Politics of Public Education and Pension Reform with Endogenous Fertility

Uchida, Yuki and Ono, Tetsuo

Seikei University, Osaka University

6 January 2024

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Yuki Uchida† Tetsuo Ono‡
Seikei University Osaka University

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Abstract

Implications of increased life expectancy on parental fertility decisions and the subsequent alteration of political influence between the younger and older generations have noteworthy consequences for government policies pertaining to education and pension. This study introduces an overlapping generations growth model that incorporates these effects, revealing that higher life expectancy leads to reduced fertility rates, a decline in the education expenditure-GDP ratio, and an increase in the pension benefit-GDP ratio. Additionally, a long-term perspective is adopted to assess the feasibility of implementing pension cuts and highlighting their optimality in terms of social welfare when significant consideration is given to future generations. Moreover, the model simulations compare the effects of immediate and gradual pension cuts on fertility, fiscal policies, and economic growth. The simulation results demonstrate that pension cuts enhance economic growth, which would have been lower in the absence of such measures, with immediate cuts exhibiting particularly notable effectiveness.

- Keywords: Fertility; Public Pension; Public Education; Probabilistic Voting; Overlapping Generations
- JEL Classification: D70, E62, H52, H55

*The authors express their gratitude towards the seminar participants at Kobe University for their valuable comments. Additionally, the authors acknowledge the financial support received from the Japan Society for the Promotion of Science through a Grant-in-Aid for Early-Career Scientists (No.18K12802, Uchida) and a Grant-in-Aid for Scientific Research (C) (No.21K01539, Ono).

†Yuki Uchida: Faculty of Economics, Seikei University, 3-3-1 Kichijoji-Kitamachi, Musashino, Tokyo 180-8633, Japan. E-mail: yuchida@econ.seikei.ac.jp.

‡Tetsuo Ono: Graduate School of Economics, Osaka University, 1-7, Machikaneyama, Toyonaka, Osaka 560-0043, Japan. E-mail: tono@econ.osaka-u.ac.jp.

1 Introduction

The declining birth rates and increasing life expectancy observed in most Organization for Economic Cooperation and Development (OECD) member countries in recent decades have increased the share of older adults in the voting population (OECD, 2016). This trend is projected to continue over the next few decades (Rouzet et al., 2019). Accordingly, pension benefits for older adults are expected to increase (Gonzalez-Eiras and Niepelt, 2012), while government spending on schemes that may not directly benefit older adults, such as public education (Poterba, 1997; Cattaneo and Wolter, 2009) could decrease. Simultaneously, an aging population reduces the willingness of the working middle-aged population to pay higher taxes to meet the government's growing pension burden (Razin et al., 2002). However, older adults may not object to education spending because of altruistic concerns for the younger generations or because such spending may enhance productivity and ensure a higher level of tax revenues (Gradstein and Kaganovich, 2004). Therefore, these opposing effects lead to the following question: how does the government allocate its limited budget to provide pension for older adults and education for the younger generation in response to population aging?

Recent studies on this topic include Gonzalez-Eiras and Niepelt (2012), Lancia and Russo (2016), Ono and Uchida (2016), and Bishnu and Wang (2017). These studies have used probabilistic voting (Lindbeck and Weibull, 1987; Persson and Tabellini, 2000) to describe intergenerational conflicts in the allocation of government revenue among the public for pension and education. In this voting environment, each office-seeking candidate proposes a policy platform to maximize the probability of winning elections, resulting in the selection of policies that maximize an objective function that weights the utility of each generation by its share of the population. The advantage of assuming probabilistic voting is that it enables us to handle votes for multiple policies and avoid the problem of voting cycles that may arise during majority voting for multiple policy variables (Persson and Tabellini, 2000).

Probabilistic voting also allows us to capture the impact of marginal changes in the population composition on the equilibrium policy through changes in generational weights of the objective function. The aforementioned studies have assumed exogenous fertility and examined the impact of its exogenous decline on equilibrium policy and resulting welfare distribution across the generations. However, fertility is endogenously determined from the optimizing behavior of households (Becker, 1991). Particularly, an increase in life expectancy affects fertility decisions of households (Ehrlich and Lui, 1991; Zhang et al., 2001; Zhang and Zhang, 2005). This consequently affects the population share or political weight of older adults during the subsequent period. Thus, as emphasized by Gonzalez-Eiras and Niepelt (2012) and Bishnu and Wang (2017), the interaction of pension and education policy choices with fertility decisions is an important issue in analyzing the determination of fiscal policies and assessing the optimality of the resulting allocation.[1]

[1] Gonzalez-Eiras and Niepelt (2012) say: "With endogenous fertility, the demographic structure would turn

To demonstrate the interaction between the determination of education and pension policies and parents' decisions on fertility, we utilize the overlapping-generation model with physical and human capital accumulation developed by Gonzalez-Eiras and Niepelt (2012) and Ono and Uchida (2016). We follow de la Croix and Doepke (2004) and extend the model by introducing the decisions regarding children, developed by Becker and Lewis (1973). Specifically, parents care about consumption, the number of children, and the human capital of their children. Parents vote for public education that affects the formation of their children's human capital and pension provisions that benefit retired older adults. Parents spend a part of their lives raising their children. Given the education and pension policies, parents choose consumption, savings, and and the number of children, to maximize their lifetime utility.[2]

In this framework, our study uncovers the dynamic interplay between current fertility choices and the determination of future pension benefits. Firstly, current fertility choices influence the political determination of future pension benefits by altering the demographic composition of future older adults. Additionally, these choices shape the labor supply of parents, impacting equilibrium market wages. Consequently, these wage fluctuations propagate through the economy, affecting savings and physical capital accumulation, thus influencing future pension benefits. Conversely, expected future pension benefits reciprocally influence the fertility choices of current parents.

By elucidating the intricate interrelationship between fertility choices and pension benefits, which has been underrepresented in the literature, we show that life expectancy directly affects parents' fertility decisions for a given set of policy variables, as shown by Ehrlich and Lui (1991) and the literature that follows them. We also show that life expectancy has an indirect effect through the political decisions regarding pension benefits for older adults, which is new to the literature. Particularly, an indirect effect occurs through the following four routes: the relative political weight of older adults, labor tax, private savings, and pension benefits. Overall, we find that the net effect is negative in the present framework. We also demonstrate that this negative effect of increased life expectancy on fertility increases the pension benefit-GDP ratio and decreases the education expenditure-GDP ratio.

Our primary contribution to the aforementioned political economy literature is demonstrating that a decrease in fertility resulting from increased life expectancy has notable implications for policy-making and economic growth. Firstly, the reduction in fertility due to extended life expectancy diminishes the representation of younger generations, thereby augmenting the political influence of older adults. Consequently, this phenomenon facilitates the implementation of policies favored by the older demographic. Secondly, the decrease in fertility elevates labor

into an endogenous state variable, rendering an analytical solution of the policy game considered in the present paper infeasible. ... We leave an analysis of these feedback effects for future research." Bishnu and Wang (2017) add: "As the intergenerational distribution of political power is tied to the demographic change, which in turn is determined by the changing pattern of fertility and longevity, a natural extension of this study is to accommodate individual choice of fertility and longevity. We leave this for future study."

[2] Life expectancy could be controllable through health investment (Grossman, 1972). However, in this study, we assume it to be exogenous, and focus on the interaction between fertility and policy choices.

supply, which subsequently drives down equilibrium wages. This exerts a detrimental effect on economic growth by inducing a reduction in savings and a deceleration in the accumulation of physical capital.

From the empirical viewpoint, the result of the positive association between life expectancy and pension expenditures is consistent with the evidence from developed countries. To curb the projected increase in future expenditures, many developed countries are working to reduce their pension benefits (OECD, 2019). However, this move could reduce consumption by older adults and consequently would not gain political support in the current environments in which older adults are gaining increasing political power. This leads us to the following second question: how would one justify the reduction in pension benefits being considered by some developed countries? One answer could be that the cuts help internalize intergenerational externalities through physical and human capital accumulation. A decrease in pension benefits reduces the tax burden on middle-aged individuals, thus increasing their savings and promoting physical capital accumulation. Moreover, a reduction in the tax burden allows them to increase their educational expenditures, which promotes human capital accumulation. These positive external effects on future generations through physical and human capital might provide benefits from a long-term perspective and therefore, justify the pension cuts.

To assess the welfare gains of pension cuts, we assume a hypothetical long-lived planner whose goal is to maximize social welfare, that is, the discounted aggregate sum of lifecycle utilities of all generations. The planner is assumed to be able to control the ceiling on pension benefits, and the resulting choices of successive short-lived governments and generations. Such a planner does not exist in the real world, but it can be viewed as approximating automatic adjustment rules that have been introduced in many OECD countries (OECD, 2012). The rules operate independently of short-sighted political processes because their purpose is to control the management of long-term pension financing.

Assuming the existence of such a planner, we examine how education and pension expenditures, labor tax, fertility, and economic growth would change if the planner introduced the ceiling. Accordingly, we derive the optimal ceiling for pension benefits to maximize social welfare. The optimal ceiling shows that the political equilibrium allocation generally achieves excessive pension benefits in terms of social welfare. This suggests the difficulty of realizing the optimal pension in a democratic society, and the need for the aforementioned automatic adjustment rules. Our results of pension ceilings provide new insights into the design of such rules.

The optimal ceiling depends on the degree to which the long-lived planner discounts future generations. Particularly, a critical value of the social discount factor represents the degree; it is optimal to set a ceiling (no ceiling) on pension when the discount factor is above (below) the critical value from the perspective of maximizing social welfare. The mechanism behind this result implies that a pension cut benefits future generations at the expense of the current older generation; thus, the planner attaches a larger weight to the benefit and a smaller weight to the cost as the discount factor grows larger. A side-effect of this result is that a pension cut invokes a trade-off between fertility and growth. A reduction in pension benefits promotes savings and economic growth but discourages fertility. This effect suggests the difficulty in reconciling the two goals of improving fertility and economic growth, which are key issues for many aging countries.

To enhance the analysis of pension reforms, we employ a model simulation that incorporates projected life-expectancy data up to 2100 for the following four distinct groups of OECD countries: synthetic rich OECD (covering all included countries), synthetic rich OECD Europe (exclusively comprising European countries), Japan, and the United States.[3] Our simulation reveals a decline in the fertility rate and the education expenditure-GDP ratio, accompanied by a temporary or marginal decrease in growth rates. Simultaneously, it indicates an increase in the labor-income tax rate and the pension benefit-GDP ratio over time. To address the issue of declining growth rates, we examine two specific scenarios for implementing pension cuts: immediate cuts to achieve the target level and gradual cuts to attain the target level. By considering rational expectations, whereby individuals accurately anticipate future reductions in pension benefits, we compare the impact of these two scenarios. Our findings demonstrate that the former scenario significantly enhances economic growth when compared with the latter.

Furthermore, we explore the scenarios under adaptive expectations, whereby individuals expect that the pension cuts implemented today will be implemented in the next period. Our analysis indicates that the shift from rational to adaptive expectations leads to a higher fertility rate as expected pension levels are higher under adaptive expectations. This moderates the political influence of older adults on fiscal policy compared to rational expectations. Additionally, changes in the expectation formation delays the onset of the effects of future pension cuts on current savings, and consequent economic growth. These results highlight the significance of considering expectation formation when investigating the effects of pension reform.

The remainder of this study is organized as follows. The next section reviews the related literature. Section 3 describes the proposed model. Section 4 characterizes political equilibrium. Section 5 considers pension reforms and provides the optimal pension ceiling. Section 6 presents a model-based simulation to predict changes in fiscal policies, fertility rates, and growth rates over time in response to projected improvements in life expectancy. Section 7 concludes with brief remarks. All the proofs are provided in the Appendix.

2 Related Literature

The literature on public education and pensions began with Pogue and Sgontz (1977), who show that pay-as-you-go (PAYG) social security incentivizes public investment in education. Such an incentive has also been indicated by Becker and Murphy (1988), who have demon-

[3]Following Gonzalez-Eiras and Niepelt (2012), we include the following countries into the set of the rich OECD countries: Australia, Canada, Denmark, France, Germany, Italy, Japan, New Zealand, Sweden, the United Kingdom, and the United States.

strated the role of PAYG social security in garnering political support from the current working population for public investment in education. Subsequent studies by Cremer et al. (1992), Kaganovich and Zilcha (1999), Pecchenino and Utendorf (1999), Boldrin and Montes (2005), Poutvaara (2006), Cremer et al. (2011), and Andersen and Bhattacharya (2017) have focused on how households behave when public education and pensions are provided by the government. Therefore, decisions on these policies through voting are abstracted away from their analyses.

Early studies on the political economy of public education and pensions include those by Bearse et al. (2001), Soares (2006), Iturbe-Ormaetxe and Valera (2012), Kaganovich and Meier (2012), Kaganovich and Zilcha (2012), and Naito (2012). A common feature of these studies is that the two-dimensional voting aspect is reduced to one dimension to simplify the analysis. In other words, they consider a vote on public education for a given pension benefit, or a vote on the allocation of tax revenue for a given tax rate. Therefore, these studies do not indicate how the size of the government (i.e., the tax rate) and the allocation of government spending between education and pensions are jointly determined through voting in the presence of generational conflict.

This problem is resolved by introducing two-dimensional voting based on altruism (Tabellini, 1991), party competition (Levy, 2005), issue-by-issue voting (Poutvaara, 2006), and reputation (Bellettini and Ceroni, 1999; Boldrin and Rustichini, 2000; Rangel, 2003). However, these studies have abstracted physical and/or human capital formation, and thus, have not examined the interaction between policy and capital formation. The concept of capital formation was first introduced by Kemnitz (2000), Gradstein and Kaganovich (2004), Holtz-Eakin et al. (2004), Tosun (2008), and Bernasconi and Profeta (2012). These studies have assumed myopic voting, in which current voters consider future policy as a given. In other words, the forward-looking decisions of voters are absent in the analysis of these studies. Therefore, they abstract from the feedback mechanism between current and future redistribution policies through physical and/or human capital accumulation, which plays a crucial role in shaping fiscal policies.

The feedback mechanism is demonstrated by Beauchemin (1998), Forni (2005), Bassetto (2008), Mateos-Planas (2008), Gonzalez-Eiras and Niepelt (2012), Song (2011), Chen and Song (2014), and Arcalean (2018).[4] In particular, the present study is closely related to Gonzalez-Eiras and Niepelt (2012), Lancia and Russo (2016), Ono and Uchida (2016), and Bishnu and Wang (2017), who have analyzed the politics of public education and pensions in the presence of a feedback mechanism in the overlapping generations model. Among these studies, Gonzalez-Eiras and Niepelt (2012) and Ono and Uchida (2016) have explored the effects of exogenously declining population growth rates on policy choices. Nonetheless, they have overlooked the dynamic interplay between policies and fertility decisions triggered by shifts in life expectancy. This study makes a valuable contribution to the existing literature by emphasizing the pivotal role

[4]The studies of multiple policy instruments other than education spending in the presence of a feedback mechanism include Hassler et al. (2003, 2005, 2007); Arawatari and Ono (2009, 2013); Song et al. (2012); Müller et al. (2016); Röhrs (2016); Arai et al. (2018); and Uchida and Ono (2021).

of this interaction in evaluating the consequences of aging on policy determinations, economic growth, and the optimality of pension reforms.

Furthermore, this study significantly deviates from the research conducted by Gonzalez-Eiras and Niepelt (2012) in the following aspect. They have investigated the influence of retirement age caps on policy choices and their implications for economic growth. Conversely, this study explores the effects of pension rules on fertility choices and the subsequent impact on policy choices and economic growth, while holding retirement age constant. Furthermore, this study examines the influence of expectation formation on future pension policies. Therefore, unlike Gonzalez-Eiras and Niepelt (2012), this study offers a novel perspective by examining the effects of fertility and expectation formation, both of which affect the political process of policymaking and economic growth.

This study contributes to the literature on aging and intergenerational conflict over policymaking through probabilistic voting, from a methodological perspective (Grossman and Helpman, 1998, Hassler et al., 2005, Gonzalez-Eiras and Niepelt, 2008, Song, 2011, Song et al., 2012, Arai et al., 2018 and Uchida and Ono, 2021). The study is, to the best of our knowledge, the first to obtain a closed-form solution of policy functions in a dynamic setting with endogenous fertility. de la Croix and Doepke (2009) and Kimura and Yasui (2009) have analyzed the politics of education when fertility is endogenous. However, their models are static in nature and thus, assume away an intertemporal interaction between fertility and policy choices via physical/human capital accumulation. The present study overcomes this limitation and demonstrates the dynamic impact of fertility on policy decisions and the resulting resource allocation across generations.

From a normative perspective, this study contributes to the literature on pension reforms. Andersen and Bhattacharya (2017), Bishnu et al. (2021) and Amol et al. (2022) have shown that in a dynamically efficient economy, a PAYG pension cut is feasible in a Pareto-improving manner. Our study shares a concern with theirs but differs with respect to the time horizon of the government's implementation of the pension cut. Their analyses and results have relied on the implicit assumption that an infinitely-lived government can calculate and implement a Pareto-improving redistribution of resources across generations. This is a common assumption in analyses of decentralization in competitive equilibrium.

In contrast to Andersen and Bhattacharya (2017), Bishnu et al. (2021) and Amol et al. (2022), our study, which belongs to the political economy literature, assumes that while the current population can compute an intergenerational redistribution of resources, there is no infinitely lived government that can commit to such a redistribution. In other words, an intergenerational resource reallocation can be performed only by a short-lived government that represents successive generations living in a current period, and this government can only reallocate resources in that period. We focus on the pension ceiling rule as a means of correcting the resource allocation decisions of such a short-lived government in the long-run perspective.

The rule, representing automatic adjustment rules that have been introduced in many OECD countries (OECD, 2012), is enacted by law, and cannot be easily modified; therefore, it can be an effective means of achieving the the desired intergenerational resource allocation in the long-term.

3 Model

The discrete time economy starts in period 0 and consists of overlapping generations. Individuals are identical within a generation and live at most for the following three periods: young, middle, and old age. They face uncertainties in the third period of life. Let $\pi \in [0, 1]$ denote the life expectancy (i.e., the probability of living in old age). This is considered idiosyncratic for all individuals and is constant across periods. Each middle-aged individual gives birth to a number of children denoted by n. The middle-aged individuals in period t is represented by N_t, and the population grows at a rate of n_{t+1} following the equation $N_{t+1} = n_{t+1}N_t$. The gross population growth rate, n_{t+1}, is determined by the fertility decisions of the middle-aged individuals.

Individuals

Individuals exhibit the following economic behavior during their life cycle. In their youth, individuals do not make any economic decisions and depend on their parents for their livelihood. In middle age, individuals work, receive market wages, pay taxes, and make fertility and saving decisions. In old age, they retire and receive and consume returns from savings.

We consider middle-aged individuals in period t. Each of them is endowed with one unit of time. Raising one child takes fraction $\phi \in (0, 1)$ of time. Each individual devotes ϕn_{t+1} units of time to raising children and supplies the remaining time, $1 - \phi n_{t+1}$, to the labor market. Each middle-aged individual obtains labor-income $(1 - \phi n_{t+1})\, w_t h_t$, where w_t is the wage rate per unit of labor and h_t is the human capital endowment. After paying tax, $\tau_t w_t h_t\,(1 - \phi n_{t+1})$, where τ_t is the period t labor-income tax rate, the individual distributes the after-tax income between consumption c_t and savings held as an annuity and invested in physical capital, s_t. Therefore, the period-t budget constraint for the middle-aged individuals becomes

$$c_t + s_t \leq (1 - \tau_t) w_t h_t\,(1 - \phi n_{t+1})\,. \tag{1}$$

The period $t + 1$ budget constraint in old age is

$$d_{t+1} \leq \frac{R_{t+1}}{\pi} s_t + b_{t+1}, \tag{2}$$

where d_{t+1} is consumption, R_{t+1} is the gross return from savings, and b_{t+1} is the PAYG public pension benefit. If an individual dies at the end of the middle-age period, their annuitized wealth is transferred via the annuity markets, to individuals who live throughout their old age. Therefore, the return on savings becomes R_{t+1}/π under the assumption of perfect annuity markets.

Children's human capital over period $t+1$, h_{t+1}, is a function of per capita government spending on public education x_t, and parents' human capital, h_t. Particularly, h_{t+1} is formulated using the following equation:

$$h_{t+1} = h(h_t, x_t) \equiv D(h_t)^{1-\eta}(x_t)^{\eta}, \tag{3}$$

where $D(>0)$ is a scale parameter and $\eta \in (0,1)$ denotes the elasticity of education technology with respect to education spending.

The following two remarks are in order. First, as in Gonzalez-Eiras and Niepelt (2012) and Lancia and Russo (2016), we abstract private education and private old-age support away from the analysis; moreover, heterogeneity within a generation is abstracted from the analysis. This simplification enables us to demonstrate precisely how the results would change when fertility choice of households is introduced in their framework. Second, we do not distinguish between spending on K-12 and higher education. Accordingly, we consider that x, an investment in public education, includes investments in both K-12 and higher education. In a real economy, the benefits of public education spending vary from person to person because some people receive higher education while others do not. This model does not explicitly depict such intra-generational heterogeneity. Instead, we focus on a representative agent to demonstrate the extent to which each individual within a generation benefits from public education investment in K-12 and higher education levels on average.

Middle-aged individuals care about consumption, c_t and d_{t+1}, their number of children, n_{t+1}, and the human capital of children, h_{t+1}. The preferences of the middle-aged in period t are specified by the following expected utility function à la de la Croix and Doepke (2003, 2004):

$$\ln c_t + \delta \ln n_{t+1}h_{t+1} + \beta\pi \ln d_{t+1}, \tag{4}$$

where $\beta \in (0,1)$ is a discount factor, and $\delta(>0)$ is the degree of preference for the children's quantity and quality.

We substitute the budget constraints (1) and (2) into the expected utility function in (4) to form the unconstrained maximization problem:

$$\max_{\{s_t, n_{t+1}\}} \ln\left((1-\tau_t)w_t h_t(1-\phi n_{t+1}) - s_t\right) + \delta \ln n_{t+1}h_{t+1} + \beta\pi \ln\left(\frac{R_{t+1}}{\pi}s_t + b_{t+1}\right).$$

By solving this problem, we obtain the following fertility, savings, and consumption functions:

$$n_{t+1} = n'(\tau_t, w_t h_t, b_{t+1}) \equiv \frac{1}{\phi}\cdot\frac{\delta}{1+\delta+\beta\pi}\cdot\frac{(1-\tau_t)w_t h_t + \frac{b_{t+1}}{R_{t+1}/\pi}}{(1-\tau_t)w_t h_t}, \tag{5}$$

$$s_t = s(\tau_t, w_t h_t, b_{t+1}) \equiv \frac{\beta\pi}{1+\delta+\beta\pi}\left[(1-\tau_t)w_t h_t - \frac{1+\delta}{\beta\pi}\cdot\frac{b_{t+1}}{R_{t+1}/\pi}\right], \tag{6}$$

$$c_t = c(\tau_t, w_t h_t, b_{t+1}) \equiv \frac{1}{1+\delta+\beta\pi}\left[(1-\tau_t)w_t h_t + \frac{b_{t+1}}{R_{t+1}/\pi}\right], \tag{7}$$

$$d_{t+1} = d'(\tau_t, w_t h_t, b_{t+1}) \equiv \frac{\beta R_{t+1}}{1+\delta+\beta\pi}\left[(1-\tau_t)w_t h_t + \frac{b_{t+1}}{R_{t+1}/\pi}\right], \tag{8}$$

where we drop the argument R_{t+1} from the expressions of $n'(\cdot)$, $s(\cdot)$, $c(\cdot)$, and $d'(\cdot)$ because R_{t+1} becomes constant, as demonstrated below. Superscript "′" in the expressions for $n'(\cdot)$ and $d'(\cdot)$ denotes the next period.

Firms

There exists a continuum of identical firms that are perfectly competitive profit maximizers. Each individual firm is indexed by i. The technology available to firm i is $Y_{it} = A_t\left(K_{it}\right)^{\alpha}\left(L_{it}\right)^{1-\alpha}$ where Y_{it}, K_{it}, and L_{it} stand for output, capital input, and labor input of firm i, respectively. Here, $A_t(>0)$ represents the general level of factor productivity, given by the individual firm, and $\alpha \in (0,1)$ is a constant parameter representing the capital share in production.

In each period, firm i chooses capital and labor to maximize its profit, $A_t\left(K_{it}\right)^{\alpha}\left(L_{it}\right)^{1-\alpha} - R_t K_{it} - w_t L_{it}$, where R_t is the gross return on physical capital and w_t is the wage rate. Firms' profit maximization leads to

$$K_{it} : R_t = \alpha A_t\left(K_{it}\right)^{\alpha-1}\left(L_{it}\right)^{1-\alpha}, \tag{9}$$

$$L_{it} : w_t = (1-\alpha)A_t\left(K_{it}\right)^{\alpha}\left(L_{it}\right)^{-\alpha}. \tag{10}$$

Capital fully depreciates within a single period.

