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Some Lectures on Macroeconomics

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28 December 2023

Online at https://mpra.ub.uni-muenchen.de/119643/
MPRA Paper No. 119643, posted 13 Jan 2024 09:02 UTC

Some Lectures on Macroeconomics

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December 28, 2023

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Contents

Introduction

We present some theoretical frameworks used in macroeconomics, mainly economic growth. Our goal is not to provide a unified model but rather to present simplified versions of models in the classic academic articles to help students familiarize with theoretical models used in macroeconomics and taught in most graduate programs.

In each section, we explain our objective and motivation illustrated by empirical data. We then build a theoretical model allowing us to address our question.

We hope that this lecture would be useful for readers (not necessarily having a good background in economics) who want to follow the graduate programs in economics. However, readers should have a background in finite-dimensional optimization and discrete-time dynamical systems.

Chapter 1

Short-term analysis: IS-LM model

This section is based on Chapters 3, 4, 5, 6 of Blanchard (2017) with slight modifications. We will present a framework (IS-LM model) to think about how output and the interest rate are determined in the short run. Our analysis will allow us to answer the question: "what would happen if the government was to change its spending and taxes or the central bank was to change money supply/interest rate?"

For the sake of simplicity, we consider a closed economy (no trade or export=import). In the short run, prices are fixed or sticky, and as we will see demand determines output. Many factors (e.g., consumer confidence, fiscal and monetary policy,...) affect demand.

1.1 The goods market and the IS curve

The demand for goods

The total demand for goods is $C + I + G + X - M$. Denote it by Z.

$$Z \equiv C + I + G + X - M \tag{1.1}$$

Since we are considering a closed economy (i.e., no trade), we have $X = M = 0$. So, the demand for goods is the sum of consumption, investment, and government spending

$$\text{Total demand: } Z \equiv C + I + G. \tag{1.2}$$

Let us discuss each of these components.

- Assume that consumption is an increasing function of the disposable income Y_D. Then, for the sake of simplicity, we can assume that disposable income equals income minus taxes: $Y_D = Y - T$. By consequence, we have

  $$C = C(Y - T). \tag{1.3}$$

  C increases when the income Y increases and/or taxes T increase.[1]

  Remark 1. *If the consumption function has the form $C = c_0 + c_1(Y - T)$, then the parameter c_1 is called the propensity to consume. (It is also called the* **marginal propensity to consume***.)*

[1] Alternatively, we can assume that consumption also depends on interest rate (this is interest rate between present and future) because consumers face a trade-off between present consumption and future consumption. If the interest rate i increases, the present consumption C decreases. In this case, we write $C = C(Y - T, i)$.

- **Investment** depends primarily on two factors: The level of sales (which is the output Y) and the interest rate i.

$$I = \underset{(+,-)}{I(Y, i)} \tag{1.4}$$

Notation + means that an increase in output leads to an increase in investment while notation − means that an increase in the interest rate leads to a decrease in investment.

- **Government spending**. Government spending G and T describe fiscal policy - the choice of taxes and spending by the government. We will take G and T as exogenous. But the reason why we assume G and T are exogenous is the following. Governments do not behave with the same regularity as consumers or firms, so there is no reliable rule we could write for G or T corresponding to the rule we wrote, for example, for consumption.

To sum up, the total demand is

$$Z = C(Y - T) + I(Y, i) + G \tag{1.5}$$

It depends on income Y, interest rate i, taxes T, and government spending G. By the way, we may write $Z = Z(Y, i, T, G)$. For a given value of the interest rate i, demand is an increasing function of output because both consumption and investment are increasing functions of output. More precisely, we can explain as follows:

- An increase in output leads to an increase in income and thus to an increase in disposable income. The increase in disposable income leads to an increase in consumption.
- An increase in output also leads to an increase in investment. This is the relation between investment and production that we have just discussed.

Equilibrium on the goods market and IS curve

The supply of the goods market is the output or production, that is Y. The equilibrium in the goods market means that the supply of goods Y be equal to the demand for goods Z. So, we obtain the following equation

$$\text{Goods market equilibrium: } Y = C(Y - T) + I(Y, i) + G \tag{1.6}$$

Given T and G, equation (1.6) represents a relationship between interest rate i and output Y.

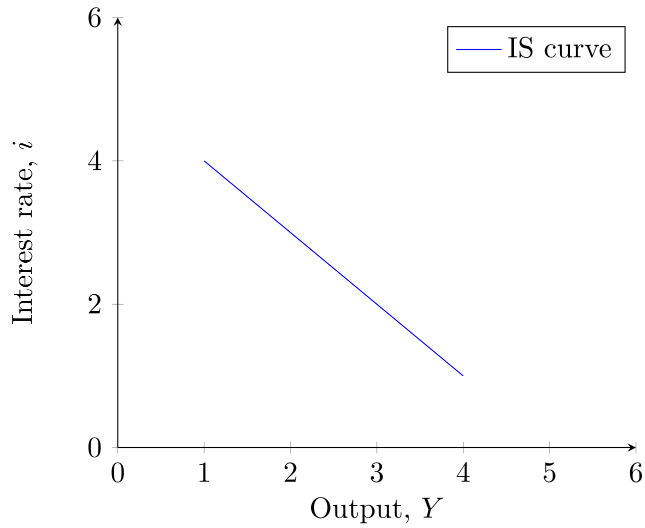
As we have seen in equation (1.5), the higher the interest rate, the lower the output. The economic intuition is that: If interest rate increases, then investment decreases. The decrease in investment leads to a decrease in output which further decreases consumption and investment. So, the relation between the interest rate and output is represented by the down-ward-sloping curve as follows.

Shifts of the IS curve

We want to answer the question: How will the relation between the interest rate and output change if taxes T and government spending G change?.

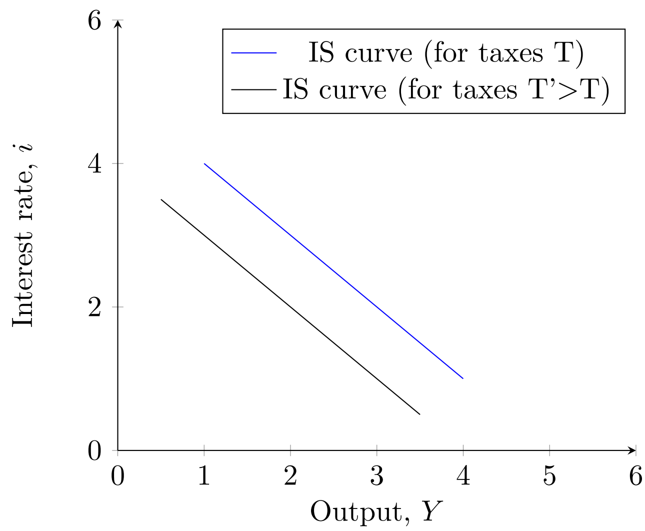
Consider an increase in taxes. We will show that the IS curve shifts to the left (see Figure 1.2). To do so, we have to check that: at a given interest rate, an increase in taxes leads to a

Figure 1.1: IS curve



decrease in equilibrium output. Indeed, at a given interest rate, disposable income $(Y - T)$ decreases, leading to a decrease in consumption $C(Y - T)$, leading in turn to a decrease in the demand for goods Z. At equilibrium, demand equals supply, hence equilibrium output decreases.

Figure 1.2: An increase in taxes shifts the IS curve to the left.



Let us summarize:

- Equilibrium in the goods market implies that an increase in the interest rate leads to a decrease in output. This relation is represented by the downward-sloping IS curve.
- Changes in factors that decrease the demand for goods given the interest rate shift the

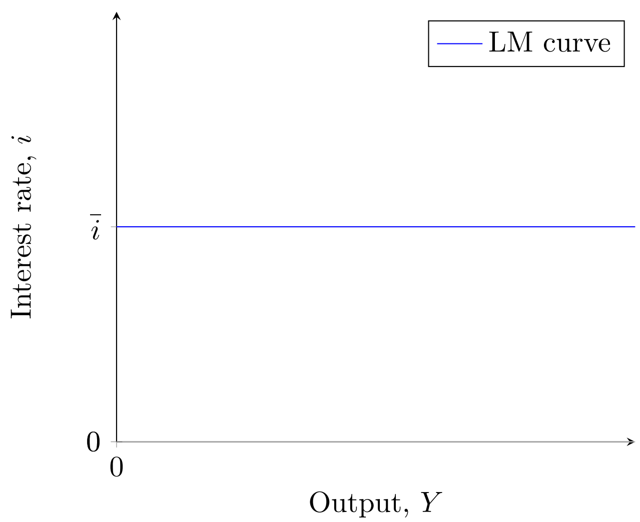
IS curve to the left. Changes in factors that increase the demand for goods given the interest rate shift the IS curve to the right.

1.2 Financial Markets and the LM Relation

The LM curve

The LM curve (LM-Liquidity Money) represents the relation between the interest rate and output. We assume that the central bank chooses an interest rate, call it, $\bar{i}$, and adjust the money supply so as as, under our assumption, the to achieve it. The LM curve is a horizontal line as in Figure 1.3.[2]

Figure 1.3: LM curve



Monetary policy in practice

The transmission of monetary policy of the ECB[3] is illustrated by Figure 1.4. The transmission of monetary policy of the Federal Reserve[4] is summarized in Figure 1.5.

In the Fed, the Federal Open Market Committee (FOMC) sets a target for the federal funds interest rate and attempts to hit the target by buying or selling government securities.[5] We refer to the Fed's purchase of government securities as expansionary monetary policy and its sale of government securities as contractionary monetary policy.

- Expansionary Monetary Policy
    1. When the Fed buys government securities through securities dealers in the bond market, it deposits the payment into the bank accounts of the banks, businesses, and individuals who sold the securities.

[2] Another version of the LM curve is describe by the following equation $M = PY \times L(i)$, where M is the money demand, P is the level of price, and L is a decreasing function of i.

[3] Source: https://www.ecb.europa.eu/mopo/intro/transmission/html/index.en.html

[4] https://www.federalreserve.gov/monetarypolicy/monetary-policy-what-are-its-goals-how-does-it-work.htm

[5] Source: https://www.stlouisfed.org/in-plain-english/a-closer-look-at-open-market-operations

Figure 1.4: Transmission mechanism of monetary policy (ECB)

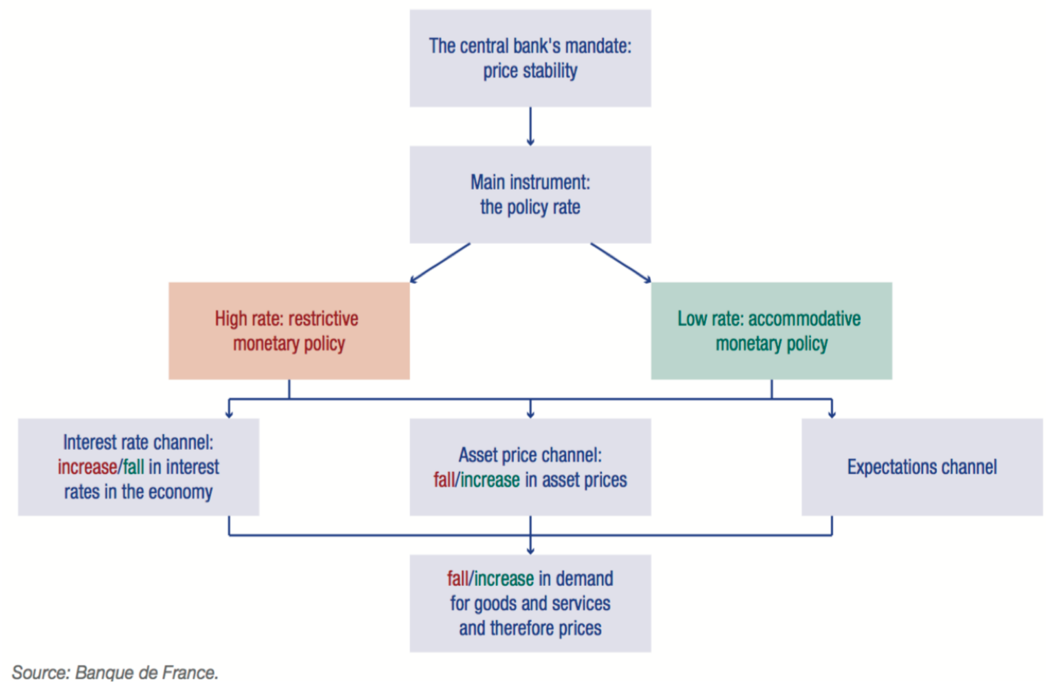
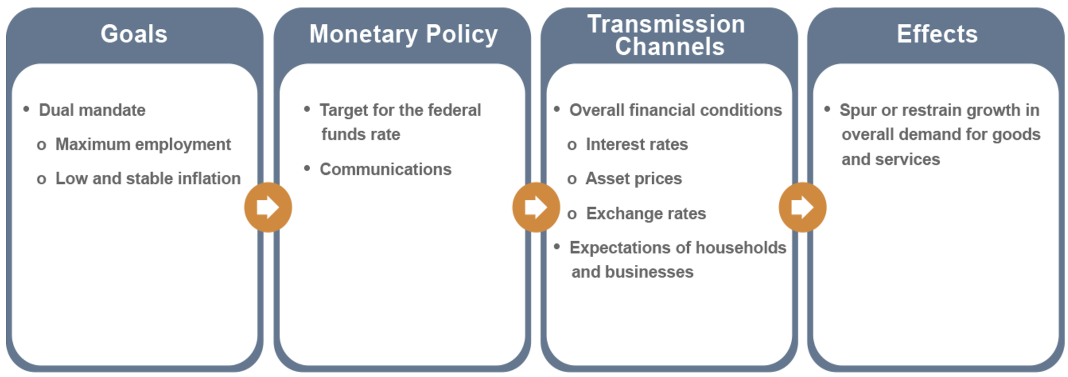


Figure 1.5: The Transmission of Monetary Policy of the Federal Reserve



2. Those deposits become part of the funds commercial banks hold at the Federal Reserve and thus part of the funds commercial banks have available to lend.
3. Because banks want to lend money, to attract borrowers they decrease interest rates, including the rate banks charge each other for overnight loans (the federal funds rate).

- Contractionary Monetary Policy
  1. When the Fed sells government securities, buyers pay from their bank accounts, which decreases the amount of funds held in their bank accounts.
  2. Banks then have less money available to lend.
  3. When banks have less money to lend, the price of lending that money - the interest rate - goes up, and that includes the federal funds rate.

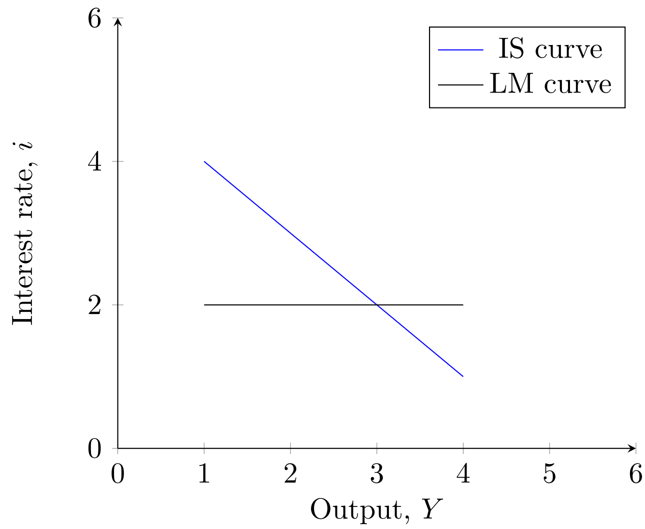
1.3 The IS-LM model and applications

We now put the IS and LM relations together. At any point in time, the supply of goods must be equal to the demand for goods (equilibrium in the goods market), and the supply of money must be equal to the demand for money (equilibrium in the money market). So, both the IS and LM relations must hold. Together, they determine both output and the interest rate.

$$\text{Goods market: } Y = C(Y-T) + I(Y,i) + G \tag{1.7}$$
$$\text{Financial market: } i = \bar{i}. \tag{1.8}$$

Graphically, we have Figure 1.6.

Figure 1.6: IS-LM model (equilibrium in both markets)



Assumption 1. *The function C is increasing. The function $I(Y,i)$ is increasing in Y and decreasing in i. Assume also they are continuously differentiable and $1-C'(Y-T)-\frac{\partial I}{\partial Y}(Y,i) > 0, \forall Y, T, i$.*

Fiscal and monetary policy

Let us summarize as follows

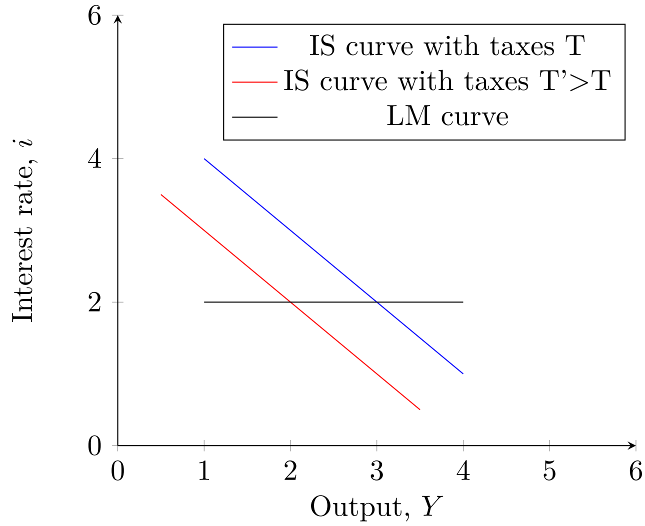
- Fiscal Policy:
  - Decrease in $G-T \iff$ fiscal contraction (or fiscal consolidation)
  - Increase in $G-T \iff$ fiscal expansion
- Monetary Policy:
  - Decrease in $i \iff$ increase in $M \iff$ monetary expansion
  - Increase in $i \iff$ decrease in $M \iff$ monetary contraction (or monetary tightening)

Application: the effects of fiscal policy

Firstly, we look at the effects of an increase of taxes. Since taxes do not appear in the LM relation, they do not shift the LM curve.

Literally, we can say that an increase in taxes shifts the IS curve to the left and leads to a decrease in the equilibrium level of output.

Figure 1.7: Using the IS-LM model: The Effects of an Increase in Taxes



The increase in taxes leads to lower disposable income, which causes people to decrease their consumption. This decrease in demand leads, in turn, to a decrease in output and income. At a given interest rate, the increase in taxes leads therefore to a decrease in output.

Looking at the components of output: The decrease in income and the increase in taxes both contribute to the decrease in disposable income and, in turn, a decrease in consumption. The decrease in output leads to a decrease in investment. Thus, both consumption and investment decrease.

Formally, we have, in equilibrium,

$$Y = C(Y - T) + I(Y, \bar{i}) + G.$$

Of course, we need to impose some conditions about $\bar{i}$, G, T so that this equation has a solution.

Then, taking derivatives of both sides, we get that

$$\frac{\partial Y}{\partial T} = C'(Y - T)\Big(\frac{\partial Y}{\partial T} - 1\Big) + \frac{\partial I}{\partial Y}\frac{\partial Y}{\partial T} \tag{1.9}$$

$$\frac{\partial Y}{\partial T}\Big(1 - C'(Y - T) - \frac{\partial I}{\partial Y}(Y, \bar{i})\Big) = -C'(Y - T) \tag{1.10}$$

$$\frac{\partial Y}{\partial T} = \frac{-C'(Y - T)}{1 - C'(Y - T) - \frac{\partial I}{\partial Y}(Y, \bar{i})} \tag{1.11}$$

So, $\frac{\partial Y}{\partial T} < 0$ if $C'(Y - T) > 0$ and $1 - C'(Y - T) - \frac{\partial I}{\partial Y}(Y, \bar{i}) > 0$.

We then find that

$$\frac{\partial C}{\partial T} = C'(Y-T)\Big(\frac{\partial Y}{\partial T} - 1\Big) = \frac{-C'(Y-T)\Big(1 - \frac{\partial I}{\partial Y}(Y,\bar{i})\Big)}{1 - C'(Y-T) - \frac{\partial I}{\partial Y}(Y,\bar{i})} < 0 \tag{1.12}$$

$$\frac{\partial I}{\partial T} = \frac{\partial I}{\partial Y}\frac{\partial Y}{\partial T} = \frac{\partial I}{\partial Y}\frac{-C'(Y-T)}{1 - C'(Y-T) - \frac{\partial I}{\partial Y}(Y,\bar{i})} < 0. \tag{1.13}$$

Consider a particular case where $C(Y-T) = c_0 + c_1(Y-T)$ where $c_0 > 0, c_1 \in (0,1)$, and $I(Y,i) = \bar{I}, \forall Y, i$. In this case, we have

$$\frac{\partial Y}{\partial T} = \frac{-c_1}{1-c_1} < 0. \tag{1.14}$$

An increase of taxes decreases the output.

Exercise 1. *Find an expression of* $\frac{\partial Y}{\partial G}$. *Provide conditions under which* $\frac{\partial Y}{\partial G} > 0$ *and interpret the result.*

Proof. We have

$$\frac{\partial Y}{\partial G} = \frac{1}{1 - C'(Y-T) - \frac{\partial I}{\partial Y}(Y,\bar{i})} \tag{1.15}$$

This is the so-called **fiscal multiplier**.

If we assume that $1 - C'(Y-T) - \frac{\partial I}{\partial Y}(Y,\bar{i}) > 0$, then $1 - C'(Y-T) - \frac{\partial I}{\partial Y}(Y,\bar{i}) \in (0,1)$ which implies that $\frac{\partial Y}{\partial G} > 1$. It means that if the government increases its spending by 1 usd, then the output increases more than 1 usd.

Consider a particular case where $C(Y-T) = c_0 + c_1(Y-T)$ where $c_0 > 0, c_1 \in (0,1)$, $I(Y,i) = \bar{I}, \forall Y, i$, we find the fiscal multiplier

$$\frac{\partial Y}{\partial G} = \frac{1}{1-c_1} = 1 + c_1 + c_1^2 + c_1^3 + \cdots \tag{1.16}$$

□

Application: the effects of a monetary expansion (i decreases)

Now turn to monetary policy. Suppose the central bank decreases the interest rate. Recall that, to do so, it increases the money supply, so such a change in monetary policy is called a monetary expansion.

A change in i does not shift the IS curve but shifts the LM curve down. The economy moves along the IS curve.

