A Dynamically Consistent Global Nonlinear Solution Method in the Sequence Space and Applications

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Solving DSGE Models Without a Law of Motion: An Ergodicity-Based Method and an Application*

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Abstract

This paper develops a novel method to solve dynamic stochastic general equilibrium models globally and accurately without specifying the law of motion. The method is based on the ergodic theorem: if a simulated path of the aggregate shock is long enough, all the possible equilibrium allocations are realized somewhere on the path. Then, the rationally expected future value function at each period on the path can be completely characterized by identifying the periods with each possible future state realization and by combining the corresponding time-specific value functions. The method provides an accurate solution even for models with highly nonlinear aggregate fluctuations. I apply this method to a heterogeneous-firm business cycle model with the corporate saving glut where the aggregate corporate cash stocks nonlinearly fluctuate. This nonlinearity passes through the dividend, leading to a state-dependent TFP shock sensitivity of consumption.

Keywords: Nonlinear business cycle, heterogeneous agents, stochastic dynamic programming, monotone function, state dependence.

JEL codes: E32, C63, D25.

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1 Introduction

This paper develops a novel method that solves dynamic stochastic general equilibrium models *without* specifying the law of motion. I name it the repeated transition method. This method globally and accurately solves a broad class of business cycle models with rich aggregate dynamics. The method is particularly useful for models with highly nonlinear aggregate fluctuations, as it does not require specifying the unknown nonlinear form of the law of motions.[1] Applying this method to a heterogeneous-firm business cycle model with the corporate saving glut, I study the macroeconomic implication of the recently observed rising corporate cash stocks.

The repeated transition method is based on the theoretical fact of the ergodic theorem: if a simulated path of the stationary aggregate shock process is long enough, all the possible equilibrium allocations should be realized on the simulated path. This fact implies that state-contingent future allocations are obtainable somewhere on the simulated path as a realized equilibrium outcome. Then, by properly identifying the period that has the corresponding outcome to each expected future state, an agent's rational expectation can be completely characterized at any period on the simulated path. In the identifying step for the corresponding periods for expected future outcomes, the law of motion does not need to be specified: the only information needed for this step is a measure of similarity among the aggregate states across the periods.

For example, suppose an agent is at time t, and a macroeconomist needs to come up with a rationally expected value function of period $t+1$. Suppose there are two possible aggregate shock realizations: G (Good) or B (Bad). For each possible shock realization $s \in \{G, B\}$ in $t+1$, I find a period in the simulated path where the endogenous aggregate states are the closest to the ones in period $t+1$, and the aggregate shock realization is s. Then, I combine the time-specific value functions

[1] Both heterogeneous-agent and representative-agent business cycle models featuring highly nonlinear dynamics can be accurately solved using the method.

from these two periods to construct the expected future value function of period $t + 1$. Due to the ergodic theorem, there almost surely exists such a period where the endogenous aggregate allocations (e.g., the distribution of individual states) are perfectly identical to the ones in period $t+1$ among the periods of the shock realization s if the simulated path is long enough. Therefore, the expected future value function at each period can be accurately constructed by combining the time-specific value functions.

When the repeated transition method is applied to a heterogeneous-agent model, the step to compare infinite dimensional objects across the periods can be a computational bottleneck. So, I suggest a sufficient statistics approach and provide a theoretical condition under which the approach leads to the exact solution. Specifically, when the time-specific value functions on the simulated path are strictly monotone in an aggregate allocation across the periods given an individual state and an aggregate shock realization, the aggregate allocation can be used as a sufficient statistic to obtain the exact solution.

The algorithm of the repeated transition method runs until the time series of the expected allocations and the simulated allocations converge. Therefore, the solution is highly accurate. When the implied law of motion is recovered from the converged equilibrium path, the law of motion features R^2 at 1 and mean squared error close to zero, even for highly nonlinear models. In terms of speed, the repeated transition method outperforms the method of Krusell and Smith (1997) in heterogeneous-agent models with non-trivial market-clearing conditions, as it does not require an extra loop for the market-clearing price.

Using the repeated transition method, I study the role of corporate cash stocks on the business cycle in a heterogeneous-firm business cycle model. The corporate saving in the model displays a starkly different pattern than the household saving in Aiyagari (1994) and Krusell and Smith (1998), as it features a flat region after the threshold of the target cash holding level. On top of this flat policy, due to the missing general equilibrium force to flatten the dynamics of aggregate cash holdings, the TFP-driven aggregate fluctuations of the cash stocks are highly nonlinear. Despite the nonlinearity, the repeated transition method accurately and globally solves the equilibrium dynamics in the model. Using the converged solution, I recover the true law of motion implied in the equilibrium dynamics. The true law of motion includes high-order polynomials of contemporaneous and lagged terms of the aggregate cash stocks.

In the calibrated model, the lagged aggregate cash stock significantly mitigates the aggregate consumption responsiveness to a negative productivity shock and intensifies the responsiveness to a positive productivity shock. Especially, the corporate cash stock gives an asymmetrically stronger insurance effect toward the negative TFP shock than the consumption boosting effect when the positive TFP shock hits. The data counterpart empirically supports this model's prediction of state dependence, and the empirical pattern is observed only after the early 1980s.[2] The fact that corporate cash holding has dramatically increased since the early 1980s partly explains why such a significant nonlinear effect is observed only after the early 1980s.

Related literature The repeated transition method is closely related to the global algorithms to solve heterogeneous-agent business cycle models (Krusell and Smith, 1997, 1998). While the methods of these papers are powerful in solving models with linear aggregate dynamics, they often fail to solve models with nonlinear dynamics due to the difficulty in correctly specifying the law of motion. Den Haan and Rendahl (2010) characterizes the law of motion by explicitly aggregating the Taylor-approximated individual policy functions, which can handle the nonlinear law of motion. In contrast, the repeated transition method accurately solves models with nonlinear dynamics without a functional approximation.

My method builds upon the method utilizing perfect-foresight impulse response

[2]The result is robust over other choices of the cutoff year around 1980.

suggested by Boppart et al. (2018). In the paper, aggregate allocations' impulse responses are obtained from the transition dynamics induced by MIT shocks to the steady-state distribution. Then, the law of motion of aggregate allocations is locally approximated around the steady state with a certainty equivalence assumption. In contrast, the repeated transition method does not assume certainty equivalence and globally solves the model. Moreover, it directly computes aggregate allocations and market-clearing prices in each period on the simulated path without specifying the law of motion. Therefore, the repeated transition method is distinguished from the solution methods based on perturbation and linearization (Reiter, 2009; Boppart et al., 2018; Ahn et al., 2018; Winberry, 2018; Childers, 2018; Auclert et al., 2019).

My method is related to the simulation-based approach of Judd et al. (2011) and Maliar et al. (2011). Their method solves a model only in the realized state space in the equilibrium, which is the ergodic set. This method significantly saves computation time, as it focuses only on a part of the entire state space. On top of this gain, the repeated transition method utilizes the information contained in the realized state space to form the agent's rational expectation in each period on the simulated path, which significantly improves the accuracy of the solution.

As the repeated transition method utilizes a single path of simulated aggregate shock that is long enough to completely represent the stochastic process, the approach is related to Kahou et al. (2021). Kahou et al. (2021) utilizes the fact that a whole economy's dynamics can be characterized by solving a finite number of agents' problems on a single Monte Carlo draw of individual shocks under the permutation-invariance condition. And the law of motion is computed using the deep-learning algorithm.

Instead of specifying the law of motion, the repeated transition method computes the path of equilibrium allocations at each point on the simulated path. Then, the true law of motion can be backed out from the converged equilibrium dynamics. My method relies only on relatively simple computational techniques but computes highly accurate solutions. Also, compared to the other methods, this method is more efficient in the computation of the models with the non-trivial market clearing conditions, as each iteration does not require clearing the market: only in the limit is the market cleared.

Roadmap Section 2 explains the repeated transition method. Section 3 validates the accuracy of the repeated transition method by comparing the computed outcome with the existing well-known results in the literature. Section 4 introduces a heterogeneous-firm business cycle model where firms save cash. Section 5 discusses the business cycle implication of corporate cash holdings predicted by the model compared to the observations from the data. Section 6 concludes.

2 Repeated transition method

2.1 A generalized model framework

To explain the repeated transition method, I introduce a generalized model framework that can nest a broad class of general equilibrium models. I denote the individual state as x and the aggregate state as X. The individual state x is composed of the endogenous individual state a and the exogenous individual state s (the idiosyncratic shocks). The aggregate state X is composed of the endogenous aggregate states Φ and the exogenous aggregate state S (the aggregate shocks). Φ can be a distribution of the individual states x in a heterogeneous-agent model or an aggregate allocation in a representative-agent model.

$$\begin{aligned} &[\text{Individual state}]: \quad x = \{a, s\} \\ &[\text{Aggregate state}]: \quad X = \{\Phi, S\} \end{aligned}$$

The idiosyncratic and aggregate shock processes are assumed to follow a Markov process represented by a transition matrix Π^s and Π^S, respectively. I denote the value function as V. All the variables with an apostrophe indicate variables in the future period. The objective function of an economic agent is composed of the contemporaneous part $f(y, x', x; X)$ and the expected future value. The agent maximizes the objective function by choosing (y, a'), where y is a vector of control variables that affects only the contemporaneous period. Then, the recursive formulation of an agent's problem is as follows:

$$
\begin{aligned}
V(x; X) = \max_{y,a'} \; & f(y, a', x; X) + \mathbb{E}m(X, X')V(a', s'; X') \\
\text{s.t.} \quad & (y, x') \in \mathcal{B}(x; X, X', q), \quad \Phi' = F(X)
\end{aligned}
$$

where $m(X, X')$ is the stochastic discount factor; $q(X, X')$ is a price bundle; $\mathcal{B}(x; X, X', q)$ is the budget constraint; $F(X)$ is the law of motion known to to an individual agent.[3] I denote the combined prices (m, q) as p. The following market clearing condition pins down the price p:

$$
[\text{Market clearing}]: \quad p(X, X') = \arg_{\widetilde{p}}\{Q^D(\widetilde{p}, X, X') - Q^S(\widetilde{p}, X, X') = 0\},
$$

where Q^D is a vector of demand; Q^S is a vector of supply. For simplicity of the illustration, I assume the aggregate shock S can take two possible values $\{G, B\}$, and the transition matrix Π^S is a 2×2 matrix.[4]

2.2 Intuition behind the method

Suppose I simulate T periods of aggregate shocks $\{S_t\}_{t=0}^T$, and hypothetically the simulated path is long enough to make almost all the possible equilibrium allocations

[3] The stochastic discount factor can be a single parameter, for example β, as in a canonical dynamic household's problems.