The productivity parameter A_t is assumed to be proportional to the per labor capital $A_t = Q\cdot\left(K_t/L_t\right)^{1-\alpha}$, where $K_t = \sum_i K_{it}$ and $L_t = \sum_i L_{it}$ represent the aggregate capital stock and labor, respectively, and $Q(>0)$ is constant. Thus, capital investment involves a technological externality of the type often used in endogenous-growth theories. This assumption, called the AK technology, results in a constant interest rate across periods, which is demonstrated below. This approach helps us obtain a closed-form solution of the model. Under this assumption, the first-order conditions in (9) and (10) are rewritten in aggregate terms as follows:

$$R_t = R \equiv \alpha Q, \tag{11}$$

$$w_t = (1-\alpha)Q\frac{K_t}{L_t}. \tag{12}$$

Government Budget Constraint

Government expenditures include public education spending and public pension payments. They are financed by taxes on labor income. The government budget constraint in period t is $\tau_t w_t L_t = \pi N_{t-1} b_t + x_t N_{t+1}$, where $\tau_t w_t L_t$ is the aggregate labor-income tax revenue, $\pi N_{t-1} b_t$ is the public pension payment, and $x_t N_{t+1}$ is the aggregate public expenditure on education.

Let $k_t \equiv K_t/N_t$ denote per capita capital. By using (12) and dividing both sides of the constraint by N_t, we obtain a per-capita expression of the government budget constraint:

$$\tau_t(1-\alpha)Qk_t = \frac{\pi b_t}{n_t} + n_{t+1}x_t. \tag{13}$$

Market Clearing

The market-clearing condition for capital is $K_{t+1} = N_t s_t$, which expresses the equality of total savings by the middle-aged individuals in period t, $N_t s_t$, to the stock of aggregate physical capital at the beginning of the period $t+1$. We rewrite the capital market-clearing condition as

$$n_{t+1}k_{t+1} = s\left(\tau_t, w_t h_t, b_{t+1}\right), \tag{14}$$

where $s(\cdot)$ is defined in (6).

The market-clearing condition for labor is

$$L_t = \left(1-\phi n_{t+1}\right) N_t h_t, \tag{15}$$

which expresses the equality of the aggregate labor demand, L_t, to the aggregate supply, $\left(1-\phi n_{t+1}\right) N_t h_t$. Using (12) and (15), we define labor income as follows:

$$w_t h_t = \bar{w}\left(n_{t+1}, k_t\right) \equiv \frac{(1-\alpha)Q}{1-\phi n_{t+1}} k_t. \tag{16}$$

Thus, the labor income, $w_t h_t = \bar{w}\left(n_{t+1}, k_t\right)$, depends on n_{t+1} and k_t, but is independent of h_t.

Using (16), we can reformulate the fertility function in (5) as $n_{t+1} = n'\left(\tau_t, \bar{w}\left(n_{t+1}, k_t\right), b_{t+1}\right)$. Solving this expression for n_{t+1} leads to:

$$n_{t+1} = n'\left(\tau_t, b_{t+1}, k_t\right) \equiv \frac{1}{\phi} \cdot \frac{\delta}{1+\delta+\beta\pi} \cdot \frac{(1-\tau_t)(1-\alpha)Qk_t + \frac{b_{t+1}}{R/\pi}}{(1-\tau_t)(1-\alpha)Qk_t + \frac{\delta}{1+\delta+\beta\pi} \cdot \frac{b_{t+1}}{R/\pi}}. \tag{17}$$

Using (16) and (17), we can also reformulate the saving function in (6) as

$$s_t = s\left(\tau_t, \bar{w}\left(n'\left(\tau_t, b_{t+1}, k_t\right), k_t\right), b_{t+1}\right),$$

or:

$$s_t = s\left(\tau_t, b_{t+1}, k_t\right) \equiv \frac{\beta\pi}{1+\beta\pi}\left[(1-\tau_t)(1-\alpha)Qk_t - \frac{1}{\beta\pi} \cdot \frac{b_{t+1}}{R/\pi}\right]. \tag{18}$$

Indirect Utility

In the present framework, the following three state variables exist in period t: physical capital, k_t, human capital, h_t, and the fertility rate, n_t. We can express the indirect utility of the middle-aged over period t, V_t^M, and that of older adults over period t, V_t^O, as functions of the three state variables, as follows:

$$\begin{aligned} V_t^M &= \ln c\left(\tau_t, \bar{w}\left(n'\left(\tau_t, b_{t+1}, k_t\right), k_t\right), b_{t+1}\right) + \delta \ln n'\left(\tau_t, b_{t+1}, k_t\right) \cdot h\left(h_t, x_t\right) \\ &\quad + \beta\pi \ln d'\left(\tau_t, \bar{w}\left(n'\left(\tau_t, b_{t+1}, k_t\right), k_t\right), b_{t+1}\right), \quad (19) \\ V_t^O &= \ln d\left(b_t, n_t, k_t\right), \quad (20) \end{aligned}$$

where $h(\cdot)$, $c(\cdot)$, $d'(\cdot)$, $\bar{w}(\cdot)$, and $n'(\cdot)$ are defined in (3), (7), (8), (16), and (17), respectively, and $d(\cdot)$, representing consumption by older adults, is defined as follows:

$$d\left(b_t, n_t, k_t\right) \equiv \frac{R}{\pi} n_t k_t + b_t.$$

4 Political Equilibrium

This section considers voting on fiscal policies. We employ probabilistic voting à la Lindbeck and Weibull (1987), in which there is electoral competition between the two office-seeking candidates. Each candidate announces a set of fiscal policies subject to the government's budget constraints. As Persson and Tabellini (2000) have demonstrated, the two candidates' platforms converge in the equilibrium to the same fiscal policy that maximizes the weighted average utility of voters.

In the current framework, both older and middle-aged adults have an incentive to vote. Thus, the political objective in period t is the weighted sum of the utilities of older and middle-aged adults; this is given by $\pi\omega V_t^O + n_t(1-\omega)V_t^M$, where $\omega \in (0,1)$, and $1-\omega$ are the political weights placed on older and middle-aged adults, respectively. A larger value of ω implies greater political power of older adults. We use the gross population growth rate n_t to adjust the weight of the middle-aged, and life expectancy (i.e., the probability of living in old age) π to adjust the weight of older adults, to reflect their population share. To obtain the intuition behind this result, we divide the objective function by $n_t(1-\omega)$ and redefine it, denoted by Ω_t, as follows:

$$\Omega_t = \frac{\pi\omega}{n_t(1-\omega)} V_t^O + V_t^M, \tag{21}$$

where V_t^M and V_t^O are defined as (19) and (20). respectively. The coefficient $\pi\omega/n_t(1-\omega)$ of V_t^O represents the relative political weight of older adults.

The political objective function in (21) suggests that the current policy choice of (τ_t, b_t, x_t) affects the future policy decisions through fertility choices and physical capital accumulation. Specifically, the current choice of τ_t, b_t, and x_t affects fertility decisions and the formation of physical capital in the next period, respectively. This influences political decision-making on pension payments, b_{t+1}, in the next period. Conversely, as seen in (6) and (17), the level of pension benefits in the next period also affects the economic decisions on savings and fertility in the current period.

To demonstrate this mutual interaction between economic and political decisions, we employ the Markov-perfect equilibrium concept, in which today's fiscal policy depends on the current payoff-relevant state variables. In the current framework, the payoff-relevant state variables in period t are the fertility rate, n_t, and physical capital, k_t; the human capital, h_t, is a payoff-irrelevant state variable because of the specification of the human capital formation function in (3) and the assumption of the logarithmic utility function in (4). Thus, the expected provision of public pension in period $t+1$, b_{t+1}, could be given by the function of the period $t+1$ state variables, k_{t+1} and n_{t+1}: $b_{t+1} = \bar{B}(k_{t+1}, n_{t+1})$. Using the notation, with z' denoting the next period z, we can define a Markov-perfect political equilibrium in the current framework as follows.

Definition 1 A *Markov-perfect political equilibrium* is a five-tuple , $(\bar{T}, \bar{B}, \bar{X}, \bar{S}, \bar{N})$, where $\bar{T} : \Re_+ \times \Re_+ \to [0,1]$ is the tax rule, $\tau = \bar{T}(k,n)$; $\bar{B} : \Re_+ \times \Re_+ \to \Re_+$ is the pension rule, $b = \bar{B}(k,n)$; $\bar{X}: \Re_+ \times \Re_+ \to \Re_+$ is the education expenditure rule, $x = \bar{X}(k,n)$; $\bar{S}: [0,1] \times \Re_+ \to \Re_+$ is the optimal private saving rule, $s = \bar{S}\left(\tau, k \mid \bar{B}\right)$; and $\bar{N}: [0,1] \times \Re_+ \to \Re_+$ is the optimal private fertility rule, $n' = \bar{N}\left(\tau, k \mid \bar{B}\right)$, such that (i) for a given τ, k, and $\bar{B}$, the optimal private saving and fertility rules are the maps, $\bar{S}$ and $\bar{N}$, respectively, that solve

$$\bar{S}\left(\tau, k \mid \bar{B}\right) = s\left(\tau, \bar{w}\left(\bar{N}\left(\tau, k \mid \bar{B}\right), k\right), \bar{B}\left(\bar{S}\left(\tau, k \mid \bar{B}\right), \bar{N}\left(\tau, k \mid \bar{B}\right)\right)\right),$$
$$\bar{N}\left(\tau, k \mid \bar{B}\right) = n'\left(\tau, \bar{B}\left(\bar{S}\left(\tau, k \mid \bar{B}\right), \bar{N}\left(\tau, k \mid \bar{B}\right)\right), k\right),$$

where $b' = \bar{B}(k', n')$ with $n' = \bar{N}\left(\tau, k \mid \bar{B}\right)$ and $n'k' = \bar{S}\left(\tau, k \mid \bar{B}\right)$; (ii) given the set of initial conditions, (k, n), and the political objective function

$$\Omega\left(b, \tau, x, n, k, h \mid \bar{B}\right) \equiv \frac{\pi\omega}{n(1-\omega)} V^O(b, n, k) + V^M\left(\tau, x, b', k, h\right),$$

where $b' = \bar{B}(k', n')$ with $n' = \bar{N}\left(\tau, k \mid \bar{B}\right)$ and $n'k' = \bar{S}\left(\tau, k \mid \bar{B}\right)$, the equilibrium fiscal policies solve

$$\left(\bar{T}(k,n), \bar{B}(k,n), \bar{X}(k,n)\right) = \arg\max \Omega\left(b, \tau, x, n, k, h \mid \bar{B}\right)$$

subject to the government budget constraint

$$\bar{T}(k,n)(1-\alpha)Qk = \frac{\pi \bar{B}(k,n)}{n} + \bar{N}\left(\bar{T}(k,n), k \mid \bar{B}\right)\bar{X}(k,n).$$

Part (i) defines the functional equations that map the current tax and physical capital stock to achieve optimal private savings and fertility, $s = \bar{S}\left(\tau, k \mid \bar{B}\right)$ and $n' = \bar{N}\left(\tau, k \mid \bar{B}\right)$. This set of rules describes the private sector's response to changes in τ under the expectation that future pensions will be set according to the equilibrium rule $\bar{B}(k', n')$. Part (ii) describes the government's problem. In each period, the government sets fiscal policies subject to its budget constraints and the private sector's response, consistent with the expectation that future governments will follow the Markov-perfect political equilibrium rule.

4.1 Characterization of Political Equilibrium

To obtain the set of policy functions in Definition 1, we conjecture the following policy function of pension benefits in the next period:

$$\begin{aligned} b' &= \frac{\frac{\pi\omega}{n'(\tau,b',k)(1-\omega)}B + C}{\frac{\pi\omega}{n'(\tau,b',k)(1-\omega)}E + F} \cdot \frac{R}{\pi} n'\left(\tau, b', k\right) k' \\ &= \frac{\frac{\pi\omega}{n'(\tau,b',k)(1-\omega)}B + C}{\frac{\pi\omega}{n'(\tau,b',k)(1-\omega)}E + F} \cdot \frac{R}{\pi} s\left(\tau, \bar{w}\left(n'\left(\tau, b', k\right), k\right), b'\right), \end{aligned} \tag{22}$$

where B, C, E, and F are constant parameters, and the equality in the the second line originates from the capital-market-clearing condition in (14). Equation (22) implies that the amount of the pension benefit, b', is set to match a certain proportion of the savings, $n'k' = s$. The proportion depends on the relative weight given to older adults, $\pi\omega/n'(1-\omega)$, in the the political objective function.

Using the first-order condition with respect to capital in (11), the conjecture in (22) is rewritten as follows:

$$\frac{\pi b' N}{Y'} = \frac{\frac{\pi\omega}{n'(\tau,b',k)(1-\omega)}B + C}{\frac{\pi\omega}{n'(\tau,b',k)(1-\omega)}E + F}\alpha.$$

This shows that public pension payments are linearly related to GDP. The conjecture that satisfies this property is based on the results of Gonzalez-Eiras and Niepelt (2008) who show the linear relation of the policy functions on GDP under an exogenous fertility rate. Our conjecture here indicates that the same property holds for the endogenous fertility rate.

The conjecture in (22) suggests that a mutual interaction exists between pension benefits, b', and the fertility rate, n'. To examine this interaction, we recall the fertility function in (5), which is rewritten as follows:

$$n' = \frac{\frac{\delta}{1+\delta+\beta\pi}\left[(1-\tau)wh + \frac{b'}{R/\pi}\right]}{\phi\,(1-\tau)\,wh} = \frac{\frac{\delta}{1+\delta+\beta\pi}\left[(1-\tau)(1-\alpha)Q\frac{k}{(1-\phi n')} + \frac{b'}{R/\pi}\right]}{\phi\,(1-\tau)\,(1-\alpha)Q\frac{k}{(1-\phi n')}}, \tag{23}$$

where the second equality comes from the labor-market-clearing condition in (16).

An increase in the fertility rate n' leads to a decrease in labor supply because of the time required for childbirth and child rearing. The decrease in labor supply leads to an increase in the equilibrium wage in the labor market, thereby improving fertility through the income effect, as seen in the numerator of the right-hand side of (23). However, an increase in wages reduces fertility through the increased opportunity cost of fertility, as seen in the denominator of the right-hand side of (23). In the current framework, the negative effect of the latter exceeds the positive effect of the former, such that the right-hand side of (23) is decreasing given the fertility rate n'.

Given the property on the right-hand side of (23), we consider the effect of an increase in pension benefits b' on the right-hand side and thus the determination of n' that satisfies (23). An increase in pension benefits b' raises the lifetime income of an individual, thereby improving fertility through the income effect. This is the economic effect of pension benefits on the fertility rate. Conversely, an increase in the fertility rate leads to a decrease in the relative political weight of older adults, as seen in the conjecture in (22), which affects the determination of pension benefits. This is the political effect of fertility on pension benefits. Thus, pension benefits and fertility interact with each other through economic and political effect.

There is also a mutual interaction between savings, s, and pension benefits, b'. This interaction is apparent in the savings function in (6). As shown in the second term in parentheses, pension benefits b' discourage middle-aged individuals from saving. However, the pension benefits encourage them to have more children, as shown in (23). This in turn leads to an increase in the labor market equilibrium wages through a decrease in the labor supply, as described above. This has a positive income effect on savings. Thus, pension benefits have two opposing effects on savings, with the net effect being negative. This represents the economic effect of pension benefits on savings. A decrease in savings leads to a decrease in pension benefits based on the conjecture of b' in (22): this is the political effect. Thus, savings and pension benefits also interact through the economic and political effects.

Considering these two interactions, we substitute (6) and (23) into (22), and solve for b' to obtain the following equation:

$$b' = G \cdot (1-\tau)\, k, \tag{24}$$

where $G(>0)$, a constant, is defined in Appendix A.1. Equation (24) shows that a higher labor-income tax rate is associated with a lower level of future expected pension benefits. An increase in the tax rate lowers the relative price of births. This incentivizes individuals to increase fertility, which in turn affects the level of pension benefits through the conjecture of b' in (22). Simultaneously, an increase in the tax rate reduces savings because it reduces disposable income. This reduces pension benefits through the conjecture of b' in (22). Thus, the net effect of both variables is negative.

We substitute the policy function of b' from (24) into the fertility function presented in (17) and consequently obtain

$$n' = \frac{1}{\phi} \cdot \frac{\delta}{1+\delta+\beta\pi} \cdot \frac{(1-\alpha)Q + \frac{G}{R/\pi}}{(1-\alpha)Q + \frac{\delta}{1+\delta+\beta\pi} \cdot \frac{G}{R/\pi}}. \tag{25}$$

Equation (25) reveals that the fertility rate remains constant across periods, independent of variations in the labor-income tax rate and state variables. This constancy implies that the direct influence of taxes on fertility is counterbalanced by the indirect effect of taxes through public pension benefits, denoted as b'. It is important to note that this result is contingent upon the assumption of a logarithmic utility function. If this assumption is relaxed, the two effects may not necessarily cancel each other out. Nonetheless, even under the current assumption, there remains an endogenous determination of fertility through individuals' utility-maximizing behavior and their responses to changes in structural parameters.

Furthermore, the constancy of the fertility rate is contingent on the assumption of constant life expectancy. Given the persistent increase in life expectancy, a prevalent trend in developed countries, each generation's weight within the political objective function undergoes transformation due to its impact on fertility choices. This transformation, in turn, influences government policy decisions. Notably, the dynamic interplay between life expectancy and policy formulation through fertility choices sets our study apart from prior research (Gonzalez-Eiras and Niepelt, 2012; Lancia and Russo, 2016). We will delve deeper into this unique aspect in Section 6 of our analysis.

Using the pension benefits in (24) and the constant fertility rate in (25), we can reformulate the political objective function in (21) as follows:

$$\Omega = \frac{\pi\omega}{n(1-\omega)} \ln d\,(b,n,k) + \ln c\,(\tau,k) + \delta \ln n' \cdot h\,(x,h) + \beta\pi \ln d'\,(\tau,k)\,,$$

where $h(x,h) \equiv D\,(h)^{1-\eta}\,(x)^{\eta}$ as in (3), and $d(\cdot)$, $c(\cdot)$, and $d'(\cdot)$ are defined as follows:

$$
\begin{aligned}
d\,(b,n,k) &\equiv \frac{R}{\pi} nk + b, \\
c\,(\tau,k) &\equiv \frac{1}{1+\delta+\beta\pi} \cdot \left[(1-\tau)\,\bar{w}\left(n',k\right) + \frac{G\cdot(1-\tau)\,k}{R/\pi} \right], \\
d'\,(\tau,k) &\equiv \frac{\beta R}{1+\delta+\beta\pi} \cdot \left[(1-\tau)\,\bar{w}\left(n',k\right) + \frac{G\cdot(1-\tau)\,k}{R/\pi} \right].
\end{aligned}
$$

Given the government budget constraint in (13), we derive the first-order conditions with respect to τ, x, and b as follows:

$$
\tau : \frac{c_\tau}{c} + \beta\pi \frac{d'_\tau}{d'} + \lambda(1-\alpha)Qk = 0, \tag{26}
$$

$$
x : \delta \frac{h'_x}{h'} - \lambda n' = 0, \tag{27}
$$

$$
b : \frac{\pi\omega}{n(1-\omega)} \cdot \frac{d_b}{d} - \lambda \frac{\pi}{n} = 0, \tag{28}
$$

where λ is the Lagrangian multiplier associated with the government budget constraint and p_q $(p = c, d', h', d; q = \tau, x, b)$ denotes the derivative of p with respect to q. Notably, in deriving the first-order conditions in (26) – (28), we use the fact that the fertility rate is constant and independent of policy variables.

According to the expressions in (26) – (28), the government chooses a policy to equate its marginal benefits with its marginal costs. A detailed interpretation of each condition is as follows. Equation (26) shows that the government chooses the labor-income tax rate, τ, to equate its marginal costs and benefits. The first term of (26) represents the marginal costs of the tax and includes the following two effects on the consumption by middle-aged individuals: First, an increase in the tax rate decreases the disposable income of the middle-aged, leading to a decrease in their consumption. Second, an increase in the tax rate leads to a decrease in pension benefits among older adults, as shown in (24), which leads to a decrease in consumption by the middle-aged individuals. The second term of (26) also shows the marginal costs of the tax on old-age consumption, and includes two similar effects as in the first term. The third term shows the marginal benefits from increased tax revenue.

Equation (27) shows that the government chooses the education expenditures, x, to equate its marginal benefits arising from human capital accumulation represented by the first term, with the marginal costs from increased spending on x, represented by the second term. Equation (28) shows that the government chooses the pension benefits, b, to equate the marginal benefits arising from increased consumption by older adults, represented by the first term, with the marginal costs from increased spending on b, represented by the second term.

Using the conditions in (26) – (28), alongside the government budget constraint in (13), we verify the conjecture in (22), and obtain the following result.

Proposition 1 *Suppose that the following condition holds:*

$$1+\delta\eta+\beta\pi < \begin{cases} \frac{\pi\omega}{n_0(1-\omega)}\cdot\frac{1-\alpha}{\alpha} & \text{for } t=0, \\ \frac{\pi\omega}{n'(1-\omega)}\cdot\frac{1-\alpha}{\alpha} & \text{for } t\geq 1. \end{cases} \tag{29}$$

There is a Markov perfect political equilibrium such that the policy functions, τ, b, and x, are expressed as follows:

$$\tau = \frac{\frac{\pi\omega}{n(1-\omega)} + \left[\delta\eta - (1+\beta\pi)\frac{\alpha}{1-\alpha}\right]}{\frac{\pi\omega}{n(1-\omega)} + (1+\delta\eta+\beta\pi)} \in [0,1], \tag{30}$$

$$b = \frac{\frac{\pi\omega}{n(1-\omega)}\frac{1-\alpha}{\alpha} - (1+\delta\eta+\beta\pi)}{\frac{\pi\omega}{n(1-\omega)} + (1+\delta\eta+\beta\pi)}\frac{R}{\pi}nk > 0, \tag{31}$$

$$x = \frac{1}{n'}\cdot\frac{\delta\eta}{\frac{\pi\omega}{n(1-\omega)} + (1+\delta\eta+\beta\pi)}Qk > 0, \tag{32}$$

where n' is the fertility rate given by

$$n' = n'(\pi) \equiv \frac{-(1+\delta+\alpha\beta\pi) + \sqrt{(1+\delta+\alpha\beta\pi)^2 + 4\alpha\beta(1+\delta\eta+\beta\pi)\frac{1-\omega}{\omega}\frac{\delta}{\phi}}}{2\alpha\beta(1+\delta\eta+\beta\pi)\frac{1-\omega}{\omega}}. \tag{33}$$

Proof. See Appendix A.1.

The condition (29) ensures $b>0$. It indicates that life expectancy has three effects on the government's provision of public pensions. First, the higher the life expectancy, the easier it is for the government to provide pension, given the greater political weight of older adults. This effect is captured by the term π on the right-hand side of (29). Second, the higher the life expectancy, the higher the weight of consumption utility in old age. This provides an incentive for the government to lower its labor-income tax rate to maintain the consumption level of the older population. This effect is captured by the term π on the left-hand side of the equation.

In period $t=0$, if the effect of the former exceeds that of the latter, pension benefits will be paid to older adults. From period $t=1$ onward, there exists a third effect in addition to the two aforementioned effects. An increase in life expectancy decreases fertility rates, as will be shown in Proposition 2 below. This increases relative political weight of older adults and strengthens the government's incentive to provide pension benefits. This effect is represented by the term n' on the right-hand side of (29). Thus, because of this additional effect, the increase in life expectancy provides a stronger incentive for the government to provide pension benefits from period 1 onward compared to period 0.

Proposition 1 implies that the fertility and policy functions have the following features: First, the fertility rate remains constant and independent of physical capital over time. This stability arises from the previously mentioned factors: the direct influence of taxes on fertility is offset by the indirect effects of taxes on public pension benefits. Second, the levels of public pension benefits, b', and public education expenditure, x, are linear functions of output, Qk. This property is necessary for generating a balanced growth path for the economy. Finally, the labor-income tax rate is independent of the state variables and is constant across periods. This property is necessary for the government's budget to be balanced each period.

4.2 Effects of Life Expectancy

The result in Proposition 1 suggests that increased life expectancy affects the choice of fertility and policy. These effects extend to economic growth. This subsection focuses on an unexpected and permanent increase in life expectancy and analyzes its effects on fertility and subsequently, on policies and growth.

First, we consider the effect of life expectancy on the fertility rate in (33), which can be summarized as follows:

Proposition 2 *A higher life expectancy is associated with a lower fertility rate as follows:* $\partial n'/\partial \pi < 0$.

