}_t$ and $\tilde{w}_t$ are household k_t and w_t, respectively. Let Γ be the subjective valuation of $\frac{\tilde{w}_t}{\tilde{k}_t}$ by a household and Γ_i be the value of $\frac{\tilde{w}_t}{\tilde{k}_t}$ of household i ($i = 1, 2, 3, \ldots, M$). Each household assesses whether it feels comfortable with its current Γ (i.e., its combination of income and capital expressed by CWR). “Comfortable” in this context means “at ease,” “not anxious,” and other similar feelings.

Let the “degree of comfortability” (DOC) represent how comfortable a household feels with its Γ. The higher the value of DOC, the more a household feels comfortable with its Γ. For each household, there will be a most comfortable CWR value because the household will feel less comfortable if CWR is either too high or too low. That is, for each household, a maximum DOC exists. Let $\tilde{s}$ be a household’s state at which its DOC is the maximum (MDC). MDC therefore indicates the state at which the combination of revenues and assets is felt most comfortable. Let $\Gamma(\tilde{s})$ be a household’s Γ when it is at $\tilde{s}$. $\Gamma(\tilde{s})$ indicates the Γ that gives a household its MDC, and $\Gamma(\tilde{s}_i)$ is household i’s Γ_i when it is at $\tilde{s}_i$.

A2.2 Homogeneous population

I first examine the behavior of households in a homogeneous population (i.e., all households are assumed to be identical).

A2.2.1 Rules

Household i should act according to the following rules:

Rule 1-1: If household i feels that the current Γ_i is equal to $\Gamma(\tilde{s}_i)$, it maintains the same level of consumption for any i.
Rule 1-2: If household i feels that the current Γ_i is not equal to $\Gamma(\tilde{s}_i)$, it adjusts its level of consumption until it feels that Γ_i is equal to $\Gamma(\tilde{s}_i)$ for any i.

A2.2.2 Steady state

Households can reach a steady state even if they behave only according to Rules 1-1 and 1-2. Let S_t be the state of the entire economy in period t and $\Gamma(S_t)$ be the value of $\frac{w_t}{k_t}$ of the entire economy at S_t (i.e., the economy's average CWR). In addition, let $\tilde{S}_{MDC}$ be the steady state at which MDC is achieved and kept constant by all households, and $\Gamma(\tilde{S}_{MDC})$ be $\Gamma(S_t)$ for $S_t = \tilde{S}_{MDC}$. Let also $\tilde{S}_{RTP}$ be the steady state under the RTP-based procedure; that is, it is the steady state in a Ramsey-type growth model in which households behave based on rational expectations generated by discounting utilities by θ, where θ (> 0) is the RTP of a household. In addition, let $\Gamma(\tilde{S}_{RTP})$ be $\Gamma(S_t)$ for $S_t = \tilde{S}_{RTP}$.

Proposition 1: If households behave according to Rules 1-1 and 1-2, and if the value of θ that is calculated from the values of variables at $\tilde{S}_{MDC}$ is used as the value of θ under the RTP-based procedure in an economy where θ is identical for all households, then $\Gamma(\tilde{S}_{MDC}) = \Gamma(\tilde{S}_{RTP})$.
Proof: See Harashima (2018).

Proposition 1 indicates that we can interpret $\tilde{S}_{MDC}$ to be equivalent to $\tilde{S}_{RTP}$. This means that both the MDC-based and RTP-based procedures can function equivalently and that CWR at MDC can be substituted for RTP as a guide for household behavior.

A2.3 Heterogeneous population

In actuality, however, households are not identical—they are heterogeneous—and if heterogeneous households behave unilaterally, there is no guarantee that a steady state other than corner solutions exists (Becker 1980; Harashima 2010, 2012a). However, Harashima (2010, 2012a) has shown that SH exists under the RTP-based procedure. In addition, Harashima (2018) has shown that SH also exists under the MDC-based procedure, although Rules 1-1 and 1-2 have to be revised, and a rule for the government should be added in a heterogeneous population.

Suppose that households are identical except for their MDCs (i.e., their values of $\Gamma(\tilde{s})$). Let $\tilde{S}_{MDC,SH}$ be the steady state at which MDC is achieved and kept constant by any household (i.e., SH in a heterogeneous population under the MDC-based procedure), and let $\Gamma(\tilde{S}_{MDC,SH})$ be $\Gamma(S_t)$ for $S_t = \tilde{S}_{MDC,SH}$. In addition, let Γ_R be a household's numerically adjusted value of Γ for SH based on its estimated value of $\Gamma(\tilde{S}_{MDC,SH})$ and several other related values. Specifically, let $\Gamma_{R,i}$ be Γ_R of household i, T be the net transfer that a household receives from the government with regard to SH, and T_i be the net transfer that household i receives (i = 1,2,3, … , M).

A2.3.1 Revised and additional rules

Household i should act according to the following rules in a heterogeneous population:

Rule 2-1: If household i feels that the current $\Gamma_{R,i}$ is equal to $\Gamma(\tilde{s}_i)$, it maintains the same level of consumption as before for any i.
Rule 2-2: If household i feels that the current $\Gamma_{R,i}$ is not equal to $\Gamma(\tilde{s}_i)$, it adjusts its level of consumption or revises its estimated value of $\Gamma(\tilde{S}_{MDC,SH})$ so that it perceives that $\Gamma_{R,i}$ is equal to $\Gamma(\tilde{s}_i)$ for any i.

