Central limit theorems for weighted quadratic forms of dependent processes with applications in specification testing

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Central Limit Theorems for Generalized U–Statistics with Applications in Econometric Specification [1]

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Abstract

In this paper, we establish some new central limit theorems for generalized U–statistics of dependent processes under some mild conditions. Such central limit theorems complement existing results available from both the econometrics literature and statistics literature. We then look at applications of the established results to a number of test problems in time series econometric models.

1. Introduction

The study of central limit theorems for random quadratic forms has a long history. For example, Hall (1984), De Jong (1987, 1990), and Fan and Li (1996) establish central limit theorems of U–statistics for the case where the random variables involved are independent. Those results have been employed quite heavily for various specification tests, such as Hong and White (1995). For the case where dependent time series are involved, existing results include Yoshihara (1976, 1989), Hjellvik, Yao and Tjøstheim (1996), Tenreiro (1997), and Fan and Li (1999) for stationary and absolutely regular processes. Along with the paper by Li (1999), the last two papers also discuss several applications of the established central limit theorems for testing independence, linearity and nonparametric significance for time series data. Recently, Gao and Anh (2000) establish a central limit theorem for a randomly quadratic form of strictly stationary mixing processes. The

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result has been applied for specification testing in nonparametric series regression. More recently, Gao and King (2004) establish some general results for such quadratic forms of strictly stationary α–mixing processes before applying them for specification testing in continuous–time diffusion models. In this paper we extend the existing results to a more general setting and then discuss several applications in specification testing problems.

Let $\{X_t : t \geq 1\}$ be a r–dimensional strictly stationary β–mixing time series data and define the following U–statistic

$$L_{0T} = \sum_{s=1}^{T} \sum_{t=1}^{T} a_{st} \phi_1(X_s, X_t), \tag{1.1}$$

where $\{a_{st}\}$ is a sequence of non–random real numbers possibly depending on T, $\phi_1(x_1, x_2)$ is symmetric function of (x_1, x_2) defined on $R^r \times R^r$, and T is the size of the time series data. Existing results are applicable to the form (1.1). In many other test problems in time series specification, however, we need to deal with the case where $\{a_{st}\}$ may also depend on the history of (X_s, X_t), such as (X_{s-u}, X_{t-u}) for $1 \leq u \leq \min(s-1, t-1)$. For example, $a_{st} = \sum_{u=1}^{\min(s-1,t-1)} A_{Tu}\ \phi_2(X_{s-u}, X_{t-u})$, where $\{A_{Tu}\}$ is a sequence of non–random real numbers and $\phi_2(\cdot, \cdot)$ is also a symmetric measurable function over $R^r \times R^r$. This motivates us to consider a generalized U–statistic of the form

$$L_T = \sum_{s=1}^{T} \sum_{t=1}^{T} \psi_T(Z_s, Z_t) \phi_1(X_s, X_t), \tag{1.2}$$

where $Z_t = (X_{t-1}, \cdots, X_1)$ and $\psi_T(Z_s, Z_t) = \sum_{u=1}^{\min(s-1,t-1)} A_{Tu}\ \phi_2(X_{s-u}, X_{t-u})$ with $\phi_2(x_1, x_2)$ being also a symmetric function of (x_1, x_2). As can be seen, L_{0T} defined in (1.1) is a special case of L_T defined in (1.2) where $\phi_2(\cdot, \cdot)$ is just a sequence of non–random real numbers.

In this paper we will then establish new central limit theorems for L_T in Section 2. The proofs of the established theorems are given in Section 3. Section 4 concludes the paper with comments on possible extensions.

2. Central Limit Theorems

Before we establish our new central limit theorems, we provide an example to motivate the proposal of such new central limit theorems.

EXAMPLE 2.1. Assume that $\{X_t : t = 1, \cdots, T\}$ is a sequence of stationary time series data with $E[X_t] = 0$, auto–covariance function $\rho(j)$, and normalized spectral density function

$$f(\omega) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \rho(j) \cos(j\omega), \ \omega \in [-\pi, \pi]. \tag{2.1}$$

To test the independence of $\{X_t\}$, we are interested in testing

$$H_{01}: \ \rho(j) = 0 \ \text{ for all } j \neq 0 \text{ versus } \ H_{11} : \rho(j) \neq 0 \ \text{ for some } j \neq 0. \tag{2.2}$$

It follows from (2.1) that testing H_{01} is equivalent to testing $f(\omega) = f_0(\omega) = \frac{1}{2\pi}$ for $\omega \in [-\pi, \pi]$. Since $f(\cdot)$ is unknown, we estimate it by

$$\widehat{f}_T(\omega) = \frac{1}{2\pi} \sum_{j=-T+1}^{T-1} K\left(\frac{j}{p}\right) \widehat{\rho}(j) \cos(j \ \omega), \ \omega \in [-\pi, \pi] \tag{2.3}$$

with $\widehat{\rho}(j) = \frac{\sum_{t=|j|+1}^{T} X_t X_{t-|j|}}{\sum_{t=1}^{T} X_t^2}$, where $p = p(T)$ is the bandwidth satisfying $\lim_{T\to\infty} p(T) = \infty$ and $\lim_{T\to\infty} \frac{p(T)}{T} = 0$, and $K(\cdot)$ is a probability kernel function.

In order to test H_{01}, we thus suggest using a test statistic of the form

$$Q(\widehat{f}, f_0) = 2\pi \int_{-\pi}^{\pi} \left(\widehat{f}(\omega) - f_0(\omega)\right)^2 d\omega. \tag{2.4}$$

It may be shown that the leading term of $Q(\widehat{f}, f_0)$ is then as follows:

$$M_T = 2 \sum_{j=1}^{T} K^2\left(\frac{j}{p}\right) \ \widehat{\rho}^2(j) \equiv \sum_{s=2}^{T} \sum_{t=2}^{T} \psi_T(Z_s, Z_t) \ X_s X_t, \tag{2.5}$$

where $\psi_T(Z_s, Z_t) = \sum_{|j|=1}^{\min(s-1,t-1)} \frac{2}{T^2} K^2\left(\frac{j}{p}\right) X_{s-|j|} \ X_{t-|j|}$.

As can be seen from (2.5), M_T is a type of generalized U–statistic with stochastic coefficients $\{\psi_T(Z_s, Z_t)\}$. Hence, the existing results available for deterministic coefficients are not applicable. In addition, it is obvious that M_T of (2.5) is a special case of L_T of

(1.2). We therefore believe that it is of general interest to establish new asymptotic distributions for L_T.

Let $\{X_t : t \geq 1\}$ be a strictly stationary time series. Assume that X_t is absolutely regular (β–mixing) with mixing coefficient $\beta(t) \leq C_\beta \, \rho^t$ defined by

$$\beta(t) = \sup_{s \geq 1} E\left[\sup_{A \in A \in I_{s+t}^{\infty}} |P(A|I_1^s) - P(A)|\right],$$

where $0 < C_\beta < \infty$ and $0 < \rho < 1$ are constants, and I_i^j denotes the σ-field generated by $\{X_t : i \leq t \leq j\}$. For $i \geq 1$, let $I_i = I_1^i$. Let A_{Tu} be a sequence of positive non–random weight functions, $Z_t = (X_{t-1}, \cdots, X_1)$, and define $\psi_T(Z_s, Z_t) = \sum_{u=1}^{\min(s-1,t-1)} A_{Tu} \; \phi_2(X_{s-u}, X_{t-u})$, where $\phi_2(\cdot, \cdot)$ is a symmetric measurable function defined on $R^r \times R^r$. Let $\phi_1(\cdot, \cdot)$ be also a symmetric measurable function defined on $R^r \times R^r$. For $1 \leq s < t \leq T$, define $\theta_{st}(u) = \phi_1(X_s, X_t)\phi_2(X_{s-u}, X_{t-u})$ and $\theta_{st} = \sum_{u=1}^{s-1} A_{Tu}\theta_{st}(u)$. Let

$$L_T = \sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{u=1}^{s-1} A_{Tu}\phi_1(X_s, X_t)\phi_2(X_{s-u}, X_{t-u}) = \sum_{t=3}^{T}\sum_{s=2}^{t-1}\theta_{st} \tag{2.6}$$

with $E[L_T] = 0$ and $\sigma_T^2 = \sum_{t=3}^{T}\sum_{s=2}^{t-1} \mathrm{var}[\theta_{st}]$.