Proof. See Appendix A.1.

To understand the mechanism behind the result in Proposition 2, recall the fertility function in (17), which expresses the privately optimal fertility rate derived from individual utility maximization for a given policy set. The expression in (17) indicates the following two types of effects of life expectancy on fertility. First is the direct effect on an individual's decision regarding fertility for a given set of policy variables; the other is the indirect effect on fertility through political decisions regarding the level of pension benefits. Fertility is independent of the labor-income tax rate, as noted in the paragraph following (25). Hence, there is no indirect effect on fertility through the labor-income tax rate. In what follows, we examine various factors that contribute to these two effects.

Firstly, we consider the direct effects of the following two routes, named as "Effect n_i" ($i = 1, 2$) for later references. First, an increase in life expectancy increases the weight of old-age consumption utility. This strengthens the incentive for individuals to save, thereby increasing the costs of raising children. This has a negative effect on fertility ("Effect n_1"). Second, an increase in life expectancy lowers the return on savings and thus reduces the incentive for individuals to save. This has a positive effect on fertility ("Effect n_2").

Next, we consider the indirect effects of the following four routes, named as "Effect n_i" ($i = 3, 4, 5, 6$). First, an increase in life expectancy increases the political weight of older adults. This works to increase the level of pension benefits through voting, having a positive effect on fertility ("Effect n_3"). Second, an increase in life expectancy increases the weight of old-age consumption utility. This reduces the labor-income tax rate, which subsequently reduces pension benefits. This negatively affects fertility ("Effect n_4"). Third, an increase in the weight of old-age consumption utility increases the incentive for individuals to save. This leads to an increase in the level of pension benefits, as observed in the policy function of pension benefits. This has a positive effect on fertility ("Effect n_5"). Finally, the per-capita pension benefits are reduced to maintain a constant pension benefit-GDP ratio against an increase in life expectancy. This negatively affects fertility ("Effect n_6"). Overall, there are six conflicting effects. However, in the current framework, the net effect is negative.

As described above, life expectancy affects the government's policy choices. This implies that life expectancy affects the pension benefit-GDP ratio, $\pi bN_{-}/QK$, and the education expenditure-GDP ratio, xN'/QK, where N_{-} and N' denote the previous and next period N, respectively. The following proposition shows the effects of life expectancy on these ratios.

Proposition 3 *An increase in life expectancy results in the following effects: (i) an increase in the pension benefit-GDP ratio; and (ii) a decrease in the education expenditure-GDP ratio:* $\partial\left(\pi bN_{-}/QK\right)/\partial\pi > 0$ *and* $\partial\left(xN'/QK\right)/\partial\pi < 0$.

Proof. See Appendix A.2.

First, we consider the effect of life expectancy on the pension benefit-GDP ratio. As described in the paragraphs following Proposition 1, there exist two positive and one negative effects of life expectancy on per capita pension benefits. Additionally, given the per capita pension benefits, an increase in life expectancy leads to an increase in total pension benefits. Overall, there exist three positive and one negative effects, and the former exceeds the latter. Hence, an increase in life expectancy leads to an increase in the pension benefit-GDP ratio.

Next, we consider the education expenditure-GDP ratio. As life expectancy increases, the political weight of older adults also increases. The decrease in fertility brought about by the increase in life expectancy further increases the relative political weight of older adults. These effects reduce education spending through voting. Furthermore, an increase in life expectancy implies an increase in the weight of the utility of old-age consumption, which lowers the current labor-income tax rate. This decreases the education spending. Considering these two negative effects, an increase in life expectancy leads to a decrease in the education expenditure-GDP ratio.

As discussed, life expectancy has the three effects on the labor-income tax rate, named as "Effect τ_i" ($i = 1, 2, 3$) for later references. They can be summarized as follows. First, an increase in life expectancy increases the weight of utility form consumption in old age for middle-aged individuals. This lowers the tax rate to maintain consumption or savings ("Effect τ_1"). Second, life expectancy increases the political weight of older adults, incentivizing the government to increase pension benefits. This has the effect of raising the tax rate ("Effect τ_2"). Finally, because an increase in life expectancy leads to a decrease in fertility rate (Proposition 2), the political weight of older adults further increases, reinforcing the second effect ("Effect τ_3"). In summary, life expectancy has two positive and one negative effect on labor-income tax rate, and the net effect is positive or negative, depending on the structural parameter values.

Finally, based on the results presented in Proposition 1, we derive the growth rate of the economy and investigate how it is affected by increased life expectancy. Accordingly, we consider the per-capita output $y = Qk$. The growth rate of per capita output is calculated as follows:

$$\frac{y'}{y} = \frac{Qk'}{Qk} = \frac{s/n'}{k}. \tag{34}$$

Equation (34) indicates that life expectancy affects growth rate via the fertility rate, n', as well as savings, s.

Proposition 4 *An increase in life expectancy increases the per-capita GDP growth rate as follows:* $\partial\left(y'/y\right)/\partial\pi > 0$.

Proof. See Appendix A.3.

As already shown, an increase in life expectancy leads to a decrease in the fertility rate. This positively affects the growth rate. To observe the growth effect through savings, recall the savings function in (18), which is restated as follows:

$$s = \frac{\beta\pi}{1+\beta\pi}\left[(1-\tau)(1-\alpha)Qk - \frac{1}{\beta\pi}\cdot\frac{b'}{R/\pi}\right].$$

Life expectancy affects savings through the following three terms: a coefficient $\beta\pi/(1+\beta\pi)$, representing the propensity to save, the labor-income tax rate, τ, and pension benefits, b'.

As life expectancy increases, the weight on the utility of old-age consumption increases. This strengthens the incentive for individuals to save, and positively affects savings. Second, an increase in life expectancy increases older adults' political weight, which strengthens the incentive for the government to increase pension payments. This raises the labor-income tax rate. On the other hand, since a higher life expectancy increases the weight of the utility of old-age consumption, the government lowers the labor-income tax rate to maintain the consumption level in old age. Finally, an increase in life expectancy has three positive and one negative effects on pension benefit determination, as discussed above. Ultimately, the sum of the positive effects exceeds the sum of the negative effects in the current framework. Thus, an increase in life expectancy increases the per capita GDP growth rate.

4.3 Discussion

To examine the model predictions outlined in Propositions 2-4, Figure 1 illustrates the correlation between life expectancy and various factors for rich OECD countries. Panel (a) displays a negative correlation with fertility, Panel (b) exhibits a positive correlation with the ratio of public pension benefits to GDP, and Panel (c) portrays a negative correlation with the ratio of public education expenditures to GDP. Notably, the model predictions of Propositions 2 and 3 align with the observation based on cross-country data in Panels (a), (b), and (c) of Figure 1. However, in Panel (d), a relatively weak negative correlation between life expectancy and the GDP growth rate per capita is observed. This finding contradicts Proposition 4, which suggests a positive correlation. The inconsistency arises because our comparative statics analysis does not differentiate between current and subsequent period life expectancies.

To gain a deeper understanding of this inconsistency, we derive the following expression for the growth rate from the current period to the next period, utilizing the savings function from

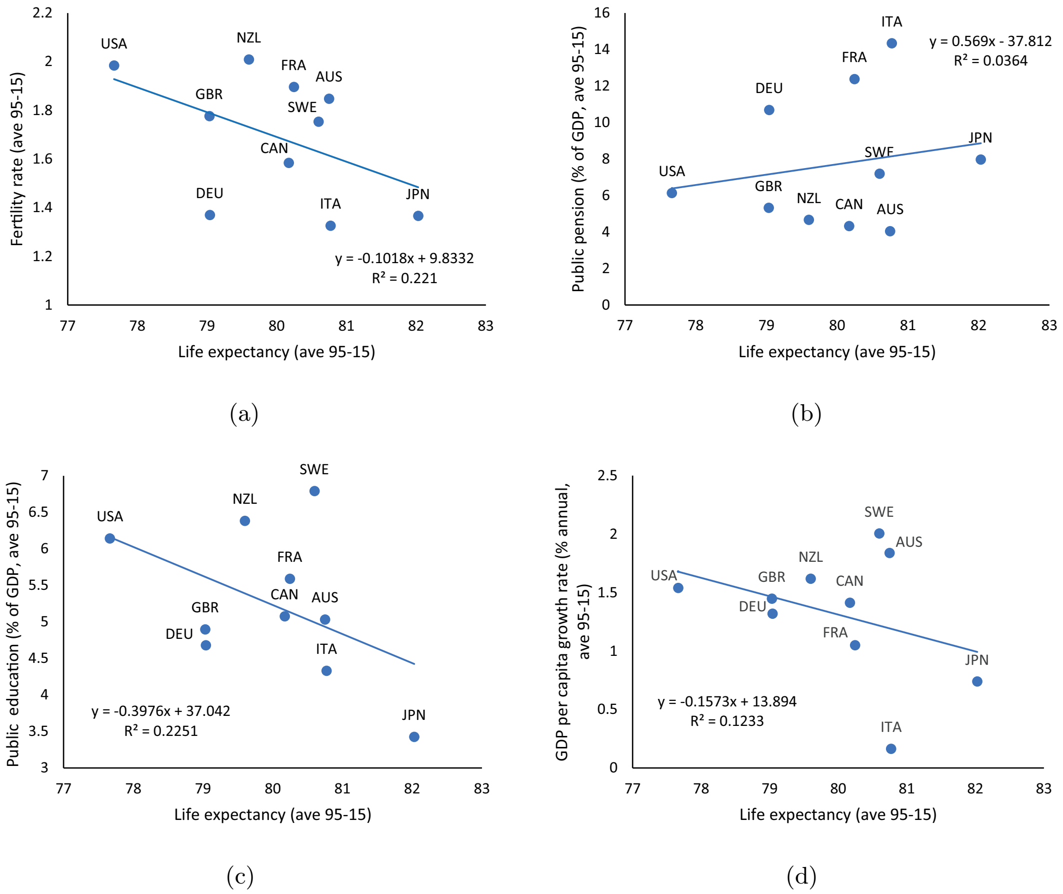

Figure 1: Association between life expectancy and fertility rates (Panel (a)), the ratio of public pension benefits to GDP (Panel (b)), the ratio of public education expenditures to GDP (Panel (c)), and the GDP growth rate per capita (Panel (d)) for rich OECD countries during 1995-2015. Note. The set of rich OECD countries includes Australia, Canada, Denmark, France, Germany, Italy, Japan, New Zealand, Sweden, the United Kingdom, and the United States.
Sources: Data on fertility rates are sourced from the United Nations Department of Economic and Social Affairs Population Division (2022) World Population Prospects 2022, Online Edition (`https://population.un.org/wpp/`) (accessed February 28, 2023). The sources of the other data are described in Appendix A.6.

(18) and the policy functions from Proposition 1:

$$\frac{k'}{k} = \frac{1}{n'\left(\pi', \pi''\right)} \cdot \frac{\beta\pi'}{1+\beta\pi'} \left[(1-\alpha)Q - \frac{1}{\beta} \cdot \frac{G\left(\pi', \pi''\right)}{R}\right] \left(1 - \tau\left(\pi, \pi'\right)\right), \tag{35}$$

where π, π', and π'' represent the life expectancies for the current period, the next period, and the period after next, respectively.

It is noteworthy that the growth rate's dependency on π, π', and π'' is mediated through four distinct terms in (35), denoted as $n'\left(\pi', \pi''\right)$, $\beta\pi'/\left(1+\beta\pi'\right)$, $G\left(\pi', \pi''\right)$, and $\tau\left(\pi, \pi'\right)$. An increase in life expectancy for the current period, π, tends to reduce the growth rate by diminishing the savings of middle-aged individuals in that period through an increased labor income tax rate due to Effects τ_2 and τ_3. In contrast, an increase in life expectancy for the next period, π', tends to stimulate the growth rate by reducing fertility, $n'\left(\pi', \pi''\right)$ through Effects n_1, n_2, n_3,

n_5 and n_6, and increasing the savings rate $\beta\pi' / (1 + \beta\pi')$. In addition, life expectancy for the next period influences pension benefits, represented by $G(\pi', \pi'')$, and also affects the labor income tax rate through Effects τ_1 and τ_3. Furthermore, life expectancy for the period after next, π'', negatively impacts the fertility rate via Effect n_4 and potentially has an uncertain effect through pension benefits. In summary, the growth effects stemming from these factors remain ambiguous.

Proposition 4 examined a scenario in which life expectancy in three successive periods increased uniformly from the same level. This analysis revealed a net positive effect of increased life expectancy on the growth rate. However, in the real world, the rate of life expectancy increase varies among countries, and the impact of increased life expectancy in one period may not align with that in another. Panel (d) of Figure 1 offers evidence suggesting that the negative effect is more pronounced in Japan and Italy, while the positive effect is stronger in Australia and Sweden. These divergent patterns will be subject to more comprehensive investigation in the model predictions and simulations detailed in Section 6.

5 Pension Reforms

The aging of the population due to the increase in life expectancy, and the associated decline in fertility (Proposition 2), will lead to an increase in the political weight of older adults. This gives the short-lived government, representing the currently living generations, an incentive to increase pension benefits and decrease education expenditure (Proposition 3). While increased pension benefits discourage individuals from saving, they also influence the behavior of the government, which ultimately leads to an increase in the GDP per capita (Proposition 4). However, future generations cannot partake in present policy-making decisions and cannot internalize the effects of current policies on their physical and human capital accumulation. This indicates a need for the improvement of social welfare, which should incorporate considerations for future generations through interventions in pension reforms.

To measure welfare gains associated with pension reforms, we consider a hypothetical long-lived planner whose goal is to maximize social welfare, which is the discounted aggregate sum of the welfare of all generations. The planner is assumed to be able to control the ceiling on pension benefits, to curb the choices of successive short-lived governments and generations. We examine how the corresponding tax rate, education expenditure, fertility rates, and economic growth rates would change if the planner introduced a ceiling on pension benefits in an economy that does not have such a ceiling, such as the one we characterized in Section 4.

For the analysis, we use a numerical approach because of the limitations of the analytical approach in the presence of a pension ceiling. In Subsection 5.1, we calibrate the model economy in such a manner that the steady-state equilibrium matches the key statistics of the average synthetic rich OECD countries over the time period 1995–2015. Following Gonzalez-Eiras and Niepelt (2012), we include the following countries into the set of the rich OECD countries:

Australia, Canada, Denmark, France, Germany, Italy, Japan, New Zealand, Sweden, the United Kingdom, and the United States. In Subsection 5.2, we derive the policy functions and the associated fertility rate in the presence of the pension ceiling, and use the calibrated parameter values to conduct some quantitative experiments. In Subsection 5.3, we derive the optimal ceiling on pension benefits in terms of maximizing social welfare, and identify when it is desirable to impose the ceiling. Appendix A.6 describes the sources of the data used in calibration.

5.1 Calibration

We take one period in the model to correspond to 30 years in the data. This assumption is standard in quantitative analyses of two- or three-period overlapping generations models (e.g., Gonzalez-Eiras and Niepelt, 2008; Lancia and Russo, 2016). The probability of living in old age, π, is derived based on the average life expectancy at birth. The average life expectancy in the rich OECD countries is 79.989 years; therefore, individuals will, on average, live $19.989(= 79.989-60)$ years into old age. In other words, individuals are expected to live 19.989/30 of their 30 years of old age, thus $\pi = 0.666$.

We fix the share of capital at $\alpha = 0.3$, which is considered a standard value in the literature (e.g., Gonzalez-Eiras and Niepelt, 2012). Our selection of R is 1.04 per year (e.g., Song et al., 2012; Lancia and Russo, 2016). The productivity parameter is $Q = 9.730$ because $Q = R/\alpha(= (1.04)^{30}/0.3)$. The opportunity cost of raising a child is approximately 20% of parents' time. (Kimura and Yasui, 2009). By assuming that the duration of parenthood is 18 years, ϕ is $0.2 \times (18/30) = 0.12$.

To determine the remaining four parameters, δ, η, ω, and β, we focus on the fertility rate, the education expenditure-GDP ratio, the pension benefit-GDP ratio, and per capita output growth rate. The fertility rate, n', is given by (33), and the two ratios, xN'/Y and $\pi bN_-/Y$, and the growth rate, y'/y, are as follows:

$$\frac{xN'}{Y} = \frac{\delta\eta}{\frac{\pi\omega}{n'(1-\omega)} + (1+\delta\eta+\beta\pi)}, \tag{36}$$

$$\frac{\pi bN_-}{Y} = \frac{\frac{\pi\omega}{n(1-\omega)}(1-\alpha) - (1+\delta\eta+\beta\pi)\alpha}{\frac{\pi\omega}{n(1-\omega)} + (1+\delta\eta+\beta\pi)}, \tag{37}$$

$$\frac{y'}{y} = \frac{k'}{k} = \frac{\alpha Q}{\left(1 - \frac{\beta\pi}{1+\beta\pi}(1-\alpha)\right)\frac{\omega}{(1-\omega)\beta} + \left(1+\frac{\delta\eta}{1+\beta\pi}\right)\alpha n}. \tag{38}$$

We use the data of the synthetic rich OECD average during 1995–2015 to solve the four equations (33), (36),and (37) and (38) for δ, η, ω, and β. Annual population growth rate data during the covered period, which is 1.0067, is taken as the target fertility rate. Therefore, the population growth rate over 30 years is considered $(1.0067)^{30}$.[5] The average education expenditure-GDP

[5]In this framework, the gross population growth rate and fertility rates are identical when π and n remain constant. The population size in period t is calculated by adding πN_{t-1}, N_t, and N_{t+1}, and that in period $t+1$ is obtained by adding πN_t, N_{t+1}, and N_{t+2}. Accordingly, the gross population growth rate from period t to $t+1$ is given by $(\pi N_t + N_{t+1} + N_{t+2}) / (\pi N_{t-1} + N_t + N_{t+1}) = (\pi + n_{t+1} + n_{t+1}n_{t+2}) / (\pi/n_t + 1 + n_{t+1})$. This equation reduces to n when $n_{t+1} = n_{t+2} = n$.

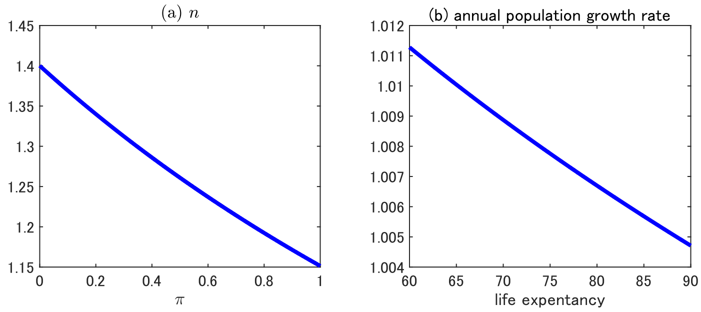

Figure 2: Panel (a) The association between π and n. Panel (b) The association between life expectancy and annual population growth rate.
Note. The life expectancy and annual population growth rate in Panel (b) are computed using $\pi \times 30 + 60$ and $(n)^{1/30}$, respectively.

ratio is 0.0523, the average pension benefit-GDP ratio is 0.0770, and the average per-capita output growth rate for 30 years is $(1.0131)^{30}$. We substitute these data and the values of α, ϕ, and π into (33), (36), (37), and (38), and solve them for δ, η, ω, and β. Accordingly, we obtain $(\delta, \eta, \omega, \beta) = (0.243, 0.573, 0.648, 0.780)$.

Based on this calibration, the impact of changes in life expectancy on fertility and the resulting population growth rate can be estimated, as demonstrated in Figure 2. Panel (a) illustrates the relationship between π, which represents the probability of living to old age, and n, which represents fertility rates, while Panel (b) illustrates the corresponding relationship between life expectancy and the annual population growth rate. The findings in the figure show that a higher life expectancy is associated with a lower fertility rate, as demonstrated in Proposition 2.

Table 1 presents the calibration of life expectancy and its correlation with population growth in the following four country groups: the synthetic rich OECD countries, synthetic rich OECD Europe (consisting exclusively of European countries), Japan, and the United States. Based on the results presented in Table 1, an increase in life expectancy corresponds to a decrease in the annual population growth rate. Particularly, in the United States, where life expectancy is comparatively low, a one-year increase in life expectancy results in a reduction of 211 individuals per million. Conversely, in Japan, where life expectancy is higher than that in the United States, a corresponding reduction is 203 individuals per million, which is lower than that in the United States. These findings suggest that the reduction in population growth because of an increase in life expectancy is less significant as life expectancy increases.

5.2 Effects of Pension Cuts

To introduce a ceiling on pension benefits into the model, we recall the pension benefits in the absence of the ceiling in (22), which is restated as follows:

$$b = P(n)\frac{R}{\pi}nk,$$

	(1)	(2)	(3)
rich OECD	79.989	-0.00020664	-206.644
Japan	82.028	-0.00020297	-202.971
the U.S.	77.664	-0.00021102	-211.022
Europe	79.937	-0.00020674	-206.740

Table 1: Column 1 displays the mean life expectancy between 1995 to 2015. Columns 2 and 3 demonstrate the predicted impact of a one-year increase in life expectancy on the rate of change in the annual population growth (Column 2) and the number of population change per million (Column 3).

where $P(n)$ is defined as

$$P(n) \equiv \frac{\frac{\pi\omega}{n(1-\omega)}B + C}{\frac{\pi\omega}{n(1-\omega)}E + F}.$$

From the equilibrium policy function of b in (31), B, C, E, and F are defined by

$$B \equiv \frac{1-\alpha}{\alpha}, C \equiv -\left(1+\delta\eta+\beta\pi\right), E \equiv 1, F \equiv 1+\delta\eta+\beta\pi.$$

Let ε and ε' denote the ratios defining the current and future pension ceilings. They are based on current and future benefit levels in the absence of a pension ceiling, respectively. In the presence of a ceiling, the current and future benefits, b and b', must satisfy the following condition:

$$b \leq \varepsilon P(n)\frac{R}{\pi}nk, \tag{39}$$

$$b' \leq \varepsilon' P(n')\frac{R}{\pi}n'k', \tag{40}$$

respectively. At this stage, we distinguish between ε and ε' to identify each different effect. In the latter part of this section, we focus on a situation in which the ceiling is stationary and $\varepsilon = \varepsilon'$ holds true. There exist the following two possible cases: (i) $\varepsilon, \varepsilon' > 1$, and (ii) $\varepsilon, \varepsilon' \leq 1$.

First, we consider the case in which the pension ceiling is set to $\varepsilon, \varepsilon' > 1$. Suppose the ceilings in (39) and (40) are non-binding. Under this assumption, we solve the government problem and obtain the policy functions as demonstrated in Proposition 1. The resulting policy functions satisfy the constraints in (39) and (40). Therefore, the constraints are non-binding when $\varepsilon, \varepsilon' > 1$.

Next, we consider the case in which the pension ceiling is set at $\varepsilon, \varepsilon' \leq 1$. Suppose that the ceilings in (39) and (40) are non-binding and solve the government problem. In this case, the derived policy functions, presented in Proposition 1, do not satisfy the constraints in (39) and (40). Thus, when $\varepsilon, \varepsilon' \leq 1$, we need to solve the government's problem by assuming that the constraints in (39) and (40) are binding. The following analysis derives the policy functions when the pension ceiling is set at $\varepsilon, \varepsilon' \leq 1$.

Suppose that when $\varepsilon, \varepsilon' \leq 1$, the future benefit is expressed as

$$b' = \varepsilon' P(n')\frac{R}{\pi}n'k'. \tag{41}$$

This conjecture is reformulated using the fertility function in (17) and saving function in (18) as follows:

$$b' = \tilde{G}(\varepsilon')(1-\tau)k, \tag{42}$$

where the definition of $\tilde{G}$ is provided in the Appendix A.4. Equation (42) is analogous to (24) in the absence of a ceiling, and thus $G = \tilde{G}$ holds if $\varepsilon' = 1$.

