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A Combinatorial Topology Approach to Arrow's Impossibility Theorem

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Abstract

Baryshnikov presented a remarkable algebraic topology proof of Arrow's impossibility theorem trying to understand the underlying reason behind the numerous proofs of this fundamental result of social choice theory. We continue this program, but focusing on combinatorial topology arguments that do not use advanced mathematics, providing a very intuitive geometric reason for Arrow's impossibility under domain restrictions. We present a geometric proof for the basis case of two voters, $\boldsymbol{n = 2}$, and three alternatives, $\boldsymbol{|X| = 3}$, based on the index lemma, that counts the absolute number of times that a closed curve in the plane travels around a point. This yields a characterization of the domain restrictions that allow non-dictatorial aggregation functions and, as a consequence, Baryshnikov's conjecture relating such domains with contractible spaces is revealed as untrue. It also exposes the geometry behind prior pivotal arguments to Arrow's impossibility. We explain why the basis case of two voters, is where this interesting geometry happens, by giving a simple proof that this case implies Arrow's impossibility for any $\boldsymbol{|X| \geq 3}$ and any finite $\boldsymbol{n \geq 2}$.

Keywords: Social choice, Arrow's impossibility theorem, combinatorial topology, domain restrictions, distributed computing, simplicial complexes, index lemma.

1 Introduction

Social choice theory is a highly developed field of interest to economics and political science, and more recently to computer science (Brandt et al., 2016). The modern field of social choice theory took off with Kenneth Arrow's remarkable 1950 result (Arrow, 1950) for the basic problem of democracy: it is impossible to aggregate the individual preferences into a single social preference, under some reasonable-looking axioms. Soon after the publication of Arrow's result alternative proofs began to emerge; starting with Inada (1954), numerous other proofs followed, and continue to be proposed until recently, e.g. Feldman and Serrano (2008); Geanakoplos (2005); SEN et al. (2014); Fey (2014). For an overview, including the importance of Arrow's result, see introductory books such as Gaertner (2009), or more advanced such as Feldman and Serrano (2006).

Motivation

Trying to understand the underlying reason behind the many proofs of Arrow's theorem, Baryshnikov (1993) presented a remarkable different approach, a topological impossibility proof. However, the goal of providing intuition about the nature of the problem of social choice is hindered by the relatively advanced algebraic topology tools used by Baryshnikov (several attempts at explaining the proof have been made Baryshnikov (1997) , Chia (2015) and Baigent (2011)).

Our goal here is to further advance the program of Baryshnikov, while making it accessible to an audience not familiar with algebraic topology. Furthermore, we aim at understanding the gap between the literature on topological social choice (Lauwers, 2000) and combinatorial proofs, which have developed largely independently. We do so by moving from algebraic topology to combinatorial topology, and in doing so discover (and benefit from) remarkable connections with distributed computing (Herlihy et al., 2013).

Contributions

First, we provide new geometric proofs of Arrow's impossibility that do not require any acquaintance with algebraic topology. The proofs give a new insight for the reason of the impossibility, based on the *index lemma*. This is a combinatorial topology result, useful to compute winding numbers. Recall that the *winding number* of a closed curve in the plane around a point is the number of times that the curve passes counterclockwise around the point minus the number of times it passes clockwise.

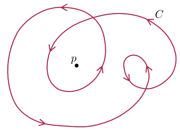


Fig. 1: C has winding number 2 around p.

The geometric argument shows that the basis case of two individuals and three alternatives is somehow special, explaining an intriguing phenomenon, appearing several times in the literature. Some papers simply treat this case only e.g. Akashi (2005), Saari (2011), Baigent (2011) and Tanaka (2009). More interestingly, some papers hint at the idea that this is the case where the interesting things happen. Baryshnikov (1993), in Section 7.1, explains that only the arguments of his proof for triples of alternatives are in fact used, and one could concentrate only on the 2-skeleton of the simplicial complex using one-dimensional (co)homology.

We show the usefulness of the combinatorial topology approach by providing a characterization of the domain restrictions of the basis case for which there is a non-dictatorial aggregation function, a problem that remains open despite a substantial amount of research (Barberà et al., 2020; Elkind et al., 2022). A very simple geometric argument for Arrow's impossibility based on a domain restriction is presented. The domain restriction analysis we present shows that contractibility of the space of preference profiles is not the reason for Arrow's impossibility, as conjectured in topological social choice (Lauwers, 2000).

With the goal of exposing the relation of our topological perspective with with previous proofs (something not done by Baryshnikov (1993)), we present a combinatorial topology perspective of the recent pivotal arguments to prove Arrow's impossibility by Geanakoplos (2005) and Yu (2012), that have received much attention.

Finally, we present a simple proof showing that Arrow's impossibility result for the basis case of two individuals and three alternatives implies the general case. This result has been shown before under the restriction of finite number of alternatives by Tang and Lin (2009) and partially by Akashi (2005), but our proof seems, in addition to be more general, more direct.

New intuition behind Arrow's impossibility and the connection with distributed computing

Very roughly, the intuition behind our approach, for the base case of two voters and three alternatives A, B and C is the following. The first step is to represent the set of possible preferences of the voters, N_I, as well as the set of possible social preferences, N_O, as geometric objects built from triangles. These objects are called 2-dimensional simplicial complexes; an introduction to combinatorial topology is in Section 2.2.

The notation N_I, N_O stands for "input" and "output" complexes, following the notion of a *task* in distributed computing. We present an introduction of the relation with distributed computing in (Herlihy et al., 2013) and in a proceedings version of this paper (Rajsbaum and Raventós-Pujol, 2022).