At the same time, the government should act according to the following rule:

Rule 3: The government adjusts T_i for some i if necessary so as to make the number of votes cast in elections in response to increases in the level of economic inequality equivalent to the number cast in response to decreases.

A2.3.2 Steady state

Even if households and the government behave according to Rules 2-1, 2-2, and 3, there is no guarantee that the economy can reach $\tilde{S}_{MDC,SH}$. However, thanks to the government's intervention, SH can be approximately achieved. Let $\tilde{S}_{MDC,SH,ap}$ be the state at which $\tilde{S}_{MDC,SH}$ is approximately achieved (an approximate SH), and $\Gamma(\tilde{S}_{MDC,SH,ap})$ be $\Gamma(S_t)$ at $\tilde{S}_{MDC,SH,ap}$ on average. Here, let $\tilde{S}_{RTP,SH}$ be the steady state that satisfies SH under the RTP-based procedure, that is, in a Ramsey-type growth model in which households that are identical except for their θs behave generating rational expectations by discounting utilities by their θs. Furthermore, let $\Gamma(\tilde{S}_{RTP,SH})$ be $\Gamma(S_t)$ for $S_t = \tilde{S}_{RTP,SH}$.

Proposition 2: If households are identical except for their values of $\Gamma(\tilde{s})$ and behave unilaterally according to Rules 2-1 and 2-2, if the government behaves according to Rule 3, and if the value of θ_i that is calculated back from the values of variables at $\tilde{S}_{MDC,SH,ap}$ is used as the value of θ_i for any i under the RTP-based procedure in an economy where households are identical except for their θs, then $\Gamma(\tilde{S}_{MDC,SH,ap}) = \Gamma(\tilde{S}_{RTP,SH})$.
Proof: See Harashima (2018).

Proposition 2 indicates that we can interpret $\tilde{S}_{MDC,SH,ap}$ as being equivalent to $\tilde{S}_{RTP,SH}$. No matter what values of T, Γ_R, and $\Gamma(\tilde{S}_{MDC,SH})$ are estimated by households, any $\tilde{S}_{MDC,SH,ap}$ can be interpreted as the objectively correct and true steady state. In addition, a government need not necessarily provide the objectively correct T_i for $\tilde{S}_{MDC,SH,ap}$ even though the $\tilde{S}_{MDC,SH,ap}$ is interpreted as objectively correct and true.

APPENDIX 3: Simulation method

A3.1 Simulation assumptions

A3.1.1 Environment

No technological progress and capital depreciation are assumed, and all values are expressed in real and per capita terms. It is assumed that there are H economies in a country, the number of households in each of economy is identical, and households within each economy are identical.

A3.1.2 Production

The production function of Economy i ($1 \leq i \leq H$) is

$$y_{i,t} = \omega_i A_t^{\alpha} k_{i,t}^{1-\alpha} \ , \tag{A3.1}$$

where ω_i is the productivity of a household in Economy i. Because α indicates the labor share, I set α = 0.65. In addition, I set $A_t = 1$ and $\omega_i = 1$ for any t and i. The initial capital a household owns is set at 1 for any household.

With $A_t = 1$ and $\omega_i = 1$, by equation (A3.1), the production of a household in Economy i in period t ($y_{i,t}$) is calculated, for any i, by

$$y_{i,t} = k_{i,t}^{1-\alpha} \ . \tag{A3.2}$$

A3.1.3 Capital

Because the marginal productivity is kept equal across economies within the country through arbitrage in markets, the amount of capital used (not owned) by each household (i.e., $k_{i,t}$) is kept identical among households in all economies in any period; that is, $k_{i,t}$ is identical for any i although the amount of capital each household owns (not uses) can be heterogeneous. Hence, by equation (A3.2), the amount of production ($y_{i,t}$) is always identical across households and economies regardless of how much capital a household in Economy i owns, when $\omega_i = 1$. In addition, for any i,

$$k_{i,t} = \frac{\sum_{i=1}^{H} \check{k}_{i,t}}{H} \ ,$$

where $\check{k}_{i,t}$ is the amount of capital a household in Economy i owns (not uses). As shown above, I set the initial capital of a household owns to be 1 (i.e., $\check{k}_{i,0} = 1$ for any i)

throughout simulations in this paper.

