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SYNTHESIS OF EQUILIBRIUM

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Synthesis of Equilibrium.

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Abstract. For a noncooperative N-player normal-form game, we introduce the concept of hybrid equilibrium (HE) by combining the concepts of Nash and Berge equilibria and Pareto maximum. Some properties of hybrid equilibria are explored and their existence in mixed strategies is established under standard assumptions of mathematical game theory (convex and compact strategy sets and continuous payoff functions). Similar results are obtained for noncooperative N-player normal-form games under uncertainty.

Keywords: *uncertainty, mixed strategies, equilibrium, saddle point, Pareto optimality*

Introduction

In 1949 twenty-one years old Princeton University postgraduate J. F. Nash suggested and proved the existence of a solution [1, 2], which subsequently became known as Nash equilibrium (NE). Nash equilibrium has been widely used in economics, military science, policy and sociology. After 45 years, J. Nash together with R. Selten and J. Harsanyi were awarded the Nobel Prize in Economic Sciences «for their pioneering analysis of equilibria in the theory of non-cooperative games.» The point is that NE has stability against arbitrary unilateral deviations of a single player, which explains its success in economic and political applications [3, 4].

Almost every issue of modern journals on operations research, systems analysis, or game theory contains papers involving the concept of Nash equilibrium. However, there are spots on the sun: an obvious drawback of NE is its pronounced selfishness, as each player seeks *to increase his own payoff only.*

The antipode of NE is the concept of Berge equilibrium (BE): each player makes every effort to maximize the payoffs of the other players, neglecting his individual interests. BE was formalized in 1985 by Zhukovskiy [5] as a possible solution of noncooperative N-player games, after a critical analysis of C. Berge's book *Théorie générale des jeux a n personnes* [6] published in 1957 (which explains the term «Berge equilibrium»). In 1995, Russian mathematician K. Vaisman defended his Candidate of Sciences Dissertation entitled «Berge equilibrium» [7] at Department of Applied Mathematics and Control Processes (St. Petersburg State University) under the scientific supervision of Zhukovskiy. This dissertation and Vaisman's early papers [8, 9] attracted the attention of researchers, first in Russia and then abroad. As of today, the number of publications related to this equilibrium has exceeded three hundreds. BE is a good mathematical model for the Golden Rule of ethics («Behave to others as you would like them to behave to you.»). BE is famed for its altruism.

Obviously, these features-selfishness and altruism — are intrinsic (in some proportion) to any individual, including a conflicting party. However, it seems delusive to expect that such a combined solution exists in pure strategies. Therefore, again employing the approach of Borel [10], von Neumann [11], Nash [1] and their followers, we will establish the existence of a combined Nash-Berge equilibrium in mixed strategies. This solution is called a hybrid equilibrium (HE). The main goal of this paper is to prove the existence of HE in mixed strategies. Also note a negative property of NE [12] and BE: the sets of both types of equilibria are internally unstable, i.e., there may exist two (NE or BE) profiles such that the payoff of each player in one of them is strictly greater than in the other. We will remove this undesirable negative feature by adding the Pareto maximality of HE with respect to all other equilibria. Thus, our formalization combines three properties, namely, a HE is

first, a Nash equilibrium;

second, a Berge equilibrium;

third, Pareto-maximal with respect to the other equilibria.

This paper proves the following result: if a noncooperative N-player normal-form game has bounded convex and closed strategy sets of players and continuous payoff functions, then there exists a HE in mixed strategies in this game.

In addition, we obtain sufficient conditions for the existence of HE that are reduced to calculation of a saddle point for a special Germeier convolution of payoff functions.

Finally, the derived results are extended to the case of noncooperative N-player normal-form games under strategic uncertainty. A proper consideration of uncertain factors yields more adequate models of real conflicts, which is testified by numerous publications in this field (recall the over 1 million research works with keywords «mathematical modeling under uncertainty» in Google Scholar).

1. Formalization of Hybrid Equilibrium

Consider the mathematical model of a conflict as a noncooperative N-player normal-form game described by an ordered triplet

$$\Gamma = \langle \mathbb{N}, \{X_i\}_{i\in\mathbb{N}}, \{f_i(x)\}_{i\in\mathbb{N}} \rangle.$$

Here $\mathbb{N} = \{1, 2, \dots, N\}$ denotes the set of players ($N > 1$); each of N players chooses his *strategy* $x_i \in X_i \subseteq \mathbb{R}^{n_i}$, thereby forming a *strategy profile*

$$x = (x_1, \dots, x_N) \in X = \prod_{i\in\mathbb{N}} X_i \subseteq \mathbb{R}^n \qquad (n = \sum_{i\in\mathbb{N}} n_i)$$

in this game; a *payoff function* $f_i(x)$ is defined on the set X, which gives the *payoff* of player i ($i \in \mathbb{N}$). At a conceptual level, each player i in the game Γ is looking for a strategy x_i that would *maximize* his payoff.

A natural approach is to define a solution of the game Γ using a pair

$$(x^*, f(x^*) = f_1(x^*), \dots, f_N(x^*)) \in X \times \mathbb{R}^N,$$

where the strategies of a profile $x^* = (x_1^*, \dots, x_N^*) \in X_1 \times \dots \times X_N = X$ are determined by an optimality principle while the components of the vector $f(x^*)$ specify the corresponding payoffs of players under these strategies. As noted by N. Vorobiev, the founder of the largest national scientific school on game theory, «. . . the practice of games shows that all the optimality principles developed so far directly or indirectly reflect the idea of a stable strategy profile that satisfies these principles. . . » [13, pp. 94]. To introduce the concept of hybrid equilibrium, we will adopt three optimality principles, namely, Nash equilibrium, Berge equilibrium (from the theory of noncooperative games) and Pareto maximum (PM, from the theory of multicriteria choice problems). Interestingly, each of these principles has its own *type of stability*: NE is stable against the unilateral deviations of any player i (i.e., the deviations of x_i from x_i^*); BE is stable against the deviations of all players except for one player i with the payoff function $f_i(x)$ (i.e., the deviations of $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$ from $(x_1^*, \dots, x_{i-1}^*, x_{i+1}^*, \dots, x_N^*)$); finally PM is stable against the deviations of all players (i.e., the deviation of the whole current profile x from the optimal solution x^*). Using the standard notation $(x||z_i) = (x_1, \dots, x_{i-1}, z_i, x_{i+1}, \dots, x_N)$ of noncooperative games, we introduce the following notions.