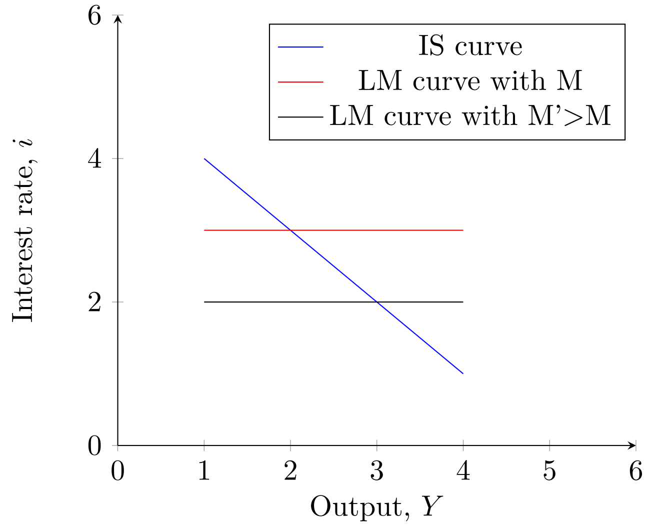
The economy moves down along the IS curve, and the equilibrium moves from point A to point A'. Output increases from Y to Y', and the interest rate decreases from i to i'.

We can explain by words as follows: The lower interest rate leads to an increase in investment and, in turn, to an increase in demand and output.

Looking at the components of output:

- The increase in output and the decrease in the interest rate both lead to an increase in investment. Formally, we have $I' = I(Y', \bar{i}') > I = I(Y, \bar{i})$.
- The increase in income leads to an increase in disposable income and, in turn, in consumption. Formally, we have $C' = C(Y'-T) > C = C(Y-T)$.

Figure 1.8: Using the IS-LM model: The Effects of a Monetary Expansion (M increases



So, both consumption and investment increase.

Example 1 (monetary-fiscal policy mix)**.** *Using the above framework, investigate the following questions.*

- *The Effects of a Combined Fiscal and Monetary Expansion.*
- *The Effects of a Combined Fiscal Consolidation and a Monetary Expansion*

Appendix: quarterly GDP vs annual GDP

In the US, the Bureau of Economic Analysis (BEA) estimates both quarterly and annual GDPs.

Let us look at the following example. According to the Bureau of Economic Analysis

– The annual GDP of the US of the year 2018 is 20,580.223 Billions of Dollars.

– The quarterly GDP of the US of the fourth quarter of 2018 is 20,897.804 Billions of Dollars.

However, this does not mean that the US produces more during the period 10/2018-12/2018 than during the period 01/2018-12/2018. The reason is that the quarterly GDP of the US is annualized (in the above example, the market valued added of the US of the fourth quarter of 2018 is equal to $\frac{20,897.804}{4}$ Billions of Dollars).

Chapter 2

Economic growth: an overview

2.1 Introduction and the importance of economic growth

Economic growth is the steady increase in aggregate output over time. Quantitatively, it is the increase in the inflation-adjusted market value of the goods and services produced by an economy over time. The rate of growth of real GDP from period t to period $t+1$ is computed by

$$\frac{Y_{t+1} - Y_t}{Y_t} \tag{2.1}$$

where Y_t represents the real GDP at the year t.

Remark 2 (Rule of 70)**.** *Growing at a constant rate of g% per year, GDP (or anything else) will double approximately every 70/g years.*

$$\textit{Rule of 70: Years to Double} = 70/g$$

Examples:

- *Growing at 2%: Double every 70/2 = 35 years.*
- *Growing at 5%: Double every 70/5 = 14 years.*
- *Growing at 7%: Double every 70/7 = 10 ans.*

With this rule, we can reverse to estimate a growth rate when observing a "time to double".

A natural question is whether economic growth matters. The reason we care about growth is that we care about the **standard of living** (Remark 2 shows an example). Looking across time, we want to know by how much the standard of living has increased. Looking across countries, we want to know how much higher the standard of living is in one country relative to another. Thus, the variable we want to focus on, and compare either over time or across countries, is output per person, rather than output itself.

We will present relations between income per capita growth and other indicators. See Figures 2.1, 2.2, 2.3, 2.4, 2.5.

Figure 2.1: Poverty and income per person. Source: https://www.gapminder.org/tools/

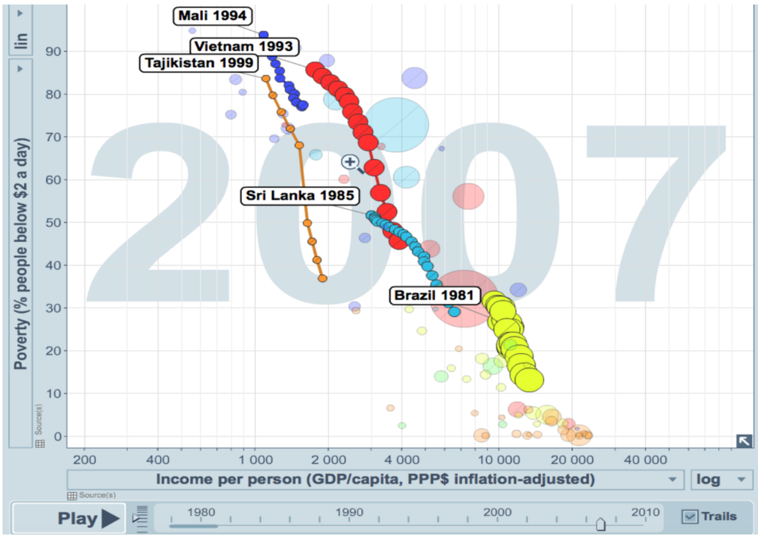
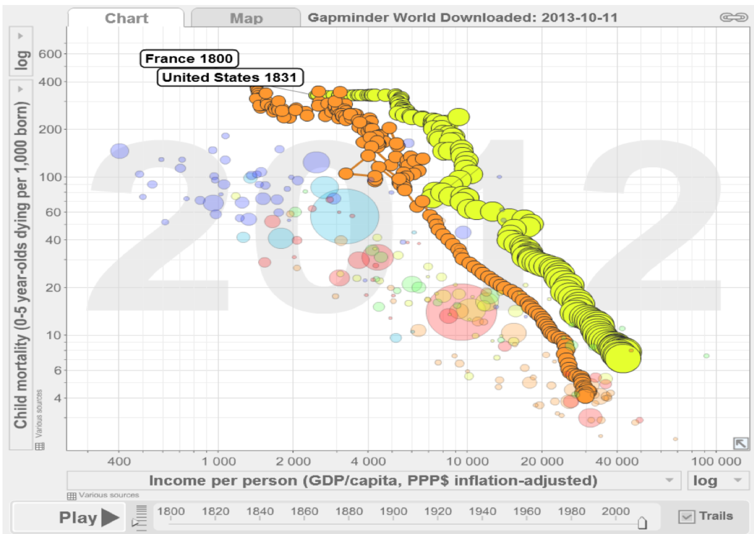


Figure 2.2: Child mortality rate and income per person. Source: https://www.gapminder.org/tools/



2.2 Stylized facts

Following Benassy-Quere et al. (2019) (Section 9), we present some stylized facts on economic growth.

1. By historical standards, fast growth in income per person is a recent phenomenon (see Figures 2.6, 2.7).

   Figure 2.6 shows the world GDP per person (in 1990 purchasing-power-parity dollars) since the start of the first millenium. Four major periods can be distinguished. From prehistory through the Middle Ages, yearly income remains at around $450 per person (in fact, it declines throughout the first millenium). It then increases to about $600 between $1400 and 1800. The true "take-off" comes with the industrial revolution in the nineteenth century and GDP per person exceeds $1500 on the eve of the World

Figure 2.3: The life expectancy and income per person. Source: https://www.gapminder.org/tools/

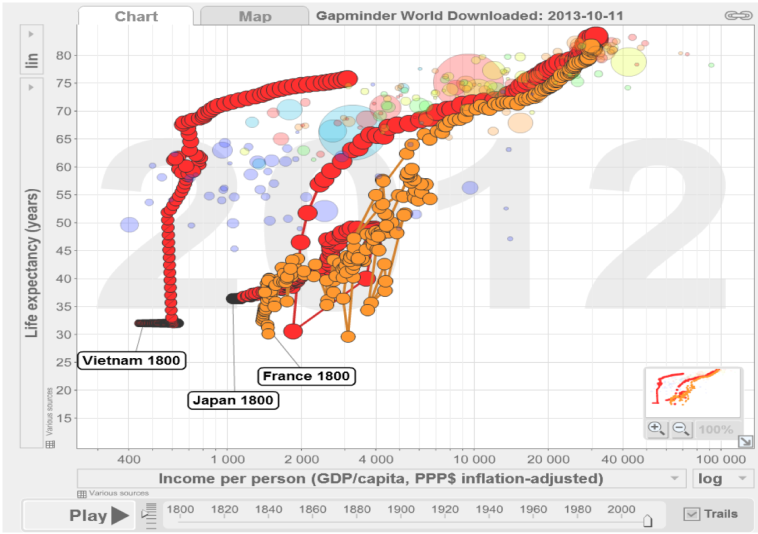
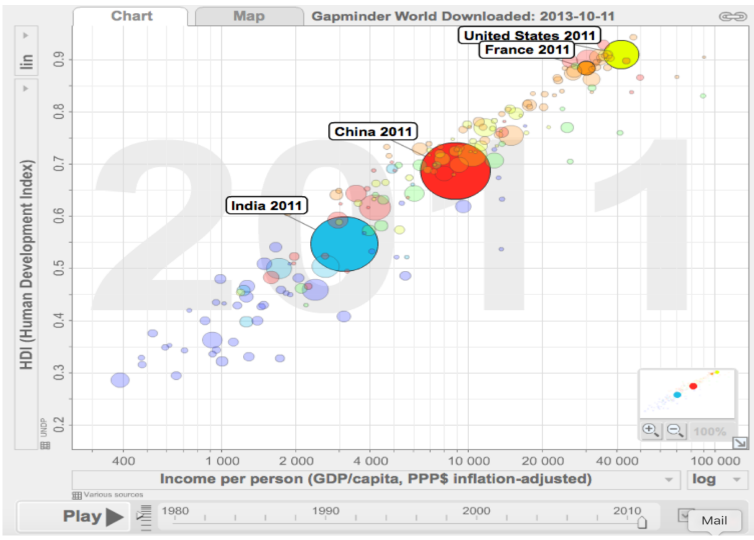


Figure 2.4: HDI and income per person. Source: https://www.gapminder.org/tools/



War I. By 2003 it reaches \$6500, having multiplied by more than five over the course of the century. Maddison expects it to reach \$11700 by 2030.

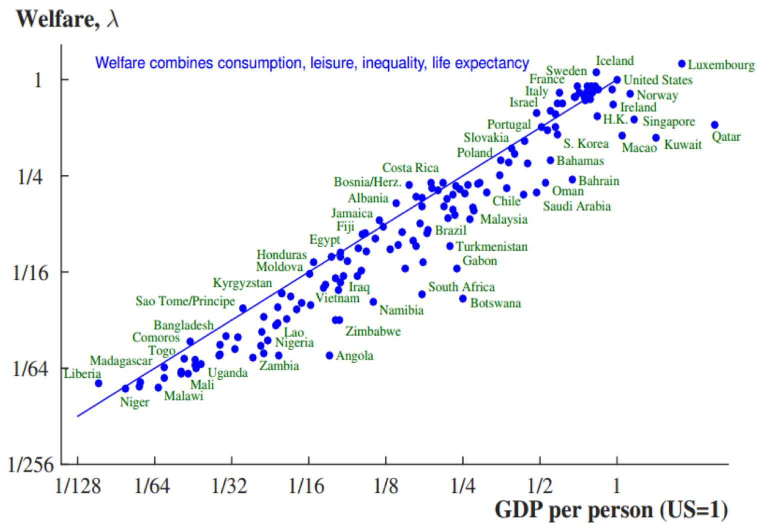
Source: Benassy-Quere et al. (2019) and Maddison (2007), http://www.ggdc.net/maddison/.

2. Along a growth path, income per person and productivity exhibit significant medium-term turning points that are not necessarily synchronous across countries at similar development levels. See Figure 2.8.

   GDP per person and productivity can thus experience significant synchronous and asynchronous inflections. Especially Europe and Japan, which had been catching up with the US standard of living since the Second World War, fell behind after the 1980s.

3. Convergence at the top is neither general nor unattainable. In the last decades, the income per person in some formerly underdeveloped countries, such as East Asian

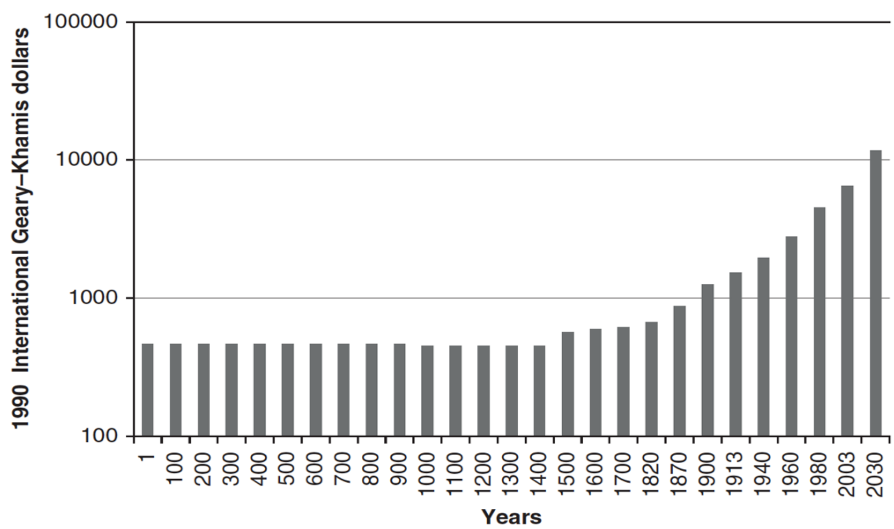
Figure 2.5: Welfare and income per person.



	Economic welfare	Income per capita
US	100.0	100.0
UK	96.6	75.2
France	91.8	67.2
Italy	80.2	66.1
Mexico	21.9	28.6
Russia	20.7	37.0
Brazil	11.1	17.2
China	6.3	10.1
India	3.2	5.6

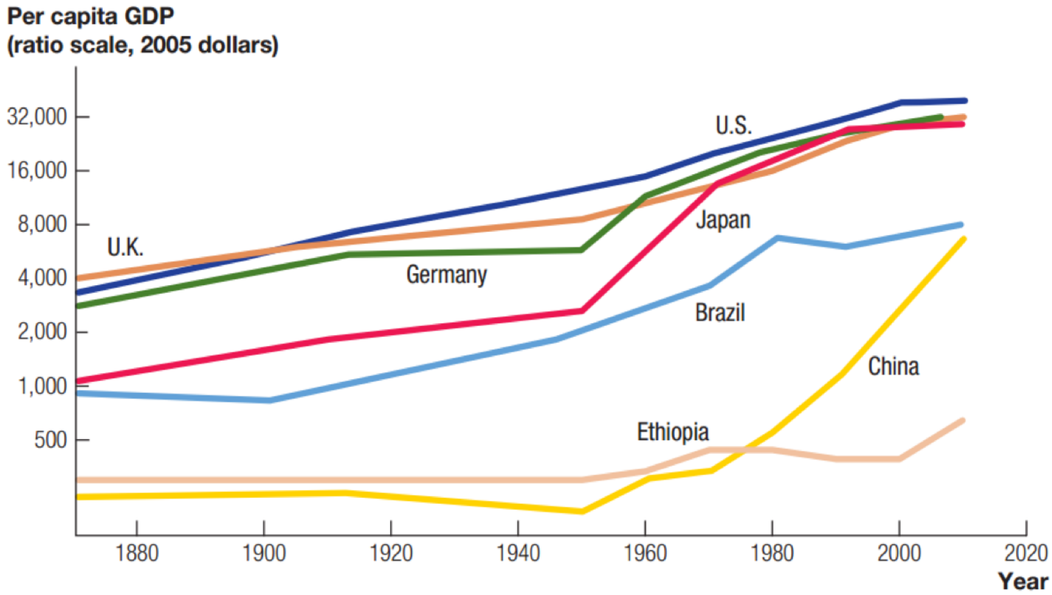
Source: Jones and Klenow (2016), Table 2, p. 2442.

Figure 2.6: Long-term evolution of world GDP per person.



countries, has caught up with that of the most advanced ones, but other countries,

Figure 2.7: Convergence at the top?



Source: Angus Maddison, "Statistics on World Population, GDP and Per Capita GDP, 1 AD–2006 AD." Observations are presented every decade after 1950 and less frequently before that as a way of smoothing the series.

Figure 2.8: Level of GDP per capita and productivity

Time	2015					
Subject		GDP per head of population		GDP per hour worked		GDP per person employed
Measure		USD, current prices, current PPPs				
Unit		US Dollar		US Dollar		US Dollar
Country						
United States		56 066.0		68.3		119 448.0
Germany		47 998.6		66.6		91 062.2
France		41 199.2		66.3		100 002.0
Euro area (19 countries)		41 186.2		58.8		92 310.2
Spain		34 726.6		51.3		87 234.9
Turkey		24 312.6	(E)	38.6		70 631.4
Russia		23 720.0	(E)	24.4	(E)	48 250.8
South Africa	(E)	13 342.5		..	(E)	46 449.9
Brazil		15 869.1		..	(E)	30 690.2
China (People's Republic of)		14 388.1		..		25 536.4

Legend:
E Estimated value

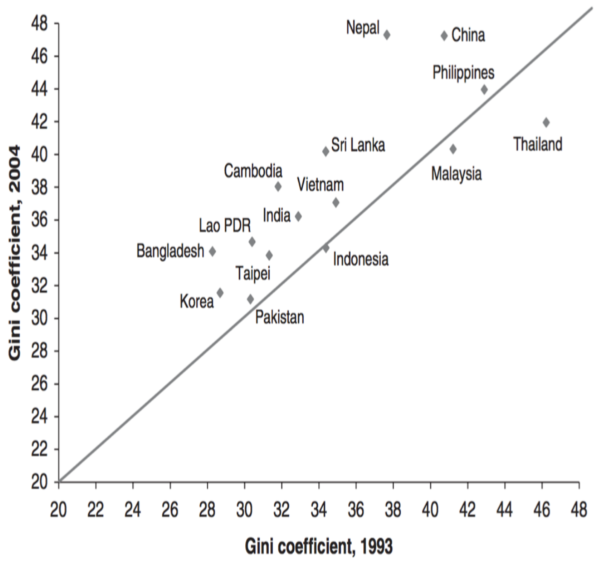
including most sub-Saharan African countries, have further diverged. See Figure 2.7.

4. Largely as a consequence of growth developments, income inequalities among world citizens increased strongly during the nineteenth and the first half of the twentieth centuries. They have stabilized since the 1990s, essentially through the rapid increase in wealth of part of the Chinese and Indian populations. See Figures 2.9, 2.10, 2.11

and 2.12.

In Figure 2.9, the higher the GINI coefficient, the more unequal the distribution of income within the country. Countries above the 45^o line have experienced widening inequality over the period.

Figure 2.9: Inequality within Asian countries, 1994-2004.



Figures 2.10, 2.11 show that, in the United States, the distribution of income controlled by the wealthiest Americans has been growing for some time now. The time-lapse shows an interesting change over the period 1979 to 2014. The bottom 90 percent of income earners, representing the vast majority of the US population, see their share of total income fall quickly over the period. At the same time, the wealthiest gain more and more control as time passes. The wealthier the income group, the faster the share of total income grows.[1]

5. Growth patterns differ over time and they can at times increase inequality within countries. See Figures 2.9, 2.10, 2.11, 2.12.

2.3 Thinking about economic growth: main determinants

As we have discussed, economic growth as the source of current income differences. Some determinants of economic growth are

- Geography (natural resources, climate, land, ...).
- Labor (labor supply, education, motivation, ...).
- Capital (machines, manufactory, road, ...).
- Technology (science, management organization, spirit, ...).

[1]Source: http://howmuch.net/articles/income-distribution-usa

Figure 2.10

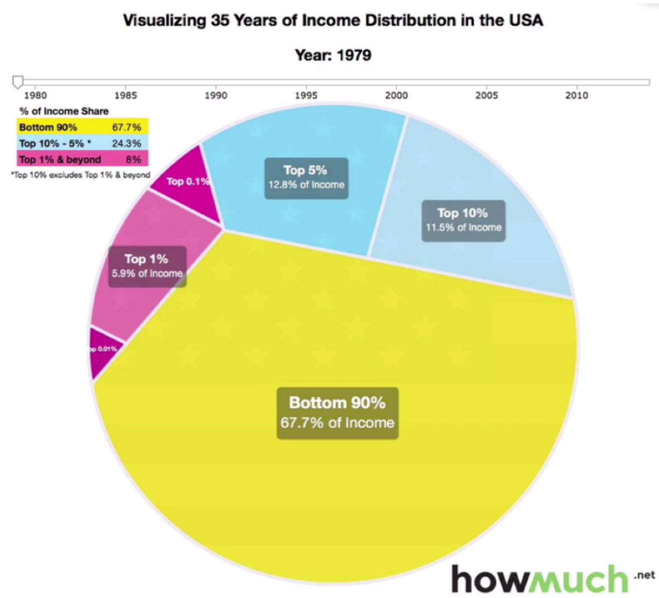
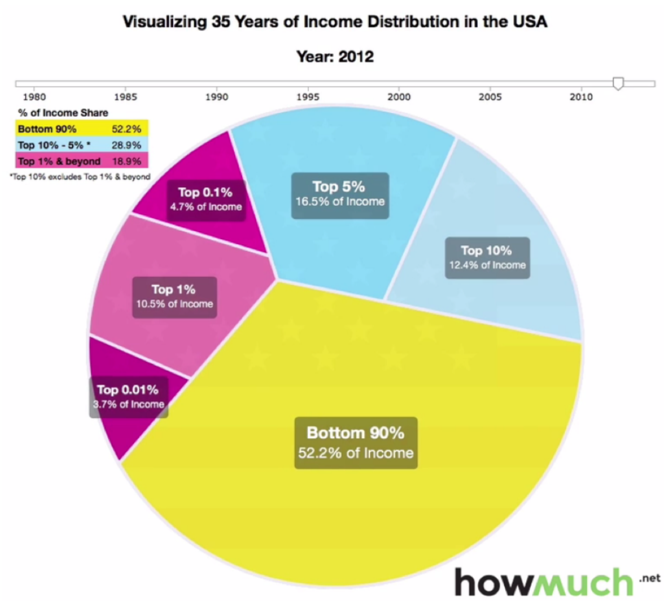


Figure 2.11



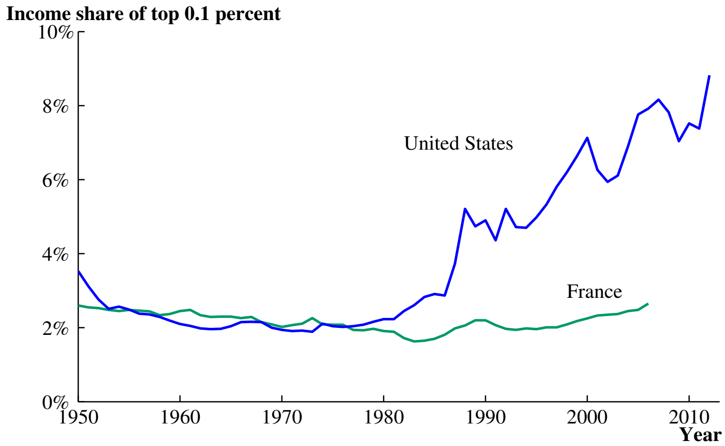
- Political institutions, property rights, and rule of law.

