# Modeling long-range dependent Gaussian processes with application in continuous-time financial models

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## Abstract

This paper considers a class of continuous–time long–range dependent Gaussian processes. The corresponding spectral density is assumed to have a general and flexible form, which covers some important and special cases. For example, the spectral density of a continuous–time fractional stochastic differential equation is included. A modelling procedure is then established through estimating the parameters involved in the spectral density by using an extended continuous–time version of the Gauss-Whittle objective function. The resulting estimates are shown to be strongly consistent and asymptotically normal. An application of the modelling procedure to the identification and modelling of a fractional stochastic volatility (FSV) is discussed in some detail.

KEYWORDS: Continuous–time model, diffusion process, long–range dependence, parameter estimation, stochastic volatility.

AMS 2000 Subject Classification: Primary 60G17; Secondary 60G10

## 1. Introduction

Continuous-time diffusion processes arise in many applications in economics, but perhaps nowhere do they play as large a role as in finance. Following the pathbreaking work of Merton (1969, 1973) and Black and Scholes (1973), the use of continuous-time diffusion processes has become a common feature of many applications, especially asset pricing models. As pointed out by Sundaresan (2001), perhaps the most significant development in the continuous–time field during the last decade has been the innovations in econometric theory and in the estimation techniques for models in continuous time. In addition, numerical methods for stochastic differential equations have provided approximate solutions to many continuous–time models and made applications of continuous–time models to practical problems possible and tractable. For this part, Kloeden and Platen (1999), Platen (1999), and Heath and Platen (2002) are some of the major developments. It should also be mentioned that there are some other modelling methods. One of these is the subordinator model method, essentially begun by Mandelbrot and Taylor (1967), and recently developed by Hurst, Platen and Rachev (1997), Mandelbrot, Fisher and Calvet (1997), and Heyde (1999).

Recent studies have indicated that data in economics and finance display long–range dependence (LRD) (see Robinson 1994, 1999; Baillie and King 1996; Comte and Renault 1996, 1998; Heyde 1999; Shiryaev 1999; and others). A fundamental example is fractional Brownian motion (fBm) $B_H$ with Hurst index $H$, $\frac{1}{2} < H < 1$. Following the introduction of the fractional Brownian motion in continuous–time by Mandelbrot and Van Ness (1968),

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the use of continuous-time long–range dependent processes has become a common feature of many applications, especially in econometrics and finance (see Baillie and King 1996; Comte and Renault 1996, 1998). This is probably due to the following two reasons. The first one is that the class of continuous time stochastic processes most commonly employed in finance can be extended to encompass long-range dependent models, which have already been used to model real financial data (see Comte and Renault 1998, p. 311). Existing studies show not only that this extension is possible, but also that it is the natural one in order to get variations (of prices or rates) which have an instantaneous variance of order less than two (but not necessarily integer). The usual short–range dependence case (diffusion processes) corresponds to the order one. This property is fundamental in the modern continuous–time finance theory (see Merton 1990, Chapter 1 for example) and corresponds to some kind of 'instantaneous unpredictability' of asset prices in the sense of Sims (1984). The second reason is more statistical. Since existing studies (see Ding, Granger and Engle 1993; Ding and Granger 1996) already suggest that some financial derivatives (the Standard & Poor (SP) 500 stock market daily closing price index for example) display some kind of LRD property, existing studies for short–range dependent processes are therefore not applicable to the LRD case.

This paper assumes that there are fractional stochastic differential equations such that their solutions are Gaussian processes having a spectral density of the form

$$\phi(\omega) = \phi(\omega, \theta) = \frac{\pi(\omega, \theta)\sigma^2}{|\omega|^{2\beta}(\omega^2 + \alpha^2)^{\gamma}}, \ \omega \in (-\infty, \infty), \tag{1.1}$$

where $\theta = (\alpha, \beta, \sigma, \gamma) \in \Theta = \left\{0 < \alpha < \infty, 0 < \beta < \frac{1}{2}, 0 < \sigma < \infty, 0 < \gamma < \infty, \beta + \gamma > \frac{1}{2}\right\}$, $\pi(\omega, \theta)$ is a continuous and positive function satisfying $0 < \lim_{\omega \to 0 \text{ or } \omega \to \pm\infty} \pi(\omega, \theta) < \infty$ for each $\theta \in \Theta$, $\alpha$ is normally involved in the drift function of the process involved, $\beta$ is the LRD parameter, $\sigma$ is involved in the diffusion function of the process considered, and $\gamma$ is normally called the intermittency parameter of the process considered. When $\pi(\omega, \theta) \equiv 1$ and $\alpha = 1$ in (1.1), the existence of such a process has been justified in Anh, Angulo and Ruiz–Medina (1999). For this case, model (1.1) corresponds to the fractional Riesz–Bessel motion (fRBm) case. The significance of fRBm is in its behaviour when $|\omega| \to \infty$. It is noted that when $\alpha = 1$, $\phi(\omega)$ of (1.1) is well-defined as $|\omega| \to \infty$ due to the presence of the component $(1 + \omega^2)^{-\gamma}$, $\gamma > 0$, which is the Fourier transform of the Bessel potential. As a result, the covariances $R(t)$ of the increments of fRBm are strong for small $|t|$. That is, large (resp. small) values of the increments tend to be followed by large (resp. small) values with probability sufficiently close to one. This is the clustering phenomenon observed in stochastic finance (see Shiryaev 1999, page 365). This phenomenon is referred to as (second-order) intermittency in the turbulence literature (see Frisch 1995).

When $\pi(\omega, \theta) = \frac{1}{\Gamma^2(1+\beta)}$ and $\gamma = 1$, model (1.1) reduces to

$$\psi(\omega) = \psi(\omega, \theta) = \frac{\sigma^2}{\Gamma^2(1 + \beta)} \frac{1}{|\omega|^{2\beta}} \frac{1}{\omega^2 + \alpha^2}, \tag{1.2}$$

which is just the spectral density of processes that are solutions of continuous–time fractional stochastic differential equations of the form

$$dX(t) = -\alpha X(t)dt + \sigma dB_\beta(t),\ X(0) = 0,\ t \in (0, \infty), \tag{1.3}$$

where $B_\beta(t)$ is general fractional Brownian motion given by $B_\beta(t) = \int_0^t \frac{(t-s)^\beta}{\Gamma(1+\beta)} dB(s)$, $B(t)$ is standard Brownian motion, and $\Gamma(x)$ is the usual $\Gamma$ function. Obviously, model (1.3) is a fractional stochastic differential equation. It is noted that the solution of (1.3) is given by

$$X(t) = \int_0^t a(t-s)dB(t),\ a(x) = \frac{\sigma}{\Gamma(1+\beta)} \left( x^\beta - \alpha \int_0^x e^{-\alpha(x-u)} u^\beta du \right),\ t \in [0, \infty). \tag{1.4}$$

Model (1.2) corresponds to that of an Ornstein–Ulhenbeck process of the form (1.3) driven by fractional Brownian motion with Hurst index $H = \beta + \frac{1}{2}$. Obviously, the process $X(t)$ of (1.4) is Gaussian.

