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A new approach to guaranteed solutions of multicriteria choice problems: Pareto consideration of Savage–Niehans risk and outcomes.

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Abstract. This article considers three new approaches to important problems of mathematical game theory and multicriteria choice.The first approach ensures payoff increase with simultaneous risk reduction in the Savage–Niehans sense in multicriteria choice problem and noncooperative games. The second approach allow us to stabilize coalitional structures in cooperative games without side payments under uncertainty. The third approach serves to integrate the selfish Nash equilibrium with the altruistic Berge equilibrium. Note that the investigations involve a special Germeier convolution of criteria and calculation of its saddle point in mixed strategies.

Keywords: *Savage–Niehans risk, minimax regret, uncertainties, multicriteria choice, Pareto consideration*

1. Prologue

This article introduces an original approach to multicriteria choice problems under uncertainty: a decision maker (DM) seeks not only to increase the guaranteed values of each criterion, but also to reduce the guaranteed risk of such increase. The approach lies at the junction of multicriteria choice problems [1, 2] and the Savage–Niehans principle of minimax regret (risk) [3, 4, 5]. More specifially, we will employ the notion of a weakly efficient estimate and the Germeier theorem [6, 7] from the theory of multicriteria choice problems and an estimated value of the regret function as the Savage–Niehans risk from the principle of minimax regret [3]. Considerations are restricted to interval-type uncertainty, i.e., the DM merely knows the limits of a range of values, without any probabilistic characteristics.We suggest a new concept — the Slater–maximal strongly–guaranteed solution in outcomes and risks (SGOR) — and establish its existence under standard assumptions of mathematical programming (continuous criteria, compact strategy sets and compact uncertainty [9] – [12]). As a possible application, the SGOR in the diversification problem of a deposit into sub–deposits in different currencies is calculated in explicit form.

2. Introduction

Consider a multicriteria choice problem under uncertainty (MCPU)

$$\Gamma_c = \langle \mathbb{N}, X, Y, f(x, y)\rangle,$$

where $\mathbb{N} = \{1, ..., N\}(N \leq 2)$ denotes the set of numbers assigned to the elements $f_i(x, y)$ of a vector criterion $f(x, y) = (f_1(x, y), ..., f_N(x, y))$; $X \subset \mathbb{R}^n$ is the set of alternatives x; $Y \subset \mathbb{R}^m$ forms the set of interval uncertainty y. For Savage–Niehans risk function design, we will also use the strategic uncertainties $y(x) : X \to Y$, denoting their set by Y^X.

At the conceptual level, it is often assumed that the DM in the problem Γ_c seeks for an alternative $x \in X$ that maximizes the values of all criteria (outcomes) under any realization of the uncertainty $y \in Y$. In this article we will also take into account N new criteria – the risk posed by increasing these outcomes. Thus, the problem setup with include N additional criteria, i.e., the Savage–Niehans risk function associated with outcome increase.

Thus, this article will justify in mathematical terms the next design method of alternatives in the MCPUs that simultaneously "hits into targets" namely, achieving higher of all outcomes under smaller risks posed by them.

The Savage–Niehans Principle of Minimax Regret

In 1939 A. Wald, a Romanian mathematician who emigrated to the USA in 1938, introduced the maxmin principle, also known as the principle of guaranteed outcome. This principle allows one to find a guaranteed outcome [13, 14] in a single–criterion choice problem under uncertainty (SCPU). Almost a decade later, German mathematician J. Niehans (1948) and American mathematician, economist and statistician L. Savage (1951) suggested the principle of minimax regret (PMR) for building guaranteed risks in the SC-PUs [4], In the modern literature, this principle is also refered to as the Savage risk or the Niehans–Savage criterion. Interestingly, during World War II Savage worked as an assistant of J. von Neumann, which surely contributed to the appearance of the PMR.Note that the autos of two most remarkable dissertations in economics and statistics are annually awarded the Savage Prize, which was established in 1971 in the USA.

For the single–criterion problem $\Gamma_1 = \langle X, Y, \phi(x,y)\rangle$, the PMR is to construct a pair $(x^r, R^r_\phi) \in X \times \mathbb{R}$ that satisfies the chain of equalities [12]

$$R^r_\phi = \max_{y\in Y} R_\phi(x^r, y) = \min_{x\in X}\max_{y\in Y} R_\phi(x,y), \tag{1}$$

where te savage–Niehans risk function has the form

$$R_\phi(x,y) = \max_{z\in X} \phi(z,y) - \phi(x,y). \tag{2}$$

The value R^r_ϕ given by (1) is called the Savage–Niehans risk in the problem Γ_1. The risk function $R_\phi(x,y)$ assessed the difference between the realized value of the criterion $\phi(x,y)$ and its best–case value $\max\limits_{z\in X} \phi(z,y)$ from the DM's view. Obviously, the DM strives to reduce $R_\phi(x,y)$ as much as possible with an appropriately chosen alternative $x \in X$, naturally expecting the strongest opposition from the uncertainty in accordance with the principle of guaranteed result (formula (1)). Therefore, adhering to (1)–(2), the DM is an optimist seeking for the best–case value $\max\limits_{x\in X} \phi(x,y)$. In contrast, the pessimistic DM is oriented towards the worth–case result — the Wald maximin solutions $(x^0, \phi^0 = \max\limits_{x\in X}\min\limits_{y\in Y} \phi(x,y) = \min\limits_{y\in Y} \phi(x^0,y))$.