In the same manner as in the absence of the ceiling, we derive the fertility and consumption functions and the associated political objective function as follows:

$$\Omega = \frac{\pi\omega}{n(1-\omega)} \ln \tilde{d}(n,k,\varepsilon) + \ln \tilde{c}\left(\tau,k,\varepsilon'\right) + \delta \ln \tilde{n}'\left(\varepsilon'\right) h(x,h) + \beta\pi \ln \tilde{d}'\left(\tau,k,\varepsilon'\right), \tag{43}$$

where $\tilde{n}'$, $\tilde{c}$, $\tilde{d}'$, and $\tilde{d}$ are

$$\tilde{n}'\left(\varepsilon'\right) \equiv \frac{1}{\phi} \cdot \frac{\delta}{1+\delta+\beta\pi} \cdot \frac{(1-\alpha)Q + \frac{\tilde{G}(\varepsilon')}{R/\pi}}{(1-\alpha)Q + \frac{\delta}{1+\delta+\beta\pi} \cdot \frac{\tilde{G}(\varepsilon')}{R/\pi}}, \tag{44}$$

$$\tilde{c}\left(\tau,k,\varepsilon'\right) \equiv \frac{1}{1+\delta+\beta\pi} \cdot \left[(1-\tau)\,\bar{w}\left(\tilde{n}'\left(\varepsilon'\right),k\right) + \frac{\tilde{G}\left(\varepsilon'\right)\cdot(1-\tau)\,k}{R/\pi}\right], \tag{45}$$

$$\tilde{d}'\left(\tau,k,\varepsilon'\right) \equiv \frac{\beta R}{1+\delta+\beta\pi} \cdot \left[(1-\tau)\,\bar{w}\left(\tilde{n}'\left(\varepsilon'\right),k\right) + \frac{\tilde{G}\left(\varepsilon'\right)\cdot(1-\tau)\,k}{R/\pi}\right], \tag{46}$$

$$d = \tilde{d}(n,k,\varepsilon) \equiv \frac{R}{\pi} nk + \varepsilon \cdot \frac{\frac{\pi\omega}{n(1-\omega)}B + C}{\frac{\pi\omega}{n(1-\omega)}E + F} \cdot \frac{R}{\pi} nk. \tag{47}$$

In each period, the government chooses a set of policy variables, (τ, x, b), to maximize Ω in (43), subject to the government budget constraint in (13). The state variables, k and n, and ceilings, ε and ε', are given. The derivation of the following solutions is presented in the Appendix A.5.

The government's choice of labor-income tax rate becomes

$$\tau = \tilde{\tau}(n,\varepsilon) \equiv \frac{\delta\eta(1-\alpha)\left[\frac{\pi\omega}{n(1-\omega)} + (1+\delta\eta+\beta\pi)\right] + \varepsilon\,(1+\beta\pi)\left[\frac{\pi\omega}{n(1-\omega)}(1-\alpha) - (1+\delta\eta+\beta\pi)\,\alpha\right]}{(1+\delta\eta+\beta\pi)\,(1-\alpha)\left[\frac{\pi\omega}{n(1-\omega)} + (1+\delta\eta+\beta\pi)\right]}. \tag{48}$$

The associated education expenditure, x, is obtained by substituting the tax rate in (48) and the pension benefits in (39) into the government budget constraint:

$$x = \tilde{X}\left(\varepsilon, n, \tilde{n}'\left(\varepsilon'\right)\right) \cdot Qk, \tag{49}$$

where $\tilde{X}(\cdot)$ is defined as:

$$\tilde{X}(\cdot) = \begin{cases} \tilde{X}\left(\varepsilon_0, n_0, \tilde{n}'\left(\varepsilon_1\right)\right) \equiv \frac{1}{\tilde{n}'(\varepsilon_1)} \cdot \hat{X}\left(\varepsilon_0, n_0\right) & \text{for } t = 0, \\ \tilde{X}\left(\varepsilon_t, \tilde{n}'\left(\varepsilon_t\right), \tilde{n}'\left(\varepsilon_{t+1}\right)\right) \equiv \frac{1}{\tilde{n}'(\varepsilon_{t+1})} \cdot \hat{X}\left(\varepsilon_t, \tilde{n}'\left(\varepsilon_t\right)\right) & \text{for } t \geq 1, \end{cases} \tag{50}$$

and $\hat{X}(\cdot)$ is defined by

$$\hat{X}(\cdot) = \begin{cases} \hat{X}\left(\varepsilon_0, n_0,\right) \equiv \frac{\delta\eta}{1+\delta\eta+\beta\pi} \cdot \frac{(1-\alpha)\left[\frac{\pi\omega}{n_0(1-\omega)} + (1+\delta\eta+\beta\pi)\right] - \alpha\varepsilon_0\left[\frac{\pi\omega}{n_0(1-\omega)}\frac{1-\alpha}{\alpha} - (1+\delta\eta+\beta\pi)\right]}{\frac{\pi\omega}{n_0(1-\omega)} + (1+\delta\eta+\beta\pi)} & \text{for } t = 0, \\ \hat{X}\left(\varepsilon_t, \tilde{n}'\left(\varepsilon_t\right)\right) \equiv \frac{\delta\eta}{1+\delta\eta+\beta\pi} \cdot \frac{(1-\alpha)\left[\frac{\pi\omega}{\tilde{n}'(\varepsilon_t)(1-\omega)} + (1+\delta\eta+\beta\pi)\right] - \alpha\varepsilon_t\left[\frac{\pi\omega}{\tilde{n}'(\varepsilon_t)(1-\omega)}\frac{1-\alpha}{\alpha} - (1+\delta\eta+\beta\pi)\right]}{\frac{\pi\omega}{\tilde{n}'(\varepsilon_t)(1-\omega)} + (1+\delta\eta+\beta\pi)} & \text{for } t \geq 1. \end{cases} \tag{51}$$

Based on the calibration described in Subsection 5.1, in Figure 3, we numerically show how the policy variables and corresponding economic growth rates are affected when the current ceiling on pension benefits, ε, is reduced by considering the future ceiling, ε', as a given for the synthetic rich OECD countries. Considering that a reduction in the ceiling implies a decrease in the financial resources needed, the labor-income tax rate decreases along with the pension benefit-GDP ratio, as illustrated in Panels (a) and (b), respectively. However, the education expenditure-GDP ratio increases, as shown in Panel (c). This is because the decrease in pension benefits leaves more financial resources available for education. Additionally, the income effect of the lower tax rate increases savings, as in Panel (d), which turns into an increase in the economic growth rate as depicted in Panel (e).

Panels (d), (e), and (f) present the effects of the future ceiling, ε'. The result in Panels (d) and (e) imply that a reduction in future pension benefits strengthens the incentive for individuals to save, which stimulates physical capital accumulation and thus increases the growth rate. The findings depicted in Panel (f) indicate that the fertility rate is not affected by the present ceiling. However, it is contingent upon the future ceiling. Specifically, a decline in future pension benefits reduces the motivation of individuals to bear children, which consequently results in a decrease in the fertility rate. We use the properties of these policy functions, the growth rate, and the fertility rate in the following analysis.

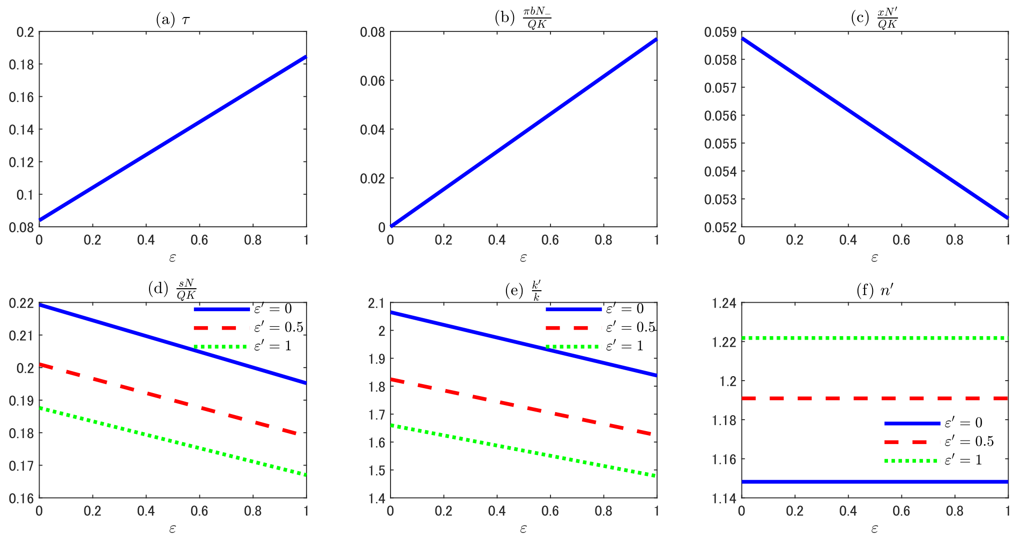

Figure 3: Impacts of the current pension ceiling on (a) the labor-income tax rate, (b) the pension benefit-GDP ratio, (c) the education expenditure-GDP ratio, (d) the saving-GDP ratio, (e) per capita capital growth rate, and (f) fertility rate. Additionally, Panels (d), (e), and (f) present three different scenarios ($\varepsilon' = 0$, 0.5, and 1.) to demonstrate the potential effects of future pension ceilings.

Before proceeding to the next subsection, note that the pension ceiling considered here is a given institution for short-lived governments. If they could choose the pension ceiling simultaneously, as the pension benefit level and public education expenditure, they would naturally choose $\varepsilon = 1$, that is, no restriction on the pension benefit. This is because this choice is optimal for maximizing their objective function. In other words, there is scope to support the introduction of pension ceilings when a society's objective function differs from that of the short-lived governments. This issue is addressed in the next subsection.

5.3 Optimal Pension Reform

As mentioned in the introduction of this section, the governments representing currently living generations do not consider intergenerational externalities in their policy choices. To solve this problem, we have introduced the pension ceiling and have shown that by manipulating the ceiling, we can control the governments' policy choices and the corresponding economic outcomes. Considering this result, we aim to maximize social welfare, which is the aggregate sum of the utility of each generation, by manipulating the pension ceiling. Accordingly, we clarify the conditions under which the equilibrium allocation fails to achieve social welfare maximization in the absence of a pension ceiling and what level of the pension ceiling should be imposed in such a case. In the following analysis, we maintain the assumption of $\varepsilon, \varepsilon' \leq 1$ to focus on the situation in which the pension ceiling has a substantial impact on governments' policy choices.

The social welfare function, denoted by SW, is defined as follows:

$$SW = V_0^O + \sum_{t=0}^{\infty} \gamma^t V_t^M, \tag{52}$$

where V_0^O is the indirect utility function of the period-0 old, V_t^M is the indirect utility of the period-t middle-aged, and $\gamma \in (0,1)$ denotes the social discount factor. The long-lived planner, whose objective is to choose the sequence of the pension ceiling, maximizes SW subject to the successive short-lived governments' choices regarding policies in (42), (48), (49), and the associated fertility and consumption functions in (44) – (47).

Substituting the constraints into (52) leads to the social welfare function being expressed as a function of the sequence of the pension ceilings, $\{\varepsilon_t\}_{t=0}^{\infty}$:

$$SW \simeq Z_0(\varepsilon_0) + \sum_{t=1}^{\infty} \gamma^{t-1} Z(\varepsilon_t), \tag{53}$$

where $Z_0(\varepsilon_0)$ and $Z(\varepsilon_t)$, representing the effects of the period-0 and -$t\,(\geq 1)$ pension ceilings, respectively, are defined by

$$Z_0(\varepsilon_0) \equiv \left(1+\varepsilon_0 \cdot \frac{\frac{\pi\omega}{n_0(1-\omega)}B+C}{\frac{\pi\omega}{n_0(1-\omega)}E+F}\right)\cdot\frac{R}{\pi}n_0$$
$$+\left\{(1+\beta\pi)+\left[(1+\beta\pi+\delta\eta)+\delta(1-\eta)\frac{\gamma\eta}{1-\gamma(1-\eta)}\right]\frac{\gamma}{1-\gamma}\right\}\ln(1-\tilde{\tau}(n_0,\varepsilon_0))$$
$$+\frac{\delta\eta}{1-\gamma(1-\eta)}\ln\hat{X}\left(\varepsilon_0,\tilde{n}'(\varepsilon_0)\right), \tag{54}$$
$$Z(\varepsilon_t) \equiv \tilde{V}(\varepsilon_t)+\frac{\delta\eta}{1-\gamma(1-\eta)}\ln\frac{1}{\tilde{n}'(\varepsilon_t)}$$
$$+\gamma\left\{(1+\beta\pi)+\left[(1+\beta\pi+\delta\eta)+\delta(1-\eta)\frac{\gamma\eta}{1-\gamma(1-\eta)}\right]\frac{\gamma}{1-\gamma}\right\}\ln(1-\tau(\tilde{n}'(\varepsilon_t),\varepsilon_t))$$
$$+\gamma\frac{\delta\eta}{1-\gamma(1-\eta)}\ln\hat{X}\left(\varepsilon_t,\tilde{n}'(\varepsilon_t)\right)$$
$$+\left[(1+\beta\pi+\delta\eta)+\delta(1-\eta)\frac{\gamma\eta}{1-\gamma(1-\eta)}\right]\frac{\gamma}{1-\gamma}$$
$$\times\ln\frac{1}{\tilde{n}'(\varepsilon_t)}\left[\frac{(1-\alpha)Q}{1-\phi\tilde{n}'(\varepsilon_t)}-\frac{1+\delta}{\beta\pi}\cdot\frac{\tilde{G}(\varepsilon_t)}{R/\pi}\right], \tag{55}$$

where $\tilde{V}(\varepsilon_t)$ is defined as follows:

$$\tilde{V}(\varepsilon_t) \equiv (1+\beta\pi)\ln\left[(1-\alpha)Q+\frac{\tilde{G}(\varepsilon_t)}{R/\pi}\right]+\delta\ln\frac{(1-\alpha)Q+\frac{\tilde{G}(\varepsilon_t)}{R/\pi}}{(1-\alpha)Q+\frac{\delta}{1+\delta+\beta\pi}\cdot\frac{\tilde{G}(\varepsilon_t)}{R/\pi}}. \tag{56}$$

The Appendix A.7 provides the derivation of (53). The initial physical and human capital, k_0 and h_0, also affect social welfare; however, they are policy-irrelevant and consequently moved away from the expression in (53),

In (54), the first term denotes the utility of consumption for older adults in period 0 and the first part of the second term, $(1+\beta\pi)\ln(1-\tilde{\tau}(n_0,\varepsilon_0))$, represents the after-tax labor income of middle-aged individuals in the same period, which in turn represents their utility of consumption in middle- and old age. The second part, $(1+\beta\pi+\delta\eta)\frac{\gamma}{1-\gamma}\ln(1-\tilde{\tau}(n_0,\varepsilon_0))$, represents the after-tax income part of the period-0 saving function and shows the impact of savings in period 0 on future generations through physical capital accumulation. The third part, $\delta(1-\eta)\frac{\gamma\eta}{1-\gamma(1-\eta)}\frac{\gamma}{1-\gamma}\ln(1-\tilde{\tau}(n_0,\varepsilon_0))$, illustrates the influence of period-0 saving (i.e., period-1 capital) on human capital through public education expenditure, subsequently impacting future generations via the accumulation of human capital. The third term represents public education expenditure in period 0, showing its impacts on future generations through human capital accumulation.

Equation (55) shares similarities with Equation (54), where the third and fourth terms in (55) correspond to the second and third terms in (54), respectively. However, the first, second, and fifth terms in (55) introduce new effects that are not present in (54). Specifically, the first term in (55) represents the dual effects of the pension ceiling in period t on pension benefits, both directly and indirectly, via the fertility rate. Particularly, the pension ceiling affects pension

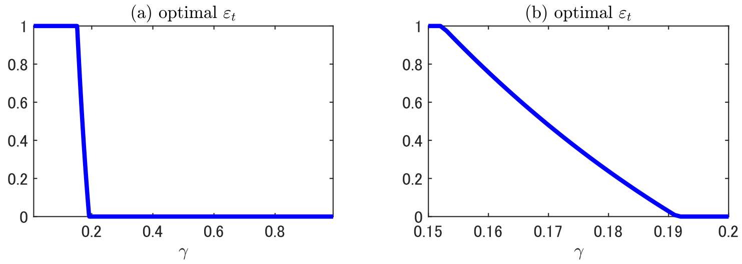

Figure 4: (a) The optimal ε_t that maximizes $Z(\varepsilon_t)$; (b) an enlarged view of the figure in Panel (a) around $\gamma = 0.152$.

benefits, $\tilde{G}(\varepsilon_t)$, the life-time income, and thus, the lifetime utility of consumption, as represented by the first term of $\tilde{V}(\varepsilon_t)$ in (56). Additionally, the pension ceiling affects pension benefits and fertility, as represented by the second term of $\tilde{V}(\varepsilon_t)$ in (56). The second term in (55) represents the impact of the pension ceiling on per capita public education spending through the fertility rate of middle-aged individuals in period t. The fifth term represents a part of the period-t saving function and a part of the period-t public education expenditure, showing their impacts on future generations through physical and human capital accumulation.

In the social welfare function of (53), the fertility rate is exogenously given in period 0, while in period 1 onwards, it becomes endogenous. Therefore, the optimal pension ceiling choice in period 0 generally differs from those in subsequent periods. Thus, maximizing the social welfare function requires a time-inconsistent plan for the pension ceiling. The optimal ceiling in period 0 becomes suboptimal when the period 1 arrives. Consideration of this time inconsistency is an interesting issue but is beyond the scope of this study. Therefore, we focus on the optimal ceiling from period 1 onwards, and consider the choice of ε_t that maximizes $Z(\varepsilon_t)$ in (55).

We derive the optimal ε_t which maximizes $Z(\varepsilon_t)$ in (55) based on the numerical approach introduced in Subsection 5.1. Figure 4 takes social discount factor γ on the horizontal axis and plots the associated optimal ε_t. This figure shows the critical value of γ, which is approximately 0.152. When γ exceeds the critical value, it is optimal to set $\varepsilon_t < 1$. The restriction or prohibition of the provision of pension benefits is required for the short-lived government's choice of pension benefits from the perspective of social welfare maximization. However, when γ is less than the critical value, it is optimal to set $\varepsilon_t = 1$ from the viewpoint of social welfare. Notably, when the planner assigns natural weights to generations, that is, $\gamma = \beta n \simeq 0.0953$, it is optimal to set at $\varepsilon_t = 1$.

To understand the mechanism underlying this outcome, it is necessary to revisit the expression $Z(\varepsilon_t)$ in (55). The act of reducing ε_t results in opposing effects on $Z(\varepsilon_t)$, as shown in Figure 5. The initial effect is negative and arises from the term $\tilde{V}(\varepsilon_t)$ in (56). The cut in ε_t reduces the pension benefits middle-aged individuals receive upon reaching old age. Consequently, this reduction in lifetime income leads to decreased consumption during middle and old age. Additionally, it reduces fertility rate, as observed in the second term of $\tilde{V}(\varepsilon_t)$. The second effect is

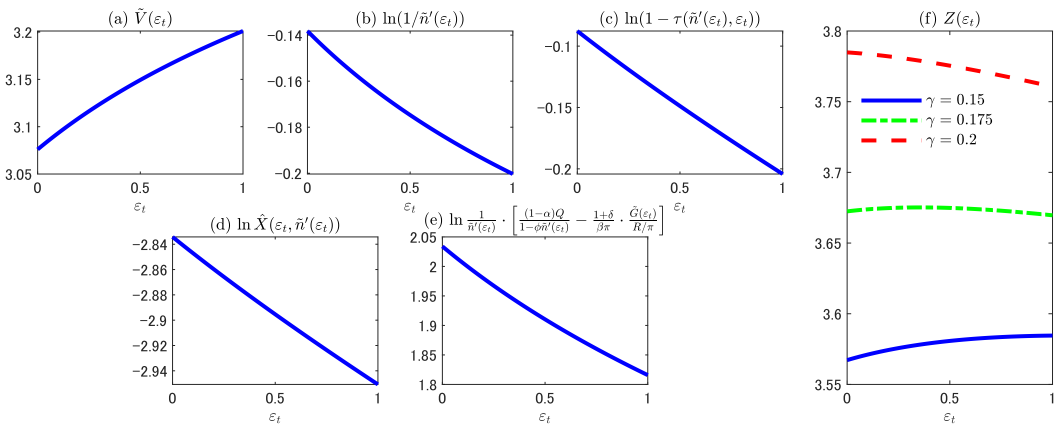

Figure 5: Effects of the pension cut on each term of $Z(\varepsilon_t)$ in (55): $\tilde{V}(\varepsilon_t)$ (Panel (a)), $\ln(1/\tilde{n}'(\varepsilon_t))$ (Panel (b)), $\ln(1-\tau(\tilde{n}'(\varepsilon_t), \varepsilon_t)$ (Panel (c)), $\ln \hat{X}(\varepsilon_t, \tilde{n}'(\varepsilon_t))$ (Panel (d)), and $\ln \frac{1}{\tilde{n}'(\varepsilon_t)} \cdot \left[\frac{(1-\alpha)Q}{1-\phi\tilde{n}'(\varepsilon_t)} - \frac{1+\delta}{\beta\pi} \cdot \frac{\tilde{G}(\varepsilon_t)}{R/\pi}\right]$ (Panel (e)). Panel (f) compares the effects of the pension cut on $Z(\varepsilon_t)$ for the three cases: $\gamma = 0.15$, 0.175, and 0.2.

positive, owing to the term $\tilde{n}'(\varepsilon_t)$. A cut in ε_t decreases fertility, thereby increasing per capita public education spending as indicated in (50). The third effect is positive and originates from the term $\ln(1-\tau(\tilde{n}'(\varepsilon_t), \varepsilon_t))$. The cut in ε_t leads to an increase in post-tax income by lowering the labor-income tax rate, which subsequently boosts consumption and physical and human capital accumulation.

The fourth effect, derived from the term $\ln \hat{X}(\varepsilon_t, \tilde{n}'(\varepsilon_t))$, exhibits a positive impact. This is achieved through the cut in ε_t, which facilitates the redistribution of government spending from pensions to education, driven by tax revenues. This effect is evident in the first argument of $\hat{X}(\varepsilon_t, \tilde{n}'(\varepsilon_t))$. Conversely, a reduction in ε_t leads to a decline in the fertility rate, $\tilde{n}'(\varepsilon_t)$, consequently amplifying the political influence of older adults on the political objection function. This, in turn, negatively affects public education expenditure. However, the numerical results demonstrate a net positive effect, resulting in an increase in public spending on education.

Finally, the fifth effect comprises three components in the term $\ln \frac{1}{\tilde{n}'(\varepsilon_t)} \cdot \left[\frac{(1-\alpha)Q}{1-\phi\tilde{n}'(\varepsilon_t)} - \frac{1+\delta}{\beta\pi} \cdot \frac{\tilde{G}(\varepsilon_t)}{R/\pi}\right]$, namely, (i) a positive effect of an increase in capital equipment per capita resulting from lower fertility, (ii) a negative effect of a decrease in wages per capita owing to higher total hours worked as a result of lower fertility, and (iii) a positive effect of an increase in capital per capita resulting from an increase in savings owing to a decrease in pension benefits. The first and third positive effects outweigh the second negative effect. In summary, the impact of reducing pension benefits on economic growth has both positive and negative effects, and its overall effect is neutral at $\gamma = 0.152$. Above (below) this threshold, the positive effects outweigh (are less than) the negative effects. This suggests that the decision to reduce pension benefits, in terms of social welfare, depends on the level of priority assigned to future generations by society.

The result of the optimal pension ceiling shows a trade-off between fertility and growth. A reduction in pension benefits promotes savings, that is, capital accumulation, by reducing the tax burden on the working middle-aged individuals. This increases the per capita GDP growth rate. However, a reduction in pension benefits affects the fertility behavior of households through policy changes, leading to a decline in the fertility rate. In many developed countries, declining fertility is one of the major policy issues. Simultaneously, dealing with an increasing pension burden, which hampers growth, is also a major policy challenge. Our results suggest the difficulty of reconciling the two goals of improving fertility and economic growth, which are likely to be urgent issues for many aging rich OECD countries.

6 Projected Changes in Fiscal Policies, Fertility, and Growth Rates

To facilitate our analysis, we utilize a model-based time series approach to predict the changes in fiscal policies, fertility, and growth rates over time in response to projected improvements in life expectancy. In Subsection 5.1, following Gonzalez-Eiras and Niepelt (2012), we have categorized the OECD countries included in our parameter estimation into four groups: synthetic rich OECD (including all covered countries), synthetic rich OECD Europe (comprising only European countries), Japan, and the United States. In this section we use life expectancy data from the United Nations World Population Prospects,[6] and examine the impact of projected life expectancy on fiscal policies, fertility, and economic growth over time, providing quantitative estimates of the predicted effects.

The analysis proceeds by computing three sequences of model predictions, each spanning a 30-year period. The first sequence includes the years 1990, 2020, 2050, and 2080; the second includes 2000, 2030, 2060, and 2090; and the third includes 2010, 2040, 2070, and 2100. To present the time series predictions, these three sequences are merged into a single time series following the procedure outlined by Gonzalez-Eiras and Niepelt (2012). Figure 6 displays the predictions of π_t in our model for the four groups based on the United Nations World Population Prospects. The other parameter values utilized in the analysis are obtained from the calibration conducted in Subsection 5.1. It should be noted that instead of using average life expectancy between $t-1$ and t, we utilize the life expectancy at time t. Furthermore, π_t for all groups, except the United States, is projected to reach the upper limit of 1 after 2070 in Japan and after 2080 in the synthetic rich OECD and synthetic rich OECD Europe. This is because, in the model, the upper limit of life expectancy is set at 90 years, whereas the estimated life expectancy exceeds 90 years after 2070 or 2080.