It is noted that the $\phi(\omega)$ of (1.1) is well–defined for both $|\omega| \to 0$ and $|\omega| \to \infty$ due to the presence of the component $(\alpha^2 + \omega^2)^{-\gamma}$, which provides some additional information for the identification and estimation of $\alpha$ and $\gamma$. For model (1.2), when $|\omega| \to 0$, $\psi(\omega) \sim \frac{1}{\Gamma^2(1+\beta)} \left(\frac{\sigma}{\alpha}\right)^2 \frac{1}{\omega^{2\beta}}$. For this case, if only information for LRD is used, it is easy to estimate the whole component $\left(\frac{\sigma}{\alpha}\right)^2$, but difficult to estimate both $\sigma$ and $\alpha$ individually. Thus, the use of information for LRD only can cause a model misspecification problem. This suggests using some additional information for the high frequency area (i.e., $|\omega| \to \infty$) for the identification and estimation of both $\alpha$ and $\gamma$ involved in model (1.1).

It should be pointed out that the processes having a spectral density of the form (1.1) can be nonstationary. As can be seen from (1.4), $X(t)$ of (1.4) is a nonstationary Gaussian process, but the spectral density $\psi(\omega)$ is a special case of from (1.1). It is worthwhile to point out that model (1.1) extends and covers many important cases, including the important case where $0 < \beta < \frac{1}{2}$ and $\gamma \geq \frac{1}{2}$. For this case, $\beta + \gamma > \frac{1}{2}$ holds automatically. Recently, Gao, Anh, Heyde and Tieng (2001) considered the special case where $\pi(\omega, \theta) \equiv 1$, $\alpha = 1$, $0 < \beta < 1/2$ and $\gamma \geq 1/2$ in (1.1). The authors were able to establish asymptotic results for estimators of $\theta$ based on discretization. See for example, Theorem 2.2 of Gao, Anh, Heyde and Tieng (2001). As a special case of model (1.1), another important case where $\pi(\omega, \theta) \equiv 1$, $\alpha = 1$, $0 < \beta < \frac{1}{2}$, $0 < \gamma < \frac{1}{2}$ but $\beta + \gamma > \frac{1}{2}$ that was not discussed in detail previously has now been included in this paper. There are two reasons to explain why the latter case is quite important. The first reason is that it is theoretically much more difficult to estimate both $\beta$ and $\gamma$ when they relate each other in the form of $\beta + \gamma > \frac{1}{2}$. As can be seen from Section 2 below, a constrained estimation procedure is needed for this case. The second reason is that one needs to consider the case where both the long–range dependence and intermittency are moderate but the collective impact of the two is quite significant.

The main contribution of this paper to the literature includes two parts. The first part is the novelty of model (1.1). The second part is that this paper proposes a novel estimation procedure for the case where in model (1.1) $0 < \beta < \frac{1}{2}$, $0 < \gamma < \frac{1}{2}$ but $\beta + \gamma > \frac{1}{2}$, which, as pointed out earlier, has not been discussed before to the best of our knowledge. In the rest of this paper, we consider estimating the parameters involved in (1.1) through using a continuous–time version of the Gauss-Whittle objective function. Both the consistency and the asymptotic normality of the estimators of the parameters are established in Theorems 2.1 and 2.2 below. As an application of the proposed estimation procedure to financial modelling, we discuss fractional interest rate and stochastic volatility models. In applications, unlike some existing studies (see Comte and Renault 1998, §6.4; Comte and Renault 1996), we do not discretize continuous-time models in order to avoid causing any approximation errors. Instead we suggest using discrete data rather than discretized values as in Section 3 of Gao, Anh and Heyde (2002). In empirical studies, this kind of strategy is quite natural as real data are normally available in a discrete form.

The organisation of this paper is as follows. Section 2 proposes our estimation and modelling procedure and the corresponding asymptotic theory. An application of the estimation procedure to the identification and modelling of fractional interest rate and stochastic volatility models is briefly mentioned in Section 3. We conclude the main parts of the paper with a concluding remark on future directions. Mathematical proofs are relegated to the Appendix.

## 2. Estimation and modelling procedure

This section establishes some kind of correspondence between form (1.1) and a corresponding class of fractional stochastic differential equations, develops an estimator of the spectral density and then constructs an estimation procedure for the parameters involved in the spectral density through using an extended continuous time Gauss–Whittle contrast function. Asymptotic consistency and normality results are then given in detail.

For a given function $g(\cdot)$ over $R^1 = (-\infty, \infty)$, define its Riemann–Liouville fractional derivative and integral by (see Podlubny 1999)

$$\mathcal{D}^\beta g(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\beta-1} g(\tau) d\tau,$$

for $\beta \in [n-1, n)$, $n = 1, 2, \ldots$, and the Riemann–Liouville fractional integral by

$$\mathcal{I}^\beta g(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} g(\tau) d\tau$$

for $\beta > 0$, respectively.

We then define a class of fractional stochastic differential equations of the form

$$\mathcal{D}^\beta \left(\alpha I + \mathcal{D}\right)^\gamma X(t) = \bar{B}(t),\ t \in [0, \infty), \tag{2.1}$$

where $\alpha$, $\beta$ and $\gamma$ are as defined in (1.1), $I$ is the identity operator, and $\bar{B}(t)$ is the standard Gaussian noise. By using some existing results (see Kleptsyna, Kloeden and Anh 1998 for the case of fractional Brownian motion based stochastic differential equations; Anh, Angulo and Ruiz–Medina 1999 for the fRBm case; and Anh, Heyde and Leonenko 2002 for the modified fractional Levy motion case), the existence of solutions to (2.1) can be justified. The solution of (2.1) is interpreted as

$$X(t) = \int_0^t A(t-s)dB(s),\ t \in [0, \infty), \tag{2.2}$$

where $A(\cdot)$ is defined by its Laplace transform: $a(x) = \frac{1}{x^\beta}\frac{1}{(x+\alpha)^\gamma}$. The inverse Laplace transform of $a(x)$ is given in Prudnikov, Brychkov and Marichev (1990).

It follows from (2.2) that the solution of (2.1) belongs to a family of Gaussian processes. Since the process $X(t)$ of (2.2) is not stationary, we denote by $Y(t)$ the "stationary version" of $X(t)$,

$$Y(t) = \int_{-\infty}^t A(t-s)dB(s),\ t \in [0, \infty). \tag{2.3}$$

Define the autocovariance function of $Y$ by $\gamma_Y(h) = \mathrm{cov}[Y(t), Y(t+h)]$ for any $h \in (-\infty, \infty)$. In the frequency domain, as the spectral density of $Y(t)$ is idential to that of the process $X(t)$ (see Proposition 6 of Comte and Renault 1996), the spectral density of $X(t)$ of (2.2) defined by the Fourier transform of $\gamma_Y(h)$: $\psi(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau}\gamma_Y(\tau)d\tau$, is given by

$$\psi(\omega) = \psi(\omega, \theta) = \frac{1}{\Gamma^2(1+\beta)}\frac{\sigma^2}{|\omega|^{2\beta}(\omega^2+\alpha^2)^\gamma},\ \omega \in (-\infty, \infty), \tag{2.4}$$

where the parameters $\theta = (\alpha, \beta, \sigma, \gamma)$ is the same as in (1.1). Thus, one can interpret that the spectral density of the form (2.4) corresponds to $X(t)$ of (2.2).

