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GUARANTEED SOLUTION FOR RISK-NEUTRAL DECISION MAKER: AN ANALOG OF MAXIMIN IN SINGLE-CRITERION CHOICE PROBLEM

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Guaranteed solution for risk-neutral decision maker: an analog of maximin in single-criterion choice problem.

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Abstract. In this article single-criterion choice problems under uncertainty (SCPUs) are considered. The principle of minimax regret and the Savage–Niehans risk function are introduced. A possible approach to solving an SCPU for a decision-maker who simultaneously seeks to increase his outcome and reduce his risk ("to kill two birds with one stone") is proposed. The explicit form of such a solution for the linear-quadratic setup of the SCPU is obtained.

Keywords: *guaranteed solution, single-criterion choice, Savage–Niehans risk, minimax regret, uncertainties*

1. Introduction

In the middle of the twentieth century, American mathematician and statistician, professor Leonard Savage (the University of Michigan) and Swiss economist, professor Jurg Niehans (the University of Zurich) independently proposed an approach to solving a single-criterion problem under uncertainty (SCPU), later called the principle of minimax regret or the Savage-Niehans principle. Along with Wald's principle of guaranteed outcome (maximin), the principle of minimax regret is crucial for guaranteed decision-making in SCPUs. The main role in this principle is played by the regret function, which determines the Savage-Niehans risk in SCPUs. In recent years, such a risk has been widely used in microeconomic analysis and applications. This article proposes a possible approach to solving SCPUs for a risk-neutral decision-maker, who simultaneously seeks to increase his outcome and reduce his risk ("to kill two birds with one stone"). The explicit form of such a solution for the linear-quadratic statement of the SCPU of a fairly general form is obtained.

2. Interval uncertainties

The mathematical model of decision-making under conflict considered below is described by the single-criterion choice problem under uncertainty (SCPU). Note that the case of interval uncertainty will be studied: the decision-maker knows only the ranges of admissible values of uncertain factors, and their probabilistic characteristics are absent, for one reason or another. The uncertainties occur due to the incomplete (inaccurate) information about the practical use of any strategies chosen by the decision maker. For example, an economic system is often subject to unexpected, difficult-to-predict disturbances, both of exogenous origin (the disruption and variation of the quantity (range) of supply, demand fluctuations for the products supplied by a given enterprise, etc.) and endogenous origin (the emergence of new technologies, breakdowns and replacement of equipment, etc.). The question naturally arises: how to take into account the presence of uncertainties when choosing strategies?

The following aspects are described in the economic literature.

First, modern economic systems are characterized by a large number of elements and functional relations between them, a high degree of dynamism, the presence of nonfunctional relations between the elements, and the action of subjective factors due to the participation of individuals or their groups in the operation of such systems; in other words, an economic system usually operates under the uncertainty of its external and internal environment.

Second, as it has been already mentioned, the sources of uncertainties in economic systems are the incomplete or insufficient information about economic processes and their conditions; random or deliberate opposition from other economic agents; random factors that cannot be predicted due to the unexpectedness of their occurrence.

Third, the uncertainties are estimated using deterministic and probabilistic-statistical approaches as well as the approaches based on fuzzy logic.

Interval uncertainties were surveyed in the books [1] – [6] and other publications.

Each type of uncertainty requires its own approach for proper consideration. In this article, the analysis will be restricted to the class of interval uncertainties: only the ranges of admissible values of uncertain factors are known, without any probabilistic characteristics. The uncertainties will be taken into account using the method proposed by V. Zhukovskiy in [7] – [11]. This method allows passing from the original single-criterion choice problem under uncertainty (SCPU) to an equivalent single-criterion choice problem without uncertainty.

3. Principle of minimax regret

Traditionally, one of the most important challenges in the mathematical theory of SCPUs is the development of optimality principles, i.e., the answer to the following questions: What behavior of the decision-maker should be considered optimal (reasonable, appropriate)? Does an optimal solution exist and how can it be constructed? This work gives a possible answer to both questions for SCPUs.

The mathematical theory of games recommends making the concept of stability the cornerstone of optimality: a player's deviation from the optimal strategy introduced below cannot improve but at the same time can worsen his payoff (as well as the associated risk).

Let us proceed to the formal statement. Consider a single-criterion choice problem under uncertainty $\Gamma^{(1)} = \langle X, Y, f(x,y)\rangle$. In $\Gamma^{(1)}$, the decision-maker chooses his alternative $x \in X \subseteq \mathbf{R}^n$, seeking to maximize the value of a scalar criterion $f(x,y)$ for all possible realizations of the uncertainty $y \in Y \subseteq \mathbf{R}^m$. Recall that only the range of admissible values of the uncertainty is known.

The presence of uncertainties leads to the set of outcomes

$$f(x,Y) = \{f(x,y) \mid \forall y \in Y\},$$

that is induced by $x \in X$. The set $f(x,Y)$ can be reduced using risks.

Risk management is a topical problem of economics: in 1990, H. Markowitz [12] was awarded the Nobel Prize in Economic Sciences "for having developed the theory of portfolio choice". What is a proper comprehension of risk? A well-known Russian expert in optimization, T. Sirazetdinov, claims that today there is no rigorous mathematical definition of risk [13, p. 31]. The monograph [14, p. 15] even suggested sixteen possible concepts of risk. Most of them require statistical data on uncertainty. However, in many cases the decision-maker does not possess such information for objective reasons.

Thus, here risks will be understood as possible deviations of realized values from the desired ones. Note that this definition (in particular, Savage-Niehans risk) is in good with the conventional notion of microeconomic; for example, see [15, pp. 40–50].

In 1939 A. Wald, a Romanian mathematician who emigrated to the USA in 1938, introduced the maximin principle, also known as the principle of guaranteed outcome [16, 17]. This principle allows finding a guaranteed outcome in a single-criterion choice problem under uncertainty (SCPU). Almost a decade later, Swiss economist J. Niehans (1948) and American mathematician, economist, and statistician L. Savage (1951) suggested the principle of minimax regret (PMR) for building guaranteed risks in the SCPUs [18, 19]. In the modern literature, this principle is also referred to as the Savage risk or the Savage–Niehans criterion. Interestingly, during World War II Savage worked as an assistant of J. von Neumann, which surely contributed to the appearance of the PMR. Note that the authors of two most remarkable dissertations in economics and statistics are annually awarded the Savage Prize, which was established in the USA as early as 1971.