[4] The applicability of the repeated transition method is not limited to a certain number of grid points for the aggregate shocks. The choice of two grid points is purely for an easy illustration.

happen on the simulated path.[5] Then, I start from guessing the following three time series: 1) value functions, $\{V_t^{(0)}\}_{t=0}^T$, 2) distributions of individual states $\{\Phi_t^{(0)}\}_{t=0}^T$, and 3) prices $\{p_t^{(0)}\}_{t=0}^T$. Using these guesses, I solve the allocations backward from the terminal period T to obtain $\{V_t^*\}_{t=0}^T$, and simulate the economy forward using the solution. The forward simulation generates the time series of the distribution of individual states $\{\Phi_t^*\}_{t=0}^T$ and prices $\{p_t^*\}_{t=0}^T$ from the market-clearing conditions. Using these, I update the guess to move on to the next iteration, $\{V_t^{(1)}, \Phi_t^{(1)}, p_t^{(1)}\}_{t=0}^T$.

Now, suppose that I've run the n^{th} iteration and that I am now at the $(n+1)^{th}$ iteration at period t after solving the problem backward from the terminal period T until period $(t+1)$. On the simulated aggregate state path, suppose that the shock realization at period $t+1$ is G: $S_{t+1} = G$. For the problem of an agent at t, a macroeconomist needs to construct a rationally expected future value function denoted as $\mathbb{E}_t\widetilde{V}_{t+1}$. However, this is a difficult task because only $V_{t+1}(\cdot, S = G)$ is available from the backward solution, while $V_{t+1}(\cdot, S = B)$ is not. This is natural as only one shock can be realized in a period. I define this unobserved values $V_{t+1}(\cdot, S = B)$ as a *counterfactual* conditional value function.

In the standard state-space-based approach, this problem is handled by replacing the time index with the distribution or sufficient statistics and specifying a law of motion in these aggregate states. Then, the counterfactual conditional value function is obtained by interpolating the unconditional value function at the predicted future state. Therefore, the accuracy of this predicted future state from the law of motion determines the accuracy of the solution. However, before obtaining the solution and simulating the economy based on the solution, it is hardly known whether the law of motion is correctly specified or not. Then, if the law of motion turns out to be incorrect, a researcher needs to restart solving the problem from scratch, coming up with a new guess about the law of motion. However, a proper guess is difficult to

[5] In theory, an infinitely-long simulation needs to be considered, but for the illustrative purpose, I consider a T-period long simulation. Later in the application, a long-enough finite simulation is used as an approximation for the infinitely-long ergodic simulation.

obtain, as there is an infinite degree of freedom in the new guess. In particular, there are two types of difficulties in this step. One is about which statistics to include in the law of motion; the other is about what functional forms to choose for the law of motion. Unless the aggregate dynamics are well-known to be log-linear, as in Krusell and Smith (1998), this problem cannot be easily resolved.

Then, I consider a new approach where the counterfactual conditional value function is obtained from the value function of another period $\widetilde{t}+1$ in which the endogenous aggregate state is exactly the same as the period $t+1$, but the counterfactual shock is realized:

$$\begin{aligned}
\Phi_{\widetilde{t}+1}^{(n)} &= \Phi_{t+1}^{(n)} \\
S_{\widetilde{t}+1} &= B \neq G = S_{t+1}
\end{aligned}$$

Then, all the aggregate states of the realized state of period $\widetilde{t}+1$ are identical to the ones in the counterfactual state of period $t+1$. Thus, the following holds:

$$V_{\widetilde{t}+1}^{(n)}(\cdot, S=B) = V_{t+1}^{(n)}(\cdot, S=B).$$

Importantly, $V_{\widetilde{t}+1}^{(n)}(\cdot, S=B)$ is the observed *factual* conditional value function available in the nth iteration. As both $V_{t+1}^{(n)}(\cdot, S=G)$ and $V_{t+1}^{(n)}(\cdot, S=B)(= V_{\widetilde{t}+1}^{(n)}(\cdot, S=B))$ are available, the rationally expected future value function $\mathbb{E}_t\widetilde{V}_{t+1}$ can be correctly constructed. Even when the aggregate shock process is discretized finer than two grid points, the rationally expected future value function can be obtained using the same procedures. Due to the ergodic theorem, if a simulated path is long enough, the existence of such period $\widetilde{t}+1$ is almost surely guaranteed.

In this new approach, a law of motion does not need to be specified to construct the rational expected future value function. As long as the period $\widetilde{t}+1$ that mimics the counterfactual realization of $t+1$ is identified, the problem can be solved. For this step, it is necessary to track the endogenous aggregate states $\{\Phi_t^{(n)}\}_{t=0}^{T}$, as it is the key identifier to locate the period $\widetilde{t}+1$. In the following section, I elaborate on the detailed steps to implement the repeated transition method.

2.3 Algorithm

I simulate a single path of exogenous aggregate shocks for a long-enough period T, $\mathbb{S} = \{S_t\}_{t=0}^{T}$, using the aggregate transition matrix Π^S. I define a time partition $\mathcal{T}(S)$ that groups periods with the same shock realization as follows.

$$\mathcal{T}_S := \{\tau | S_\tau = S\} \subseteq \{0, 1, 2, ..., T\} \text{ for } S \in \{B, G\}.$$

The pseudo algorithm of the repeated transition method is as follows:

Step 1. Guess on the paths of the value functions, the state distributions, and the prices. $\{V_t^{(n)}, \Phi_t^{(n)}, p_t^{(n)}\}_{t=0}^{T}$.[6]

Step 2. Solve the model backward from the terminal period T in the following sub-steps. The explanation is based on an arbitrary period t. Without a loss of generality, I assume $S_t = G$ and $S_{t+1} = G$:

2-a. Find $\widetilde{t}+1$ where the endogenous aggregate allocation in period is identical to the one in period $t+1$, but the shock realization is different from period $t+1$ ($S_{\widetilde{t}+1} = B$):[7]

$$\widetilde{t}+1 = \arg \inf_{\tau \in \mathcal{T}_B} ||\Phi_\tau^{(n)} - \Phi_{t+1}^{(n)}||_\infty,$$

2-b. Compute the expected future value function as follows:

$$\mathbb{E}_t \widetilde{V}_{t+1} = \pi_{G,G} V_{t+1}^{(n)} + \pi_{G,B} V_{\widetilde{t}+1}^{(n)}$$

[6] In practice, I use the stationary equilibrium allocations for all periods as the initial guess.

[7] Such $\widetilde{t}+1$ might not be unique. However, any of such $\widetilde{t}+1$ is equally good to be used in the next step.

2-c. Using $\mathbb{E}_t \widetilde{V}_{t+1}$ and $p_t^{(n)}$, solve the individual agent's problem at the period t. Then, I obtain the solution $\{V_t^*, a_{t+1}^*\}$

After the taking these sub-steps for $\forall t$, $\{V_t^*, a_{t+1}^*\}_{t=0}^T$ are available.

Step 3. Using $\{a_{t+1}^*\}_{t=0}^T$, simulate forward the time series of the distribution of the individual states $\{\Phi_t^*\}_{t=0}^T$ starting from $\Phi_0^* = \Phi_0^{(n)}$.[8]

Step 4. Using $\{\Phi_t^*\}_{t=0}^T$, all the aggregate allocations over the whole path such as $\{K_t^*\}_{t=0}^T$ can be obtained. Using the market-clearing condition, compute the time series of the implied prices $\{p_t^*\}_{t=0}^T$.[9]

Step 5. Check the distance between the implied prices and the guessed prices.

$$\sup_{BurnIn \le t \le T-BurnIn} ||p_t^* - p_t^{(n)}||_\infty < tol$$

Note that the distance is measured after excluding the burn-in periods at the beginning and the end of the simulated path. This is an adjustment to handle a potential bias from the imperfect guesses on the terminal period's value function $V_T^{(n)}$ and the initial period's distribution $\Phi_0^{(n)}$.

If the distance is smaller than the tolerance level, the algorithm is converged.

[8]In this step, I use the non-stochastic simulation method (Young, 2010).

[9]It is worth noting that the prices here are not the market-clearing prices that are determined from the interactions between demand and supply. Rather, they are the prices implied by the market-clearing condition given either demand or supply fixed:

$$p_t^* = \arg_{\widetilde{p}}\{Q^D(p_t^{(n)}, X_t, X_{t+1}) - Q^S(\widetilde{p}, X_t, X_{t+1}) = 0\} \text{ or}$$
$$p_t^* = \arg_{\widetilde{p}}\{Q^D(\widetilde{p}, X_t, X_{t+1}) - Q^S(p_t^{(n)}, X_t, X_{t+1}) = 0\}$$

In Section 3, I use this method to solve the model in Khan and Thomas (2008). In the computation method used in Khan and Thomas (2008), a market-clearing price needs to be computed in an additional loop due to the non-trivial market-clearing condition. The implied price cannot replace the market-clearing price in this method, as the misspecified price prediction rule can lead to a divergent law of motion of the aggregate allocation. In contrast, due to the missing market clearing step, the repeated transition method significantly saves computation time. I discuss further the computational gain in Section 3.

Otherwise, I make the following updates on the guess:[10]

$$p_t^{(n+1)} = p_t^{(n)}\psi_1 + p_t^*(1-\psi_1)$$

$$V_t^{(n+1)} = V_t^{(n)}\psi_2 + V_t^*(1-\psi_2)$$

$$\Phi_t^{(n+1)} = \Phi_t^{(n)}\psi_3 + \Phi_t^*(1-\psi_3)$$

for $\forall t \in \{0, 1, 2, 3, ..., T\}$. With the updated guess $\{V_t^{(n+1)}, \Phi_t^{(n+1)}, p_t^{(n+1)}\}_{t=0}^T$, I go back to Step 1.