  North (1990) offers the following definition: Institutions are the rules of the game in a society or, more formally, are the humanly devised constraints that shape human interaction.

Other variables that are usually found to have a significant long-term impact on per-capita GDP are (see Barro and Sala-i-Martin (1995), Chapter 12, for a survey):

- The functioning of markets (degree of competition, distortions introduced by state interventions, corruption).

Figure 2.12: **Inequality (Source: Piketty and Saez)**



- Macroeconomic stability (and, in particular, price stability).
- Political stability (absence of wars, coups, or frequent power shifts between opposite camps).

Chapter 3

Growth models with exogenous saving rate

As we have seen there are many factors determining economic growth and per-capita GDP. We will present several economic growth models to understand the determinants of economic growth. In this chapter, for the sake of simplicity, we focus on models where the saving rate is constant over time.

Consider a closed economy. Denote Y_t the output, K_t the capital stock, S_t the savings, I_t the investment and the labor force L_t of the economy at date t. Assume that

$$\begin{aligned} &C_t + S_t = Y_t \\ &I_t = S_t \\ &K_{t+1} = (1-\delta)K_t + I_t \text{ (capital accumulation)} \end{aligned}$$

where $\delta \in (0,1)$ represents the depreciation rate of capital.[1]

Next, we introduce the saving and production functions

$$\begin{aligned} &S_t = sY_t \text{ (constant saving rate)} \\ &Y_t = F(K_t, L_t, A_t) \text{ (production function)} \end{aligned}$$

where A_t represents the productivity.

Remark 3. *We may consider another form of the production function*

$$Y_t = A_t F(B_t K_t, D_t L_t)$$

where A_t is the total factor of productivity, B_t is the capital quality and D_t is the labor quality, K_t is the capital stock, L_t is the labor quantity (number of workers, for example).

We are interested in the evolution of the economy, specially the rate of growth of output. Before doing that, we introduce the notions of balanced growth path and steady state.

Definition 1. *(1) The sequences (Y_t, K_t, S_t) is a balanced growth path of our model if they are a solution to our model and they can be represented as follows:*

$$Y_t = Y^*(1+g_y)^t, \quad K_t = K^*(1+g_k)^t, \quad S_t = S^*(1+g_s)^t, \forall t,$$

where $Y^* > 0, K^* > 0, S^* > 0$, and $g_y > -1, g_k > -1, g_s > -1$.

(2) The (Y, K, S) is a steady state of our model if they are positive and the sequence (Y_t, K_t, S_t) with $(Y_t, K_t, S_t) = (Y, K, S)$, $\forall t$, is a solution to our model

Note that growth rates g_y, g_k, g_s may be negative.

[1] High depreciation rates are observed in some developing countries. For instance, according to Bu (2006), for machinery and equipment in Ghana, the depreciation rate exceeds 0.5 for half of the firms during 1992–1993.

3.1 The Harrod model

Let us start with the Harrod model (Harrod, 1939). Consider an infinite horizon closed economy starting with an initial capital stock $k_0 > 0$:

$$\textbf{Harrod Model:} \qquad \begin{aligned} &c_t + S_t = Y_t \\ &I_t = S_t \\ &k_{t+1} = k_t(1-\delta) + I_t \\ &S_t = sY_t \\ &Y_t = A_t k_t, \end{aligned}$$

where c_t, S_t, I_t are consumption, savings, investment at date t ($t = 0, 1, \ldots, +\infty$), $s \in (0,1)$ is the exogenous saving rate, k_t is the physical capital at date t ($k_0 > 0$ is given), $\delta \in [0,1]$ is the capital depreciation rate, Y_t is the output.

Remark 4. *The production function in this model ($Y_t = A_t k_t$) can be interpreted in several ways: (i) it is a special case of the general form of Cobb-Douglas function with $\beta = 0$, (2) the labor N_t has an exogenous rate of growth $N_t = N_0(1+n)^t$. In this case the TFP becomes $A_t N_0^\beta (1+n)^{\beta t}$, (3) if $\beta = 1 - \alpha$, the function can be written as:*

$$\frac{Y_t}{N_t} = A_t \left(\frac{k_t}{N_t}\right)^\alpha,$$

i.e., we consider the output per capita as function of capital per capita.

From the above system, we obtain that, for any $t \geq 0$,

$$\begin{aligned} Y_t &= A_t((1-\delta)k_{t-1} + sY_{t-1}) \\ \text{and } \frac{\Delta Y_t}{Y_t} &= \frac{A_{t+1}}{A_t}(1-\delta) + sA_{t+1} - 1 \\ \text{where } \Delta Y_t &\equiv Y_{t+1} - Y_t. \end{aligned}$$

Therefore, we have the following result.

Proposition 1. *Consider the above Harrod model. Suppose $A_t \to A > 0$ when t tends to infinity. We have that:*

1. $\frac{\Delta Y_t}{Y_t} \to sA - \delta$.
2. *If* $sA - \delta > 0$ *then* $Y_t \to +\infty$.
3. *If* $sA - \delta < 0$ *then* $Y_t \to 0$.

According to this result, the economy may grow or collapse, depending to the TFP A: if A is high enough ($A > \delta/s$), then we have economic growth without bounds.

3.2 The Solow-Swan model

We now consider a model à la Solow (Solow, 1956, 1957; Swan, 1956). This model is quite similar to the Harrod Model, excepted the production function.

Solow-Swan Model:

$$\begin{aligned}
&C_t + S_t = Y_t\\
&I_t = S_t\\
&K_{t+1} = (1-\delta)K_t + I_t \text{ (capital accumulation)}\\
&S_t = sY_t \text{ (constant saving rate)}\\
&Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}, \alpha \in (0,1)\\
&A_t = a_0(1+\gamma)^t\\
&L_t = L_0(1+n)^t. \text{ (population)}
\end{aligned}$$

Here $\gamma > -1$ is the rate of growth of the TFP A_t while $n > -1$ is the rate of growth of the labor force. Both of them are assumed to be exogenous.

From the above system, we obtain that, for any $t \geq 0$,

$$\begin{aligned}
Y_t &= a_0(1+\gamma)^t K_t^{\alpha} L_t^{1-\alpha}\\
\frac{Y_{t+1}}{Y_t} &= (1+\gamma)(1+n)^{1-\alpha}\Big(\frac{K_{t+1}}{K_t}\Big)^{\alpha}\\
K_{t+1} &= (1-\delta)K_t + sa_0(1+\gamma)^t K_t^{\alpha} L_t^{1-\alpha}.
\end{aligned}$$

Denote

1. $k_t \equiv \frac{K_t}{L_t}$ the capital stock per capita at date t.
2. $y_t \equiv \frac{Y_t}{L_t}$ the income per capita at date t.

When the rate of growth of TFP is constant

Assume that $\gamma = 0$. We have

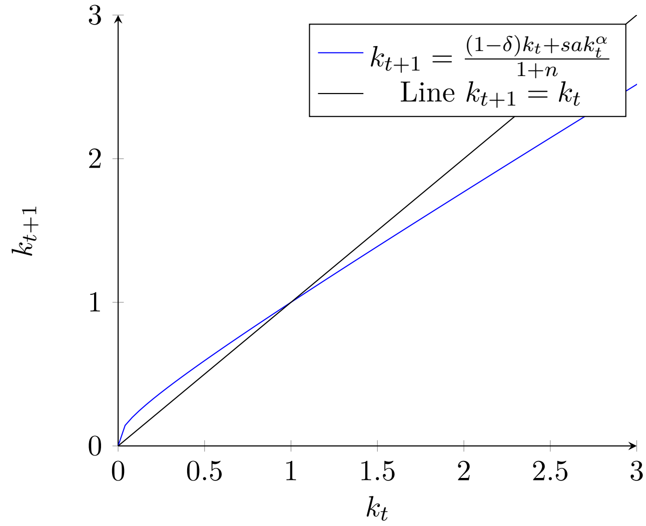
$$\begin{aligned}
K_{t+1} &= (1-\delta)K_t + sa_0 K_t^{\alpha} L_t^{1-\alpha}\\
\rightarrow (1+n)\frac{K_{t+1}}{L_{t+1}} &= (1-\delta)\frac{K_t}{L_t} + sa_0(\frac{K_t}{L_t})^{\alpha}\\
(1+n)k_{t+1} &= (1-\delta)k_t + sa_0 k_t^{\alpha}.
\end{aligned}$$

In this case, we can prove that for any $k_0 > 0$, the sequence k_t converges.

Proposition 2. *Consider the above Solow model with $\gamma = 0$. Assume that $s \in (0,1), a_0 > 0, \alpha \in (0,1)$.*

1. *If $1+n > 1-\delta$, then k_t, y_t converge a positive value k and y, where k is determined by $(1+n)k = (1-\delta)k + sa_0 k^{\alpha}$.*
2. *If $1+n \leq 1-\delta$ (extreme case - the population decreases to zero), then the capital stock per capita $k_t = \frac{K_t}{L_0(1+n)^t}$. converges to infinity. Moreover, $\frac{k_{t+1}}{k_t}$ is decreasing in time and $\lim_{t\to\infty} \frac{k_{t+1}}{k_t} = \frac{1-\delta}{1+n} \geq 1$.*

Figure 3.1: Dynamics of capital stock per capita



Proof. 1. If $1+n > 1-\delta$, k_t, y_t converges a positive value k and y, where k is determined by $(1+n)k = (1-\delta)k + sa_0 k^\alpha$, i.e., $k = \left(\frac{sa_0}{n+\delta}\right)^{\frac{1}{1-\alpha}}$.

   We can prove that k is uniquely determined.

   (a) If $k_0 = k$, then $k_t = k$ for any t.

   (b) If $k_0 < k$, then we can prove that $k_{t+1} > k_t$ and $k_t < k$ for any t. Then, k_t converges.

   (c) If $k_0 > k$, then we can prove that $k_{t+1} < k_t$ and $k_t > k$ for any t. Then, k_t converges.

2. If $1+n \leq 1-\delta$, then $k_{t+1} > k_t$ for any t. So, k_t converges. Since it cannot converge to a finite value, it must converge to infinity. Moreover, since $\frac{k_{t+1}}{k_t} = \frac{1}{1+n}\Big(1-\delta + sa_0 k_t^{\alpha-1}\Big)$, we see that $\frac{k_{t+1}}{k_t}$ is decreasing in time and $\lim_{t\to\infty} \frac{k_{t+1}}{k_t} = \frac{1-\delta}{1+n}$.

□

Let us focus on the case $1+n > 1-\delta$. Observe that the capital stock per capita at the steady state

$$k = \Big(\frac{sa_0}{n+\delta}\Big)^{\frac{1}{1-\alpha}} \tag{3.1}$$

$$y = a_0\Big(\frac{sa_0}{n+\delta}\Big)^{\frac{\alpha}{1-\alpha}} \tag{3.2}$$

Note that both y and k are increasing in the saving rate s but decreasing in the rate of population growth n and the depreciation rate δ.

Moreover, we can compute the rates of growth

$$\frac{y_{t+1}}{y_t} = \frac{a_0 k_{t+1}^\alpha}{a_0 k_{t+1}^\alpha} = (\frac{k_{t+1}}{k_t})^\alpha$$

$$\frac{k_{t+1}}{k_t} = \frac{1}{1+n}\Big(1-\delta + sa_0 k_t^{\alpha-1}\Big).$$

By consequence, we get that

1. If $k_0 = k$, then $k_t = k$ and $\frac{y_{t+1}}{y_t} = 1$ for any t.
2. If $k_0 < k$, then we can see that $\frac{k_{t+1}}{k_t}$ and $\frac{y_{t+1}}{y_t}$ are decreasing in time and converge to 1.
3. If $k_0 > k$, then we can see that $\frac{k_{t+1}}{k_t}$ and $\frac{y_{t+1}}{y_t}$ are increasing in time and converge to 1.

General case

We now consider the general case where $\gamma > -1$.

We write by $K_t = (1+g)^t X_t$ where g will be determined later. So, $K_{t+1} = K_t(1-\delta) + sA_tK_t^\alpha L_t^{1-\alpha}$ becomes

$$\begin{aligned}(1+g)^{t+1}X_{t+1} &= (1-\delta)(1+g)^t X_t + sa_0(1+\gamma)^t\big((1+g)^t X_t\big)^\alpha\big(L_0(1+n)^t\big)^{1-\alpha}\\ \iff X_{t+1} &= \frac{(1-\delta)X_t + sa_0L_0^{1-\alpha}X_t^\alpha\Big(\frac{(1+\gamma)(1+n)^{1-\alpha}}{(1+g)^{1-\alpha}}\Big)^t}{1+g}.\end{aligned}$$

Now, we choose g so that $(1+\gamma)(1+n)^{1-\alpha} = (1+g)^{1-\alpha}$. Thus

$$X_{t+1} = \frac{(1-\delta)X_t + sa_0L_0^{1-\alpha}X_t^\alpha}{1+g}.$$

Using the same argument as in Proposition 2, we can prove the convergence of X_t. Therefore, we obtain the following result:

Proposition 3. *Consider the above Solow model. The sequence (X_t) converges. Moreover, $\frac{K_{t+1}-K_t}{K_t} \to g$ and $\frac{Y_{t+1}-Y_t}{Y_t} \to g$ where g satisfies*

$$1+g = (1+n)(1+\gamma)^{\frac{1}{1-\alpha}}.$$

The long-term rate of growth g of the output depends strongly on the rate of growth of the TFP A_t. The higher A, the higher the rate of growth g.

Exercise 2. *Investigate the growth rates per capita $\frac{k_{t+1}-k_t}{k_t}$ and $\frac{y_{t+1}-y_t}{y_t}$. Do they converge? If yes, how do the limits depend on n and γ? Provide interpretations.*

Proof. We have

$$\begin{aligned}&\frac{k_{t+1}}{k_t} = \frac{\frac{K_{t+1}}{L_{t+1}}}{\frac{K_t}{L_t}} = \frac{1}{1+n}\frac{K_{t+1}}{K_t} = \frac{1}{1+n}\frac{(1+g)^{t+1}X_{t+1}}{(1+g)^tX_t} = \frac{1}{1+n}\frac{K_{t+1}}{K_t} = \frac{1+g}{1+n}\frac{X_{t+1}}{X_t}\\ \Rightarrow &\lim_{t\to\infty}\frac{k_{t+1}}{k_t} = \frac{1}{1+n}\lim_{t\to\infty}\frac{K_{t+1}}{K_t} = (1+\gamma)^{\frac{1}{1-\alpha}}.\end{aligned}$$

□

Chapter 4

Macroeconomics with micro-foundations

4.1 Finite horizon models

4.1.1 Intertemporal choice (smooth consumption)

An agent living for two periods (present and future, represented by 0 and 1) wants to choose consumption allocation (c_0, c_1) to maximize her(his) utility $U(c_0, c_1)$. This agent is endowed e_0, e_1 units of good at date 0 and 1 respectively. She(he) can borrow or lend by using a financial asset.

To sum up, the maximization problem of this agent is the following:

$$\max \Big\{U(c_0, c_1)\Big\} \tag{4.1}$$

$$c_0, c_1 \geq 0 \tag{4.2}$$

$$p_0c_0 + q_0a_0 \leq p_0e_0 \tag{4.3}$$

$$p_1c_1 \leq p_1e_1 + p_1a_0. \tag{4.4}$$

Here, the agent buys a_0 units of financial asset at date 0. Then, at date 0, she(he) receives a_0 units of good at date 1 (it means that the payoff is is in terms of good). The payoff value (in terms of money) is p_1a_0 at date 1.

Since there is no additional constraint (borrowing constraint, for example) on a_0, we can rewrite the constraints as follows (prove this as an exercise):

$$c_0, c_1 \geq 0$$

$$p_0c_0 + q_0c_1 \leq p_0e_0 + q_0e_1$$

Let us define r by $q_0 = \frac{1}{1+r}$. Then r represents the real return. We have the intertemporal budget constraint, where all terms are in

$$\underbrace{c_0 + \frac{c_1}{1+r}}_{\text{Present value of consumptions}} \leq \underbrace{e_0 + \frac{e_1}{1+r}}_{\text{Present value of endowments}} \tag{4.5}$$

Note that both sides are in terms of good at date 0.

Proposition 4. *Assume that $U(c_0, c_1) = u(c_0) + \beta u(c_1)$ where the function u is strictly concave, increasing, continuously differentiable, $u'(0) = \infty$. Parameter β represents the rate of time preference.*

At optimum, we can prove that

$$q_0 = \frac{\beta u'(c_1)}{u'(c_0)}$$
$$q_0 u'(e_0 - q_0 a_0) = \beta u'(e_1 + a_0).$$

We can provide conditions under which a_0 is a decreasing function of q_0. We can also identify conditions under which the agent lends $a_0 > 0$ or borrows ($a_0 < 0$).

Let us consider a particular case with logarithmic utility function $u(c) = ln(c)$. In this case, we find that

$$q_0 a_0 = \frac{\beta e_0 - q e_1}{1+\beta}.$$

If we write $q_0 = \frac{1}{1+r}$, then we have

$$a_0 = \frac{1+r}{1+\beta}(\beta e_0 - \frac{e_1}{1+r})$$

Write $\beta = \frac{1}{1+\rho}$.

Observe that: $a_0 > 0$ if and only if

$$r > (1 + \frac{e_1 - e_0}{e_0})(1+\rho) = (1+g)(1+\rho).$$

$g = \frac{e_1 - e_0}{e_0}$ can be interpreted as the rate of growth of endowments. By the way, the agent invests if and only if the return rate is high enough.

4.1.2 Intertemporal choice with productive investment

An agent living for two periods (present and future, represented by 0 and 1) wants to choose consumption allocation (c_0, c_1) and physical capital k_1 to maximize her(his) utility $U(c_0, c_1)$

$$\max_{(c_0,c_1,k_1)} U(c_0, c_1) \tag{4.6}$$
$$\text{subject to: } c_1 \geq 0, c_1 \geq 0, k_1 \geq 0 \tag{4.7}$$
$$c_0 + k_1 \leq w_0 \tag{4.8}$$
$$c_1 \leq w_1 + F(k_1) \tag{4.9}$$

where w_0, w_1 are given and strictly positive.

Assumption 2. *The function F is assumed to be strictly increasing, concave, continuously differentiable and $F(0) = 0$.*

Assume that $U(c_0, c_1) = u(c_0) + \beta u(c_1)$ where the function u is strictly concave, increasing, continuously differentiable, $u'(0) = \infty$. Parameter β represents the rate of time preference.

We would like to solve this problem to understand the optimal value of (c_0, c_1, k_1). Notice that in this setup, k_t can be also interpreted as investment.

First, it is easy to see that the Slater condition is satisfied. So, we can write the Lagrangian

$$L = U(c_0, c_1) + \mu_0 c_0 + \mu_1 c_1 + \mu_k k_1 + \lambda_0(w_0 - c_0 - k_1) + \lambda_1(w_1 + F(k_1) - c_1) \tag{4.10}$$

Since $u'(0) = \infty$, we have $c_0 > 0, c_1 > 0$ at optimum (@reader: why?), which implies that $\mu_0 = \mu_1 = 0$. Therefore, the first-order conditions become

$$\frac{\partial U(c_0, c_1)}{\partial c_0} - \lambda_0 \Leftrightarrow u'(c_0) = \lambda_0$$
$$\frac{\partial U(c_0, c_1)}{\partial c_1} - \lambda_1 \Leftrightarrow \beta u'(c_1) = \lambda_1$$
$$\mu_k - \lambda_0 + \lambda_1 F'(k_1) = 0 \Leftrightarrow \lambda_0 = \lambda_1 F'(k_1) + \mu_k$$
$$\mu_k k_1 = 0, \mu_k \geq 0$$

So, we obtain that $u'(c_0) = \beta F'(k_1)u'(c_1) + \mu_k$. At optimum, we must have $c_0 + k_1 = w_0$ and $c_1 = w_1 + F(k_1)$. Thus, we get that

$$u'(w_0 - k_1) = \beta F'(k_1)u'(w_1 + F(k_1)) + \mu_k \tag{4.11}$$

Notice that at this stage, we do not require that $F'(0) = \infty$.

There are two cases.

1. $k_1 > 0$. In this case, we have $\mu_k = 0$ and hence k_1 is determined by

$$H(k_1) \equiv u'(w_0 - k_1) - \beta F'(k_1)u'(w_1 + F(k_1)) = 0 \tag{4.12}$$

The function H is strictly increasing in k_1. $H(0) = u'(w_0) - \beta F'(0)u'(w_1)$ while $H(w_0) = \infty$ because $u'(0) = \infty$. So, the existence of a strictly positive solution k_1 requires that

$$u'(w_0) - \beta F'(0)u'(w_1) < 0.$$

It means that the productivity $F'(0)$, the rate of time preference β, the endowment at initial date w_0 are high and the endowment at date 1 is low.