For the single-criterion choice problem $\Gamma^{(1)} = \langle X, Y, f(x,y)\rangle$, the principle of minimax regret is to construct a pair $\left(x^r, R_f^r\right) \in X \times \mathbf{R}$ that satisfies the chain of equalities

$$R_f^r = \max_{y\in Y} R_f\left(x^r, y\right) = \min_{x\in X}\max_{y\in Y} R_f(x,y) \tag{1}$$

where the Savage–Niehans risk function has the form

$$R_f(x,y) = \max_{z\in X} f(z,y) - f(x,y) \tag{2}$$

The value R_f^r given by (1) is called the Savage-Niehans risk in the problem $\Gamma^{(1)}$. The risk function $R_f(x,y)$ assesses the difference between the realized value of the criterion $f(x,y)$ and its best-case value $\max_{z\in X} f(z,y)$ from the DM's view. Obviously, the DM strives for reducing $R_f(x,y)$ as much as possible with an appropriately chosen alternative $x \in X$, naturally expecting the strongest opposition from the uncertainty in accordance with the principle of guaranteed outcome; see formula (1). Therefore, following (1) and (2), the DM is an optimist who seeks for the best-case value $\max_{x\in X} f(x,y)$. In contrast, the pessimistic DM is oriented towards the worst-case outcome — the Wald maximin solution $(x^0, f^0 = \max_{x\in X}\min_{y\in Y} f(x,y) = \min_{y\in Y} f\left(x^0, y\right))$.

In the sequel, assume that the DM in the problem $\Gamma^{(1)}$ is optimistic: he constructs the Savage-Niehans risk function (2) for $f(x,y)$. Note two important aspects as follows. First, the criterion $f(x,y)$ from $\Gamma^{(1)}$ has its own risk $R_f(x,y)$; see (2). Second, the DM tries to choose alternatives $x \in X$ in order to reduce the risk $R_f(x,y)$, expecting any realization of the strategic uncertainty $y(\cdot) \in Y^X, y(x) : X \to Y$.

Remark 1. The models $\Gamma^{(1)}$ naturally arise, e.g., in economics: a seller in a market is interested to maximize his profits under import uncertainty.

In many publications on macroeconomics [15, 6], all decision-makers are divided into three categories: risk-averse, risk-neutral, and risk-seeking. In this appendix, the DM is assumed to be a risk-neutral person and, as it has been mentioned above, an optimist.

4. Hierarchical interpretation of principle of minimax regret

Consider two hierarchical interpretations as follows. The first arises when the Savage–Niehans risk function $R_f(x,y) = \max_{z\in X} f(z,y) - f(x,y)$ is constructed, whereas the second when the solution $\left(x^r, R_f^r\right) \in X \times \mathbf{R}$ of the problem $\Gamma^{(1)}$ for the risk-seeking DM is obtained.

5. Hierarchical interpretation of Savage-Niehans risk function design

Hierarchical games represent a mathematical model of a conflict with a fixed sequence of moves and information exchange between its parties [20, p.477]. In Russia, the intensive research of hierarchical games was initiated in the second half of the 20 th century by Yu. Germeier [21, 22] (the founder of the Department of Operations Research at the Faculty of Computational Mathematics and Cybernetics, Moscow State University) and then continued by his scholars. Hierarchical two-player games describe the interaction between the upper (Leader) and lower (Follower) levels of the hierarchy. Such games have a given sequence of moves, i.e., an order in which each player chooses his strategies and (possibly) reports them to the partner.

An important element of hierarchical games is to choose the class of admissible strategies depending on the information available to the players. In the theory of hierarchical games, the informational extension of the game was rigorously formulated in [23]. In a particular case, this extension leads the so-called strategic uncertainties, i.e., m-dimensional vector functions $y(x) : X \to Y, y(\cdot) \in Y^X$, which are used along with pure uncertainties $y \in Y$ in the game $\Gamma^{(1)}$.

Now, let us discuss the hierarchical interpretation of risk function design for the SCPU $\Gamma^{(1)}$. Assume that the lower-level player (Follower) can apply only his pure strategy $y \in Y$, whereas the upper-level player (Leader) can adopt "any conceivable information" [23, p.353]. Thus, further analysis will be confined to the Follower's pure strategies $y \in Y$ and the Leader's counterstrategies $x(y) : Y \to X,\ x(\cdot) \in X^Y$, i.e., the set of functions $x(y)$ with Y as the domain of definition and X as the codomain. For risk function design, consider the two-level two-stage hierarchical game

$$\Gamma_R = \left\langle X^Y, Y, f(x,y) \right\rangle$$

In this game, the first move is made by Follower (the lower-level player), who reports his admissible pure strategies to the upper level.

The second move belongs to Leader (the upper-level player), who performs the following actions. First, he analytically constructs the counterstrategy

$$x(y) \in Y(x) = \operatorname{Arg} \max_{x\in X} f(x,y)\ \forall y \in Y,$$

i.e., finds the scalar function $f[y] = f(x(y), y) = \max_{x \in X} f(x, y)$; second, he designs the Savage–Niehans risk function

$$R_f(x, y) = f[y] - f(x, y).$$

Solution of choice problem $\Gamma^{(I)}$ for risk-seeking DM

Assume that the Savage–Niehans risk function has the explicit form $R_f(x, y) = \max_{z \in X} f(z, y) - f(x, y)$, and the problem is to construct a pair $\left(x^r, R_f^r\right) \in X \times \mathbf{R}$ defined as the solution of the SCPU $\Gamma^{(1)}$ for the risk-seeking DM:

$$R_f^r = \min_{x \in X} \max_{y \in Y} R_f(x, y) = \max_{y \in Y} R_f\left(x^r, y\right)$$

In the problem $\Gamma^{(1)}$, suppose that Leader applies only a pure alternative (strategy) $x \in X$, whereas the other player (Follower) can adopt any conceivable information [24, 25], including his knowledge of the strategy $x \in X$, to form his strategy (uncertainty) as a function $y(x) : X \to Y, y(\cdot) \in Y^X$. (This hypothesis is well known as the informational discrimination of Leader.) As a result, the criterion in the choice problem $\Gamma^{(1)}$ is defined as the scalar function $f(x, y(x))$.

Recall that in the theory of differential games, the functions $y(\cdot) \in Y^X$ (the set of m-dimensional vector functions with the domain of definition X and the codomain Y) are called counterstrategies. The problem $\Gamma^{(1)}$ in which counterstrategies describe the behavior of uncertain factors is called the minimax game [24, 25].

Thus, consider the hierarchical two-level three-stage game of two players (Leader and Follower) in which, in contrast to Γ_R, Leader and Follower use a pure strategy $x \in X$ and a counterstrategy $y(x) : X \to Y, y(\cdot) \in Y^X$, respectively.