(ψ_1, ψ_2, ψ_3) are the parameters of convergence speed in the algorithm. If ψ_i is high, then the algorithm conservatively updates the guess, leaving the algorithm to converge slowly. If the equilibrium dynamics are almost linear, as in Krusell and Smith (1998), I found uniformly setting ψ_i around 0.8 guarantees convergence at a fairly high convergence speed. However, if a model is highly nonlinear, as in the baseline model in Section 4, the convergence speed needs to be controlled to be substantially slower than the one in the linear models. This is because the nonlinearity can lead to a sudden jump in the realized allocations during the iteration if a new guess is too dramatically changed from the last guess. A heterogeneous updating rule $\psi_i \neq \psi_j$ $(i \neq j)$ is also helpful in cases where the dynamics of certain allocations are particularly more nonlinear than the others.

As can be seen from the convergence criterion in Step 5, the algorithm stops when the expected allocation paths are close enough to the simulated allocation paths. Therefore, once the convergence is achieved, the solution is guaranteed to be highly accurate. If the accuracy is measured in R^2 or in the mean-squared errors, as in

[10] In highly nonlinear aggregate dynamics, I have found that the log-convex combination updating rule marginally dominates the standard convex combination updating rule in terms of convergence speed. The log-convex combination rule is as follows:

$$log(p_t^{(n+1)}) = log(p_t^{(n)})\psi_1 + log(p_t^*)(1-\psi_1)$$

Krusell and Smith (1998), the repeated transition method features R^2 at 1, and its mean-squared error becomes negligibly different than zero.

After the equilibrium allocations are computed over the in-sample path $\mathbb{S}$, I estimate the implied law of motion from the in-sample allocations. The law of motion can potentially take any nonlinear form. Then, using the fitted law of motion, equilibrium allocations are computed over out-of-sample paths of simulated aggregate shocks.

2.4 A sufficient statistic approach

In the algorithm explained in the previous section, Step 2-a is the most demanding step as it needs to find a period $\widetilde{t}+1$ that is identical to period $t+1$ in terms of the distribution. Therefore, the similarity of the distributions across the periods needs to be measured, which is a computationally costly process.

However, if there are sufficient statistics that can perfectly represent a period's endogenous aggregate state, such as aggregate capital stock in Krusell and Smith (1998), the computational efficiency can be substantially improved.[11] This is because I can find period $\widetilde{t}+1$ by only comparing the distance between these sufficient statistics instead of the distributions. For example, in Krusell and Smith (1998), the aggregate capital is the sufficient statistics, which makes Step 2-a easier:

$$\widetilde{t}+1 = \arg \inf_{\tau \in \mathcal{T}_B} ||K_{\tau}^{(n)} - K_{t+1}^{(n)}||_{\infty},$$

As the algorithm relies on the ergodic theorem, a sufficiently long period of simulation is needed for accurate computation. However, in practice, the simulation still ends in finite periods. Therefore, the period $\widetilde{t}+1$ that shares exactly identical sufficient statistics as period t+1 might not exist. For this hurdle, the following adjusted versions of Step 2-a and Step 2-b help improve the accuracy of the solution:

[11] I discuss under which condition the sufficient statistics approach can be used in Section 2.5

2-a$'$. Find $\widetilde{t}^{up} + 1$ where the endogenous aggregate allocation is closest to the one in period $t+1$ from above, but the shock realization is different from period $t+1$:

$$\widetilde{t}^{up} + 1 = \arg \inf_{\tau \in \mathcal{T}_B \text{ s.t. } K_\tau^{(n)} \geq K_{t+1}^{(n)}} ||K_\tau^{(n)} - K_{t+1}^{(n)}||_\infty,$$

Similarly, find $\widetilde{t}^{dn} + 1$ where the endogenous aggregate allocation is closest to the one in period $t+1$ from below, but the shock realization is different from period $t+1$:

$$\widetilde{t}^{dn} + 1 = \arg \inf_{\tau \in \mathcal{T}_B \text{ s.t. } K_\tau^{(n)} < K_{t+1}^{(n)}} ||K_\tau^{(n)} - K_{t+1}^{(n)}||_\infty,$$

Then, I have $K_{\widetilde{t}^{up}+1}^{(n)}$ and $K_{\widetilde{t}^{dn}+1}^{(n)}$ that are closest to $K_{t+1}^{(n)}$ from above and below, respectively. Using these two, I compute the weight ω to be used in the convex combination of value functions in the next step:

$$\omega = \frac{K_{t+1}^{(n)} - K_{\widetilde{t}^{dn}+1}^{(n)}}{K_{\widetilde{t}^{up}+1}^{(n)} - K_{\widetilde{t}^{dn}+1}^{(n)}}$$

2-b$'$. Compute the expected future value function as follows:

$$\mathbb{E}_t \widetilde{V}_{t+1} = \pi_{G,G} V_{t+1}^{(n)} + \pi_{G,B} \left(\omega V_{\widetilde{t}^{up}+1}^{(n)} + (1-\omega) V_{\widetilde{t}^{dn}+1}^{(n)} \right)$$

Step 2-a' and Step 2-b' construct a synthetic counterfactual conditional value function by the convex combination of the two value functions that are for the most similar periods to period $t+1$. These adjusted steps help accurately solve the problem in relatively short periods of simulation. For example, the model in Krusell and Smith (1998) can be accurately solved using only $T = 500$ periods of simulation (except for 100 burn-in periods at the beginning and the end of the simulated path).

2.5 A sufficient condition for the sufficient statistic approach

In this section, I analyze under which condition the sufficient statistic can replace the entire distribution in the repeated transition method to allow the sufficient statistics approach (Section 2.4). In Krusell and Smith (1998), the law of motion in the entire distribution is sharply approximated by the law of motion in the aggregate capital stock. This is one example where a sufficient statistic can completely represent the infinite-dimensional object. Likewise, various research in the literature has considered sufficient statistics to overcome the curse of dimensionality, but there has been little theoretical explanation of when such an approximation can be used. Proposition 1 provides a sufficient condition for using sufficient statistics in the repeated transition method.

Proposition 1 (A sufficient condition for the sufficient statistic approach)**.**
For a sufficiently large T, if there exists a time series of an aggregate allocation $\{e_t\}_{t=0}^T$ such that for each time partition $\mathcal{T}_S = \{t|S_t = S\}$, $\forall S \in \{B, G\}$ and for $\forall(a, z)$,

$$(i) \quad e_{\tau_0} < e_{\tau_1} \iff V_{\tau_0}^{(n)}(a, z) < V_{\tau_1}^{(n)}(a, z) \text{ for any } \tau_0, \tau_1 \in \mathcal{T}_S$$

or

$$(ii) \quad e_{\tau_0} < e_{\tau_1} \iff V_{\tau_0}^{(n)}(a, z) > V_{\tau_1}^{(n)}(a, z) \text{ for any } \tau_0, \tau_1 \in \mathcal{T}_S$$

then x_t is the sufficient statistics of the endogenous aggregate state Φ_t for $\forall t$. In other words, for $\forall t \in \mathcal{T}_S$,

$$\arg \inf_{\tau \in \mathcal{T}_S} ||\Phi_\tau^{(n)} - \Phi_t^{(n)}||_\infty = \arg \inf_{\tau \in \mathcal{T}_S} ||e_\tau - e_t||_\infty.$$

Proof.
See Online Appendix. ∎

Proposition 1 states that if a time series $\{e_t\}_{t=0}^T$ monotonically ranks the level of the corresponding period's value function for each individual state, e_t is the sufficient statistic of time period t in the repeated transition method. The intuition behind the proposition is as follows. Suppose a situation where a researcher is searching for a value function to build a rationally expected future value function. If a time index of the correct counterfactual period to use is explicitly given as τ to a researcher, then the researcher can easily identify which value function to use, as all value functions are indexed by time. So, in this case, V_τ is trivially the one to use.

Now instead of τ, suppose the level of e_τ is known to the researcher. Then, similar to the prior situation where τ is known, the researcher can identify which value function to use because the ranking information uniquely pins down the corresponding value function due to the strict monotonicity. For example, if two periods τ_0 and τ_1 share the same level of e_t, thus $e_{\tau_0} = e_{\tau_1}$, then the strict monotonicity says $V_{\tau_0} = V_{\tau_1}$. If this is not the case ($V_{\tau_0} \neq V_{\tau_1}$), then either the ergodicity or strict monotonicity assumption is violated, and this is the key idea of the proof.

To summarize the theoretical results in this section, once the ranking information across the different periods' value functions is known, one can exactly pin down which period's value function to use. This is also a practically desired feature for the implementation, as the strict monotonicity of value functions in the sufficient statistic makes it feasible to smoothly interpolate the value functions along the sufficient statistic (2-a' and 2-b' in Section 2.4).

The sufficient condition provides a theoretical ground to understand how a sufficient statistic approach works in the repeated transition method. In the quantitative analysis of the baseline model in Section 5.4, the monotonicity is quantitatively validated for the converged solution. However, the sufficient condition is not constructive for the algorithm as it cannot be checked prior to the implementation: the condition can be verified only after the solution converges. Also, the sufficient statistics in the repeated transition method do not imply that these statistics are only allocations to be considered in the law of motion in the state-space-based approach. This is because the former may not include sufficient information about the inter-temporal dynamics in the endogenous aggregate state variables.[12]

3 Accuracy of the repeated transition method

This section compares the equilibrium allocations obtained from the repeated transition method and the ones from the methods of Krusell and Smith (1998) and Krusell and Smith (1997). In the computation, parameters are set as in the benchmark model in Krusell and Smith (1998) without idiosyncratic shocks in the patience parameter β. Both of the algorithms are designed to stop when the largest absolute difference between the simulated average capital stock and the expected average capital stock is less than 10^{-6}.

In the converged solution, the mean squared difference in the solutions between the repeated transition method and Krusell and Smith (1998) method is around $2 * 10^{-4}$. It takes around 20 minutes for the repeated transition method to converge under the convergence speed parameter $\psi_1 = \psi_2 = \psi_3 = 0.8$; it takes around 20 mins for Krusell and Smith (1998) algorithm.[13] The convergence speed might change depending on the updating weight.