The second step is to observe that the aggregation map F that decides the social output, induces a simplicial map f from N_I to N_O. In Section 2.3 we reformulate Arrow's problem in terms of requirements about the simplicial map f. We use techniques from combinatorial topology such as the index lemma (see Section 2.2.1) to address the Arrovian problem. The mathematics used is elementary: essentially only basic parity counting operations are needed. Interestingly, the index lemma is also behind the distributed computing impossibilities related to weak symmetry breaking e.g. Castañeda and Rajsbaum (2010) and Goubault et al. (2019).

Organization

First we present the statement of Arrow's theorem, an introduction to combinatorial topology, and how to model Arrow's theorem using combinatorial topology, in Section 2. We provide a proof of Arrow's theorem ($|X| = 3$ and $n = 2$) using the index lemma in Section 3. In Section 4 we study restriction domains. A domain restriction is used in section 4.1 to prove Arrow's impossibility with a very simple intuitive geometric argument. In Section 4.2 a domain restriction is described that does allow for a non-dictatorial aggregation, in spite of having a non-contractible restriction. In Section 4.3 we present the characterization of non-dictatorial domain restrictions. In Section 5 we use combinarotial topology to prove Arrow's theorem explaining the classic pivotal arguments. In Section 7 we present the conclusions. At the end of the paper an Appendix includes an exhaustive discussion of the generalization of Arrow's theorem from the base case we have studied to the general case, and technical details about the proofs.

2 Arrow's impossibility theorem statement: classic and geometric formulations

We start by recalling Arrow's theorem in Section 2.1, we then present a quick introduction to combinatorial topology in Section 2.2 and finally the combinatorial topology restatement of Arrow's model in Section 2.3.

2.1 Classic formulation

Let X be a set of alternatives and $|X| \geq 3$. The set of all strict total orders of X is denoted by W. Let $n \geq 2$ denote the (finite) number of voters, and W^n be the set of profiles of preferences. Thus, $\mathbf{R} = (R_1, \ldots, R_n) \in W^n$ is a profile, where each $R_i \in W$ is the order on X preferred by the i-th voter. An *aggregation map* F is a function from W^n to W that maps each profile of W^n to a unique order in W. For example, if $X = \{A, B, C\}$ and $R_i \in W$, $A\,R_i\,B\,R_i\,C$ denotes that the i-th voter prefers A over B, and B over C. When no confusion arises, simply by ABC.

A classic form of Arrow's impossibility theorem states that whenever the set X of possible alternatives has at least 3 elements, there is no aggregation map F from W^n to W satisfying three reasonable axioms. Formally:

Theorem 1 (Arrow's impossibility theorem) *Let* $|X| \geq 3$ *and* $n \geq 2$. *There is no aggregation map* $F : W^n \to W$ *satisfying the following conditions:*

1. Unanimity. *If alternative, a, is ranked strictly higher than b for all orderings* $R_1, \ldots, R_n$, *then a is ranked strictly higher than b by* $F(R_1, \ldots, R_n)$.
2. Independence of irrelevant alternatives. *For two preference profiles* $\mathbf{R}$ *and* $\mathbf{S}$ *such that for all individuals i, alternatives a and b have the same order in* R_i *as in* S_i, *alternatives a and b have the same order in* $F(\mathbf{R})$ *as in* $F(\mathbf{S})$.
3. Non-dictatorship. *There is no individual k whose strict preferences always prevail. That is, there is no* $k \in \{1, \ldots, n\}$ *such that for all* $\mathbf{R} \in W^n$, *a ranked strictly higher than b by* R_k *implies a ranked strictly higher than b by* $F(\mathbf{R})$, *for all a and b.*

Some formulations of Arrow's impossibility theorem allow ties in the rankings (Arrow, 1951; Fishburn, 1970; Yu, 2015). In this sense, it could seem that the framework we present here is not as general as it might be. However, this is not the case (Baryshnikov, 1993, Lemma 1), and indeed previous proofs (Baryshnikov, 1993; Lauwers, 2000; Baigent, 2011) of Arrow's impossibility often assume, as we do, strict orders.

2.2 Combinatorial topology

Algebraic topology is a deep and highly developed branch of mathematics, studying algebraic invariants of topological spaces, such as homology groups. When the spaces are composed of individual cells attached to each other in a simple way, we have combinatorial topology, which has been gaining importance more recently as more and more applications are discovered, and the fact that such invariants can be computable. Here we use only elementary notions that can be found in books such as Henle (1994) and Herlihy et al. (2013), for more advanced treatments see Kozlov (2008) and Stillwell (1980).

A *simplicial complex* is a family of sets that is closed under taking subsets, that is, every subset of a set in the family is also in the family. The elements of the sets are called *vertices*. A set of the simplicial complex is called a *simplex*, and its *dimension* is d if it has $d+1$ elements; we say it is a d-simplex. In this paper we consider only simplicial complexes of dimension 2, meaning that each simplex contains at most 3 elements.

A simplicial complex is a purely combinatorial object, it can be seen as a generalization of a graph; in our case, in addition to edges consisting of pairs of vertices, we allow also triangles consisting of triples of vertices. As in graph theory, it is sometimes useful to embed a simplicial complex in Euclidean space. A simplicial complex can represent a discretization of a geometric object, in the case of dimension 2, a triangulation. We may think of the simplices of size 3 as triangles, the simplices of size 2 as edges, and simplices of size 1 as points, as illustrated, for example, in Figure 2 and Figure 3.