The first move is made by Leader, who reports his admissible strategies $x \in X$ to the lower level.

The second move is made by Follower, who analytically constructs $y(x)$ in accordance with

$$\max_{y(\cdot) \in Y^X} R_f(x, y) = R_f(x, y(x)) = R_f[x] \ \forall x \in X$$

assuming that the vector function $y(x)$ is unique (e.g., for a scalar function $R_f(x, y)$ that is strictly concave in y for each $x \in X$), and then reports $R_f[x]$ to the upper level.

The third move is made by Leader, who constructs a strategy $x^r \in X$ such that $\min_{x \in X} R_f[x] = R_f\left[x^r\right] = R_f^r$

This three-move game-theoretic framework completely matches the concept of the Leader's guaranteed outcome in the problem $\Gamma^{(1)}$ (in the Germeier sense) if the Follower's payoff function considered in [10, 11, 27] is replaced by $-R_f(x, y)$. Moreover, in the game Γ_R, Leader can calculate the Follower's response and immediately implement the third move if he knows the behavioral rule of the opponent. Once again, note that the analog and modification of this three-move framework is convenient to design the guaranteed solution in outcomes and risks for the risk-seeking DM, both in noncooperative and cooperative conflicts.

Remark 2. The minimax solution for the risk-seeking DM is determined by the pair $\left(x^r, R_f^r = \min_{x\in X} \max_{y(\cdot)\in Y^X} R_f(x,y) = \max_{y\in Y} R_f\left(x^r, y\right)\right)$ for two solutions as follows:
(1) For each alternative $x \in X$, the inner maximum $\max_{y\in Y} R_f(x,y) = R_f(x, y(x)) = R_f[x]$ (see move 2) gives the greatest Savage-Niehans risk of the form

$$R_f[x] = \max_{y\in Y} R_f(x,y) \geq R_f(x,y) \forall y \in Y.$$

In other words, $R_f(x,y)$ cannot exceed $R_f[x]$ for all $y \in Y$, and hence $R_f[x]$ can be considered the DM's guarantee obtained by choosing the alternative x. Note that due to (2), $R_f(x,y) \geq 0$; therefore, the Savage–Niehans risk function takes the values $R_f(x,y) \in [0, R_f[x]]$ for all $(x,y) \in X \times Y$.
(2) Like any DM, the risk-seeking one would like to implement his decisions (the choice of $x \in X$) with the smallest risk (ideally, zero!). This aspect explains his third move.

Therefore, in the problem $\Gamma^{(1)}$ the risk-seeking DM is suggested to use the alternative x^r to obtain the smallest (minimum) guarantee $R_f\left[x^r\right] = R_f\left(x^r, y\left(x^r\right)\right) \geq R\left(x^r, y\right) \forall y \in Y$. The same technique can be applied to formalize the strongly-guaranteed solution in outcomes and risks of the problem $\Gamma^{(1)}$.

Here is an important result from operations research that concerns informed uncertainties and strategies.

Lemma 1. *If in the choice problem $\Gamma^{(1)} = \langle X, Y, f(x,y)\rangle$ the sets X and Y are compact and the criterion $f(x,y)$ is continuous on $X \times Y$, then the maximum (minimum) function* $\max_{x\in X} f(x,y) \left(\min_{y\in Y} f(x,y)\right)$ *is continuous on $Y(X)$.*

Lemma 1 is a well-known fact that can be found in almost any textbook on operations research; for example, see [27].

Remark 3. Lemma 1 implies the continuity of the risk function (2) on $X \times Y$ (of course, only if in the problem $\Gamma^{(1)}$ the sets X and Y are compact and the criterion $f(x,y)$ is continuous on $X \times Y$.)

Remark 4. Assume that in the problem $\Gamma^{(1)}, X \in \operatorname{comp} \mathbf{R}^n, Y \in \operatorname{comp} \mathbf{R}^m$, and $f(\cdot) \in C(X \times Y)$. Then there exists the guaranteed solution in risks $\left(x^r, R_f^r\right)$ of this problem.

Really, the Savage–Niehans risk function $R_f(x,y)$ (2) is continuous on $X \times Y$ (see Remark 2). In this case, by Lemma 1 the function $\max_{y\in Y} R_f(x,y) = R_f[x]$ is also continuous on X. (There exists a Borel measurable counterstrategy (selector) $y(x): X \to Y$ such that

$$\max_{y\in Y} R_f(x,y) = R_f(x, y(x)) = R_f[x] \forall x \in X$$

and $R_f[x]$ is continuous on X). According to the Weierstrass extreme-value theorem, on a compact set X a continuous function $R_f[x]$ achieves minimum at the point $x^r \in X$. If both sets X and Y are compact and the function $f(x, y)$ is continuous, then the guaranteed solution in risks $\left(x^r, R_f^r\right)$ defined by (1) exists.

Thus, using x^r, the risk-seeking DM obtains a guarantee in risks $R_f^r \geq R_f\left(x^r, y\right) \forall y \in Y$, and for all $x \in X$ this guarantee will be smallest among all other guarantees $R_f[x] \geq R_f(x, y)$ for all alternatives $x \in X$. Such a procedure is characteristic of the risk-seeking DM. In this article, we will consider a similar procedure for the risk-neutral DM.

New approach to SCPU for risk-neutral DM: Preliminaries

Let us utilize the approach proposed for noncooperative games in [29]. For this purpose, from the SCPU $\Gamma^{(1)}$ we will pass to the problem of guarantees without any uncertainties.

At conceptual level, the DM's goal so far has been to choose an appropriate alternative maximizing his outcome. But this is not enough for the risk-neutral DM! He seeks for an alternative that would not only increase his outcome but also reduce his risk, as much as possible. Recall that the DM forms the Savage–Niehans risk function $R_f(x, y)$((2)), the value of which is called the DM's risk, and the Savage-Niehans risk R_f^r itself is determined by the chain of equalities (1). The pair $\left(x^r, R_f^r\right)$ is the solution of the choice problem $\Gamma^{(1)}$ for the risk-seeking DM: the value $R_f(x, y)$ characterizes his risk when choosing and implementing the alternative $x \in X$, which he strives to minimize simultaneously with outcome improvement. In this context, two questions arise naturally:
(1) How can we combine the two objectives of the decision-maker (outcome increase with simultaneous risk reduction) using only one criterion?
(2) How can we implement these objectives in a single alternative, in such a way that uncertainty is also accounted for?

How to combine DM's desire to increase outcome and reduce risks?