Figure 1 plots the expected path (Predicted) and the simulated path (Realized) of aggregate capital K_t obtained from the repeated transition method and the simulated path from Krusell and Smith (1998).[14] The expected path refers to $\{V_t^{(n)}, \Phi_t^{(n)}, p_t^{(n)}\}_{t=0}^T$ in Section 2.3, and the simulated path indicates $\{V_t^*, \Phi_t^*, p_t^*\}_{t=0}^T$. As can be seen from all three lines hardly distinguished from each other, the repeated transition method computes almost identical equilibrium allocations as Krusell and Smith (1998) algorithm at a similar speed. This is because the log-linear specification

[12] When I fit the nonlinear aggregate dynamics of sufficient statistics obtained from the repeated transition method to the parametric/non-parametric law of motion in Section 5.3, the fittest specification includes multiple lagged terms of the sufficient statistics. However, the sufficient statistics for each time period in the repeated transition method is just a single-dimensional aggregate allocation.

[13] This computation is done in 2015 MacBook Pro laptop with a 2.2 GHz quad-core processor

[14] This figure is motivated from the fundamental accuracy plot suggested in Den Haan (2010).

almost perfectly captures the actual law of motion in Krusell and Smith (1998). Thus, their algorithm with the log-linear specification can accurately compute the solution at high speed.

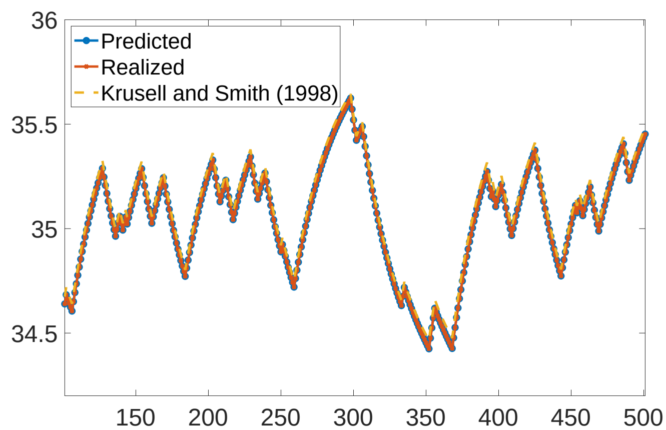


Figure 1: Computed dynamics of aggregate wealth (Krusell and Smith, 1998)

Notes: The figure plots the time series of the aggregate wealth (capital) K_t in the model of Krusell and Smith (1998). The line with a round tick mark is the predicted wealth time series (n^{th} guess) $\{K_t^{(n)}\}_{t=100}^{500}$. The line with the square tick mark is the realized wealth time series $\{K_t^*\}_{t=100}^{500}$. The dashed line is the predicted wealth time series implied by the law of motion in Krusell and Smith (1998).

However, the repeated transition method outperforms Krusell and Smith (1997) algorithm when the market-clearing condition is not trivial, as in the model of Khan and Thomas (2008).[15] This is because the non-trivial market-clearing condition requires an extra loop to find an exact market-clearing condition in each iteration, while the repeated transition method does not.

I solve the model in Khan and Thomas (2008) using both the repeated transition method and the Krusell and Smith (1997) algorithm with an external loop for the non-trivial market-clearing condition.[16] Figure 2 plots the dynamics of price p_t

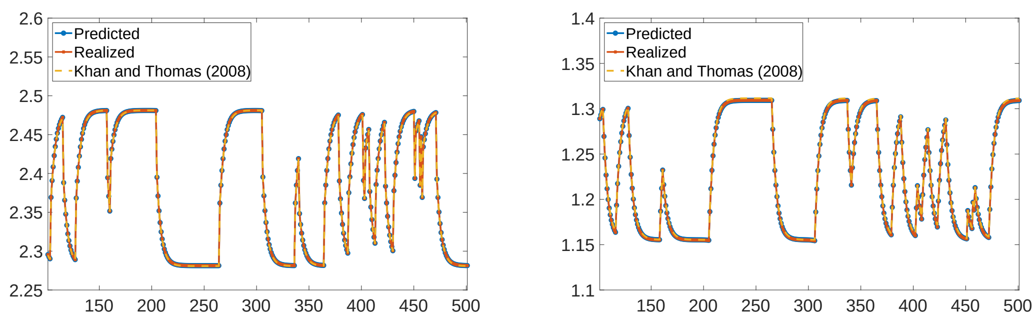
[15]Krusell and Smith (1997) algorithm is a variant of the algorithm in Krusell and Smith (1998), which is applicable to models with non-trivial market-clearing conditions. Khan and Thomas (2008) uses this algorithm.

[16]Both of the algorithms are designed to stop when the following criterion is satisfied:

$$\max\{\sup_t\{||p_t^* - p_t^{(n)}||\}, \sup_t\{||K_t^* - K_t^{(n)}||\}\} < 10^{-6}$$

and aggregate capital stock K_t computed from the repeated transition method and Krusell and Smith (1997) algorithm. For the allocations computed from the repeated transition method, both the predicted time series and the realized time series are plotted. As shown in the figure, all three lines display almost identical dynamics of the price and the aggregate allocations. The mean squared difference in the solutions between the repeated transition method and Khan and Thomas (2008) is less than 10^{-5}.

In the application of the repeated transition method, I use $\psi_1 = \psi_2 = \psi_3 = 0.9$ for the updating rule, which is higher than the previous application. The reason for using this conservative updating rule is because the model in Khan and Thomas (2008) features a strong general equilibrium effect; dramatic updates in the price might lead to divergence. The repeated transition method took around 20 minutes to converge on average, while Krusell and Smith (1997) algorithm converged in around 5 to 6 hours on average. The convergence speed might change depending on the updating weight.



(a) Price p_t $(= 1/C_t)$ (b) Aggregate capital stocks K_t

Figure 2: Computed dynamics of aggregate allocations (Khan and Thomas, 2008)

Notes: The figure plots the time series of the price p_t the aggregate wealth (capital) K_t in the model of Khan and Thomas (2008). In both panels, the line with a round tick mark is the predicted time series (n^{th} guess) $\{p_t^{(n)}, K_t^{(n)}\}_{t=100}^{500}$; the line with the square tick mark is the realized time series $\{p_t^*, K_t^*\}_{t=100}^{500}$; the dashed line is the predicted time series implied by the law of motion.

The terminal condition is slightly different from the one in Step 5 of Section 2.3. Likewise, the terminal condition can be flexibly adjusted based on different combinations of $V_t^{(n)}, \Phi_t^{(n)}$, and $p_t^{(n)}$.

4 Application: A heterogeneous-firm real business cycle model with the corporate saving glut

In this section, I analyze the business cycle implications of the rising corporate cash holding using the heterogeneous-firm real business cycle model. There are two reasons why the method is applied to this particular model. The first is the rising importance of the corporate cash holding in the U.S. economy. Figure 3 plots the time series of the aggregate cash holding to GDP ratio, where nominal cash holding is from the Flow of Fund data, and the nominal GDP is from NIPA.[17] As seen from the sharply rising trend in the ratio, the corporate cash holding has risen substantially faster than the output.[18] Have these rising corporate cash holdings affected the business cycles in the U.S.? This paper investigates the answer to this question through the lens of the business cycle model with the corporate saving glut.

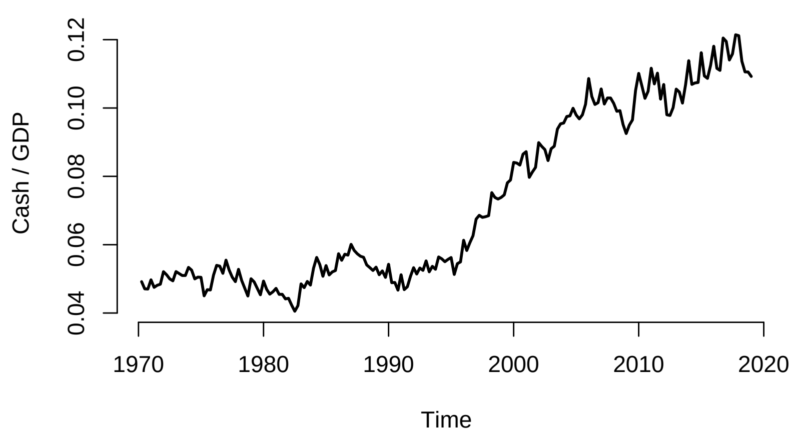


Figure 3: The time-series of the corporate cash-to-GDP ratio

Notes: The figure plots the time series of the corporate cash-to-GDP ratio. The corporate cash stock data is from the Flow of Funds of the Federal Reserve Board, and the nominal GDP is from National Income and Product Accounts (NIPA) from the Bureau of Economic Analysis (BEA).

[17] The calculation of the aggregate corporate cash stock is explained in Appendix C.

[18] The corporate finance literature has investigated reasons for rising corporate cash holding. However, analyzing these different reasons is out of the scope of this paper.

The second reason is the sharp contrast in the saving patterns between the models with the heterogeneous firms and the heterogeneous households. Due to the external financing cost, which generates a precautionary motivation for holding cash stock, the firms' saving pattern partly mimics the household's savings. However, as the internal financing does not trigger any adjustment cost, the contemporaneous component of the objective function is concave only in the limited part of the domain, unlike the counterpart in the household's utility maximization problem. Therefore, a satiation point exists for hoarding cash, leaving a flat region in the inter-temporal saving policy function. This flat policy function is the starkest difference between the baseline model and the heterogeneous-household models, which is to be investigated further in detail.

4.1 Technology

There is a continuum of measure one of ex-ante homogeneous firms that hoard cash and produces business outputs. For simplicity, I assume a firm operates using only labor input. This can be understood as an equivalent setup to a model where a firm uses both capital and labor inputs, but the optimal capital demand decision is already frictionlessly internalized in the labor demand decision. Consistent with this explanation, I set the labor share (equivalent to the span of control parameter) at $\gamma = 0.85$ in the quantitative analysis, which is greater than the standard labor share and captures internalized capital demand decision.

The business output is produced by the following Cobb-Douglas production function:

$$f(n_{it}, z_{it}; A_t) = z_{it} A_t n_{it}^{\gamma}$$

where n_{it} is labor demand; $\gamma < 1$ is the span of control parameter; z_{it} and A_t are idiosyncratic and aggregate productivities, respectively. Each firm needs to pay a fixed operation cost $\xi > 0$ in each period.