Note that when $\gamma = 1$, the spectral density of (2.4) reduces to form (1.2). For this case, $X(t)$ of (2.2) reduces to (1.4). Equations (2.1)–(2.4) therefore support the existence of a spectral density of the form (1.1) and its corresponding fractional stochastic differential equations.

Thus, it is quite natural to view form (1.1) as an extension of (2.4) and to assume that there are fractional stochastic differential equations such that their solutions $X(t)$ are Gaussian processes having a spectral density of the form (1.1). We denote by $Y(t)$ the stationary version of $X(t)$ in case the process $X(t)$ itself is nonstationary. One normally replaces the process $X(t)$ by either a stationary version such as $Y(t)$ in (2.3) or a wavelet transformed version (see Gao, Anh and Heyde 2002, p. 300). We also assume without loss of generality that both $X(t)$ and $Y(t)$ are defined on $[0, \infty)$. It is easily seen that the estimation and modelling procedure remains true when both $X(t)$ and $Y(t)$ are defined on $(-\infty, \infty)$. This paper considers only the case of $0 < \beta < \frac{1}{2}$, as the case of $\beta = 0$ implies that model (1.1) is the spectral density of solutions of stochastic diffusion equations. For any given $\omega \in (-\infty, \infty)$, we define the following estimator of $\phi(\omega) = \phi(\omega, \theta)$ by

$$I_N^Y(\omega) = \frac{1}{2\pi N}\left|\int_0^N e^{-i\omega t}Y(t)dt\right|^2, \tag{2.5}$$

where $N > 0$ is the upper bound of the interval $[0, N]$, on which each $Y(t)$ is observed. Throughout this paper, the stochastic integrals are limits in mean square of appropriate Riemann sums. It is noted that form (2.5) for the continuous–time case is an extension of the usual periodogram for the discrete case (see Brockwell and Davis 1990). For discrete time processes, some asymptotic results have already been established for periodogram estimators (see §10 of Brockwell and Davis 1990).

Before establishing the main results of this paper, we need to introduce the following assumption.

**Assumption 2.1**. (i) Assume that Gaussian processes having a spectral density of the form (1.1) are solutions of fractional stochastic differential equations. In addition, suppose that each solution has a stationary Gaussian version when the solution itself is nonstationary.

(ii) Assume that $\pi(\omega, \theta)$ is a positive and continuous function in both $\omega$ and $\theta$, bounded away from zero and chosen to satisfy

$$\int_{-\infty}^{\infty} \phi(\omega, \theta)\ d\omega < \infty \quad \text{and} \quad \frac{\partial}{\partial \theta} \left( \int_{-\infty}^{\infty} \log \left( \phi(\omega, \theta) \right) \frac{d\omega}{1+\omega^2} \right) = 0 \quad \text{for} \quad \theta \in \Theta.$$

In addition, $\pi(\omega, \theta)$ is a symmetric function in $\omega$ satisfying $0 < \lim_{\omega \to 0} \pi(\omega, \theta^*) < \infty$ and $0 < \lim_{\omega \to \pm\infty} \pi(\omega, \theta^*) < \infty$ for each given $\theta^* \in \Theta$.

(iii) Let $\theta_0$ be the true value of $\theta$, and $\theta_0$ be in the interior of $\Theta_0$, a compact subset of $\Theta$. For any small $\epsilon > 0$, if $\epsilon < ||\theta - \theta_0|| < \frac{1}{4}$ then

$$\int_{-\infty}^{\infty} \frac{\phi(\omega, \theta_0)}{\phi(\omega, \theta)} \frac{1}{1+\omega^2}\ d\omega < \infty,$$

where $|| \cdot ||$ denotes the Euclidean norm.

**Remark 2.1**. (i) Assumption 2.1(i) assumes only that the processes having a spectral density of the form (1.1) are Gaussian processes, which can be solutions of fractional stochastic differential equations. For example, the process $X(t)$ given in (1.4) is the solution of equation (1.3), and the spectral density of the solution is given by (1.2). In other words, it is not the aim of this paper both the existence of such fractional stochastic differential equations and the correspondence between (1.1) and the equations are discussed rigorously. Instead we focus on the modelling part of such Gaussian processes through estimating the parameters involved in a spectral density of the form (1.1). As can be seen, moreover, the proposed modelling procedure does not depend on whether the process involved is stationary or not. This is mainly because (i) the parameters remain unchanged when the nonstationary process concerned is replaced by a stationary version (see the forms of $\phi(\omega)$ of p. 300 and $f(\omega)$ of (2.2) in Gao, Anh and Heyde 2002) or (ii) the nonstationary process concerned and its stationary version can have the same spectral density (see equations (2.3) and (2.4) above for example).

(ii) Assumption 2.1 allows a lot of flexibility in choosing the form of $\pi(\omega, \theta)$, which includes not only the LRD parameter $\beta$ and the intermittency parameter $\gamma$, but also the parameters–of–interest, $\alpha$ and $\sigma$. The last two parameters, as can be seen from models (1.2) and (1.3), have some financial interpretations: $\alpha$ represents the speed of the fluctuations of an interest rate data set while $\sigma$ is a measure for the order of the magnitude of the fluctuations of an interest rate data set around zero for example. In general, $\pi(\omega, \theta)$ represents some kind of magnitude of the process involved. Assumption 2.1(ii) assumes $\phi(\omega, \theta)$ is also normalized so that

$$\frac{\partial}{\partial \theta}\left(\int_{-\infty}^{\infty} \log\left(\phi(\omega, \theta)\right) \frac{d\omega}{1+\omega^2}\right) = 0.$$

This extends similar conditions introduced by Fox and Taqque (1986) and then generalized by Heyde and Gay (1993).

(iii) Assumption 2.1 holds in many cases. For example, when $\pi(\omega, \theta) = \frac{1}{\Gamma^2(1+\beta)}$ and $\gamma \equiv 1$ or $\pi(\omega, \theta) \equiv 1$, Assumption 2.1 holds automatically.