In the sequel, we will consider that the DM in the problem Γ_c is an optimist: for each element $f_i(x,y)\,(i \in \mathbb{N})$ of the vector criterion $f(x,y)$, he forms a corresponding Savage–Niehans risk function:

$$R_i(x,y) = \max_{z\in X} f_i(z,y) - f_i(x,y)\ (i \in \mathbb{N}). \tag{3}$$

Note two important aspects as follows. First, each criterion $f_i(x,y)$ from Γ_c has its own risk $R_i(x,y)$ (see (3)). Second, the DM tries to choose alternatives $x \in X$ so as to reduce all risks $R_i(x,y)$, expecting any realization of the strategic uncertainty $y(\cdot) \in Y^X,\ y(x): X \to Y$.

Remark 1. The models Γ_c arise naturally, e.g., in economics: a seller in a market is interested in maximizing his profits under import uncertainty.

The uncertainties present in the problem Γ_1 lead to the sets

$$\phi(x,Y) = \{\phi(x,y)\,|\,\forall y \in Y\},$$

which are induced by an alternative $x \in X$. The set $\phi(x, Y)$ can be reduced using risks. What is a proper comprehension of risk? A well–known Russian expert in optimization, T. Sirazetdinov [16], claims that today there is no rigorous mathematical definition of risk. The monograph [17, p. 15] even suggested sixteen possible concepts of risk. Most of them require statistical data of uncertainty. However, in many cases the DM does not posses such information for objective reasons.

Thus, here risks will be understood as possible deviations of relized values from the disired ones. Note that this definition (in particular, the Savage–Niehans risk) is in good agreement with the conventional notion of microeconomic risk; e.g., see [19, pp. 40–50].

Risk management is a topical problem of economics: in 1990, H. Markowitz was awarded the Nobel Prize in Economic Sciences "for having developed the theory of portfolio choice" . In this article, the idea of his approach will be extended to the multicriteria choice problems and conflicts under uncertainty. In publications on microeconomics (e.g., see [18, p.5], [19, p.103]) all decision makers are divided into three categories: risk–averse, risk–neutral and risk–seeking. Further the DM is assumed to be a risk–neutral and, of course, an optimist.

Strong Guarantees and Transition from Γ_c to 2N–Criteria Choice Problem

For each of the N criteria $f_i(x, y)$ $(i \in \mathbb{N})$, construct the corresponding risk function $R_i(x, y)$ using formulas (3), thereby extending the MCPU Γ_c to the $2N$–criteria choice problem

$$\langle \mathbb{N}, X, Y, \{f_i(x, y), -R_i(x, y)\}_{i\in N}\rangle. \tag{4}$$

In (4) the sets $\mathbb{N}, X$ and Y are the same as in Γ_c, while the vector criterion $f(x, y)$ has an additional term in the form of the N–dimensional vector $-R(x, y) = (-R_1(x, y), \ldots, -R_N(x, y))$. Here the minus sign reflects a uniform effect of any alternative $x \in X$ on each criterion $f_i(x, y)(i \in \mathbb{N})$. More specifically, in problem (4) the DM chooses an alternative $x \in X$ in order to increase as much as possible the value of each element $f_i(x, y)$ and $-R_i(x, y)(i \in \mathbb{N})$ of the two N–dimensional vectors $f(x, y)$ and $-R(x, y)$. Moreover, the DM must expect any realization of uncertainty $y \in Y$ (note that an increase of $-R_i(x, y)$ is equivalent to a decrease of $R_i(x, y)$ due to the minus sign and $R_i(x, y) \geq 0$).

Now, consider the strong guarantees of criteria. In a series of papers [21, 22] the authors suggested three methods to take the uncertain factors into account — an analog of saddle point [22] and two analogs of maximin [21], namely, strong and vector guarantees. Note that strong guarantee is used below, while vector guarantee was applied.

Definition 1. A scalar function $f_i[x]$ is called a strong guarantee of a criterion $f_i(x,y) : X \to Y$ if, for each $x \in X$,

$$f_i[x] \le f_i(x,y) \quad \forall y \in Y \quad (i \in \mathbb{N}).$$

Remark 2. Obviously, the function $f_i[x] = \min_{y \in Y} f_i(x,y)\ \forall x \in X$ is a strong guarantee of $f_i(x,y)$. Hence, we have an explicit design method for the strong guarantees of all $2N$ criteria from (4).

Let us find the strong guarantees $R_i[x]$ of the risk functions $R_i(x,y)$ given by (3). This will be done in three steps as follows.

First, define

$$\psi_i(y) = \max_{z \in X} f_i(z,y) \quad \forall y \in Y \quad (i \in \mathbb{N}).$$

Second, construct the Savage–Niehans risk function

$$R_i(x,y) = \psi_i(y) - f_i(x,y) \quad (i \in \mathbb{N}).$$

Third, calculate the strong guarantee $\min_{y \in Y} [-R_i(x,y)]$, i.e.,

$$R_i[x] = \max_{y \in Y} R_i(x,y) \quad (i \in \mathbb{N}).$$

Note, that the DM seeks to minimize the risk $R_i(x,y)$ with an appropriate alternative $x \in X$ under any realization of the uncertainty $y \in Y$.