Recall that, according to the principle of minimax regret, the DM's risk is defined by the value of the Savage–Niehans risk function $R_f(x, y) = \max_{z \in X} f(z, y) - f(x, y)$, where $f(x, y)$ denotes the DM's criterion in the choice problem $\Gamma^{(1)}$. Thus, to construct the risk function $R_f(x, y)$ for the DM, first the dependent maximum $f[y] = \max_{x \in X} f(x, y) \forall y \in Y$ needs to be found. To calculate $f[y]$, following the theory of two-level hierarchical games, assume the discrimination of the lower-level player, who forms the uncertainty $y \in Y$ and sends this information to the upper level for constructing a counterstrategy $x(y) : Y \to X$ such that

$$\max_{x \in X} f(x, y) = f(x(y), y) = f[y] \; \forall y \in Y.$$

The set of such strategies is denoted by X^Y. (Actually, this set consists of n-dimensional vector functions $x(y) : Y \to X$ with the domain of definition Y and the codomain X). Thus, to construct the first term in (2) at the upper level of the hierarchy, we have to solve the single-criterion choice problem $\big(X^Y, Y, f(x,y)\big)$ for each uncertainty $y \in Y$; here X^Y is the set of counterstrategies $x(y) : Y \to X$. The problem itself consists in determining the scalar function $f[y]$ defined by

$$f[y] = \max_{x(\cdot)\in X^Y} f(x,y) \forall y \in Y. \tag{3}$$

Then, the Savage–Niehans risk functions are constructed by formula (2).

Hereinafter, the collection of all compact sets of Euclidean space $\mathbf{R}^k$ is denoted by comp $\mathbf{R}^k$, and if a scalar function $\psi(x)$ on the set X is continuous, we write $\psi(\cdot) \in C(X)$.

The main role in this paragraph will be played by the following result.

Proposition 1. If $X \in \operatorname{comp}\mathbf{R}^n, Y \in \operatorname{comp}\mathbf{R}^m$, and $f(\cdot) \in C(X \times Y)$, then
(1) the maximum function $\max_{x\in X} f(x,y)$ is continuous on Y;
(2) the minimum function $\min_{y\in Y} f(x,y)$ is continuous on X.

Corollary 1. *If in the choice problem* $\Gamma^{(1)}$ *the sets* $X \in \operatorname{comp}\mathbf{R}^n$ *and* $Y \in \operatorname{comp}\mathbf{R}^m$ *and the function* $f(\cdot) \in C(X \times Y)$*, then the Savage–Niehans risk function* $R_f(x,y)$ *is continuous on* $X \times Y$*. (Also, see Remark 3.)*

Let us proceed with the strongly-guaranteed outcome and risk in the SCPU $\Gamma^{(1)}$. In a series of papers [10, 11], three different ways to account for uncertain factors of decision-making in conflicts under uncertainty were proposed. Our analysis below will be confined to one of them presented in [11], based on the following method. We associate with the criterion $f(x,y)$ in the problem $\Gamma^{(1)}$ its strong guarantee $f[x] = \min_{y\in Y} f(x,y)$. As a consequence, choosing his alternatives $x \in X$, the DM ensures an outcome $f[x] \le f(x,y)\ \forall y \in Y$ under any realized uncertainty $y \in Y$. Such a strongly-guaranteed outcome $f[x]$ seems natural for the interval uncertainties $y \in Y$ addressed in this appendix, because no additional probabilistic characteristics of y (except for information on the admissible set $Y \subseteq \mathbf{R}^m$) are available. Proposition 1, in combination with Corollary 1 as well as the continuity of $f(x,y)$ and $R_f(x,y)$ on $X \times Y$, leads to the following result.

Proposition 2. If in the SCPU $\Gamma^{(1)}$ the sets X and Y are compact and the criterion $f(x,y)$ is continuous on $X \times Y$, then the strongly-guaranteed outcome

$$f[x] = \min_{y\in Y} f(x,y) \tag{4}$$

and the strongly-guaranteed risk

$$R_f[x] = \max_{y\in Y} R_f(x,y) \tag{5}$$

are scalar functions that are continuous on X.

Remark 5. First, the meaning of the guaranteed outcome $f[x]$ from (4) is that, for any $y \in Y$, the realized outcome $f(x, y)$ is not smaller than $f[x]$. In other words, using his alternative $x \in X$ in the choice problem $\Gamma^{(1)}$, the DM ensures an outcome $f(x, y)$ of at least $f[x]$ under any uncertainty $y \in Y$. Therefore, the strongly-guaranteed outcome $f[x]$ gives a lower bound for all possible outcomes $f(x, y)$ occurring when the uncertainty y runs through all admissible values from Y. Second, the strongly-guaranteed risk $R_f[x]$ also gives an upper bound for ail Savage–Niehans risks $R_f(x, y)$ that can be realized under any uncertainties $y \in Y$. Really, from (5) it immediately follows that

$$R_f[x] \geq R_f(x, y) \forall y \in Y.$$

Thus, adhering to his alternative $x \in X$, the DM obtains the strong guarantee in outcomes $f[x]$, and simultaneously the strong guarantee in risks $R_f[x]$.

Transition from single-criterion choice problem under uncertainty $\Gamma^{(1)}$ to bi-criteria vector optimization problem

The DM's desire to increase his outcome and simultaneously reduce his risk is described well by the new mathematical model of a bi-criteria choice problem under uncertainty with the two-component vector criterion

$$\Gamma_2 = \langle X, Y, \{f(x, y), -R_f(x, y)\}\rangle .$$

In this model, the sets X and Y are the same as in $\Gamma^{(1)}$. The novelty consists in the transition from the one-component criterion $f(x, y)$ to the two-component criterion $\{f(x, y), -R_f(x, y)\}$, in which $R_f(x, y)$ is the Savage-Niehans risk function for the DM. In the problem Γ_2, the DM chooses an alternative $x \in X$ in order to increase as much as possible the values of both criteria simultaneously, which explains the minus sign of $R_f(x, y)$. Moreover, the DM must expect any realization of the uncertainty $y \in Y$. Note that due to $R_f(x, y) \geq 0$, for all $(x, y) \in X \times Y$ an increase of $-R_f(x, y)$ is equivalent to a decrease of $R_f(x, y)$.

The uncertainty $y \in Y$ in the choice problem Γ_2 is of the interval type. This feature compels the DM to use the available information about the uncertainty, i.e., the limits of its range, being guided by the strongly-guaranteed outcome $f[x]$ (4) and the strongly-guaranteed risk $R_f[x]$ (5). Therefore, it seems natural to pass from $\Gamma^{(1)}$ to the two-component vector optimization problem without uncertainty

$$\Gamma_2^g = \langle X, \{f[x], -R_f[x]\}\rangle$$

in which the DM chooses an appropriate alternative $x \in X$ for maximizing both criteria $f[x]$ and $-R_f[x]$ simultaneously.