As pointed out earlier, the main objective of this section is to estimate the parameters involved in (1.1) for modelling purposes. As can be seen from the literature, parameter estimation for Gaussian random processes and fields based on various forms of Whittle's estimation procedure has a long history. See for example, Fox and Taqqu (1986) used an objective function of the form

$$L_Z(\theta) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\left\{\log(f(\omega, \theta)) + \frac{I_Z(\omega)}{f(\omega, \theta)}\right\}\, d\omega$$

for a stationary Gaussian process $Z$ having $f(\omega, \theta)$ as its spectral density defined on $[-\pi, \pi]$, where $I_Z(\omega)$ is the conventional periodogram estimator of $f(\omega, \theta)$. Dahlhaus (1989) and Robinson (1995) suggested using some discretized versions of the Whittle objective function for parameter estimation for self–similar and long-range dependent Gaussian processes. Heyde and Gay (1993) considered using smoothed periodograms for estimation of processes and fields without Gaussian assumptions, and the resulting estimation procedure is asymptotically equivalent to the Whittle objective function based estimation method.

When considering the case where the spectral density $\phi(\omega)$ of (1.1) is defined on $(-\infty, \infty)$, the weight function $\frac{1}{1+\omega^2}$ is required to ensure that the corresponding objective function:

$$L_N^Y(\theta) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\left\{\log(\phi(\omega, \theta)) + \frac{I_N^Y(\omega)}{\phi(\omega, \theta)}\right\}\frac{d\omega}{1+\omega^2} \tag{2.6}$$

is well-defined. This is mainly because $\lim_{\omega\to\pm\infty} \frac{\log(\phi(\omega,\theta))}{1+\omega^2} = 0$. As can be seen from this limitation, the weight function $\frac{1}{1+\omega^2}$ is still suitable for the case of $\alpha \neq 1$. The second part of (2.6) is also well-defined due to Assumption 2.1(iii). Equation (2.6) can also be justified by applying the entropy theory discussed in Dym and McKean (1976).

This paper then considers using $L_N^Y(\theta)$ to derive asymptotically consistent estimators for $\theta_0$, the true value of $\theta$. Due to the form of

$$\Theta = \left\{\theta : 0 < \alpha < \infty, 0 < \beta < \frac{1}{2}, 0 < \sigma < \infty, 0 < \gamma < \infty, \beta + \gamma > \frac{1}{2}\right\},$$

we need to consider the following two different cases:

- Case I: $\Theta_1 = \left\{\theta :\ 0 < \alpha < \infty,\ 0 < \sigma < \infty,\ 0 < \beta < \frac{1}{2},\ \frac{1}{2} \leq \gamma < \infty\right\}$; and
- Case II: $\Theta_2 = \left\{\theta :\ 0 < \alpha < \infty,\ 0 < \sigma < \infty,\ 0 < \beta < \frac{1}{2},\ 0 < \gamma < \frac{1}{2},\ \text{but } \beta + \gamma > \frac{1}{2}\right\}$.

Obviously, $\Theta_1 \subset \Theta$ and $\Theta_2 \subset \Theta$.

For Case I, the minimum contrast estimator of $\theta$ is defined by

$$\bar{\theta}_N = \arg \min_{\theta \in \Theta_{10}} L_N^Y(\theta), \tag{2.7}$$

where $\Theta_{10}$ is a compact subset of $\Theta_1$.

For Case II, we introduce the following Lagrangian function

$$M_N^Y(\theta) = L_N^Y(\theta) - \lambda g(\theta),$$

where $\lambda$ is the multiplier and $g(\theta) = \beta + \gamma - \frac{1}{2}$. The minimization problem:

$$\text{Minimising } \; L_N^Y(\theta), \text{ subject to } \theta \in \Theta_2$$

can now be transferred to the following minimization problem:

$$\hat{\theta}_N = \arg \min_{\theta \in \Theta_{20}} M_N^Y(\theta), \tag{2.8}$$

where $\Theta_{20}$ is a compact subset of $\Theta_2$. It should be noted that Case I corresponds to $\lambda = 0$ and that Case II corresponds to $\lambda \neq 0$. To aviod abusing the notation of $\theta_0$, we denote the true value of $\theta \in \Theta_1$ by $\theta_{10}$, and the true value of $\theta \in \Theta_2$ by $\theta_{20}$ throughout the rest of this section.

To state the following results, one also needs to introduce the following conditions.

**Assumption 2.2**. (i) For any real function $f(\cdot) \in L^2(-\infty, \infty)$,

$$\int_{-\infty}^{\infty} \frac{f^2(\omega, \theta_0)}{(1+\omega^2)^2} \left( \frac{\partial \log(\phi(\omega, \theta))}{\partial \theta} \right)^{\tau} \left( \frac{\partial \log(\phi(\omega, \theta))}{\partial \theta} \right) |_{\theta=\theta_0} \, d\omega < \infty,$$

where $\theta_0 = \theta_{10}$ or $\theta_{20}$ and $\frac{\partial \log(\phi(\omega,\theta))}{\partial \theta} = \left( \frac{\partial \log(\phi(\omega,\theta))}{\partial \alpha}, \frac{\partial \log(\phi(\omega,\theta))}{\partial \beta}, \frac{\partial \log(\phi(\omega,\theta))}{\partial \sigma}, \frac{\partial \log(\phi(\omega,\theta))}{\partial \eta} \right)^{\tau}$.

(ii) For $\theta \in \Theta$,

$$\Sigma(\theta) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \left( \frac{\partial \log(\phi(\omega, \theta))}{\partial \theta} \right) \left( \frac{\partial \log(\phi(\omega, \theta))}{\partial \theta} \right)^{\tau} \frac{1}{(1+\omega^2)^2} \, d\omega < \infty.$$

(iii) The inverse matrix, $\Sigma^{-1}(\theta_0)$, of $\Sigma(\theta_0)$ exists, where $\theta_0 = \theta_{10}$ or $\theta_{20}$.

**Assumption 2.3**. Assume that $K(\theta, \theta_0)$ is convex in $\theta$ on an open set $\mathcal{C}(\theta_0)$ containing $\theta_0$, where $\theta_0 = \theta_{10}$ or $\theta_{20}$ and

$$K(\theta, \theta_0) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\{ \frac{\phi(\omega, \theta_0)}{\phi(\omega, \theta)} - 1 - \log \left( \frac{\phi(\omega, \theta_0)}{\phi(\omega, \theta)} \right) \right\} \frac{d\omega}{1+\omega^2}.$$

**Remark 2.2**. (i) Assumption 2.2(i) is required for an application of a continuous–time central limit theorem to the proof of the asymptotic normality. Assumption 2.2(ii)(iii) is similar to those for the discrete case. See for example, Condition (A2) of Heyde and Gay

(1993). Assumptions 2.2 and 2.3 simplify some existing conditions for continuous–time models. See for example, Conditions 2.1 and 2.2 of Gao, Anh and Heyde (2002).

(ii) Assumption 2.2 holds in many cases. For example, when $\pi(\omega, \theta) = \frac{1}{\Gamma^2(1+\beta)}$ and $\gamma \equiv 1$ or $\pi(\omega, \theta) \equiv 1$, Assumption 2.2 holds automatically.