Whenever the functions $f_i[x]$ and $-R_i[x]$ described in the remark exist, they are strong guarantees of $f_i(x,y)$ and $-R_i(x,y)$, respectively. Indeed, for each $x \in X$, we have the implications

$$[f_i[x] = \min_{y \in Y} f_i(x,y)] \Rightarrow [f_i[x] \le f_i(x,y) \quad \forall y \in Y],$$
$$[-R_i[x] = \min_{y \in Y} (-R_i(x,y))] \Rightarrow [-R_i[x] \le -R_i(x,y) \quad \forall y \in Y].$$

The existence of $f_i[x]$ and $R_i[x]$ follows from a well–known result in operation research, which was mentioned earlier.

Lemma 1. *(see* [23, p.54]*)*
If the sets X and Y are compact and the criteria $f_i(x,y)$ are continuous on $X \times Y$, then the functions $f_i[x] = \min_{y \in Y} f_i(x,y)$ and $\psi_i[y] = \max_{z \in X} f_i(z,y)$ are continuous on X and Y, respectively.

From this point onwards, comp $\mathbb{R}^n$ stands for the set of all compact sets from space $\mathbb{R}^n$. In addition, if $f_i(x,y)$ is continuous on $X \times Y$, we will write $f_i(x,y) \in C(X \times Y)$.

Remark 3. If in the MCPU Γ_c the criterion $f_i(x,y) \in C(X \times Y)$, $X \in \mathrm{comp}\mathbb{R}^n$ and $Y \in \mathrm{comp}\mathbb{R}^m$, then the Savage–Niehans risk function $R_i(x,y)$ $(i \in \mathbb{N})$ definded by (3) is continuous on $X \times Y$. Indeed, the continuity of $\psi_i[y] = \max_{z \in X} f_i(z,y)$ follows from Lemma 1, and hence by (3) the function $R_i(x,y) = \psi[y] - f_i(x,y)$ $(i \in \mathbb{N})$ is also continuous.

Remark 4. The Savage–Niehans risk function (3) characterizes the deviation of the criterion $f_i(x,y)$ from the desired value $\max_{z\in X} f_i(z,y)$. This stimulates the DM's choice of an alternative $x \in X$ that would reduce as much as possible the difference $R_i(x,y)$ from (3) or, equivalently, maximize $-R_i(x,y)$.

Let us associate with the initial MCPU Γ_c the $2N$–criteria choice problem (4). Once again, at a conceptual level the DM in problem (4) seeks for an alternative $x \in X$ under which all the $2N$ criteria $f_i(x,y)$ and $-R_i(x,y)$ ($i \in \mathbb{N}$) would take the greatest values possible under any realization of the uncertainty $y \in Y$.

Formalization of a Guaranteed Solution in Outcomes and Risks for Problem Γ_c

The MCPUs are well–described in the literature (in particular, we refer to the monograph [24]). The specifics of the interval–type uncertainty y figuring in the problem Γ_c compel the DM to use in (4) the available information (the limits of the range of values). In this article, our analysis will be confined to the strong guarantees $f_i[x]$ and $-R_i[x]$ of the criteria $f_i(x,y)$ and $-R_i(x,y)$, respectively. Therefore, it seems natural to pass from the MCPU (4) to the multicriteria choice problem of guarantees without uncertainty

$$\Gamma^g = \langle X, \{f_i[x], -R_i[x]\}_{i\in\mathbb{N}}\rangle.$$

The criteria $f_i[x]$ and $-R_i[x]$ in Γ^g are closely related in terms of optimization: the criterion $R_i[x]$ is used for assessing the DM's risk posed by the outcome $f_i[x]$ so that an increase in the difference $f_i[x] - R_i[x]$ leads to a higher guaranteed outcome $f_i[x]$ and (or) a lower guaranteed risk $R_i[x]$. Conversely, a decrease in this difference leads to a lower guaranteed outcome $f_i[x]$ and (or) a higher risk $R_i[x]$. The DM is interested in the maximization of $f_i[x]$ with simultaneous minimization of $R_i[x]$ for each $i \in \mathbb{N}$. Therefore, we will associate with the original $2N$–criteria choice problem Γ^g the auxiliary N–criteria choice problem

$$\Gamma^a = \langle X, \{F_i[x] = f_i[x] - R_i[x]\}_{i\in\mathbb{N}}\rangle. \tag{5}$$

For a formalization of the optimal solution in guaranteed outcomes and risks for the problem Γ_c, we will use a concept of vector maximum from the theory of multicriteria choice problems [3]. A first optimal solution of this type was introduced in 1909 by Italian economist and sociologist V. Pareto, (1848–1923), and subsequently it because known as Pareto maximum.

The analysis below will employ the concept of Slater maximum, which includes the Pareto maximum as a particular case. Perhaps this concept appeared in the Russian literature after the translation of a paper by Hurwitz [25].

Definition 2. An alternative $x^S \in X$ is called Slater–maximal (weakly efficient) in the N–criteria choice problem (5) if the system of strict inequalities

$$F_i[x] > F_i[x^S] \quad (i \in \mathbb{N})$$

is inconsistent for any $x \in X$.