In what follows we will establish a new central limit theorem for L_T. As can be seen, such a central limit theorem covers existing cases (see Yoshihara 1989), including the case where $\psi_T(\cdot, \cdot)$ is a non–random symmetric function of s and t.

THEOREM 2.1. *Let $\{X_t : 1 \leq t \leq T\}$ be a r–dimensional strictly stationary and absolutely regular (β–mixing) time series. Let $\phi_i(\cdot, \cdot)$ be symmetric Borel functions defined on $R^r \times R^r$ for $i = 1, 2$. Assume that for any fixed $x, z \in R^r$, $t \geq 1$ and $1 \leq u \leq t-1$, $E[\phi_1(x, X_t)] = 0 = E[\phi_2(z, X_{t-u})]$. For any $1 \leq u \leq s-1$, let $\xi_s(u) = (X_s, X_{s-u})$. For $1 \leq i < j < k < l \leq T$ and $1 \leq u \leq T-1$, let us now define $P_4(\xi_i, \xi_j, \xi_k, \xi_l)$, $P_3(\xi_i, \xi_j, \xi_k)$, $P_2(\xi_i, \xi_j)$, and $P_1(\xi_i)$ as the probability measures of $(\xi_i(u), \xi_j(u), \xi_k(u), \xi_l(u))$, $(\xi_i(u), \xi_j(u), \xi_k(u))$, $(\xi_i(u), \xi_j(u))$ and $\xi_i(u)$, respectively. For some small constant $0 < \delta < 1$, let*

$$M_{T1} = \max_{1 \leq u_1, u_2 \leq T-1,\, 1 \leq i < j < k \leq T} \max\left\{E\left[|\theta_{ik}(u_1)\theta_{jk}(u_2)|^{1+\delta}\right], \int |\theta_{ik}(u_1)\theta_{jk}(u_2)|^{1+\delta} dP_1(\xi_i) dP_2(\xi_j, \xi_k)\right\},$$

$$
\begin{aligned}
M_{T21} &= \max_{1\le u_1,u_2\le T-1,\ 1\le i<j<k\le T} \max\left\{E\left[|\theta_{ik}(u_1)\theta_{jk}(u_2)|^{2(1+\delta)}\right]\right\},\\
M_{T22} &= \max_{1\le u_1,u_2\le T-1,\ 1\le i<j<k\le T} \max\left\{\int |\theta_{ik}(u_1)\theta_{jk}(u_2)|^{2(1+\delta)} dP_1(\xi_i)dP_2(\xi_j,\xi_k)\right\},\\
M_{T23} &= \max_{1\le u_1,u_2\le T-1,\ 1\le i<j<k\le T} \max\left\{\int |\theta_{ik}(u_1)\theta_{jk}(u_2)|^{2(1+\delta)} dP_1(\xi_i)dP_1(\xi_j)dP_1(\xi_k)\right\}\\
M_{T2} &= \max\left\{M_{T21}, M_{T22}, M_{T23}\right\},\\
M_{T3} &= \max_{1\le u_1,u_2\le T-1,\ 1\le i<j<k\le T} E\left[|\theta_{ik}(u_1)\theta_{jk}(u_2)|^{2}\right],\\
M_{T4} &= \max_{1\le u_1,u_2\le T-1,} \max_{\substack{1\le i,j,k,l\le 2T\\ i,j,k,l \ \textit{different}}} \left\{\max_{P_u}\int |\theta_{ij}(u_1)\theta_{kl}(u_2)|^{2(1+\delta)} dP_u\right\},
\end{aligned}
$$

where the maximization over P_u in the equation for M_{T4} is taken over the four probability measures $P_4(\xi_i,\xi_j,\xi_k,\xi_l)$, $P_1(\xi_i)P_3(\xi_i,\xi_j,\xi_k)$, $P_1(\xi_i)P_1(\xi_j)P_2(\xi_k,\xi_l)$, and $P_1(\xi_i)P_1(\xi_j)P_1(\xi_k)P_1(\xi_l)$;

$$
\begin{aligned}
M_{T51} &= \max_{1\le u_1,u_2\le T-1,\ 1\le i<j<k\le T} \max\left\{E\left[\left|\int \theta_{ik}(u_1)\theta_{jk}(u_2)dP_1(\xi_i)\right|^{2(1+\delta)}\right]\right\},\\
M_{T52} &= \max_{1\le u_1,u_2\le T-1,\ 1\le i<j<k\le T} \max\left\{\int\left|\int \theta_{ik}(u_1)\theta_{jk}(u_2)dP_1(\xi_i)\right|^{2(1+\delta)} dP_1(\xi_j)dP_1(\xi_k)\right\},\\
M_{T5} &= \max\left\{M_{T51}, M_{T52}\right\},\\
M_{T6} &= \max_{1\le u_1,u_2\le T-1,\ 1\le i<j<k\le T} E\left[\left|\int \theta_{ik}(u_1)\theta_{jk}(u_2)dP_1(\xi_i)\right|^{2}\right],\\
M_{T7} &= \max_{1\le u\le T-1,\ 1\le i<j\le T} E\left[|\theta_{ij}(u)|^{2(1+\delta)}\right].
\end{aligned}
$$

In addition, suppose that $\sum_{u=1}^{T-1} A_{Tu} < \infty$. Let

$$
\begin{aligned}
M_T &= \max\left\{T^2 M_{T1}^{\frac{1}{1+\delta}}, T^2 M_{T5}^{\frac{1}{2(1+\delta)}}, T^2 M_{T6}^{\frac{1}{2}}, T^2 M_{T7}^{\frac{1}{1+\delta}}\right\} \quad \textit{and}\\
N_T &= \max\left\{T^{\frac{3}{2}} M_{T2}^{\frac{1}{2(1+\delta)}}, T^{\frac{3}{2}} M_{T3}^{\frac{1}{2}}, T^{\frac{3}{2}} M_{T4}^{\frac{1}{2(1+\delta)}}\right\}. \qquad (2.7)
\end{aligned}
$$

If $\lim_{T\to\infty} \frac{\max\{M_T,N_T\}}{\sigma_T^2} = 0$, then

$$
\frac{1}{\sigma_T}\sum_{1\le s<t\le T} \theta_{st} \to_D N(0,1) \quad \text{as} \quad T\to\infty.
$$

Let $\psi_T(Z_s,Z_t) \equiv 1$ and $\xi_t = X_t$, Theorem 2.1 reduces to the following corollary. Its proof follows from that of Theorem 2.1.