A subset of a simplex is called a *face*. Notice that if a triangle is in the complex, so are its three 1-dimensional faces (edges), and its three 0-dimensional faces (vertices), because a complex is closed under containment.

Additionally, we say that a simplicial complex is *chromatic* with colours in a set $\mathcal{C}$ if every vertex is labeled with an element from $\mathcal{C}$.

A *simplicial map* is a function from the vertices of one simplicial complex K to the vertices of another simplicial complex K', that preserves simplices: it sends sets of vertices that belong to a simplex of K, to sets of vertices that belong to a simplex of K'; thus, it respects the simplicial structure. A simplicial map is a discrete version of a continuous map. A simplical map from K to K' is *chromatic* if K and K' are chromatic with respect the same colour set $\mathcal{C}$ and the map preserves the colours. That is, every vertex of K and its image are labeled with the same colour.

2.2.1 Index lemma

Quoting from Henle (1994),

> *"The combinatorial method is used not only to construct complicated figures from simple ones but also to deduce properties of the complicated from the simple. In combinatorial topology it is remarkable that the only machinery needed to make these deductions is the elementary process of counting!"*

The index lemma illustrates this point. Here we describe the basic version of Henle (1994).

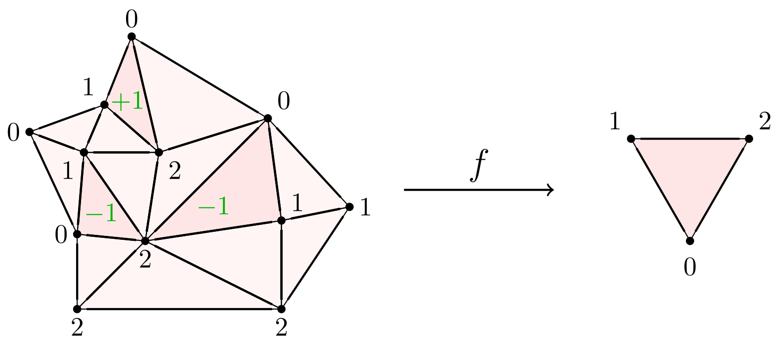


Fig. 2: Simplicial complex from Henle (1994) illustrating the index lemma, highlighting the three complete triangles.

Consider a simplicial complex, consisting of a polygon of any number of sides, triangulated; an example is in Figure 2. The vertices are coloured arbitrarily, with colours in $\mathcal{C} = \{0, 1, 2\}$. The *content* C is the number of triangles labelled $0, 1, 2$, counted by orientation: it counts $+1$ if its labels read 012 in a counterclockwise direction around the triangle, and counts -1 if they clockwise around the triangle. The *index* I is the number of edges labeled 01 around the boundary[1] of the polygon counted by orientation: and edge counts $+1$ if it reads 01 counterclockwise around the polygon, and -1 if it reads 01 clockwise. In the figure, $I = C = -1$. The index lemma says that this is always the case. Formally:

Theorem 2 (Index Lemma) *Let K be an oriented, chromatic and triangulated polygon with colours $\{0, 1, 2\}$. Then its content C is equal to its index I.*

This simplicial complex from Figure 2 illustrates the index lemma, highlighting the three complete triangles.

The miracle of the index lemma is that the proof is a very simple parity counting argument (see Theorem 9), despite the fact of being at the core of the study of vector fields and other areas (Henle, 1994). It is a generalization of Sperner's lemma (which is a discrete analogue to Brouwer's fixed point theorem (see Loera et al. (2019))). For a general formulation of the index lemma see Fan (1967).

The see the winding number interpretation, we think of the coloring of the vertices of the triangulated polygon K as a simplicial map f to the complex K' that consists of a single 2-dimensional simplex $\{0, 1, 2\}$, together with all its faces. The index lemma counts the number of times the boundary of K is wrapped around the boundary of K'. In Figure 2, if one follows the colors on the boundary of the triangulated polygon along the triangle on the right of the figure, one can see that it wraps around the triangle once, counterclockwise. While on the inside, 3 triangles are mapped on top of the triangle on the right, two with negative orientation and one with positive orientation.

In Section 3 we need a simple generalization (Theorem 9 in the Appendix) to prove Arrow's theorem: while the boundary of the complex consists of exterior edges belonging to a single triangle, each interior edge belongs to an even number of triangles (at least 2). As opposed to Sperner's lemma, the index lemma requires the complex to be orientable (Definition 2 in Appendix **??**). An example of a such an orientable complex, is the triangulated torus in Figure 3. It has no exterior edges. After removing one triangle, say cfg, the boundary consists of the edges of this triangle. An example a complex that is not orientable is a triangulation of the Möbius strip in Figure 3.

2.3 Combinatorial topology from Arrow's theorem

In this section we will transform the Arrovian model described in Section 2.1 into the topological model we will study in the following sections. We will follow the construction introduced by Baryshnikov (1993).

[1] In the subsequent pages we use boundary as usually in topology (see e.g. (Henle, 1994, Section 4)).

Fig. 3: Arrows indicate edges that are identified. The triangulated torus on the left has 10 vertices. The triangulated Möbius strip on the right has only 6 vertices, the boundary consists of a cycle of 6 edges: ab, bc, cd, de, ef and fa.

We depart from the sets of orderings W, profiles W^n and aggregation maps $F : W^n \to W$, and we will obtain a simplicial complex N_O representing the set of social preferences, a simplicial complex N_I representing the set of individual profiles and a simplicial map $f : N_I \to N_O$ representing the aggregation map F. Next, we will introduce such mathematical objects following the same order.