Remark 5. By definition 2, an alternative $x^* \in X$ is not Slater–maximal in problem (5) if there exists an alternative $\overline{x} \in X$ satisfying the N inequalities

$$F_i[\overline{x}] > F_i[x^*] \quad (i \in \mathbb{N}).$$

Proposition 1. If

$$\min_{i \in \mathbb{N}} F_i[x^S] = \max_{x \in X} \min_{i \in \mathbb{N}} F_i[x], \tag{6}$$

then the alternative $x^S \in X$ is Slater–maximal in problem (5).

Proof. By equality (6) and Remark 5, for any alternative $x \in X$ there exists a number $j \in \mathbb{N}$ such that $[F_j[x] \leq F_j[x^S]] \Rightarrow$ [the system of inequalities $F_j[x] > F_j[x^S]$ $(i \in \mathbb{N})$ is inconsistent] $\Rightarrow$ [x^S is Slater–maximal in problem (5)]. □

Theorem 1. *If $f(\cdot) \in C(X \times Y)$ and the sets X and Y are compact, then there exists a Slater–maximal alternative $x^S \in X$ in problem* (5)*.*

Proof. Using Lemma 1, we have

$$[f_i(\cdot) \in C(X \times Y), i \in \mathbb{N}] \Rightarrow [f_i[x] \in C(X), i \in \mathbb{N}],$$

and, in accordance with Remark 3, $R_i(\cdot) \in C(X \times Y)(i \in \mathbb{N})$. Then, again by Lemma 1, $\min\limits_{i \in \mathbb{N}} F_i[x] = \min\limits_{i \in \mathbb{N}} (f_i[x] - R_i[x]) \in C(X)(i \in \mathbb{N})$. Since the continuous function $\min\limits_{i \in \mathbb{N}} F_i[x]$ definded on the compact set X achieves maximum its at some point $x^S \in X$, we arrive at (6), and now the conclusion follows from Proposition 1. □

Definition 3. A triplet $(x^S, f[x^S], R[x^S])$ is called a strongly–guaranted solution in outcomes and risks (SGOR) of the MCPU Γ_c if
(1) $f_i[x] = \min\limits_{y \in Y} f_i(x, y), R_i[x] = \max\limits_{y \in Y} R_i(x, y)$ $(i \in \mathbb{N})$;
(2) the alternative x^S is Slater–maximal in problem (5).

Recall that

$$\begin{gathered} f[x] = (f_1[x], \ldots, f_N[x]), \quad R[x] = (R_1[x], \ldots, R_N[x]), \\ R_i[x] = \max_{y \in Y} R_i(x, y), R_i(x, y) = \max_{z \in X} f_i(z, y) - f_i(x, y) \ (i \in \mathbb{N}). \end{gathered} \tag{7}$$

Why is the strongly–guaranteed solution in outcomes and risks (SGOR) a good solution for the MCPU Γ_c?

First, it provides an answer to the indigenous Russian question: "What is to be done?" . The decision maker is suggested to choose the alternative x^S from the triplet $(x^S, f[x^S], R[x^S])$.

Second, for all $i \in \mathbb{N}$, this alternative x^S yields outcomes $f_i(x^S, y)$ that are not smaller than $f_i[x^S]$ with a risk $R_i(x^S, y)$ not exceeding $R_i[x]$ under any realization of the uncertainty $y \in Y$. In other words, x^S establishes lower bounds on the outcomes realized under $x = x^S$ and also upper bounds on the risks posed by them.

Third, the situation x^S implements the largest (Slater–maximal) outcomes and corresponding "minus" risks, i.e., there is no other situation $x \neq x^S$ in which all outcome guarantees $f_i[x^S]$ would increase and, at the same time, all risk guarantees $R_i[x^S]$ would decrease.

In fact, the second and third properties considered together give some analog of the maximin alternative in the single–criterion problem Γ_1 under uncertainty if the inner minimum and outer maximum in maximin are replaced by $\min\limits_{y \in Y} F_i(x, y)(i \in \mathbb{N})$ and Slater maximum, respectively. There are two lines further investigations in this field. In accordance with the first direction, one should substitute Slater maximality with Pareto, Borwein, Geoffrion optimality or conical optimality, and then establish connections between such different solutions. The second direction proceeds from the DM's desire for higher profits under the lowest guarantees in the sense of Definition 2. Consequently, it is possible to replace scalar minimum (from the inner minimum in maximin) by one of the listed vector minima, thereby increasing the guarantees for some $i \in \mathbb{N}$.

Also, it seems interesting to build a bridge between such solutions; some research efforts were made in the monograph [24].

Remark 6. Definition 2 suggests a constructive method of SGOR design. It consists of four steps as follows.
Step 1. Using $f_i(x, y)$, find $\max\limits_{z \in X} f_i(z, y) = \psi_i[y]$ and construct the Savage–Niehans risk function $R_i(x, y) = \psi_i[y] - f_i(x, y)$ for the criterion $f_i(x, y)(i \in \mathbb{N})$.
Step 2. Evaluate the outcome guarantees $f_i[x] = \min\limits_{y \in Y} f_i(x, y)$ and also the risk guarantees $R_i[x] = \max\limits_{y \in Y} R_i(x, y)(i \in \mathbb{N})$.
Step 3. For the auxiliary N–criteria choice problem of guarantees Γ^a, calculate the Slater–maximal alternative x^S. At this step, we may take advantage of Proposition 2 or perform transition to the concept of Pareto optimality. For the sake of completeness, we recall this concept.