A3.1.4 Incomes

The capital income of a household in Economy *i* in period *t* ($x_{K,t}$) is calculated by

$$x_{K,i,t} = r_t \check{k}_{i,t} \ ,$$

where r_t is the real interest rate in period *t* and

$$r_t = \frac{\partial k_{i,t}}{\partial y_{i,t}} \ . \tag{A3.3}$$

Hence, by equations (A3.1) and (A3.3), the real interest rate r_t is calculated by

$$r_t = (1-\alpha) k_{i,t}^{-\alpha} = (1-\alpha) \left(\frac{\sum_{i=1}^{H} \check{k}_{i,t}}{H} \right)^{-\alpha} \ .$$

The labor income of a household in Economy *i* in period *t* ($x_{L,i,t}$) is calculated by extracting its capital income from its production such that

$$x_{L,i,t} = y_{i,t} - r_t k_{i,t} = y_{i,t} - r_t \frac{\sum_{i=1}^{H} \check{k}_{i,t}}{H} \ .$$

Because the amount of capital used and the amount of labor inputted by a household is identical for any household in any economy when $\omega_i = 1$, household labor income is identical across economies. Note that if productivity ($\omega_{i.t}$) is heterogeneous among economies, production and labor income differ in proportion to their productivities. Note also that in a homogeneous population, the labor income becomes equal to $\alpha y_{i,t}$ for any household.

A3.1.5 Savings

Household savings in Economy *i* in period *t* ($s_{i,t}$) are calculated by

$$s_{i,t} = x_{L,i,t} + x_{K,i,t} - c_{i,t} \ .$$

In period t + 1, these savings ($s_{i,t}$) are added to the capital the household owns, and therefore,

$$\check{k}_{i,t+1} = \check{k}_{i,t} + s_{i,t} \ .$$

A3.2 Cconsumption formula

A3.2.1 Consumption formula in a homogeneous population

For a simulation to be implemented, the consumption formula that describes how a household adjusts its consumptions needs to be set beforehand. However, under the MDC-based procedure, there is no strict consumption formula for households. A household just has to behave roughly feeling and guessing (i.e., not exactly calculating) its CWR and CWR at MDC in each period. It increases its consumption somewhat if it feels that $\Gamma(\tilde{s}_i)$ is larger than $\Gamma_{i,t}$ and decreases its consumption somewhat if it feels the opposite way. The amount of the increase/decrease will differ by period. In this sense, the actual formula of consumption under the MDC-based procedure is lax and vague; therefore, it is difficult to set a strict consumption formula with a mathematical functional form.

Nevertheless, if we consider the average consumption over some periods (i.e., moving averages), it will be possible to describe a mathematical form of the consumption formula because households will behave in a similar manner on average. Considering this nature, I introduce the following simple consumption formula because it seems to simply but correctly capture the behavior of households under the MDC-based procedure on average. Please note that that this consumption formula is not the only possible choice. Other, possibly more complex and subtle, functional forms could be chosen.

Consumption formula 1: The consumption of a household in Economy *i* in period *t* is

$$c_{i,t} = \left(x_{L,i,t} + x_{K,i,t}\right)\left(\frac{\Gamma(\tilde{s}_i)}{\Gamma_{i,t}}\right)^{\gamma} \ , \tag{A3.4}$$

where $\Gamma_{i,t}$ is the CWR of household in Economy *i* in period *t* and γ is a parameter.

Because

$$\theta_i = \left(\frac{1-\alpha}{\alpha}\right)\Gamma(\tilde{s}_i) \ , \tag{A3.5}$$

as shown in Harashima (2018, 2021, 2022a), by equation (A3.5), equation (A3.4) is equal to

$$c_{i,t} = \left(x_{L,i,t} + x_{K,i,t}\right)\left(\frac{\theta_i}{\Gamma_{i,t}\frac{1-\alpha}{\alpha}}\right)^{\gamma} .$$

Athough a household is set to precisely follow equation (A3.4) in the simulations, in reality, they do not behave by calculating equation (A3.4). Furthermore, they are not even aware of Consumption formula 1 itself and cannot know the exact numerical value of each $\Gamma(\tilde{s}_i) = \theta_i\alpha/(1-\alpha)$. Instead, households feel and guess whether they should increase or decrease consumption considering their income and wealth.

That is, Consumption formula 1 is set only for the convenience of calculation in the simulation. It seems to well capture the essence of household behavior in that it increases or decreases consumption depending on a household's feelings with regard to $\Gamma_{i,t}$ and $\Gamma(\tilde{s}_i)$. In this context, the value of parameter γ represents the average adjustment velocity of increase or decrease in consumption.

Consumption formula 1 means that a household's consumption is roughly equal to the sum of its incomes $(x_{L,i,t} + x_{K,i,t})$. The reason for this equality is that there is no technological progress and capital depreciation, so savings stay around zero at the stabilized (steady) state. As mentioned above, the adjustment velocity of consumption in each period is determined by the value of γ in equation (A3.4). As the value of γ is larger, a stabilized (steady) state can be achieved more quickly (if it can be achieved). In this paper, I set the value of γ to be 0.5.

A3.2.2 Consumption formula in a heterogeneous population

As shown in Harashima (2018, 2021, 2022a), in a heterogeneous population, a household behaving under the MDC-based procedure does not use its CWR ($\Gamma_{i,t}$) to make decisions about its consumption. Instead, it uses an adjusted value of CWR considering the behaviors of other heterogeneous households and the government because the entire economic state of the country depends on these heterogeneous behaviors in a heterogeneous population. Accordingly, in a heterogeneous population, Consumption formula 1 has to be modified to accommodate the adjusted CWR. Let $\Gamma_{R,i,t}$ be the adjusted value of $\Gamma_{i,t}$ of a household in Economy i in period t and $\Gamma(S_t)$ be the CWR of the country (i.e., the aggregate capital-wage ratio).