Definition 1. A strategy profile $x^e = (x_1^e, \dots, x_i^e, \dots, x_N^e) \in X$ is called a Nash equilibrium in the game Γ if

$$\max_{x_i\in X_i} f_i(x^e||x_i) = f_i(x^e) \qquad (i \in \mathbb{N}). \tag{1}$$

Definition 2. A strategy profile $x^B = (x_1^B, \dots, x_i^B, \dots, x_N^B) \in X$ is called a Berge equilibrium in the game Γ if

$$\max_{x \in X} f_i(x||x_i^B) = f_i(x^B) \qquad (i \in \mathbb{N}). \tag{2}$$

Let us associate with the game Γ the N-criteria choice problem

$$\Gamma_c = \langle X, f(x) \rangle,$$

where the set of alternatives X coincides with the set of strategy profiles X in the game Γ and the vector criterion has the form $f(x) = (f_1(x), \dots, f_N(x))$, consisting of the payoff functions $f_i(x)$ of all players $i \in \mathbb{N}$ in the game Γ.

Definition 3. An alternative (here a strategy profile $x \in X$) is Slater (Pareto)-maximal in the problem Γ_c if, for all $x \in X$, the system of inequalities $f_i(x) > f_i(x^*)$ $(i \in \mathbb{N})$ $(f_i(x) \geq f_i(x^P)$ $(i \in \mathbb{N})$, respectively), with at least one strict inequality, is inconsistent.

Corollary 1. *The following sufficient condition of Pareto maximality is obvious: if*

$$\max_{x \in X} \sum_{i \in \mathbb{N}} f_i(x) = \sum_{i \in \mathbb{N}} f_i(x^*) \qquad \forall x \in X, \tag{3}$$

then the strategy profile x^* is Pareto-maximal in the problem Γ_c.

Now, we introduce the central concept.

Definition 4. A pair $(x^*, f(x^*)) \in X \times \mathbb{R}^N$ is called a Pareto hybrid equilibrium (PHE) in the game Γ if the strategy profile x^* is simultaneously a Nash equilibrium and a Berge equilibrium in this game, and also a Pareto-maximal alternative in the multicriteria choice problem Γ_c, i.e., the PHE x^* satisfies the following three conditions:

$$\begin{gathered} \max_{x_i \in X_i} f_i(x^*||x_i) = f_i(x^*) \qquad (i \in \mathbb{N}), \\ \max_{x \in X} f_i(x||x_i^*) = f_i(x^*) \qquad (i \in \mathbb{N}), \\ x^* \text{ is Pareto-maximal in } \Gamma_c. \end{gathered} \tag{4}$$

Remark 1. By Corollary 1, a strategy profile x^* is a PHE in the game Γ if it simultaneously satisfies the three optimality conditions (1)–(3).

Remark 2. By analogy with Definition 4, we may easily introduce the concept of Slater hybrid equilibrium (SHE), by simply replacing the Pareto maximality of x^* with its Slater maximality in the problem Γ_c.

2. Properties of Hybrid Equilibria

Hereinafter, cocomp $\mathbb{R}^n$ stands for the set of convex and compact subsets of $\mathbb{R}^n$ and we write $\phi(\cdot) \in C(X)$ if $\phi(\cdot)$ is a continuous scalar function defined on X.

In this section, the game Γ is assumed to satisfy the conditions

$$X_i \in \text{cocomp}\, \mathbb{R}^{n_i}, \quad f_i(\cdot) \in C(X) \qquad (i \in \mathbb{N}). \tag{5}$$

Property 1. *Under conditions (5), any PHE in the game* Γ *is simultaneously a SHE; the set of all SHE is compact in* $X \times \mathbb{R}^N$ *(possibly, empty).*

Property 1 directly follows from the fact that a Pareto-maximal alternative in the choice problem Γ_c is also Slater-maximal (in general, the converse is not true), while the set of Slater-maximal alternatives X^S in Γ_c is nonempty and compact in X [14, pp. 142].

The sets of Nash and Berge equilibria, X^e and X^B, in the game Γ are also compact in X (perhaps, empty) if assumptions (5) hold. In this case, the intersection of the three compact sets $X^S \bigcap X^e \bigcap X^B = X^*$ is also a compact set in X (again, it may be empty). The compactness of $f(X^*) = \{f(x) | x \in X^*\}$ is an immediate consequence of the continuity of the payoff functions $f_i(x)$ on X $(i \in \mathbb{N})$.

Note that, generally speaking, the set of PHE can be noncompact due to the noncompactness of the set of all Pareto-maximal alternatives X^P in the choice problem Γ_c. Also keep in mind the inclusion $f(X^P) \subseteq f(X^S)$.

Property 2. *Under assumptions (5), the PHE* x^* *satisfies the individual rationality condition, i.e.,*

$$\begin{aligned} f_i(x^*) &\geq \max_{x_i \in X_i} \min_{x_{\mathbb{N}\setminus\{i\}} \in X_{\mathbb{N}\setminus\{i\}}} f_i(x_i, x_{\mathbb{N}\setminus\{i\}}) = \\ &= \min_{x_{\mathbb{N}\setminus\{i\}} \in X_{\mathbb{N}\setminus\{i\}}} f_i(x_i^0, x_{\mathbb{N}\setminus\{i\}}) = f_i^0 \qquad (i \in \mathbb{N}), \end{aligned} \tag{6}$$

where $x = (x_1, \ldots, x_i, \ldots, x_N) = (x_i, x_{\mathbb{N}\setminus\{i\}}), x_{\mathbb{N}\setminus\{i\}} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$ *and* $X_{\mathbb{N}\setminus\{i\}} = \prod_{j \in \mathbb{N}\setminus\{i\}} X_j$ $(\mathbb{N} \setminus \{i\} = 1, \ldots, i-1, i+1, \ldots, N)$.

Indeed, each Nash equilibrium x^* in the game Γ has property (6) (individual rationality), i.e., $f_i(x^*) \geq f_i^0$ $(i \in \mathbb{N})$, where x_i^0 and f_i^0 are the maximin strategy and the payoff of player i, respectively.

Remark 3. As illustrated by Vaisman's counter-example [56, pp. 68–69], individual rationality generally fails for a Berge equilibrium x^B in the game Γ.

Property 3. *A PHE* x^* *is collectively rational in a cooperative* N*-player game without side payments. This is a consequence of the Pareto maximality of the alternative* x^* *in the choice problem* Γ_c.