The log of idiosyncratic productivity shock process $\{z_{it}\}$ follows an AR(1) process:

$$log(z_{it+1}) = \rho_z log(z_{it}) + \epsilon_{it+1}, \quad \epsilon_{it+1} \sim_{i.i.d} N(0, \sigma_z)$$

For computation, the idiosyncratic productivity process is discretized by the Tauchen method.[19] The stochastic aggregate productivity process is from Krusell and Smith (1998):[20]

$$\Gamma_A = \begin{bmatrix} 0.8750 & 0.1250 \\ 0.1250 & 0.8750 \end{bmatrix}$$
$$A_t \in \{A_B, A_G\} = \{0.99, 1.01\}.$$

4.2 External financing cost

A firm earns operating profit and decides how much to distribute as a dividend d_{it} to equity holders (a representative household). The remaining part of the operating profit after the dividend payout is used to adjust cash holding, $ca_{t+1}/(1 + r^{ca}) - ca_t$. The future cash holding is discounted at an internal discount rate $r^{ca} > 0$ as cash is not traded in the market across the firms. r^{ca} is an exogenous parameter and is assumed to be lower than the market interest rate r_t. The cash holding level is assumed to be non-negative $ca_t \geq 0$. Thus, the model features a standard incomplete market assumption with the borrowing limit as in Aiyagari (1994).

If a dividend is determined to be negative, then a firm is financing through an external source, which incurs extra pecuniary cost $C(d_{it})$ (Jermann and Quadrini,

[19] I discretize it using equally-spaced nine grid points within the two-standard deviation range around the mean.

[20] The repeated transition method works for a finer discretization than two grid points. For example, Lee (2022) uses a finer discretization (five grid points) for the repeated transition method. However, to preserve the symmetry between the corporate cash-holding model and the household saving model (Krusell and Smith, 1998), I assume the same aggregate productivity process.

2012; Riddick and Whited, 2009). This external financing cost is specified as follows:

$$C(d_{it}) := \frac{\mu}{2}\mathbb{I}\{d_{it} < 0\}d_{it}^2$$

Thus, the net dividend is $d_{it} - \frac{\mu}{2}\mathbb{I}\{d_{it} < 0\}d_{it}^2$. It is worth noting that this net dividend function belongs to $\mathbb{C}^1$ class and concave as it smoothly changes the slope at $d_{it} = 0$ without a kink. Therefore, the standard theory of the concave utility of the household seamlessly applies to the model.

If there is no external financing cost, hoarding cash is not the desired option for a firm because it is more expensive than receiving the dividend $\left(\frac{1}{1+r^{ca}} > \frac{1}{1+r_t}\right)$. However, due to the presence of an external financing cost, a firm has a precautionary motivation to hoard cash, saving for rainy days (low z_t or low A_t). Therefore, the firms smooth their dividend payout in equilibrium. Consistent with this, rich empirical evidence has been documented for corporate dividend smoothing behavior in the corporate finance literature (Leary and Michaely, 2011; Bliss et al., 2015). Especially, Leary and Michaely (2011) empirically showed that cash-rich firms smoothen their dividend significantly more than others. The equilibrium patterns in this model can match these empirical patterns.

4.3 Recursive formulation

At the beginning of each period, a firm i is given with a cash holding ca_{it} and an idiosyncratic productivity level z_{it}. Thus, the individual state variable x_{it} is as follows:

$$x_{it} = \{ca_{it}, z_{it}\}$$

All firms rationally expect the future and are aware of the full distribution of the firm-level state variables. The aggregate state variable X_t is as follows:

$$X_t = \{A_t, \Phi_t\}$$

where A_t is the aggregate productivity, and Φ_t is the distribution of the individual state variable x_{it}.

The recursive formulation of a firm's problem is as follows:

$$\begin{aligned}
\text{[Firm]} \quad & J(x;X) = \max_{ca',d} \quad d - C(d) + \mathbb{E}(q(X,X')J(ca',z';X')) \\
& \text{s.t.} \quad d + \frac{ca'}{1+r^{ca}} = \pi(z;A,\Phi) + ca \\
& \qquad ca' \geq 0, \quad \Phi' = G(\Phi, A) \\
\text{[Operating profit]} \quad & \pi(z;A,\Phi) := \max_n zAn^\gamma - w(A,\Phi)n - \xi \\
\text{[Idiosyncratic productivity]} \quad & z' = G_z(z) \text{ (AR(1) process)} \\
\text{[External financing cost]} \quad & C(d) := \frac{\mu}{2}\mathbb{I}(d<0)d^2 \\
\text{[Aggregate state]} \quad & X := \{A, \Phi\}
\end{aligned}$$

where J is the value function of a firm; d is dividend; ca and z are cash holding and idiosyncratic productivity, respectively; A is the aggregate productivity; Φ is the distribution of the individual state variables; w and q are wage and stochastic discount factor which are functions of aggregate state variables $X = \{A, \Phi\}$.

4.4 A representative household and market clearing

I close the model by introducing a stand-in household that holds equity as wealth and saves on equity. The household consumes and supplies labor and rationally expects the future aggregate states. The income sources of the household are labor work and dividends from equity.

The recursive formulation of the representative household's problem is as follows:

$$
\begin{aligned}
V(a;X) = \max_{c,a',l_H} \quad & log(c) - \eta l_H + \beta \mathbb{E}^{A'} V(a';X') \\
\text{s.t. } & c + \int \Gamma_{A,A'} a' q(X,X') dX' = w(X) l_H + a \\
& G(X) = \Phi' \\
& G_A(A) = A' \quad \text{(AR(1) process)}
\end{aligned}
$$

where V is the value function of the household; a is wealth; c is consumption; a' is a future saving level; l_H is labor supply; w is wage, and q is the stochastic discount factor. The household is holding the equity of firms as their wealth.

The recursive competitive equilibrium is defined based on the following market-clearing conditions:

$$
\begin{aligned}
\text{(Labor market)} \quad & l_H(X) = \int n(ca,z;X) d\Phi \\
\text{(Equity market)} \quad & a(X) = \int (J(ca,z;X) + C(d(ca,z;X))) d\Phi
\end{aligned}
$$

The external financing costs and the aggregate firm values jointly form the supply of equity. In the market clearing condition, the supply meets the demand for equity in the form of household wealth.

The model does not assume a centralized market for cash holding. Therefore, r^{ca} is not endogenously determined in the market. This is a realistic assumption, as a firm's cash holding is not tradable across firms. I interpret this setup as the cash holding return is determined by each firm's idiosyncratic financing status independently from the centralized capital market condition. r^{ca} is the average level of the idiosyncratic financing cost.[21]

[21] For simplicity, the model is abstract from the heterogeneity in the financing cost. All the exogenous heterogeneity is loaded on the heterogeneous productivities.

4.5 The role of market incompleteness and the financial friction: Comparison with Aiyagari (1994)

In this section, I study the individual firm's cash hoarding patterns in the stationary equilibrium. This analysis is essential to understand why the model would feature highly nonlinear dynamics under aggregate uncertainty. Due to the external financing cost, a firm saves cash out of precautionary motivation. However, there exists a target cash level, which the optimal cash holding does not go beyond. This is because after the target level of cash, holding cash becomes costlier: if a firm holds a large stock of cash, then the future risk of external financing is almost perfectly hedged while carrying additional cash bears only a smaller return than the market interest rate. Proposition 2 states the existence of the target cash level.[22]

Proposition 2 (The existence of the target cash-holding level)**.**
Suppose policy functions are non-trivial: $ca'(ca, z) > 0$ *and* $d(ca, z) > 0$ *for some* $ca > 0$*, given* z*. Then, there exists* $\overline{ca}(z) > 0$ *such that* $ca'(ca, z) \leq \overline{ca}(z)$ *for* $\forall ca \geq 0$*.*

Proof. See Online Appendix. ■

Therefore, the future cash holding policy function features the flat region after the target cash holding level.[23] Figure 4 shows the future cash holding policy function for the lowest and highest productivity firms. For the highest productivity firms, due to the persistence of the productivity shock, the target cash holding level is lowest: they are the least concerned firms about the future external financing cost. As can be seen from the figure, the policy function crosses the 45-degree ray. That is, the highest productivity firms with little cash increase the cash holding until they reach

[22] It is worth noting that the implication of Proposition 2 is different from Proposition 4 of Aiyagari (1993). Proposition 4 of Aiyagari (1993) implies that a household with a excessively large wealth would gradually decrease the wealth. In contrast, Proposition 2 implies that a firm with a excessively large cash stock would immediately reduce the cash stock to the target level.

[23] The existence of the target cash holding level is similar to the prediction of the consumption buffer stock model (Carroll, 1997). Despite the flat region, there is no kink in the policy function, as the objective function is smooth over the entire domain ($\mathbb{C}^1$ class).

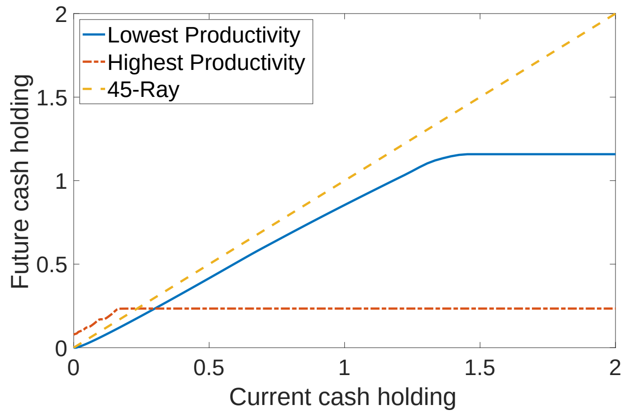


Figure 4: Cash holding policy function of individual firms

Notes: The solid line plots the future cash-holding policy function of the lowest-productivity firms. The dash-dotted line plots the future cash-holding policy function of the highest-productivity firms. The dotted line is a 45-degree ray.

the target level. On the other hand, the lowest productivity firms' target cash holding level is the highest, but their low profit makes them decrease their future cash holding. Therefore, the lowest productivity firms' policy function does not cross the 45-degree ray.