(iii) It should be pointed out that Assumption 2.3 holds automatically for the case where $\pi(\omega, \theta) \equiv 1$, as the matrix $\mathcal{K}(\theta) = \{k_{ij}(\theta)\}_{\{1 \le i,j \le 4\}}$ is positive semi–definite for every $\theta \in \mathcal{C}(\theta_0)$, an open convex set containing $\theta_0$, where $k_{ij}(\theta) = \frac{\partial^2}{\partial \theta_i \partial \theta_j} K(\theta, \theta_0)$, in which $\theta_1 = \alpha$, $\theta_2 = \beta$, $\theta_3 = \sigma$ and $\theta_4 = \gamma$. For the detailed verification, one needs to use Theorem 4.5 of Rockafeller (1970). This suggests that Assumption 2.3 is a natural condition.

(iv) In general, in order to ensure the existence and uniqueness (at least asymptotically) of $\hat{\theta}_N$, the convexity imposed in Assumption 2.3 is necessary. Previously, this type of condition has not been mentioned in detail, mainly because the convexity condition holds automatically in some special cases as pointed out in Remark 2.2(iii). For our model (1.1), as the form of $\phi(\omega, \theta)$ is very general, Assumption 2.3 is needed for rigorousness consideration.

We now state the following results for Case I and Case II in Theorems 2.1 and 2.2 respectively.

**Theorem 2.1** (Case I). (i) *Assume that Assumptions 2.1–2.3 with $\theta_0 = \theta_{10}$ hold. Then*

$$\mathrm{P}\left(\lim_{N\to\infty} \bar{\theta}_N = \theta_{10}\right) = 1.$$

(ii) *In addition, if the true value $\theta_{10}$ of $\theta$ is in the interior of $\Theta_{10}$, then as $N \to \infty$*

$$\sqrt{N}(\bar{\theta}_N - \theta_{10}) \to_D N\left(0, \Sigma^{-1}(\theta_{10})\right),$$

*where $\Sigma^{-1}(\theta_{10})$ is as defined above.*

**Theorem 2.2** (Case II). (i) *Assume that Assumptions 2.1–2.2 with $\theta_0 = \theta_{20}$ hold. In addition, let $\hat{\theta}_N$ converge to $\theta_{20}$ with probability one and the true value $\theta_{20}$ of $\theta$ be in the interior of $\Theta_{20}$. Then as $N \to \infty$*

$$\sqrt{N}(\hat{\theta}_N - \theta_{20}) \to_D N\left(0, A\Sigma^{-1}(\theta_{20})A\right),$$

*where the $4 \times 4$ matrix $A$ is given by*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

(ii) *Assume that Assumptions 2.1–2.3 with $\theta_0 = \theta_{20}$ hold. Then*

$$\mathrm{P}\left(\lim_{N\to\infty} \hat{\theta}_N = \theta_{20}\right) = 1.$$

The proofs of Theorems 2.1 and 2.2 are relegated to the Appendix.

**Remark 2.3**. (i) Theorem 2.1 establishes both the strong consistency and the asymptotic normality of $\bar{\theta}_N$ of (2.7). Previously, we were unable to establish some similar results for the continuous–time stationary case where $\pi(\omega, \theta) \equiv 1$, $\alpha = 1$, $0 < \beta < \frac{1}{2}$ and $\gamma \geq \frac{1}{2}$. Instead, we established an asymptotic normality result based on discretization. See for example, Theorem 2.2 of Gao, Anh, Heyde and Tieng (2001).

(ii) Theorem 2.1 extends and complements some existing results for both the discrete and continuous time cases. For example, Comte and Renault (1996, 1998), Gao, Anh, Heyde, and Tieng (2001), and Gao, Anh and Heyde (2002). As can be seen, strong consistency and asymptotic normality results of the estimators of the parameters involved in (1.1) do not depend on the use of discretised values of the process under consideration. It should also be pointed out that the use of continuous–time models can avoid the problem of misspecification for parameters. Moreover, the estimation procedure fully makes the best use of all the information available, and therefore can clearly identify and estimate all the four parameters involved.

**Remark 2.4**. Theorem 2.2 establishes the asymptotic consistency results for the case where $\theta \in \Theta_2$. The corresponding estimation procedure for the important class of models is now applicable to the case where the LRD parameter $\beta$ satisfies $0 < \beta < \frac{1}{2}$, the intermittency parameter $\gamma$ satisfies $0 < \gamma < \frac{1}{2}$, but the pair $(\beta, \gamma)$ satisfies the condition: $\beta + \gamma > \frac{1}{2}$. Some practical problems that we were unable to solve can now be dealt with. One needs to point out that the strong consistency of $\hat{\theta}_N$ is necessary for the establishment of the asymptotic normality and that Assumption 2.3 may only be one of the few necessary conditions for the strong consistency. Due to this reason, we impose the strong consistency directly for the establishment of the asymptotic normality.

In Section 3 below, we have a detailed look at applications of the proposed estimation and modelling procedure to a class of continuous–time long–range dependent financial models.

## 3. Application in financial models with LRD

This section includes the following two parts. The first part looks at a class of interest rate models with LRD. An application of the modelling procedure to a fractional stochastic volatility model is mentioned briefly in part two.

Assume that an interest rate data set $\{r_t\}$ satisfies model (1.3) given by

$$dr(t) = -\alpha r(t)dt + \sigma dB_\beta(t), \ t \in (0, \infty), \tag{3.1}$$

where $\alpha$ represents the speed of the fluctuations of the interest rate, $\sigma$ represents the volatility, and $\beta$ is the LRD parameter. It follows from (1.4) that the solution of (3.1) is given by

$$r(t) = \int_0^t a(t-s)dB(t), \ a(x) = \frac{\sigma}{\Gamma(1+\beta)} \left( x^\beta - \alpha \int_0^x e^{-\alpha(x-u)} u^\beta du \right),$$

where $B(t)$ is the standard Brownian motion. Obviously, when $\beta \equiv 0$, model (3.1) is a popular short-term interest rate model proposed by Vasicek (1977) and then studied by many authors. This indicates that model (3.1) is a natural extension of the short-term interest rate model to the LRD case. Recently, Comte and Renault (1996, p. 124) have already pointed out the theoretical necessity for studying such fractional stochastic interest models. In empirical studies, our preliminary results suggest that the US Federal interest rate data set from January 1963 through December 1998 (the data set is available from the author upon request) may display some kind of long-range dependence and intermittency.

Let $x(t) = \int_{-\infty}^{t} a(t-s)dB(s)$ be the "stationary version" of $r(t)$. It is shown that the spectral density of $r(t)$ can be defined as the same as that for the stationary process $x(t)$. The spectral density is then defined by

$$\psi(\omega) = \psi(\omega, \theta) = \frac{\sigma^2}{\Gamma^2(1+\beta)} \frac{1}{|\omega|^{2\beta}} \frac{1}{\omega^2 + \alpha^2}, \tag{3.2}$$

where the parameters are as defined in model (1.3). For this case, the intermittency parameter $\gamma$ equals to one. As pointed out earlier, the structure of $\psi(\omega, \theta)$ is quite interesting in terms of the involvement of the drift parameter $\alpha$, the LRD parameter $\beta$, and the volatility parameter $\sigma$. Once the parameters are estimated, model (3.1) can then be used for practical modelling purposes. Before using the estimation procedure and the asymptotic results, one needs to verify Assumptions 2.1 and 2.2 hold for model (3.1).