Remark 4. Individual rationality imposes certain requirements to alliances (coalitions) with other players: player i joins a coalition only if his payoff guaranteed by the coalition is not smaller than the maximin value f_i^0 , which can be achieved by this player independently using the maximin strategy x_i^0 .

Collective rationality drives all players to the largest payoffs (in the vector sense!) — the Pareto maxima.

As x^* is a Nash equilibrium, each player seeks to maximize his payoff.

Berge equilibrium matches an altruistic aspiration of each player to maximize the payoffs of all other players.

Let us note that, the first two requirements (individual and collective rationality) are among the standard criteria of «good» solutions for cooperative N-player games without side payments. At the same time, the properties brought by the Nash and Berge equilibria are new for such games, which (we believe) makes the novel concept of PHE an efficient, «good» solution for the game Γ.

To formulate sufficient conditions for the existence of PHE in the game Γ, we will ensure Pareto maximality in terms of Definition 3 by satisfying equality (3). The sufficient conditions will be based on the original approach from [15]. Let us introduce an N-dimensional vector $z = (z_1, \dots, z_N) \in X$ and the Germeier convolution [16, 17] of the form

$$\begin{gathered}
\phi_i(x,z) = f_i(z||x_i) - f_i(z) \qquad (i \in \mathbb{N}),\\
\phi_{i+N}(x,z) = f_i(x||z_i) - f_i(z) \qquad (i \in \mathbb{N}),\\
\phi_{2N+1}(x,z) = \sum_{j\in\mathbb{N}} f_j(x) - \sum_{j\in\mathbb{N}} f_j(z),\\
\psi(x,z) = \max_{r=1,\dots,2N+1} \phi_r(x,z).
\end{gathered} \tag{7}$$

A saddle point $(x^0, z^*) \in X \times X$ of the scalar function $\psi(x,z)$ (7) is given by the chain of inequalities

$$\psi(x,z^*) \le \psi(x^0,z^*) \le \psi(x^0,z) \qquad \forall x \in X,\ z \in X. \tag{8}$$

Theorem 1. *If (x^0, z^*) is a saddle point of the function $\phi(x,y)$ (8) in the zero-sum two-player game*

$$\Gamma_a = \langle X, Z = X, \psi(x,z)\rangle,$$

then the maximin strategy $z^* \in X$ is a PHE of the game Γ.

Proof. Indeed, formula (7) with $z = x^0$ gives $\psi(x^0, x^0) = 0$. Then, by transitivity,

$$\psi(x, z^*) \leq 0 \qquad \forall x \in X.$$

Using the fact that $\max\limits_{r=1,\ldots,2N+1} \phi_r(x, z^*) \leq 0$ $\forall x \in X$ and (7), we arrive at a set of $2N + 1$ inequalities of the form

$$\begin{aligned} f_i(z^*||x_i) \leq f_i(z^*) &\qquad \forall x_i \in X_i \ (i \in \mathbb{N}), \\ f_i(x||z_i^*) \leq f_i(z^*) &\qquad \forall x \in X \ (i \in \mathbb{N}), \\ \sum_{j\in\mathbb{N}} f_j(x) \leq \sum_{j\in\mathbb{N}} f_j(z^*) &\qquad \forall x \in X. \end{aligned}$$

Here the first N inequalities make $z^* \in X$ a Nash equilibrium in the game Γ (see (1)); the second group of inequalities ensures that z^* is a Berge equilibrium as dictated by (2); finally, the last, $(2N + 1)$th inequality means that z^* is a Pareto-maximal alternative in the choice problem Γ_c.

□

Remark 5. By Theorem 1, the construction of a PHE reduces to the calculation of a saddle point (x^0, z^*) for the Germeier convolution $\psi(x, z)$ (7). Thus, we have developed a *constructive method* of PHE design in the game Γ, which consists of the following steps:

first, define the scalar function $\psi(x, z)$ using formulas (7);

second, find a saddle point (x^0, z^*) of the function $\psi(x, z)$ (see the chain of inequalities (8));

third, calculate the values $f_i(z^*)$ $(i \in \mathbb{N})$.

Then the pair $(z^*, f(z^*) = (f_1(z^*), \ldots, f_N(z^*)))$ is a PHE in the game Γ: each player $i \in \mathbb{N}$ should apply his strategy from the profile z^*, thereby obtaining the payoff $f_i(z^*)$.

Remark 6. The whole complexity of constructing a PHE in the game Γ lies in calculation of the saddle point (x^0, z^*) (8) for the Germeier convolution $\psi(x, z) = \max\limits_{r=1,\ldots,2N+1} \phi_r(x, z)$ (7). The reason is that the maximization of a finite number of functions $\phi_r(x, z)$ $(r = 1, \ldots, 2N+1)$ spoils the differentiability and concavity (or convexity) of the functions $\phi_r(x, z)$, despite the fact that it preserves the continuity of this function on the product $X \times Z$ of the compact sets X and Z. Here we face a situation well described by C. Hermite: «I turn with terror and horror from this lamentable scourge of continuous functions with no derivatives». Thus, it is necessary to develop numerical calculation methods for the saddle point (x^0, z^*) of the Germeier convolution $\max\limits_{r=1,\ldots,2N+1} \phi_r(x, z)$. Unfortunately, to this date we were not able to find any literature devoted to this field of research. In particular, the saddle point calculation problem was not solved at the International Conference on Constructive Nonsmooth Analysis and Related Topics (CNSA-2017, St. Petersburg, May 22–27, 2017) dedicated to the Memory of Professor V. Demyanov.

One must be a rather optimistic person to look for a game Γ (especially with an explicit form of the payoff function) in which a PHE in pure strategies $x_i^* \in X_i$ $(i \in \mathbb{N})$ exists (by Definition 4, the desired strategy profile x^* must be simultaneously a Nash equilibrium and a Berge equilibrium in the game Γ and also a Pareto-maximal alternative in the corresponding choice problem). Thus, employing the approach of Borel [10], von Neumann [11], Nash [1] and their followers, we will extend the set X_i of pure strategies x_i to a set of mixed strategies. Then we will establish the existence of appropriately formalized mixed strategy profiles in the game Γ that satisfy the three requirements of hybrid equilibrium.

As before, cocomp $\mathbb{R}^{n_i}$ stands for the set of all convex and compact (closed and bounded) subsets of the Euclidean n_i-dimensional space $\mathbb{R}^{n_i}$ while $f_i(\cdot) \in C(X)$ means that the scalar function $f_i(x)$ is continuous on X.