2. $k_1 = 0$. Condition (4.11) becomes

$$u'(w_0) - \beta F'(0)u'(w_1) \leq 0.$$

To sum up, we obtain the following result:

Proposition 5. *Assume that above conditions hold. Assume also that $u'(0) = \infty$. The optimal choice k_1 is strictly positive if and only if*

$$u'(w_0) - \beta F'(0)u'(w_1) < 0$$

In such a case, k_1 is the unique solution to the equation $H(k_1) = 0$.

It is useful to consider some particular cases.

1. Assume that $u(c) = ln(c)$, $F(k) = Ak^\alpha$ where $\alpha \in (0, 1)$, and $w_1 = 0$. In this case, $F'(0) = \infty$. We have

$$\frac{1}{w_0 - k_1} = \beta\alpha A k_1^{\alpha-1} \frac{1}{w_1 + Ak_1^\alpha} = \beta\alpha A k_1^{\alpha-1} \frac{1}{Ak_1^\alpha} \tag{4.13}$$
$$\Leftrightarrow k_1 = \frac{\alpha\beta}{1 + \alpha\beta} w_0 \tag{4.14}$$

The investment k_1 is increasing in the initial endowment w_0 and the rate of time preference.

2. Assume that $u(c) = ln(c)$, $F(k) = Ak$, $w_1 > 0$. We have that:

   (a) If $w_1 < \beta A w_0$, then k_1 is strictly positive and we can compute that $k_1 = \frac{\beta A w_0 - w_1}{A(1+\beta)}$. The investment is increasing in the productivity A, the rate of time preference β, the initial endowment but decreasing in the endowment in the future.

   (b) If $w_1 \geq \beta A w_0$, then $k_1 = 0$. The intuition: when the productivity, the initial endowment, the rate of time preference are low, but the endowment in the future is high, we do not need to save/invest.

Comparative statics. Let conditions in Proposition 5 be satisfied. Then, the optimal physical capital k_1 is determined by

$$u'(w_0 - k_1) - \beta F'(k_1) u'(w_1 + F(k_1)) = 0 \tag{4.15}$$

Notice that $w_0, w_1,\ \beta$ are exogenous parameters while k_1 is endogenous and depends on $w_0, w_1,\ \beta$.

Denote $f(k_1, w_0, w_1, \beta) \equiv u'(w_0 - k_1) - \beta F'(k_1) u'(w_1 + F(k_1))$. Observe that the function f is stricIty increasing in k_1 and w_1, but strictly decreasing in w_0 and β

Assume that u' and F' are continuously differentiable. Applying the implicit functions theorem, the optimal value k_1 determined by $f(k_1, w_0, w_1, \beta) = 0$ can be expressed as a differentiable function of w_0, w_1, β. We write $k_1 = k_1(w_0, w_1, \beta)$.

We now look at the role of the initial endowment w_0. Taking the derivative with respect to w_0 of both sides of the equation $f(k_1, w_0, w_1, \beta) = 0$, we have

$$\frac{\partial f}{\partial k_1}(k_1, w_0, w_1, \beta) \frac{\partial k_1}{\partial w_0}(w_0, w_1, \beta) + \frac{\partial f}{\partial w_0}(k_1, w_0, w_1, \beta) = 0$$

Since $\frac{\partial f}{\partial k_1} > 0$ and $\frac{\partial f}{\partial w_0} < 0$, we get that $\frac{\partial k_1}{\partial w_0}(w_0, w_1, \beta) > 0$. It means that the optimal value k_1 is increasing in the initial endowment.

4.1.3 Intertemporal choice with productive investment: a general equilibrium approach

We now present a decentralized version of the model in Section 4.1.2 There are two economic agents in this model: a household and a firm. The household maximization problem is

$$\max_{(c_0, c_1, k_1)} U(c_0, c_1) \tag{4.16}$$

$$\text{subject to: } c_0 \geq 0, c_1 \geq 0, k_1 \geq 0 \tag{4.17}$$

$$c_0 + k_1 \leq w_0 \tag{4.18}$$

$$c_1 \leq w_1 + r_1 k_1 + (1 - \delta) k_1 + (Profit) \tag{4.19}$$

At date 0, the endowment of household is given. At date 1, he(she) receives a profit because he(she) is the owner of the firm.

r_1 is the rental rate of physical capital. The firm rents physical capital at the rate r_1.

The profit of the firm is endogenously determined by

$$Profit \equiv \max_{K_1 \geq 0} f(K_1) - r_1 K_1 \tag{4.20}$$

Definition 2. *An intertemporal equilibrium is a list* $(c_0, c_1, k_1, K_1, r_1)$ *satisfying the following conditions:*

1. *Given r_1, the allocation (c_0, c_1, k_1) is a solution to the problem of the household and K_1 is a solution to the firm's problem.*

2. *The rental market clears: $K_1 = k_1$ (demand=supply).*

In equilibrium, we have

$$
\begin{aligned}
u'(c_0) &= \lambda_0 \\
\beta u'(c_1) &= \lambda_1 \\
\mu_k - \lambda_0 + \lambda_1(r_1 + 1 - \delta) = 0 \Leftrightarrow \lambda_0 &= \lambda_1(r_1 + 1 - \delta) + \mu_k \\
\mu_k k_1 &= 0, \mu_k \geq 0
\end{aligned}
$$

The first order condition of the firm gives $F'(K_1) = r_1$. So, $c_1 = w_1 + r_1 k_1 + (1-\delta)k_1 + F(K_1) - r_1 k_1$. Since $k_1 = K_1$, we get $c_1 = w_1 + (1-\delta)k_1 + f(K_1)$.

We can prove that the outcomes of this model coincides to those of the model in Section 4.1.2 with $F(K) = (1-\delta)K + f(K)$.

Application: Endowment effect

Consider a particular case where $w_1 = 0$, $u(c) = ln(c)$, $\delta = 1$ and $F(k) = Ak^\alpha$, where $\alpha \in (0, 1)$. In this case, we find that

$$k_1 = \frac{\alpha\beta}{1+\alpha\beta} w_0 \tag{4.21}$$

$$\text{rental return: } r_1 = \alpha A \Big(\frac{\alpha\beta}{1+\alpha\beta} w_0\Big)^{\alpha-1} \tag{4.22}$$

The capital level k_1 is increasing in the initial endowment. However, the rental return is decreasing in w_0.

Assume that due to some reasons (a war, for example), the initial endowment goes down. Then, the capital supply k_1 does down and rental return goes up. Producers have to pay more because the supply goes dow.

Application: Externalities

The conclusion in Section 4.1.3 is no longer the case if there are externalities.

We continue to assume that $u(c) = ln(c)$, $\delta = 1$, $f(k) = Ak^\alpha$ where $\alpha \in (0, 1)$.

We have $r_1 = f'(K_1) = \alpha A K_1^{\alpha-1}$. Since $f'(0) = \infty$, we have $k_1 > 0$ at equilibrium.

$$H(k_1) \equiv u'(w_0 - k_1) - \beta r_1 u'(w_1 + f(k_1)) = 0 \tag{4.23}$$

$$\frac{1}{w_0 - k_1} = \beta r_1 \frac{1}{w_1 + Ak_1^\alpha} \tag{4.24}$$

We now assume that there is an externality which represents by the assumption that A depends on the aggregate capital K_1: $A = a(K_1)$.[1] In this case, in equilibrium, we have

$$
\begin{aligned}
r_1 &= \alpha a(k_1) k_1^{\alpha-1} \\
\frac{1}{w_0 - k_1} &= \beta\alpha a(k_1) k_1^{\alpha-1} \frac{1}{w_1 + a(k_1)k_1^\alpha} \\
(w_0 - k_1)\beta\alpha a(k_1) k_1^{\alpha-1} &= w_1 + a(K_1) k_1^\alpha
\end{aligned}
$$

[1] See also Golosov et al. (2014) who present a dynamic stochastic general-equilibrium (DSGE) model with an externality from using fossil energy.

Consider a particular case where $w_1 = 0$, we have $\frac{1}{w_0 - k_1} = \beta\alpha\frac{1}{k_1}$ and hence we find the same level of physical capital $k_1 = \frac{\alpha\beta}{1+\alpha\beta}w_0$. However, the output is

$$Y_1 = f(k_1) = a(k_1)k_1^\alpha = a(\frac{\alpha\beta}{1+\alpha\beta}w_0)\Big(\frac{\alpha\beta}{1+\alpha\beta}w_0\Big)^\alpha$$

The output in the case without externalities is $Y_1^* = A\Big(\frac{\alpha\beta}{1+\alpha\beta}w_0\Big)^\alpha$.

Observe that

$$Y_1^* \geq Y_1 \Longleftrightarrow A \geq a\Big(\frac{\alpha\beta}{1+\alpha\beta}w_0\Big).$$

We consider two cases.

1. **Positive externality**: It means that the aggregate capital has a positive externality on the individual firm's productivity. Assume that $a(K) = A + \gamma K$ with $\gamma > 0$. (This is the idea of Romer (1986)'s endogenous growth model.) In this case, we observe that the aggregate output in the case without externalities is lower than the aggregate output of the economy with externality.

2. **Negative externality**: It means that the aggregate capital has a negative externality on the individual firm's productivity (pollution, for example). Assume that $a(K) = Max(A - \gamma K, 0)$ with $\gamma > 0$. In this case, we observe that the aggregate output in the case without externalities is higher than the aggregate output of the economy with externality.

4.1.4 Negative externality, tax, and optimality

Let us consider the previous model with externality. Assume that there is a negative externality. Assume that we are in a competitive economy. The government sets a tax on the output of the firms and uses the revenue from this tax to help the households.

The representative household's maximization problem is

$$\max_{(c_0,c_1,k_1)} U(c_0, c_1) \tag{4.25}$$

$$\text{subject to: } c_0 \geq 0, c_1 \geq 0, k_1 \geq 0 \tag{4.26}$$

$$c_0 + k_1 \leq w_0 \tag{4.27}$$

$$c_1 \leq w_1 + r_1k_1 + (1-\delta)k_1 + (Profit) + T \tag{4.28}$$

where T is the transfer from the government.

The profit of the firm is now determined by

$$Profit \equiv \max_{K_1 \geq 0}(1-\tau)f(K_1) - r_1K_1 \tag{4.29}$$

Definition 3. *Given $\tau \in [0, 1)$, an intertemporal equilibrium is a list $(c_0, c_1, k_1, K_1, r_1, T)$ satisfying the following conditions:*

1. *Given r_1, the allocation (c_0, c_1, k_1) is a solution to the problem of the household and K_1 is a solution to the firm's problem.*

2. *The rental market clears: $K_1 = k_1$ (demand=supply).*

3. *$T = \tau f(K_1)$.*

Again, assume that $u(c) = ln(c)$, $w_1 = 0$, $\delta = 1$, $f(k) = Ak^{\alpha}$ where $\alpha \in (0,1)$, and there is an externality $A = a(k)$. In equilibrium, we find that

$$k_1 = k_1(\tau) \equiv \frac{\alpha\beta(1-\tau)}{1+\alpha\beta(1-\tau)} w_0.$$

The problem of the social planner is

$$\max_{(c_0,c_1,k_1)} ln(c_0) + \beta ln(c_1) \tag{4.30}$$

$$\text{subject to: } c_1 \geq 0, c_1 \geq 0, k_1 \geq 0 \tag{4.31}$$

$$c_0 + k_1 \leq w_0 \tag{4.32}$$

$$c_1 \leq a(k_1)k_1^{\alpha} \tag{4.33}$$

Under mild assumptions, we have the first-order condition

$$\frac{1}{w_0 - k_1} = \beta \frac{a'(k_1)k_1^{\alpha} + a(k_1)\alpha k_1^{\alpha-1}}{a(k_1)k_1^{\alpha}}. \tag{4.34}$$

Assume that this equation has a unique solution, denoted by k_1^s, and that k_1^s is also the unique solution of the social planner's problem.

Proposition 6. *Under above assumptions, if the government sets the tax rate* τ

$$1 - \tau = \frac{k_1^s}{\alpha\beta(w_0 - k_1^s)},$$

then we have $k_1^s = \frac{\alpha\beta(1-\tau)}{1+\alpha\beta(1-\tau)} w_0$*, i.e.,* $k_1^s = k_1(\tau)$.

This means that the capital k_1 in the decentralized economy $k_1(\tau)$ equals the capital level k_1^s chosen by the social planner. In other words, this tax recovers the optimality of the allocation given that the objective of the government is the welfare of households.

4.2 Infinite horizon models à la Ramsey

Although the Harrod and Solow models help us to explain the role of TFP, they have two limits: (1) the rate of saving is exogenous and (2) the rate of growth of the output is also exogenous. The following graphic, taken from the World Bank database, shows the evolution of the saving rate in some countries.

With the Ramsey model, we can endogenize the rate of saving but we do not resolve the question of the exogeneity of the rate of growth of the output. This question will be resolved with endogenous growth models in Section 5.4.

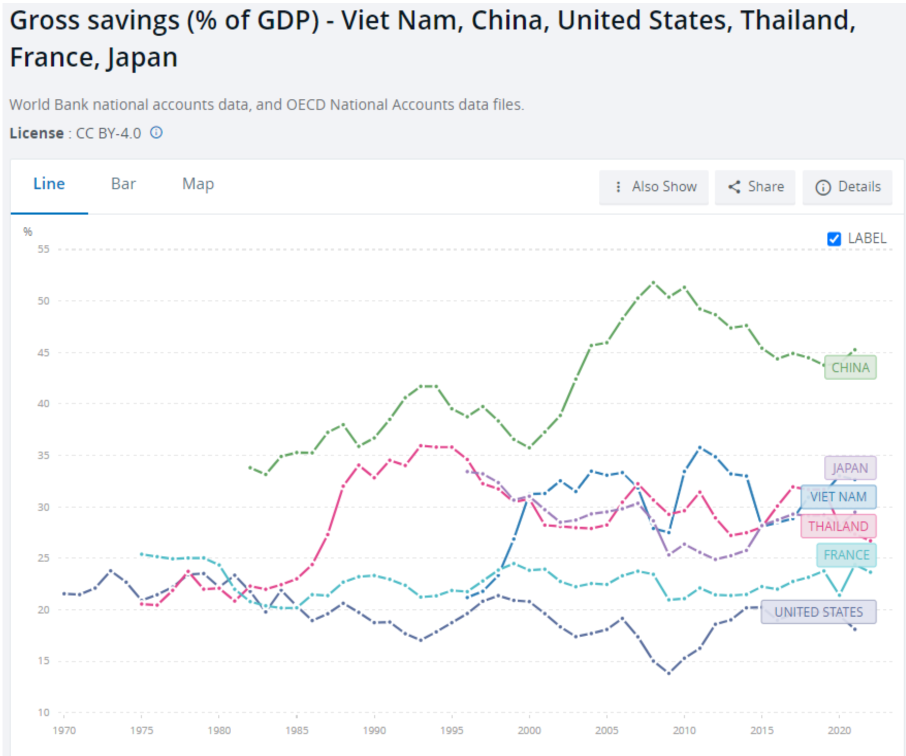
However, before presenting endogenous growth models, we introduce a Ramsey model. We assume there exists a representative consumer who lives for an infinite number of periods. She/he maximizes her/his intertemporal utility under sequential constraints

$$\textbf{Ramsey model:} \quad \max_{(c_t,k_{t+1},I_t)} \sum_{t=0}^{+\infty} \beta^t u(c_t)$$

$$\text{subject to: } c_t + I_t \leq F_t(k_t)$$

$$k_{t+1} = k_t(1-\delta) + I_t,$$

where $k_0 > 0$ is given, $\beta \in (0,1)$ represents the rate of time preference.



Remark 5. *As in the Harrod Model, in the Ramsey Model, implicitly, either we consider the number of workers is exogenous and has an exogenous rate of growth, or we consider in fact output per capita and capital per capita.*

Assumption 3. *The instantaneous utility function u is strictly increasing, strictly concave, twice continuously differentiable, $u'(0) = +\infty$.*

The production function F_t is concave, strictly increasing, twice continuously differentiable, and $F_t(0) = 0$.

Assume also that

$$\sup_{(c_t, k_{t+1}) \geq 0} \Big\{ \sum_{t=0}^{+\infty} \beta^t u(c_t) : c_t + k_{t+1} \leq F_t(k_t) + k_t(1-\delta) \Big\} < +\infty.$$

Note that we allow that the function F is time-dependent.[2]

Exercise 3. $\sum_{t=0}^{+\infty} \beta^t u(c_t) = \infty$ *if*

$$\begin{aligned} u(c_t) &= c^\sigma \\ F_t(k) &= Ak \\ k_{t+1} &= s(A+1-\delta)k_t \\ c_t &= (1-s)(A+1-\delta)k_t \\ \beta\big(s(A+1-\delta)\big)^\sigma &> 1. \end{aligned}$$

However, $\sum_{t=0}^{+\infty} \beta^t u(c_t) < \infty$ *if*

$$\begin{aligned} u(c_t) &= ln(c) \\ F_t(k) &= Ak, \forall t, \text{ of } F_t(k) = Ak^\alpha, \alpha \in (0,1), \forall t. \end{aligned}$$

For more details, see Le Van and Pham (2016).

[2] See Le Van and Dana (2003) for a detailed presentation of optimal growth models.

We will implicitly assume that the Ramsey problem exists a solution. To prove this result, the idea is to show that we are maximizing a continuous function in a compact set (with the product topology, i.e., we say $\lim_{n\to\infty}(x^n) = (x)$ if $\lim_{n\to\infty} x_t^n = x_t, \forall t$). A detailed proof (when F is stationary) can be found in Le Van and Dana (2003).

In the following, we focus on the properties of the solution of the Ramsey problem.

Proposition 7. *1. Given the initial capital stock k_0, there exists a unique solution (c, k) to the optimal growth model.*

2. If $k_0 > 0$, then $c_t > 0, k_t > 0$ for any t.

Proof. **Part 1**. First, observe that, since u is strictly increasing, we have $c_t + k_{t+1} = F_t(k_t) + k_t(1-\delta), \forall t$ for any solution (c, k).

Now, assume that there are two solutions $(c, k), (c', k')$. Assume that $(c, k) \neq (c', k')$. Then there exist a date t such that $c_t \neq c'_t$. Indeed, if $c_t = c'_t, \forall t$, we have $k_1 = k'_1$ because $k_0 = k'_0$ and $c_0 = c'_0$. By induction, we have $k_t = k'_t, \forall t$. It means that $(c, k) = (c', k')$ which is a contradiction.

So, let t such that $c_t \neq c'_t$.

Let $\lambda \in (0, 1)$. Define $(c(\lambda), k(\lambda))$ by $(c(\lambda), k(\lambda)) = \lambda(c, k) + (1-\lambda)(c', k')$. This allocation is feasible (exercise!). Since $c_t \neq c'_t$ and the function u is strictly concave, we have $u(\lambda c_t + (1-\lambda)c'_t) > \lambda u(c_t) + (1-\lambda)u(c'_t)$. We also have $u(\lambda c_s + (1-\lambda)c'_s) \geq \lambda u(c_s) + (1-\lambda)u(c'_s)$, $\forall s$. By consequence, we get that

$$\sum_s \beta^s u(\lambda c_s + (1-\lambda)c'_s) > \lambda \sum_s \beta^s u(c_s) + (1-\lambda) \sum_s \beta^s u(c'_s) = \sum_s \beta^s u(c_s)$$

which is a contradiction.

Part 2. Let $k_0 > 0$. It is easy to prove that the consumption path $(0, 0, \ldots, 0, \ldots)$ is not optimal because u is strictly increasing. So, there exists a date t_1 such that $c_{t_1} > 0$.

Claim: If $c_{t+1} > 0$, then $c_t > 0$.

Proof of claim: Let $c_{t+1} > 0$. Suppose that $c_t = 0$, then by increasing c_1 and decreasing c_{t+1}, we get a new allocation which can give an intertemporal utility higher than the maximum intertemporal utility (a contradiction). (This is an exercise for Readers.)

So, our claim is true. Applying this claim, we obtain that $c_t > 0$ for any $t \leq t_1$.

Since $c_0 > 0$, we can prove that $k_1 > 0$ (exercise). Then, $k_1 > 0$ implies that $c_1 > 0$. By induction, we get that $c_t > 0, k_t > 0$ for any t.

□

Proposition 8. *1. Necessary condition: If $(c_t^*, k_{t+1}^*)_{t\geq 0}$ is an optimal solution of the above Ramsey problem, then we have the so-called Euler equation:*

$$u'(c_t^*) = \beta u'(c_{t+1}^*)(1 - \delta + F'_{t+1}(k_{t+1})). \tag{4.35}$$

2. Sufficient condition: If

$$\text{Euler condition: } u'(c_t^*) = \beta u'(c_{t+1}^*)(1 - \delta + F'_{t+1}(k_{t+1})), \forall t \tag{4.36}$$

$$\text{Transversality condition: } \lim_{T\to\infty} \beta^T u(c_T^*) k_{T+1}^* = 0 \tag{4.37}$$

then $(c_t^*, k_{t+1}^*)_{t\geq 0}$ is an optimal solution of the above Ramsey problem.

Proof. **1. Necessary condition**.

Fix $t \geq 0$ and consider another allocation $(c_s, k_{i,s+1})_s$ given by $c_s = c^*_s, \forall s \notin \{t+1\}$, $k_s = k^*_t, \forall s \neq t+1$, and $(c_t, c_{t+1}, k_{i,t_1})$ is determined by

$$\underbrace{c^*_t - \epsilon}_{c'_t} + \underbrace{k^*_{t+1} + \epsilon}_{k'_{t+1}} = F_t(k^*_t) + (1-\delta)k^*_t$$

$$\underbrace{c^*_{t+1} + F_t(k^*_{t+1} + \epsilon) - F_t(k^*_{t+1}) + (1-\delta)\epsilon}_{c'_{t+1}} + k^*_{t+2} = F_t(k^*_{t+1} + \epsilon) + (1-\delta)(k^*_{t+1} + \epsilon),$$

where $\epsilon > 0$ is low enough so that $c^*_t - \epsilon > 0$.