It is important to note that the data on π_t of the previous year in each column, that is, 2080, 2090, and 2100, is solely used to calculate n_t at $t = 2050$, 2060, and 2070, respectively. To obtain the fertility rate n_t and policy variables for the last three years of 2080, 2090, and 2100, life expectancy π_t data for 2110, 2120, and 2130, respectively, is required. However, since the

[6] See Appendix A.6 for the source of the data.

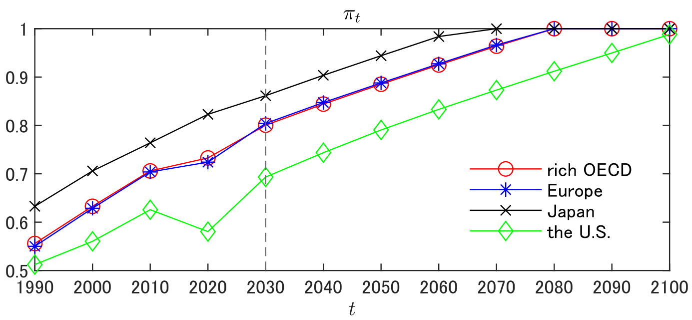

Figure 6: The predictions of π_t for the synthetic rich OECD, the synthetic rich OECD Europe, Japan, and the United States.
Note: The figure plots measured values up to 2020 and projected values after 2030.

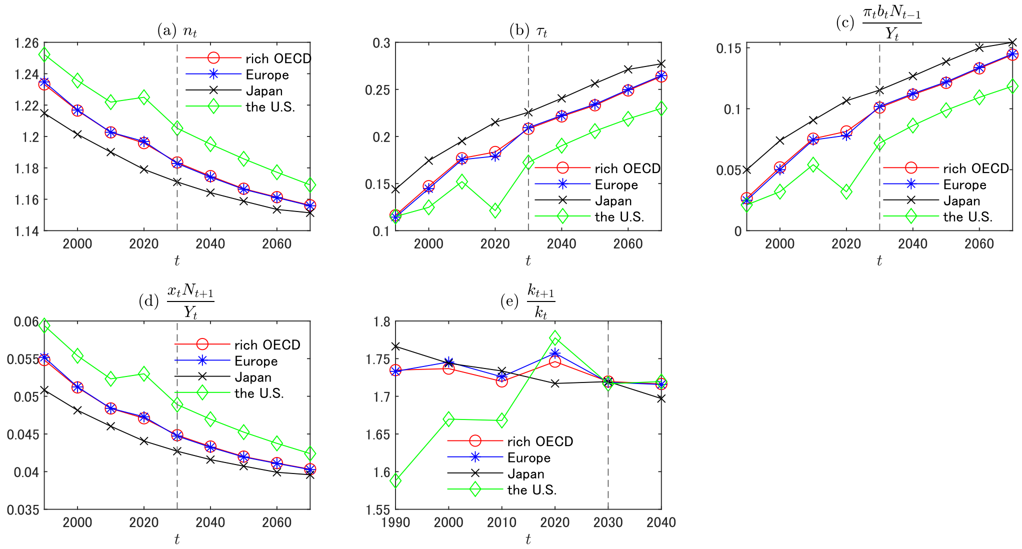

Figure 7: Projected changes in fertility rate (Panel (a)), the labor-income tax rate (Panel (b)), the pension benefit-GDP ratio (Panel (c)), the education expenditure-GDP ratio (Panel (d)), and the growth rate of per capita capital (Panel (e)) in the absence of pension benefit cuts.

available life expectancy estimates are limited to the year 2100, it is not possible to calculate the fertility rates and policy functions for the last three years. Therefore, we forecast fertility rate, policy functions, and the growth rate for each column up to one period before the final year.

Figure 7 illustrates the projections of fertility rates, fiscal policies, and growth rates for the four groups of countries in the absence of pension benefit cuts. The figure shows that the fertility rate and education expenditure-GDP ratio will decrease, while the labor-income tax rate and pension benefit-GDP ratio will increase over time. This projection aligns with model-based qualitative predictions in Propositions 2 and 3. However, the growth rate of per-capita physical capital is projected to decrease for Japan and slightly decrease for the synthetic rich OECD and synthetic rich OECD Europe, with a temporary decline also observed in the United States. This finding contrasts with the result of Proposition 4, which states that the growth rate increases as life expectancy increases, based on comparative statics that assume equal life expectancy increases between the current and future periods. The numerical analysis indicates that the degree of increase in life expectancy differs between periods, leading to different forecasts of economic growth rates.

To better understand the mechanism behind the difference in results, we refer to the equation for the growth rate of capital per capita in (35). We introduce the pension ceiling into this equation and explicitly specify the time subscript to facilitate subsequent analysis. Consequently, Equation (35) undergoes the following reformulation:

$$\frac{k_{t+1}}{k_t} = \frac{1}{n_{t+1}\left(\pi_{t+1}, \pi_{t+2}, \varepsilon_{t+1}\right)} \cdot \frac{\beta \pi_{t+1}}{1+\beta \pi_{t+1}} \left[(1-\alpha) Q - \frac{1}{\beta} \cdot \frac{\tilde{G}\left(\pi_{t+1}, \pi_{t+2}, \varepsilon_{t+1}\right)}{R} \right] \left(1 - \tau_t \left(\pi_t, \pi_{t+1}, \varepsilon_t\right)\right). \tag{57}$$

This equation indicates that an increase in life expectancy affects the per-capita economic growth rate based on four factors. The first factor is represented by the term $1/n_{t+1}$, which is the fertility rate decline that results in an increase in capital equipment per capita, positively affecting the economic growth rate. The second factor, represented by the term $\beta\pi_{t+1}/\left(1+\beta\pi_{t+1}\right)$, is an increase in saving, as individuals are incentivized to save more because of their increased life expectancy. The third factor, represented by the term $\tilde{G}\left(\varepsilon_{t+1}\right)$, is an increase in pension benefits, which discourages individuals from saving, and negatively affects the growth rate. The fourth factor, represented by the term $(1-\tau_t)$, is an increase in the labor-income tax rate, which reduces savings and has a negative effect on the growth rate. Therefore, an increase in life expectancy has two positive and two negative effects on economic growth. In Japan, where life expectancy is rapidly increasing, the negative effects outweigh the positive effects, leading to a decline in economic growth. Contrastingly, in the United States, where life expectancy has temporarily declined from 2010 to 2020, the positive effects outweigh the negative effects, resulting in a temporary increase in economic growth from 2030 to 2040.

We examine the potential impact of a reform aimed at reducing pension benefits by lowering the pension ceiling (ε_t). Thus, we analyze the effects of the reform on fertility, fiscal policies, and economic growth. The reform is modeled under two different scenarios: Scenario A involves a reduction of ε_t from 1 to 0.75 after 2030, while Scenario B involves a gradual reduction of ε_t by 0.05 every 10 years after 2030, resulting in a 5% reduction in pension benefits every decade. In both scenarios, the pension ceiling in 2070 is fixed at 0.75. It is important to note that our proposed reform is very drastic, whereas actual pension reforms tend to be more gradual. Despite the extreme nature of our reform scenario, we analyze and compare large pension benefit cut scenarios for the purpose of exposition.

The impact of pension benefit cuts on fertility, fiscal policies, and economic growth depends on individuals' expectations regarding future pension benefit cuts. In this study, we assume two types of expectation formation and analyze each case. The first is rational expectation formation (see Table 2). We assume that people rationally form expectations and make optimal decisions regarding pension benefit reductions to be implemented 30 years from now, that is, in 2030, 2040, and 2050, in the years 2000, 2010, and 2020, respectively. Thus, people make utility-maximizing

	ε_{2030}	ε_{2040}	ε_{2050}	ε_{2060}	ε_{2070}
scenario A	0.75	0.75	0.75	0.75	0.75
scenario B	0.95	0.9	0.85	0.8	0.75
	$\mathbb{E}_{2000}\varepsilon_{2030}$	$\mathbb{E}_{2010}\varepsilon_{2040}$	$\mathbb{E}_{2020}\varepsilon_{2050}$	$\mathbb{E}_{2030}\varepsilon_{2060}$	$\mathbb{E}_{2040}\varepsilon_{2070}$
scenario A	0.75	0.75	0.75	0.75	0.75
scenario B	0.95	0.9	0.85	0.8	0.75

Table 2: Rows 1 and 2 display the ε_t sequences in Scenarios A and B, respectively. Rows 3 and 4 exhibit the expected ε_t sequences in Scenarios A and B, respectively, under rational expectations. $\mathbb{E}_{t-1}\varepsilon_t$ denotes the individuals' expectation of ε_t in period $t-1$. This notation also applies in Table 3.
Note: The following holds from 1990 to 2020: $\mathbb{E}_{1960}\varepsilon_{1990} = \mathbb{E}_{1970}\varepsilon_{2000t=}\mathbb{E}_{1980}\varepsilon_{2010=}\mathbb{E}_{1990}\varepsilon_{2010} = 1$. This also applies to Table 3.

decisions by considering the reduction in annual benefits that will be implemented after 30 years. The second is adaptive expectation formation (see Table 3). People expect that the pension benefit cuts implemented at each point in time will be implemented in the subsequent period as well. In other words, the pension ceiling that individuals expect 30 years from now at 2000, 2010, and 2020 is $\varepsilon_t = 1$. In 2030 and 2040, when the pension benefit reductions are implemented, individuals' 30-year forecasts are 0.95 and 0.9, respectively.

	ε_{2030}	ε_{2040}	ε_{2050}	ε_{2060}	ε_{2070}
Scenario A	0.75	0.75	0.75	0.75	0.75
Scenario B	0.95	0.9	0.85	0.8	0.75
	$\mathbb{E}_{2000}\varepsilon_{2030}$	$\mathbb{E}_{2010}\varepsilon_{2040}$	$\mathbb{E}_{2020}\varepsilon_{2050}$	$\mathbb{E}_{2030}\varepsilon_{2060}$	$\mathbb{E}_{2040}\varepsilon_{2070}$
Scenario A	1	1	1	0.75	0.75
Scenario B	1	1	1	0.95	0.9

Table 3: Rows 1 and 2 display the ε_t sequences in Scenarios A and B, respectively. Rows 3 and 4 exhibit the expected ε_t sequences in Scenarios A and B, respectively, under adaptive expectations.

Figures 8, 9, 10, and 11 present the simulation results for the synthetic rich OECD, synthetic rich OECD Europe, Japan, and the United States, respectively, under rational expectations. Each figure comprises the following three scenarios: the baseline case with no pension benefit cut, Scenario A with a reduction of ε_t to 0.75 introduced in 2030 and continued permanently, and Scenario B with a reduction of ε_t by 0.05 every 10 years after 2030, aiming to reach a target level of 0.75. The figures reveal several key observations, which are common across all four groups of countries.

First, a reduction in pension benefits leads to a decrease in the fertility rate (Panel (a) in each figure). Secondly, a decrease in the labor-income tax rate occurs with a reduction in pension benefits because of a decrease in the required tax revenue (Panel (b)). Third, the combination of the reduction in pension benefits and the decline in the fertility rate results in a decline in the pension benefit-GDP ratio (Panel (c)), but the upward trend in the ratio is maintained due to the projected increase in life expectancy. Fourth, the education expenditure-GDP ratio increases as pension benefits decrease. This is because the government can reallocate financial resources that were previously used for pension benefits towards public education spending (Panel (d)). Fifth, a decrease in pension benefits enhances economic growth, as the disincentive to save is reduced by the pension benefit cut (Panel (e)). Moreover, Scenario A, involving an immediate 25% reduction in benefits, exerts a more substantial impact on the three fiscal variables and fertility compared to Scenario B, which gradually reduces benefits in 5% increments.

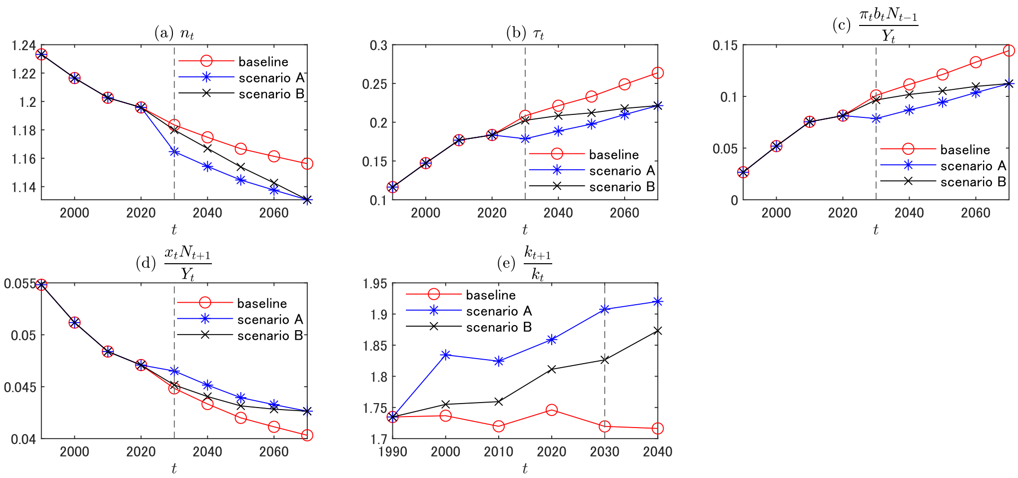

Figure 8: Simulation results of the three scenarios, baseline with no pension benefit cut, scenario A, and scenario B for the synthetic rich OECD, under rational expectations. Panel (a) shows projected changes in fertility. Panels (b), (c), (d), and (e) show the labor-income tax rate, pension benefit-GDP ratio, education expenditure-GDP ratio, and the growth rate of per capita capital, respectively.

Figures 12, 13, 14, and 15 present the simulation results for the synthetic rich OECD, synthetic rich OECD Europe, Japan, and the United States, respectively, under adaptive expectations. Each figure consists of three scenarios, similar to the case of rational expectations. Changing the hypothesis from rational to adaptive expectations results in the following effects. With adaptive expectation formation, any alteration in the pension rule has no immediate effect until the change is implemented. This is because expectation formation is denoted by $\mathbb{E}_t\varepsilon_{t+1} = \varepsilon_t$. For example, when the pension cuts begin in 2030, individuals in 2000 expect no pension cuts in the next period, i.e., 2030. Consequently, under adaptive expectation formation, the impact of pension reform is visible from 2030 onwards. This is in contrast to the results under rational expectations formation, where the effect is evident one period before the pension cuts begin. This is because individuals anticipate future pension cuts in 2030, which influences their behavior in 2000.

Considering this property, the effects of changing the hypothesis from rational to adaptive expectations on fertility, fiscal policies, and economic growth are as follows: First, under adaptive expectations, a future pension rule ε_{t+1} is expected to be higher than that under rational expectations, thereby resulting in a lower estimated pension reductions. Therefore, adaptive

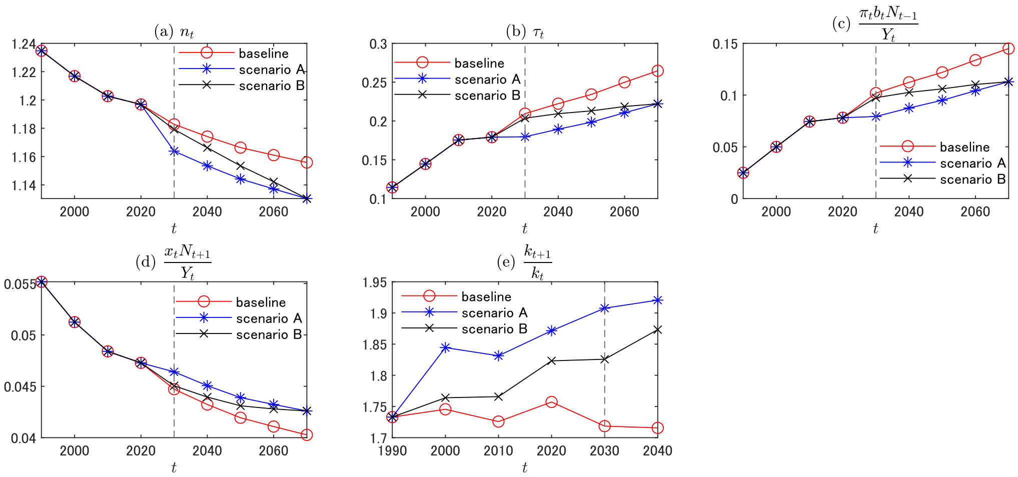

Figure 9: Simulation results of the three scenarios, baseline with no pension benefit cut, scenario A, and scenario B for the synthetic rich OECD Europe, under rational expectations. Panel (a) shows projected changes in fertility. Panels (b), (c), (d), and (e) show the labor-income tax rate, pension benefit-GDP ratio, education expenditure-GDP ratio, and the growth rate of per capita capital, respectively.

expectation formation leads to higher fertility rates than rational expectation formation.

Second, we focus on the labor-income tax rate identified in (48). It is observed that the tax rate for the current period remains unaffected by the subsequent period's pension rule. Thus, the formation of expectations regarding future pension rules does not influence the determination of the current period's tax rate. Similarly, the pension benefit-GDP ratio is independent of the next period's pension rule. This independence property can be observed from the ratio formula, which is

$$\frac{\pi_t b_t N_{t-1}}{Y_t} = \frac{\pi_t b_t N_{t-1}}{QK_t} = \frac{\pi_t b_t N_{t-1}}{Q(K_t/N_t)N_t} = \frac{\pi_t b_t}{Qk_t n_t} = \frac{\pi_t \varepsilon_t P\left(n_t\right) \frac{R}{\pi_t} n_t k_t}{Qk_t n_t} = \alpha \varepsilon_t P\left(n_t\right).$$

Third, the education expenditure-GDP ratio is given by

$$\frac{x_t N_{t+1}}{Y_t} = \frac{x_t N_{t+1}}{QK_t} = \frac{x_t N_{t+1}/N_t}{Qk_t} = \frac{x \tilde{n}'\left(\mathbb{E}_t \varepsilon_{t+1}\right)}{Qk_t} = \tilde{X}\left(\varepsilon_t, n_t, \tilde{n}'\left(\mathbb{E}_t \varepsilon_{t+1}\right)\right) \cdot \tilde{n}'\left(\mathbb{E}_t \varepsilon_{t+1}\right).$$

Expectation formation with respect to future pension rules affects the education expenditure-GDP ratio in the current period through two routes via the fertility rate. One is a direct effect on the fertility rate and the other is an indirect effect affecting $\tilde{X}\left(\varepsilon_t, n_t, \tilde{n}'\left(\mathbb{E}_t \varepsilon_{t+1}\right)\right)$ through the fertility rate. Given the definition of $\tilde{X}\left(\varepsilon_t, n_t, \tilde{n}'\left(\mathbb{E}_t \varepsilon_{t+1}\right)\right)$ in (50), the two effects cancel each other out. Thus, the formation of expectations regarding future pension rules does not affect the education expenditure-GDP ratio.

It is important to note that the fertility rate n_t in the current period is exclusively determined by the pension rule established for that same period, as indicated in Equation (44). Consequently, the pension benefit-GDP ratio for the current period is solely contingent upon the implemented pension rule, regardless of the anticipated rule for the subsequent period. This

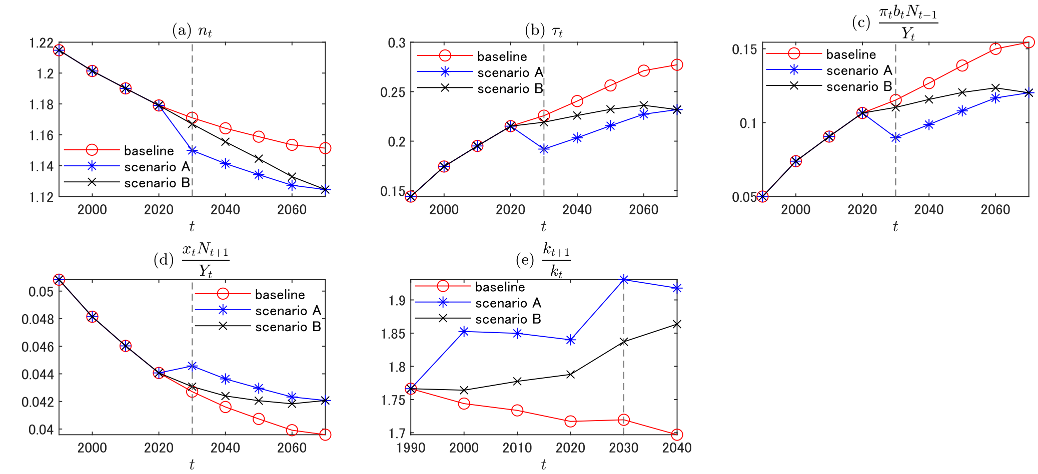

Figure 10: Simulation results of the three scenarios, baseline with no pension benefit cut, scenario A, and scenario B for Japan, under rational expectations. Panel (a) shows projected changes in fertility. Panels (b), (c), (d), and (e) show the labor-income tax rate, pension benefit-GDP ratio, education expenditure-GDP ratio, and the growth rate of per capita capital, respectively.

observation confirms that the choice between rational or adaptive expectation formation does not affect the formulation of fiscal policies. However, employing adaptive expectation formation results in a higher value of n_t compared to rational expectation formation. This consequently diminishes the political influence exerted by older adults and restricts the impact of population aging on fiscal policies.

Finally, the effect of expectation formation on the growth rate should be considered. According to Equation (57), the growth rate of capital per capita, with the expectation formation of the ceiling, is given by the following equation:

$$\frac{k_{t+1}}{k_t} = \frac{1}{\tilde{n}'\left(\mathbb{E}_t\varepsilon_{t+1}\right)} \cdot \frac{\beta\pi_{t+1}}{1+\beta\pi_{t+1}} \left[(1-\alpha)Q - \frac{1}{\beta\pi_{t+1}} \cdot \frac{1}{R/\pi_{t+1}} \tilde{G}\left(\mathbb{E}_t\varepsilon_{t+1}\right)\right] \left(1-\tilde{\tau}\left(n_t,\varepsilon_t\right)\right).$$

This equation indicates that the formation of expectations affects the growth rate from current period to the next through the population growth rate, $\tilde{n}'\left(\mathbb{E}_t\varepsilon_{t+1}\right)$, and pension benefits, $\tilde{G}\left(\mathbb{E}_t\varepsilon_{t+1}\right)$. Under adaptive expectation formation, any change in the pension rule of the next period will not affect the growth rate because the expectation formation is $\mathbb{E}_t\varepsilon_{t+1} = \varepsilon_t$. Therefore, when pension cuts begin in 2030, people will expect the same pension cuts in the next period, 2060, as in 2030, and this expectation will influence their behavior in 2030. Consequently, under adaptive expectation formation, the impact on economic growth will only be noticeable from 2030 onwards. This contrasts with the result of rational expectations formation, where the effect on the growth rate becomes apparent one period before the pension cuts begin, as people anticipate future pension cuts, which affects their behavior.

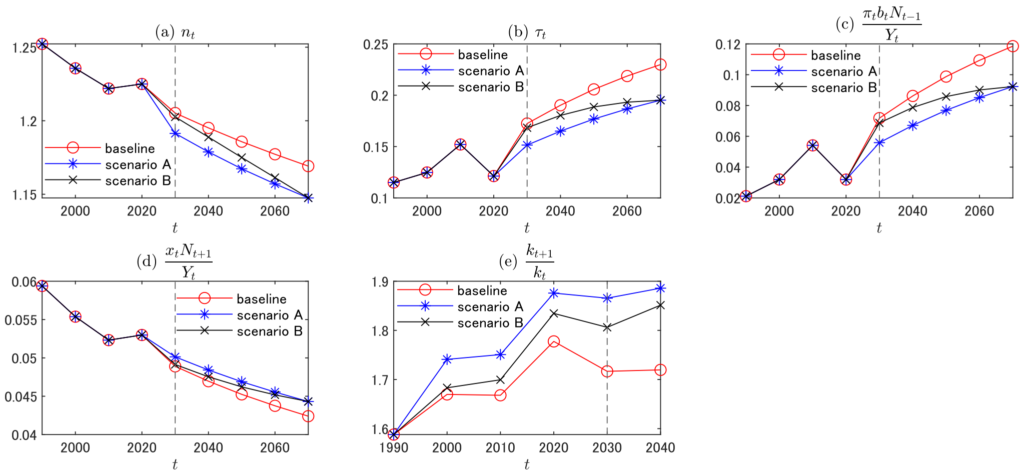

Figure 11: Simulation results of the three scenarios, baseline with no pension benefit cut, scenario A, and scenario B for the United States, under rational expectations. Panel (a) shows projected changes in fertility. Panels (b), (c), (d), and (e) show the labor income tax rate, pension benefit-GDP ratio, education expenditure-GDP ratio, and the growth rate of per capita capital, respectively.