Corollary 2.1. *Let $\{\xi_t : 1 \le t \le T\}$ be a r–dimensional strictly stationary and β–mixing time series. Let $\phi_1(\cdot,\cdot)$ be a symmetric Borel function defined on $R^r \times R^r$. Assume that for any fixed $x, y \in R^r$, $E[\phi_1(\xi_1, y)] = E[\phi_1(x, \xi_1)] = 0$. Let $\theta_{st} = \phi_1(\xi_s, \xi_t)$ with*

$$E\left[\theta_{st}\right] = 0 \quad \textit{and} \quad \sigma_{0T}^2 = \sum_{1\le s<t\le T} \mathrm{var}[\theta_{st}].$$

For some small constant $0 < \delta < 1$, let

$$\begin{aligned}
M_{T11} &= \max_{1\le i<j<k\le T} \max\left\{E|\theta_{ik}\theta_{jk}|^{1+\delta}\right\},\\
M_{T12} &= \max_{1\le i<j<k\le T} \max\left\{\int |\theta_{ik}\theta_{jk}|^{1+\delta} dP(\xi_i) dP(\xi_j, \xi_k)\right\},\\
M_{T21} &= \max_{1\le i<j<k\le T} \max\left\{E|\theta_{ik}\theta_{jk}|^{2(1+\delta)}\right\},\\
M_{T22} &= \max_{1\le i<j<k\le T} \max\left\{\int |\theta_{ik}\theta_{jk}|^{2(1+\delta)} dP(\xi_i) dP(\xi_j, \xi_k)\right\},\\
M_{T23} &= \max_{1\le i<j<k\le T} \max\left\{\int |\theta_{ik}\theta_{jk}|^{2(1+\delta)} dP(\xi_i, \xi_j) dP(\xi_k)\right\},\\
M_{T24} &= \max_{1\le i<j<k\le T} \max\left\{\int |\theta_{ik}\theta_{jk}|^{2(1+\delta)} dP(\xi_i) dP(\xi_j) dP(\xi_k)\right\},\\
M_{T3} &= \max_{1\le i<j<k\le T} E|\theta_{ik}\theta_{jk}|^2,\\
M_{T4} &= \max_{\substack{1 < i,j,k \le 2T \\ i,j,k \text{ different}}} \left\{\max_P \int |\theta_{1i}\theta_{jk}|^{2(1+\delta)} dP\right\},
\end{aligned}$$

where the maximization over P in the equation for M_{T4} is taken over the probability measures $P(\xi_1, \xi_i, \xi_j, \xi_k)$, $P(\xi_1)P(\xi_i, \xi_j, \xi_k)$, $P(\xi_1)P(\xi_{i_1})P(\xi_{i_2}, \xi_{i_3})$, and $P(\xi_1)P(\xi_i)P(\xi_j)P(\xi_k)$, where (i_1, i_2, i_3) is the permutation of (i, j, k) in ascending order;

$$\begin{aligned}
M_{T51} &= \max_{1\le i<j<k\le T} \max\left\{E\left|\int \theta_{ik}\theta_{jk}\theta_{ik}\theta_{jk} dP(\xi_i)\right|^{2(1+\delta)}\right\},\\
M_{T52} &= \max_{1\le i<j<k\le T} \max\left\{\int \left|\int \theta_{ik}\theta_{jk}\theta_{ik}\theta_{jk} dP(\xi_i)\right|^{2(1+\delta)} dP(\xi_j) dP(\xi_k)\right\},\\
M_{T6} &= \max_{1\le i<j<k\le T} E\left|\int \theta_{ik}\theta_{jk} dP(\xi_i)\right|^2, \quad M_{T7} = \max_{1\le i<j<T} E\left[|\theta_{ij}|^{2(1+\delta)}\right].
\end{aligned}$$

Let

$$M_{T1} = \max_{1\le i\le 2}\{M_{T1i}\}, \; M_{T2} = \max_{1\le i\le 4}\{M_{T2i}\}, \; M_{T5} = \max_{1\le i\le 2}\{M_{T5i}\}.$$

Assume that all the M_{Ti} are finite. Let

$$\begin{aligned} M_T &= \max\left\{T^2 M_{T1}^{\frac{1}{1+\delta}}, T^2 M_{T5}^{\frac{1}{2(1+\delta)}}, T^2 M_{T6}^{\frac{1}{2}}, T^2 M_{T7}^{\frac{1}{1+\delta}}\right\}, \\ N_T &= \max\left\{T^{\frac{3}{2}} M_{T2}^{\frac{1}{2(1+\delta)}}, T^{\frac{3}{2}} M_{T3}^{\frac{1}{2}}, T^{\frac{3}{2}} M_{T4}^{\frac{1}{2(1+\delta)}}\right\}. \end{aligned}$$

If $\lim_{T\to\infty} \frac{\max\{M_T, N_T\}}{\sigma_{0T}^2} = 0$, *then*

$$\frac{1}{\sigma_{0T}} \sum_{1\le s<t\le T} \phi_1(\xi_s, \xi_t) \to_D N(0,1) \text{ as } T\to\infty.$$

Corollary 2.1 improves some corresponding results of Hjellvik, Yao and Tjøstheim (1996), and Fan and Li (1999) for the β–mixing case by avoiding using the martingale difference condition: $E[\phi_1(X_i, X_j)\Omega_0^{j-1}] = 0$ for any $i < j$, where Ω_i^j denotes the σ–field generated by $\{X_s : i \le s \le j\}$. As discussed in Section 3 below, the replacement of the martingale condition would make Corollary 2.1 directly applicable to establish asymptotically normal tests for density specification.

Before we prove Theorem 2.1 in Section 4 below, we explain why the conditions of Theorem 2.1 are justifiable in Section 3.

3. Examples and Applications

Example 3.1. Consider a time series regression model of the form

$$Y_t = g(U_t) + e_t, \tag{3.1}$$

where $\{e_t\}$ is a sequence of martingale differences, $\{U_t\}$ is a strictly stationary time series, and $g(\cdot)$ is a smooth but unknown function defined over R^d. In the literature of time series econometric specification testing, focus has been on the construction of various tests for testing whether $g(\cdot)$ can be specified parametrically. In general, the leading term of such a test is a U–statistic. When using a kernel function based test, we may have a test statistic of the form (2.6) with $X_t = (U_t, e_t)$ (see Hong and Kao 2004; Hong and Lee 2005),

$$\phi_1(X_s, X_t) = e_s \ K\left(\frac{U_s - U_t}{h}\right) \ e_t \quad \text{and} \quad \phi_2(X_{s-u}, X_{t-u}) = \phi_2(U_{s-u}, U_{t-u}) \tag{3.2}$$

for $u = 1, \cdots, \min(s-1, t-1)$, where $K(\cdot)$ is a probability kernel function, h is a bandwidth parameter satisfying certain conditions, and $\phi_2(\cdot, \cdot)$ is a bounded function. In addition, the non–random weight function may be chosen as $A_{Tu} = W\left(\frac{u}{T}\right)$ where $W(\cdot)$ is a smooth function satisfying $\int W^2(x)dx < \infty$. Under certain conditions on $\{e_t\}$, we may verify that the conditions of Theorem 2.1 are all satisfied. In detail, we can verify one part of (2.7) as follows.

Let M_T and N_T be defined as in Theorem 2.1. We now verify only the following condition

$$\frac{T^2 M_{T1}^{\frac{1}{1+\delta}}}{\sigma_T^2} \to 0 \ \text{ as } T \to \infty. \tag{3.3}$$

The others follow similarly.