For the practical design of the strongly-guaranteed outcome and risk in Γ_2^g, we will employ the mathematical theory of vector optimization, e.g., from [28], with its different approaches and results. Consider an optimal solution of multicriteria problems introduced in 1909 by Italian economist and sociologist V. Pareto [30]. For the problem Γ_2^g, the Pareto maximality (efficiency) of an alternative x^P is reduced to the inconsistency of the system of two inequalities $f[x] \geq f\left[x^P\right], -R_f[x] \geq -R_f\left[x^P\right] \forall x \in X$, in which at least one inequality is strict. This leads to the following notion.

Definition 1. A triplet $\left(x^P, f\left[x^P\right], R_f\left[x^P\right]\right)$ is called a Pareto-maximal strongly-guaranteed solution in outcomes and risks (PSGOR) of the problem Γ_2^g if
(1) the alternative x^P is Pareto-maximal in the problem Γ_2^g;
(2) $f\left[x^P\right]$ is the value of the strongly-guaranteed outcome $f[x] = \min_{y\in Y} f(x,y)$ in the problem Γ_2^g for $x = x^P$;
(3) $R_f\left[x^P\right]$ is the value of the strongly-guaranteed risk $R_f[x] = \max_{y\in Y} R_f(x,y)$ in the problem Γ_2^g for $x = x^P$.

Remark 6. Definition 1 may also involve other optimality principles (Pareto, Geoffrion, Borwein, cone, A-optimality). All these principles as well as connections between different vector optimal solutions were considered in [31].

According to the definition of Pareto maximality,
(1) if x^P is a Pareto-maximal alternative, then for $\bar{x} \neq x^P, \bar{x} \in X$ an increase of value of one criterion will inevitably reduce the value of the other;
(2) there exists no alternative $x \in X$ for which the values of both criteria will increase in comparison with their values for $x = x^P$.

Perhaps the term "Slater maximality" appeared in the Russian literature after the translation [32] of a paper by Hurwitz.

If Pareto optimality is replaced by Slater maximality (weak efficiency), then Definition 1 takes the following form.

Definition 2. A triplet $\left(x^S, f\left[x^S\right], R_f\left[x^S\right]\right)$ is called a Slater-strongly-guaranteed solution in outcomes and risks of the problem Γ_2^g if
(1) the alternative $x^S \in X$ is Slater-maximal in the problem Γ_2^g, i.e., for any $x \in X$ the system of two strict inequalities

$$f[x] > f\left[x^S\right], -R_f[x] > -R_f\left[x^S\right]$$

is inconsistent;
(2) $f\left[x^S\right]$ is the value of the strongly-guaranteed outcome in the problem Γ_2^g for $x = x^S$;
(3) $R_f\left[x^S\right]$ is the value of the strongly-guaranteed risk in the problem Γ_2^g for $x = x^S$.

Any efficient (Pareto-maximal) alternative is also weakly efficient, which follows directly from Definitions 1 and 2. Generally speaking, the converse is false. Also, property (2) of Remark 6 remains valid for the Slater-strongly-guaranteed solution in outcomes and risks of the problem $\Gamma^{(1)}$. The next result seems quite obvious.

Proposition 3. If in the problem Γ_2^g there exists an alternative $x^P \in X$ and values $\alpha, \beta \in (0,1)$ such that x^P maximizes the scalar function $\Phi[x] = \alpha f[x] - \beta R_f[x]$, i.e.,

$$\Phi\left[x^P\right] = \max_{x \in X} \left(\alpha f[x] - \beta R_f[x]\right) \tag{6}$$

then x^P is the Pareto-maximal alternative in the problem Γ_2^g; in other words, for any $x \in X$ the system of two inequalities

$$f[x] \geq f\left[x^P\right], \quad R_f[x] \leq R_f\left[x^P\right] \tag{7}$$

with at least one strict inequality, is inconsistent. (Here $\alpha = \beta = 1$.)

Remark 7. The combination of the criteria (4) and (5) in the form $\Phi[x] = \alpha f[x] - \beta R_f[x]$ is of interest for two reasons. First, even if for $\bar{x} \neq x^P$ we have an increase of the guaranteed outcome $f[\bar{x}] > f\left[x^P\right]$, then due to the Pareto maximality of x^P and the fact that $R_f[\bar{x}] \geq 0$ such an improvement of the guaranteed outcome $f[\bar{x}] > f\left[x^P\right]$ will inevitably lead to an increase of the guaranteed risk $R_f[\bar{x}] > R_f\left[x^P\right]$; conversely, for the same reasons, a reduction of the guaranteed risk $R_f[\bar{x}] < R_f\left[x^P\right]$ will lead to a reduction of the guaranteed outcome $f[\bar{x}] < f\left[x^P\right]$ (both cases are undesirable for the DM). Therefore, the replacement of the bi-criteria choice problem Γ_2^g with the single-criterion choice problem $(X, \Phi[x] = \alpha f[x] - \beta R_f[x])$ matches well the DM's desire to increase $f[x]$ and simultaneously reduce $R_f[x]$. Second, since $R_f[x] \geq 0$ and $\alpha, \beta \in (0,1)$, an increase of the difference $\alpha f[x] - \beta R_f[x]$ also matches the DM's desire to increase the guaranteed outcome $f[x]$ and simultaneously reduce the guaranteed risk $R_f[x]$.

Now, let us answer the second question: how can we combine both objectives of the DM in a single alternative taking into account the existing interval uncertainty? To do this, from the problem $\Gamma^{(1)}$ we will pass sequentially to choice problems Γ_1, Γ_2, and Γ_3 :

$$\begin{gathered}\Gamma_1 = \langle X, Y, \{f(x,y), -R_f(x,y)\}\rangle \\ \Gamma_2 = \langle X, \{f[x], -R_f[x]\}\rangle \\ \Gamma_3 = \langle X, \{\Phi[x] = f[x] - R_f[x]\}\rangle\end{gathered} \tag{8}$$

In all the three choice problems, $x \in X \subseteq \mathbf{R}^n$ denotes the alternative chosen by the DM; $y \in Y \subseteq \mathbf{R}^m$ are uncertainties; the DM's criterion $f(x,y)$ is defined on the pairs $(x,y) \in X \times Y$; in (2), $R_f(x,y)$ means the Savage–Niehans risk function. In the choice problem Γ_1, the criterion has two components - the original criterion $f(x,y)$ of the problem $\Gamma^{(1)}$ and the risk function $R_f(x,y)$ of (2). In the choice problem Γ_2, the original criterion $f(x,y)$ and the risk function $R_f(x,y)$ are replaced by their guarantees $f[x] = \min_{y \in Y} f(x,y)$ and $R_f[x] = \max_{y \in Y} R_f(x,y)$, respectively. Finally, in the choice problem Γ_3, the linear convolution of the guarantees $f[x]$ and $-R_f[x]$ (see Proposition 3) is used instead of the two-component criterion.