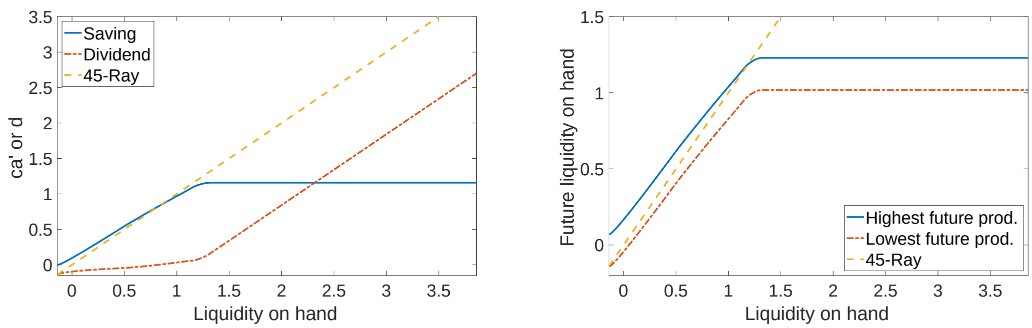
The flat region in the cash-holding policy function is starkly contrasted with the wealth accumulation pattern of households in Aiyagari (1994). To make a direct comparison feasible, I define the liquidity on hand, which is a firm-side counterpart of the total resource in Aiyagari (1994):

$$\text{Liquidity on hand}_t := \underbrace{\pi_t}_{\text{Liquidity from operating profit}} + \underbrace{ca_t}_{\text{Cash}}.$$

Figure 5 plots the saving and dividend policies in panel (a) and the future liquidity on hand in panel (b) as functions of the liquidity on hand. This figure is the firm-side counterpart of Figure I in Aiyagari (1994).[24] For a sharp illustration, I only plot the policy functions for a firm at the lowest productivity ($z = min\mathbb{Z}$). For both firms in

[24]Dividend d is the counterpart of consumption c, and future cash holding ca' is the counterpart of future wealth a_{t+1}.

my model and households in Aiyagari (1994), there are cases where the borrowing limit is bound, leaving the saving policy flat near the borrowing limit. However, that region spans only a tiny range for the firm side, so it is not visible in panel (a) of Figure 5.[25] If a firm does not hold enough liquidity on hand, an extra dollar increase in liquidity goes to both saving and dividends. Especially a greater amount goes to the saving than dividends. However, if a firm holds enough liquidity on hand, additional liquidity is solely paid out as a dividend, as can be seen from the parallel dividend curve to the 45-degree line for the high liquidity on hand region. In panel (b), the future liquidity on hand displays a flat region for the high level of the current liquidity on hand, similar to the dynamics of the cash policy function in Figure 4.



(a) Allocation of liquidity on hand (b) Evolution of liquidity on hand

Figure 5: Liquidity on hand and the policy functions (when $z = min\mathbb{Z}$)

Notes: This figure is the firm-side counterpart of Figure I in Aiyagari (1994).

The nonlinear policy function at the firm level generates nonlinearity in the business cycle. Especially depending on the fraction of firms that are located over the flat policy region, the aggregate cash dynamics vary. Therefore, the aggregate fluctuations in this model crucially depend on the endogenous state Φ, the distribution of individual states.

[25]Households behave in a similar pattern in Krusell and Smith (1998). However, in both Aiyagari (1994) and Krusell and Smith (1998), the fraction of these constrained households are small as well. Especially, this is one of the major reasons why the aggregate dynamics in Krusell and Smith (1998) display only a negligible nonlinearity.

5 Quantitative analysis

In this section, I quantitatively analyze the recursive competitive equilibrium allocations computed from the repeated transition method. For easier computation, I first normalize the firm's value function by contemporaneous consumption c_t following Khan and Thomas (2008). I define a price $p_t := 1/c_t$ and the normalized value function $\tilde{J}_t := p_t J_t$. From the intra-temporal and inter-temporal optimality conditions of households, I have $w_t = \eta/p_t$ and $q_t = \beta p_{t+1}/p_t$. Thus, p_t is the only price to characterize the equilibrium. Then, the equilibrium price p_t is determined from the following variant of the non-trivial market clearing condition:[26]

$$p = \arg_{\widetilde{p}} \left\{ c(\widetilde{p}) - \int \left[d(x; X, \widetilde{p}) + w(X, \widetilde{p}) l(x; X, \widetilde{p}) \right] d\Phi = 0 \right\}.$$

This is a fixed-point problem and computationally costly to solve. Instead, the repeated transition method uses the implied price p^*, which is obtained as follows:

$$\begin{aligned} p^* &= \arg_{\widetilde{p}} \left\{ c(\widetilde{p}) - \int \left[d(x; X, p^{(n)}) + w(X, p^{(n)}) l(x; X, p^{(n)}) \right] d\Phi = 0 \right\} \\ &= 1 / \int \left[d(x; X, p^{(n)}) + w(X, p^{(n)}) l(x; X, p^{(n)}) \right] d\Phi, \end{aligned}$$

where $p^{(n)}$ is the guessed price in the n^{th} iteration. I take the sufficient statistics approach described in Section 2.4, and the aggregate cash holdings CA_t (the first moment of the distribution of cash holding) is the sufficient statistics. I validate this approach by showing Proposition 1 is satisfied in Section 5.4.

5.1 Calibration

The model's key parameters are the external financing cost parameter μ and the operating cost parameter ξ. The external financing cost μ is identified from the

[26]This condition is derived from combining the household's budget constraint and the equity market clearing condition. For a sharp illustration, the time subscript is lifted.

corporate cash-to-output ratio. In the moment calculation, the aggregate cash stock is obtained from the Flow of Funds of the Federal Reserve Board, and the aggregate output (GDP) is from the National Income and Product Accounts (NIPA) from the Bureau of Economic Analysis (BEA).[27] In the model, as μ increases, the corporate cash-to-output ratio increases due to the heightened precautionary motivation.

Parameters	Target Moments	Data	Model	Level
μ	Corporate cash holding/Output (%)	10.00	9.28	0.40
ξ	Consumption/Output (%)	66.00	64.02	0.15
η	Labor supply hours	0.33	0.34	3.90

Table 1: Calibration target and parameters

Notes: The third column reports the target moment obtained from the data. The corporate cash is from the Flow of Funds; output and consumption are from NIPA. The fourth column reports the calibrated model's corresponding moments. The last column reports the identified parameter levels.

The identifying moment of the operating cost parameter ξ is the consumption-to-output ratio. The consumption data is from NIPA.[28] As operating cost increases, the ratio decreases due to the reduced dividends. The calibrated parameters and the corresponding moments are summarized in Table 1. The other fixed parameters are summarized in Appendix B.

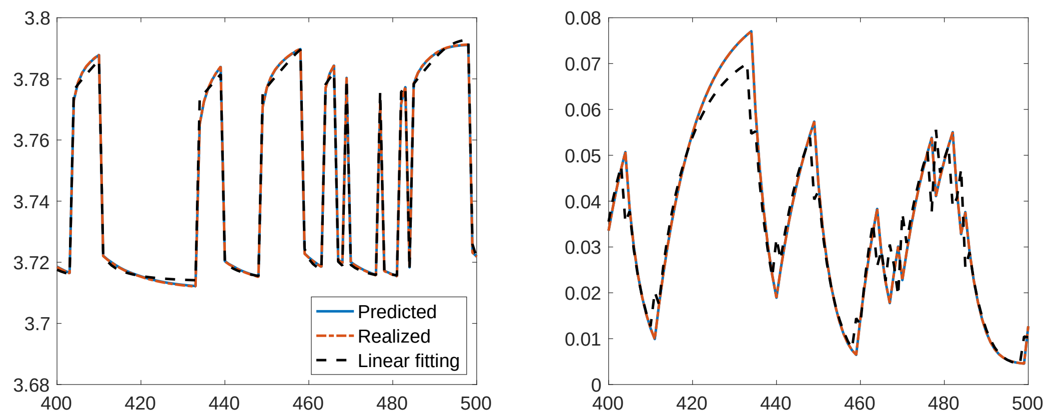
5.2 Nonlinear business cycle

Using the repeated transition method, I compute the recursive competitive equilibrium allocations over the simulated path of aggregate shocks. Using the aggregate cash stock, I take the sufficient statistics approach described in Section 2.4. The dynamics of the aggregate cash stocks are highly nonlinear for two reasons. First, the individual firm's cash holding policy function becomes flat for high levels of individual

[27] This condition is driven by. The detailed definition of aggregate cash holding is available in Appendix C.

[28] Consumption includes both durable and non-durable consumptions.

cash stocks, as described in Section 4.5. Second, the general equilibrium effect does not strongly affect each firm's cash holding demand. It is because the price of cash holding is exogenously fixed at r^{ca} as the cash is not allowed to be traded across the firms.



(a) Price p_t $(= 1/C_t)$ (b) Aggregate cash stocks CA_t

Figure 6: Aggregate fluctuations in the baseline model

Notes: The figure plots the time series of the price p_t the aggregate cash stock CA_t in the baseline model. In both panels, the solid line is the predicted time series (n^{th} guess) $\{p_t^{(n)}, CA_t^{(n)}\}_{t=400}^{500}$; the dash-dotted line is the realized time series $\{p_t^*, CA_t^*\}_{t=400}^{500}$; the dashed line is the predicted time series implied by the linear law of motion.

Figure 6 plots a part of the simulated path of the price p_t (panel (a)) and aggregate corporate cash holding CA_t (panel (b)) obtained from both the repeated transition method and the log-linear specification of the law of motion. The solid line plots the expected allocations (guess from the n^{th} iteration), and the dot-dashed line plots the realized allocations (simulation based on the policy in $(n+1)^{th}$ iteration) in the repeated transition method. The dashed line represents the dynamics of the allocations in the log-linear specification of the law of motion. To obtain the parameters in the log-linear specification, I fit the equilibrium allocations from the repeated transition method into the log-linear specification, and the result is as follows:

$$log(CA_{t+1}) = -0.5742 + 0.9061 * log(CA_t), \quad \text{if } A_t = A_B, \text{ and } R^2 = 0.9971, \ MSE = 0.0017$$
$$log(CA_{t+1}) = -0.8949 + 0.6829 * log(CA_t), \quad \text{if } A_t = A_G, \text{ and } R^2 = 0.9823, \ MSE = 0.0039$$
$$log(p_t) = 1.3232 - 0.0018 * log(CA_t), \quad \text{if } A_t = A_B, \text{ and } R^2 = 0.8828, \ MSE = 0.0000$$
$$log(p_t) = 1.3093 - 0.0011 * log(CA_t), \quad \text{if } A_t = A_G, \text{ and } R^2 = 0.8928, \ MSE = 0.0000$$

In the repeated transition method, the expectation path and the realized path converge as can be seen from the figure. Therefore, R^2s are 1 and the mean squared errors are as small as 10^{-6} for both p_t and CA_t. On the other hand, the log-linear fitting results in low R^2 and high mean squared errors. This results show that the aggregate fluctuations in this economy are highly nonlinear.