7 Conclusion

This study investigated the following two key questions: (1) How does the government allocate its limited budget between pension provisions for older adults and education for the younger generation in response to population aging in each period? (2) From an economic perspective, what justifies the reduction in pension benefits by developed countries? To answer these questions, we employed an overlapping-generation model that incorporates physical and human capital accumulation. We further enhanced the model by introducing parental decisions regarding fertility. By endogenizing fertility choices, our approach offers insights into the interplay between political decisions on education and pension expenditures and parental fertility decisions.

Past studies on the political economy of education and pensions have typically treated the population growth rate as an exogenous variable, focusing on the influence of changes in this external factor (specifically, a decline in the population growth rate) on policy formulation. In contrast, our study endogenously models households' fertility decisions and demonstrates their substantial impact on government policy choices and resource allocations. Additionally, our study provides predictions regarding fiscal policies, fertility patterns, and economic growth over time, considering anticipated changes in life expectancy. However, our study assumes an inelastic labor supply, thereby necessitating the need to address the endogenization of labor supply. By extending our analysis, we can examine the influence of household fertility and labor supply decisions on policymaking. We anticipate that the approach outlined in this study will serve as a foundation for addressing this matter in future research.

Besides the endogenization of labor supply, there are several remaining issues to be addressed

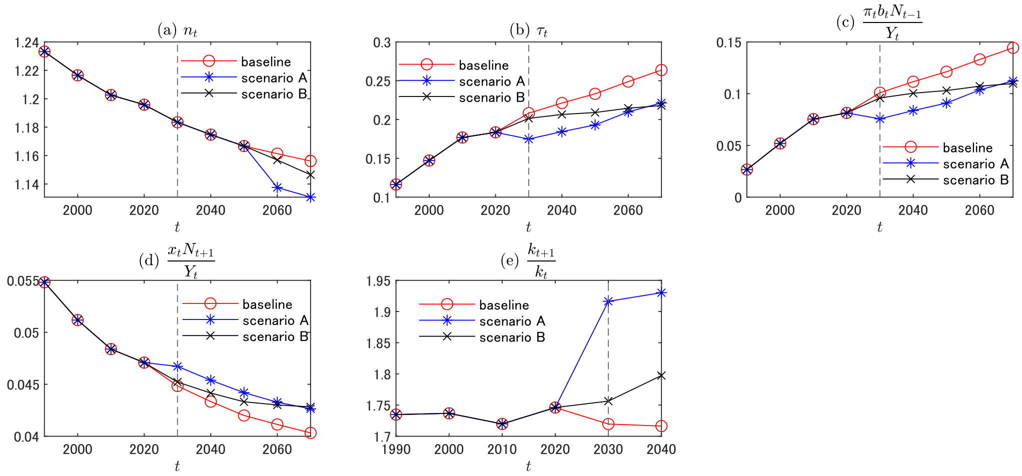

Figure 12: Simulation results of the three scenarios, baseline with no pension benefit cut, scenario A, and scenario B for the synthetic rich OECD, under adaptive expectations. Panel (a) shows projected changes in fertility. Panels (b), (c), (d), and (e) show the labor-income tax rate, pension benefit-GDP ratio, education expenditure-GDP ratio, and the growth rate of per capita capital, respectively.

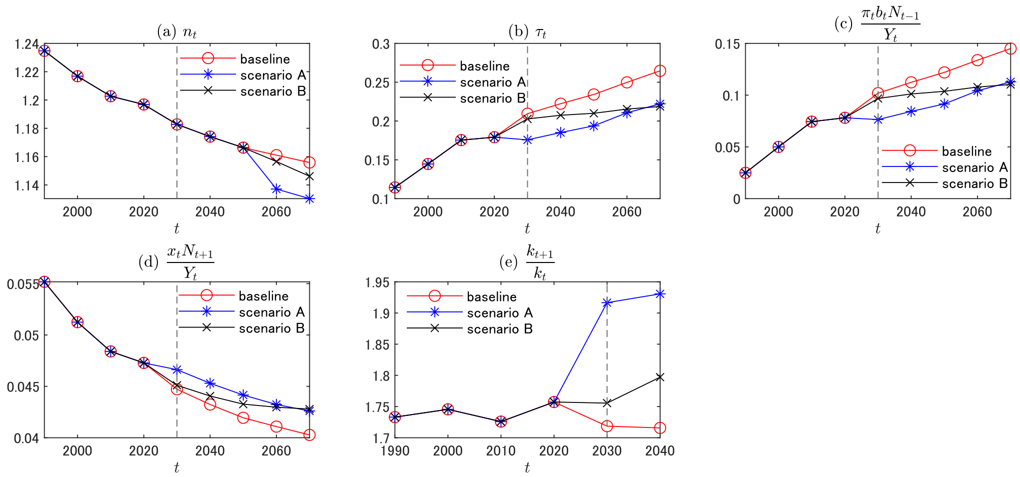

Figure 13: Simulation results of the three scenarios, baseline with no pension benefit cut, scenario A, and scenario B for the synthetic rich OECD Europe, under adaptive expectations. Panel (a) shows projected changes in fertility. Panels (b), (c), (d), and (e) show the labor-income tax rate, pension benefit-GDP ratio, education expenditure-GDP ratio, and the growth rate of per capita capital, respectively.

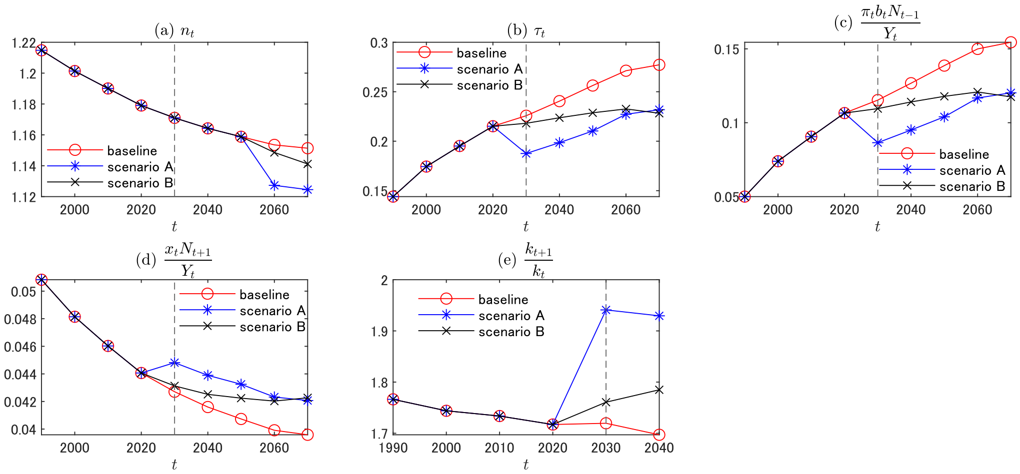

Figure 14: Simulation results of the three scenarios, baseline with no pension benefit cut, scenario A, and scenario B for Japan, under adaptive expectations. Panel (a) shows projected changes in fertility. Panels (b), (c), (d), and (e) show the labor-income tax rate, pension benefit-GDP ratio, education expenditure-GDP ratio, and the growth rate of per capita capital, respectively.

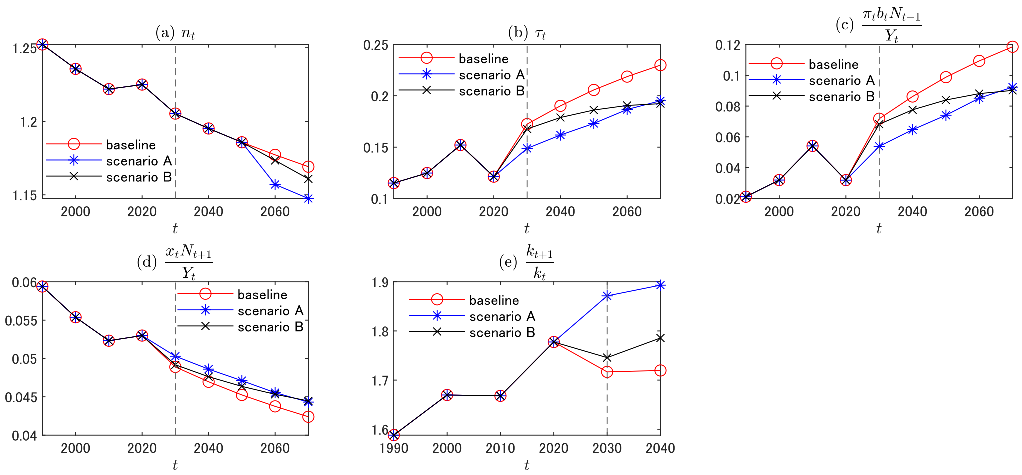

Figure 15: Simulation results of the three scenarios, baseline with no pension benefit cut, scenario A, and scenario B for the United States, under adaptive expectations. Panel (a) shows projected changes in fertility. Panels (b), (c), (d), and (e) show the labor income tax rate, pension benefit-GDP ratio, education expenditure-GDP ratio, and the growth rate of per capita capital, respectively.

in this study. The first is the generalization of the functional form. Every result of this study hinges on the assumptions of the logarithmic utility function and the Cobb-Douglas production function. In particular, the comparative statics result would change if we instead assumed the utility function with a constant inter-temporal elasticity of substitution. The second is the financial costs of child rearing. The present framework focuses on the opportunity costs of child rearing and assumes away the financial costs because they are not intrinsically different under the assumption of logarithmic utility. The introduction of financial costs together with the generalization of the utility function, may provide several new insights. The third is a childcare subsidy which is implemented in many developed countries. This creates a conflict of interest between older and middle-aged adults and may provide an additional impact when combined with the first two extensions. These extensions may be of substantial value, but are left for future work.

A Appendices

A.1 Proofs of Propositions 1 and 2

Recall the conjecture of b' in (22), which is restated as

$$b' = \frac{\frac{\pi\omega}{n'(1-\omega)}B + C}{\frac{\pi\omega}{n'(1-\omega)}E + F}\frac{R}{\pi}s. \tag{A.1}$$

In addition, with the use of (5) and (16), the saving function in (6) is rewritten as

$$s = \frac{\beta\pi}{1+\beta\pi}\left[(1-\tau)(1-\alpha)Qk - \frac{1}{\beta\pi}\cdot\frac{b'}{R/\pi}\right]. \tag{A.2}$$

We substitute the fertility function into (17) and save function in (A.2) to the conjecture that b' in (A.1) to obtain

$$b' = \frac{\phi^{\frac{1+\delta+\beta\pi}{\delta}}\frac{(1-\tau)(1-\alpha)Qk+\frac{\delta}{1+\delta+\beta\pi}\cdot\frac{b'}{R/\pi}}{(1-\tau)(1-\alpha)Qk+\frac{b'}{R/\pi}}\frac{\pi\omega}{1-\omega}B + C}{\phi^{\frac{1+\delta+\beta\pi}{\delta}}\frac{(1-\tau)(1-\alpha)Qk+\frac{\delta}{1+\delta+\beta\pi}\cdot\frac{b'}{R/\pi}}{(1-\tau)(1-\alpha)Qk+\frac{b'}{R/\pi}}\frac{\pi\omega}{1-\omega}E + F}\cdot\frac{R}{\pi}\cdot\frac{\beta\pi}{1+\beta\pi}\cdot\left[(1-\tau)(1-\alpha)Qk - \frac{1}{\beta\pi}\cdot\frac{b'}{R/\pi}\right], \tag{A.3}$$

or,

$$G_1\cdot\left(b'\right)^2 + G_2(1-\tau)k\cdot b' + G_3\cdot((1-\tau)k)^2 = 0, \tag{A.4}$$

where G_1, G_2, and G_3 are defined as:

$$\begin{aligned}
G_1 &\equiv \frac{\delta}{1+\delta+\beta\pi}\cdot\frac{1}{R/\pi}\cdot\left[\left(\frac{\pi\omega}{1-\omega}E + \frac{F}{\phi}\right) + \left(\frac{\pi\omega}{1-\omega}B + \frac{C}{\phi}\right)\frac{1}{1+\beta\pi}\right],\\
G_2 &\equiv (1-\alpha)\,Q\left\{\left(\frac{\pi\omega}{1-\omega}E + \frac{\delta}{1+\delta+\beta\pi}\cdot\frac{F}{\phi}\right)\right.\\
&\quad\left.+\frac{\beta\pi}{1+\beta\pi}\cdot\left[-\frac{\delta}{1+\delta+\beta\pi}\cdot\left(\frac{\pi\omega}{1-\omega}B + \frac{C}{\phi}\right) + \frac{1}{\beta\pi}\cdot\left(\frac{\pi\omega}{1-\omega}B + \frac{\delta}{1+\delta+\beta\pi}\cdot\frac{C}{\phi}\right)\right]\right\},\\
G_3 &\equiv (-1)\left(\frac{\pi\omega}{1-\omega}B + \frac{\delta}{1+\delta+\beta\pi}\cdot\frac{C}{\phi}\right)\cdot\frac{R}{\pi}\cdot\frac{\beta\pi}{1+\beta\pi}\cdot\left[(1-\alpha)\,Q\right]^2.
\end{aligned}$$

Assuming $G_1 \neq 0$, we solve (A.4) for b' and obtain

$$b' = G(1-\tau)k = \frac{-G_2 + \sqrt{(G_2)^2 - 4G_1G_3}}{2G_1}(1-\tau)k, \tag{A.5}$$

where G is

$$G \equiv \frac{-G_2 + \sqrt{(G_2)^2 - 4G_1G_3}}{2G_1}.$$

We substitute V_t^M into (19) and V_t^O in (20) into Ω_t in (21), and obtain

$$\begin{aligned}
\Omega \simeq{}& \frac{\pi\omega}{n(1-\omega)}\ln\left(\frac{R}{\pi}nk + b\right) + (1+\delta+\beta\pi)\ln\left[(1-\tau)(1-\alpha)Qk + \frac{b'}{R/\pi}\right]\\
&-\delta\ln\left[(1-\tau)(1-\alpha)Qk + \frac{\delta}{1+\delta+\beta\pi}\cdot\frac{b'}{R/\pi}\right] + \delta\eta\ln x.
\end{aligned} \tag{A.6}$$

We use the notation $\simeq$ in (21) because irrelevant terms are omitted from the expressions for Ω_t. We further substitute (A.5) and the government budget constraint in (13) into the political objective function of (A.6), and obtain

$$\Omega \simeq \frac{\pi\omega}{n(1-\omega)} \ln\left(\frac{R}{\pi}nk + b\right) + (1+\beta\pi)\ln(1-\tau) + \delta\eta \ln\left[\tau(1-\alpha)Qk - \frac{\pi b}{n}\right]. \tag{A.7}$$

The first-order conditions with respect to b and τ are:

$$b: \frac{\pi\omega}{n(1-\omega)} \cdot \frac{1}{\frac{R}{\pi}nk + b} - \delta\eta \frac{\frac{\pi}{n}}{\tau(1-\alpha)Qk - \frac{\pi b}{n}} \leq 0, \tag{A.8}$$

$$\tau: \frac{(-1)(1+\beta\pi)}{1-\tau} + \frac{\delta\eta(1-\alpha)Qk}{\tau(1-\alpha)Qk - \frac{\pi b}{n}} = 0. \tag{A.9}$$

Equation (A.9) is rewritten as

$$\tau = \frac{\delta\eta(1-\alpha)Qk + (1+\beta\pi)\frac{\pi b}{n}}{(1+\delta\eta+\beta\pi)(1-\alpha)Qk}. \tag{A.10}$$

Substitution of (A.10) into (A.8) leads to the policy function of pension benefits:

$$b = \frac{\frac{\pi\omega}{n(1-\omega)}\frac{(1-\alpha)}{\alpha} - (1+\delta\eta+\beta\pi)}{\frac{\pi\omega}{n(1-\omega)} + (1+\delta\eta+\beta\pi)} \frac{R}{\pi}nk, \tag{A.11}$$

verifying the conjecture of (22). Equation (A.11) indicates that a public pension is provided, that is, $b > 0$ holds if the fertility rate falls below the following critical value:

$$b > 0 \Leftrightarrow 1 + \delta\eta + \beta\pi < \frac{\pi\omega}{n(1-\omega)} \cdot \frac{1-\alpha}{\alpha}. \tag{A.12}$$

We express this condition in (29).

We derive the policy function of τ by substituting (A.11) into (A.9):

$$\tau = \frac{\frac{\pi\omega}{n(1-\omega)} + \left[\delta\eta - (1+\beta\pi)\frac{\alpha}{1-\alpha}\right]}{\left[\frac{\pi\omega}{n(1-\omega)} + (1+\delta\eta+\beta\pi)\right]}. \tag{A.13}$$

$\tau < 1$ is obtained immediately from the expression in (A.13). $\tau > 0$ holds if the sign of the numerator in (A.13) is positive; that is, if

$$\frac{\pi\omega}{n(1-\omega)} + \left[\delta\eta - (1+\beta\pi)\frac{\alpha}{1-\alpha}\right] > 0,$$

or,

$$1 + \beta\pi - \delta\eta\frac{1-\alpha}{\alpha} < \frac{\pi\omega}{n(1-\omega)} \cdot \frac{1-\alpha}{\alpha}. \tag{A.14}$$

(A.14) holds if (A.12) (that is, (29)) holds true. Thus, we obtain $\tau \in [0,1]$ if (29) holds.

Recall that the fertility rate for a given set of policy variables is (17). We must replace τ and b' in (17) with n and k using the policy functions in (A.11) and (A.13). Taking one period lag of (A.11):

$$b' = \frac{\frac{\pi\omega}{n\prime 1-\omega)}\frac{1-\alpha}{\alpha} - (1+\delta\eta+\beta\pi)}{\frac{\pi\omega}{n'(1-\omega)} + (1+\delta\eta+\beta\pi)}\frac{R}{\pi}n'k' = \frac{z_1(n')}{z_0(n')}\frac{R}{\pi}n'k', \tag{A.15}$$

where $z_0(\cdot)$ and $z_1(\cdot)$ are defined as:

$$z_0(n') \equiv \frac{\pi\omega}{n'(1-\omega)} + (1+\delta\eta+\beta\pi)\,, \tag{A.16}$$

$$z_1(n') \equiv \frac{\pi\omega}{n'(1-\omega)}\frac{1-\alpha}{\alpha} - (1+\delta\eta+\beta\pi)\,. \tag{A.17}$$

The policy function b' in (A.15) is reformulated as follows:

$$\begin{aligned} b' &= \frac{z_1(n')}{z_0(n')}\frac{R}{\pi}s \\ &= \frac{z_1(n')}{z_0(n')}\frac{R}{\pi}\frac{\beta\pi}{1+\beta\pi}\left[(1-\tau)(1-\alpha)Qk - \frac{1}{\beta\pi}\cdot\frac{b'}{R/\pi}\right], \end{aligned}$$

where the equality in the first line comes from (14), and the equality in the second line originates from (A.2). By rearranging the terms, we have

$$\begin{aligned} \left[1 + \frac{z_1(n')}{z_0(n')}\frac{R}{\pi}\frac{\beta\pi}{1+\beta\pi}\frac{1}{\beta\pi}\frac{1}{R/\pi}\right]b' &= \frac{z_1(n')}{z_0(n')}\frac{R}{\pi}\frac{\beta\pi}{1+\beta\pi}(1-\tau)(1-\alpha)Qk \\ &= \frac{z_1(n')}{z_0(n')}\frac{R}{\pi}\frac{\beta\pi}{1+\beta\pi}\frac{1+\beta\pi}{(1-\alpha)z_0(n)}(1-\alpha)Qk, \end{aligned}$$

where the equality in the second line is derived from (A.13). Thus, we obtain

$$b' = \frac{\frac{z_1(n')}{z_0(n')}\frac{R}{\pi}\frac{\beta\pi}{(1-\alpha)z_0(n)}}{1+\frac{z_1(n')}{z_0(n')}\frac{1}{1+\beta\pi}}(1-\alpha)Qk. \tag{A.18}$$

We substitute (A.13) and (A.18) into the fertility function in (17) and obtain the following equation that characterizes the equilibrium fertility rate,

$$n' = \frac{1}{\phi}\cdot\frac{\delta}{1+\delta+\beta\pi}\cdot\frac{1+\frac{z_1(n')}{z_0(n')}}{1+\frac{z_1(n')}{z_0(n')}\cdot\frac{1+\delta}{1+\delta+\beta\pi}}. \tag{A.19}$$

We reformulate the expression in (A.19) by substituting $z_0(\cdot)$ in (A.16) and $z_1(\cdot)$ in (A.17) and rearranging the terms to obtain

$$n' = \frac{\delta}{\phi}\cdot\frac{1}{(1+\delta+\alpha\beta\pi)+\alpha\beta\pi\,(1+\delta\eta+\beta\pi)\,\frac{n'(1-\omega)}{\pi\omega}}. \tag{A.20}$$

Equation (A.20) shows that there exists a unique $n' > 0$ satisfying (A.20). Note that (A.20) is rewritten as a quadratic equation for n' :

$$\alpha\beta\,(1+\delta\eta+\beta\pi)\,\frac{1-\omega}{\omega}\,(n')^2 + (1+\delta+\alpha\beta\pi)\,n' - \frac{\delta}{\phi} = 0. \tag{A.21}$$

Thus, we can solve this equation for n as

$$n' = n'(\pi) \equiv \frac{-(1+\delta+\alpha\beta\pi) + \sqrt{(1+\delta+\alpha\beta\pi)^2 + 4\alpha\beta(1+\delta\eta+\beta\pi)\frac{1-\omega}{\omega}\frac{\delta}{\phi}}}{2\alpha\beta(1+\delta\eta+\beta\pi)\frac{1-\omega}{\omega}}. \tag{A.22}$$

The left-hand side of (A.21) increases in π and n', indicating that a higher π value is associated with a lower n' in (A.21). Thus, we have $\partial n'/\partial \pi < 0$.

To obtain the policy function of x, we substitute the policy functions of b in (A.11) and τ in (A.13) into the budget constraints in (13). Then, we have

$$x = \frac{1}{n'(\pi)} \cdot \frac{\delta\eta}{\frac{\pi\omega}{n(1-\omega)} + (1+\delta\eta+\beta\pi)} Qk > 0, \tag{A.23}$$

where $n' = n'(\pi)$ is the fertility rate in equation (A.22).

■

A.2 Proof of Proposition 3

The aggregate expenditure on public pensions is $\pi b N_-$, and the aggregate GDP is QK. Thus, the pension benefit-GDP ratio is

$$\begin{aligned}\frac{\pi b N_-}{QK} &= \frac{\pi b}{Qkn} \\ &= \frac{\frac{\pi\omega}{n(1-\omega)}(1-\alpha) - (1+\delta\eta+\beta\pi)\alpha}{\frac{\pi\omega}{n(1-\omega)} + (1+\delta\eta+\beta\pi)},\end{aligned}$$

where the second equality comes from the policy function b in (31). After manipulation, we obtain

$$\frac{\pi b N_-}{QK} = \begin{cases} (1-\alpha) - \left[\frac{1}{\frac{1+\delta\eta}{\pi\omega}+\frac{\beta}{\omega}} \cdot \frac{1}{n(1-\omega)} + 1\right]^{-1} & \text{for } t = 0, \\ (1-\alpha) - \left[\frac{1}{\frac{1+\delta\eta}{\pi\omega}+\frac{\beta}{\omega}} \cdot \frac{1}{n'(\pi)(1-\omega)} + 1\right]^{-1} & \text{for } t \geq 1. \end{cases}$$

As $\partial n'/\partial \pi < 0$, we have $\partial\left[\pi b N_-/QK\right]/\partial \pi > 0$.

The education expenditure-GDP ratio is

$$\begin{aligned}\frac{xN'}{QK} &= \frac{xn'}{Qk} \\ &= \frac{\delta\eta}{\frac{\pi\omega}{n(1-\omega)} + (1+\delta\eta+\beta\pi)},\end{aligned}$$

where the second equality comes from the policy function x in (32). Thus, we have

$$\frac{xN'}{QK} = \begin{cases} \frac{\delta\eta}{\frac{\pi\omega}{n(1-\omega)}+(1+\delta\eta+\beta\pi)} & \text{for } t = 0, \\ \frac{\delta\eta}{\frac{\pi\omega}{n'(\pi)(1-\omega)}+(1+\delta\eta+\beta\pi)} & \text{for } t \geq 1. \end{cases}$$

As $\partial n'/\partial \pi < 0$, we have $\partial\left(xN'/QK\right)/\partial \pi < 0$.