Let $a_{st} = K\left(\frac{U_s - U_t}{h}\right)$, $b_{st}(u) = \phi_2(X_{s-u}, X_{t-u}) = \phi_2(U_{s-u}, U_{t-u})$ and $\psi_{st}(u) = a_{st} b_{st}(u)$. It follows that for some $0 < \delta < 1$, $1 \le i < j < k \le T$ and $1 \le u_1, u_2 \le T$,

$$E\left[|\psi_{ik}(u)\psi_{jk}(u)|^{1+\delta}\right] = E\left[|e_i e_j e_k^2 a_{ik} b_{ik}(u_1) a_{jk} b_{jk}(u_2)|^{1+\delta}\right]$$

$$\begin{aligned} &\le \left\{E\left[|e_i e_j e_k^2|^{2(1+\delta)(1+\delta_2)}\right]\right\}^{\frac{1}{2(1+\delta_2)}} \left\{E\left[|a_{ij} a_{ik} b_{ik}(u_1) b_{jk}(u_2)|^{(1+\delta)(1+\delta_1)}\right]\right\}^{\frac{1}{(1+\delta_1)}} \\ &\le C_1 \left\{E\left[|a_{ij} a_{ik}|^{(1+\delta)(1+\delta_1)}\right]\right\}^{\frac{1}{(1+\delta_1)}}, \end{aligned} \tag{3.4}$$

assuming the boundedness of $\phi_2(\cdot, \cdot)$ and $\left\{E\left[|e_i e_j e_k^2|^{2(1+\delta)(1+\delta_2)}\right]\right\}^{\frac{1}{2(1+\delta_2)}}$, where $C_1 > 0$ is a constant, $0 < \delta_1 < 1$ and $0 < \delta_2 < 1$ are chosen such that $\frac{1}{1+\delta_1} + \frac{1}{2(1+\delta_2)} = 1$ and $\frac{1+\delta}{3-\delta} < \delta_1 < \frac{1-\delta}{1+\delta}$. We therefore have that

$$1 < \zeta_1 = (1+\delta)(1+\delta_2) < 2 \ \text{ and } \ 1 < \zeta_2 = (1+\delta)(1+\delta_1) < 2.$$

For convenience, we use $\zeta = \zeta_2$ and ignore the small order $o(1)$ throughout the rest of verification. For the given $1 < \zeta < 2$ and T sufficiently large, we obtain

$$\begin{aligned} M_{T11} &= E\left|a_{ik}a_{jk}\right|^{\zeta} \\ &= \int\int\int \left|K\left(\frac{u-w}{h}\right)\right|^{\zeta}\left|K\left(\frac{v-w}{h}\right)\right|^{\zeta} f(u,v,w)dudvdw \\ &= h^{2d}\int\int\int |K(x)K(y)|^{\zeta} f(z+xh, z+yh, z)dxdydz \\ &= C_2 h^{2d}, \end{aligned} \tag{3.5}$$

under certain conditions on $K(\cdot)$, where $f(x,y,z)$ is the joint density function of (U_i, U_j, U_k) and C_2 is a constant.

Similarly, we may show that as $T \to \infty$

$$\sigma_T^2 = \sum_{t=3}^{T}\sum_{s=2}^{t-1} \mathrm{var}[\theta_{st}] = C_3 T^2 h^d. \tag{3.6}$$

where $C_3 > 0$ is a constant.

Thus, as $T \to \infty$

$$\frac{T^2 M_{T11}^{\frac{1}{1+\delta}}}{\sigma_T^2} = C_4 \frac{T^2\left(h^{2d}\right)^{1/\zeta}}{T^2 h^d} = h^{\frac{(2-\zeta)d}{\zeta}} \to 0. \tag{3.7}$$

Hence, equations (3.4)–(3.7) show that (3.3) holds for the first part of M_{T1}. The proof for the second part of M_{T1} follows similarly.

This shows that the conditions of Theorem 2.1 are verifiable.

Example 3.2. Let $\{X_t\}$ be a sequence of strictly stationary time series with the marginal density function being given by $\pi(\cdot)$. Our interest in this example is to test whether there a parametric density function $\pi(x, \theta_0)$ indexed by θ_0 such that

$$H_{02}: \ \pi(x) = \pi(x, \theta_0) \quad \text{versus} \quad H_{12}: \ \pi(x) = \pi_1(x, \theta_1) \tag{3.8}$$

for all x and some $\theta_0 \in \Theta_0$, where $\pi_1(x, \theta_1)$ is another parametric density function indexed by $\theta_1 \in \Theta_1$, and both Θ_0 and Θ_1 are parameter spaces.

Let $X_1, \cdots, X_T$ be the observations. Similarly to Gao and King (2004), we propose using a test statistic of the form

$$\widehat{N}_T = \widehat{N}_T(h) = Th \int \left(\widehat{\pi}(x) - \widetilde{\pi}(x, \widetilde{\theta})\right)^2 \widehat{\pi}(x) dx, \tag{3.9}$$

where

$$\widehat{\pi}(x) = \frac{1}{Th} \sum_{t=1}^{T} K\left(\frac{x - X_t}{h}\right) \quad \text{and} \quad \widetilde{\pi}(x, \widehat{\theta}) = \sum_{t=1}^{T} w_t(x) \pi(X_t, \widetilde{\theta}), \tag{3.10}$$

where $K(\cdot)$ is the probability kernel function, h is the bandwidth parameter, $\widetilde{\theta}$ is an $\sqrt{T}$–consistent estimator of θ_0, and

$$w_t(x) = w_t(x, h) = \frac{1}{Th} K\left(\frac{x - X_t}{h}\right) \frac{s_2(x) - s_1(x)(x - X_t)}{s_2(x)s_0(x) - s_1^2(x)}, \tag{3.11}$$

in which $s_r(x) = \frac{1}{Th} \sum_{s=1}^{T} K\left(\frac{x - X_t}{h}\right) (x - X_s)^r$ for $r = 0, 1, 2$.

In oder to continue our discussion, we introduce the following notation:

$$\begin{aligned}
\epsilon_t(x) &= K\left(\frac{x - X_t}{h}\right) - E\left[K\left(\frac{x - X_t}{h}\right)\right], \\
\theta_{st} &= \theta(X_s, X_t) = (Th)^{-1} \int \epsilon_s(u) \epsilon_t(u) \pi(u) \, du, \\
N_{0T} &= N_{0T}(h) = \sum_{s=1}^{T} \sum_{t=1}^{T} \theta_{st}.
\end{aligned}$$

It can be easily shown that for any $x, y \in R^1 = (-\infty, \infty)$,

$$E\left[\theta(x, X_t)\right] = E\left[\theta(X_s, y)\right] = 0 \tag{3.12}$$

while the martingale condition is not satisfied. It may also be shown that $N_{0T}(h)$ is the leading term of $\widehat{N}_T(h)$. Thus, the asymptotic normality of a suitably normalized version of $\widehat{N}_T(h)$ follows from an application of Corollary 2.1.

We would like to point out that Corollary 2.1 is also applicable for establishing asymptotical distributions for other nonparametric kernel tests, such as the ones in Hong and White (2005).

4. Proofs

The following two technical lemmas have already been used in the proof of Theorem 2.1. The two lemmas are of general interest and can be used for other nonparametric estimation and testing problems associated with the β–mixing condition.