Definition 4. An alternative $x^P \in X$ is called Pareto–maximal (efficient) in problem (5) if for any alternatives $x \in X$ the system of inequalities

$$F_i[x] \geq F_i[x^P] \quad (i \in \mathbb{N}),$$

with at least one strict inequality, is inconsistent.

Note that, first, by Definitions 2 and 3, every Pareto–maximal alternative is also Slater–maximal (the converse generally fails); second, by Karlin's lemma [26] , an alternative $x^P \in X$ that satisfies the condition

$$\max_{x\in X} \sum_{i\in\mathbb{N}} \alpha_i F_i[x] = \sum_{i\in\mathbb{N}} \alpha_i F_i[x^P] \tag{8}$$

for some $\alpha_i = \text{const} > 0$ is Pareto–maximal for problem (5).

For the bi–criteria choice problem, letting $\alpha_1 = \alpha_2 = 1$ in (8) gives the equality

$$\max_{x\in X} (F_1[x] + F_2[x]) = F_1[x^s] + F_2[x^s] \tag{9}$$

for obtaining a Pareto–maximal (hence, Slater–maximal) alternative x^S.

Step 4. Using x^S, evaluate the guarantees $f_i[x^S]$ and $R_i[x^S](i \in \mathbb{N})$ and compile the two N–dimensional vectors $f[x^S] = (f_1[x^S], \ldots, f_N[x^S])$ and $R[x^S] = (R_1[x^S], \ldots, R_N[x^S])$.

The resulting triplet $(x^S, f[x^S], R[x^S])$ is the desired SGOR, which complies with Definition 3, i.e., for each criterion $f_i(x, y)(i \in \mathbb{N})$ the alternative x^S leads to a guaranteed outcome $f_i[x^S]$ with a guaranteed Savage–Niehans risk $R_i[x^S]$.

Risks and Outcomes for Diversification of a Deposit into Sub–deposits Currencies

As mentioned earlier, in economics all decision makers are divided [26] – [32] into three categories: risk–averse, risk–neutral and risk–seeking. In this work we will solve the problem of diversification of a one–year deposit into sub–deposits in national and foreign currency for a risk–neutral person. Note that a similar problem was addressed in the paper [1, p.9], and the results established therein differ from those below. The case is that the Slater solutions generaly form a set of distinct elements. Like in [1], the analysis in this article involves different elements of the same set.

Let us proceed to the diversification problem. The amount of money in a deposit diversified into two sub–deposits (in national and foreign currency) accumulated by the end of the year can be represented as $\phi(x, y) = x(1 + r) + \dfrac{(1 - x)}{k}(1 + d)y$; see [30, pp.58–60] and also the explanations below. This leads to the single–criterion choice problem $\Gamma_1 = \langle X, Y, \phi(x, y)\rangle$, which was studied in [1]. In particular, the guaranteed solutions for risk–averse, risk–neutral and risk–seeking persons was found. In contrast to the paper [1] dealing with the single–criterion choice problem with the criterion $\phi(x, y)$, in this work we will consider a bi–criteria analog of the problem Γ_1 with the criteria

$$f_1(x) = x(1 + r), \quad f_2(x, y) = \frac{(1 - x)}{k}(1 + d)y. \tag{10}$$

The first criterion concerns the annual income for the sub–deposit in national currency from an investment x, while the second concerns the annual income for the sub–deposit in foreign currency from the residual investment $1 - x$. In formula (10), r and d denote the interest rates for the sub–deposits in national and foreign currency , respectively; k and y are the exchange rates (to the national value) at the beginning and at the end of the year, respectively; finally, $x \in [0, 1]$ specifies a proposition in which the main deposit is dedived into the sub–deposits. Thus, x is the part corresponding to the national sub–deposit, while the other part $1 - x$ is converted into foreign currency, $\frac{1-x}{k}$, and then allocated to the corresponding sub–deposit. At the end of the year, it is converted back into national currency, $\frac{(1-x)}{k}(1+d)y$; the resulting amount of money makes up $f_1(x) + f_2(x, y)$. The decision maker (depositor) has to determine the part x under which the resulting amount of money is as large as possible. It must be taken in account that the future exchange rate y is usually unknown. However, we will assume a range of its possible fluctuations, i.e., $y \in [a, b]$, where the constants $b > a > 0$ are given or a priory known.