By the optimality $(c^*_t, k^*_{t+1})_t$, we have

$$\begin{aligned}
\beta_t u(c^*_t) + \beta_{t+1} u(c^*_{t+1}) \geq & \beta_{i,t} u(c_t) + \beta_{t+1} u(c_{t+1}) \\
\beta_{i,t} u(c^*_t) + \beta_{i,t+1} u(c^*_{t+1}) \geq & \beta_{i,t} u(c^*_t - \epsilon) + \beta_{t+1} u\Big(c^*_{t+1} + F_t(k^*_{t+1} + \epsilon) - F_t(k^*_{t+1}) + (1-\delta)\epsilon\Big) \\
\beta_{i,t} \frac{u(c^*_t) - u(c^*_t - \epsilon)}{\epsilon} \geq & \beta_{i,t+1} \frac{u\Big(c^*_{t+1} + F_t(k^*_{t+1} + \epsilon) - F_t(k^*_{t+1}) + (1-\delta)\epsilon\Big) - u(c^*_{t+1})}{F_t(k^*_{t+1} + \epsilon) - F_t(k^*_{t+1}) + (1-\delta)\epsilon} \times \\
& \times \frac{F_t(k^*_{t+1} + \epsilon) - F_t(k^*_{t+1}) + (1-\delta)\epsilon}{\epsilon}
\end{aligned}$$

Let ϵ tend to zero, we get that $u'(c^*_t) \geq \beta u'(c^*_{t+1})(1 - \delta + F'_{t+1}(k_{t+1}))$.

Since $k^*_{i,t+1} > 0$, we can do as above but with $\epsilon < 0$ (chose ϵ so that $k^*_{t+1} + \epsilon > 0$) and get that $u'(c^*_t) \leq \beta u'(c^*_{t+1})(1 - \delta + F'_{t+1}(k_{t+1}))$. Therefore, we have the equality.

2. Sufficient condition.

Consider a feasible allocation $(c_s, k_{s+1})_s$. We have $c_t + k_{t+1} \leq F_t(k_t) + k_t(1-\delta), \forall t$. This implies that $c_t \leq F_t(k_t) + k_t(1-\delta) - k_{t+1}, \forall t$.

We can prove that, $\forall T$,

$$\sum_{t=0}^{T} \Big(\beta_{i,t} u(c^*_t) - \beta_{i,t} u(c_t)\Big) \geq \sum_{t=0}^{T} \beta^t u'(c^*_t)(c_t - c'_t) \geq -\beta^T u'(c^*_T) k^*_{T+1}.$$

□

Proposition 9 (Proposition 2.4.4 in Le Van and Dana (2003))**.** *Define the function $f(x) = (1-\delta)x + F(x)$.*

Assume that $u : \mathbb{R}_+ \to \mathbb{R}_+$ be twice continuously differentiable, strictly increasing, strictly concave, $u(0) = 0$, $u'(0) = +\infty$.

Assume that the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ be twice continuously differentiable, strictly increasing, strictly concave, $f(0) = 0$, $f'(+\infty) < 1$, $M \equiv f'(0) \leq +\infty$.

Let $k_0 > 0$.

Then, we have that

1. *The optimal capital stock (k_t) with $k_0 > 0$ is monotonic.*
2. *If $f'(0) \leq \frac{1}{\beta}$, then the optimal path (k_t) converges to zero.*
3. *If $f'(0) > \frac{1}{\beta}$, then the optimal path (k_t) converges to k^s determined by $f'(k^s) = \frac{1}{\beta}$.*

Remark 6. *The value k^s is called the "steady state capital stock".*

In general, finding solutions of the Ramsey problem is not easy. We will consider two examples where we can explicitly compute the optimal paths and rate of growth.

Example 2 (Logarithmic utility function and AK production function)**.** *Suppose* $u(c) = \ln(c)$, $F_t(k) = A_t k$. *Let us denote* $A'_t = A_t + 1 - \delta$. *We can prove (exercise) that the optimal path* (k_t) *is given by* $k_{t+1} = \beta(1 - \delta + A_t)k_t \; \forall t$. *Then the optimal output* Y^*_t *satisfies*

$$Y^*_t = \beta^t (A'_0 A'_1 \dots A'_t) Y_0,$$

with $Y_0 = A'_0 k_0$. *The optimal rate of saving is*

$$s^*_t = \frac{\beta A_t + (1-\delta)(\beta - 1)}{A_t} \leq \beta < 1,$$

which is increasing in A_t. *Moreover, if* $A_t \leq A_{t+1}$, *then* $s^*_t \leq s^*_{t+1}$.

We can also compute the rate of growth by $\frac{Y^*_{t+1}}{Y^*_t} = \beta(A_t + 1 - \delta)$.

Now suppose $A_t \to A > 0$ *as* $t \to +\infty$. *In this case* $\frac{Y^*_{t+1}}{Y^*_t} \to \beta(A + 1 - \delta)$ *and* $s^*_t \to s = \frac{\beta A + (1-\delta)(\beta-1)}{A}$. *Let us look at two cases:*

- *If* $\beta(A + 1 - \delta) > 1 \Leftrightarrow sA - \delta > 0$ *then* $\frac{Y^*_{t+1}}{Y^*_t} \to +\infty$.
- *If* $\beta(A + 1 - \delta) < 1 \Leftrightarrow sA - \delta < 0$ *then* $\frac{Y^*_{t+1}}{Y^*_t} \to 0$.

We get the same results as in the Harrod model: the TFP plays a crucial role on the economic growth.

Example 3 (Logarithmic utility function and Cobb-Douglas production function)**.** *Assume that* $u(c) = \ln(c), F_t(k) = Ak^\alpha, \alpha \in (0,1)$, *and* $\delta = 1$. *In this case, we can prove that the optimal path is given by* $k_{t+1} = \beta\alpha A k_t^\alpha \; \forall t \geq 0$, *and the saving rate is* $\alpha\beta$. *Therefore, the optimal output is (exercise)*

$$y^*_{t+1} = A^{\frac{1-\alpha^{t+2}}{1-\alpha}} (\alpha\beta)^{\frac{\alpha - \alpha^{t+2}}{1-\alpha}} k_0^{\alpha^{t+2}}.$$

When t *goes to infinity, the output* y^*_{t+1} *converges to a steady state*

$$y^s = A^{\frac{1}{1-\alpha}} (\alpha\beta)^{\frac{\alpha}{1-\alpha}}.$$

There is no growth in the long run. It is due to the fact the production function is of strictly decreasing returns to scale. However, observe when A *increases, the steady state becomes higher.*

Chapter 5

Endogenous growth

5.1 How to increase TFP and obtain economic growth?

Determinants of TFP

So far the TFP A_t in period t seems to be a blackbox in a production function of the type

$$y_t = A_t k_t^\alpha N_t^\beta,$$

where k_t, N_t are the number of machines and the number of workers. In this modeling, we do not take into account the quality of the machines, nor the skill of the workers.

Now, following Le Van and Pham (2022), we write the production function as follows

$$y_t = a m_t (\mathcal{K}_t)^\alpha (\mathcal{N}_t)^\beta,$$

where m_t is the quality of the management, the macroeconomic environment (stability, law rule), $\mathcal{K}_t$ is the effective capital stock, $\mathcal{N}_t$ is the effective labor. Let ζ_t denote the technology embedded in the machines, θ_t denote the working time, h_t the human capital (education, training, health) of the workers. We then have

$$\mathcal{K}_t = \zeta_t k_t \text{ and } \mathcal{N}_t = \theta_t h_t N_t.$$

The production function now is $y_t = A_t k_t^\alpha N_t^\beta$ where the TFP is $A_t \equiv \left[a m_t \zeta_t^\alpha (\theta_t h_t)^\beta\right]$. If we assume θ_t depends positively on wages or bonus (incentive mechanism) then

$$y_t = A_t k_t^\alpha N_t^\beta \tag{5.1}$$

$$\text{where the TFP} \qquad A_t = \left[a m_t \zeta_t^\alpha \left(\theta(w_t) h_t\right)^\beta\right] \tag{5.2}$$

The TFP A_t is not anymore a black box. If we invest in the quality of management,[1] in technology, in training, education, health and if the salaries of the workers are sufficiently incentive, we will have a high TFP. Using endogenous growth models (Lucas, 1988; Romer, 1990), we can prove that there may be economic growth even with strictly decreasing returns to scale production function.

[1] Bloom et al. (2013) ran a management field experiment on large Indian textile firms and provided free consulting on management practices to randomly chosen treating plants. By comparing the performance of these plants to a set of control plants, they found that adopting these management practices raised the TFP by 17% in the first year.

5.2 Ideas, innovation, population

We consider an endogenous growth model à la Romer (1990) but we assume constant saving rate. The first block of the model is similar to Solow (1957):

$$\begin{aligned}
&C_t + S_t = Y_t\\
&I_t = S_t\\
&K_{t+1} = (1-\delta)K_t + I_t\\
&S_t = sY_t \text{ (constant saving rate)}\\
&L_t = L_0 N^t \text{ (population).}
\end{aligned}$$

The second block of the model is inspired by Romer (1990): we assume that the output and the productivity are determined by

$$Y_t = A_t^{\alpha_a} K_t^{\alpha_k} L_{yt}^{\alpha_l}, \alpha_i \in (0,1), \forall i = a, k, l \tag{5.3}$$

$$A_{t+1} = d_a A_t + G(A_t, \lambda_a L_t) = d_a A_t + h A_t^{\gamma_a} (\lambda_a L_t)^{\gamma_l} \tag{5.4}$$

The final good production process uses 3 inputs: physical capital K_t, labor L_{yt} and knowledge A_t.

Here, we interpret A_t as the stock of knowledge which is non-rival. The knowledge at date $t+1$ is produced from the knowledge at date t and labor (researcher, engineer, for instance).

The labor force L_t is divided into two parts: $L_{yt} = (1-\lambda_a)L_t$ for the production of the final good and $L_{at} = \lambda_a L_t$ for the production of knowledge. Here, we assume that $\lambda_a \in [0,1]$. If $\lambda_a = 0$, then we recover the standard Solow model.

Proposition 10. *Consider the above model. Assume that (Y_t, K_t, S_t, A_t) is a balanced growth path:*

$$Y_t = Y^*(1+g_y)^t, \quad K_t = K^*(1+g_k)^t, \quad S_t = S^*(1+g_s)^t, \quad A_t = A^*(1+g_a)^t, \forall t.$$

Then, we have that

$$g_y = g_a \tag{5.5}$$

$$(1+g_a)^{\gamma_a - 1}(1+n)^{\gamma_l} = 1 \tag{5.6}$$

$$(1+g_k)^{\gamma_a} = (1+g_a)^{\gamma_a}(1+n)^{\gamma_l} \tag{5.7}$$

$$(1+g_a)A^* = d_a A^* + h(\lambda_a L_0)^{\gamma_a}(A^*)^{\gamma_a} \tag{5.8}$$

$$(1+g_k)K^* = (1-\delta)K^* + s(\lambda_l L_0)^{\alpha_l}(A^*)^{\alpha_a}(K^*)^{\alpha_k}. \tag{5.9}$$

Proof. Let (Y_t, K_t, S_t, A_t) be a balanced growth path. Since $L_{yt} = (1-\alpha)L_t$, the capital accumulative gives

$$\begin{aligned}
K_{t+1} &= (1-\delta)K_t + sA_t^{\alpha_a} K_t^{\alpha_k} L_{yt}^{\alpha_l}\\
&= (1-\delta)K_t + sA_t^{\alpha_a} K_t^{\alpha_k} ((1-\alpha)L_t)^{\alpha_l}\\
&= (1-\delta)K_t + sA_t^{\alpha_a} K_t^{\alpha_k} ((1-\alpha)L_0 N^t)^{\alpha_l}
\end{aligned}$$

Hence, we get a system

$$\begin{aligned}
\big(K^*(1+g_k)\big)^{t+1} &= (1-\delta)K^*(1+g_k)^t + s\big(A^*(1+g_a)^t\big)^{\alpha_a}\big(K^*(1+g_k)^t\big)^{\alpha_k}((1-\alpha)L_0N^t)^{\alpha_l}\\
A^*(1+g_a)^{t+1} &= d_a A^*(1+g_a)^t + h\big(A^*(1+g_a)^t\big)^{\gamma_a}(\lambda_a L_0 N^t)^{\gamma_l}
\end{aligned}$$

Then, by using the same argument in Proposition 3, we obtain our result. □

Note that we need to impose several assumptions to ensure the existence of balanced growth path. Indeed, condition $A^* > 0$ requires that $1 + g_a > d_a$. Let $\gamma_a < 1$. In this case, condition $1 + g_a > d_a$ becomes

$$(1+n)^{\gamma_l} = (1+g_a)^{1-\gamma_a} > d_a^{1-\gamma_a}.$$

So, the population growth rate must not be too low. For the role of population growth, see also Malthus (1978), Kremer (1993), Jones (2022).

Exercise 4 (Frankel (1962))**.** *Let us study the following system.*

Frankel (1962)'s model:

$$\begin{aligned}
&C_t + S_t = Y_t\\
&I_t = S_t\\
&K_{t+1} = (1-\delta)K_t + I_t \textit{ (capital accumulation)}\\
&S_t = sY_t \textit{ (constant saving rate)}\\
&Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}, \alpha \in (0,1)\\
&A_t = A(K_t, L_t) = a_0 \frac{K_t^{\gamma}}{L_t^{\gamma_l}}\\
&L_t = L_0 N^t, \quad N = 1+n > 0, \textit{ (population)}.
\end{aligned}$$

Find the balanced growth path of the Frankel model. How do growth rates depend on parameters? Find conditions under which the balanced growth path is stable.

Proof. Hints: From this system, we get that

$$K_{t+1} = (1-\delta)K_t + s a_0 K_t^{\alpha+\gamma} (L_0 N^t)^{1-\alpha-\gamma_l}.$$

□

5.3 Role of investment in training and education

We investigate the role of investment in training and education. For the sake of tractability, we consider a static model (see Barro and Sala-i-Martin (1995), Lucas (1988) among other for infinite-horizon growth models with human capital). The social planner maximizes the intertemporal utility $u(c_0) + \beta u(c_1)$ subject to the following constraints

$$c_0 + k_1 + G \leq w_0 \tag{5.10a}$$
$$L_1 \leq H(L_0, G) \tag{5.10b}$$
$$c_1 \leq F(k_1, L_1) \tag{5.10c}$$
$$c_0, c_1, k_1, G \geq 0 \tag{5.10d}$$

where $w_0 > 0$ is the initial output and L_0 is the original labor force.

We assume that $u' > 0 > u''$ and $u'(0) = \infty$. This ensures that both consumptions are strictly positive at optimal.

We assume that $H(L_0, G) = L_0 + hG$ where $h > 0$ represents the efficiency of investment in training of high-skilled workers or human capital. Assume that $L_0 > 0$. L_0 is equal to $H(L_0, 0)$ which is the labor effective when there is no high-skilled worker.

Assume that $F(K, L) = AK^{\alpha}L^{\beta}$.

The maximization problem of the social planner can be rewritten as follows:

$$\max_{(c_0,c_1,S} u(c_0)+\beta u(c_1) \tag{5.11}$$

$$c_0+S\leq w_0 \tag{5.12}$$

$$c_1\leq G(S) \tag{5.13}$$

$$c_0,c_1,S\geq 0 \tag{5.14}$$

where the payoff function $G(S)$ is defined by

$$G(S)\equiv \max_{(k_1,G)}\Big\{Ak_1^\alpha(L_0+hG)^\gamma\Big\} \tag{5.15}$$

$$k_1+G\leq S,\quad k_1,G\geq 0 \tag{5.16}$$

5.3.1 Training or not training?

Let us consider the intermediate problem. This is called *static problem*.

$$G(S)\equiv \max_{(k_1,G)}\Big\{Ak_1^\alpha(L_0+hG)^\gamma\Big\} \tag{5.17}$$

$$k_1+G\leq S,\quad k_1,G\geq 0 \tag{5.18}$$

By using the standard method, we obtain the following result.

Lemma 1. *Let $H(L_0,G)=L_0+hG$. Under these above specifications, we have that*

1. *If $\gamma hS\leq \alpha L_0$, then $G=0$.*
2. *If $\gamma hS>\alpha L_0$, then*

$$G=\frac{\gamma hS-\alpha L_0}{h(\alpha+\gamma)} \tag{5.19}$$

$$k_1=\frac{\alpha}{\alpha+\gamma}\frac{L_0+hS}{h}. \tag{5.20}$$

According to this result, given the savings S, it is optimal to invest in training high-skilled workers if and only if $\gamma hS>\alpha L_0$. So, all γ,h,S,L_0 matter.

5.3.2 Optimal allocation

We now look at the problem (5.10). Notice that at optimum, we have $c_0>0,c_1>0,k_1>0$ but G may be zero.

The Lagrange function is

$$\begin{aligned}L&=u(c_0)+\beta u(c_1)+\lambda_0(w_0-c_0-k_1-G)+\lambda_1(F(k_1,H(L_0,G))-c_1)+\mu_gG\\&=u(c_0)+\beta u(c_1)+\lambda_0(w_0-c_0-k_1-G)+\lambda_1\Big(Ak_1^\alpha(L_0+hG)^\beta-c_1)+\mu_gG\end{aligned}$$

At optimum, we have

$$\begin{aligned}u'(c_0)-\lambda_0&=0\\u'(c_1)-\lambda_1&=0\\(k_1):\quad -\lambda_0+\alpha\lambda_1Ak_1^{\alpha-1}(L_0+hG)^\beta&=0\\(G):\quad -\lambda_0+h\beta\lambda_1Ak_1^{\alpha-1}(L_0+hG)^{\beta-1}+\mu_g&=0\\\mu_g\geq 0,\mu_gG&=0.\end{aligned}$$

By solving this system, we get that:

Proposition 11. *Assume that $u(c) = ln(c)$.*

1. *If $\gamma\beta h w_0 \leq (1+\alpha\beta)L_0$, then $G = 0$.*
2. *If $\gamma\beta h w_0 > (1+\alpha\beta)L_0$, then*

$$k_1 = \frac{\alpha\beta}{1+\alpha\beta+\gamma\beta}(w_0 + \frac{L_0}{h}) \tag{5.21}$$

$$G = \frac{1}{h}\Big(\frac{\gamma\beta h}{1+\alpha\beta+\gamma\beta}w_0 - \frac{1+\alpha\beta}{1+\alpha\beta+\gamma\beta}L_0\Big). \tag{5.22}$$

Note that both the quantity G and the ratio $\frac{G}{w_0}$, i.e., the proportion of investment in humain capital to the aggregate income are increasing in the income. This is consistent with data.

5.4 A simple endogenous growth model à la Ramsey

We present a simple infinite-horizon endogenous growth model studied in Le Van and Pham (2022). There is a social planer who maximizes the population's intertemporal utility $\sum_{t=0}^{\infty}\beta^t u(c_t)$ subject to sequential constraints: $c_t + S_{t+1} = G_t F(k_t)\ \forall t \geq 0$, where c_t, S_{t+1} are consumption, savings.

We now assume that the saving S_{t+1} is shared in investment in physical capital k_{t+1} and in investment T_{t+1} in TFP, i.e., $k_{t+1} + T_{t+1} = S_{t+1}$. G_{t+1} is a function of T_{t+1} and we write $G(T_{t+1})$. We rewrite the model as follows

$$\begin{aligned}
&\max \sum_{t=0}^{\infty} \beta^t u(c_t)\\
\text{for } t \geq 1 \qquad & c_t + S_{t+1} = H(S_t) \equiv \max\{G(T_t)F(k_t) : T_t + k_t = S_t, \text{ and } T_t, k_t \geq 0\}\\
& \text{where } k_{t+1} + T_{t+1} = S_{t+1}.
\end{aligned}$$

For the sake of tractability, we assume that $F(k) = k^{\alpha}, \alpha \in (0,1), G(T) = (\lambda T+1)^{\xi}, \xi > 0$, and $\lambda > 0$.[2] The parameter ξ measures the quality of the TFP investment technology. The higher ξ the more efficient the TPF investment. The parameter λ measures the utilization of T_t. For instance λ is small because of diversion of T_t.

Static analysis. We firstly look at the static problem and the properties of the function H. Under our specifications, we have $H(S_t) \equiv \max\{(\lambda T_t + 1)^{\xi} k_t^{\alpha} : T_t + k_t = S_t, \text{ and } T_t, k_t \geq 0\}$. Solving this problem is equivalent to solving the following problem whose objective function is strictly concave

$$\max\{\xi ln(\lambda T_t + 1) + \alpha ln(k_t) : T_t + k_t = S_t, \text{ and } T_t, k_t \geq 0\}.$$

(T_t, k_t) is an optimum point if and only if there are non-negative values μ_1, μ_2 such that

$$\frac{\alpha}{k_t} = \mu_1, \quad \xi\frac{\lambda}{\lambda T_t + 1} + \mu_2 = \mu_1, \quad \mu_2 T_t = 0.$$

If $T_t = 0$ at optimal, then we have $\lambda\xi = \mu_1 - \mu_2 \leq \mu_1 = \alpha/k_t = \alpha/S_t$. Thus, we have $S_t \leq \alpha/(\lambda\xi)$.

[2] Here, we implicitly assume that u is continuously differentiable, strictly increasing, concave, $u'(0) = \infty$ and $\sum_{t=0}^{\infty}\beta^t u(D_t) < \infty$ where the sequence (D_t) is defined by $D_0 = H(S_0)$, $D_{t+1} = H(D_t)$.

If $T_t > 0$ at optimal, the FOC implies that $\frac{\alpha}{k_t} = \xi\frac{\lambda}{\lambda T_t+1}$, i.e., $(\lambda T_t + 1)\alpha = \xi\lambda k_t = \xi\lambda(S_t - T_t)$. So, we can compute that

$$T_t = \frac{\xi\lambda S_t - \alpha}{\lambda(\alpha+\xi)}, \quad k_t = \frac{\alpha(\lambda S_t + 1)}{\lambda(\alpha+\xi)}$$
$$H(S_t) = \Big(\frac{\xi(\lambda S_t+1)}{\alpha+\xi}\Big)^{\xi}\Big(\frac{\alpha(\lambda S_t+1)}{\lambda(\alpha+\xi)}\Big)^{\alpha} = \frac{\xi^\xi\alpha^\alpha}{(\alpha+\xi)^{\alpha+\xi}}\frac{(\lambda S_t+1)^{\alpha+\xi}}{\lambda^\alpha}.$$

Of course, $T_t > 0$ is equivalent to $\xi\lambda S_t - \alpha > 0$.