Lemma 4.1. *Suppose that I_m^n are the σ-fields generated by a stationary β-mixing process ξ_i with mixing coefficient $\beta(i)$. For some positive integers m let $\eta_i \in I_{s_i}^{t_i}$ where $s_1 < t_1 < s_2 < t_2 < \cdots < t_m$ and suppose $t_i - s_i > \tau$ for all i. Assume further that*

$$||\eta_i||_{p_i}^{p_i} = E|\eta_i|^{p_i} < \infty,$$

for some $p_i > 1$ for which

$$Q = \sum_{i=1}^{l} \frac{1}{p_i} < 1.$$

Then

$$\left| E\left[\prod_{i=1}^{l} \eta_i\right] - \prod_{i=1}^{l} E[\eta_i] \right| \leq 10(l-1)\alpha(\tau)^{(1-Q)} \prod_{i=1}^{l} ||\eta_i||_{p_i}.$$

Proof. See Theorem 5.4 of Roussas and Ionnides (1987).

Lemma 4.2. (i) *Let $\psi(\cdot,\cdot,\cdot)$ be a symmetric Borel function defined on $R^r \times R^r \times R^r$. Let the process ξ_i be defined as in Lemma 3.1. Assume that for any fixed $x, y \in R^r$, $E[\psi(\xi_1,\xi_2,\xi_3)] = E[\psi(\xi_1,x,y)] = 0$. Then*

$$E\left\{\sum_{1\leq i<j<k\leq T} \psi(\xi_i,\xi_j,\xi_k)\right\}^2 \leq CT^3 M^{\frac{1}{1+\delta}},$$

where $0 < \delta < 1$ is a small constant, $C > 0$ is a constant independent of T and the function ψ, $M = \max\{M_1, M_2, M_3\}$, and

$$M_1 = \max_{1<i<j\leq T} \max\left\{E|\psi(\xi_1,\xi_i,\xi_j)|^{2(1+\delta)}, \int |\psi(\xi_1,\xi_i,\xi_j)|^{2(1+\delta)} dP(\xi_1)dP(\xi_i,\xi_j)\right\},$$

$$M_2 = \max_{1<i<j\leq T} \max\left\{\int |\psi(\xi_1,\xi_i,\xi_j)|^{2(1+\delta)} dP(\xi_j)dP(\xi_1,\xi_i)\right\},$$

$$M_3 = \max_{1<i<j\leq T} \max\left\{\int |\psi(\xi_1,\xi_i,\xi_j)|^{2(1+\delta)} dP(\xi_1)dP(\xi_i)dP(\xi_j)\right\}.$$

(ii) *Let* $\phi(\cdot,\cdot)$ *be a symmetric Borel function defined on* $R^r \times R^r$. *Let the process* ξ_i *be defined as in Lemma 4.1. Assume that for any fixed* $x \in R^r$, $E[\psi(\xi_1,\xi_2)] = E[\phi(\xi_1,x)] = 0$. *Then*

$$E\left\{\sum_{1\le i<j\le T} \phi(\xi_i,\xi_j)\right\}^2 \le CT^2 M_4^{\frac{1}{1+\delta}},$$

where $\delta > 0$ *is a constant,* $C > 0$ *is a constant independent of* T *and the function* ϕ, *and*

$$M_4 = \max_{1<i<j\le T} \max\left\{E|\phi(\xi_1,\xi_i)|^{2(1+\delta)}, \int |\phi(\xi_1,\xi_i)|^{2(1+\delta)} dP(\xi_1)dP(\xi_i)\right\}.$$

Proof: See Lemma C.2 of Gao and King (2004).

Lemma 4.3. *Let* $\phi(\cdot,\cdot)$ *be a symmetric Borel function defined on* $R^r \times R^r$. *Let the process* ξ_i *be defined as in Lemma 4.1. Assume that for any fixed* $x \in R^r$ *and* $j \ge 1$, $E[\psi(\xi_1,\xi_2)] = E[\phi(x,\xi_j)] = 0$. *Then for* $1 \le i < j \le T$,

$$|E\left[\phi(\xi_i,\xi_j)|I_i\right]| \le C\beta^{\frac{\delta}{1+\delta}}(j-i)\ \left(E\left[|\phi(\xi_i,\xi_j)|^{1+\delta}\right]\right)^{\frac{1}{1+\delta}},$$

where $0 < \delta < 1$ *is some constant such that* $\max_{1\le i<j\le T} E\left[|\phi(\xi_i,\xi_j)|^{1+\delta}\right] < \infty$.

Proof: See Yoshihara (1989) or Theorem 5.5 of Roussas and Ionnides (1987).

Proof of Theorem 2.1: For simplicity, we denote A_{Tu} by A_u throughout the proof. Let I_t be a σ–field generated by $\{X_s : 1 \le s \le t\}$. For a given constant $0 < \rho_0 \le \frac{1}{4}$, choose $q = [T^{\rho_0}]$ as the largest integer part of T^{ρ_0}. Obviously, $\sum_{T=1}^{\infty} e^{-d_0 q_T} < \infty$ for any given $d_0 > 0$. Recall the notation of $\theta_{st}(u) = \phi_1(X_s,X_t)\phi_2(X_{s-u},X_{t-u})$ and define

$$\phi_{st}(u) = \theta_{st}(u) - E\left[\theta_{st}(u)|I_{t-q}\right] \quad \text{and} \quad \psi_{st}(u) = E\left[\theta_{st}(u)|I_{t-q}\right]. \tag{4.1}$$

Observe that

$$\begin{aligned} L_T &= \sum_{t=3}^{T}\sum_{s=2}^{t-1}\sum_{u=1}^{s-1} A_u \phi_1(X_s,X_t)\phi_2(X_{s-u},X_{t-u}) \\ &= \sum_{t=q+3}^{T}\sum_{s=2}^{t-q}\sum_{u=1}^{s-1} A_u \phi_{st}(u) + \sum_{t=q+3}^{T}\sum_{s=2}^{t-q}\sum_{u=1}^{s-1} A_u \psi_{st}(u) \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{t=3}^{T} \sum_{s=t+1-q}^{t-1} \sum_{u=1}^{s-1} A_u \phi_{st}(u) + \sum_{t=3}^{T} \sum_{s=t+1-q}^{t-1} \sum_{u=1}^{s-1} A_u \psi_{st}(u) \\
&\equiv \sum_{j=1}^{4} L_{jT}. \qquad (4.2)
\end{aligned}
$$

To establish the asymptotic distribution of L_T, it suffices to show that as $T \to \infty$

$$
\frac{L_{1T}}{\sigma_T} \to N(0,1) \quad \text{and} \quad \frac{L_{jT}}{\sigma_T} \to_p 0 \;\; \text{for} \;\; j = 2, \cdots, 4. \qquad (4.3)
$$

Let $\phi_{st} = \sum_{u=1}^{s-1} \phi_{st}(u)$ and $V_t = \sum_{s=2}^{t-q} \phi_{st}$. Then $E[V_t | I_{t-q}] = 0$. This implies that $\{V_t\}$ is a sequence of martingale differences with respect to I_{t-q}. We now start proving the first part of (4.3). Applying a central limit theorem for martingale sequences (see Theorem 1 of Chapter VIII of Pollard 1984), in order to prove the first part of (4.3), it suffices to show that

$$
\frac{1}{\sigma_T^2} \sum_{t=2}^{T} V_t^2 \to_p 1 \quad \text{and} \quad \frac{1}{\sigma_T^4} \sum_{t=2}^{T} E\left[V_t^4\right] \to 0. \qquad (4.4)
$$