Consider again the noncooperative N-player game Γ without side payments. Without special mention, assume that the elements of the ordered triplet Γ satisfy requirements (5), i.e.,

$$X_i \in \operatorname{cocomp} \mathbb{R}^{n_i}, \quad f_i(\cdot) \in C(X) \qquad (i \in \mathbb{N}).$$

For each compact set $X_i \subset \mathbb{R}^{n_i}$ $(i \in \mathbb{N})$, consider the Borel σ-algebra $\mathfrak{B}(X_i)$. Further, consider the Borel σ-algebra $\mathfrak{B}(X)$ for the set $X = \prod_{i \in \mathbb{N}} X_i$ of all strategy profiles, such that $\mathfrak{B}(X)$ contains all Cartesian products of elements from the Borel σ-algebras $\mathfrak{B}(X_i)$ $(i \in \mathbb{N})$.

Within the framework of mathematical game theory, a mixed strategy $\nu_i(\cdot)$ of player i is identified with a *probability measure on the compact set* X_i. By definition [18, p. 271], in the notations of [19, p. 284] a probability measure is a *nonnegative* scalar function $\nu_i(\cdot)$ defined on the Borel σ-algebra $\mathfrak{B}(X_i)$ that satisfies the following two conditions:

1. $\nu_i\left(\bigcup_k Q_k^{(i)}\right) = \bigcup_k \nu_i\left(Q_k^{(i)}\right)$ for any sequence $\{Q_k^{(i)}\}_{k=1}^{\infty}$ of pairwise disjoint elements from $\mathfrak{B}(X_i)$ (*countable additivity*);

2 $\nu_i(X_i) = 1$ (normalization), which implies $\nu_i\left(Q^{(i)}\right) \leq 1$ for all $Q^{(i)} \in \mathfrak{B}(X_i)$.

Denote by $\{\nu_i\}$ the set of all mixed strategies of player i $(i \in \mathbb{N})$.

The product measures $\nu(dx) = \nu_1(dx_1) \cdots \nu_N(dx_N)$, treated in the sense of the well-known definitions from [18, p. 370] (and in the notations of [19, p. 123]), are probability measures on the strategy profile set X. Let $\{\nu\}$ be the set of such probability measures (strategy profiles). Once again, we emphasize that in the construction of the product measure $\nu(dx)$, the role of the σ-algebra of all subsets of the set $X_1 \times \cdots \times X_N = X$ is played by the *smallest* σ-algebra $\mathfrak{B}(X)$ that contains all Cartesian products $Q^{(1)} \times \cdots \times Q^{(N)}$, where $Q^{(i)} \in \mathfrak{B}(X_i)$ $(i \in \mathbb{N})$. The wellknown properties of probability measures [18, p. 254] imply that the sets of all possible measures $\nu_i(dx_i)$ $(i \in \mathbb{N})$ and $\nu(dx)$ are *weakly* *closed and weakly compact* (see [18, pp. 212, 254]). As applied, e.g., to $\{\nu\}$, this means that from any infinite sequence $\{\nu^{(k)}\}$ $(k = 1, 2, \ldots)$ one can extract a subsequence $\{\nu^{(k_j)}\}$ $(j = 1, 2, \ldots)$ which *weakly converges* to a measure $\nu^{(0)}(\cdot) \in \{\nu\}$. In other words, for any continuous scalar function $\psi(x)$ on X,

$$\lim_{j\to\infty} \int\limits_X \psi(x)\nu^{(k_j)}(dx) = \int\limits_X \psi(x)\nu^{(0)}(dx)$$

and $\nu^{(0)}(\cdot) \in \{\nu\}$. Due to the continuity of $\psi(x)$, the $\int\limits_X \psi(x)\nu(dx)$ (the expectations) are well defined; by Fubini's theorem,

$$\int\limits_X \phi(x)\nu(dx) = \int\limits_{X_1} \cdots \int\limits_{X_N} \phi(x)\nu_N(dx_N)\cdots\nu_1(dx_1),$$

and the order of integration can be interchanged.

Let us associate with the game Γ in pure strategies its *mixed extension*

$$\widetilde{\Gamma} = \langle \mathbb{N}, \{\nu_i\}_{i\in\mathbb{N}}, \{f_i[\nu] = \int\limits_X f[x]\nu(dx)\}_{i\in\mathbb{N}}\rangle, \tag{9}$$

where, like in Γ, $\mathbb{N}$ is the set of players while $\{\nu_i\}$ is the set of mixed strategies $\nu_i(\cdot)$ of player i; in game (9), each conflicting party $i \in \mathbb{N}$ chooses its mixed strategy $\nu_i(\cdot) \in \{\nu_i\}$, thereby forming a mixed strategy profile $\nu(\cdot) \in \{\nu\}$; the payoff function of each player i, i.e., the expectation

$$f_i[\nu] = \int\limits_X f_i[x]\nu(dx),$$

is defined on the set $\{\nu\}$.

For game (9), the notion of a PHE x^* (see Definition 4) has the following analog.

Definition 5. A mixed strategy profile $\nu^*(\cdot) \in \{\nu\}$ is called a hybrid equilibrium (HE) in the mixed extension (9) (equivalently, a hybrid equilibrium in mixed strategies in the game Γ) if

1. $\nu^*(\cdot)$ is a Nash equilibrium in the game $\widetilde{\Gamma}$, i.e.,

$$\max_{\nu_i(\cdot)\in\{\nu_i\}} f_i(\nu^*||\nu_i) = f_i(\nu^*) \qquad (i \in \mathbb{N}); \tag{10}$$

2. $\nu^*(\cdot)$ is a Berge equilibrium in game (9), i.e.,

$$\max_{\nu_{\mathbb{N}\setminus\{i\}}(\cdot)\in\{\nu_{\mathbb{N}\setminus\{i\}}\}} f_i(\nu||\nu_i^*) = f_i(\nu^*) \qquad (i \in \mathbb{N}); \tag{11}$$

3. $\nu^*(\cdot)$ is a Pareto-maximal alternative in the N-criteria choice problem

$$\widetilde{\Gamma}_c = \langle \{\nu\}, \{f_i(\nu)\}_{i\in\mathbb{N}} \rangle,$$

i.e., for all $\nu(\cdot) \in \{\nu\}$, the system of inequalities

$$f_i(\nu) \geq f_i(\nu^*) \qquad (i \in \mathbb{N}),$$

with at least one strict inequality, is inconsistent.