Summing up, we obtain the following result:

Lemma 2. *Let $S_t > 0$ be given. Under above specifications, we have that:*

1. *If $S_t \leq \frac{\alpha}{\xi\lambda}$ then $T_t = 0$. It is not optimal to invest in TFP when S_t is small. In this case $S_t = k_t$ and $H(S_t) = S_t^\alpha$.*
2. *If $S_t > \frac{\alpha}{\xi\lambda}$ then $T_t > 0$. (If S_t is high enough then it is worthwhile to invest in TFP.) In this case*
$$H(S_t) = a_h\frac{(\lambda S_t+1)^{\alpha+\xi}}{\lambda^\alpha},$$
where $a_h \equiv \frac{\xi^\xi\alpha^\alpha}{(\alpha+\xi)^{\alpha+\xi}}$ depending on (α,ξ).

The function H is increasing in λ when $S > \frac{\alpha}{\xi\lambda}$. The lower the level of diversion, the higher the total output.

Notice that the function $H(S)$ is increasing return to scale and convex for any $S > \alpha/(\xi\lambda)$. This is one way to introduce increasing return to scale technology is growth models (see Romer (1986) for more detailed discussions).

Observe also that in point 2 of Lemma 2, the ratio of investment to the total savings is $\frac{T_t}{S_t} = \frac{\xi\lambda - \frac{\alpha}{S_t}}{\lambda(\alpha+\xi)}$ is increasing in S_t.

The following graphic shows the evolution of investment in R&D in some countries.

Dynamic analysis. We now show the dynamics of the optimal path. It is easy to see that the optimal path (S_t) is monotonic. We then have the convergence of optimal paths.[3]

Proposition 12. *Assume that $\beta\alpha^\alpha\xi^{1-\alpha}\lambda^{1-\alpha} > 1$ and $\alpha+\xi \geq 1$. Then any optimal sequence $\{S_t^*\}_t$, and hence any optimal sequence of outputs $\{y_t^* = H(S_t^*)\}$ converge to infinity.*[4] *By consequence, there is a date τ such that the country invests in TFP from date τ on (i.e., $T_t > 0\ \forall t \geq \tau$).*

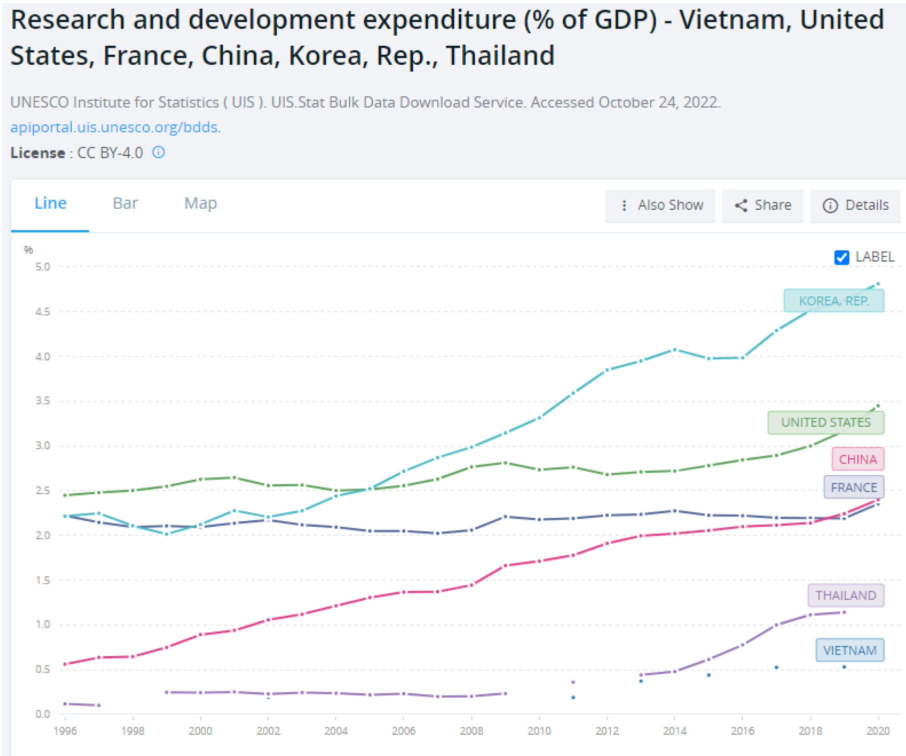
According to our result, if the utilization of investment in technology (parameter λ) and the quality of the TFP investment technology (parameter ξ) are high, and we have increasing return to scale ($\alpha+\xi \geq 1$) technology, we get growth without bounds.

[3] We do not provide a full analysis in this paper. However, more dynamic properties may be obtained by adopting the method in Kamihigashi and Roy (2007), Bruno et al. (2009).

[4] Proof: If $S < \frac{\alpha}{\xi\lambda}$, then we have $H'(S) = \alpha S^{\alpha-1} > \alpha\big(\frac{\alpha}{\xi\lambda}\big)^{\alpha-1} = \alpha^\alpha\xi^{1-\alpha}\lambda^{1-\alpha}$.
If $S > \frac{\alpha}{\xi\lambda}$, then we have

$$H'(S) = a_h(\alpha+\xi)\lambda\frac{(\lambda S_t+1)^{\alpha+\xi-1}}{\lambda^\alpha} > a_h(\alpha+\xi)\lambda\frac{(\lambda\frac{\alpha}{\xi\lambda}+1)^{\alpha+\xi-1}}{\lambda^\alpha} = \alpha^\alpha\xi^{1-\alpha}\lambda^{1-\alpha}. \tag{5.23}$$

Since $\beta\alpha^\alpha\xi^{1-\alpha}\lambda^{1-\alpha} > 1$, by applying Proposition 4.6. in Kamihigashi and Roy (2007), we have that every optimal path increasingly converges to infinity.



The rate of growth $(\frac{y^*_{t+1}}{y^*_t} - 1)$ is now endogenous. It is obtained by an optimal share between investing in physical capital and investing in HC, Technology, Management Quality, incentive mechanisms. For that reason, we call these types of models Endogenous Growth Models.

The above results (Lemma 2 and Proposition 12) above deserve some comments.

- The country will wait until some date τ, when the optimal output generates enough saving $S_\tau > \frac{\alpha}{\xi\lambda}$, before investing in TFP.
- If the diversion of the T_t is high (i.e. λ is low), the country may never invest in TFP and will not have growth.
- If λ is lower (the diversion exists), the date τ becomes larger. The country has to wait longer before starting to invest in TFP.

Chapter 6

Finance and macroeconomics

6.1 Productive sector or financial market

The financial market has been considered as one of main causes of economic recession or/and fluctuation. But, does financial market always cause an economic recession? What is the role of financial market on the productive sector?

We address these questions by using a very simple model (see Le Van and Pham (2016) among other for an infinite-horizon model). Consider an agent whose initial endowment is S. Agent has two choices to invest: to produce or to invest in financial asset. She may produce $AF(K)$ units of consumption good by using K units of physical capital. If she buys a units of financial asset with price q, she will receive ξa units of consumption good, where ξ is the dividend of the financial asset.

$$\max_{K,a\geq 0} AF(K) + \xi a \tag{6.1}$$

$$K + qa \leq S \tag{6.2}$$

At optimum, we must have $K + qa = S$. So, the problem is equivalent to the following one:

$$\max_{0\leq K\leq S} AF(K) + \frac{\xi}{q}(S-K) = \max_{0\leq K\leq S} AF(K) - \frac{\xi}{q}K + \frac{\xi}{q}S. \tag{6.3}$$

From this, we can easily obtain the following result.

Proposition 13. *Assume that F is concave, strictly increasing, continuously differentiable. Let $S > 0, q > 0, A > 0, \xi > 0$.*

(i) If $AF'(0) \leq \frac{\xi}{q}$, agent does not produce, i.e., $K = 0$ and $a = S$.

(ii) If $AF'(S) \geq \frac{\xi}{q}$, agent does not invest in financial asset, i.e., $a = 0$ and $K = s$.

(iii) If $AF'(S) \leq \frac{\xi}{q} \leq AF'(0)$, agent produces and invests in financial asset. K is determined by $AF'(K) = \frac{\xi}{q}$ and $a = S - K$.

The intuition is very clear: We invest in the highest return asset. Point (i) says that we do not produce if the maximum return of the productive sector is less than the return of the financial sector. The main implication of Proposition 13 is that the productive sector will be disappeared if its productivity is low.

We can define the notion of economic recession: there is no investment in the productive sector (or the input level is lower than some threshold).

According to this concept, we can see the importance of both productivity and financial return on economic recession.

6.2 Credit constraints

We now introduce credit constraints and investigate its effects on the firm's decision and on the whole economy.[1]

6.2.1 Individual maximization problem with credit constraints

Let us consider an agent who has S units of good at the initial date. This good can be consumed or used to produce. There is no uncertainty. What he want is to invest in order to maximize its wealth at the next date.

On the one hand, he can produce with technology $F(K)$.

On the other hand, he can also invest in financial asset with gross return is r. If he want to borrow an amount a units of consumption good then: (i) at next date, he must pay back ra, (ii) there is a borrowing constraint (because the financial market is imperfect) under which the agent cannot borrow more than a fraction $f < 1$ of his project's outcome $F(K)$, where f is an exogenous parameter which is the borrowing limit of the agent.

The maximization problem of the agent is

$$(P) \qquad \pi_1 = \max_{K,a} \left[F(K) - ra\right] \tag{6.4}$$

$$\text{subject to} \qquad 0 \leq K \leq S + a \tag{6.5}$$

$$ra \leq fF(K). \tag{6.6}$$

If we interpret this agent as a country, the parameter f can be interpreted as an index of the country's financial development.

The following table from the Enterprise Surveys (2018)'s panel datasets suggests that borrowing and collateral constraints matter for the development of firms.

Economy	Proportion of loans requiring collateral (%)	Value of collateral needed for a loan (% of the loan amount)	Percent of firms not needing a loan	Percent of firms whose recent loan application was rejected	Proportion of investments financed internally (%)
All Countries	79.1	205.8	46.4	11.0	71.0
East Asia & Pacific	82.6	238.4	50.7	6.4	77.8
Europe & Central Asia	78.7	191.9	54.3	10.9	72.4
Latin America & Caribbean	71.3	198.5	45.0	3.1	62.7
Middle East & North Africa	77.4	183.0	51.8	10.2	71.1
South Asia	81.1	236.0	44.7	14.4	73.9
Sub-Saharan Africa	85.3	214.8	37.4	15.3	73.9

Remark 7. *Let us compare our credit constraint with those in the literature. Matsuyama (2007) (Section 2) considers a model with heterogeneous agents, which corresponds to our model with $k_i = 1$, $S_i = w, a_i = 1 - w$. However, different from our setup, investment projects in Matsuyama (2007) are non-divisible.*

It should be noticed that constraint (6.6) is different from condition (3) in Kiyotaki and Moore (1997). Indeed, Kiyotaki and Moore (1997) assume that the borrower's repayment does not exceed the market value of her land quantity while we assume that the repayment does not exceed the market value of the borrower's project.

[1] The literature on this topic is quite large. We can mention, among others, Kiyotaki and Moore (1997), Kiyotaki (1998), Buera and Shin (2013) Midrigan and Xu (2014), Moll (2014), Le Van and Pham (2016), Pham (2022) who work in infinite-horizon general equilibrium models.

Some authors (Buera and Shin, 2013; Moll, 2014) set $k_i \leq \theta w_i$, where $w_i \geq 0$ is the agent i's wealth and interpret that θ measures the degree of credit frictions (credit markets are perfect if $\theta = \infty$ while $\theta = 1$ corresponds to financial autarky, where all capital must be self-financed by entrepreneurs). In our framework, S_i plays a similar role of wealth w_i in Buera and Shin (2013), Moll (2014). Notice that Buera and Shin (2013), Moll (2014) do not allow to intertemporal borrowing as in our framework.

Another way to introduce credit constraint is to set $a_i \leq \theta k_i$. This corresponds to constraint (3) in Midrigan and Xu (2014).

We consider two particular production functions.

1. **Linear technology**. Assume that the technology is linear $F(K) = AK$. We study the individual problem (P).

   Lemma 3 (individual problem)**.** *Assume that $F_i(K) = A_i K$. The solution for agent i's maximization problem is described as follows.*

   (a) If $r > A_i$, then agent i does not produce goods and invest all her initial wealth in the financial market: $k_i = 0, a_i = -S_i$.

   (b) If $A_i = r$, then the solutions for the agent's problem are all allocations (k_i, a_i) such that $-S_i \leq a_i \leq f_i k_i$ and $k_i = a_i + S_i$.

   (c) If $A_i > r > f_i A_i$, then agent i borrows from the financial market and the borrowing constraint is binding.

   $$k_i = \frac{r}{r - f_i A_i} S_i, \quad a_i = \frac{f_i A_i}{r - f_i A_i} S_i. \tag{6.7}$$

   (d) If $r \leq f_i A_i$, there is no solution.[2]

   Proof. At optimal, we have $k_i = S_i + a_i$. Indeed, if $k_i < S_i + a_i$, we can increase k_i a little bit, keep a_i unchanged, and hence and get a higher value of $F(k_i) - r a_i$. Therefore, we have $k_i = S_i + a_i$. So, $A_i k_i - r a_i = A_i S_i + a_i(A - r)$. Budget and credit constraint become $a_i \geq -S_i$ and $(r - f_i A_i) a_i \leq f_i A_i S_i$.

   We consider different cases.

   (a) If $r > A_i$, then $r > f_i A_i$ (because $f_i < 1$). So, our problem constraints becomes $-S_i \leq a_i \leq \frac{f_i A_i S_i}{r - f_i A_i}$. Since $A_i k_i - r a_i = A_i S_i + a_i(A_i - r)$ and $A_i - r < 0$, the optimal value of a_i must be $-S_i$ which implies that $k_i = 0$.

   (b) If $A_i = r$, then $A_i k_i - r a_i = A_i S_i + a_i(A_i - r) = 0$ for any $a_i \geq -S_i$ and $r a_i \leq f_i A_i k_i$. So, the solutions for the agent's problem are all allocation (k_i, a_i) such that $-S_i \leq a_i \leq f_i k_i$ and $k_i = a_i + S_i$.

   (c) If $A_i > r > f_i A_i$, then the problem's constraints become $a_i \geq -S_i$ and $a_i \leq a_i \leq \frac{f_i A_i S_i}{r - f_i A_i}$. Since $A_i k_i - r a_i = A_i S_i + a_i(A_i - r)$ and $A_i - r > 0$, the optimal value of a_i must be $\frac{f_i A_i S_i}{r - f_i A_i}$ which implies that $k_i = \dfrac{r}{r - f_i A_i} S_i$.

   (d) If $r \leq f_i A_i$, there is no solution. Indeed, if $r \leq f_i A_i$, then agent i may choose $a_i = +\infty$ and $k_i = +\infty$ and have $c_i = +\infty$.

[2] However, in the case of Cobb-Douglas technology, we have $f'(\infty) = 0$, then the feasible set is compact, which implies that the individual problem always has a solution.

□

When the productivity is low, i.e., $A < r$ then $K = 0$, $a = S$. This agent invests in financial asset, not to produce. When $A > r > fA$, the agent borrows with maximum level in the sense that the borrowing constraint is binding.

Corollary 1. *(**Collateral effect**) Assume that $A > r > fA$. At optimum, we have*

$$\frac{\partial K}{\partial f} > 0, \frac{\partial^2 K}{\partial^2 f} < 0 \tag{6.8}$$

$$\frac{\partial a}{\partial f} > 0, \frac{\partial^2 a}{\partial^2 f} < 0 \tag{6.9}$$

$$\frac{\partial \pi_1}{\partial f} > 0, \frac{\partial^2 \pi_1}{\partial^2 f} < 0. \tag{6.10}$$

2. **Cobb-Douglas technology** Assume that $F(K) = AK^\alpha$ where $\alpha \in (0,1)$. The maximization problem of the agent is

$$(P) \qquad \pi_1 = \max_{K,a} \left[AK^\alpha - ra\right] \tag{6.11}$$

$$\text{subject to} \qquad 0 \leq K \leq S + a \tag{6.12}$$

$$ra \leq fAK^\alpha. \tag{6.13}$$

Different from the linear technology case, we always have that $K > 0$ at optimum since $F'(0) = \infty$.

The characterization of the solution is presented in the following result.

Proposition 14. *(a) If $1 > \frac{f}{\alpha} + \left(\frac{r}{\alpha AS^{\alpha-1}}\right)^{\frac{1}{1-\alpha}}$ then the borrowing constraint is binding. In this case, the solution is given by*

$$K^{1-\alpha} - \frac{S}{K^\alpha} = \frac{fA}{r}, \quad a = K - S. \tag{6.14}$$

(b) If $1 < \frac{f}{\alpha} + \left(\frac{r}{\alpha AS^{\alpha-1}}\right)^{\frac{1}{1-\alpha}}$ then the borrowing constraint is not binding. In this case, the solution is given by

$$K = \left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}} \quad \text{i.e.,} \quad \alpha AK^{\alpha-1} = r, \tag{6.15}$$

$$a = K - S. \tag{6.16}$$

Agent i lends ($a \leq 0$) if $r \geq \alpha AS^{\alpha-1}$, and borrows ($a \geq 0$) if $r \leq \alpha AS^{\alpha-1}$.

Proof. See Pham and Pham (2021). □

Corollary 2. *(**Collateral effect**) Assume that $1 > \frac{f}{\alpha} + \left(\frac{r}{\alpha AS^{\alpha-1}}\right)^{\frac{1}{1-\alpha}}$. At optimum, we have*

$$\frac{\partial K}{\partial f} > 0, \frac{\partial a}{\partial f} > 0, \frac{\partial \pi_1}{\partial f} > 0. \tag{6.17}$$

The role of financial development f

We have a direct consequence: if the borrowing limit f is high enough in the sense that $f > \alpha$ then the borrowing constraint is not binding whenever the level of r, S, A.

The more interesting case is when the borrowing limit is not so high, i.e., $f < \alpha$. Note that $\alpha A S^{\alpha-1} = F'(S) \leq F'(K)$ for any $K \leq S$. Hence, $\alpha A S^{\alpha-1}$ is the lowest marginal productivity if the agent cannot borrow. The intuition is that the higher the marginal productivity, the more amount the agent want to borrow. The borrowing constraint is binding when the marginal productivity is high enough.

We consider an economy with $1 - \left(\frac{r}{\alpha A S^{\alpha-1}}\right)^{\frac{1}{1-\alpha}} > 0$, i.e. $\alpha A S^{\alpha-1} > r$. This condition means that the marginal productivity is greater the interest rate. This condition is satisfied if the TFP A is high enough. According to Proposition 14 the borrowing constraint is binding if and only if the borrowing limit is lower than a critical threshold: $f < \bar{f} \equiv \alpha\Big(1 - \left(\frac{r}{\alpha A S^{\alpha-1}}\right)^{\frac{1}{1-\alpha}}\Big)$.

We see that

$$\frac{\partial \bar{f}}{\partial \bar{r}} < 0, \frac{\partial \bar{f}}{\partial \bar{A}} > 0, \frac{\partial \bar{f}}{\partial \bar{S}} < 0 \tag{6.18}$$

6.2.2 General equilibrium

Definition 4. *Assume that there are m agents. A list of allocations and gross interest rate $((k_i, a_i)_{i=1}^m, r)$ is a general equilibrium if it satisfies two conditions:*

1. *For each i,, given r, the allocation (k_i, a_i) is a solution to the maximization problem of the agent i is*

$$(P_i) \qquad \pi_i = \max_{k_i, a_i} \Big[F_i(k_i) - r a_i\Big] \tag{6.19}$$

$$\textit{subject to} \qquad 0 \leq k_i \leq S_i + a_i \tag{6.20}$$

$$r a_i \leq f_i F_i(k_i). \tag{6.21}$$

2. *The financial market clears:*

$$\sum_{i=1}^{m} a_i = 0. \tag{6.22}$$

Proposition 15 (characterization of general equilibrium)**.** *Assume that $f_i \in [0,1)$ and $F_i(k) = A_i k$. Assume that $A_1 > A_2$. There are only three different cases (each having a unique equilibrium and with the productive agent being the borrower).*

1. *If $f_1 \leq \frac{A_2}{A_1}\frac{S_2}{S_1+S_2}$, then the borrowing constraint of agent 1 is binding and there exists a unique equilibrium characterized by:*

$$\textit{Interest rate:} \quad r = A_2$$

$$\textit{Physical capital:} \quad k_1 = \frac{A_2}{A_2 - f_1 A_1} S_1, \quad k_2 = -\frac{f_1 A_1}{A_2 - f_1 A_1} S_1 + S_2$$

$$\textit{Financial asset:} \quad a_1 = \frac{f_1 A_1}{A_2 - f_1 A_1} S_1, \quad a_2 = -\frac{f_1 A_1}{A_2 - f_1 A_1} S_1;$$

The aggregate output and consumption of each agent are:

$$Y = A_2S_2 + A_1S_1\frac{A_2 - f_1A_2}{A_2 - f_1A_1}, \quad c_1 = A_1S_1\frac{A_2 - f_1A_2}{A_2 - f_1A_1}, \quad c_2 = A_2S_2.$$

2. *If* $\frac{A_2}{A_1}\frac{S_2}{(S_1+S_2)} < f_1 < \frac{S_2}{S_1+S_2}$, *then the borrowing constraint of agent 1 is binding, and there exists a unique equilibrium characterized by:*

$$\begin{aligned} \textit{Interest rate:} \quad & r = f_1A_1\Big(1 + \frac{S_1}{S_2}\Big) \\ \textit{Physical capital:} \quad & k_1 = S_1 + S_2, \quad k_2 = 0 \\ \textit{Financial asset:} \quad & a_1 = S_2, \quad a_2 = -S_2. \end{aligned}$$

The aggregate output and consumption of each agent are:

$$Y = A_1(S_1 + S_2), \quad c_1 = A_1(1 - f_1)(S_1 + S_2), \quad c_2 = f_1A_1(S_1 + S_2).$$

3. *If* $f_1 \geq \frac{S_2}{S_1+S_2}$, *then the borrowing constraint is not binding, and there exists a unique equilibrium characterized by:*

$$\begin{aligned} \textit{Interest rate:} \quad & r = A_1 \\ \textit{Physical capital:} \quad & k_1 = S_2 + S_1, \quad k_2 = 0 \\ \textit{Financial asset:} \quad & a_1 = S_2; \quad a_2 = -S_2 \end{aligned}$$

The aggregate output and consumption of each agent are:

$$Y = A_1(S_1 + S_2), \quad c_1 = A_1S_1, \quad c_2 = A_1S_2.$$

Proof. See Pham and Pham (2021). □

According to this result, we can present explicit formulas of equilibrium interest rate and aggregate output:

$$r = r(f_1) \equiv \begin{cases} A_2 & \text{if} \quad f_1 \leq f_1^* \\ f_1A_1\Big(1 + \dfrac{S_1}{S_2}\Big) & \text{if} \quad f_1^* < f_1 < f_1^{**} \\ A_1 & \text{if} \quad f_1 \geq f_1^{**} \end{cases}, \tag{6.23}$$

$$Y = Y(f_1) \equiv \begin{cases} A_2S_2 + A_1S_1\dfrac{A_2 - f_1A_2}{A_2 - f_1A_1} & \text{if} \quad f_1 \leq f_1^* \\ A_1(S_1 + S_2) & \text{if} \quad f_1^* < f_1. \end{cases} \tag{6.24}$$

We see that the interest rate r is in $[A_2, A_1]$ and Y belongs to $[A_1S_1 + A_2S_2, A_1(S_1 + S_2)]$. Both the interest rate r and the aggregate output Y are increasing in the credit limit f_1. So, the financial development matters for the economic development.