To verify (4.4), we first need to calculate some useful quantities. Recall the definition of V_t and observe that

$$
\begin{aligned}
V_t^2 &= \sum_{s=2}^{t-q} \sum_{u=1}^{s-1} A_u^2 \phi_{st}^2(u) + 2 \sum_{s=3}^{t-q} \sum_{u=2}^{s-1} \sum_{v=1}^{u-1} A_u A_v \phi_{st}(u) \phi_{st}(v) + 2 \sum_{s_1=3}^{t-q} \sum_{s_2=2}^{s_1-1} \sum_{u=1}^{s_2-1} A_u^2 \phi_{s_1 t}(u) \phi_{s_2 t}(u) \\
&+ 4 \sum_{s_1=4}^{t-q} \sum_{s_2=3}^{s_1-1} \sum_{u=2}^{s_2-1} \sum_{v=1}^{u-1} A_u A_v \phi_{s_1 t}(u) \phi_{s_2 t}(v) + 2 \sum_{s_1=3}^{t-q} \sum_{s_2=2}^{s_1-1} \sum_{u=s_2}^{s_1-1} \sum_{v=1}^{s_2-1} A_u A_v \phi_{s_1 t}(u) \phi_{s_2 t}(v), \\
E[V_t^2] &= \sum_{s=2}^{t-q} \sum_{u=1}^{s-1} A_u^2 E[\phi_{st}^2(u)] + 2 \sum_{s=3}^{t-q} \sum_{u=2}^{s-1} \sum_{v=1}^{u-1} A_u A_v E\left[\phi_{st}(u) \phi_{st}(v)\right] + 2 \sum_{s_1=3}^{t-q} \sum_{s_2=2}^{s_1-1} \sum_{u=1}^{s_2-1} A_u^2 E\left[\phi_{s_1 t}(u) \phi_{s_2 t}(u)\right] \\
&+ 4 \sum_{s_1=4}^{t-q} \sum_{s_2=3}^{s_1-1} \sum_{u=2}^{s_2-1} \sum_{v=1}^{u-1} A_u A_v E\left[\phi_{s_1 t}(u) \phi_{s_2 t}(v)\right] + 2 \sum_{s_1=3}^{t-q} \sum_{s_2=2}^{s_1-1} \sum_{u=s_2}^{s_1-1} \sum_{v=1}^{s_2-1} A_u A_v E\left[\phi_{s_1 t}(u) \phi_{s_2 t}(v)\right], \\
\sum_{t=q+3}^{T} E[V_t^2] &= \sum_{t=q+3}^{T} \sum_{s=2}^{t-q} \sum_{u=1}^{s-1} A_u^2 E[\phi_{st}^2(u)] + 2 \sum_{t=q+4}^{T} \sum_{s=3}^{t-q} \sum_{u=2}^{s-1} \sum_{v=1}^{u-1} A_u A_v E\left[\phi_{st}(u) \phi_{st}(v)\right] \\
&+ 2 \sum_{t=q+4}^{T} \sum_{s_1=3}^{t-q} \sum_{s_2=2}^{s_1-1} \sum_{u=1}^{s_2-1} A_u^2 E\left[\phi_{s_1 t}(u) \phi_{s_2 t}(u)\right] + 4 \sum_{t=q+5}^{T} \sum_{s_1=4}^{t-q} \sum_{s_2=3}^{s_1-1} \sum_{u=2}^{s_2-1} \sum_{v=1}^{u-1} A_u A_v E\left[\phi_{s_1 t}(u) \phi_{s_2 t}(v)\right] \\
&+ 2 \sum_{t=q+4}^{T} \sum_{s_1=3}^{t-q} \sum_{s_2=2}^{s_1-1} \sum_{u=s_2}^{s_1-1} \sum_{v=1}^{s_2-1} A_u A_v E\left[\phi_{s_1 t}(u) \phi_{s_2 t}(v)\right] \equiv \sigma_{1T}^2 + \sum_{j=1}^{4} \Delta_{jT}. \qquad (4.5)
\end{aligned}
$$

We now show that as $T \to \infty$

$$\sigma_{1T}^2 = \sigma_T^2 \,(1 + o(1)) \;\text{ and }\; \Delta_{jT} = o\left(\sigma_T^2\right) \quad \text{for} \quad j = 1, \cdots, 4. \tag{4.6}$$

By Lemma 4.1 listed in the Appendix (with $\eta_1 = \phi_{st}(u)$, $\eta_2 = \phi_{st}(v)$, $l = 2$, $p_i = 2(1+\delta)$ and $Q = \frac{1}{1+\delta}$),

$$E\left|\phi_{st}(u)\phi_{st}(v)\right| \le 10 M_{T1}^{\frac{1}{1+\delta}} \beta^{\frac{\delta}{1+\delta}}(u-v).$$

Therefore,

$$\sum_{u=2}^{s-1}\sum_{v=1}^{u-1} A_u A_v E\left|\phi_{st}(u)\phi_{st}(v)\right| \le 10 M_{T1}^{\frac{1}{1+\delta}} \sum_{u=2}^{s-1}\sum_{v=1}^{u-1} A_u A_v \beta^{\frac{\delta}{1+\delta}}(u-v) \le C M_{T1}^{\frac{1}{1+\delta}} \tag{4.7}$$

using $\sum_{u=2}^{T-1}\sum_{v=1}^{u-1}\beta^{\frac{\delta}{1+\delta}}(u-v)A_uA_v < \infty$. This, together with the conditions of Theorem 2.1, impliles that $\Delta_{2T} = o\left(\sigma_T^2\right)$ as $T \to \infty$. The verification of (4.6) for $j = 1$ follows similarly using $\sum_{u=1}^{T-1} A_{Tu}^2 < \infty$. For $j = 3, 4$, one needs to use Lemma 3.2 twice to deal with the case where $s_1 \neq s_2$ and $u \neq v$. The verification of the first part of (4.6) follows similarly using both Lemmas 4.1 and 4.3.

We now start to verify the first part of (4.4). Let $\sigma_{st}^2(u) = E[\phi_{st}^2(u)]$. Observe that

$$\begin{aligned}
E\left(\sum_{t=q+3}^{T} V_t^2 - \sigma_{1T}^2\right)^2 \;&\le\; 2E\left\{\sum_{t=q+3}^{T}\sum_{s=2}^{t-q}\sum_{u=1}^{s-1} A_u^2\left[\phi_{st}^2(u) - \sigma_{st}^2(u)\right]\right\}^2 \\
&+\; 8E\left\{\sum_{t=q+4}^{T}\sum_{s_1=3}^{t-q}\sum_{s_2=2}^{s_1-1}\sum_{u=1}^{s_2-1} A_u^2 \phi_{s_1t}(u)\phi_{s_2t}(u)\right\}^2 \\
&+\; 32E\left\{\sum_{t=q+5}^{T}\sum_{s_1=4}^{t-q}\sum_{s_2=3}^{s_1-1}\sum_{u=2}^{s_2-1}\sum_{v=1}^{u-1} A_uA_v \phi_{s_1t}(u)\phi_{s_2t}(v)\right\}^2 \\
&+\; 8E\left\{\sum_{t=q+4}^{T}\sum_{s_1=3}^{t-q}\sum_{s_2=1}^{s_1-1}\sum_{u=s_2}^{s_1-1}\sum_{v=1}^{s_2-1} A_uA_v \phi_{s_1t}(u)\phi_{s_2t}(v)\right\}^2 \\
&\equiv\; Q_{1T} + Q_{2T} + Q_{3T} + Q_{4T}.
\end{aligned} \tag{4.8}$$