Remark 8. Let us discuss the advantages of the solution formalized by Definitions 1 and 2. First, recall that economists divide all decision-makers into three categories: risk-averse, risk-neutral, and risk-seeking. In Definitions 1 and 2, the DM is assumed to be a risk-neutral person, who simultaneously considers the outcome and associated risk. Second, this solution imposes a lower bound on the outcomes and also an upper bound on the risks, $f[x] \leq f\left(x^P, y\right) \forall y \in Y$ and $R_f[x] \geq R_f\left(x^P, y\right) \forall y \in Y$, respectively. Note that the existence and continuity of the guarantees $f[x]$ and $R_f[x]$ are based on the hypotheses $X \in \text{comp } \mathbf{R}^n, Y \in \text{comp } \mathbf{R}^m$, and $f(\cdot) \in C(X \times Y)$; see Proposition 1. Third, an improvement of the Pareto-maximal guaranteed outcome (in comparison with $f\left[x^P\right]$) will inevitably increase the guaranteed risk (in comparison with $R_f\left[x^P\right]$); conversely, a reduction of the risk will inevitably decrease the guaranteed payoff.

Remark 9. Definitions 1 and 2 suggest a constructive method of SGPOR design. It consists of four steps as follows.

Step I. Using $f(x, y)$, find $f[y] = \max_{x \in X} f(x, y)$ and construct the Savage–Niehans risk function $R_f(x, y) = f[y] - f(x, y)$ for the criterion $f(x, y)$.

Step II. Evaluate the strong guarantee in outcomes $f[x] = \min_{y \in Y} f(x, y)$ and also the strong guarantee in risks $R_f[x] = \max_{y \in Y} R_f(x, y)$.

Step III. For the auxiliary choice problem Γ_2, calculate the Pareto-maximal alternative x^P. At this step, Proposition 3 is of assistance.

Then the Pareto-maximal alternative in the auxiliary choice problem Γ_3 is x^P for which

$$\max_{x \in X} \left(f[x] - R_f[x]\right) = f\left[x^P\right] - R_f\left[x^P\right]. \tag{9}$$

Step IV. Using x^P, evaluate the strong guarantees $f\left[x^P\right]$ and $R_f\left[x^P\right]$.

The resulting triplet $\left(x^P, f\left[x^P\right], R_f\left[x^P\right]\right)$ is the requisite SGPOR, which complies with Definition 1, i.e., for the original criterion $f(x, y)$ the alternative x^P leads to a guaranteed outcome $f\left[x^P\right]$ with a guaranteed Savage–Niehans risk $R_f\left[x^P\right]$.

Explicit form of Savage-Niehans risk for linear-quadratic SCPU

Consider the linear-quadratic single-criterion choice problem under uncertainty

$$\Gamma_{lq} = \left\langle \mathbf{R}^n, \mathbf{R}^m, f(x, y) \right\rangle,$$

in which the set of alternatives x coincides with the n-dimensional Euclidean space $\mathbf{R}^n$, the set of uncertainties y is $\mathbf{R}^m$, and the linear-quadratic criterion is given by

$$f(x, y) = x'Ax + 2x'By + y'Cy + 2a'x + 2c'y + d.$$

Here A and C are constant and symmetric matrices of dimensions $n \times n$ and $m \times m$, respectively; B is rectangular constant matrix of dimensions $n \times m$; a and c are constant vectors of dimensions n and m, respectively; finally, d is a constant. As before, the prime denotes transposition. In the problem Γ_{lq}, the DM chooses an appropriate alternative

$x \in \mathbf{R}^n$ in order to maximize the linear-quadratic criterion $f(x, y)$ and simultaneously minimize a risk function under any possible realizations of the uncertainty $y \in \mathbf{R}^m$.

The problem is to design an explicit form of the Savage–Niehans risk function for the linear-quadratic choice problem Γ_{lq} (see Remark 9) and then to obtain the SGPOR. Hereinafter, for a square constant matrix A of dimensions $n \times n$, the inequality $A > 0$ ($A < 0$) means that the quadratic form with the matrix A is positive definite (negative definite, respectively). Also, the following notations will be used below: 0_n as a zero vector of dimension n;

$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$ as the gradient of a scalar function $f(x, y)$ with respect to x under a fixed vector y;

$\frac{\partial^2 f}{\partial x^2} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$ as a Hessian of a scalar function $f(x, y)$ with respect to x under a fixed vector y;

$\det A$ as the determinant of a matrix A;

E_n as an identity matrix of dimensions $n \times n$.

Direct calculations show that

$$\frac{\partial}{\partial x}(x'Ax) = 2Ax, \frac{\partial}{\partial x}(2x'By) = 2By, \frac{\partial}{\partial x}(2a'x) = 2a, \frac{\partial^2}{\partial x^2}(x'Ax) = 2A.$$

Well, let us construct an explicit form of the Savage-Niehans risk function $R_f(x, y)$ for the linear-quadratic choice problem Γ_{lq}; see Stage I from Remark 9.

Step I. Explicit-form design of the the Savage-Niehans risk function $R_f(x, y)$ for the problem Γ_{lq}.