A3.2.2.1 Consumption formula 2

Unilateral behavior implies that a household behaves supposing that other households must behave in the same manner as it does. In other words, it assumes that other households' preferences are almost identical to its preferences, or at least, its preferences are not exceptional but roughly the same as the preferences of the average household

(Harashima, 2018). If all households behaved in the same manner as a household in Economy *i* did, the real interest rate (r_t) would be equal to the household's $\Gamma_{R,i,t}(1-\alpha)/\alpha$ and eventually converge at its $\Gamma(\tilde{s}_i)(1-\alpha)/\alpha$. Hence, if a household in Economy *i* behaves unilaterally in a heterogeneous population, it feels and guesses that its $\Gamma_{R,i,t}$ $(1-\alpha)/\alpha$ is roughly identical to the real interest rate (r_t). That is, the real interest rate will be used as $\Gamma_{R,i,t}(1-\alpha)/\alpha$, and $r_t\alpha/(1-\alpha)$ will be used as its adjusted CWR ($\Gamma_{R,i,t}$).

Therefore, even if a unilaterally behaving household's raw (unadjusted) CWR is accidentally equal to its CWR at MDC, the household does not feel that it is at its MDC unless at the same time r_t is accidentally equal to its $\Gamma(\tilde{s}_i)(1-\alpha)/\alpha$. The household will instead feel that the value of r_t will soon change, and accordingly, its raw (unadjusted) CWR will also change soon. That is, it feels and guesses that the entire economic state of the country is not yet stabilized because r_t is not equal to its $\Gamma(\tilde{s}_i)(1-\alpha)/\alpha$. As a result, the household will still continue to change its consumption to accumulate or diminish capital (see Lemma 2 in Harashima, 2018).

Considering the above-shown nature of the adjusted CWR, Consumption formula 1 can be modified to Consumption formula 2 to use in simulations with a heterogeneous population.

Consumption formula 2: In a heterogeneous population, the consumption of a household in Economy *i* in period *t* is

$$c_{i,t} = \left(x_{L,i,t} + x_{K,i,t}\right)\left(\frac{\Gamma(\tilde{s}_i)}{\Gamma_{R,i,t}}\right)^{\gamma}$$

$$= \left(x_{L,i,t} + x_{K,i,t}\right)\left(\frac{\Gamma(\tilde{s}_i)}{r_t\dfrac{\alpha}{1-\alpha}}\right)^{\gamma} = \left(x_{L,i,t} + x_{K,i,t}\right)\left(\frac{\Gamma(\tilde{s}_i)\dfrac{1-\alpha}{\alpha}}{r_t}\right)^{\gamma} \tag{A3.6}$$

and equivalently, by equations (A3.5) and (A3.6),

$$c_{i,t} = \left(x_{L,i,t} + x_{K,i,t}\right)\left(\frac{\theta_i}{r_t}\right)^{\gamma} .$$

As with $\Gamma_{i,t}$ in Consumption formula 1, the use of r_t in equation (A3.6) does not mean that households always actually behave by paying attention to r_t. What Consumption formula 2 means is that, on average, unilaterally behaving households will feel and guess that r_t represents their adjusted CWRs.

Under the RTP-based procedure, a household changes its consumption according to

$$\frac{\dot{c}_{i,t}}{c_{i,t}}=\varepsilon^{-1}(r_t - \theta_i) \ ,$$

where ε is the degree of relative risk aversion. That is, a household changes its consumption by comparing r_t and its $\theta_i = \Gamma(\tilde{s}_i)(1-\alpha)/\alpha$. The household changes consumption as r_t increasingly differs from $\theta_i = \Gamma(\tilde{s}_i)(1-\alpha)/\alpha$. This household's behavior under the RTP-based procedure is very similar to that according to Consumption formula 2, which means that the formula is basically consistent with a household's behavior under the RTP-based procedure.

In addition, in a homogeneous population, r_t is always equal to a homogenous household's $\Gamma_{i,t}(1-\alpha)/\alpha$ because all households behave in the same manner. Hence, equation (A3.4) is practically identical to equation (A3.6) (i.e., Consumption formula 1 is practically identical to Consumption formula 2) because $\Gamma_{i,t}$ in equation (A3.4) can be replaced with $r_t \frac{\alpha}{1-\alpha}$.

A3.2.2.2 Consumption formula 2-a

In Consumption formula 2, a household is supposed to feel that its preferences are not exceptional and almost the same as the preferences of the average household, but it may not actually feel that way. It may instead feel that its preferences are different from those of the average household. In this case, the household will not only feel its preferences are different, but it will also have to guess how far its preferences are from the average (i.e., by how much its adjusted CWR is different from the real interest rate).