Here and in the sequel,

$$\begin{gathered}\nu_{\mathbb{N}\setminus\{i\}}(dx_{\mathbb{N}\setminus\{i\}}) = \nu_1(dx_1)\cdots\nu_{i-1}(dx_{i-1})\nu_{i+1}(dx_{i+1})\cdots\nu_N(dx_N),\\ (\nu||\nu_i^*) = \nu_1(dx_1)\cdots\nu_{i-1}(dx_{i-1})\nu_i^*(dx_i)\nu_{i+1}(dx_{i+1})\cdots\nu_N(dx_N),\\ \nu^*(dx) = \nu_1^*(dx_1)\cdots\nu_N^*(dx_N);\end{gathered}$$

in addition, denote by $\{\nu^*\}$ the set of hybrid equilibria $\nu^*(\cdot)$, i.e., the set of strategy profiles that satisfy the three requirements of Definition 5.

Let us state several results used below for proving the existence of HE in mixed strategies. The following sufficient condition of Pareto maximality is obvious.

Proposition 1. A mixed strategy profile $\nu^*(\cdot) \in \{\nu\}$ is a Pareto-maximal alternative in the choice problem $\Gamma_c = \langle \{\nu\}, \{f_i(\nu)\}_{i\in\mathbb{N}} \rangle$ if

$$\max_{\nu(\cdot)\in\{\nu\}} \sum_{i\in\mathbb{N}} f_i(\nu) = \sum_{i\in\mathbb{N}} f_i(\nu^*). \tag{12}$$

Proposition 2. Consider the game Γ under conditions (5), i.e., the sets X_i are convex and compact and the payoff functions $f_i(x)$ are continuous on $X = X_1 \times \cdots \times X_N$. Let

$\{\nu^e\}$ be the set of Nash equilibria $\nu^e(\cdot)$ that satisfy (10) with $\nu^*(\cdot)$ replaced by $\nu^e(\cdot)$;

$\{\nu^B\}$ be the set of Berge equilibria $\nu^B(\cdot)$ that satisfy (11) with $\nu^*(\cdot)$ replaced by $\nu^B(\cdot)$;

$\{\nu^P\}$ be the set of alternatives $\nu^P(\cdot)$ that satisfy (12) with $\nu^*(\cdot)$ replaced by $\nu^P(\cdot)$ (i.e., ν^P is a Pareto-maximal alternative in mixed strategies in the N-criteria choice problem $\langle \{\nu\}, \{f_i(\nu)\}_{i\in\mathbb{N}} \rangle$).

Then the set $\{\nu^*\}$ of hybrid equilibria $\nu^*(\cdot)$ in the mixed extension $\widetilde{\Gamma}$ of the game Γ is a weakly compact subset of the set of mixed strategy profiles $\{\nu\}$ in the game $\Gamma\{\nu^*\}$ (may be empty).

Proof. Under conditions (5), we have $\{\nu^e\} \neq \emptyset$ as shown by Gliksberg's theorem [30]. Next, the fact $\{\nu^B\} \neq \emptyset$ has been established in the preceding sections of our book. The non-emptiness of the set of Pareto-maximal alternatives, $\{\nu^P\} \neq \emptyset$, can be proved in analogous manner. The intersection of a finite number of weakly compact sets (in our case, three) is also weakly compact, possibly empty.

□

Corollary 2. *Under conditions (5), the set*

$$f(\{\nu^*\}) = \bigcup_{\nu(\cdot)\in\{\nu^*\}} f(\nu), \quad f = (f_1, \dots, f_N),$$

is compact (bounded and closed) in the N-dimensional Euclidean criterion space $\mathbb{R}^N$.

Theorem 2 below establishes the implication (5) $\Rightarrow$ $\{\nu^*\} \neq \emptyset$, which is the central result of Sect. 2.

Proposition 3. Consider game (9) under conditions (5). Then the function $\phi_r(x, z)$ in the formula

$$\psi(x, z) = \max_{r=1,\dots,2N,2N+1} \phi_r(x, z) \tag{13}$$

satisfies the inequality

$$\begin{aligned} \max_{r=1,\dots,2N,2N+1} \int\limits_{X\times X} \phi_r(x, z)\mu(dx)\nu(dz) \leq \\ \leq \int\limits_{X\times X} \max_{r=1,\dots,2N,2N+1} \phi_r(x, z)\mu(dx)\nu(dz) \end{aligned} \tag{14}$$

for any $\mu(\cdot) \in \{\nu\}$ and $\nu(\cdot) \in \{\nu\}$, where

$$\begin{gathered} \phi_i(x, z) = f_i(x||z_i) - f_i(z) \qquad (i \in \mathbb{N}), \\ \phi_j(x, z) = f_j(z||x_i) - f_j(z) \qquad (j \in \{N+1, \dots, 2N\}), \\ \phi_{2N+1}(x, z) = \sum_{i\in\mathbb{N}} [f_i(x) - f_i(z)]. \end{gathered} \tag{15}$$

This proposition was proved in [12].

Remark 7. In fact, formula (3) generalizes the well-known property of maximization: the maximum of a sum does not exceed the sum of the maxima.

Let us state an interesting fact from operations research, which plays a crucial role in the proof of Theorem 2. Consider $2N+1$ scalar functions $\phi_r(x,z)$ $(r=1,\dots,2N,2N+1)$, where $z=(z_1,\dots,z_N)\in Z=X$ and $\phi_j(x,z)$ $(j=1,\dots,2N+1)$ are defined by (15).

Proposition 4. If $2N+1$ scalar functions $\phi_j(x,z)$ $(j=1,\dots,2N+1)$ are continuous on the product $X\times(Z=X)$ of compact sets, then the function

$$\psi(x,z)=\max_{j=1,\dots,2N+1}\phi_j(x,z)$$

is also continuous on $X\times Z$.

The proof of a more general result can be found in many textbooks on operations research, e.g., [20].

Finally, let us establish the central result of Sect. 2 — the existence of a hybrid equilibrium (HE) in mixed strategies under conditions (5).