Proposition 4. In the linear-quadratic choice problem Γ_{lq} with a matrix $A < 0$, the Savage–Niehans risk function has the form

$$R_f(x, y) = -(x'A + y'B' + a')\, A^{-1}(Ax + By + a).$$

Proof. An n-dimensional vector function $x(y)$ with the domain of definition $\mathbf{R}^m$ and the codomain $\mathbf{R}^n$ such that

$$\max_{z \in \mathbf{R}^n} f(z, y) = f(x(y), y) \forall y \in \mathbf{R}^m,$$

exists under the sufficient conditions

$$\left.\frac{\partial f(x, y)}{\partial x}\right|_{x=x(y)} = 2Ax(y) + 2By + 2a = 0_n;\ \forall y \in \mathbf{R}^m$$

$$\left.\frac{\partial^2 f(x, y)}{\partial x^2}\right|_{x=x(y)} = 2A < 0.$$

The second condition (inequality) holds due to $A < 0$; from the first condition (identity) it follows that

$$x(y) = -A^{-1}(By + a).$$

Substituting $x = x(y)$ into $f(x, y)$ gives

$$\max_{z\in\mathbf{R}^n} f(z, y) = f(x(y), y) = (y'B' + a')\,A^{-1}(By + a) - 2\,(y'B' + a')\,A^{-1}By + y'Cy$$
$$- 2a'A^{-1}(By + a) + 2c'y + d = -\,(y'B' + a')\,A^{-1}(By + a) + y'Cy + 2c'y + d$$
$$= y'\left[C - B'A^{-1}B\right] y + 2\left(c' - a'A^{-1}B\right) y + \left(d - a'A^{-1}a\right)$$

As a result, the Savage–Niehans risk function can be written as

$$R_f(x, y) = f(x(y), y) - f(x, y) = -x'Ax - 2x'By - 2a'x - y'B'A^{-1}By$$
$$- 2a'A^{-1}By - a'A^{-1}a = -\,(x'A + y'B' + a')\,A^{-1}(Ax + By + a)$$

The proof of this proposition is complete. □

Step II. Construct the function $R_f[x] = \max_{y\in\mathbf{R}^m} R_f(x, y)$.

Proposition 5. In the linear-quadratic choice problem Γ_{lq} with matrices

$$A < 0, \det B \neq 0,$$

the strong guarantee in risks is

$$R_f[x] = \max_{y\in\mathbf{R}^m} R_f(x, y) \equiv 0 \forall x \in \mathbf{R}^n$$

Proof. First of all, the condition $\det B \neq 0$ implies that B is a square matrix, i.e., $n = m$. For finding $R_f[x]$, define an n-dimensional vector function $y(x) : \mathbf{R}^n \to \mathbf{R}^n$ such that

$$\max_{y\in\mathbf{R}^m} R_f(x, y) = R_f(x, y(x)) = R_f[x]\ \forall x \in \mathbf{R}^n.$$

Recall the sufficient conditions of maximum for $y = y(x) : \mathbf{R}^n \to \mathbf{R}^n$:

$$\left.\frac{\partial R_f(x, y)}{\partial y}\right|_{y=y(x)} = -2B'x - 2B'A^{-1}By(x) - 2B'A^{-1}a = 0_m\ \forall x \in \mathbf{R}^n,$$
$$\left.\frac{\partial^2 R_f(x, y)}{\partial y^2}\right|_{y=y(x)} = -2B'A^{-1}B > 0. \tag{10}$$

Since $A < 0$ and $\det B \neq 0$, the following chain of implications is the case:

$$A^{-1} < 0 \Longrightarrow B'A^{-1}B < 0 \Longrightarrow -B'A^{-1}B > 0 \Longrightarrow -2B'A^{-1}B > 0.$$

(In other words, the second condition of (10) is satisfied.)

In view of

$$\left(B'A^{-1}B\right)^{-1} = B^{-1}A\,(B')^{-1},$$

the first condition of (10) gives

$$y(x) = -\left(B'A^{-1}B\right)^{-1}\left(B'x + B'A^{-1}a\right) = -B^{-1}A\left(x + A^{-1}a\right) = -B^{-1}(Ax + a).$$

Then, substituting $y = y(x)$ into $R_f[x]$ yields

$$R_f[x] = R_f(x, y(x)) = -\left(x'A - x'A - a' + a'\right) A^{-1}(Ax - Ax - a + a) \equiv 0\ \forall x \in \mathbf{R}^n,$$

which finally establishes the identity $R_f[x] \equiv 0 \forall x \in \mathbf{R}^n$. □

Continuing Step II (from Remark 9), we find the strong guarantee in outcomes $\min_{y\in Y} f(x, y)$ for

$$f(x, y) = x'Ax + 2x'By + y'Cy + 2a'x + 2c'y + d,$$

that is, $f[x] = \min_{y\in\mathbf{R}^m} f(x, y)$, in the case $A < 0, C > 0$.

Lemma 2. [33] *For any positive definite matrix C of dimensions $n \times n$, there exists a unique positive definite matrix S of dimensions $n \times n$ such that $S^2 = C$. The matrix S is called the square root of the matrix C and denoted by $C^{\frac{1}{2}}$. Moreover, the eigenvalues of the matrix C are the squares of the eigenvalues of the matrix $C^{\frac{1}{2}}$.*

Lemma 3. *For a symmetric matrix $C > 0$ of dimensions $n \times n$, $C^{-1} = [S^2]^{-1} = [S^{-1}]^2$.*

Proof. Indeed, for $S = C^{\frac{1}{2}}$ it follows that

$$C = S \cdot S = S^2 \Longrightarrow C^{-1} = [S \cdot S]^{-1} = S^{-1} \cdot S^{-1} = \left[S^{-1}\right]^2.$$

□

Lemma 4.

$$A < 0 \wedge C > 0 \Rightarrow (A - BCB') < 0\ \forall B \in \mathbf{R}^{n\times m}$$

where $\mathbf{R}^{n\times m}$ is the set of constant matrices of dimensions $n \times m$.

Proof. Really,

$$C > 0 \Rightarrow C^{-1} > 0 \Rightarrow BC^{-1}B' \geq 0\ \forall B \in \mathbf{R}^{n\times m} \Rightarrow -BC^{-1}B' \leq 0\ \forall B \in \mathbf{R}^{n\times m}$$
$$\Rightarrow A - BC^{-1}B' < 0\ \forall B \in \mathbf{R}^{n\times m}.$$

□

Proposition 6. If $A < 0$ and $C > 0$, then

$$f[x] = \min_{y\in\mathbf{R}^m} f(x, y) = x'\left[A - BC^{-1}B'\right]x + 2x'\left[a - BC^{-1}c\right] + d - c'C^{-1}C. \qquad (11)$$

Proof. According to Lemma 2, there exists a matrix S such that $C = S^2$; moreover, $C > 0 \Longrightarrow S > 0 \wedge S = S'$. Due to $S^{-1}S^{-1} = C^{-1}$ (Lemma 3), SS $= C$, and $S^{-1}S = E_n$, it follows that

$$f(x, y) = x'Ax + 2x'By + y'Cy + 2a'x + 2c'y + d = \left\|S^{-1}B'x + Sy + S^{-1}c\right\|^2$$
$$- x'BC^{-1}B'x - 2x'BC^{-1}c - c'C^{-1}c - y'Cy - 2x'By - 2c'y + x'Ax + 2x'By$$
$$+ 2a'x + d \geq x'\left[A - BC^{-1}B'\right]x + 2x'\left[a - BC^{-1}c\right] + d - c'C^{-1}c = f[x]$$

for all $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$, because $\| \cdot \| \geq 0$ by the properties of the Euclidean norm. Using the definition of the strong guarantee in outcomes,

$$f(x, y) \geq f[x] \forall x \in \mathbf{R}^n, y \in \mathbf{R}^m,$$

we finally arrive in (11). □

Steps III-IV (construction of the Pareto-maximal alternative x^P in the problem Γ_2 (5) and calculation of $f\left[x^P\right]$).