In the following, we first show that as $T \to \infty$

$$Q_{3T} = o\left(\sigma_T^4\right). \tag{4.9}$$

The proofs for Q_{2T} and Q_{4T} follow similarly. Using Lemma 3.1 again, one can show that as $T \to \infty$

$$\begin{aligned}
Q_{3T} &= 32E\left\{\sum_{t=q+5}^{T}\sum_{s_1=4}^{t-q}\sum_{s_2=3}^{s_1-1}\sum_{u=2}^{s_2-1}\sum_{v=1}^{u-1} A_u A_v \phi_{s_1t}(u)\phi_{s_2t}(v)\right\}^2 \\
&\leq 32\sum_{u_1,u_2}\sum_{v_1,v_2} A_{u_1}A_{u_2}A_{v_1}A_{v_2}\left(\sum_{t_1\neq t_2}\sum_{s_1\neq s_2}\sum_{r_1\neq r_2}\left|E\left[\phi_{s_1t_1}(u_1)\phi_{s_2t_1}(v_1)\phi_{r_1t_2}(u_2)\phi_{r_2t_2}(v_2)\right]\right|\right) \\
&\leq 32\max\left\{M_T^2, N_T^2\right\}\cdot\max\left\{\left(\sum_{u=1}^{T-1}A_u\right)^4, \left(\sum_{u=1}^{T-1}A_u^2\right)^2\right\} = o\left(\sigma_T^4\right)
\end{aligned}$$

under the conditions of Theorem 2.1.

Let $C_\phi(u_1,u_2) = \int \phi_{12}^2(u_1)\phi_{34}^2(u_2)dP_1(\xi_1(u_1))dP_1(\xi_2(u_1))dP_1(\xi_3(u_2))dP_1(\xi_4(u_2))$, where $P_1(\xi_i(u))$ denotes the probability measure of $\xi_i(u)$.

Using Lemma 4.1 repeatedly, we have that for different i,j,k,l

$$\begin{aligned}
\sup_{u_1,u_2}\left|E\left[\phi_{ij}^2(u_1,u_2)\phi_{kl}^2(u_1,u_2)\right] - C_\phi(u_1,u_2)\right| &\leq 10\left\{\beta(\Delta(i,j,k,l))\right\}^{1-\frac{1}{1+\delta}} M_{T4}^{\frac{1}{1+\delta}} \\
&= 10M_{T4}^{\frac{1}{1+\delta}}\left\{\beta(\Delta(i,j,k,l))\right\}^{\frac{\delta}{1+\delta}}, \quad (4.10)
\end{aligned}$$

where $\Delta(i,j,k,l)$ is the minimum increment in the sequence which is the permutation of i,j,k,l in ascending order.

Similar to (4.10), one can have for all different i,j,k,l

$$\max_{u_1,u_2}\left|\sigma_{ij}^2(u_1)\sigma_{kl}^2(u_2) - C_\phi(u_1,u_2)\right| \leq 10M_{T4}^{\frac{1}{1+\delta}}\left\{\beta(\Delta(i,j,k,l))\right\}^{\frac{\delta}{1+\delta}}. \quad (4.11)$$

Therefore, using (4.10) and (4.11),

$$\begin{aligned}
Q_{1T} &= 2E\left\{\sum_{t=q+3}^{T}\sum_{s=2}^{t-q}\sum_{u=1}^{s-1} A_u^2\left[\phi_{st}^2(u) - \sigma_{st}^2(u)\right]\right\}^2 \\
&\leq 2\sum_{u_1,u_2} A_{u_1}^2A_{u_2}^2\left(\sum_{t_1,t_2}\sum_{s_1,s_2}\left|E\left[\phi_{ij}^2(u_1)\phi_{kl}^2(u_2)\right] - \sigma_{ij}^2(u_1)\sigma_{kl}^2(u_2)\right|\right) \\
&\leq 2\sum_{u_1,u_2} A_{u_1}^2A_{u_2}^2\left(\sum_{t_1,t_2}\sum_{s_1,s_2}\left|E\left[\phi_{ij}^2(u_1)\phi_{kl}^2(u_2)\right] - C_\phi(u_1,u_2)\right| + \left|C_\phi(u_1,u_2) - \sigma_{ij}^2(u_1)\sigma_{kl}^2(u_2)\right|\right) \\
&\leq \left\{O\left(T^3M_{T4}^{\frac{1}{1+\delta}}\right) + O\left(T^3M_{T3}\right)\right\}\cdot\left(\sum_{u=1}^{T-1}A_u^2\right)^2 = o(\sigma_T^4). \quad (4.12)
\end{aligned}$$

It now follows from (4.8)–(4.12) that for any $\epsilon > 0$

$$P\left\{\left|\frac{1}{\sigma_{1T}^2}\sum_{t=q+3}^{T}V_t^2 - 1\right| \geq \epsilon\right\} \leq \frac{1}{\sigma_T^4\epsilon^2}E\left[\sum_{t=q+3}^{T}V_t^2 - \sigma_{1T}^2\right]^2 \to 0. \tag{4.13}$$

Thus, the first part of (4.4) is proved.

Recall that

$$\begin{aligned} V_t^2 &= \sum_{s=2}^{t-q}\sum_{u=1}^{s-1}A_u^2\phi_{st}^2(u) + 2\sum_{s_1=3}^{t-q}\sum_{s_2=2}^{s_1-1}\sum_{u=1}^{s_2-1}A_u^2\phi_{s_1t}(u)\phi_{s_2t}(u) + 2\sum_{s=3}^{t-q}\sum_{u=2}^{s-1}\sum_{v=1}^{u-1}A_uA_v\phi_{st}(u)\phi_{st}(v) \\ &+ 4\sum_{s_1=4}^{t-q}\sum_{s_2=3}^{s_1-1}\sum_{u=2}^{s_2-1}\sum_{v=1}^{u-1}A_uA_v\phi_{s_1t}(u)\phi_{s_2t}(v) + 2\sum_{s_1=3}^{t-q}\sum_{s_2=2}^{s_1-1}\sum_{u=s_2}^{s_1-1}\sum_{v=1}^{s_2-1}A_uA_v\phi_{s_1t}(u)\phi_{s_2t}(v) \\ &\equiv \sum_{j=1}^{5}V_{tj}. \end{aligned}$$

Since there are some similarlities between $E[V_{tj}V_{tk}]$ for different j,k, we may need only to deal with some of the terms. For example, we now apply Lemma 4.1 to deal with $E[V_{t4}V_{t5}]$ for $2 \leq t \leq T$.

$$\begin{aligned} |E[V_{t4}V_{t5}]| &= 8\left|\sum_{s_1=4}^{t-q}\sum_{s_2=3}^{s_1-1}\sum_{u_1=2}^{s_2-1}\sum_{v_1=1}^{u_1-1}\sum_{r_1=3}^{t-q}\sum_{r_2=2}^{r_1-1}\sum_{u_2=r_2}^{r_1-1}\sum_{v_2=1}^{r_2-1}A_{u_1}A_{v_1}A_{u_2}A_{v_2}\phi_{s_1t}(u_1)\phi_{s_2t}(v_1)\phi_{r_1t}(u_2)\phi_{r_2t}(v_2)\right| \\ &\leq 8\sum_{u_1,u_2}\sum_{v_1,v_2}A_{u_1}A_{v_1}A_{u_2}A_{v_2}\left(\sum_{s_1,s_2}\sum_{r_1,r_2}|E[\phi_{s_1t}(u_1)\phi_{s_2t}(v_1)\phi_{r_1t}(u_2)\phi_{r_2t}(v_2)]|\right) \end{aligned} \tag{4.14}$$

It is easy to see that

$$\begin{aligned} &\int|\phi_{s_1t}(u_1)\phi_{s_2t}(v_1)\phi_{r_1t}(u_2)\phi_{r_2t}(v_2)|^{1+\delta}\,dP \\ &\leq \left\{\int|\phi_{s_1t}(u_1)\phi_{s_2t}(v_1)|^{2(1+\delta)}\,dP\int|\phi_{r_1t}(u_2)\phi_{r_2t}(v_2)|^{2(1+\delta)}\,dP\right\}^{1/2} \\ &\leq M_{T4}. \end{aligned} \tag{4.15}$$

Similar to (4.10), one can have for any $(s_1,s_2) \neq (r_1,r_2)$,

$$|E[\phi_{s_1t}(u_1)\phi_{s_2t}(v_1)\phi_{r_1t}(u_2)\phi_{r_2t}(v_2)]| \leq 10M_{T4}^{\frac{1}{1+\delta}}\left\{\beta(\Delta(s_1,s_2,r_1,r_2))\right\}^{\frac{\delta}{1+\delta}}, \tag{4.16}$$

where $\Delta(\cdot)$ is as defined in (4.10).