Theorem 2. *If in the game Γ the sets $X_i\in$ cocomp $\mathbb{R}^{n_i}$ and $f_i(\cdot)\in C(X)$ $(i\in\mathbb{N})$, then there exists a hybrid equilibrium in mixed strategies in this game.*

Proof. Consider an auxiliary zero-sum two-player game

$$\Gamma^a=\langle\{1,2\},\{X,Z=X\},\psi(x,z)\rangle.$$

In the game Γ^a, the set X of strategies x chosen by player 1 (seeking to maximize $\psi(x,z)$) coincides with the set of strategy profiles of the game Γ; the set Z of strategies z chosen by player 2 (seeking to minimize $\psi(x,z)$) coincides with X. A solution of the game Γ^a is a *saddle point* $(x^0,z^B)\in X\times X$; for all $x\in X$ and each $z\in X$, it satisfies the chain of inequalities

$$\psi(x,z^B)\le\psi(x^0,z^B)\le\psi(x^0,z).$$

□

Now, associate with the game Γ^a its mixed extension

$$\widetilde{\Gamma}^a=\langle\{1,2\},\{\mu\},\{\nu\},\psi(\mu,\nu)\rangle,$$

where $\{\nu\}$ and $\{\mu\}=\{\nu\}$ denote the sets of mixed strategies $\nu(\cdot)$ and $\mu(\cdot)$ of players 1 and 2, respectively. The payoff function of player 1 is the expectation

$$\psi(\mu,\nu)=\int\limits_{X\times X}\psi(x,z)\mu(dx)\nu(dz).$$

The solution of the game $\widetilde{\Gamma}^a$ is also a saddle point (μ^0, ν^*) defined by the two inequalities

$$\psi(\mu, \nu^*) \leq \psi(\mu^0, \nu^*) \leq \psi(\mu^0, \nu), \tag{16}$$

for any $\nu(\cdot) \in \{\nu\}$ and $\mu(\cdot) \in \{\nu\}$.

Sometimes, the pair (μ^0, ν^*) is also called the *solution of the game* Γ^a *in mixed strategies.*

Applying Gliksberg's [21] existence theorem of a mixed strategy Nash equilibrium for a noncooperative game of $N \geq 2$ players to the zero-sum two-player game Γ^a, we obtain the following result. In the game Γ^a, suppose the set $X \subset \mathbb{R}^n$ is nonempty, convex and compact and the payoff function $\psi(x, z)$ of player 1 is continuous on $X \times X$ (note that the continuity of $\psi(x, z)$ is assumed in Proposition 4). Then the game Γ^a has a solution (μ^0, ν^*) defined by (16), i.e., there exists a saddle point in mixed strategies in this game.

Using (13), inequalities (16) can be written as

$$\begin{aligned} &\int\limits_{X\times X} \max_{j=1,\ldots,2N+1} \phi_j(x, z)\mu(dx)\nu^*(dz) \leq \\ &\leq \int\limits_{X\times X} \max_{j=1,\ldots,2N+1} \phi_j(x, z)\mu^0(dx)\nu^*(dz) \leq \\ &\leq \int\limits_{X\times X} \max_{j=1,\ldots,2N+1} \phi_j(x, z)\mu^0(dx)\nu(dz) \end{aligned} \tag{17}$$

for all $\nu(\cdot) \in \{\nu\}$ and $\mu(\cdot) \in \{\nu\}$. Using the measure $\nu_i(dz_i) = \mu_i^0(dx_i)$ $(i \in \mathbb{N})$ (and hence $\nu(dz) = \mu^0(dx)$) in the expression

$$\psi(\mu^0, \nu) = \int\limits_{X\times X} \max_{j=1,\ldots,2N+1} \phi_j(x, z)\mu^0(dx)\nu(dz),$$

we obtain $\psi(\mu^0, \mu^0) = 0$ due to (13). Similarly, $\psi(\nu^*, \nu^*) = 0$, and then it follows from (16) that

$$\psi(\mu^0, \nu^*) = 0.$$

The condition $\psi(\mu^0, \mu^0) = 0$ and the chain of inequalities (16) by transitivity give

$$\psi(\mu, \nu^*) = \int\limits_{X\times X} \max_{j=1,\ldots,2N+1} \phi_j(x, z)\mu(dx)\nu^*(dz) \leq 0 \qquad \forall \mu(\cdot) \in \{\nu\}.$$

By Proposition 3, we then have

$$0 \geq \int\limits_{X\times X} \max_{j=1,\dots,2N+1} \phi_j(x,z)\mu(dx)\nu^*(dz) \geq$$
$$\geq \max_{j=1,\dots,2N+1} \int\limits_{X\times X} \phi_j(x,z)\mu(dx)\nu^*(dz).$$

Therefore, for all $j = 1, \dots, 2N+1$,

$$\int\limits_{X\times X} \phi_j(x,z)\mu(dx)\nu^*(dz) \leq 0 \qquad \forall \mu(\cdot) \in \{\nu\}. \tag{18}$$

Consider three cases as follows.

Case I ($j = N, ..., 2N$) Here, by (18), (15) and the normalization of $\mu(\cdot)$, we obtain

$$0 \geq \int\limits_{X\times X} \phi_{N+i}(x,z)\mu^0(dx)\nu(dz) = \int\limits_{X\times X} [f_i(z||x_i) - f_i(z)]\mu^0(dx)\nu(dz) =$$
$$= \int\limits_{X\times X} f_i(z||x_i)\mu^0(dx)\nu(dz) - \int\limits_{X} f_i(z)\mu^0(dx) \int\limits_{X} \nu(dz) =$$
$$= f_i(\mu^0||\nu_i) - f_i(\mu^0) \qquad \forall \nu(\cdot) \in \{\nu\} \qquad (i \in \mathbb{N}).$$

By (10), $\mu^0(\cdot)$ is a Nash equilibrium in the game $\widetilde{\Gamma}$ (equivalently, a Nash equilibrium in mixed strategies in the game Γ).

Case II ($j = 1, \dots, N$) Again, using (18), (15) and the normalization of $\nu(\cdot)$,

$$0 \geq \int\limits_{X\times X} \phi_i(x,z)\mu(dx)\nu^*(dz) = \int\limits_{X\times X} [f_i(x||z_i) - f_i(z)]\mu(dx)\nu^*(dz) =$$
$$= \int\limits_{X\times X_i} f_i(x||z_i)\mu(dx)\nu_i^*(dz) - \int\limits_{X} f_i(z)\mu(dz) \int\limits_{X} \nu^*(dz) =$$
$$= f_i(\mu||\nu_i^*) - f_i(\nu^*) \qquad \forall \mu(\cdot) \in \{\nu\} \;\; (i \in \mathbb{N}).$$

In view of (11), the mixed strategy profile $\nu^*(\cdot)$ is a Berge equilibrium in the game Γ, by Definition 5.