6.2.3 A model with exogenous borrowing limit

To provide a sharper comparison in terms of equilibrium outcomes between our model and the setup with exogenous borrowing limit, we replace constraint (6.6) by $a_i \leq \bar{a}_i$. The problem of agent i now becomes

$$(Q_i): \quad \pi_i = \max_{(k_i,a_i)} [F_i(k_i) - ra_i] \text{ subject to: } 0 \leq k_i \leq S_i + a_i \tag{6.25a}$$

$$\text{and } a_i \leq \bar{a}_i. \tag{6.25b}$$

Recall that in the problem (P_i) with credit constraint (6.6), the bound of a_i depends on the future value of the investment project and on the interest rate. Consequently, agent i cannot borrow if her project is not productive. By contrast, under exogenous borrowing limit setup (the problem (Q_i)), agent i can always borrow an amount $\bar{a}_i$ whether she has a project or not.

Notice that $a_i \geq -S_i \ \forall i$. At optimal, we must have $k_i = S_i + a_i$. Then, $\pi_i = A_iS_i + (A_i - r)a_i$. Consequently, we obtain the following result which is similar to Lemma 3.

Lemma 4 (individual problem)**.** *The solution of the problem (Q_i) is given by the following.*

1. *If $A_i < r$, then agent i does not produce goods and invest all her wealth in the financial market: $k_i = 0, a_i = -S_i$.*
2. *If $A_i > r$, then agent i borrows from the financial market and the borrowing constraint is binding: $a_i = \bar{a}_i, k_i = S_i + \bar{a}_i$.*
3. *If $A_i = r$, then the solutions for the agent's problem include all sets (k_i, a_i) such that $-S_i \leq a_i \leq \bar{a}_i$ and $k_i = a_i + S_i$.*

Notice that, in each case, the allocation does not depend on the interest rate. This is the main difference between this model and one with credit constraint.

Proposition 16 (general equilibrium with two agents)**.** *Assume that there are two agents with production function $F_i(k) = A_ik_i \ \forall i$ and $A_1 > A_2$.*

1. *If $S_2 > \bar{a}_1$, then $r = A_2$.*
2. *If $S_2 = \bar{a}_1$, then any $r \in [A_2, A_1]$ is an equilibrium interest rate. Equilibrium indeterminacy arises.*
3. *If $S_2 < \bar{a}_1$, then $r = A_1$. (High borrowing limit $\bar{a}_1$.)*

Proof. See Pham and Pham (2021). Since $\sum_i a_i = 0$, Lemma 4 implies that $r \in [A_2, A_1]$. There are three cases.

If $r = A_2$, then $r < A_1$ which implies that $a_1 = \bar{a}_1$ and $k_1 = S_1 + \bar{a}_1$. By using market clearing condition, we have $a_2 = -\bar{a}_1$, and hence $k_2 = S_2 - \bar{a}_1$. Therefore, we need condition $S_2 - \bar{a}_1 \geq 0$.

If $r = A_1$, then $r > A_2$ which implies that $k_2 = 0$, $a_2 = -S_2$. By using market clearing condition, we have $a_1 = S_2$, and hence $k_1 = S_1 + S_2$. We also need $a_1 \leq \bar{a}_1$, i.e., $S_2 \leq \bar{a}_1$.

If $r \in (A_2, A_1)$, then $a_1 = \bar{a}_1$ and $k_1 = S_1 + \bar{a}_1$. By using market clearing condition, we have $a_2 = -\bar{a}_1$, and hence $k_2 = S_2 - \bar{a}_1$. However, $A_2 < r$ implies that $k_2 = 0$. Therefore, we need condition $S_2 - \bar{a}_1 = 0$.

It is easy to verify that, if $S_2 - \bar{a}_1 = 0$, then any $r \in [A_2, A_1]$ is an equilibrium interest rate. In this case, we have

$$c_2 = A_2k_2 - ra_2 = A_2S_2 + (A_2 - r)a_2 = rS_2 \tag{6.26}$$
$$c_1 = A_1k_1 - ra_1 = A_1S_1 + (A_1 - r)a_1 = A_1S_1 + (A_1 - r)S_2. \tag{6.27}$$

Observe that c_1 is decreasing in r.

It should be noticed that multiple equilibria arises but it is not totally generic because we need $S_2 = \bar{a}_1$.

□

According to Proposition 16, we can compute individual consumptions and the aggregate

$$Y = \begin{cases} A_2S_2 + A_1S_1 + (A_1 - A_2)\bar{a}_1 & \text{if} \quad \bar{a}_1 < S_2 \\ A_1 S & \text{if} \quad \bar{a}_1 \geq S_2 \end{cases}$$

$$c_2 = \begin{cases} A_2S_2 & \text{if} \quad \bar{a}_1 < S_2 \\ r\bar{a}_1 & \text{if} \quad \bar{a}_1 = S_2 \\ A_1S_2 & \text{if} \quad \bar{a}_1 \geq S_2 \end{cases} \quad \text{and } c_1 = \begin{cases} A_1S_1 + (A_1 - A_2)\bar{a}_1 & \text{if} \quad \bar{a}_1 < S_2 \\ A_1S_1 + (A_1 - r)\bar{a}_1 & \text{if} \quad \bar{a}_1 = S_2 \\ A_1S_1 & \text{if} \quad \bar{a}_1 \geq S_2 \end{cases}$$

where $r \in [A_2, A_1]$.

There are two main differences between the model with exogenous borrowing limits and that with credit constraints (6.6).

- According to Proposition 16, multiple equilibria arises (but it is not totally generic because it only happens when $\bar{a}_1 = S_2$). However, our model with credit constraint (6.6) has a unique equilibrium.
- Both individual consumptions are increasing in exogenous borrowing limits $\bar{a}_1$. However, our model with credit constraint (6.6), the consumption of borrower has an inverted U–sharp form and that of lender is increasing in credit limit. The intuition is that the borrowing amount that an agent can borrow are exogenous in the problem (Q_i) while it is endogenous and depends on the interest rate r in the problem (P_i) with credit constraint (6.6).

These points suggest that the forms of borrowing constraints (credit constraint or exogenous borrowing limit) matter for the equilibrium analysis.

6.3 When do we do default?

There is an agent living in two periods: 0 and 1. At initial period, agent has a debt level a_{-1}. Agent maximizes her utility

$$u(c_0) + \beta u(c_1). \tag{6.28}$$

At date $t = 0, 1$, agent has $w_t > 0$ units of consumption good.
Agent has 2 choices: to default or to repay debt.

1. If agent does default, i.e., she does not pay debt, and she cannot participate to the financial market but she can only consume her endowment. In this case, her utility is $U(endowment) = u(w_0) + \beta u(w_1)$.
2. If agent repays her debt, she can smooth her consumption by selling or buying an asset with price q_0. If agent buys/sells 1 unit of financial asset at date 0 with price q_0 then at date 1, she will receive/repay 1 unit of consumption good. There is no borrowing constraint. In this case, agent chooses c_0, c_1, a_0 to maximize her utility

$$\max_{c_0,c_1} u(c_0) + \beta u(c_1) \tag{6.29}$$

$$c_0 + a_{-1} \leq w_0 + q_0 a_0 \tag{6.30}$$

$$c_0 + a_0 \leq w_1 \tag{6.31}$$

If we write $q_0 = \dfrac{1}{1+r}$, then r is the net interest rate.

Assume that $u(c) = ln(c)$. Then the solution of this maximization problem (if exist) is given by

$$c_0 = \frac{w_0 + q_0 w_1 - a_{-1}}{1+\beta} \tag{6.32}$$

$$c_1 = \frac{\beta c_0}{q_0}. \tag{6.33}$$

Agent chooses to repay her debt if and only if

$$u(c_0) + \beta u(c_1) \geq u(w_0) + \beta u(w_1) \tag{6.34}$$

$$[\frac{w_0 + q_0 w_1 - a_{-1}}{1+\beta}]^{1+\beta} \beta^\beta \geq w_0 (q_0 w_1)^\beta. \tag{6.35}$$

This condition is equivalent to

$$a_{-1} \leq D := w_0 + q_0 w_1 - (1+\beta) w_0^{\frac{1}{1+\beta}} [\frac{q_0 w_1}{\beta}]^{\frac{\beta}{1+\beta}}.$$

Proposition 17. *There exists a level D^* such that if $a_{-1} > D^*$ then agent choose to do default.*

$$D^* = \sup \{D : \text{ The utility of problem (2) is greater than U(endowment)}\}$$

Proposition 18. *Assume that r is large enough then $\frac{\partial D}{\partial w_0} > 0$, $\frac{\partial D}{\partial w_1} < 0$, i.e., default level D increases when w_0 increase or w_1 decreases.*

Chapter 7

International macroeconomics

7.1 International macroeconomics: a general equilibrium approach

The general equilibrium theory is widely used in international macroeconomics. In this section, we present a general equilibrium model with a finite number of countries to investigate the role of globalization and the direction of capital flow.

Definition 5. *Consider a world model with m countries. Assume that there are two periods. Each country i has endowments $w_{i,0}$ and $w_{0,1}$ at period 0 and 1 respectively.*

A general equilibrium is a positive list $(r, (c_{i,0}, c_{i,1}, k_i, a_i)_{i=1}^m)$ satisfying the following conditions:

(1) Given r, for each i, the allocation $(c_{i,0}, c_{i,1}, k_i, a_i)$ is a solution to the agent (country) i' maximization problem:

$$\begin{aligned} & \max_{(c_{i,0},c_{i,1},k_i,a_i)} \Big[u_i(c_{i,0}) + \beta u_i(c_{i,1})\Big] && (7.1) \\ \textit{budget constraints}: \quad & c_{i,0} + k_i + a_i \leq w_{i,0}, && (7.2) \\ & c_{1,i} \leq w_{i,1} + F_i(k_i) + r a_i && (7.3) \\ & k_i \geq 0. && (7.4) \end{aligned}$$

(2) Market clearing conditions

$$\begin{aligned} \sum_{i=1}^m (c_{i,0} + k_i) &= \sum_{i=1}^m w_i && (7.5) \\ \sum_{i=1}^m c_{i,1} &= \sum_{i=1}^m \Big(F_i(k_i) + w_{i,1}\Big) && (7.6) \\ \textit{(Financial market)}: \sum_{i=1}^m a_i &= 0. && (7.7) \end{aligned}$$

Computing explicitly the equilibrium is not easy. We will present some special cases. At equilibrium, under standard conditions, we have the first order conditions

$$\begin{aligned} u_i'(c_{i,0}) &= \lambda_{i,0} && (7.8) \\ \beta_i u_i'(c_{i,1}) &= \lambda_{i,1} && (7.9) \\ \lambda_{i,0} &= \lambda_{i,1} F_i'(k_i) + \lambda(k_i) && (7.10) \\ \lambda(k_i) k_i &= 0 && (7.11) \\ \lambda_{i,0} &= r \lambda_{i,1} && (7.12) \end{aligned}$$

We work under the following specification:

Assumption 4. $u_i(c) = ln(c)$. $F_i(k) = A_i k^{\alpha_i}$.

Remark 8. *Here, we assume that the international financial market is perfect (there is no borrowing constraint). We may introduce borrowing constraints as in Section 6.2.*

According to the FOCs, we see that $F_i(k_i) = r$ for any i. Hence $k_i = \left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}}$.

We also have $\lambda_{i,0} = r\lambda_{i,1}$ which becomes $w_{i,1} + A_i k_i^{\alpha_i} + ra_i = r\beta_i(w_{i,0} - k_i - a_i)$. Hence we obtain

$$\frac{w_{i,1} + A_i k_i^{\alpha_i} + r\beta_i k_i}{1+\beta_i} + ra_i = r\frac{\beta_i w_{i,0}}{1+\beta_i}. \tag{7.13}$$

Combining with $k_i = \left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}}$, we get

$$\frac{w_{i,1}}{1+\beta_i} + \frac{(1+\alpha_i\beta_i)}{1+\beta_i}\frac{\alpha_i^{\frac{\alpha_i}{1-\alpha_i}} A_i^{\frac{1}{1-\alpha_i}}}{r^{\frac{\alpha_i}{1-\alpha_i}}} + ra_i = r\frac{\beta_i w_{i,0}}{1+\beta_i}. \tag{7.14}$$

Therefore, we have:

Proposition 19. *The international interest rate r is determined by*

$$\sum_{i=1}^{m}\frac{w_{i,1}}{1+\beta_i}\frac{1}{r} + \sum_{i=1}^{m}\frac{(1+\alpha_i\beta_i)}{1+\beta_i}\frac{\alpha_i^{\frac{\alpha_i}{1-\alpha_i}} A_i^{\frac{1}{1-\alpha_i}}}{r^{\frac{1}{1-\alpha_i}}} = \sum_{i=1}^{m}\frac{\beta_i w_{i,0}}{1+\beta_i}. \tag{7.15}$$

Comparative statics. It is easy to see that, for each i, the interest rate is increasing in endowment $e_{i,1}$ at the second date and agent' TFP A_i, decreasing in endowment $e_{i,0}$ at initial date and rate of time preference β_i.

Corollary 3 (Endowment effects in an exchange world). *Assume that $A_i = 0$ for every i. The world interest rate is determined by*

$$\sum_{j=1}^{m}\frac{w_{j,1}}{1+\beta_j}\frac{1}{r} = \sum_{j=1}^{m}\frac{\beta_j w_{j,0}}{1+\beta_j} \Leftrightarrow r = \frac{\sum_{j=1}^{m}\frac{w_{j,1}}{1+\beta_j}}{\sum_{j=1}^{m}\frac{\beta_j w_{j,0}}{1+\beta_j}} \tag{7.16}$$

We can also compute a_i and $c_{i,0}$

$$a_i = \frac{\beta_i w_{i,0}}{1+\beta_i} - \frac{w_{i,1}}{1+\beta_i}\frac{1}{r} \tag{7.17a}$$

$$c_{i,0} = \frac{w_{i,0}}{1+\beta_i} + \frac{w_{i,1}}{1+\beta_i}\frac{1}{r} \tag{7.17b}$$

$$c_{i,1} = r\frac{\beta_i w_{i,0}}{1+\beta_i} + \frac{\beta_i w_{i,1}}{1+\beta_i}. \tag{7.17c}$$

Let us consider two countries $i \neq j$, we have

$$\frac{\partial c_{i,0}}{\partial w_{j,0}} > 0, \quad \frac{\partial c_{i,1}}{\partial w_{j,0}} < 0. \tag{7.17d}$$

Let us mention an implication of this result. Assume that there is a disaster in the country j that makes its endowment $w_{j,0}$ at date 0 decreases. Assume also that there is no international aid. In this case, the equilibrium consumption of other countries at date 0 decreases but that at date 1 increases because the equilibrium interest rate increases.

However, if a disaster affects the initial endowment of both countries. Then, its effect on these two countries' consumption would be different; it strontly depends on the disaster's size.

Corollary 4 (The world interest rate in an exchange world). *Assume that $A_i = 0, \beta_i = \beta$ for every i. The world interest rate is determined by*

$$r = \frac{1}{\beta}\frac{\sum\limits_{i=1}^{m} w_{i,1}}{\sum\limits_{i=1}^{m} w_{i,0}} \tag{7.18}$$

Interpretation: In the exchange world, we write $\beta = \frac{1}{1+i}$ where i is the interest rate of time preference and can be interpreted as riskless interest rate.

Let us denote $Y_t := \sum\limits_{i=1}^{m} w_{i,t}$ is the world output at date t. We write $\frac{Y_1}{Y_0} = 1+\frac{Y_1-Y_0}{Y_0} = 1+R$, where R can be interpreted as the world growth rate. Then the world interest rate is

$$r = (1+i)(1+R). \tag{7.19}$$

We recover a version of the Fisher formula.

Development level and direction of capital flows

Corollary 5. *Assume that $w_{i,1} = 0$ and $\alpha_i = \alpha \in (0,1)$ for every i. Assume that borrowing constraints are not binding. Then, the equilibrium price is determined by*

$$\sum_{i=1}^{m} \frac{(1+\alpha\beta_i)\alpha^{\frac{\alpha}{1-\alpha}} A_i^{\frac{1}{1-\alpha}}}{1+\beta_i} = r^{\frac{1}{1-\alpha}} \sum_{i=1}^{m} \frac{\beta_i w_i}{1+\beta_i}. \tag{7.20}$$

Corollary 6. *Assume that $w_{i,1} = 0, \beta_i = \beta, \alpha_i = \alpha$ for every i. We continue to consider equilibrium where borrowing constraints are not binding. We have*

$$\frac{(1+\alpha\beta)\alpha^{\frac{\alpha}{1-\alpha}}}{1+\beta} \sum_{i=1}^{m} A_i^{\frac{1}{1-\alpha}} = r^{\frac{1}{1-\alpha}} \sum_{i=1}^{m} w_i. \tag{7.21}$$

In this case, the direction of capital flows is given by:

$$a_i < 0 \Leftrightarrow \frac{w_i}{\sum\limits_{j=1}^{m} w_j} < \frac{A_i^{\frac{1}{1-\alpha}}}{\sum\limits_{j=1}^{m} A_j^{\frac{1}{1-\alpha}}}. \tag{7.22}$$

Corollary 7 (The role of countries' size). *Assume that $w_{i,1} = 0, \beta_i = \beta, \alpha_i = \alpha$ for every i. We continue to consider equilibrium where borrowing constraints are not binding.*

Assume that there are two countries H and F. We have

$$a_H < 0 \Leftrightarrow \frac{w_H}{w_F} < \Big(\frac{A_H}{A_F}\Big)^{\frac{1}{1-\alpha}}. \tag{7.23}$$

In particular, if $w_H = 0.25 w_F$ and $\alpha = 1/3$, then the country H borrows if and only if $A_H > 0.4 A_F$.

Implication. This result leads to several implications in international macroeconomics. Assume that there are two countries H and F with $e_{i,1} = 0, \beta_i = \beta, \alpha_i = \alpha$ for each $i = H, F$. We continue to consider equilibrium where borrowing constraints are not binding. We have that

$$a_H < 0 \Leftrightarrow \frac{e_H}{e_F} < \Big(\frac{A_H}{A_F}\Big)^{\frac{1}{1-\alpha}} \tag{7.24}$$

$$\Leftrightarrow \alpha A_H e_H^{\alpha-1} < \alpha A_F e_F^{\alpha-1} \tag{7.25}$$

When the two countries have the same initial endowment (i.e., $e_H = e_F$), then the country with higher level of TFP will borrow from the country with lower level of TFP. If we interpret that the country having higher level of TFP is the more developed country and the other is the less developed country, then our result suggests that capital moves to the more developed country.

When the two countries have the same TFP (i.e., $A_H = A_F$), the country with lower level of initial endowment will borrow. If we interpret that the country with higher level of initial endowment as the developed country and the other is the less developed country, then our result suggests that capital moves to the less developed country.

These two above points about capital flows are not opposite since the development level of a country is characterized by a vector of many indices, not only one unique parameter. In our simple example, the development level of country i is characterized by the vector (e_i, A_i) of initial endowment and TFP. By the way, our theoretical result suggests that we should be careful when interpreting the relationship between the development level of a country and the direction of capital flows.

We point out here that when we consider two countries with different levels of endowments and TFPs, the analysis would be more complicated. Let us consider an example. Let α be $1/3$, a reasonable parameter. Assume that, at initial date, the country F is four times richer that the country H, i.e. $e_F = 4e_H$. In this case, the country H borrows if and only if $A_H > 0.4A_F$.

7.2 Foreign aid and economic growth in a Solow model

7.2.1 Framework

The official development assistance (ODA) is important for low income countries as shown in the following graphics taken from the World Bank's website.

We consider a model à la Solow with exogenous saving rate but we introduce foreign aid in order to investigate the effects of foreign aid on the economic development of the recipient country. As usual, we assume that

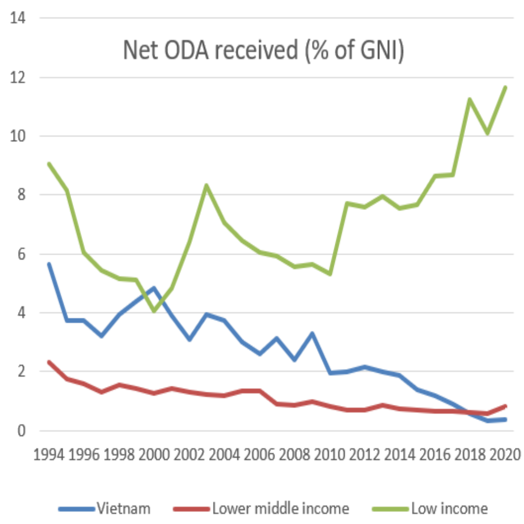
$$c_t + S_t = Y_t \tag{7.26a}$$

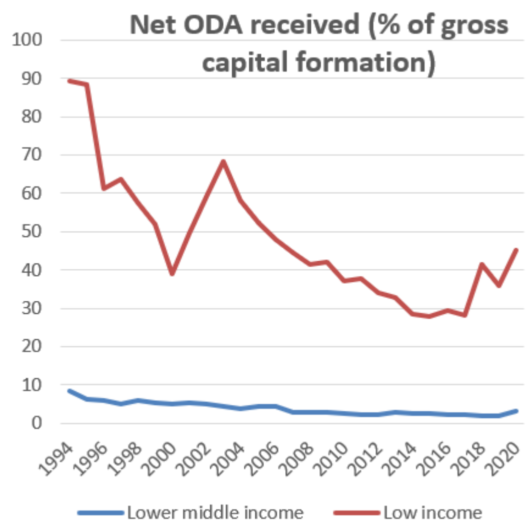
$$S_t = sY_t \tag{7.26b}$$

$$Y_t = F(k_t) \tag{7.26c}$$

$$k_{t+1} = (1-\delta)k_t + I_t \tag{7.26d}$$

where c_t, S_t, I_t are consumption, saving, investment at date t ($t = 0, 1, \ldots, +\infty$), $s \in (0, 1)$ is the exogenous saving rate, k_t is the physical capital at date t ($k_0 > 0$ is given), $\delta \in [0, 1]$ is the capital depreciation rate, Y_t is the output.