One important reason for the nonlinearity is the missing general equilibrium effect on cash. If the price of cash is determined in the competitive market, the dynamics of aggregate cash stocks are smoothed. For example, when there is a surge of cash holding demand, the price of cash holding goes up to mitigate the surge and vice versa for the case of decreasing cash holding demand. In many of the models in the literature, this flattening force from the general equilibrium has been proven to be powerful enough to guarantee the log-linear specification as the true law of motion. One example is Khan and Thomas (2008), where the micro-level lumpiness is smoothed out by real interest rate dynamics. However, due to the missing general equilibrium effect, the log-linear prediction rule fails to capture the true law of motion in this paper.

On top of the nonlinearity, there is another complication in the model that the prototype method of Krusell and Smith (1998) cannot simply address: there is a non-trivial market-clearing condition with respect to price p_t. Krusell and Smith (1997) suggests an algorithm to solve this problem by considering an external loop in the algorithm that solves market-clearing price p_t in each iteration. This algorithm is known to successfully solve the log-linear models with non-trivial market-clearing conditions, such as Khan and Thomas (2008). However, due to the extra loop in each iteration, the algorithm entails high computation costs. In contrast, the repeated transition method tracks the implied price instead of the market clearing price on the simulated path. Therefore, the method does not require an extra loop for computing market-clearing price, so it saves a great amount of computation time.

5.3 Recovering the true nonlinear law of motion

In this section, I recover the true law of motion from the converged equilibrium outcomes over the simulated path. Then, I test the validity of the true law of motion by fitting the law of motion into the out-of-sample simulated path.

Specifically, the following laws of motion are studied:

$$
\begin{aligned}
CA_{t+1} &= G_{CA}(CA_t, CA_{t-1}, CA_{t-2}, \ldots, CA_{t-n}; A_t) \\
p_t &= G_p(CA_t, CA_{t-1}, CA_{t-2}, \ldots, CA_{t-n}; A_t)
\end{aligned}
$$

Table 2 reports the goodness of fitness (R^2) of the different specifications. The first five rows report the fitness when only the contemporaneous aggregate cash stock CA_t is considered up to different polynomial orders. When a single argument is considered without the higher-order polynomials, R^2 gets as low as 0.8956 for G_{CA} if the contemporaneous productivity state is G.[29] As the more higher-order polynomials are included, the better the fitness becomes. However, the fitness of the specification of G_p stops improving after a certain threshold. This shows that the true law of motion can be recovered only by including further historical allocations.

The bottom seven rows of Table 2 report the fitness of the law of motion when the additional lagged terms of the aggregate cash stock are considered. Up to the third order polynomials are included for each of lagged terms on top of the polynomial

[29] This pure linear specification is different from the log-linear specification in Section 5.2.

terms of the contemporaneous cash stocks up to the fifth order.[30] As more lagged terms are considered in the law of motion, the fitness improves, especially in G_p. However, only after the polynomials of the seven-period lagged aggregate cash stock are included in the law of motion, the accurate law of motion is recovered.

			Goodness of fitness: R^2			
	# of lagged	order	$CA_{t+1}: Good$	$CA_{t+1}: Bad$	$p_t: Good$	$p_t: Bad$
Contemp.	0	1	0.8956	0.9452	0.9922	0.9966
	0	2	0.9839	0.9952	0.9927	0.9976
	0	3	0.9973	0.9995	0.9930	0.9976
	0	4	0.9993	0.9999	0.9932	0.9976
	0	5	0.9996	1.0000	0.9933	0.9976
Add. history	1	3	0.9999	1.0000	0.9987	0.9979
	2	3	0.9999	1.0000	0.9997	0.9984
	3	3	0.9999	1.0000	0.9998	0.9987
	4	3	0.9999	1.0000	0.9998	0.9991
	5	3	0.9999	1.0000	0.9998	0.9994
	6	3	0.9999	1.0000	0.9998	0.9996
	7	3	0.9999	1.0000	0.9998	0.9997

Table 2: The fitness of the law of motion across different specifications

Notes: The first column (# of lagged) reports the number of lagged terms included in the specification. The second column (order) reports the highest order of polynomials considered. When lagged terms are included, contemporaneous terms are considered up to the fifth polynomial for all specifications. The third and fourth column reports the R^2 of the law of motion of CA_{t+1} for Good and Bad shock realizations. The fifth and sixth column reports the R^2 of the law of motion of p_t for Good and Bad shock realizations.

This exercise shows the substantial nonlinearity of the aggregate fluctuations in this model. In the repeated transition method, the contemporaneous aggregate cash stock is used as a sufficient statistic of each period's cross-section. However, this does *not* imply that the true aggregate law of motion is a function of only the contemporaneous cash stock. The contemporaneous cash stock is rather a labeling of each period that correctly sorts the rankings of the value functions across the periods in the repeated transition method (Proposition 1). In Section 5.4, the monotonicity of value functions in the contemporaneous cash stock is verified.

[30]The results only negligibly change over different order choices.

I validate the recovered law of motion by fitting it on the out-of-sample simulated path.[31] Specifically, I solve the model on another simulated path to obtain the converged equilibrium dynamics using the repeated transition method and compare the dynamics with the implied dynamics in the recovered true law of motion on the in-sample path. Figure 7 plots p_t (panel (a)) and CA_t (panel (b)) for 1) predicted time series (solid line), 2) realized time series (dot-dashed line), 3) time series implied by the recovered in-sample law of motion (solid line with ticks), and 4) time series implied by the linear law of motion (dashed line). The predicted time series and the realized time series are indistinguishably close to each other due to the convergence requirement of the repeated transition method. The time series implied by the recovered law of motion also closely track the converged equilibrium dynamics, validating the specification. The goodness of fitness (R^2) in the time series implied by the recovered law of motion is greater than 0.999 for both G_{CA} and G_p in all shock realizations.

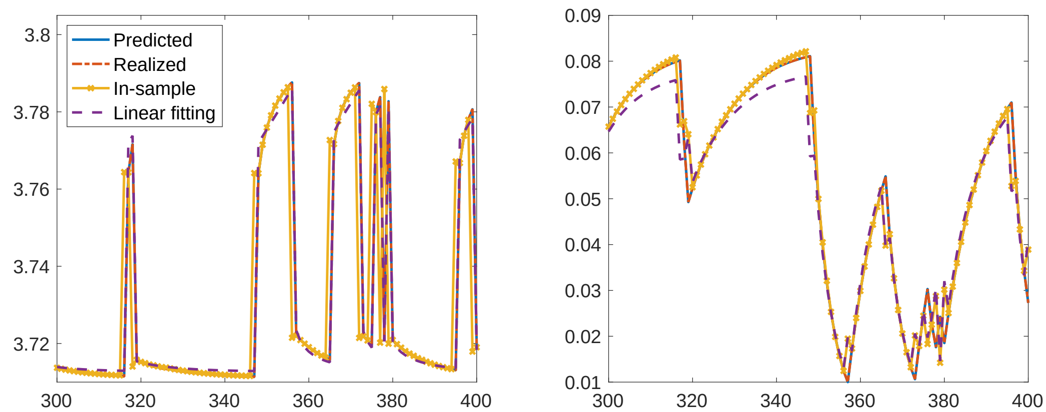
5.4 Monotonicity of the value function

In this section, I check the monotonicity of the value function in the aggregate cash stock. The monotonicity is the sufficient condition for the aggregate cash stock to be used as a sufficient statistic (Proposition 1). Specifically, I check whether the value function is strictly monotone in the aggregate cash stock CA_t for individual state variables (a_t, z_t) and aggregate shock realizations A_t.

Figure 8 plots the level of the value function for different individual states for the aggregate shock realization $A_t = A_B$. Four different levels of individual cash stock ca_t are considered, and the corresponding figures are panels (a),(b),(c), and (d).[32] Each panel is the scatter plot of value function levels for the five different levels of the

[31] The recovered true of motion refers to the specification that considers up to the seven-period lagged aggregate cash stocks, where R^2 is the highest in Table 2.

[32] The four different levels of ca_t are 1e-8, 0.09, 0.23, and 0.47



(a) Price p_t $(= 1/C_t)$ (b) Aggregate cash stocks CA_t

Figure 7: Fitting into the out-of-sample path

Notes: The figure plots the time series of the price p_t the aggregate cash stock CA_t in the baseline model for an out-of-sample simulated path of the aggregate shocks. In both panels, the solid line is the predicted time series (n^{th} guess) $\{p_t^{(n)}, CA_t^{(n)}\}_{t=300}^{400}$; the dash-dotted line is the realized time series $\{p_t^*, CA_t^*\}_{t=300}^{400}$; the line with the star tick mark is the predicted time series implied by the exact law of motion recovered from the in-sample path; the dashed line is the predicted time series implied by the linear law of motion.