Case III ($j = 2N+1$) Again, using (18), (15) and the normalization of $\nu(\cdot)$ and $\mu(\cdot)$, we have

$$0 \geq \int\limits_{X \times X} \left[\sum_{r \in \mathbb{N}} f_r(x) - \sum_{r \in \mathbb{N}N} f_r(z) \right] \mu(dx)\nu^*(dz) =$$

$$= \int\limits_X \sum_{r \in \mathbb{N}} f_r(x)\mu(dx) \int\limits_X \nu^*(dz) - \int\limits_X \mu(dx) \int\limits_X \sum_{r \in \mathbb{N}} f_r(z)\nu^*(dz) =$$

$$= \sum_{r \in \mathbb{N}} f_r(\mu) - \sum_{r \in \mathbb{N}} f_r(\nu^*) \qquad \forall \mu(\cdot) \in \{\nu\}.$$

By Proposition 1 and (12), the mixed strategy profile $\nu^*(\cdot) \in \{\nu\}$ of the game Γ is a Pareto-maximal alternative in the multicriteria choice problem

$$\widetilde{\Gamma}_c = \langle \{\nu\}, \{f_i(\nu)\}_{i \in \mathbb{N}} \rangle.$$

Thus, we have proved that the mixed strategy profile $\nu^*(\cdot)$ in the game Γ is simultaneously a Nash equilibrium and a Berge equilibrium that satisfies Pareto maximality. Hence, by Definition 5, the mixed strategy profile $\nu^*(\cdot)$ is a hybrid equilibrium in the game Γ.

3. Hybrid Equilibrium in Games Under Uncertainty

Let us augment the mathematical model of a conflict

$$\Gamma = \langle \mathbb{N}, \{X_i\}_{i \in \mathbb{N}}, \{f_i(x)\}_{i \in \mathbb{N}} \rangle$$

by including the influence of uncertain factors $y \in Y$. Assume that these factors take arbitrary values from given ranges without any probability characteristics (e.g., the distribution of y on Y is absent for some reasons). Once again, we emphasize that a proper consideration of uncertainties gives a more adequate description of the decision-making process in economics, ecology, sociology, management, trade, policy, security, and so on. Uncertain factors occur due to incomplete (inaccurate) knowledge about the realizations of strategies chosen by conflicting parties. «There is no such uncertainty as a sure thing.» (R. Burns)[1]. For example, an economic system is subject to almost unpredictable *exogenous disturbances* (forces of nature, disruption of supplies, low qualification or incompetence of economic partners, counteractions of rivals) as well as *endogenous disturbances* (breakdown and failure of industrial equipment, unplanned additional cost and losses of materials, innovations suggested by employees, etc.). New technologies and also anthropogenic and weather changes may cause uncertainty in ecological systems; in mechanical systems, among the sources of uncertainty are weather conditions. «The only

[1]Robert Burns, (1759–1796), was a national poet of Scotland, who wrote lyrics and songs in Scots and in English.

thing that makes life possible is permanent, intolerable uncertainty; not knowing what comes next.» (Ursula K. Le Guin)[2]. Possible approaches to take the effect of uncertain factors into account were the subject of investigations [22, 23] initiated in 2013, which resulted in the book [24]. In this paper, we will use elementary methods to deal with uncertainty.

Consider a noncooperative N-player normal form game under uncertainty

$$\langle \mathbb{N}, \{X_i\}_{i\in\mathbb{N}}, Y, \{f_i(x,y)\}_{i\in\mathbb{N}}\rangle. \tag{19}$$

Compared with the game Γ (which shares the first two components of its ordered triplet with game (19), namely, $\mathbb{N} = \{1, 2, \ldots, N\}$ and the set X_i of pure strategies x_i of player i, $i \in \mathbb{N}$), in this game we have an additional set $Y \subset \mathbb{R}^m$ of uncertain factors y and payoff functions $f_i(x,y)$ that depend on y.

Game (19) runs as follows. Each player $i \in \mathbb{N}$ chooses his individual strategy $x_i \in X_i \subset \mathbb{R}^{n_i}$ $(i \in \mathbb{N})$, which gives a strategy profile $x = (x_1, \ldots, x_N) \in X = \prod_{j\in\mathbb{N}} X_j \subset \mathbb{R}^n$ $(n = \sum_{j\in\mathbb{N}} n_j)$ in this game. Regardless of their choice, an arbitrary uncertainty $y \in Y$ figures in (19). For each player i $(i \in \mathbb{N})$, a payoff function $f_i(x,y)$ is defined on all such pairs $(x,y) \in X \times Y$. At a conceptual level, each player i seeks to maximize his *payoff* $f_i(x,y)$ under any unpredictable realization of the uncertainty $y \in Y$. This last requirement calls for estimating the set

$$f_i(x,Y) = \bigcup_{y\in Y} f_i(x,y)$$

for each player i $(i \in \mathbb{N})$. In turn, for such a multivalued function $f_i(x,Y)$ $(i \in \mathbb{N})$, it is necessary to choose another function $f_i[x]$ that would act as *a guarantee* for any element $f_i(x,y)$ from the set $f_i(x,Y)$. As defined by the Merriam-Webster dictionary, **guarantee is an assurance for the fulfillment of a condition**. A most obvious guarantee for player i in game (19) is the so-called *strong guarantee* [22], provided by the scalar function

$$f_i[x] = \min_{y\in Y} f_i(x,y). \tag{20}$$

Indeed, it follows from (20) that, for each strategy profile $x \in X$,

$$f_i[x] \le f_i(x,y) \qquad \forall y \in Y,$$

i.e., in each strategy profile $x \in X$ the value $f_i(x,y)$ is not smaller than the guarantee $f_i[x]$ under any realization of the uncertainty $y \in Y$.

[2]Ursula K. Le Guin, original name Ursula Kroeber, (1929–2018), was an American writer best known for tales of science fiction and fantasy

Proposition 5. If a scalar function $F(x, y)$ is continuous on the product $X \times Y$ of convex and compact sets X and Y, then the function $f[x] = \min_{y \in Y} F(x, y)$ is continuous on X.

Therefore, all the N strong guarantees $f_i[x]$ (20) are continuous on X under the assumptions $X_i \in \text{comp}\, \mathbb{R}^{n_i}$ $(i \in \mathbb{N})$, $Y \in \text{comp}\, \mathbb{R}^m$ and $f_i(\cdot) \in C(X \times Y)$.