Note that Assumption 2.1 holds automatically for $\psi(\omega, \theta)$. To verify Assumption 2.2, one needs to use

$$\frac{\partial \log(\psi(\omega, \theta))}{\partial \theta} = \left( -\frac{2\alpha}{\omega^2 + \alpha^2}, -2\frac{\Gamma'(1+\beta)}{\Gamma(1+\beta)} - \log(w^2), \frac{2}{\sigma} \right)^{\tau},$$

where $\Gamma'(1+\beta) = \beta \int_0^\infty e^{-x} x^{\beta-1} dx$. For this case, Assumption 2.2(i) holds automatically while Assumption 2.2(ii) follows from a simple algebric derivation.

As for model (3.2), $\gamma = 1$ and therefore the parameter space is $\Theta_1$, Theorem 2.1 implies the following theorem for model (3.1).

**Theorem 3.1**. (i) *Let $\theta_{10}$ be the true value of $\theta$. Then as $N \to \infty$*

$$\mathrm{P}\left( \lim_{N\to\infty} \bar{\theta}_N = \theta_{10} \right) = 1.$$

(ii) *In addition, if the true value $\theta_{10}$ of $\theta$ is in the interior of $\Theta_{10}$, then as $N \to \infty$*

$$\sqrt{N}(\bar{\theta}_N - \theta_{10}) \to_D N\left(0, \Sigma^{-1}(\theta_{10})\right),$$

*where*

$$\Sigma(\theta) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \left( \frac{\partial \log(\psi(\omega, \theta))}{\partial \theta} \right) \left( \frac{\partial \log(\psi(\omega, \theta))}{\partial \theta} \right)^{\tau} \frac{1}{(1+\omega^2)^2} d\omega,$$

*in which* $\frac{\partial \log(\psi(\omega,\theta))}{\partial \theta} = \left( -\frac{2\alpha}{\omega^2+\alpha^2}, -2\frac{\Gamma'(1+\beta)}{\Gamma(1+\beta)} - \log(w^2), \frac{2}{\sigma} \right)^{\tau}$.

Model (3.1) and Theorem 3.1 are applicable to pricing some other financial derivatives, such as the S&P 500 stock market daily closing price index, although it is assumed that model (3.1) is a stochastic interest rate model.

In the following, we consider fractional stochastic volatility models and then determine the stochastic volatility models through estimating the parameters involved in the models.

Let $S(t)$ denote the price of the underlying asset and $v(t)$ be its instantaneous volatility. In both theory and practice, the volatility function $v(t)$ is an unobserved process. Thus, the determination of $v(t)$ is of both theoretical and practical interest, in particular for the fractional case. Consider a class of fractional stochastic volatility models of the form (see Comte and Renault 1998)

$$d\ln(S(t)) = v(t)dB(t), \tag{3.3}$$

$$d\ln(v(t)) = -\alpha \ln(v(t))dt + \sigma dB_\beta(t), \tag{3.4}$$

where $\alpha$ and $\beta$ are as defined in (3.1) without introducing any new parameters, $B(t)$ is the standard Brownian motion, and $B_\beta(t)$ is fractional Brownian motion as defined before.

To make sure that the modelling procedure for model (3.1) is applicable to models (3.3) and (3.4), one may consider using the following schemes:

- For real data application, one can use a set of discrete observations of $S(t)$ and a set of simulated values of $B(t)$ to obtain the corresponding observations of $v(t)$, and then estimate the parameters involved in (3.4) based on the observations of $v(t)$ and the fact that the process $X(t) = \ln(v(t))$ has a spectral density of the form (3.2).

- For simulation, one needs to use a set of simulated values of $X(t) = \ln(v(t))$ generated from its solution given by (1.4) with an initial value for $\theta = (\alpha, \beta, \sigma)$, and then estimate the parameters involved in (3.4) based on the simulated values and the fact that the process $X(t) = \ln(v(t))$ has a spectral density of the form (3.2).

In some detailed empirical studies and applications, models (3.1)–(3.4) are being used to check whether there is some kind of long–range dependence and intermittency property in both the interest rate data and the S&P 500 data mentioned above. In addition, the real data sets analysed in Example 3.2 of Gao, Anh, Heyde and Tieng (2001) and in Example 3.1 of Gao, Anh, Heyde (2002) will be reexamined by using model (1.1). The details are very lengthy and will be reported elsewhere.

**Remark 3.2**. As can be seen from Anh, Heyde and Leonenko (2002), general fractional Lévy processes also have applications in finance. Before establishing Theorems 2.1 and 2.2, the author of this paper was trying to establish some corresponding results for stochastic differential equations driven by fractional Lévy motion. However, the author was unable to overcome some theoretical difficulties. These topics are left for future research.

## Appendix

As the proof of Theorem 2.1 is a special case of that of Theorem 2.2, we provide only a detailed proof for Theorem 2.2. For the proof of Theorem 2.1, we give an outline. For the simplicity of notation, we use $\theta_0$ to replace $\theta_{20}$ throughout the proof of Theorem 2.2.

**Proof of Theorem 2.2(i)**. Recall that

$$M_N^Y(\theta) = L_N^Y(\theta) - \lambda g(\theta). \tag{A.1}$$

Note that

$$\frac{\partial M_N^Y(\theta)}{\partial\theta}|_{\theta=\hat{\theta}_N} - \frac{\partial M_N^Y(\theta)}{\partial\theta}|_{\theta=\theta_0} = \left[\frac{\partial^2}{\partial\theta^2}M_N^Y(\theta)|_{\theta=\theta_N^*}\right](\hat{\theta}_N - \theta_0), \tag{A.2}$$

where $||\theta_N^* - \theta_0|| \leq ||\hat{\theta}_N - \theta_0||$. If $\hat{\theta}_N$ lies in the interior of $\Theta_2$, we have $\frac{\partial M_N^Y(\theta)}{\partial\theta}|_{\theta=\hat{\theta}_N} = 0$ with probability one. If $\hat{\theta}_N$ lies on the boundary of $\Theta_2$, then the assumption that $\theta_0$ is in the interior of $\Theta_2$ and the strong consistency of $\hat{\theta}_N$ to $\theta_0$ imply that $\frac{\partial M_N^Y(\theta)}{\partial\theta}|_{\theta=\hat{\theta}_N} = 0$ holds with probability approaching one. Thus for large $N$

$$\frac{\partial M_N^Y(\theta_0)}{\partial\theta} = \frac{\partial L_N^Y(\theta_0)}{\partial\theta} - \lambda\frac{\partial g(\theta_0)}{\partial\theta} = -\frac{\partial^2}{\partial\theta^2}M_N^Y(\theta_N^*)(\hat{\theta}_N - \theta_0) = -\frac{\partial^2}{\partial\theta^2}L_N^Y(\theta_N^*)(\hat{\theta}_N - \theta_0), \tag{A.3}$$

noting that

$$\frac{\partial g(\theta)}{\partial\alpha} = 0, \ \frac{\partial g(\theta)}{\partial\beta} = \frac{\partial g(\theta)}{\partial\gamma} = 1, \text{ and } \frac{\partial g(\theta)}{\partial\sigma} = 0. \tag{A.4}$$