Define the set $U^{\overline{\boldsymbol{\sigma}}}_{\alpha\beta}$, with $\alpha, \beta \in X$ and $\overline{\boldsymbol{\sigma}} \in \{+,-\}^n$ as the subset of profiles of W^n where for each voter i, α is ranked higher than β if $\overline{\boldsymbol{\sigma}}(i) = +$, and otherwise, β is ranked higher than α.

$$U^{\overline{\boldsymbol{\sigma}}}_{\alpha\beta} = \{\mathbf{R} \in W^n : \alpha R_i \beta \text{ if } \overline{\boldsymbol{\sigma}}(i) = +,\ \beta R_i \alpha \text{ if } \overline{\boldsymbol{\sigma}}(i) = -\}$$

Notice that $U^{\overline{\boldsymbol{\sigma}}}_{\alpha\beta}$ defines the same set of social preferences as $U^{-\overline{\boldsymbol{\sigma}}}_{\beta\alpha}$. We can define a simplicial complex whose vertices V are $U^{\overline{\boldsymbol{\sigma}}}_{\alpha\beta}$. A set of vertices of V forms a simplex iff their intersection (namely, of the social preferences corresponding to each of the vertices $U^{\overline{\boldsymbol{\sigma}}}_{\alpha\beta}$) is nonempty.

Definition 1 The *input complex* N_I is the simplicial complex whose set of vertices V consists of all $U^{\overline{\boldsymbol{\sigma}}}_{\alpha\beta}$, $\alpha, \beta \in W$, and $\boldsymbol{\sigma} \in \{+,-\}^2$,

The *output complex* N_O is the simplicial complex whose set of vertices V consists of all $U^{\overline{\boldsymbol{\sigma}}}_{\alpha\beta}$, $\alpha, \beta \in W$, and $\boldsymbol{\sigma} \in \{+,-\}^1$.

For both N_I and N_O, a set of vertices of V forms a simplex iff their intersection is nonempty.

The complex N_O is depicted on the rights side of Figure 4 taking $X = \{A, B, C\}$. Notice that $U^+_{\alpha\beta}$ denotes the same set as $U^-_{\beta\alpha}$. The complex N_O is chromatic: its colours are the (non-ordered) pair of alternatives placed on the subindex of every vertex. For instance, the vertex $U^+_{\alpha\beta}$ is coloured by $\{\alpha, \beta\}$. A facet is a 2-simplex $\left\{U^{\sigma_0}_{\alpha_0\beta_0}, U^{\sigma_1}_{\alpha_1\beta_1}, U^{\sigma_2}_{\alpha_2\beta_2}\right\}$, which represents the strict order that is compatible with its three vertices, that is, the strict order contained in $U^{\sigma_0}_{\alpha_0\beta_0} \cap U^{\sigma_1}_{\alpha_1\beta_1} \cap U^{\sigma_2}_{\alpha_2\beta_2}$. Consider for example the triangle ABC in Figure 4, and its two vertices U^+_{AB} and U^+_{BC}. Notice that $U^+_{AB} = \{ABC, ACB, CAB\}$, and $U^+_{BC} = \{ABC, BAC, BCA\}$. These two vertices form an edge of N_O because their intersection is not empty. Moreover, it belongs to a single triangle, because the third vertex is unique, $U^-_{CA} = \{ABC, ACB, BAC\}$. Indeed, the three vertices intersect in a unique order, ABC.

Moreover, there are exactly two *hollow triangles* in N_O, that is, three (distinctly coloured vertices) which do not form a simplex. The external one,

$\{U^+_{AB}, U^+_{BC}, U^+_{CA}\}$, would require a preference preferring A over B, B over C and C over A (a Condorcet cycle), and the central one, $\{U^-_{AB}, U^-_{BC}, U^-_{CA}\}$, the converse. Furthermore, the boundary edges that belong to a single triangle are those that, by transitivity, imply the third vertex, e.g. the edge $\{U^+_{AB}, U^+_{BC}\}$ implies the third vertex, U^-_{CA}.

Remark 1 For simplicity, we always denote the six vertices of N_O by the representatives U^+_{AB}, U^-_{AB}, U^+_{BC}, U^-_{BC}, U^+_{CA} and U^-_{CA}, as in the figure. In Section 3 we will need all vertices in the same boundary to share the same sign.

Remark 2 Consider two *adjacent* 2-simplices, intersecting in an edge. The strict order associated with one simplex and the one associated with the other simplex are equal, modulo permuting two consecutive elements in the strict order. For example, the facet corresponding to ABC and the one corresponding to ACB are adjacent: they are equal modulo the permutation of B and C. This fact will be used in the proof of Section 5.

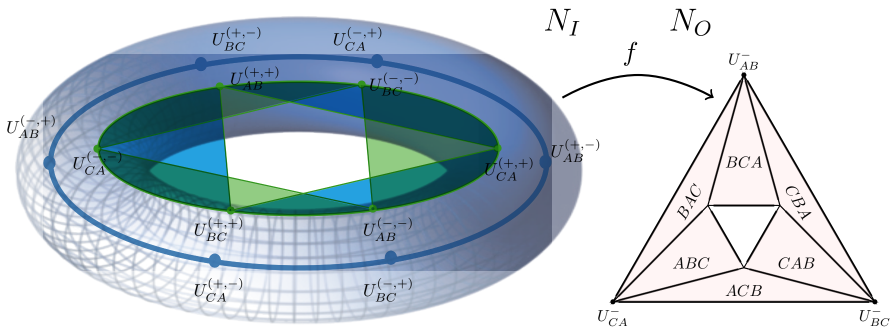


Fig. 4: On the left, N_I is a torus with 12 additional triangles that form four boundary hollow triangles. Here only 6 of them are shown together with their 2 hollow triangles (attached to the green-dashed cycle); the other 6 triangles are omitted for clarity, they are attached to the blue-dotted cycle. Instead, N_O is homeomorphic to a cylinder with two hollow boundary triangles.