For example, a household in Economy *i* may feel and guess that its adjusted CWR is

$$\Gamma_{R,i,t} = \frac{\alpha}{1-\alpha}\ (r_t + \chi_i) \qquad \text{(A3.7)}$$

instead of $\Gamma_{R,i,t} = r_t \frac{\alpha}{1-\alpha}$ in Consumption formula 2, where χ_i is a constant and $\chi_i \neq \chi_j$ for any *i* and *j*. χ_i represents the magnitude of how much a household in Economy *i* feels it is different from the average household. I refer to a modified version of Consumption formula 2 in which $r_t \frac{\alpha}{1-\alpha}$ is replaced with $\frac{\alpha}{1-\alpha}\ (r_t + \chi_i)$ shown in equation (A3.7) as Consumption formula 2-a. In this case, a household in Economy *i* behaves feeling that

$$\Gamma_{R,i,t} = \frac{\alpha}{1-\alpha}\ (r_t + \chi_i) = \Gamma_{i,t} \qquad \text{(A3.8)}$$

holds at a stabilized (steady) state that will be realized at some point in the future.

A3.2.2.3 Consumption formula 2-b

In both Consumption formulae 2 and 2-a, the raw (unadjusted) CWR is not included and therefore plays no role. Nevertheless, a household may utilize a piece of information derived from its raw (unadjusted) CWR because past behaviors may contain some useful information for guiding future behavior. As indicated in Section A3.2.2.2, χ_i is a parameter that indicates how far a household is from the average household. In general, the value of the parameter should be adjusted if households obtain any new and additional pieces of information. This implies that a piece of information derived from the raw (unadjusted) CWR may be used to adjust the value of parameter χ_i.

For example, a household in Economy *i* may use its raw (unadjusted) CWR ($\Gamma_{i,t}$) to adjust the value of χ_i such that

$$\chi_{i,t} = \chi_{i,t-1} + \zeta_i \left(\Gamma_{i,t} \frac{1-\alpha}{\alpha} - r_{t-1} - \chi_{i,t-1} \right) , \tag{A3.9}$$

where $\chi_{i,t}$ is χ_i in period *t*, and ζ_i is a positive constant and its value is close to zero. Equation (A3.9) means that a household in Economy *i* increases the value of $\chi_{i,t}$ a little if its raw (unadjusted) CWR is higher than its adjusted CWR ($r_{t-1} + \chi_{i,t-1}$) in the previous period and vice versa. It fine-tunes $\chi_{i,t}$ in this manner because it feels that equation (A3.8) will eventually hold at some point in the future, as shown in Section A3.2.2.2. The value of ζ_i is close to zero because $\Gamma_{i,t}$ is highly likely to be almost equal to $\Gamma_{i,t-1}$, and therefore, the guess of $\chi_{i,t}$ in period *t* will not change largely from that of $\chi_{i,t-1}$ in period $t-1$. I refer to the modified version of Consumption formula 2-a in which χ_i is replaced with $\chi_{i,t}$ shown in equation (A3.9) as Consumption formula 2-b.

A3.3 Rule of government transfer

Although governments implement transfers among households in complex and subtle manners, a simple bang-bang (two-step) control is adopted in simulations in this paper as the rule of government transfer for simplicity. In addition, government transfers in each period are assumed to be added to or extracted from the capital of each relevant household in the next period.

In simulations with government transfers, it is assumed for simplicity that there are two economies (Economies 1 and 2) in a country, the economies are identical except for each $\Gamma(\tilde{s}_i)(1-\alpha)/\alpha = \theta_i$, and all households in each economy are identical. Let κ be the $\check{k}_{1,t}$ that a government aims for to force a household in Economy 1 to own capital at a stabilized (steady) state (i.e., κ is the target value set by the government). Under these conditions, the bang-bang (two-step) control of government transfers is set as follows.

Transfer rule: The amount of government transfers from a household in Economy 1 to a household in Economy 2 in period t is T_{low} if $\check{k}_{1,t}$ is lower than κ and T_{high} if $\check{k}_{1,t}$ is higher than κ, where T_{low} and T_{high} are constant amounts of capital predetermined by the government.

In the simulations, I set T_{low} to be -0.1 and T_{high} to be 0.5. The value of κ is varied in each simulation depending on what stabilized (steady) state the government is aiming to achieve. Note that because of the discontinuous control signal in bang-bang (two-step) control, flow variables may show discontinuous zigzag paths but stock variables can move relatively smoothly. These zigzag paths may look unnatural, but they are generated only because of the bang-bang (two-step) control method that is adopted for simplicity.

Even if a household knows about the existence of government transfers, it still behaves based on Consumption formula 2 (or 2-a and 2-b) with no government transfer. That is, a household uses $x_{L,i,t} + x_{K,i,t}$, not $x_{L,i,t} + x_{K,i,t}$ + government transfers (T_{low} or T_{high}), as the "base" consumption in determining whether it should increase or decrease its consumption. This behavior superficially may mean that a household does not consider government transfers in the process of adjusting its CWR. However, it is implicitly assumed that a household knows that government transfers exist and that they are an exogenous factor. Therefore, the household feels that the transfers should be removed from the elements that it can change or control freely. Furthermore, it is implicitly assumed that a household correctly knows the exact amount of government transfers.

However, these assumptions may be oversimplifications, and they can be relaxed to allow for incorrect guesses on the amount of government transfers. This relaxation enables a household to use $x_{L,i,t} + x_{K,i,t}$ + government transfers (T_{low} or T_{high}) instead of $x_{L,i,t} + x_{K,i,t}$ in determining its consumption.