■

A.3 Proof of Proposition 4

The aggregate output is QK; therefore, the per-capita output is $QK/N = Qk$. Thus, the gross growth rate of per capita output is $Qk'/Qk = k'/k$. To compute k'/k, we recall the capital market-clearing condition in (14), which we can rewrite as

$$\begin{aligned} n'k' &= \frac{\beta\pi}{1+\beta\pi}\left[(1-\tau)(1-\alpha)Qk - \frac{1}{\beta\pi}\cdot\frac{b'}{R/\pi}\right] \\ &= \frac{\beta\pi}{1+\beta\pi}\left[\frac{1+\beta\pi}{(1-\alpha)z_0(n)}(1-\alpha)Qk - \frac{1}{\beta\pi}\cdot\frac{1}{R/\pi}\cdot\frac{\frac{z_1(n')}{z_0(n')}\frac{Q}{\pi}\frac{\beta\pi}{1+\beta\pi}\frac{1+\beta\pi}{(1-\alpha)z_0(n)}(1-\alpha)Qk}{1+\frac{z_1(n')}{z_0(n')}\frac{Q}{\pi}\frac{\beta\pi}{1+\beta\pi}\frac{1}{\beta\pi}\frac{1}{R/\pi}}\right], \end{aligned}$$

where the first equality is derived using (A.2) and the second equality is derived by using (A.13) and (A.18). We rearrange the terms and substitute $n' = n'(\pi)$ in (A.22) into the above expression and obtain

$$\frac{k'}{k} = \frac{\alpha Q}{\left(1 - \frac{\beta\pi}{1+\beta\pi}(1-\alpha)\right)\frac{\omega}{(1-\omega)\beta} + \left(1+\frac{\delta\eta}{1+\beta\pi}\right)\alpha n'(\pi)}. \tag{A.24}$$

The first term in the denominator on the right-hand side decreases in π, and the second term in the denominator also decreases in π because $\partial n'/\partial\pi < 0$. Thus, we obtain $\partial(k'/k)/\partial\pi > 0$.

■

A.4 Derivation of (42)

Recall the conjecture for b' in (41). By using the fertility function in (17) and the saving function in (18), we reformulate (41) as follows:

$$\begin{aligned} b' = \varepsilon' \cdot &\frac{\frac{\omega}{1-\omega}B\left[(1-\tau)(1-\alpha)Qk + \frac{\delta}{1+\delta+\beta\pi}\cdot\frac{b'}{R/\pi}\right] + C\frac{1}{\phi}\cdot\frac{\delta}{1+\delta+\beta\pi}\cdot\left[(1-\tau)(1-\alpha)Qk + \frac{b'}{R/\pi}\right]}{\frac{\omega}{1-\omega}E\left[(1-\tau)(1-\alpha)Qk + \frac{\delta}{1+\delta+\beta\pi}\cdot\frac{b'}{R/\pi}\right] + F\frac{1}{\phi}\cdot\frac{\delta}{1+\delta+\beta\pi}\cdot\left[(1-\tau)(1-\alpha)Qk + \frac{b'}{R/\pi}\right]} \\ \times &\frac{R}{\pi}\frac{\beta\pi}{1+\beta\pi}\left[(1-\tau)(1-\alpha)Qk - \frac{1}{\beta\pi}\cdot\frac{b'}{R/\pi}\right]. \end{aligned}$$

This is further reformulated as

$$\tilde{G}_1(\varepsilon')\cdot(b')^2 + \tilde{G}_2(\varepsilon')\cdot(1-\tau)kb' + \tilde{G}_3(\varepsilon')\cdot[(1-\tau)k]^2 = 0, \tag{A.25}$$

where $\tilde{G}_1(\varepsilon')$, $\tilde{G}_2(\varepsilon')$, and $\tilde{G}_3(\varepsilon')$ are defined as:

$$\begin{aligned} \tilde{G}_1(\varepsilon') &\equiv \frac{\delta}{1+\delta+\beta\pi}\frac{1}{R/\pi}\left[\left(\frac{\pi\omega}{1-\omega}E + \frac{F}{\phi}\right) + \varepsilon'\left(\frac{\pi\omega}{1-\omega}B + \frac{C}{\phi}\right)\frac{1}{1+\beta\pi}\right], \\ \tilde{G}_2(\varepsilon') &\equiv (1-\alpha)Q\left\{\left(\frac{\pi\omega}{1-\omega}E + \frac{\delta}{1+\delta+\beta\pi}\frac{F}{\phi}\right)\right. \\ &\quad \left. +\varepsilon'\frac{\beta\pi}{1+\beta\pi}\left[-\frac{\delta}{1+\delta+\beta\pi}\left(\frac{\pi\omega}{1-\omega}B + \frac{C}{\phi}\right) + \frac{1}{\beta\pi}\left(\frac{\pi\omega}{1-\omega}B + \frac{\delta}{1+\delta+\beta\pi}\frac{C}{\phi}\right)\right]\right\}, \\ \tilde{G}_3(\varepsilon') &\equiv (-1)\varepsilon'\left(\frac{\pi\omega}{1-\omega}B + \frac{\delta}{1+\delta+\beta\pi}\frac{C}{\phi}\right)\frac{R}{\pi}\frac{\beta\pi}{1+\beta\pi}\left[(1-\alpha)Q\right]^2. \end{aligned}$$

Solving (A.25) for b' yields:

$$b' = \tilde{G}\left(\varepsilon'\right) \cdot (1-\tau)k,$$

where $\tilde{G}\left(\varepsilon'\right)$ is defined as

$$\tilde{G}\left(\varepsilon'\right) \equiv \frac{-\tilde{G}_2\left(\varepsilon'\right) + \sqrt{\left(\tilde{G}_2\left(\varepsilon'\right)\right)^2 - 4\tilde{G}_1\left(\varepsilon'\right)\tilde{G}_3\left(\varepsilon'\right)}}{2\tilde{G}_1\left(\varepsilon'\right)}.$$

■

A.5 Derivation of (48) and (49)

We substitute the fertility and consumption functions shown in subsection 5.2 into the political objective function in (43), and obtain

$$\begin{aligned}\Omega &= \frac{\pi\omega}{n(1-\omega)} \ln \tilde{d}\left(n,k,\varepsilon\right) + \ln \tilde{c}\left(\tau,k,\varepsilon'\right) + \delta \ln \tilde{n}'\left(\varepsilon'\right) h(x,h) + \beta\pi \ln \tilde{d}'\left(\tau,k,\varepsilon'\right) \\ &\simeq \frac{\pi\omega}{n(1-\omega)} \ln \left(1 + \varepsilon \cdot \frac{\frac{\pi\omega}{n(1-\omega)}B + C}{\frac{\pi\omega}{n(1-\omega)}E + F}\right) + (1+\beta\pi)\ln(1-\tau) + \delta\eta \ln x.\end{aligned}$$

Using the government budget constraint $\tau(1-\alpha)Qk = \pi b/n + n'x$, we can reformulate the expression of Ω as:

$$\begin{aligned}\Omega &\simeq \frac{\pi\omega}{n(1-\omega)} \ln \left(1 + \varepsilon \cdot \frac{\frac{\pi\omega}{n(1-\omega)}B + C}{\frac{\pi\omega}{n(1-\omega)}E + F}\right) + (1+\beta\pi)\ln(1-\tau) \\ &\quad + \delta\eta \ln \frac{1}{\tilde{n}'\left(\varepsilon'\right)} \left[\tau(1-\alpha) - \varepsilon \frac{\frac{\pi\omega}{n(1-\omega)}B + C}{\frac{\pi\omega}{n(1-\omega)}E + F}\alpha\right].\end{aligned}$$

The first-order condition with respect to τ is expressed as

$$\frac{1+\beta\pi}{1-\tau} = \frac{\delta\eta(1-\alpha)}{\tau(1-\alpha) - \varepsilon \frac{\frac{\pi\omega}{n(1-\omega)}B+C}{\frac{\pi\omega}{n(1-\omega)}D+E}\alpha},$$

which is rewritten as

$$\tau = \tilde{\tau}\left(n,\varepsilon\right) \equiv \frac{\delta\eta(1-\alpha)\left[\frac{\pi\omega}{n(1-\omega)} + (1+\delta\eta+\beta\pi)\right] + \varepsilon(1+\beta\pi)\left[\frac{\pi\omega}{n(1-\omega)}(1-\alpha) - (1+\delta\eta+\beta\pi)\alpha\right]}{(1+\delta\eta+\beta\pi)(1-\alpha)\left[\frac{\pi\omega}{n(1-\omega)} + (1+\delta\eta+\beta\pi)\right]}. \tag{A.27}$$

Substituting (A.27) and (44) into the government budget constraint in (13) yields :

$$\tilde{\tau}\left(n,\varepsilon\right)(1-\alpha)Qk = \frac{\pi}{n}\varepsilon \cdot \frac{\frac{\pi\omega}{n(1-\omega)}B + C}{\frac{\pi\omega}{n(1-\omega)}D + E} \cdot \frac{R}{\pi}nk + \tilde{n}'\left(\varepsilon'\right)x,$$

or,

$$x = \tilde{X}\left(\varepsilon, n, \tilde{n}'\left(\varepsilon'\right)\right) \cdot Qk, \tag{A.28}$$

where $\tilde{X}(\cdot)$ is defined as in (50).

To verify the conjecture in (41), we substitute derived the policy functions in (A.27) and (A.28) in the government budget constraint in (13), and obtain

$$\begin{aligned}
\tau(1-\alpha)Qk &= \frac{\pi b}{n} + n'x \\
&\Rightarrow \tilde{\tau}(n,\varepsilon)(1-\alpha)Qk = \frac{\pi b}{n} + \tilde{n}'(\varepsilon')\tilde{X}\left(\varepsilon, n, \tilde{n}'(\varepsilon')\right) Qk \\
&\Rightarrow \frac{\pi b}{n} = \tilde{\tau}(n,\varepsilon)(1-\alpha)Qk - \tilde{n}'(\varepsilon')\tilde{X}\left(\varepsilon, n, \tilde{n}'(\varepsilon')\right) Qk \\
&\Rightarrow b = \left[\tilde{\tau}(n,\varepsilon)\frac{1-\alpha}{\alpha} - \frac{1}{\alpha}\tilde{n}'(\varepsilon')\tilde{X}\left(\varepsilon, n, \tilde{n}'(\varepsilon')\right)\right]\frac{R}{\pi}nk; \text{ since } R = \alpha Q.
\end{aligned}$$

Thus, the conjecture in (41) is verified if the following expression holds:

$$\varepsilon P(n) = \left[\tilde{\tau}(n,\varepsilon)\frac{1-\alpha}{\alpha} - \frac{1}{\alpha}\tilde{n}'(\varepsilon')\tilde{X}\left(\varepsilon, n, \tilde{n}'(\varepsilon')\right)\right],$$

or,

$$\varepsilon P(n) = \tilde{\tau}(n,\varepsilon)\frac{1-\alpha}{\alpha} - \frac{1}{\alpha}\tilde{n}'(\varepsilon')\tilde{X}\left(\varepsilon, n, \tilde{n}'(\varepsilon')\right). \tag{A.29}$$

The left-hand side of (A.29) is

$$\begin{aligned}
\varepsilon P(n) &= \varepsilon\frac{\frac{\pi\omega}{n(1-\omega)}B + C}{\frac{\pi\omega}{n(1-\omega)}E + F} \\
&= \varepsilon\frac{\frac{\pi\omega}{n(1-\omega)}\frac{1-\alpha}{\alpha} - (1+\delta\eta+\beta\pi)}{\frac{\pi\omega}{n(1-\omega)} + (1+\delta\eta+\beta\pi)},
\end{aligned}$$

and the right side of (A.29) is

$$\begin{aligned}
&\tilde{\tau}(n,\varepsilon)\frac{1-\alpha}{\alpha} - \frac{1}{\alpha}\tilde{n}'(\varepsilon')\tilde{X}\left(\varepsilon, n, \tilde{n}'(\varepsilon')\right) \\
&= \frac{\delta\eta(1-\alpha)\left[\frac{\pi\omega}{n(1-\omega)} + (1+\delta\eta+\beta\pi)\right] + \varepsilon\left(1+\beta\pi\right)\left[\frac{\pi\omega}{n(1-\omega)}(1-\alpha) - (1+\delta\eta+\beta\pi)\right]}{(1+\delta\eta+\beta\pi)(1-\alpha)\left[\frac{\pi\omega}{n(1-\omega)} + (1+\delta\eta+\beta\pi)\right]}\frac{1-\alpha}{\alpha} \\
&- \frac{1}{\alpha}\tilde{n}'(\varepsilon')\frac{1}{\tilde{n}'(\varepsilon')}\frac{\delta\eta}{1+\delta\eta+\beta\pi}\frac{\frac{\pi\omega}{n(1-\omega)}(1-\alpha)\left(1-\varepsilon\right) + (1+\delta\eta+\beta\pi)(1-\alpha\left(1-\varepsilon\right))}{\frac{\pi\omega}{n(1-\omega)} + (1+\delta\eta+\beta\pi)} \\
&= \frac{1}{(1+\delta\eta+\beta\pi)(1-\alpha)\left[\frac{\pi\omega}{n(1-\omega)} + (1+\delta\eta+\beta\pi)\right]} \\
&\times\left\{\frac{\pi\omega}{n(1-\omega)}\left[\delta\eta\frac{1-\alpha}{\alpha} + \varepsilon\left(1+\beta\pi\right)\frac{1-\alpha}{\alpha} - \frac{\delta\eta}{\alpha}(1-\alpha)\left(1-\varepsilon\right)\right]\right. \\
&\left.+(1+\delta\eta+\beta\pi)\left[\delta\eta\frac{1-\alpha}{\alpha} - \varepsilon\left(1+\beta\pi\right) - \frac{\delta\eta}{\alpha}(1-\alpha\left(1-\varepsilon\right))\right]\right\} \\
&= \frac{\frac{\pi\omega}{n(1-\omega)}\varepsilon\frac{1-\alpha}{\alpha}(1+\delta\eta+\beta\pi) - (1+\delta\eta+\beta\pi)\varepsilon(1+\delta\eta+\beta\pi)}{(1+\delta\eta+\beta\pi)\left[\frac{\pi\omega}{n(1-\omega)} + (1+\delta\eta+\beta\pi)\right]} \\
&= \varepsilon\frac{\frac{\pi\omega}{n(1-\omega)}\frac{1-\alpha}{\alpha} - (1+\delta\eta+\beta\pi)}{\frac{\pi\omega}{n(1-\omega)} + (1+\delta\eta+\beta\pi)},
\end{aligned}$$

where the first equality is obtained by using $\tilde{\tau}(n,\varepsilon)$ in (A.27) and $\tilde{X}\left(\varepsilon, n, \tilde{n}'(\varepsilon')\right)$ in (A.28). Thus, we find that (A.29) holds, thus verifying the conjecture in (41).

■

A.6 Sources of Data

We source data on the average life expectancy and the average population growth rate in rich OECD countries from the United Nations Department of Economic and Social Affairs Population Division (2022) World Population Prospects 2022, Online Edition (`https://population.un.org/wpp/`) (accessed March 8, 2023). Data on the average education expenditure-GDP ratio are sourced from World Development Indicators (WDI) (`https://datatopics.worldbank.org/world-development-indicators/`) (accessed March 2, 2023). Public education expenditure includes expenditure funding through transfers from international sources to general government. General government refers to local, regional, and central governments.

Data on the pension benefit-GDP ratio are sourced from the OECD (2023), "Pension spending" (indicator), `https://doi.org/10.1787/a041f4ef-en` (accessed on March 2, 2023). Pension spending is defined as all cash expenditure (including lump-sum payments) on old age and survivors' pensions. Data on GDP per capita are from International Comparison Program, World Bank | World Development Indicator Database, World Bank | Eurostat-OECD (`https://datatopics.worldbank.org/world-development-indicators/`) (accessed March 9, 2023). GDP per capita is based on purchasing power parity (PPP).

■

A.7 Derivation of Social Welfare functions

The social welfare function is the sum of the lifecycle utilities of all current and future generations

$$SW = V_0^o + \sum_{t=0}^{\infty} \gamma^t V_t^M, \tag{A.30}$$

where $\gamma \in (0,1)$ is the planner discount factor. To write the social welfare function in terms of the sequence of pension ceilings, $\{\varepsilon_t\}_{t=0}^{\infty}$, we begin by expressing the policy function and fertility rate as a function of both the ceiling and the capital stock. Recall the pension benefits in the presence of a ceiling in (41), which is restated as follows:

$$b_t = \varepsilon_t \cdot \frac{\frac{\pi\omega}{n_t(1-\omega)}B + C}{\frac{\pi\omega}{n_t(1-\omega)}E + F} \cdot \frac{R}{\pi} n_t k_t, \tag{A.31}$$

$$b_{t+1} = \varepsilon_{t+1} \cdot \frac{\frac{\pi\omega}{n_{t+1}(1-\omega)}B + C}{\frac{\pi\omega}{n_{t+1}(1-\omega)}E + F} \cdot \frac{R}{\pi} n_{t+1} k_{t+1}. \tag{A.32}$$

For the tractability of the following analysis, the period is specified as subscripts. From (42), (A.32) is reformulated as

$$b_{t+1} = \tilde{G}\left(\varepsilon_{t+1}\right)(1-\tau_t)k_t. \tag{A.33}$$

The fertility function in (44) for a specified period is written as follows:

$$n_{t+1} = \begin{cases} n_0 & \text{for } t = -1, \\ \tilde{n}'\left(\varepsilon_{t+1}\right) & \text{for } t \geq 0, \end{cases} \tag{A.34}$$

where $n_0(>-1)$ is the exogenously given initial condition.

From (A.34), we can write the tax rate in (48) with each period specified as

$$\tau_t = \tilde{\tau}(n,\varepsilon) = \begin{cases} \tilde{\tau}(n_0,\varepsilon_0) & \text{for } t=0, \\ \tilde{\tau}(\tilde{n}'(\varepsilon_t),\varepsilon_t) & \text{for } t\geq 1. \end{cases} \tag{A.35}$$

Education expenditure x_t, given by (49), is

$$x_t = \tilde{X}(\cdot)Qk_t = \begin{cases} \frac{1}{\tilde{n}'(\varepsilon_1)}\cdot\hat{X}(\varepsilon_0,\tilde{n}'(\varepsilon_0))Qk_0 & \text{for } t=0, \\ \frac{1}{\tilde{n}'(\varepsilon_{t+1})}\cdot\hat{X}(\varepsilon_t,\tilde{n}'(\varepsilon_t))Qk_t & \text{for } t\geq 1, \end{cases} \tag{A.36}$$

where the second equality comes from (50).

We utilize (A.31), (A.34), (A.35) and (A.36) to reformulate the indirect utility function of middle-aged individuals in (19). Recall (19), which is restated as

$$V_t^M \simeq \ln\left[(1-\tau_t)w_th_t + \frac{b_{t+1}}{R/\pi}\right] + \delta\ln\frac{(1-\tau_t)(1-\alpha)Qk_t + \frac{b_{t+1}}{R/\pi}}{(1-\tau_t)(1-\alpha)Qk_t + \frac{\delta}{1+\delta+\beta\pi}\cdot\frac{b_{t+1}}{R/\pi}}D(h_t)^{1-\eta}(x_t)^\eta$$
$$+\beta\pi\ln\left[(1-\tau_t)w_th_t + \frac{b_{t+1}}{R/\pi}\right]. \tag{A.37}$$

Using (11), (16) and (17), we can reformulate the expression in (A.37) as:

$$V_t^M \simeq (1+\delta+\beta\pi)\ln\left[(1-\tau_t)(1-\alpha)Qk_t + \frac{b_{t+1}}{R/\pi}\right]$$
$$-\delta\ln\left[(1-\tau_t)(1-\alpha)Qk_t + \frac{\delta}{1+\delta+\beta\pi}\cdot\frac{b_{t+1}}{R/\pi}\right] + \delta\eta\ln x_t + \delta(1-\eta)\ln h_t. \tag{A.38}$$

We substitute (A.33) into (A.38) and obtain

$$V_t^M \simeq (1+\delta+\beta\pi)\ln\left[(1-\tau_t)(1-\alpha)Qk_t + \frac{\tilde{G}(\varepsilon_{t+1})(1-\tau_t)k_t}{R/\pi}\right]$$
$$-\delta\ln\left[(1-\tau_t)(1-\alpha)Qk_t + \frac{\delta}{1+\delta+\beta\pi}\cdot\frac{\tilde{G}(\varepsilon_{t+1})(1-\tau_t)k_t}{R/\pi}\right] + \delta\eta\ln x_t + \delta(1-\eta)\ln h_t$$
$$= (1+\delta+\beta\pi)\ln\left[(1-\alpha)Q + \frac{\tilde{G}(\varepsilon_{t+1})}{R/\pi}\right] - \delta\ln\left[(1-\alpha)Q + \frac{\delta}{1+\delta+\beta\pi}\cdot\frac{\tilde{G}(\varepsilon_{t+1})}{R/\pi}\right]$$
$$+(1+\beta\pi)\ln(1-\tau_t)k_t + \delta\eta\ln x_t + \delta(1-\eta)\ln h_t,$$

or,

$$V_t^M \simeq \tilde{V}(\varepsilon_{t+1}) + (1+\beta\pi)\ln(1-\tau_t)k_t + \delta\eta\ln x_t + \delta(1-\eta)\ln h_t, \tag{A.39}$$

where $\tilde{V}(\varepsilon_{t+1})$ is defined as

$$\tilde{V}(\varepsilon_{t+1}) \equiv (1+\delta+\beta\pi)\ln\left[(1-\alpha)Q + \frac{\tilde{G}(\varepsilon_{t+1})}{R/\pi}\right] - \delta\ln\left[(1-\alpha)Q + \frac{\delta}{1+\delta+\beta\pi}\cdot\frac{\tilde{G}(\varepsilon_{t+1})}{R/\pi}\right]. \tag{A.40}$$

We further substitute (A.34), (A.35) and (A.36) into (A.39) and rearrange the terms to obtain

$$V_t^M \simeq \begin{cases} \begin{array}{c} \tilde{V}(\varepsilon_1) + (1+\beta\pi)\ln(1-\tilde{\tau}(n_0,\varepsilon_0)) + \delta\eta\ln\frac{1}{\tilde{n}'(\varepsilon_1)}\hat{X}(\varepsilon_0,n_0) \\ +(1+\beta\pi+\delta\eta)\ln k_0 + \delta(1-\eta)\ln h_0 \end{array} & \text{for } t=0, \\ \begin{array}{c} \tilde{V}(\varepsilon_{t+1}) + (1+\beta\pi)\ln(1-\tau(\tilde{n}'(\varepsilon_t),\varepsilon_t)) + \delta\eta\ln\frac{1}{\tilde{n}'(\varepsilon_{t+1})}\cdot\hat{X}(\varepsilon_t,\tilde{n}'(\varepsilon_t)) \\ +(1+\beta\pi+\delta\eta)\ln k_t + \delta(1-\eta)\ln h_t \end{array} & \text{for } t\geq 1. \end{cases} \tag{A.41}$$

With (A.41), we can compute $\sum_{t=0}^{\infty} \gamma^t V_t^M$ as

$$\begin{aligned}\sum_{t=0}^{\infty} \gamma^t V_t^M \simeq & (1+\beta\pi)\ln(1-\tilde{\tau}(n_0,\varepsilon_0)) + \delta\eta \ln \hat{X}(\varepsilon_0, n_0) + (1+\beta\pi+\delta\eta)\ln k_0 + \delta(1-\eta)\ln h_0 \\ & + \sum_{t=1}^{\infty} \gamma^{t-1} \Bigg\{ \tilde{V}(\varepsilon_t) + \delta\eta \ln \frac{1}{\tilde{n}'(\varepsilon_1)} \\ & \gamma \left[(1+\beta\pi)\ln(1-\tau(\tilde{n}'(\varepsilon_t), \varepsilon_t)) + \delta\eta \ln \hat{X}(\varepsilon_t, \tilde{n}'(\varepsilon_t)) \right] \Bigg\} \\ & + (1+\beta\pi+\delta\eta)\sum_{t=1}^{\infty} \gamma^t \ln k_t + \delta(1-\eta)\sum_{t=1}^{\infty}\gamma^t \ln h_t. \end{aligned} \tag{A.42}$$

The next task is to reformulate the terms $\sum_{t=1}^{\infty} \gamma^t \ln k_t$ and $\sum_{t=1}^{\infty} \gamma^t \ln h_t$ appeared in (A.42) in terms of $\{\varepsilon_t\}_{t=0}^{\infty}$. The term $\sum_{t=1}^{\infty} \gamma^t \ln k_t$ is reformulated, using the capital market clearing condition, in the following way. From the capital market-clearing condition,

$$\begin{aligned} k_{t+1} &= \frac{s_t}{n_{t+1}} \\ &= \frac{1}{n_{t+1}} \cdot \frac{\beta\pi}{1+\delta+\beta\pi} \left[(1-\tau_t)\frac{(1-\alpha)Q}{1-\phi n_{t+1}} k_t - \frac{1+\delta}{\beta\pi} \cdot \frac{b_{t+1}}{R/\pi} \right] \\ &= \frac{1}{n_{t+1}} \cdot \frac{\beta\pi}{1+\delta+\beta\pi} \left[\frac{(1-\alpha)Q}{1-\phi n_{t+1}} - \frac{1+\delta}{\beta\pi} \cdot \frac{\tilde{G}(\varepsilon_{t+1})}{R/\pi} \right] (1-\tau_t) k_t. \end{aligned} \tag{A.43}$$