This approach allows us to associate with game (19) under uncertainty the game of guarantees (without uncertainty)

$$\Gamma^g = \langle \mathbb{N}, \{X_i\}_{i \in \mathbb{N}}, \{f_i[x]\}_{i \in \mathbb{N}} \rangle, \tag{21}$$

which coincides with the game from Sect. 1 provided that $f_i(x)$ is replaced by the strong guarantee $f_i[x] = \min_{y \in Y} f_i(x, y)$.

In contrast to (19), here the performance of each player i is assessed using the strong guarantee $f_i[x]$ instead of the payoff function $f_i(x, y)$ itself (this seems quite natural for considering arbitrary realizations $y \in Y$).

Then the following analog of Definition 4 can be suggested for the game under uncertainty (19) with the strong guarantees (20).

Definition 6. A pair $(x^P, f[x^P] = (f_1[x^P], \ldots, f_N[x^P])) \in X \times \mathbb{R}^N$ is called a strongly-guaranteed Pareto hybrid equilibrium in game (19) if

1. the strong guarantees $f_i[x]$ (20) are continuous on X;
2. the strategy profile x^P is simultaneously a Nash equilibrium and a Berge equilibrium in the game of guarantees (21), i.e.,

   $$\max_{x_i \in X_i} f_i[x^P || x_i] = f_i[x^P] \qquad (i \in \mathbb{N}),$$

   and

   $$\max_{x \in X} f_i[x || x_i^P] = f_i[x^P] \qquad (i \in \mathbb{N}),$$

   respectively;
3. the strategy profile x^P is a Pareto-maximal alternative in the N-criteria choice problem $\langle X, \{f_i[x]\}_{i \in \mathbb{N}} \rangle$.

Similarly to Definition 5, we introduce an analog of Definition 6 with a feature that the players use mixed strategies $\nu_i(\cdot)$ $(i \in \mathbb{N})$ in game (19).

Definition 7. A mixed strategy profile $\nu^P(\cdot) \in \{\nu\}$ is called a strongly-guaranteed Pareto hybrid equilibrium in mixed strategies in game (19) if

1. for each player i $(i \in \mathbb{N})$, there exists the strong guarantee

$$f_i[x] = \min_{y \in Y} f_i(x, y)$$

that is continuous on X;

2. ν^P is simultaneously a Nash equilibrium and a Berge equilibrium in game (9), i.e., equalities (10) and (11) hold with $\nu^*(\cdot)$ replaced by $\nu^P(\cdot)$;
3. ν^P in game (9) is a Pareto-maximal alternative in the N-criteria choice Problem $\widetilde{\Gamma}_c = \langle \{\nu\}, \{f_i[\nu]\}_{i \in \mathbb{N}} \rangle$.

Finally, the combination of Proposition 5 and Theorem 1 directly leads to the following result on the existence of a strongly-guaranteed Pareto hybrid equilibrium in mixed strategies.

Theorem 3. *Consider game (19) with convex and compact sets X_i $(i \in \mathbb{N})$, compact set Y, and payoff functions $f_i(x, y)$ $(i \in \mathbb{N})$ continuous on $X \times Y$. Then there exists a strongly-guaranteed Pareto hybrid equilibrium in mixed strategies in this game.*

Remark 8. Our analysis in Sect. 3 has been confined to the strong guarantees $f_i[x] = \min_{y \in Y} f_i(x, y)$ $(i \in \mathbb{N})$ as the smallest ones. It is possible to adopt the so-called vector guarantees: the components of an N-dimensional vector $f[x] = (f_1[x], \ldots, f_N[x])$ form *a vector guarantee* for an N-dimensional vector $f(x, y) = (f_1(x, y), \ldots, f_N(x, y))$ if, for all $y \in Y$ and each $x \in X$, the N strict inequalities

$$f_i(x, y) < f_i[x] \qquad (i \in \mathbb{N})$$

are inconsistent. In other words, the vector guarantee $f[x]$ cannot be reduced simultaneously in all the components by choosing $y \in Y$. In terms of vector optimization, for each alternative $x \in X$ the vector $f[x]$ is a Slater minimum (weakly efficient) solution in the N-criteria choice problem $\Gamma(x) = \langle Y, f(x, y) \rangle$.

In the same fashion, using other concepts of vector optima (minima in the sense of Pareto, Geoffrion, Borwein, cone optimality), we may introduce a whole collection of vector guarantees. These guarantees have the remarkable property that their values, first, are not smaller than the corresponding components of the strong guarantee vector $f[x]$ (20) but, second, can be large. Recall that the goal is to increase the payoffs of players (in particular, by increasing their guarantees!). In this respect, the listed vector guarantees are preferable to their strong counterparts. However, one should keep in mind an important aspect: transition from the game under uncertainty (19) to the game of guarantees Γ^g (with subsequent application of Theorem 1) is possible only if the new payoff functions $f_i[x]$ $(i \in \mathbb{N})$ in the game Γ^g are continuous. This continuity can be ensured in the following way.

Let $X_i \in \text{comp}\,\mathbb{R}^{n_i}$, $Y \in \text{comp}\,\mathbb{R}^m$ and $f_i(\cdot) \in C(X \times Y)$ $(i \in \mathbb{N})$ in game (19). In addition, require that for each $x \in X$ at least one $f_j(x, y)$ $(j \in \mathbb{N})$ is *strictly convex* in y on the set Y. Then the minimum in

$$\min_{y \in Y} f_j(x, y) = f_j[x] \tag{22}$$

is achieved at a unique point $y^*(x)$ for each $x \in X$, and the m-dimensional vector function $y^*(x)$ itself is continuous on X. In this case, the superposition of the continuous functions $f_i(x, y)$ and $y^*(x)$ implies the continuity of all scalar functions $f_i[x] = f_i(x, y^*(x))$ $(i \in \mathbb{N})$. We finalize the design of Γ^g with the following fact. Assume for each $x \in X$ the same function $f_j[x]$ is implemented by the minimum in (22). Then for all $x \in X$ the N-dimensional vector $f[x] = (f_1[x], \ldots, f_N[x])$ is a Slater-minimal alternative in the current N-criteria choice problem $\Gamma(x) = \langle Y, \{f_i(x, y)\}_{i \in \mathbb{N}} \rangle$. In other words, it is impossible to find $\bar{y} \in Y$ such that $f_i(x, \bar{y}) < f_i[x]$ $(i \in \mathbb{N})$. A detailed treatment of these issues for Slater, Pareto, Geoffrion, Borwein, and cone optimality will be given in our future publications.

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