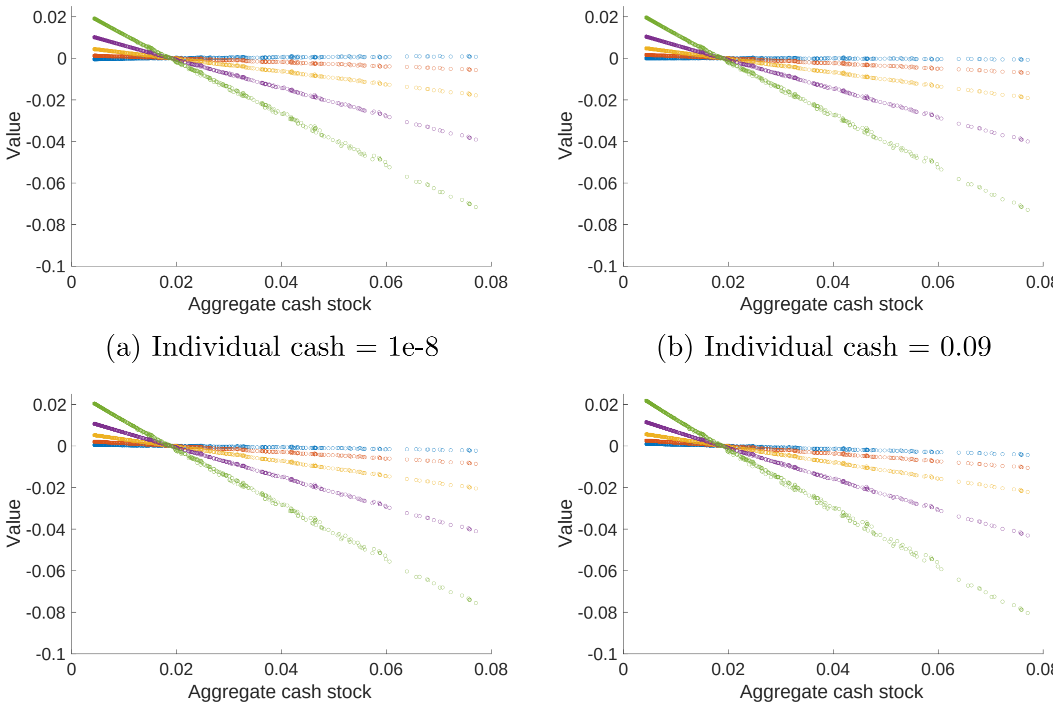
idiosyncratic productivity z_t.[33] The horizontal axis is the aggregate cash stock CA_t and the vertical axis is the level of value function V_t. As can be seen from the figure, the scatter plot of the values forms five different linear lines. This shows that the value functions are strictly monotone in the aggregate cash, validating the qualification of the aggregate cash as the sufficient statistics in the repeated transition method.

5.5 Macroeconomic implications and empirical evidence

In this section, I analyze the role of corporate cash holdings on aggregate consumption fluctuations using the baseline model and support the model prediction from the empirical evidence.

In the model, the aggregate productivity A_t can take one of two values $\{A_B, A_G\}$, and it follows a persistent process. I define the negative aggregate productivity shock

[33]The five different levels of z_t are 0.7950, 0.8916, 1, 1.1215, and 1.2579.



(a) Individual cash = 1e-8

(b) Individual cash = 0.09

(c) Individual cash = 0.23

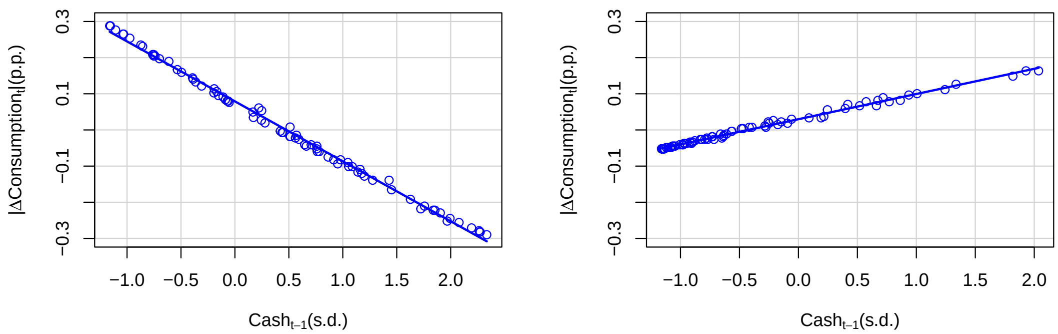
(d) Individual cash = 0.47

Figure 8: Monotonicity of value function in aggregate cash stock

Notes: The figure plots the level of time-specific value functions with the corresponding aggregate cash stock in the horizontal axis for different idiosyncratic productivity levels. Each panel from (a) through (d) is for different cash stock at the firm level.

as a TFP shift from A_G to A_B and the positive aggregate productivity shock as a TFP shift from A_B to A_G. In the baseline model, depending on the cash stock a firm holds, the responsiveness of the firm-level dividends to the exogenous aggregate TFP shock changes. For example, when a firm is short of cash, a negative TFP shock in productivity makes a firm reduce dividends further than it would do when it has abundant cash stocks. It is because the firm with little cash needs to not only pay out dividends but save cash out of concern for the future.

Then, due to this dividend channel, the responsiveness of household consumption becomes dependent on the aggregate cash stock. Figure 9 plots the relationship between the consumption responsiveness over the aggregate cash stocks separately for negative aggregate shock (panel (a)) and positive aggregate shock (panel (b)). In this



(a) Negative shock

(b) Positive shock

Figure 9: State-dependent shock responses of consumption

Notes: The figure plots consumption responses to a negative (panel (a)) and positive (panel (b)) aggregate TFP shock normalized by the average level (in percentage points) with the lagged aggregate cash stocks in the horizontal axis.

model, the magnitude of aggregate shock is uniform at $|A_G - A_B| (= 2\%$ TFP shock).[34] Therefore, If the consumption shock responses are different across the periods, it is due to the endogenous state dependence of the responsiveness rather than the shock magnitude variation. Then, I separately collect the periods of negative aggregate shock and positive aggregate shock and compute the consumption c_t and one-period-ahead aggregate cash stock CA_{t-1} for each period. As can be seen from Figure 9, the consumption responsiveness decreases in the aggregate cash stock for the negative aggregate shock. From a similar intuition with the opposite direction, the consumption responsiveness increases in the aggregate cash stock for the positive aggregate shock.

Table 3 reports the regression coefficient when the observations in Figure 9 are fitted into the linear regression. The numbers in the bracket are the standard errors. When the lagged aggregate cash stock increases by one standard deviation, the consumption responsiveness to the negative aggregate TFP shock (-2% TFP shock) significantly decreases by 0.17 percentage points. For the positive aggregate TFP shock (+2% TFP shock), the consumption responsiveness decreases by 0.07 percent-

[34]The aggregate shock is defined as a shift from one productivity to the other.

	Dep. Var.: $|log(c_t)|$ $(p.p.)$	
	Neg.	Pos.
	(1)	(2)
$Cash_{t-1}(s.d.)$	-0.166	0.07
	(0.001)	(0.001)
Constant	Yes	Yes
Observations	83	84
R^2	0.996	0.994

Table 3: State-dependent consumption responses to negative and positive shocks

Notes: The table reports the results of the regression of consumption responses to a negative and positive aggregate TFP shock on the lagged aggregate cash stocks using the simulated data. The first column is for the negative aggregate TFP shock, and the second is for the positive aggregate TFP shock. The numbers in the bracket are standard errors.

age points when the lagged aggregate cash stock increases by one standard deviation.

Therefore, the aggregate cash holding gives a consumption buffer against a negative aggregate shock by smoothing the dividend stream in the simulated data. Also, the aggregate cash holding helps a positive productivity shock solely pass down to the consumption. And the consumption buffer effect against the negative TFP shock is significantly stronger than the consumption boosting effect for the positive TFP shock in the model. I support this model prediction from the macro-level data. The data is the quarterly frequency and covers from 1951 to 2018. Consumption and the total dividend of the corporate sector are from NIPA; the aggregate cash holding and the total asset holding are obtained from the Flow of Funds.[35]

Table 4 reports the data counterpart of the results in Table 3, separately for the periods before 1980 (the first two columns) and after 1980 (the last two columns).[36] For this analysis, the consumption variations derived from the TFP variation are controlled by consumption residualization over the polynomials of TFP up to the fourth order.[37] As can be seen from Figure 3, since around 1980, the corporate cash stock has rapidly increased. This rising corporate cash holding has brought a change

[35]All time series are detrended using the HP filter with a frequency parameter at 1600.
[36]The result is not sensitive to the choice of the cutoff year.
[37]I use the utilization-adjusted TFP from Fernald (2014).

	Dependent variables:			
	$\lvert log(c_t) \rvert$ $(p.p.)$ before 1980		$\lvert log(c_t) \rvert$ $(p.p.)$ after 1980	
	Neg. (1)	Pos. (2)	Neg. (3)	Pos. (4)
$Cash_{t-1}(s.d.)$	-0.108	0.036	-0.226	0.164
	(0.09)	(0.072)	(0.085)	(0.09)
Constant	Yes	Yes	Yes	Yes
Observations	63	49	77	79
R^2	0.023	0.005	0.086	0.041

Table 4: State-dependent consumption responses to negative and positive shocks in data

Notes: The table reports the results of the regression of consumption responses to a negative and positive aggregate TFP shock on the lagged aggregate cash stocks using the macro-level data. The first column is for the negative aggregate TFP shock before 1980; the second is for the positive aggregate TFP shock before 1980; the third is for the negative aggregate TFP shock after 1980; the last is for the positive aggregate TFP shock after 1980. The numbers in the bracket are standard errors.

in the relationship between corporate cash stock and consumption. In the pre-1980 periods, consumption responsiveness was not dependent on the one-period-ahead corporate cash stock. In contrast, in the post-1980 periods, consumption responsiveness becomes significantly dependent on the corporate cash stock. When the lagged aggregate cash stock increases by one standard deviation, the consumption responsiveness to the negative aggregate shock significantly decreases by 0.23 percentage points. For the positive aggregate shock, the consumption responsiveness increases by 0.16 percentage points when the lagged aggregate cash stock increases by one standard deviation.[38] In Appendix A, I show that this effect in the data is driven by the dividend channel. These results tightly support the model prediction.

[38] These two estimates are not statistically different from the model-side estimates in Table 3.

6 Concluding remarks

This paper develops a novel method to solve dynamic stochastic general equilibrium models without specifying a law of motion. The method utilizes the ergodic theorem's prediction that if the simulated path is long enough, all the possible equilibrium outcomes are realized on the path. Therefore, the rationally expected future value function can be perfectly recovered at each period on the path by identifying counterfactual state-contingent outcomes on the path. The algorithm runs until the expected path converges to the realized path, so the solution is highly accurate. I apply this method to a heterogeneous-firm business cycle model with the corporate saving glut. Despite the highly nonlinear aggregate dynamics, the method accurately solves the model. In the model, when the economy is with large corporate cash stocks, consumption responds less sensitively to a negative TFP shock and more strongly to a positive TFP shock than it would otherwise do. This model prediction is well supported by the macro-level data.

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