APPENDIX 4:
An Asymptotically Non-Scale Endogenous Growth Model

A4.1 Production of technologies

Outputs (production) Y_t are the sum of consumption C_t, the increase in capital K_t, and the increase in technologies A_t in period t are such that

$$Y_t = C_t + \dot{K}_t + v\dot{A}_t$$

and thus,

$$\dot{k}_t = y_t - c_t - \frac{v\dot{A}_t}{L_t} - n_t k_t$$

where $y_t = \frac{Y_t}{L_t}$, $c_t = \frac{C_t}{L_t}$, $k_t = \frac{K_t}{L_t}$, L_t is labor input, $n_t \left(= \frac{\dot{L}_t}{L_t} \right)$ is the population growth rate in period t, and in addition, $v(>0)$ is a constant, and a unit of K_t and v^{-1} of a unit of A_t are equivalent; that is, they are produced using the same quantities of inputs (capital, labor, and technology). This means that technologies are produced with capital, labor, and technology in the same way as consumer goods and services and capital.

Because balanced growth paths are the focal point of this paper, Harrod-neutral technical progress is assumed. Hence, the production function is $Y_t = K_t^{1-\alpha}(A_t L_t)^{\alpha}$; thus,

$$y_t = A_t^{\alpha} k_t^{1-\alpha} \quad .$$

It is assumed for simplicity that the population growth rate (n_t) is constant and not negative such that $n_t = n \geq 0$.

A4.2 Substitution between investments in capital and technologies

For any period,

$$m = \frac{M_t}{L_t} \tag{A4.1}$$

where M_t is the number of firms (which are assumed to be identical) and m (> 0) is a constant. Equation (A4.1) presents a natural assumption that the population and number of firms are proportional to each other. Equation (A4.1) therefore indicates that any firm consists of the same number of employees regardless of L_t. Note that, unlike the arguments in Young (1998), Peretto (1998), Aghion and Howitt (1998), and Dinopoulos and Thompson (1998), M_t is not implicitly assumed to be proportional to the number of sectors or researchers in the economy (see also Jones, 1999). Equation (A4.1) merely indicates that the average number of employees per firm in an economy is independent of the population. Hence, M_t is not essential for the amount of production of A_t. As will be shown by equations (A4.2) and (A4.3), production of A_t does not depend on the number of researchers but on investments in technologies. In contrast, M_t plays an important role in the amount of uncompensated knowledge spillovers.

The constant m implicitly indicates that the size of a firm is, on average, unchanged even if the population increases. This assumption can be justified by Coase (1937) who argued that the size of a firm is limited by the overload of administrative information. In addition, Williamson (1967) argued that there can be efficiency losses in larger firms (see also Grossman and Hart, 1986 and Moore, 1992). Their arguments equally imply that there is an optimal firm size that is determined by factors that are basically independent of population.

Next, for any period,

$$\frac{\partial Y_t}{\partial K_t} = \frac{\varpi}{M_t^{1-\rho}} \frac{\partial Y_t}{\partial (v A_t)} \quad ; \tag{A4.2}$$

thus,

$$\frac{\partial y_t}{\partial k_t} = \frac{\varpi L_t^{\rho}}{m^{1-\rho} v} \frac{\partial y_t}{\partial A_t} \tag{A4.3}$$

is always kept, where $\varpi(>1)$ and $\rho(0 \le \rho < 1)$ are constants. The parameter ρ describes the effect of uncompensated knowledge spillovers, and the parameter ϖ indicates the effect of patent protection. With patents, incomes are distributed not only to capital and labor but also to technology. For simplicity, the patent period is assumed to be indefinite, and no capital depreciation is assumed.

Equations (A4.2) and (A4.3) indicate that returns on investing in capital and technology for the investing firm are kept equal. The driving force behind the equations is that firms exploit all opportunities and select the most profitable investments at all times. Through arbitrage, this behavior leads to equal returns on investments in capital and technology. With substitution between investments in capital and technology, the model exhibits endogenous balanced growth. Because

$$\frac{\varpi L_t^{\rho}}{m^{1-\rho}v}\frac{\partial y_t}{\partial A_t}=\frac{\partial y_t}{\partial k_t}\Leftrightarrow\frac{\varpi L_t^{\rho}\alpha}{m^{1-\rho}v}A_t^{\alpha-1}k_t^{1-\alpha}=(1-\alpha)A_t^{\alpha}k_t^{-\alpha}\ ,$$

$$A_t=\frac{\varpi L_t^{\rho}\alpha}{m^{1-\rho}v(1-\alpha)}k_t$$

by equations (A4.3), which indicates that $\frac{A_t}{k_t}$ = constant for L_t^{ρ} = constant, and the model can therefore show balanced endogenous growth.

A4.3 Uncompensated knowledge spillovers

Equations (A4.2) and (A4.3) also indicate that the investing firm cannot obtain all of the returns on its investment in technology. That is, although investment in technology increases Y_t, the investing firm's returns are only a fraction of the increase in Y_t, such that $\frac{\varpi}{M_t^{1-\rho}}\frac{\partial Y_t}{\partial(vA_t)}$, because knowledge spills over to other firms without compensation and other firms possess complementary technologies.