Consequently, one can show that as $T \to \infty$

$$|E\left[V_{t4}V_{t5}\right]| \leq \left\{O\left(T^3 M_{T4}^{\frac{1}{1+\delta}}\right) + O\left(T^2 M_{T3}\right)\right\} \cdot \max\left\{\left(\sum_{u=1}^{T-1} A_u\right)^4, \left(\sum_{u=1}^{T-1} A_u^2\right)^2\right\} = o\left(\sigma_T^4\right). \tag{4.17}$$

Thus, the second part of (4.4) follows from

$$\sum_{t=q+3}^{T} E\left[V_t^4\right] = \sum_{j=1}^{5}\sum_{k=1}^{5}\sum_{t=q+1}^{T} E\left[V_{tj}V_{tk}\right] = O\left(T^3 M_{T4}^{\frac{1}{1+\delta}}\right) = o(\sigma_T^4). \tag{4.18}$$

This finishes the proof, and therefore the first part of (4.3) is proved. We now finish the proof of the second part of (4.3).

Applying Lemma 4.3 implies that as $T \to \infty$

$$\begin{aligned} E\left|L_{2T}\right| &\leq \sum_u A_u \sum_{t=q+2}^{T}\sum_{s=2}^{t-q} E\left|E\left[\theta_{st}(u)|I_{t-q}\right]\right| \\ &\leq C\left(\sum_{u=1}^{T-1} A_u\right) \cdot \left(T M_{T7}^{\frac{1}{2(1+\delta)}}\right) = o(\sigma_T) \end{aligned} \tag{4.19}$$

using the conditions of Theorem 2.1.

The second part of (4.3) for L_{4T} follows from the conditions of Theorem 2.1 and

$$\begin{aligned} E\left|L_{4T}\right| &\leq \sum_u A_u \sum_{t=2}^{T}\sum_{t-s\leq q-1} E\left(E\left[\left|\theta_{st}(u)\right| | I_{t-q}\right]\right) \\ &= \sum_u A_u \sum_{t=2}^{T}\sum_{s=t+1-q}^{t-1} E\left[\left|\theta_{st}(u)\right|\right] \leq \left(\sum_{u=1}^{T-1} A_u\right) \cdot \left(T M_{T7}^{\frac{1}{2(1+\delta)}}\right) = o(\sigma_T). \end{aligned} \tag{4.20}$$

We finally prove the second part of (4.3) for L_{3T}. Similar to (4.7), using Lemma 4.1, one can show that as $T \to \infty$

$$\begin{aligned} \left|\sum_{t=2}^{T}\sum_{s_1=t+1-q}^{t-1}\sum_{s_2\neq s_1, s_2=t+1-q}^{t-1} E\left[\phi_{s_1 t}\phi_{s_2 t}\right]\right| &\leq \sum_{t=2}^{T}\sum_{s_1=t+1-q}^{t-1}\sum_{s_2\neq s_1, s_2=t+1-q}^{t-1} E\left[\left|\phi_{s_1 t}\phi_{s_2 t}\right|\right] \\ &\leq o\left(T^2 q\ M_{T3}\right), \\ \left|\sum_{t_1=3}^{T}\sum_{t_2=t_1+1-q}^{t_1-1}\sum_{s_1=t_1+1-q}^{t_1-1}\sum_{s_2=t_2+1-q}^{t_2-1} E\left[\phi_{s_1 t_1}\phi_{s_2 t_2}\right]\right| &\leq o\left(T^2 q^2\ M_{T3}\right). \end{aligned} \tag{4.21}$$

Using $E[\phi_{st}|I_{t-q}] = 0$ and (4.21) imply that as $T \to \infty$

$$\begin{aligned}
E\left[L_{3T}^2\right] &= \sum_{t=2}^{T}\sum_{s=t+1-q}^{t-1} E\left[\phi_{st}^2\right] + \sum_{t=2}^{T}\sum_{s_1=t+1-q}^{t-1}\sum_{s_2\neq s_1, s_2=t+1-q}^{t-1} E\left[\phi_{s_1 t}\phi_{s_2 t}\right] \\
&+ 2\sum_{t_1=3}^{T}\sum_{t_2=2}^{t_1-q}\sum_{s_1=t_1+1-q}^{t_1-1}\sum_{s_2=t_2+1-q}^{t_2-1} E\left[\phi_{s_1 t_1}\phi_{s_2 t_2}\right] \\
&+ 2\sum_{t_1=3}^{T}\sum_{t_2=t_1+1-q}^{t_1-1}\sum_{s_1=t_1+1-q}^{t_1-1}\sum_{s_2=t_2+1-q}^{t_2-1} E\left[\phi_{s_1 t_1}\phi_{s_2 t_2}\right] \\
&= \sum_{t=2}^{T}\sum_{s=t+1-q}^{t-1} E\left[\phi_{st}^2\right] + \sum_{t=2}^{T}\sum_{s_1=t+1-q}^{t-1}\sum_{s_2\neq s_1, s_2=t+1-q}^{t-1} E\left[\phi_{s_1 t}\phi_{s_2 t}\right] \\
&= \sum_{u=1}^{T-1}\sum_{v=1}^{T-1} A_u A_v \sum_{t=2}^{T}\sum_{s=t+1-q}^{t-1} E\left[\phi_{st}(u)\phi_{st}(v)\right] + \sum_{t=2}^{T}\sum_{s_1=t+1-q}^{t-1}\sum_{s_2\neq s_1, s_2=t+1-q}^{t-1} E\left[\phi_{s_1 t}\phi_{s_2 t}\right] \\
&+ 2\sum_{t_1=3}^{T}\sum_{t_2=t_1+1-q}^{t_1-1}\sum_{s_1=t_1+1-q}^{t_1-1}\sum_{s_2=t_2+1-q}^{t_2-1} E\left[\phi_{s_1 t_1}\phi_{s_2 t_2}\right] \\
&= O\left(Tq^2 M_{T3}^{\frac{1}{2}}\right) = o\left(\sigma_T^2\right).
\end{aligned}$$

This therefore completes the proof of Theorem 2.1.

5. Conclusions

In this paper, we have established some new results for central limit theorems for U–statistics of weakly dependent processes. Theorem 2.1 is useful for establishing asymptotic distributions for nonparametric estimators and test statistics computed using the weakly dependent β–mixing data. In addition, we have demonstrated the conditions of Theorem 2.1 are justifiable. We also show that the weakened condition: $E[\phi_1(X_1, y)] = E[\phi_1(x, X_1)] = 0$ for all x and y would make Corollary 2.1 directly applicable to establish asymptotically normal test statistics for density function specification.

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