The complex N_I is much bigger than N_O. A schematic representation is in Figure 4 and Figure 6. Notice, in the remark below, that analogous observations to the ones we made for N_O hold for N_I as well.

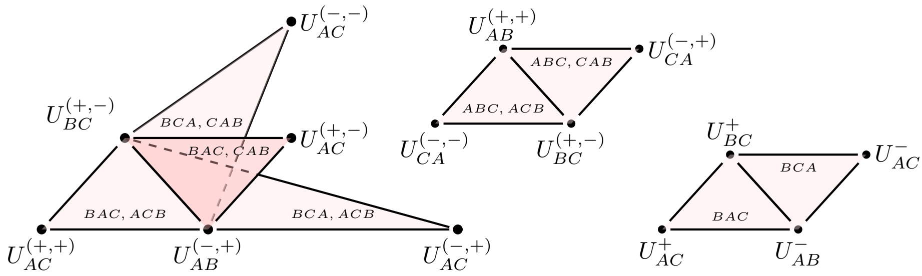


Fig. 5: Four triangles of N_I, then two, and finally two of N_O, intersecting in an edge, because they agree on two pairwise preferences, AB and BC.

Remark 3 First, in the case $|X| = 3$ and $n = 2$, whereas each 2-simplex of N_O is a preference, in N_I each 2-simplex is represented by two individual preferences. Second, consider two adjacent 2-simplices (intersecting in an edge) of N_I. The individual preferences associated with one simplex and those associated with the other simplex are equal, modulo permuting the preference of two alternatives, x, y, of one or two voters, without changing the preferences of other alternatives. For example, in Figure 5, the triangles (BAC, ACB) and (BCA, CAB) are adjacent, because the preferences of both voters over A and C are exchanged, and only over A and C. This fact will be a keystone of the proof of Section 5.

Finally, we have the following two propositions.

Proposition 3 *Given an aggregation map $F : W^2 \to W$ satisfying independence of irrelevant alternatives, there is an induced aggregation chromatic simplicial map $f : N_I \to N_O$, and viceversa, any chromatic simplicial map $f : N_I \to N_O$ induces an aggregation map $F : W^2 \to W$ satisfying independence of irrelevant alternatives.*

Proof Let $n = 2$, but the proof holds for any n. Given $U_{\alpha\beta}^{\boldsymbol{\sigma}}$, since it represents a subset of profiles in W^n defined purely by the orderings between α and β, $f\left(U_{\alpha\beta}^{\boldsymbol{\sigma}}\right)$ is defined as $U_{\alpha\beta}^{\sigma}$ with the sign σ determined by the ordering of α and β on the social aggregation of any of the profiles in $U_{\alpha\beta}^{\boldsymbol{\sigma}}$[2]. The images of the higher dimensional simplices of N_I can be defined by extension. We only need such simplices to be in N_O. However, this is immediate because a simplex in N_I exists whenever the intersection of their vertices contains at least one profile. The image of such a profile must belong to the intersection of the images of those vertices, since the image of a profile is determined by the ordering of pairs of alternatives.

[2] If we assume independence of irrelevant alternatives together with unanimity, it can be defined as $f\left(U_{\alpha\beta}^{\boldsymbol{\sigma}}\right) = \left\{F(\mathbf{R}) \in W : \mathbf{R} \in U_{\alpha\beta}^{\boldsymbol{\sigma}}\right\}$.

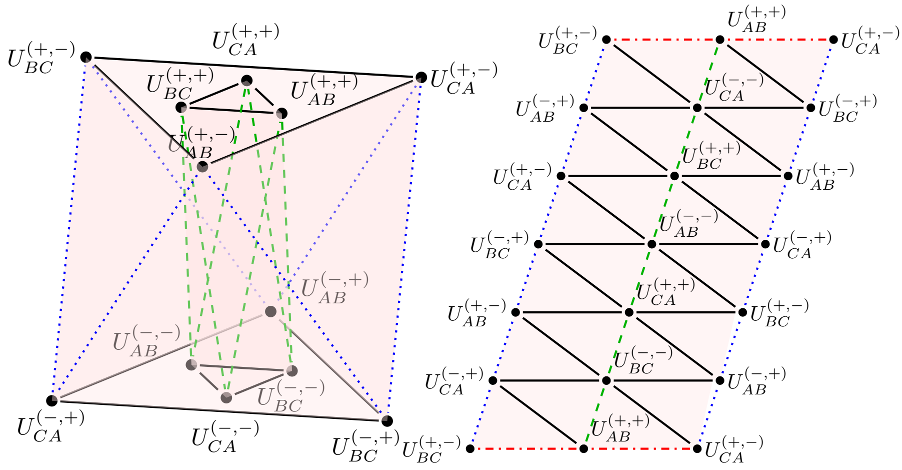


Fig. 6: When $|X| = 3$ and $n = 2$, the complex N_I can be built using two (cylindrical) copies of N_O placed one inside the other (on the left side of the figure). The inner cylinder are the unanimous profiles, whereas the outer one are the profiles where the voters have opposite preferences. Additionally, both cylinders are joined through the torus in the right (the torus is folded by identifying vertices according to the coloured and patterned edges), so the total number of vertices of N_I is 12.