It now follows from (A.2) and (A.4) that

$$\frac{\partial L_N^Y(\theta)}{\partial\theta}|_{\theta=\hat{\theta}_N} - \frac{\partial L_N^Y(\theta)}{\partial\theta}|_{\theta=\theta_0} = \left[\frac{\partial^2}{\partial\theta^2}L_N^Y(\theta)|_{\theta=\theta_N^*}\right](\hat{\theta}_N - \theta_0). \tag{A.5}$$

Let $B(\theta) = \frac{\partial^2}{\partial\theta^2}L_N^Y(\theta)$ and write

$$B(\theta) = \begin{pmatrix} B_1(\theta)^\tau \\ B_2(\theta)^\tau \\ B_3(\theta)^\tau \\ B_4(\theta)^\tau \end{pmatrix} = \begin{pmatrix} b_{11}(\theta) & b_{12}(\theta) & b_{13}(\theta) & b_{14}(\theta) \\ b_{21}(\theta) & b_{22}(\theta) & b_{23}(\theta) & b_{24}(\theta) \\ b_{31}(\theta) & b_{32}(\theta) & b_{33}(\theta) & b_{34}(\theta) \\ b_{41}(\theta) & b_{42}(\theta) & b_{43}(\theta) & b_{44}(\theta) \end{pmatrix}. \tag{A.6}$$

It now follows from (A.5) and (A.6) that

$$\frac{\partial L_N^Y(\hat{\theta}_N)}{\partial\beta} - \frac{\partial L_N^Y(\theta_0)}{\partial\beta} = B_2(\theta_N^*)^\tau(\hat{\theta}_N - \theta_0) = \sum_{i=1}^4 b_{2i}(\theta)\left(\hat{\theta}_{iN} - \theta_{i0}\right) \tag{A.7}$$

and

$$\frac{\partial L_N^Y(\hat{\theta}_N)}{\partial\gamma} - \frac{\partial L_N^Y(\theta_0)}{\partial\gamma} = B_4(\theta_N^*)^\tau(\hat{\theta}_N - \theta_0) = \sum_{i=1}^4 b_{4i}(\theta)\left(\hat{\theta}_{iN} - \theta_{i0}\right), \tag{A.8}$$

where $\hat{\theta}_{iN}$ and $\theta_{i0}$ are the $i$–th elements of $\hat{\theta}_N = (\hat{\theta}_{1N}, \ldots, \hat{\theta}_{4N})^\tau$ and $\theta_0 = (\theta_{10}, \ldots, \theta_{40})^\tau$, respectively.

Using (A.1) and $\frac{\partial M_N^Y(\theta)}{\partial\theta}|_{\theta=\hat{\theta}_N} = 0$, we have for large enough $N$

$$0 = \frac{\partial M_N^Y(\theta)}{\partial\theta}|_{\theta=\hat{\theta}_N} = \frac{\partial L_N^Y(\theta)}{\partial\theta}|_{\theta=\hat{\theta}_N} - \lambda\frac{\partial g(\theta)}{\partial\theta}|_{\theta=\hat{\theta}_N}. \tag{A.9}$$

This implies that for large enough $N$,

$$\lambda \cdot \left(\frac{\partial g(\hat{\theta}_N)}{\partial\theta}\right)^\tau \frac{\partial g(\hat{\theta}_N)}{\partial\theta} = \left(\frac{\partial g(\hat{\theta}_N)}{\partial\theta}\right)^\tau \frac{\partial L_N(\hat{\theta}_N)}{\partial\theta},$$

which, using (A.4), gives

$$\lambda = \frac{1}{2}\left[\frac{\partial L_N(\hat\theta_N)}{\partial\beta} + \frac{\partial L_N(\hat\theta_N)}{\partial\gamma}\right]. \tag{A.10}$$

Equations (A.7)–(A.10) then imply

$$\lambda = \frac{1}{2}\left[\frac{\partial L_N(\theta_0)}{\partial\beta} + \frac{\partial L_N(\theta_0)}{\partial\gamma}\right] + \frac{1}{2}\left[B_2(\theta_N^*)^\tau(\hat\theta_N - \theta_0) + B_4(\theta_N^*)^\tau(\hat\theta_N - \theta_0)\right]. \tag{A.11}$$

Substituting (A.11) into (A.3) given by

$$\frac{\partial L_N^Y(\theta_0)}{\partial\theta} - \lambda\frac{\partial g(\theta_0)}{\partial\theta} = -B(\theta_N^*)(\hat\theta_N - \theta_0),$$

one can obtain

$$A\frac{\partial L_N^Y(\theta_0)}{\partial\theta} = -AB(\theta_N^*)\left(\hat\theta_N - \theta_0\right), \tag{A.12}$$

where $A$ is as defined in Theorem 2.2.

Thus, in order to prove that as $N\to\infty$, $\sqrt{N}(\hat\theta_N - \theta_0) \to_D N\left(0, A\Sigma^{-1}(\theta_0)A\right)$, it suffices to show that as $N\to\infty$

$$\sqrt{N}\frac{\partial L_N^Y(\theta_0)}{\partial\theta} \to N\left(0, \Sigma(\theta_0)\right) \tag{A.13}$$

and

$$B(\theta_N^*) = \frac{\partial^2}{\partial\theta^2}L_N^Y(\theta_N^*) \to \Sigma(\theta_0) \ \text{ with probability one.} \tag{A.14}$$

The proof of (A.14) follows from the strong consistency of $\hat\theta_N$ to $\theta_0$. The proof of (A.13) follows from

$$\sqrt{N}E\left[\frac{\partial L_N^Y(\theta_0)}{\partial\theta}\right] \to 0 \ \text{ as } N\to\infty \tag{A.15}$$

and

$$\sqrt{N}\left(\frac{\partial L_N^Y(\theta_0)}{\partial\theta} - E\left[\frac{\partial L_N^Y(\theta_0)}{\partial\theta}\right]\right) \to_D N\left(0, \Sigma(\theta_0)\right) \ \text{ as } N\to\infty. \tag{A.16}$$