Broadly speaking, there are two types of uncompensated knowledge spillovers: intra-sectoral knowledge spillovers (MAR externalities: Marshall, 1890; Arrow, 1962; Romer, 1986) and inter-sectoral knowledge spillovers (Jacobs' externalities: Jacobs, 1969). MAR theory assumes that knowledge spillovers between homogenous firms are the most effective and that spillovers will primarily emerge within sectors. As a result, uncompensated knowledge spillovers will be more active if the number of firms within a sector is larger. On the other hand, Jacobs (1969) argues that knowledge spillovers are most effective among firms that practice different activities and that diversification (i.e., a variety of sectors) is more important in influencing spillovers. As a result, uncompensated knowledge spillovers will be more active if the number of sectors in the economy is larger. If all sectors have the same number of firms, an increase in the number of firms in the economy results in more knowledge spillovers in any case, as a result of either MAR or Jacobs externalities.

As uncompensated knowledge spillovers increase, the investing firm's returns on investment in technology decrease. $\frac{\partial Y_t}{\partial A_t}$ indicates the total increase in Y_t in the economy by an increase in A_t, which consists of increases in both outputs of the firm that invested in the new technologies and outputs of other firms that utilize the newly invented technologies, regardless of whether the firms obtained the technologies by compensating the originating firm or through uncompensated knowledge spillovers. If the number of firms increases and uncompensated knowledge spillovers increase, the compensated fraction in $\frac{\partial Y_t}{\partial A_t}$ that the investing firm can obtain becomes smaller, as do its returns on the investment in technology. The parameter ρ describes the magnitude of this effect. If ρ = 0, the investing firm's returns are reduced at the same rate as the increase of the number of firms. $0 < \rho < 1$ indicates that the investing firm's returns diminish as the number of firms increase but not to the same extent as when $\rho = 0$.

Both types of externalities predict that uncompensated knowledge spillovers will increase as the number of firms increases, and scale effects have not actually been observed (Jones, 1995a), which implies that scale effects are almost canceled out by the effects of MAR and Jacobs externalities. Thus, the value of ρ is quite likely to be very small. From the point of view of a firm's behavior, a very small ρ appears to be quite natural. Because firms intrinsically seek profit opportunities, newly established firms work as hard as existing firms to profit from knowledge spillovers. An increase in the number of firms therefore indicates that more firms are trying to obtain the investing firm's technologies.

Because of the non-rivalness of technology, all firms can equally benefit from uncompensated knowledge spillovers, regardless of the number of firms. Because the size of firms is independent of population and thus constant, each firm's ability to utilize the knowledge that has spilled over from each of the other firms will not be reduced by an increase in population. In addition, competition over technologies will increase as the number of firms increases, and any firm will completely exploit all opportunities to utilize uncompensated knowledge spillovers as competition increases. Hence, it is quite likely that the probability that a firm can utilize a unit of new technologies developed by each of the other firms without compensation will be kept constant even if the population and the number of firms increase. As a result, uncompensated knowledge spillovers will increase eventually to the point that they increase at the same rate as the increase in the number of firms.

The investing firm's fraction of $\frac{\partial Y_t}{\partial A_t}$ that it can obtain will thereby be reduced at the same rate as the increase in the number of firms, which means that ρ will naturally decrease to zero as a result of firms' profit-seeking behavior. Based on $\rho = 0$,

$$\frac{\partial Y_t}{\partial K_t} = \frac{\varpi}{M_t} \frac{\partial Y_t}{\partial (vA_t)}$$

by equations (A4.2) and (A4.3); thus,

$$\frac{\partial y_t}{\partial k_t} = \frac{\varpi}{mv} \frac{\partial y_t}{\partial A_t}$$

is always maintained.

Complementary technologies also reduce the fraction of $\frac{\partial Y_t}{\partial A_t}$ that the investing firm can obtain. If a new technology is effective only if it is combined with other technologies, the returns on investment in the new technology will belong not only to the investing firm but also to the firms that possess the other technologies. For example, an innovation in computer software technology generated by a software company increases the sales and profits of computer hardware companies. The economy's productivity increases because of the innovation but the increased incomes are attributed not only to the firm that generated the innovation but also to the firms that possess complementary technologies. A part of $\frac{\partial Y_t}{\partial A_t}$ leaks to these firms, and the leaked income is a kind of rent revenue that unexpectedly became obtainable because of the original firm's innovation. Most new technologies will have complementary technologies. Because of both complementary technologies and uncompensated knowledge spillovers, the fraction of $\frac{\partial Y_t}{\partial A_t}$ that an investing firm can obtain on average will be very small; that is, ϖ will be far smaller than M_t except when M_t is very small.