Notice that the simplicial map f is chromatic, in the sense that it sends vertices of N_I labeled $\alpha\beta$ to vertices of N_O also labeled with the same alternatives, $\alpha\beta$. The other direction of the claim is analogous. $\square$

We say that f is a *projection* if there is a voter k such that for all vertices $U^{\overline{\sigma}}_{\alpha\beta}$ of N_I, $f\left(U^{\overline{\sigma}}_{\alpha\beta}\right) = U^{\overline{\sigma}(k)}_{\alpha\beta}$. That is, f return the preference of the k-th voter.

Taking into account the discussion above, we can state:

Proposition 4 *Let $F : W^n \to W$ be an aggregation map satisfying independence of irrelevant alternatives and $f : N_I \to N_O$ be its induced simplicial aggregation map. Then, F satisfies unanimity iff f satisfies unanimity. Moreover, F is dictatorial iff f is a projection.*

Proof If F satisfies unanimity, it sends profiles where everybody prefers α over β to a social preference where α is preferred over β. Then f sends the vertices where everybody prefers α over β, denoted $U^{(+,\cdots,+)}_{\alpha\beta}$, to vertices where α is preferred over β in the social choice, denoted $U^{+}_{\alpha\beta}$. Thus, we say that the simplicial map f

satisfies unanimity if it is such that for all vertices $U_{\alpha\beta}^{(+,\cdots,+)}$ of N_I, it holds that $f\left(U_{\alpha\beta}^{(+,\cdots,+)}\right) = U_{\alpha\beta}^{+}$.

By the definition of projection itself, F being dictatorial is equivalent to f being a projection. □

These two propositions assure that Arrow's impossibility theorem (Theorem 1) for $|X| = 3$ and $n = 2$ is equivalent to Theorem 5 in the next section. Moreover, we defer to Section 6 a pair of lemmas that allows generalizing the impossibility from the base case ($|X| = 3$ and $n = 2$) to the general case ($|X| \geq 3$ and $n \geq 2$).

3 Impossibility proof based on the index lemma

In this section we will prove the Theorem 5 below, equivalent to Arrow's impossibility theorem when $|X| = 3$ and $n = 2$, using the index lemma. This lemma is stated and discussed in detail in Section **??**.

Theorem 5 *If $f : N_I \to N_O$ is a chromatic simplicial map that satisfies unanimity then f is a projection.*

Proof Let $f : N_I \to N_O$ be a simplicial map such that for all vertices $U_{\alpha\beta}^{(+,+)}$ of N_I, it holds that $f\left(U_{\alpha\beta}^{(+,+)}\right) = U_{\alpha\beta}^{+}$. We will use f to define an additional coloring of the vertices of N_I with colors $\mathcal{C} = \{0, 1, 2\}$[3].

In order to define the coloring of the vertices of N_I, first we colour them with $\{+1, -1\}$ according to the image of every vertex by f. That is, we label $U_{\alpha\beta}^{\boldsymbol{\overline{\sigma}}}$ with $+1$ iff $f\left(U_{\alpha\beta}^{\boldsymbol{\overline{\sigma}}}\right) \in N_O$ has the superindex $+$, and otherwise with -1. We call it the sign of $U_{\alpha\beta}^{\boldsymbol{\overline{\sigma}}}$ and it is denoted by $s\left(U_{\alpha\beta}^{\boldsymbol{\overline{\sigma}}}\right)$.

Second, we color every vertex of N_I with one colour p in $\mathcal{C} = \{0, 1, 2\}$ following the rule:

$$p\left(U_{\alpha\beta}^{\boldsymbol{\overline{\sigma}}}\right) = ID\left(U_{\alpha\beta}^{\boldsymbol{\overline{\sigma}}}\right) + s\left(U_{\alpha\beta}^{\boldsymbol{\overline{\sigma}}}\right) \qquad (mod\, 3) \tag{1}$$

where the *identifiers* are defined as $ID\left(U_{AB}^{\boldsymbol{\overline{\sigma}}}\right) = 0$, $ID\left(U_{BC}^{\boldsymbol{\overline{\sigma}}}\right) = 1$ and $ID\left(U_{CA}^{\boldsymbol{\overline{\sigma}}}\right) = 2$ (for every $\boldsymbol{\overline{\sigma}} \in \{+, -\}^n$).

We show in Section **??** that N_I is orientable and we can use the index lemma.

Notice that a cycle of three vertices is 3-coloured if and only if the sign of all of them is the same[4]. This implies that the content $C = 0$ because no 2-simplex in N_I can be mapped to one of the holes in N_O.

We conclude from the index lemma that $I = 0$, on the boundary of N_I, which consists of 4 combinations of Condorcet cycles (see Figure 7). The contribution to the index from the unanimity cycles is $+2$ (see Figure 8).

Since the contribution of the unanimity cycles is $+2$ and $I = 0$, the two remaining contributions to I have to be -1 for each one of the remaining boundary components.

[3] We remark that this new coloring is different from the one used to introduce N_I in Definition 1.

[4] The key of this statement relies on the three vertices of any triangle having different identifiers. So, the combination of two distinct signs with two distinct identifiers always induces the same colour p.

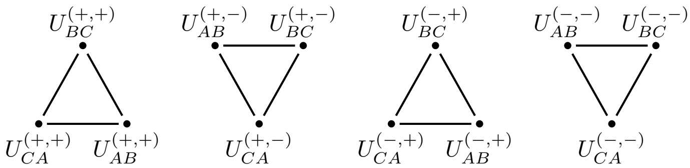


Fig. 7: N_I has four boundary components generated by Condorcet cycles. A single triangle intersects each boundary edge since each pair of vertices determines the third one by transitivity.