As it has been established (see Proposition 5), in the linear-quadratic problem Γ_{lq} with

$$A < 0, m = n, \det B \neq 0, \tag{12}$$

the strong guarantee in risks is $R_f[x] = 0$ for all $x \in \mathbf{R}^n$. Hence, this is also the case for the Pareto-maximal alternative x^P in the problem Γ_3 (8). Therefore, the Pareto-maximal alternative in the linear-quadratic problem Γ_{lq} with the matrices (12) and $C < 0$ can be reduced to the maximization of $f[x]$, i.e.,

$$\max_{x \in \mathbb{R}^n} f[x] = f\left[x^P\right]. \tag{13}$$

Proposition 7. In the linear-quadratic problem Γ_{lq} with

$$A < 0, C > 0, m = n, \det B \neq 0$$

the Pareto-maximal strongly-guaranteed solution is given by

$$x^P = -\left[A - BC^{-1}B'\right]^{-1}\left(a - BC^{-1}c\right) \tag{14}$$

$$f\left[x^P\right] = -\left(a' - c'C^{-1}B'\right)\left[A - BC^{-1}B'\right]^{-1}\left(a - BC^{-1}c\right) + d - c'C^{-1}c. \tag{15}$$

Proof. The alternative x^P defined by (13) exists under the sufficient conditions

$$\left.\frac{\partial f[x]}{\partial x}\right|_{x=x^P} = 2\left[A - BC^{-1}B'\right]x^P + 2\left(a - BC^{-1}c\right) = 0_n, \tag{16}$$

$$\left.\frac{\partial^2 f[x]}{\partial x^2}\right|_{x=x^P} = 2\left[A - BC^{-1}B'\right]^{-1} < 0. \tag{17}$$

Note that (17) is satisfied due to Lemma 4 and $A < 0, C > 0$. In view of $A - BC^{-1}B' < 0$, equality (16) implies

$$x^P = -\left[A - BC^{-1}B'\right]^{-1}\left(a - BC^{-1}c\right).$$

Substituting this alternative x^P into (11) gives

$$\begin{aligned} f\left[x^P\right] &= \left(a' - c'C^{-1}B'\right)\left[A - BC^{-1}B'\right]^{-1}\left[A - BC^{-1}B'\right] \cdot \left[A - BC^{-1}B'\right]^{-1} \\ &\times \left(a - BC^{-1}c\right) - 2\left(a' - c'C^{-1}B'\right)\left[A - BC^{-1}B'\right]^{-1}\left(a - BC^{-1}c\right) + d - c'C^{-1}c \\ &= -\left(a' - c'C^{-1}B'\right)\left[A - BC^{-1}B'\right]^{-1}\left(a - BC^{-1}c\right) + d - c'C^{-1}c. \end{aligned}$$

□

Remark 10. Thus, the following result has been obtained for the class of linear-quadratic SCPUs Γ_{lq} : if the criterion in the linear-quadratic problem

$$\Gamma_{lq} = \langle \mathbf{R}^n, \mathbf{R}^m, f(x,y) = x'Ax + 2x'By + y'Cy + 2a'x + 2c'y + d \rangle$$

satisfies the conditions $A < 0, C > 0$, and $\det B \neq 0$, then the triplet $\left(x^P, f\left[x^P\right], R_f\left[x^P\right]\right)$, where

$$\begin{gathered} x^P = -\left[A - BC^{-1}B'\right]^{-1}\left(a - BC^{-1}c\right) \\ f\left[x^P\right] = -\left(a' - c'C^{-1}B'\right)\left[A - BC^{-1}B'\right]^{-1}\left(a - BC^{-1}c\right) + d - c'C^{-1}c \end{gathered} \tag{18}$$

and

$$R_f\left[x^P\right] = 0$$

is the Pareto-maximal strongly-guaranteed solution of Γ_{lq}.

This result has the following interpretation in terms of game theory: choosing the alternative x^P (18) in the linear-quadratic SCPU Γ_{lq}, the DM obtains the strongly-guaranteed outcome $f\left[x^P\right]$ (18) with the (minimum possible) zero risk $R_f\left[x^P\right] = 0$ (i.e., surely!). Note that by Lemma 4 a considerable part of this outcome is

$$-\left(a' - c'C^{-1}B'\right)\left[A - BC^{-1}B'\right]^{-1}\left(a - BC^{-1}c\right) > 0.$$

Conclusions

The simplest conflict under uncertainty is "the game with nature," where a person (player) has to choose an optimal action (strategy) for a given criterion (e.g., profit). Moreover, each action is accompanied by incomplete or inaccurate information (uncertainty) about the results (outcome) of such an action.

This raises the question of risk associated with the resultants. Here an area of intensive research is focused on a special type of uncertainties (interval), for which the only available information is the ranges of their admissible values, without any probabilistic characteristics. An example of such uncertainties is the diversification problem of a deposit into sub-deposits in different currencies [29].

In Russia, interval uncertainties were called "bad uncertainties" due to the unpredictability of their realizations [34]–[36]. The effect of such uncertainties can be assessed using the Savage–Niehans function for a particular alternative or strategy is a measure of risk.

In this article a solution of the single-criterion choice problem under uncertainty (SCPU) that takes into account, first, the effect of such uncertainties and, second, the DM's desire to increase the outcome and simultaneously reduce the associated risk has been presented. More specifically, the concept of a strong guarantee from [10, 11] has been adopted for introducing a new approach that considers all the three factors of decision-making (uncertainty, outcome, and risk). This approach has been reduced to the construction of the game of guarantees, which contains no uncertainties. For the game of guarantees, a corresponding bi-criteria optimization problem has been designed and solved. In the future, a different approach based on vector guarantees [10, 11] can be used. For a fairly general class of linear-quadratic SCPUs, the new approach proposed above has anyway yielded an explicit form of the strongly-guaranteed solution in outcomes and risks in which the guaranteed risk (and hence any Savage–Niehans risk) is 0.

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