A4.4 The optimization problem

Because $A_t = \frac{\varpi \alpha}{mv(1-\alpha)} k_t$, then $y_t = \left(\frac{\varpi \alpha}{mv}\right)^{\alpha} (1-\alpha)^{-\alpha} k_t$ and $\dot{A}_t = \frac{\varpi}{mv} \dot{k}_t \left(\frac{\alpha}{1-\alpha}\right)$. Hence, $\dot{k}_t = y_t - c_t - \frac{v\dot{A}_t}{L_t} - n_t k_t = \left(\frac{\varpi \alpha}{mv}\right)^{\alpha} (1-\alpha)^{-\alpha} k_t - c_t - \frac{\varpi}{mL_t} \dot{k}_t \left(\frac{\alpha}{1-\alpha}\right) - n_t k_t$ and

$$\dot{k}_t = \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi \alpha} \left[\left(\frac{\varpi \alpha}{mv}\right)^{\alpha} (1-\alpha)^{-\alpha} k_t - c_t - n_t k_t\right] . \tag{A4.4}$$

As a whole, the optimization problem of the representative household is to maximize the expected utility

$$E\int_0^\infty u(c_t)\exp(-\theta t)dt$$

subject to equation (A4.4) where $u(\bullet)$ is a constant relative risk aversion (CRRA) utility function and E is the expectation operator.

A4.5 Growth rate and transversality condition

Let Hamiltonian H be

$$H = u(c_t)\exp(-\theta t) + \lambda_t \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}k_t - c_t - n_t k_t\right],$$

where λ_t is a costate variable. The optimality conditions for the optimization problem shown in the previous section are

$$\frac{\partial u(c_t)}{\partial c_t}\exp(-\theta t) = \lambda_t \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha} \tag{A4.5}$$

$$\dot{\lambda}_t = -\frac{\partial H}{\partial k_t} \tag{A4.6}$$

$$\dot{k}_t = \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}k_t - c_t - n_t k_t\right] \tag{A4.7}$$

$$\lim_{t\to\infty}\lambda_t k_t = 0 \quad . \tag{A4.8}$$

By equation (A4.6),

$$\dot{\lambda}_t = -\lambda_t \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t\right] \quad . \tag{A4.9}$$

Hence, by equations (A4.5) and (A4.9), the growth rate of consumption is

$$\frac{\dot{c}_t}{c_t} = \varepsilon^{-1}\left\{\frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t\right] - \theta\right\} \quad ,$$

where $\varepsilon = -\frac{c_t u''}{u'}$. Note that usually $\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t > 0$, so this is the case examined in this paper.

By equation (A4.7), $\frac{\dot{k}_t}{k_t} = \frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t - \frac{c_t}{k_t}\right]$, and by equation (A4.9), $\frac{\dot{\lambda}_t}{\lambda_t} = -\frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha} - n_t\right]$. Hence,

$$\frac{\dot{\lambda}_t}{\lambda_t} + \frac{\dot{k}_t}{k_t} = -\frac{mL_t(1-\alpha)}{mL_t(1-\alpha)+\varpi\alpha}\left(\frac{c_t}{k_t}\right) .$$

Therefore, if $\frac{c_t}{k_t} > 0$ for any period, then $\frac{\dot{\lambda}_t}{\lambda_t} + \frac{\dot{k}_t}{k_t} < 0$, and transversality condition (A4.8) is satisfied. Conversely, if $\frac{c_t}{k_t} = 0$ for any period after a certain period, the transversality condition is not satisfied.

A4.6 Balanced growth path

There is a balanced growth path on which all the optimality conditions are satisfied.

Lemma: If and only if $\lim_{t\to\infty}\frac{\dot{c}_t}{c_t} = \lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$, all the conditions (equations [A4.5]–[A4.8]) are satisfied.

Proof: See Harashima (2013).

Rational households will set an initial consumption that leads to the growth path that satisfies all the conditions. The Lemma therefore indicates that, given an initial A_0 and k_0, rational households will set the initial consumption c_0 so as to achieve the growth path that satisfies $\lim_{t\to\infty}\frac{\dot{c}_t}{c_t} = \lim_{t\to\infty}\frac{\dot{k}_t}{k_t}$, while firms will adjust k_t so as to achieve $\frac{\partial Y_t}{\partial K_t} = \frac{\varpi}{M_t}\frac{\partial Y_t}{\partial(vA_t)}$. With this household behavior, the growth rates of technology, per capita output, consumption, and capital converge at the same rate.

Proposition: If all of the optimality conditions (equations [A4.5]–[A4.8]) are satisfied,

$$\lim_{t\to\infty}\frac{\dot{y}_t}{y_t} = \lim_{t\to\infty}\frac{\dot{A}_t}{A_t} = \lim_{t\to\infty}\frac{\dot{c}_t}{c_t} = \lim_{t\to\infty}\frac{\dot{k}_t}{k_t} .$$

Proof: Proof: See Harashima (2013).

By Proposition and Lemma, the balanced growth path is

$$\lim_{t\to\infty}\frac{\dot{y}_t}{y_t}=\lim_{t\to\infty}\frac{\dot{A}_t}{A_t}=\lim_{t\to\infty}\frac{\dot{c}_t}{c_t}=\lim_{t\to\infty}\frac{\dot{k}_t}{k_t}=\varepsilon^{-1}\left[\left(\frac{\varpi\alpha}{mv}\right)^{\alpha}(1-\alpha)^{-\alpha}-n-\theta\right] .$$

This balanced growth path can be seen as a natural extension of the steady state in the conventional Ramsey growth model with exogenous technology.

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