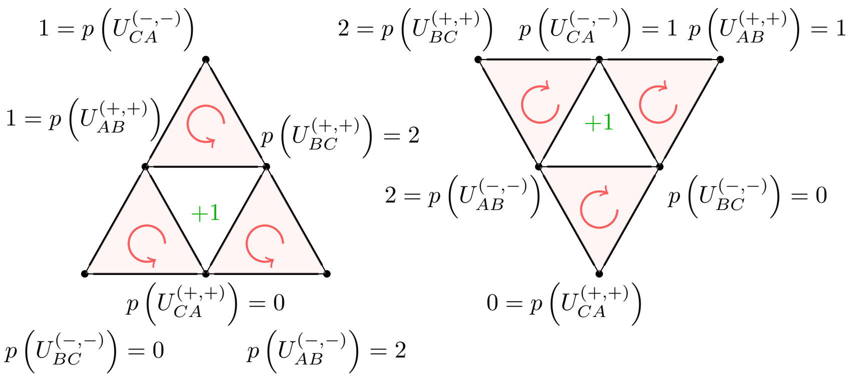


Fig. 8: The contribution of these two boundary components to the index is +2 according with the orientation defined in Proposition 10, Figure 18 in Appendix **??**.

So, we can conclude that both have to be tricoloured and mapped by f to the boundary of N_O.

Both of these boundary components cannot be mapped to the same boundary of N_O because if it were the case all of the non-unanimous vertices would be mapped in the same boundary. Then, for instance, the simplex $\left\{U_{AB}^{(-,+)}, U_{BC}^{(-,+)}, U_{CA}^{(+,-)}\right\}$ would be mapped to one of the hollow triangles of N_O (see Figure 9).

Finally, we have all the information we need about the images of the 12 vertices of N_I to state that f is a projection. Recall that the images of the first and the fourth boundaries in Figure 7 are determined by the unanimity. If the second boundary is mapped to the inner boundary of N_O (and the third in the outer), it is straightforward to check that f is the projection over the first component. In contrast, if the second boundary is mapped to the outer boundary of N_O (and the third on the inner), then f is the projection over the second component. $\square$

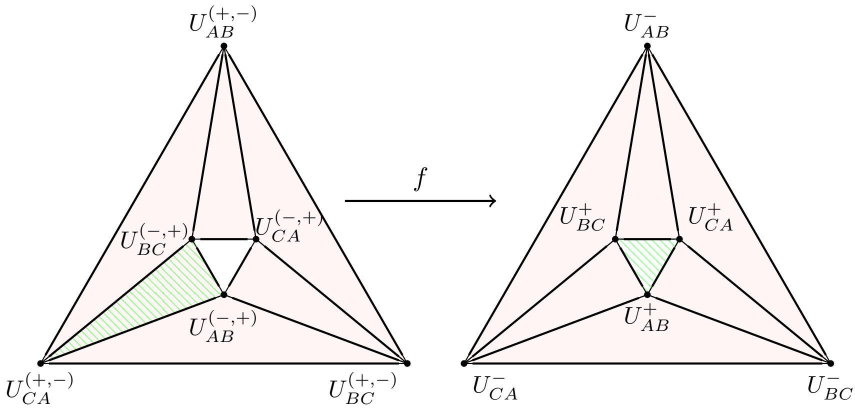


Fig. 9: If these two boundary components of N_I are mapped to the same boundary component of N_O, then $U_{AB}^{(-,+)}$, $U_{BC}^{(-,+)}$ and $U_{CA}^{(+,-)}$ are also mapped to the same boundary component. That is, a hollow triangle.

For completeness we have included in Appendix 6 a discussion and two lemmas that generalizes the Arrow's impossibility theorem in the base case ($|X| = 3$ and $n = 2$) to the general case Theorem 1.

4 Applying the combinatorial topology approach to domain restrictions

Arrow's impossibility applies to *universal domains*, where all possible individual preferences are considered. There is an extensive literature on the subject of domain restrictions, going back at least to Black (1948), Arrow (1951) and their famous *single-peaked* domain restriction, where the alternatives to be ranked lie on a one-dimensional axis and voters prefer values that are close to their favorite value. The research area is still very active today, some recent surveys are Barberà et al. (2020) and Elkind et al. (2022). Researchers have proved that it is possible to avoid Arrow's impossibility on various non-universal domains, including generalizations of single-peakedness, see, e.g. Gaertner (2002) and Le Breton and Weymark (2011) and the previous surveys for many examples. However, there is no general rule characterizing the domains in which aggregation is possible.

We illustrate here how the combinatorial topology approach can shed light on this topic. We present a very intuitive proof of Arrow's impossibility using domain restrictions in Section 4.1. We provide a characterization of the domain restrictions of the basis case in which non-dictatorial aggregation is possible in Section 4.3. We also discuss the role of contractibility of the restricted domain, showing it is not what determines the possibility of avoiding Arrow's impossibility, in Section 4.2.

Remarkably, considering task solvability under restricted domains has been thoroughly studied in distributed computing since Mostéfaoui et al. (2003).

4.1 Arrow's impossibility using domain restrictions

We start with a domain restriction that exposes clearly a geometric reason for Arrow's impossibility, related to winding numbers, providing another proof of Theorem 5. It is the basis of the characterization of the domain restrictions in which non-dictatorial aggregation is possible of Section 4.3.