Following Chenery and Strout (1966), we assume that the total investment of the recipient country equals the sum of its savings and foreign aid:

$$I_t = S_t + a_t. \tag{7.27}$$

Here, foreign aid is used to complement the domestic investment. From the above relationships, we get $c_t = (1 - s)F(k_t)$ and

$$k_{t+1} = (1 - \delta)k_t + sF(k_t) + a_t \tag{7.28}$$

Assume that the foreign aid at period t, denoted by a_t, will be disappeared once the physical capital of the recipient country is high enough. Formally, we set the following rule:

$$a_t = a \text{ if } k_t < b_1, \text{ and } a_t = 0 \text{ if } k_t \geq b_1. \tag{7.29}$$

Here, we do not consider the foreign aid in form of loans. For the effects of loans, see Le Van, Pham and Pham (2023).

We set the following assumption that plays a key role on our analyses.

Assumption 5. *Assume that $F(k) = Af((k - b_0)^+)$ where $b_0 > 0$ and f is a strictly increasing, concave function with $f(0) = 0$.*

It means that the the economy has to pay a fixed cost before the production process takes place. The existence of this fixed cost may due to the lack of infrastructure.

7.2.2 Avoiding a collapse with a permanent aid

With foreign aid, the dynamics of physical capital becomes

$$k_{t+1} = \begin{cases} (1-\delta)k_t + sAf(k_t - b_0)^+) + a_t & \text{if } k_t < b_1 \\ (1-\delta)k_t + sAf(k_t - b_0)^+) & \text{if } k_t \geq b_1 \end{cases} \tag{7.30}$$

Notice that the dynamical system of physical capital is not continuous (at the point b_1). Providing a full characterization of this system is not easy.

Let k_a^1 be determined by

$$k = (1 - \delta)k + a. \tag{7.31}$$

It means that $k_a^1 = \frac{a}{\delta}$.

Denote k_a^2 be the smallest solution of the following system:

$$k_a^2: \quad k = (1-\delta)k + sAf(k - b_0) + a \text{ and } k > b_0 \tag{7.32}$$

Proposition 20. *Let Assumption 5 be satisfied. Assume that* $\max_{k\geq 0} sAf((k-b_0)^+) - \delta k < 0$. *Assume also that* $b_1 > k_a^2 > b_0 > k_a^1$.

1. *Assume that there is no foreign aid. Then the economy collapses for any initial value* k_0 *(formally,* $\lim_{t\to\infty} k_t = 0$ *for any* $k_0 > 0$*).*
2. *Assume that there is foreign aid.* k_t *does not converge to zero (the economy never collapses). Moreover, if* $0 < k_0 \leq k_a^2$*, then* k_t *converges to* k_a^1 *and* $a_t = a\ \forall t$.

In this context, we are considering a poor country having a low TFP. If there is no fixed cost, we recover a standard Solow growth model: the capital path converges to a steady state whatever the initial value of capital.

The situation is different when there is a fixed cost in the production. Without foreign aid, its economy will collapse because its production process is not efficient. Interestingly, a positive amount of foreign aid helps this country to overcome the fixed cost b_0 and the GDP may converge to a strictly positive steady state.

Proof of Proposition 20. We have

$$k_{t+1} = (1-\delta)k_t + sF(k_t) + a_t \tag{7.33}$$

Point 1. Without aid, we have

$$k_{t+1} = (1-\delta)k_t + sAf((k_t - b_0)^+) < (1-\delta)k_t + \delta k_t = k_t \forall t \geq 0 \tag{7.34}$$

So, the sequence k_t converges to K satisfied $K = (1-\delta)K + sAf((K-b_0)^+)$. This implies that $K = 0$.

Point 2 (with foreign aid). We have the dynamical system

$$k_{t+1} = (1-\delta)k_t + sF(k_t) + a_t \tag{7.35}$$

$$= \begin{cases} (1-\delta)k_t + a & \text{if } k_t < b_0 \\ (1-\delta)k_t + sAf(k_t - b_0) + a & \text{if } b_0 \leq k_t < b_1 \\ (1-\delta)k_t + sAf(k_t - b_0) & \text{if } k_t \geq b_1 \end{cases} \tag{7.36}$$

It is easy to see that $k_t \geq min(a, (1-\delta)b_1 + sAf(b_1 - b_0)) > 0\ \forall t \geq 1$. So, the sequence k_t never converges to zero.

We now suppose that $0 < k_0 \leq k_a^2$. We will prove, by induction, that $k_t < k_a^2\ \forall t$. Assume that $k_t < k_a^2$, then $k_t < b_1$. So, we have

$$k_{t+1} = \begin{cases} (1-\delta)k_t + a & \text{if } k_t < b_0 \\ (1-\delta)k_t + sAf(k_t - b_0) + a & \text{if } b_0 \leq k_t < b_1 \end{cases} \tag{7.37}$$

$$\leq (1-\delta)k_a^2 + sAf(k_a^2 - b_0) + a = k_a^2. \tag{7.38}$$

So, we get that $k_t < k_a^2\ \forall t$. Hence $k_t < b_1\ \forall t$. So, the system becomes

$$k_{t+1} = \begin{cases} (1-\delta)k_t + a & \text{if } k_t < b_0 \\ (1-\delta)k_t + sAf(k_t - b_0) + a & \text{if } b_0 \leq k_t \end{cases} \tag{7.39}$$

which is continuous. We will prove that k_t converges to k_a^1. To do so, we consider two cases.

Case 1: $k_0 \leq k_a^1$. In this case, we have $k_1 = (1-\delta)k_0 + a \geq k_0$ because $a = \delta k_a^1 \geq \delta k_0$. Moreover, we have $k_1 = (1-\delta)k_0 + a \leq (1-\delta)k_a^1 + a = k_a^1$. By induction, we can check that $k_t \in [k_0, k_a^1]$ and (k_t) is a increasing sequence. Therefore, it converges. By consequence, it converges to k_a^1.

Case 2: $k_a^1 < k_0 \leq k_a^2$. We firstly prove that $k_a^1 < k_t \leq k_a^2$ $\forall t$. Indeed, without loss of generality, we prove this for $t = 1$. We have

$$k_1 = \begin{cases} (1-\delta)k_0 + a & \text{if } k_0 < b_0 \\ (1-\delta)k_0 + sAf(k_0 - b_0) + a & \text{if } b_0 \leq k_0 \end{cases} \tag{7.40}$$

$$> (1-\delta)k_a^1 + a = k_a^1 \tag{7.41}$$

and

$$k_1 = \begin{cases} (1-\delta)k_0 + a & \text{if } k_0 < b_0 \\ (1-\delta)k_0 + sAf(k_0 - b_0) + a & \text{if } b_0 \leq k_0 \end{cases} \tag{7.42}$$

$$< (1-\delta)k_a^2 + sAf(k_a^2 - b_0) + a = k_a^2. \tag{7.43}$$

We secondly prove that (k_t) is a decreasing sequence. Without loss of generality, we prove this for $t = 1$.

If $k_0 < b_0$, we have

$$k_1(1-\delta)k_0 + a = k_0 + a - \delta k_0 \leq k_0 + a - \delta k_a^1 = k_0. \tag{7.44}$$

If $k_0 \geq b_0$, we have

$$k_1 = (1-\delta)k_0 + sAf(k_t - b_0) + a = k_0 + sAf(k_0 - b_0) + a - \delta k_0 \tag{7.45}$$

$$\leq k_0 + sAf(k_a^2 - b_0) + a - \delta k_a^2 = k_0. \tag{7.46}$$

To sum up, k_t must converge to some value. It is easy to see that this value must be k_a^1. □

7.2.3 Escaping poverty trap with foreign aid only at the early stages

We now consider a different situation: when there is no foreign aid, the dynamics of capital path is given by the following graphic (where we can see that there are positive two steady states). We have that

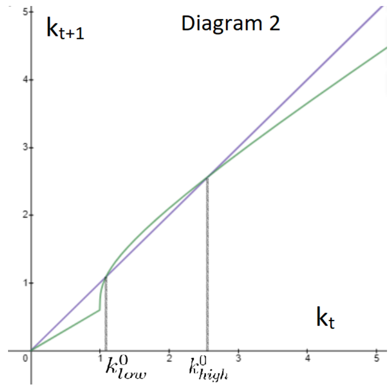
$$k_{t+1} = \begin{cases} (1-\delta)k_t & \text{if } k_t < b_0 \\ (1-\delta)k_t + sAf(k_t - b_0) & \text{if } b_0 \leq k_t \end{cases} \tag{7.47}$$

1. If $k_0 < k_{low}$, it is easy to see that $k_t < k_{t+1}$ $\forall t$. So, k_t converges to zero.

2. If $k_0 > k_{low}$, we can see that $k_0 \leq (\geq) k_{high}$, then k_t increasingly (decreasingly) converges to k_{high}.

Our result is formalized as follows.

Proposition 21 (without foreign aid)**.** *Let Assumption 5 be satisfied. Assume that* $f'(0) = \infty$*,* $f'(\infty) = 0$*, and* $b_0 > 0$*. Assume also that* $\max_{k\geq 0} sAf((k-b_0)^+) - \delta k > 0$*. Then the equation* $K = (1-\delta)K + sAf(K-b_0)^+)$ *has two solutions,* $k_{low} < k_{high}$*, and these solutions are higher than* b_0*.*



1. *If the initial capital k_0 is strictly lower than k_{low}, then k_t converges to zero.*

2. *If the initial capital k_0 is strictly higher than k_{low}, then k_t converges to k_{high}.*

According to Proposition 21, the value k_{low} can be interpreted as a poverty trap.

We now assume that the country receives an aid. The following result shows the role of this aid in fighting poverty trap.

Proposition 22 (poverty trap and foreign aid)**.** *Let Assumption 5 be satisfied. Assume that $f'(0) = \infty$, $f'(\infty) = 0$, and $b_0 > 0$. Assume also that $\max_{k \geq 0} sAf((k - b_0)^+) - \delta k > 0$.*

Then the equation $K = (1-\delta)K + sAf(K - b_0)^+)$ has two solutions, $k_{low} < k_{high}$, and these solutions are higher than b_0.

Assume that the country receives an amount of aid given by (7.29). Assume that $b_1 < b_0$.

Assume that the initial capital is lower that the threshold b_1, i.e., $k_0 < b_1$. Then $a_0 = a > 0$ and we have that:

1. *If $(1-\delta)b_1 + sY_0 + a < k_{low}$, then $k_t \leq k_{low}$ $\forall t$. Moreover, if $a < \delta b_1$ and $k_0 < a/\delta$, then k_t converges to a/δ and the aid amount is always positive: $a_t = a$ $\forall t \geq 1$.*

2. *If $(1-\delta)k_0 + a > k_{low}$, then k_t converges to k_{high} (which is higher than k_{low}). Moreover, the foreign aid is vanished from date 1: $a_t = 0$ $\forall t \geq 1$.*

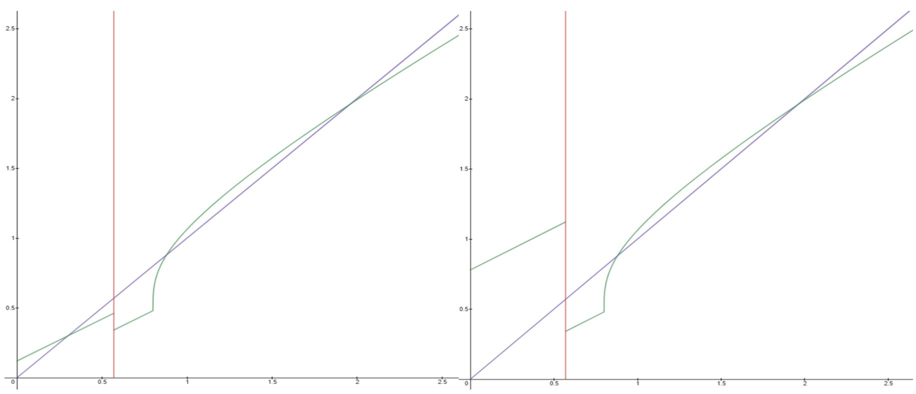


Figure 7.1: LHS: a low foreign aid (point 1 of Proposition 22). RHS: a high foreign aid (point 2 of Proposition 22).

Proof of Proposition 22. Since $b_1 < b_0$, the dynamics of capital path becomes

$$k_{t+1} = \begin{cases} (1-\delta)k_t + a & \text{if } k_t < b_1 \\ (1-\delta)k_t & \text{if } b_1 \leq k_t < b_0 \\ (1-\delta)k_t + sAf(k_t - b_0) & \text{if } b_0 \leq k_t \end{cases} \tag{7.48}$$

1. If $(1-\delta)b_1 + sY_0 + a < k_{low}$, we have $k_1 = (1-\delta)k_0 + sY_0 + a \leq (1-\delta)b_1 + sY_0 + a < k_{low}$.

   If $k_1 \geq b_0$, then $k_2 = (1-\delta)k_1 + sAf(k_1 - b_0) < (1-\delta)k_{low} + sAf(k_{low} - b_0) = k_{low}$.

   If $k_1 \in [b_1, b_0)$, then $k_2 = (1-\delta)k_1 < (1-\delta)k_{low} < k_{low}$.

   If $k_1 \leq b_1$, then $k_2 = (1-\delta)k_1 + a < (1-\delta)b_1 + a < k_{low}$.

   By using induction argument, we obtain that $k_t < k_{low}$ $\forall t$.

   We now consider the case where $k_0 < a/\delta$. Since $a/\delta < b_1$, tt is easy to prove that k_t increasingly converges to a/δ.

2. If $(1-\delta)k_0 + a > k_{low}$, then $k_1 = (1-\delta)k_0 + a > k_{low} > b_0$. First, we prove that $k_t > k_{low}$ $\forall t \geq 1$. Indeed, this is true for $t = 1$. Suppose that it is true until t, i.e., $k_t > k_{low}$. Consider date $t+1$ we have

   $$k_{t+1} = (1-\delta)k_t + sAf(k_t - b_0) > (1-\delta)k_{low} + sAf(k_{low} - b_0) = k_{low}. \tag{7.49}$$

   By consequence, we get that $k_t > k_{low}$ $\forall t \geq 1$.

   Second, it is easy prove that k_t converges to k_{high}.

□

7.3 FDI, new industry, and economic development

A simple example

In developing countries, FDI has been also viewed as an important factor in the economic growth as shown by the following graphic (source: The World Bank).[1]

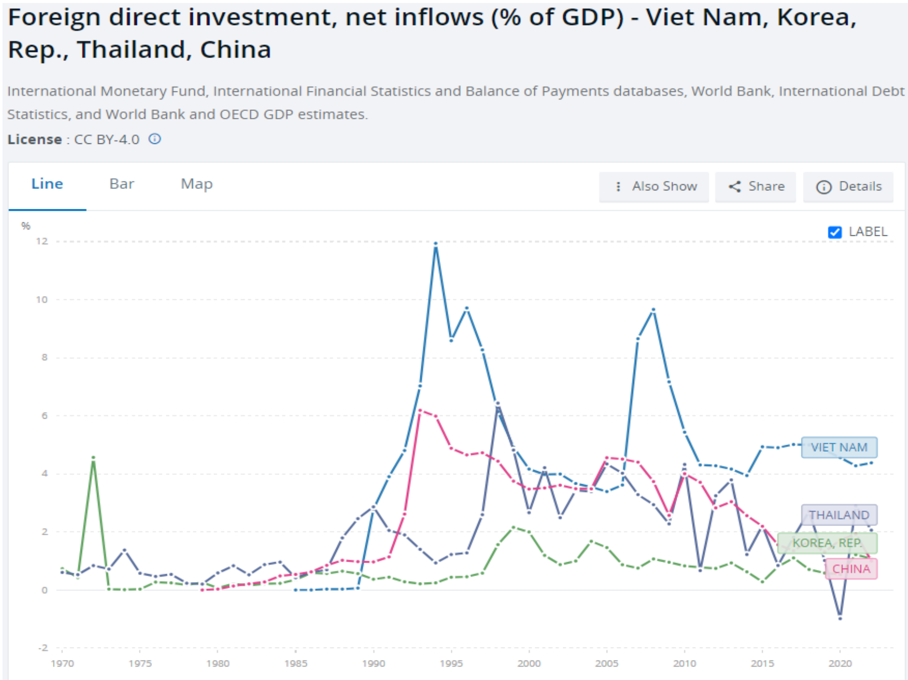
However, is attracting FDI spillovers the key to developing of their own industries? Should the host country develop a new industry or continue to work for multinational firms? What are the roles of different macroeconomic variables such as development level, FDI spillovers, return of training, and heterogeneity of firms.

To address these questions, we present here a very simple model (see Nguyen-Huu and Pham (2023) for an infinite-horizon models with multinational and domestic firms where we explain how a host country can benefit from FDI spillovers). Assume that we have L units of labor. We get salary with wage w (in term of consumption good) if we work for multinational firm. There is a fixed cost $\bar{L}$ if we want to create a new firm in this industry. For simplicity, I begin by assuming that labor is the unique input and the production function of our firm is $F(L_d) = A_d(L_d - \bar{L})^+$, where A is the productivity. I formalize the problem by the following simple model

$$\max_{L_d, L_e} p_d A_d (L_d - \bar{L})^+ + wL_e \tag{7.50}$$

$$L_d + L_e \leq L, \tag{7.51}$$

[1]See Harrison and Rodriguez-Clare (2010) for a complete review.



where L_d is labor utilized to produce the consumption good, L_e is labor working for the multinational firm.

We have the following result showing the optimal quantities of labor L_e, L_d.

Proposition 23. *(i) Assume that $L \leq \bar{L}$, we have $L_d = 0$ for every A_d.*

(ii) Assume that $L > \bar{L}$.

(ii.a) If $p_d A_d(L - \bar{L}) \leq wL$, $L_d = 0$.

(ii.b) If $p_d A_d(L - \bar{L}) > wL$, we have $L_d = L > \bar{L}$ and $L_e = 0$.

Point (i) shows that if the initial labor cannot cover the fixed cost, no domestic firm cannot be created in this industry for every level of the productivity A_d. Point (ii) says that even the initial labor is greater than the fixed cost, we invest in this industry if and only if the productivity A_d reaches a critical threshold $\dfrac{wL}{L - \bar{L}}$. Moreover, the multinational firm may be eliminated by the domestic one.

Appendix A

Karush–Kuhn–Tucker theorem for finite dimensional maximization problems

In many economic models, we need to maximize a function subject to several constraints (physical constraint, financial constraint, legal constraint, ...). We present here the result concerning the following maximization problem (P):

$$(P) \qquad \text{Maximize } f_0(x) \text{ under the constraints } \begin{cases} f_i(x) \leq 0, \ \forall i \in I \\ g_i(x) \leq 0, \ \forall i \in J \\ g_i(x) = 0, \ \forall i \in K. \end{cases}$$

where $f_0 : \mathbb{R}^n \to \mathbb{R}$ is a **concave function**, I, J and K are finite and possibly empty sets, for all $i \in I$, f_i is convex, non-affine function from $\mathbb{R}^n$ into $\mathbb{R}$, for all $i \in J \cup K$, g_i is a non-null affine function.

Definition 6. *The Lagrangian of Problem (P) is the function $L : \mathbb{R}^p_+ \times \mathbb{R}^q_+ \times \mathbb{R}^{r-q} \times \mathbb{R}^n \to \mathbb{R}$ defined by: for all $(\lambda, \mu, x) = ((\lambda_i)_{i\in I}, (\mu_i)_{i\in J}, (\mu_i)_{i\in K}, x)$,*

$$L(\lambda, \mu, x) = f_0(x) - \sum_{i\in I} \lambda_i f_i(x) - \sum_{i\in J} \mu_i g_i(x) - \sum_{i\in K} \mu_i g_i(x).$$

where $p = card(I), q = card(J)$ and $r = card(J \cup K)$.

Theorem 7 (Kuhn-Tucker theorem (maximization problem))**.** *Assume that functions f_i, g_i are differentiable.*

Assume that Slater Condition is satisfied for Problem (P). Then $\overline{x}$ is a solution to (P) if, and only if, there exists coefficients $(\lambda_i)_{i\in I}, (\mu_i)_{i\in J\cup K}$ which, together with $\overline{x}$, satisfy Kuhn-Tucker Conditions for Problem (P), i.e.,

1. $\forall i \in I,\ \lambda_i \geq 0, \lambda_i f_i(\overline{x}) = 0,$
   $\forall i \in J,\ \mu_i \geq 0, \mu_i g_i(\overline{x}) = 0.$

2. $$f_0(\overline{x}) - \sum_{i\in I} \lambda_i f_i(\overline{x}) - \sum_{i\in J\cup K} \mu_i g_i(\overline{x}) \geq f_0(x) - \sum_{i\in I} \lambda_i f_i(x) - \sum_{i\in J\cup K} \mu_i g_i(x) \tag{A.1}$$

   for any x.

   If f_0, $(f_i)_{i\in I}, (g_i)_{i\in J\cup K}$ are differentiable, then condition (A.1) is replaced by

   $$Df_0(\overline{x}) = \sum_{i\in I} \lambda_i Df_i(\overline{x}) + \sum_{i\in J\cup K} \mu_i Dg_i(\overline{x}) \tag{A.2}$$

Proof. See Florenzano and Le Van (2001). □

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