The mathematical model of the bi–criteria deposit diversification problem can be written as an ordered triplet

$$\Gamma_2 = \langle X = [0, 1], Y = [a, b], \{f_i(x, y)\}_{i=1,2} \rangle, \tag{11}$$

where the functions $f_i(x, y)$ are defined by (10); the set $X = [0, 1]$ consists of the DM's alternatives x; $Y = [a, b]$ is the set of uncertainties y; finally, $f_i(x, y)$ denote the DM's utility function (criteria), and their values are called outcomes. In the terminology of operations research, Γ_2 is a single–criterion choice problem under uncertainty. The DM's desire to take into account the existing uncertain factors has a close connection with risk — "possible deviation of some variables from the desired values". We will use the Savage–Niehans risk function. For problem (11), consider three cases as illustrated at Fig.1, namely,

$$(1)\ k\frac{1+r}{1+d} \leq a;$$
$$(2)\ k\frac{1+r}{1+d} \geq b;$$
$$(3)\ a < k\frac{1+r}{1+d} < b.$$

Fig. 1. The possible arrangements of the point $k\frac{1+r}{1+d}$ and the interval $[a, b]$ on axis y

Cases 1 and 2. Recall that Γ_2 is a bi–criteria problem under uncertainty. We will solve it using Definition 4, which is based on the concept of Pareto optimality.

Proposition 2. In cases 1 and 2, the SGOR in the problem Γ_2 has the explicit form

$$(x^S, f[x^S], R[x^S]) = (x^S; f_1[x^S], f_2[x^S]; R_1[x^S], R_2[x^S])$$
$$= \begin{cases} (0; 0, \frac{1+d}{k}a; 1+r, 0), & \text{if } k\frac{1+r}{1+d} \leq a \\ (1; 1+r, 0; 0, \frac{1+d}{k}b), & \text{if } k\frac{1+r}{1+d} \geq b. \end{cases} \quad (12)$$

That is, in case 1 the DM invests everything in the foreign currency sub–deposit, obtaining with zero risk a guaranteed minimum amount of $\frac{1+d}{k}a$ at the end of the year; in case 2, he invests everything in the national currency sub–deposit, obtaining with zero risk a guaranteed minimum amount of $1+r$ at the end of the year. In both cases, the guaranteed minimum amounts are obtained with zero risk under any exchange rate functions $y \in [a, b]$ during the year.

Proof. We carry out the proof in two steps. In first step, following Remark 6, we construct the resulting $2N$–criteria choice problem of guarantees Γ^g and then the N–criteria choice problem (5). In the second step, for this problem (5), we find the Slater–maximal alternative x^S using Proposition 1 and then calculate the explicit form of the SGOR for the bi–criteria choice problem (11).

First step. In (11), the criteria are given by

$$f_1(x, y) = f_1(x) = x(1+r), \quad f_2(x, y) = \frac{(1-x)}{k}(1+d)y.$$

Sub-step 1. Using (3), construct the Savage–Niehans risk function

$$R_1(x, y) = [\max_{z\in[0,1]} f_1(z)] - (1+r)x = (1+r) - x(1+r) = (1-x)(1+r),$$
$$R_2(x, y) = [\max_{z\in[0,1]} f_2(z)] - (1-x)\frac{1+d}{k}y = \frac{1+d}{k}y - (1-x)\frac{1+d}{k}y = xy\frac{1+d}{k}.$$

Sub-step 2. Now, calculate the strong guarantees in outcomes and risks

$$f_1[x] = \min_{y\in[a,b]} x(1+r) = x(1+r),$$
$$f_2[x] = \min_{y\in[a,b]} (1-x)\frac{1+d}{k}y = (1-x)\frac{1+d}{k}a,$$
$$R_1[x] = \max_{y\in[a,b]} R_1(x, y) = (1-x)(1+r),$$
$$R_2[x] = \max_{y\in[a,b]} R_2(x, y) = x\frac{1+d}{k}b.$$

Sub-step 3. The quad–criteria choice problem of guarantees takes the form

$$\Gamma^g = \langle X = [0, 1], \{f_i[x], -R_i[x]\}_{i=1,2}\rangle.$$

Step 2 also allows us to define the criteria

$$F_1[x] = f_1[x] - R_1[x] = x(1+r) - (1-x)(1+r) = (2x-1)(1+r),$$
$$F_2[x] = f_2[x] - R_2[x] = (1-x)\frac{1+d}{k}a - x\frac{1+d}{k}b = \frac{1+d}{k}a - \frac{1+d}{k}(a+b)x$$

in the auxiliary bi–criteria problem (5)

$$\Gamma^a = \langle X = [0,1], \{F_i[x]\}_{i=1,2}\rangle.$$

Second step. Sub-step 4. Maximize the sum of criteria

$$\max_{[0,1]} (F_1[x] + F_2[x]) = F_1[x^S] + F_2[x^S].$$

The resulting Pareto–maximal (ergo, Slater–maximal) alternative x^S is

$$F[x^S] = \max_{[0,1]} F[x], \tag{13}$$

where

$$\begin{aligned} F[x] = F_1[x] + F_2[x] &= (2x-1)(1+r) + \frac{1+d}{k}a - \frac{1+d}{k}(a+b)x \\ &= x[2(1+r) - \frac{1+d}{k}(a+b)] - (1+r) + \frac{1+d}{k}a \\ &= \frac{1+d}{k}\{[2\gamma - (a+b)]x - \gamma + a\}, \end{aligned}$$

and $\gamma = \dfrac{1+r}{1+d}k$. The function $F[x]$ under maximization is linear in x and defined on the interval $[0,1]$. Therefore, it achieves maximum at one of the endpoints of this interval, i.e., either at $x = 0$, or at $x = 1$. For $x = 0$, we have $F[0] = \dfrac{1+d}{k}(\alpha - \gamma)$; for $x = 1$, $F[1] = \dfrac{1+d}{k}(\gamma - b)$.