Note that the proof of (A.16) follows from

$$\sqrt{N}\int_{-\infty}^{\infty}\left\{I_N^Y(\omega) - E\left[I_N^Y(\omega)\right]\right\}D(\omega,\theta)d\omega \to_D N(0,\Sigma(\theta_0)), \ \text{as } N\to\infty, \tag{A.17}$$

where $\Sigma(\theta) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{1}{(1+\omega^2)^2}\left\{\frac{\partial}{\partial\theta}\log(\phi(\omega,\theta))\right\}\left\{\frac{\partial}{\partial\theta}\log(\phi(\omega,\theta))\right\}^\tau d\omega$ and

$$D(\omega,\theta) = \frac{-1}{1+\omega^2}\frac{\partial\phi^{-1}(\omega,\theta)}{\partial\theta} = \frac{-1}{1+\omega^2}\left(\frac{\partial\phi^{-1}(\omega,\theta)}{\partial\alpha}, \frac{\partial\phi^{-1}(\omega,\theta)}{\partial\beta}, \frac{\partial\phi^{-1}(\omega,\theta)}{\partial\sigma}, \frac{\partial\phi^{-1}(\omega,\theta)}{\partial\gamma}\right)^\tau.$$

To finish the proof of (A.15), one needs to use the following result

$$E\left[I_N^Y(\omega)\right] = \phi(\omega,\theta_0)\left(1 + O\left(\frac{1}{N}\right)\right). \tag{A.18}$$

The proof of (A.18) is similar to that of (2.9) of Gao, Anh and Heyde (2002).

By using Assumption 2.1(ii) and (A.18), we have

$$\sqrt{N}E\left[\frac{\partial L_N^Y(\theta_0)}{\partial\theta}\right] = \sqrt{N}\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial\theta}\log\left(\phi(\omega,\theta_0)\right) + E\left[I_N^Y(\omega)\right]\cdot\frac{\partial}{\partial\theta}\phi^{-1}(\omega,\theta_0)\right)\frac{d\omega}{1+\omega^2}$$

$$= \sqrt{N} \int_{-\infty}^{\infty} E\left[I_N^Y(\omega)\right] \cdot \phi^{-1}(\omega, \theta_0) \frac{\partial}{\partial \theta} \log\left(\phi(\omega, \theta_0)\right) \frac{d\omega}{1+\omega^2}$$
$$= \sqrt{N}\left(1 + O\left(\frac{1}{N}\right)\right) \cdot \int_{-\infty}^{\infty} \frac{\partial}{\partial \theta} \log\left(\phi(\omega, \theta_0)\right) \frac{d\omega}{1+\omega^2} = 0.$$

The proof of (A.17) is similar to (A.28)–(A.36) of Gao, Anh and Heyde (2002), noting that $D(\omega, \theta)$ involved in (A.17) is now a vector of four components.

**Proof of Theorem 2.2(ii)**: Before proving $\hat{\theta}_N \to \theta_0$ with probability one, we need to show that as $N \to \infty$

$$L_N^Y(\theta) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\{ \log(\phi(\omega, \theta)) + \frac{I_N^Y(\omega)}{\phi(\omega, \theta)} \right\} \frac{d\omega}{1+\omega^2} \to L^Y(\theta)$$
$$\equiv \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\{ \log(\phi(\omega, \theta)) + \frac{\phi(\omega, \theta_0)}{\phi(\omega, \theta)} \right\} \frac{d\omega}{1+\omega^2} \quad (A.19)$$

holds with probability one for $\epsilon < |\lambda| \cdot ||\theta - \theta_0|| < \frac{1}{4}$ with any given $\epsilon > 0$ and the defined multiplier $\lambda$ introduced in (A.1). It is obvious that (A.19) follows from the result that

$$\int_{-\infty}^{\infty} \left\{ \frac{I_N^Y(\omega)}{\phi(\omega, \theta)} \right\} \frac{d\omega}{1+\omega^2} \to \int_{-\infty}^{\infty} \left\{ \frac{\phi(\omega, \theta_0)}{\phi(\omega, \theta)} \right\} \frac{d\omega}{1+\omega^2}$$

holds with probability one. This corresponds to some existing results for the case where the spectral density of long–range dependent Gaussian processes is defined on $[-\pi, \pi]$ (see Lemma 1 of Fox and Taqqu 1986).

As the proof of (A.19) is very similar to that of (A.21) of Gao, Anh and Heyde (2002), we shall not repeat the proof here. It now follows from (A.19) that for $\theta \neq \theta_0$,

$$L_N^Y(\theta) - L_N^Y(\theta_0) \to K(\theta, \theta_0), \quad \text{with probability one}, \quad (A.20)$$

where $K(\theta, \theta_0)$ defined in Assumption 2.3 satisfies

$$K(\theta, \theta_0) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\{ \frac{\phi(\omega, \theta_0)}{\psi(\omega, \theta)} - 1 - \log\left(\frac{\phi(\omega, \theta_0)}{\phi(\omega, \theta)}\right) \right\} \frac{d\omega}{1+\omega^2} > 0 \quad (A.21)$$

using the fact that for $x > 0$ and $x \neq 1$, $x - 1 > \log(x)$.

Thus, for any small enough $\epsilon > 0$, large enough $N$ and $|\lambda| \cdot ||\theta - \theta_0|| > \epsilon$

$$L_N^Y(\theta) - L_N^Y(\theta_0) > 0 \ \text{ with probability one.} \quad (A.22)$$

To finish the proof of Theorem 2.2(ii), in view of (A.19)–(A.22), one needs to show that for any small enough $\epsilon > 0$, large enough $N$ and $|\lambda| \cdot ||\theta - \theta_0|| > \epsilon$

$$K(\theta, \theta_0) > \lambda\left[(\beta - \beta_0) + (\gamma - \gamma_0)\right]. \quad (A.23)$$

The proof of (A.23) follows from the convexity of $K(\theta, \theta_0)$ in $(\beta, \gamma)$ imposed in Assumption 2.3. This is because the Hessian matrix $\mathcal{H}(\theta) = \{h_{ij}(\theta)\}_{\{1 \leq i,j \leq 2\}}$ is positive semi–definite for every $(\beta, \gamma) \in \mathcal{C}(\beta_0, \gamma_0)$, an open convex set containing the pair $(\beta_0, \gamma_0)$, where

$$h_{11}(\theta) = \frac{\partial^2}{\partial \beta^2} K(\theta, \theta_0), \ h_{12}(\theta) = h_{21}(\theta) = \frac{\partial^2}{\partial \beta \partial \gamma} K(\theta, \theta_0), \ \text{and } h_{22}(\theta) = \frac{\partial^2}{\partial \gamma^2} K(\theta, \theta_0).$$

See Theorem 4.5 of Rockafeller (1970). This finally completes the proof of Theorem 2.2(ii).

**Proof of Theorem 2.1**: The proof of Theorem 2.1(i) follows from (A.20), (A.21) and

$$\lim_{N \to \infty} \inf_{||\theta - \theta_0|| > \delta} \left[ L_N^Y(\theta) - L_N^Y(\theta_0) \right] > 0 \ \text{ with probability one for any given } \delta > 0.$$

As pointed out earlier, Theorem 2.1(ii) corresponds to the case of $\lambda = 0$. Thus, the proof of Theorem 2.1(ii) follows from (A.16). This finishes the proof of Theorem 2.1.

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