The second line originates from the savings function in (18) and the effective wage income in (16). The third line is from equation (A.33). By substituting (A.34) and (A.35) into (A.43), we obtain

$$k_{t+1} = \tilde{K}(\cdot)\, k_t, \tag{A.44}$$

where $\tilde{K}(\cdot)$ is defined as follows:

$$\tilde{K}(\cdot) = \begin{cases} \tilde{K}(\varepsilon_0, n_0, \varepsilon_1, \tilde{n}'(\varepsilon_1)) \equiv \tilde{K}_0(\varepsilon_1) \cdot \tilde{K}_{-1}(\varepsilon_0) \\ = \underbrace{\frac{1}{\tilde{n}'(\varepsilon_1)} \cdot \frac{\beta\pi}{1+\delta+\beta\pi}\left[\frac{(1-\alpha)Q}{1-\phi\tilde{n}'(\varepsilon_1)} - \frac{1+\delta}{\beta\pi}\cdot\frac{\tilde{G}(\varepsilon_1)}{R/\pi}\right]}_{\equiv \tilde{K}_0(\varepsilon_1)} \cdot \underbrace{(1-\tilde{\tau}(n_0,\varepsilon_0))}_{\equiv \tilde{K}_{-1}(\varepsilon_0)} & \text{for } t=0, \\ \tilde{K}(\varepsilon_t, \tilde{n}'(\varepsilon_t), \varepsilon_{t+1}, \tilde{n}'(\varepsilon_{t+1})) \equiv \tilde{K}_0(\varepsilon_{t+1}) \cdot \tilde{K}_{-1}(\varepsilon_t) \\ = \underbrace{\frac{1}{\tilde{n}'(\varepsilon_{t+1})} \cdot \frac{\beta\pi}{1+\delta+\beta\pi}\left[\frac{(1-\alpha)Q}{1-\phi\tilde{n}'(\varepsilon_{t+1})} - \frac{1+\delta}{\beta\pi}\cdot\frac{\tilde{G}(\varepsilon_{t+1})}{R/\pi}\right]}_{\equiv \tilde{K}_0(\varepsilon_{t+1})} \cdot \underbrace{(1-\tilde{\tau}(\tilde{n}'(\varepsilon_t),\varepsilon_t))}_{\equiv \tilde{K}_{-1}(\varepsilon_t)} & \text{for } t \geq 1. \end{cases} \tag{A.45}$$

With (A.45), we can reformulate the term $\sum_{t=1}^{\infty}\gamma^t \ln k_t$ in (A.42) as follows:

$$\begin{aligned}\sum_{t=1}^{\infty}\gamma^t \ln k_t &= \gamma \ln \tilde{K}_0(\varepsilon_1)\tilde{K}_{-1}(\varepsilon_0) + \gamma^2 \ln \tilde{K}_0(\varepsilon_2)\tilde{K}_{-1}(\varepsilon_1)\tilde{K}_0(\varepsilon_1)\tilde{K}_{-1}(\varepsilon_0)\\ &+ \gamma^3 \ln \tilde{K}_0(\varepsilon_3)\tilde{K}_{-1}(\varepsilon_2)\tilde{K}_0(\varepsilon_2)\tilde{K}_{-1}(\varepsilon_1)\tilde{K}_0(\varepsilon_1)\tilde{K}_{-1}(\varepsilon_0) + \cdots\\ &= \gamma\left(1+\gamma+\gamma^2+\cdots\right)\ln \tilde{K}_{-1}(\varepsilon_0)\\ &+ \gamma\left(1+\gamma+\gamma^2+\cdots\right)\ln \tilde{K}_0(\varepsilon_1) + \gamma^2\left(1+\gamma+\gamma^2+\cdots\right)\ln \tilde{K}_{-1}(\varepsilon_1)\\ &+ \gamma^2\left(1+\gamma+\gamma^2+\cdots\right)\ln \tilde{K}_0(\varepsilon_2) + \gamma^3\left(1+\gamma+\gamma^2+\cdots\right)\ln \tilde{K}_{-1}(\varepsilon_2)\\ &+ \cdots\\ &= \frac{\gamma}{1-\gamma}\ln \tilde{K}_{-1}(\varepsilon_0) + \frac{\gamma}{1-\gamma}\sum_{t=1}^{\infty}\gamma^{t-1}\left(\tilde{K}_0(\varepsilon_t) + \gamma\tilde{K}_{-1}(\varepsilon_t)\right). \end{aligned} \tag{A.46}$$

The term $\sum_{t=1}^{\infty}\gamma^t \ln h_t$ in (A.42) is reformulated, using the human capital formation function, in the following way. From (3), we have

$$\begin{aligned}\ln h_t &= \ln D\left(h_{t-1}\right)^{1-\eta}\left(x_{t-1}\right)^{\eta}\\ &= \ln D\left[D\left(h_{t-2}\right)^{1-\eta}\left(x_{t-2}\right)^{\eta}\right]^{1-\eta}\left(x_{t-1}\right)^{\eta}\\ &= \ln D\left\{D\left[h_{t-2}\left(D\left(h_{t-3}\right)^{1-\eta}\left(x_{t-3}\right)^{\eta}\right)\right]^{1-\eta}\left(x_{t-2}\right)^{\eta}\right\}^{1-\eta}\left(x_{t-1}\right)^{\eta}\\ &= \cdots,\end{aligned}$$

or,

$$\ln h_t \simeq \eta(1-\eta)^{t-1}\ln x_0 + \eta(1-\eta)^{t-2}\ln x_1 + \cdots + \eta(1-\eta)\ln x_{t-2} + \eta \ln x_{t-1}. \tag{A.47}$$

With (A.36) and (A.44), the expression in (A.47) is further reformulated as follows:

$$\begin{aligned}\ln h_t \simeq\ & \eta(1-\eta)^{t-1}\ln\frac{1}{\tilde{n}'(\varepsilon_1)}\hat{X}\left(\varepsilon_0,\tilde{n}'(\varepsilon_0)\right) + \eta(1-\eta)^{t-2}\ln\frac{1}{\tilde{n}'(\varepsilon_2)}\hat{X}\left(\varepsilon_1,\tilde{n}'(\varepsilon_1)\right)\tilde{K}\left(\varepsilon_0,n_0,\varepsilon_1,\tilde{n}'(\varepsilon_1)\right)\\ &+ \eta(1-\eta)^{t-3}\ln\frac{1}{\tilde{n}'(\varepsilon_3)}\hat{X}\left(\varepsilon_2,\tilde{n}'(\varepsilon_2)\right)\tilde{K}\left(\varepsilon_0,n_0,\varepsilon_1,\tilde{n}'(\varepsilon_1)\right)\tilde{K}\left(\varepsilon_1,\tilde{n}'(\varepsilon_1),\varepsilon_2,\tilde{n}'(\varepsilon_2)\right)\\ &\ \vdots\\ &+ \eta(1-\eta)\ln\frac{1}{\tilde{n}'(\varepsilon_{t-1})}\hat{X}\left(\varepsilon_{t-2},\tilde{n}'(\varepsilon_{t-2})\right)\tilde{K}\left(\varepsilon_0,n_0,\varepsilon_1,\tilde{n}'(\varepsilon_1)\right)\cdot\cdots\cdot\tilde{K}\left(\varepsilon_{t-3},\tilde{n}'(\varepsilon_{t-3}),\varepsilon_{t-2},\tilde{n}'(\varepsilon_{t-2})\right)\\ &+ \eta\ln\frac{1}{\tilde{n}'(\varepsilon_t)}\hat{X}\left(\varepsilon_{t-1},\tilde{n}'(\varepsilon_{t-1})\right)\tilde{K}\left(\varepsilon_0,n_0,\varepsilon_1,\tilde{n}'(\varepsilon_1)\right)\cdot\cdots\cdot\tilde{K}\left(\varepsilon_{t-2},\tilde{n}'(\varepsilon_{t-2}),\varepsilon_{t-1},\tilde{n}'(\varepsilon_{t-1})\right),\end{aligned}$$

or,

$$
\begin{aligned}
\ln h_t \simeq & \left[\eta\left(1-\eta\right)^{t-1}\ln\frac{1}{\tilde{n}'\left(\varepsilon_1\right)}\hat{X}\left(\varepsilon_0,\tilde{n}'\left(\varepsilon_0\right)\right)+\eta\left(1-\eta\right)^{t-2}\ln\frac{1}{\tilde{n}'\left(\varepsilon_2\right)}\hat{X}\left(\varepsilon_1,\tilde{n}'\left(\varepsilon_1\right)\right)\right.\\
&\left.\cdots+\eta\left(1-\eta\right)\ln\frac{1}{\tilde{n}'\left(\varepsilon_{t-1}\right)}\hat{X}\left(\varepsilon_{t-2},\tilde{n}'\left(\varepsilon_{t-2}\right)\right)+\eta\ln\frac{1}{\tilde{n}'\left(\varepsilon_t\right)}\hat{X}\left(\varepsilon_{t-1},\tilde{n}'\left(\varepsilon_{t-1}\right)\right)\right]\\
&+\left[\eta\left(1-\eta\right)^{t-2}+\eta\left(1-\eta\right)^{t-3}+\cdots+\eta\left(1-\eta\right)+\eta\right]\ln\tilde{K}\left(\varepsilon_0,n_0,\varepsilon_1,\tilde{n}'\left(\varepsilon_1\right)\right)\\
&+\left[\eta\left(1-\eta\right)^{t-3}+\eta\left(1-\eta\right)^{t-2}+\cdots+\eta\left(1-\eta\right)+\eta\right]\ln\tilde{K}\left(\varepsilon_1,\tilde{n}'\left(\varepsilon_1\right),\varepsilon_2,\tilde{n}'\left(\varepsilon_2\right)\right)\\
&\vdots\\
&+\left[\eta\left(1-\eta\right)+\eta\right]\ln\tilde{K}\left(\varepsilon_{t-3},\tilde{n}'\left(\varepsilon_{t-3}\right),\varepsilon_{t-2},\tilde{n}'\left(\varepsilon_{t-2}\right)\right)\\
&+\eta\ln\tilde{K}\left(\varepsilon_{t-2},\tilde{n}'\left(\varepsilon_{t-2}\right),\varepsilon_{t-1},\tilde{n}'\left(\varepsilon_{t-1}\right)\right). \qquad \text{(A.48)}
\end{aligned}
$$

Using (A.48), we can rewrite $\sum_{t=1}^{\infty}\gamma^t\ln h_t$ as

$$
\begin{aligned}
\sum_{t=1}^{\infty}\gamma^t\ln h_t \simeq & \gamma\eta\ln\frac{1}{\tilde{n}'\left(\varepsilon_1\right)}\hat{X}\left(\varepsilon_0,\tilde{n}'\left(\varepsilon_0\right)\right)\\
&+\gamma^2\left[\eta\left(1-\eta\right)\ln\frac{1}{\tilde{n}'\left(\varepsilon_1\right)}\hat{X}\left(\varepsilon_0,\tilde{n}'\left(\varepsilon_0\right)\right)+\eta\ln\frac{1}{\tilde{n}'\left(\varepsilon_2\right)}\hat{X}\left(\varepsilon_1,\tilde{n}'\left(\varepsilon_1\right)\right)+\eta\ln\tilde{K}\left(\varepsilon_0,n_0,\varepsilon_1,\tilde{n}'\left(\varepsilon_1\right)\right)\right]\\
&+\gamma^3\left[\eta\left(1-\eta\right)^2\ln\frac{1}{\tilde{n}'\left(\varepsilon_1\right)}\hat{X}\left(\varepsilon_0,\tilde{n}'\left(\varepsilon_0\right)\right)+\eta\left(1-\eta\right)\ln\frac{1}{\tilde{n}'\left(\varepsilon_2\right)}\hat{X}\left(\varepsilon_1,\tilde{n}'\left(\varepsilon_1\right)\right)\right.\\
&\left.+\eta\ln\frac{1}{\tilde{n}'\left(\varepsilon_3\right)}\hat{X}\left(\varepsilon_2,\tilde{n}'\left(\varepsilon_2\right)\right)+\left[\eta\left(1-\eta\right)+\eta\right]\ln\tilde{K}\left(\varepsilon_0,n_0,\varepsilon_1,\tilde{n}'\left(\varepsilon_1\right)\right)+\eta\ln\tilde{K}\left(\varepsilon_1,\tilde{n}'\left(\varepsilon_1\right),\varepsilon_2,\tilde{n}'\left(\varepsilon_2\right)\right)\right]\\
&+\cdots\\
\simeq & \left[\gamma\eta+\gamma^2\eta\left(1-\eta\right)+\gamma^3\eta\left(1-\eta\right)^2+\cdots\right]\\
&\times\left[\ln\frac{1}{\tilde{n}'\left(\varepsilon_1\right)}\hat{X}\left(\varepsilon_0,\tilde{n}'\left(\varepsilon_0\right)\right)+\gamma\ln\frac{1}{\tilde{n}'\left(\varepsilon_2\right)}\hat{X}\left(\varepsilon_1,\tilde{n}'\left(\varepsilon_1\right)\right)+\gamma^2\frac{1}{\tilde{n}'\left(\varepsilon_3\right)}\hat{X}\left(\varepsilon_2,\tilde{n}'\left(\varepsilon_2\right)\right)+\cdots\right]\\
&+\gamma\left[\gamma\eta+\gamma^2\left(\eta\left(1-\eta\right)+\eta\right)+\gamma^3\left(\eta\left(1-\eta\right)^2+\eta\left(1-\eta\right)+\eta\right)+\cdots\right]\\
&\times\left[\ln\tilde{K}\left(\varepsilon_0,n_0,\varepsilon_1,\tilde{n}'\left(\varepsilon_1\right)\right)+\gamma\ln\tilde{K}\left(\varepsilon_1,\tilde{n}'\left(\varepsilon_1\right),\varepsilon_2,\tilde{n}'\left(\varepsilon_2\right)\right)+\gamma^2\ln\tilde{K}\left(\varepsilon_2,\tilde{n}'\left(\varepsilon_2\right),\varepsilon_3,\tilde{n}'\left(\varepsilon_3\right)\right)+\cdots\right]\\
\simeq & \frac{\gamma\eta}{1-\gamma\left(1-\eta\right)}\left[\ln\frac{1}{\tilde{n}'\left(\varepsilon_1\right)}\hat{X}\left(\varepsilon_0,\tilde{n}'\left(\varepsilon_0\right)\right)+\gamma\ln\frac{1}{\tilde{n}'\left(\varepsilon_2\right)}\hat{X}\left(\varepsilon_1,\tilde{n}'\left(\varepsilon_1\right)\right)+\gamma^2\frac{1}{\tilde{n}'\left(\varepsilon_3\right)}\hat{X}\left(\varepsilon_2,\tilde{n}'\left(\varepsilon_2\right)\right)+\cdots\right]\\
&+\frac{\gamma^2\eta}{1-\gamma\left(1-\eta\right)}\left[\ln\tilde{K}_0\left(\varepsilon_1\right)\tilde{K}_{-1}\left(\varepsilon_0\right)+\gamma\ln\tilde{K}_0\left(\varepsilon_2\right)\tilde{K}_{-1}\left(\varepsilon_1\right)+\gamma^2\ln\tilde{K}_0\left(\varepsilon_3\right)\tilde{K}_{-1}\left(\varepsilon_2\right)+\cdots\right],
\end{aligned}
$$

or,

$$
\begin{aligned}
\sum_{t=1}^{\infty}\gamma^t\ln h_t \simeq & \frac{\gamma\eta}{1-\gamma\left(1-\eta\right)}\left[\ln\hat{X}\left(\varepsilon_0,\tilde{n}'\left(\varepsilon_0\right)\right)+\frac{\gamma}{1-\gamma}\ln\tilde{K}_{-1}\left(\varepsilon_0\right)\right]\\
&+\frac{\gamma\eta}{1-\gamma\left(1-\eta\right)}\sum_{t=1}^{\infty}\gamma^{t-1}\left[\ln\frac{1}{\tilde{n}'\left(\varepsilon_t\right)}+\gamma\ln\hat{X}\left(\varepsilon_t,\tilde{n}'\left(\varepsilon_t\right)\right)+\frac{\gamma}{1-\gamma}\left(\ln\tilde{K}_0\left(\varepsilon_t\right)+\gamma\ln\tilde{K}_{-1}\left(\varepsilon_t\right)\right)\right]. \qquad \text{(A.49)}
\end{aligned}
$$

With (A.46), (A.49), and the definition of $\tilde{K}_0\left(\cdot\right)$ and $\tilde{K}_{-1}\left(\cdot\right)$ in (A.45), we can rewrite (A.42) as

$$
\begin{aligned}
\sum_{t=0}^{\infty}\gamma^t V_t^M \simeq\ & (1+\beta\pi)\ln(1-\tilde{\tau}\left(n_0,\varepsilon_0\right)) + \delta\eta\ln\hat{X}\left(\varepsilon_0,n_0\right) \\
& + \left[(1+\beta\pi+\delta\eta)+\delta\left(1-\eta\right)\frac{\gamma\eta}{1-\gamma\left(1-\eta\right)}\right]\frac{\gamma}{1-\gamma}\ln\tilde{K}_{-1}\left(\varepsilon_0\right) \\
& + \delta\left(1-\eta\right)\frac{\gamma\eta}{1-\gamma\left(1-\eta\right)}\ln\hat{X}\left(\varepsilon_0,\tilde{n}'\left(\varepsilon_0\right)\right) \\
& + \sum_{t=1}^{\infty}\gamma^{t-1}\left\{\tilde{V}\left(\varepsilon_t\right)+\delta\eta\ln\frac{1}{\tilde{n}'\left(\varepsilon_1\right)}\right. \\
& + \gamma\left[(1+\beta\pi)\ln(1-\tau\left(\tilde{n}'\left(\varepsilon_t\right),\varepsilon_t\right)) + \delta\eta\ln\hat{X}\left(\varepsilon_t,\tilde{n}'\left(\varepsilon_t\right)\right)\right] \\
& + (1+\beta\pi+\delta\eta)\frac{\gamma}{1-\gamma}\left(\ln\tilde{K}_0\left(\varepsilon_t\right)+\gamma\ln\tilde{K}_{-1}\left(\varepsilon_t\right)\right) \\
& \left. +\delta\left(1-\eta\right)\frac{\gamma\eta}{1-\gamma\left(1-\eta\right)}\left[\ln\frac{1}{\tilde{n}'\left(\varepsilon_t\right)}+\gamma\ln\hat{X}\left(\varepsilon_t,\tilde{n}'\left(\varepsilon_t\right)\right)+\frac{\gamma}{1-\gamma}\left(\ln\tilde{K}_0\left(\varepsilon_t\right)+\gamma\ln\tilde{K}_{-1}\left(\varepsilon_t\right)\right)\right]\right\},
\end{aligned}
$$

or,

$$
\begin{aligned}
\sum_{t=0}^{\infty}\gamma^t V_t^M \simeq\ & (1+\beta\pi)\ln(1-\tilde{\tau}\left(n_0,\varepsilon_0\right)) + \left[\delta\eta+\delta\left(1-\eta\right)\frac{\gamma\eta}{1-\gamma\left(1-\eta\right)}\right]\ln\hat{X}\left(\varepsilon_0,n_0\right) \\
& + \left[(1+\beta\pi+\delta\eta)+\delta\left(1-\eta\right)\frac{\gamma\eta}{1-\gamma\left(1-\eta\right)}\right]\frac{\gamma}{1-\gamma}\ln(1-\tilde{\tau}\left(n_0,\varepsilon_0\right)) \\
& + \sum_{t=1}^{\infty}\gamma^{t-1}\left\{\tilde{V}\left(\varepsilon_t\right)+\delta\eta\ln\frac{1}{\tilde{n}'\left(\varepsilon_1\right)}\right. \\
& + \gamma\left[(1+\beta\pi)\ln(1-\tau\left(\tilde{n}'\left(\varepsilon_t\right),\varepsilon_t\right)) + \delta\eta\ln\hat{X}\left(\varepsilon_t,\tilde{n}'\left(\varepsilon_t\right)\right)\right] \\
& + \left[(1+\beta\pi+\delta\eta)+\delta\left(1-\eta\right)\frac{\gamma\eta}{1-\gamma\left(1-\eta\right)}\right]\frac{\gamma}{1-\gamma} \\
& \times\left[\ln\frac{1}{\tilde{n}'\left(\varepsilon_t\right)}\left(\frac{(1-\alpha)Q}{1-\phi\tilde{n}'\left(\varepsilon_t\right)}-\frac{1+\delta}{\beta\pi}\cdot\frac{\tilde{G}\left(\varepsilon_t\right)}{R/\pi}\right)+\gamma\ln(1-\tau\left(\tilde{n}'\left(\varepsilon_t\right),\varepsilon_t\right))\right] \\
& \left. +\delta\left(1-\eta\right)\frac{\gamma\eta}{1-\gamma\left(1-\eta\right)}\left[\ln\frac{1}{\tilde{n}'\left(\varepsilon_t\right)}+\gamma\ln\hat{X}\left(\varepsilon_t,\tilde{n}'\left(\varepsilon_t\right)\right)\right]\right\}. \qquad (A.50)
\end{aligned}
$$

The term $\frac{\beta\pi}{1+\delta+\beta\pi}$ included in $\tilde{K}_0\left(\varepsilon_t\right)$ is dropped from the above expression because it is irrelevant to the decision on ε_t under the assumption of the logarithmic utility function.

Equation (A.50) is further reformulated as follows:

$$
\begin{aligned}
\sum_{t=0}^{\infty}\gamma^t V_t^M \simeq & \left\{(1+\beta\pi)+\left[(1+\beta\pi+\delta\eta)+\delta(1-\eta)\frac{\gamma\eta}{1-\gamma(1-\eta)}\right]\frac{\gamma}{1-\gamma}\right\}\ln(1-\tilde{\tau}(n_0,\varepsilon_0)) \\
& + \frac{\delta\eta}{1-\gamma(1-\eta)}\ln\hat{X}(\varepsilon_0,n_0) \\
& + \sum_{t=1}^{\infty}\gamma^{t-1}\left\{\tilde{V}(\varepsilon_t)+\frac{\delta\eta}{1-\gamma(1-\eta)}\ln\frac{1}{\tilde{n}'(\varepsilon_1)}\right. \\
& + \gamma\left\{(1+\beta\pi)+\left[(1+\beta\pi+\delta\eta)+\delta(1-\eta)\frac{\gamma\eta}{1-\gamma(1-\eta)}\right]\frac{\gamma}{1-\gamma}\right\}\ln(1-\tau(\tilde{n}'(\varepsilon_t),\varepsilon_t)) \\
& + \gamma\frac{\delta\eta}{1-\gamma(1-\eta)}\ln\hat{X}\left(\varepsilon_t,\tilde{n}'(\varepsilon_t)\right) \\
& + \left[(1+\beta\pi+\delta\eta)+\delta(1-\eta)\frac{\gamma\eta}{1-\gamma(1-\eta)}\right]\frac{\gamma}{1-\gamma} \\
& \left.\times\ln\frac{1}{\tilde{n}'(\varepsilon_t)}\cdot\frac{\beta\pi}{1+\delta+\beta\pi}\left[\frac{(1-\alpha)Q}{1-\phi\tilde{n}'(\varepsilon_t)}-\frac{1+\delta}{\beta\pi}\cdot\frac{\tilde{G}(\varepsilon_t)}{R/\pi}\right]\right\}.
\end{aligned}
\tag{A.51}
$$

From (20), the indirect utility in period 0 becomes

$$
V_0^o = \ln\left(\frac{R}{\pi}n_0k_0+b_0\right) = \ln\left(\frac{R}{\pi}+\varepsilon_0\frac{\frac{\pi\omega}{n_0(1-\omega)}B+C}{\frac{\pi\omega}{n_0(1-\omega)}E+F}\frac{R}{\pi}\right)n_0k_0. \tag{A.52}
$$

Using (A.51) and (A.52), we define $Z_0(\varepsilon_0)$ and $Z(\varepsilon_t)$ as in (54) and (55), respectively. We can then reformulate the the social welfare function in (A.30), as:

$$
\begin{aligned}
SW &= V_0^o + \sum_{t=0}^{\infty}\gamma^t V_t^M \\
&\simeq Z_0(\varepsilon_0)+Z(\varepsilon_1)+\gamma Z(\varepsilon_2)+\gamma^2 Z(\varepsilon_3)+\ldots \\
&= Z_0(\varepsilon_0)+\sum_{t=1}^{\infty}\gamma^{t-1}Z(\varepsilon_t).
\end{aligned}
$$

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