Lemma 2. *The next implication is valid*

$$[a \geq \gamma] \Rightarrow [F[0] > F[1]].$$

Proof. Indeed

$$\begin{aligned} [a \geq \gamma] \Leftrightarrow \Big[[\frac{a+a}{2} \geq \gamma] \Rightarrow [\frac{a+b}{2} > \gamma]\Big] &\Rightarrow [a - \gamma > \gamma - b] \\ \Rightarrow [F[0] = \frac{1+d}{k}(a-\gamma) > F[1] = \frac{1+d}{k}(\gamma - b)]. \end{aligned}$$

□

In a similar fashion, we can easily establish

Lemma 3. *The implication*

$$[\gamma \geq b] \Rightarrow [F[0] < F[1]]$$

is valid.

Proof. Indeed,

$$[\gamma \geq b] \Leftrightarrow \Big[[\gamma \geq \frac{b+b}{2}] \Rightarrow [\gamma > \frac{b+a}{2}]\Big] \Rightarrow [\gamma - b > a - \gamma]$$
$$\Rightarrow [F[1] = \frac{1+d}{k}(\gamma - b) > F[0] = \frac{1+d}{k}(a - \gamma)].$$

□

By lemmas 2 and 3 , the maximum in (13) is achieved
(a) at $x^S = 0$ if $a \geq \gamma$;
(b) at $x^S = 1$ if $\gamma \geq b$.

The corresponding guarantees are calculated using this result and Sub-step 2:

$$f_1[0] = 0, f_2[0] = \frac{1+d}{k}a, R_1[0] = 1 + rR_2[0] = 0;$$
$$f_1[1] = 1 + r, f_2[0] = 0, R_1[0] = 1, R_2[0] = \frac{1+d}{k}b.$$

Recall that $\gamma = \dfrac{1+r}{1+d}k$, and the proof of Proposition 2 is complete. □

Let us make a few of remarks. First, $R_1[0] = 1 + r$ (the Savage–Niehans risk). The value $R_2[0] = \dfrac{1+d}{k}b$ has a similar meaning. Second, Proposition 2 was proved in the paper [29] using a different technique.

Finally, consider case 3. Here we will utilize, first, the results of Sub–step 3 of Proposition 2, in particular, the bi–criteria choice problem

$$\Gamma^a = \langle X = [0, 1], \{F_i[x]\}_{i=1,2}\rangle,$$

where

$$F_1[x] = (2x - 1)(1 + r),$$
$$F_2[x] = \frac{1+d}{k}a - \frac{1+d}{k}(a + b)x; \tag{14}$$

second, the sufficient conditions (6) for the existence of the alternative x^S (see Proposition 1), writing them for the deposit diversification problem (11) as

$$\min_{i=1,2} F_i[x^S] = \max_{x\in[0,1]} \min_{i=1,2} F_i[x].$$

Proposition 3. If $a < \dfrac{1+r}{1+d}k < b$, the SGOR in the problem Γ_2 has the form

$$(x^S, f[x^S], R[x^S]) = (x^S; f_1[x^S], f_2[x^S]; R_1[x^S], R_2[x^S])$$
$$= \Big(\frac{\gamma + a}{2\gamma + a + b}; \frac{(\gamma + a)(1 + r)}{2\gamma + a + b}, \frac{\gamma + b}{2\gamma + a + b}\frac{1+d}{k}a; \tag{15}$$
$$(1 + r)\frac{\gamma + b}{2\gamma + a + b}, b\frac{1+d}{k}\frac{\gamma + a}{2\gamma + a + b}\Big).$$

Proof. Draw the graphs of the two functions $F_1[x]$ and $F_2[x]$ from (14). These functions are linear in x and defined on the interval $[0,1]$ (a compact set); see Fig.2.

In Fig. 2 the function $\min\limits_{i=1,2}\{F_1[x], F_2[x]\}$ is indicated by the bold line, see the angle ABC. For $\max\limits_{x\in[0,1]}\min\limits_{i=1,2}\{F_1[x], F_2[x]\}$, the point B satisfies the equality

$$F_1[x^S] = F_2[x^S]$$

or, using (14)

$$x^S\left[2(1+r)+\frac{1+d}{k}(a+b)\right] = 1+r+\frac{1+d}{k}a.$$

With the notation $\gamma = \dfrac{1+r}{1+d}k$, it can be written as

$$x^S[2\gamma+a+b] = \gamma + a,$$

which gives

$$x^S = \frac{\gamma+a}{2\gamma+a+b}, \quad 1-x^S = \frac{\gamma+b}{2\gamma+a+b}.$$

Using the formulas of Sub-step 2, we calculate the strong guarantees in outcomes and risks:

$$f_1[x^S] = (1+r)\frac{\gamma+a}{2\gamma+a+b}, \qquad f_2[x^S] = \frac{1+d}{k}a\frac{\gamma+b}{2\gamma+a+b},$$

$$R_1[x^S] = (1+r)\frac{\gamma+b}{2\gamma+a+b}, \qquad R_2[x^S] = \frac{1+d}{k}b\frac{\gamma+a}{2\gamma+a+b}.$$

□

Thus, we have established the following result (see Proposition 3). If $a < \dfrac{1+r}{1+d}k < b$, the strongly–guaranteed solution in outcomes and risks of the deposit diversification problem has form (15). It suggests the DM to invest the part $\dfrac{\gamma+a}{2\gamma+a+b}$ in the national currency sub–deposit and the residual part $\dfrac{\gamma+b}{2\gamma+a+b}$ in the foreign currency sub–deposit. At the end of the year, the DM will obtain the amount $(1+r)\dfrac{\gamma+a}{2\gamma+a+b}$ for the national currency sub–deposit with the Savage–Niehans risk $(1+r)\dfrac{\gamma+b}{2\gamma+a+b}$ and the amount $\dfrac{1+d}{k}a(1+r)\dfrac{\gamma+b}{2\gamma+a+b}$ (after conversion in national currency) for the foreign currency sub–deposit with the Savage–Niehans risk $\dfrac{1+d}{k}b\dfrac{\gamma+a}{2\gamma+a+b}$ under the exchange rate fluctuations $y\in[a,b]$ during the year.

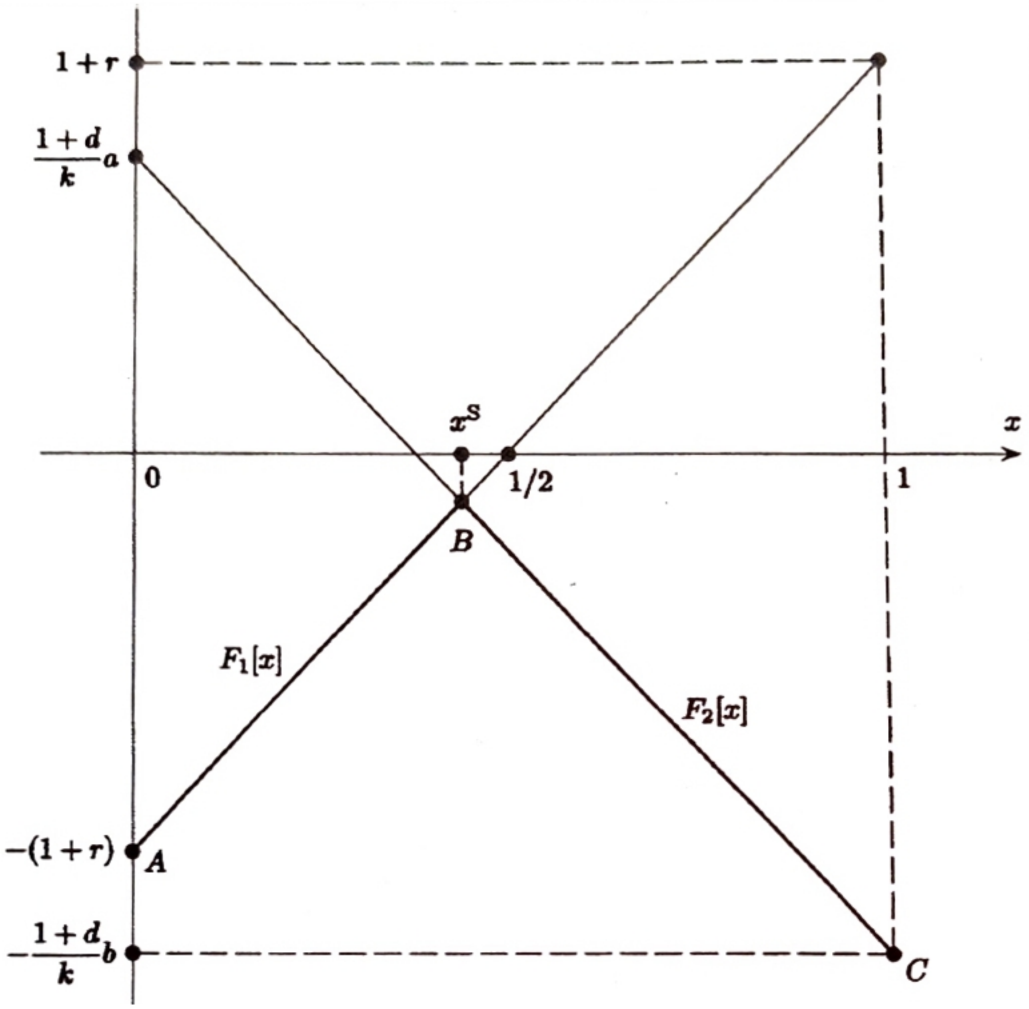

Fig. 2. Graphs of functions defined by (14).

Remark 7. If $\frac{1+r}{1+d}k \leq a$ (case 1), the DM is recommended to invest everything in the foreign currency sub–deposit, because at the end of the year he will obtain the guaranteed minimum income $\frac{1+d}{k}a$ with zero risk (Proposition 2).

If $\frac{1+r}{1+d}k \geq b$ (case 2), the DM is recommended to invest everything in the national currency sub–deposit, which will yield him the guaranteed minimum income $(1+r)$ with zero risk at the end of the year (Proposition 2).

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