Recall the torus on the right of Figure 6. It consists of all the social profiles of N_I where the two voters disagree in either 1 or 2 of their pairwise preferences. The torus is depicted again in Figure 10, where in the top-left triangle, the profile is (ABC, ACB), and there is disagreement in only one pairwise preference, BC, since the first voter prefers B over C and the second prefers C over B. In the following triangle on the left, the profile is (BAC, ACB), with two pairwise disagreements, on BC and on AB.

The torus is made of two triangulated cylinders, joined by the blue-dotted circle and by the green-dashed circle. The left cylinder is called C_1 and the right one is C_2. They are symmetric, if one exchanges the voter 1 and voter 2 in C_1 one gets C_2. Namely, the top-left triangle of C_1 is (ABC, ACB), and the symmetric triangle in C_2 is (ACB, ABC). Similarly for the next triangle of C_1, (BAC, ACB), its symmetric triangle on C_2 is (ACB, BAC).

Consider C_1 as a domain restriction of N_I, in Figure 10. It is obtained by removing the cylinder C_2 from the torus on the right of Figure 6, and removing also both of the concentric cylinders on the left of the figure, corresponding to unanimous profiles and those where the voters have opposite preferences. In Figure 10 all the triangles of C_2 are removed from the torus: from top to bottom, the triangles (CAB, ABC), (ACB, ABC), etc. Only the triangles on the left remain, which form the cylinder C_1. Notice that N_O is also a cylinder, except that the cylinder C_1 is subdivided into 12 triangles while N_O consists of 6 triangles.

Now, Arrow's geometric impossibility becomes clear: C_1 is wrapped once around N_O, and the wrapping is determined by the green-dashed cycle in C_1, due to unanimity. In Figure 10 the image of the green-dashed cycle in C_1 on N_O is shown. This implies that the blue-dotted cycle, which is parallel to the green-dashed cycle, also has to wrap once around the cylinder, going in the same direction. There are two options for the aggregation function, labeled on the blue-dotted edges; to map the first (from top to bottom) blue-dotted edge to the edge 2 or to 5, the next one to 3 or 0 in N_O, and so on. In the first option the first voter is the dictator, in the second option the second voter is (in either case, the blue-dotted cycle goes on top of the green-dashed cycle of N_O).

The argument can be formalized using the index lemma, as illustrated in Figure 11. Consider the orientation in the figure, where one can see that the

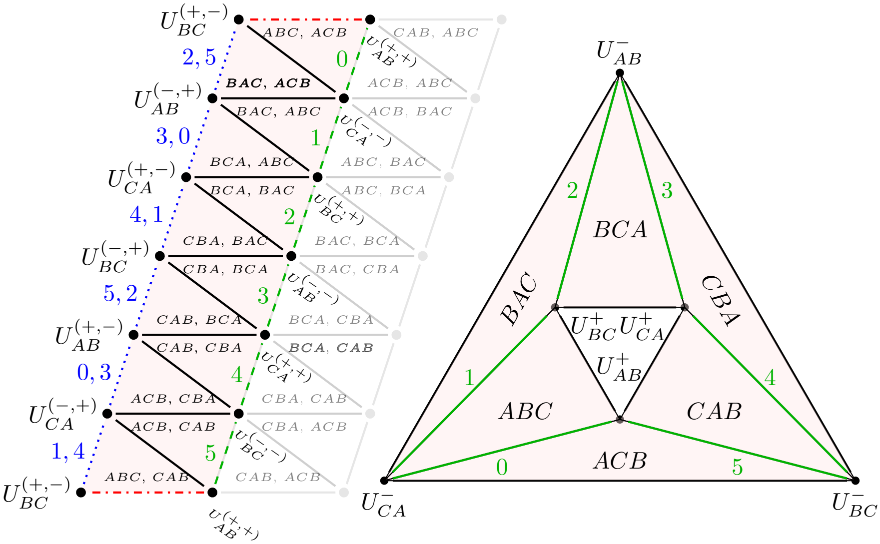


Fig. 10: On the left is C_1, a domain restriction on N_I, resulting in a cylinder and how the green-dashed cycle is mapped to N_O. Inside of each triangle of C_1 is the corresponding individual preference; the top triangle is ABC, ACB, the next one BAC, ACB, and so on. The blue-dotted cycle has two labels on each of its edges; the first one is the social choice where the first voter is the dictator, from top to bottom, $2, 3, 4, 5, 0, 1$. With the second labels, the second voter is the dictator.

boundary of C_1 consists of two cycles, where the green-dashed cycle gets an induced orientation downwards, while the blue-dotted cycle gets it upwards. The colors of the vertices of C_1 are defined following the same procedure as in the proof of Theorem 5.

As in the proof of Theorem 5, no triangle of C_1 is colored with 3 colors, because there is no triangle with 3 colors in N_O to which f could be mapped to. Hence we have that $C = 0$. Then, notice that the contribution to the index I of the green-dashed cycle is $+1$, since exactly one of its edges is colored $\overrightarrow{01}$ (the edge labeled 5, and with positive direction). Since $I = C$, the contribution of the blue-dotted cycle must be -1. Finally, there are only two of its edges that could be coloured $\overrightarrow{10}$ (upwardly oriented) as indicated in the figure.

We can repeat the same procedure for the edges $\overrightarrow{12}$ instead of $\overrightarrow{01}$. As before, there are only two edges in the dotted-blue cycle which can compensate the index: $\left\{U_{CA}^{(+,-)}, U_{BC}^{(-,+)}\right\}$ or $\left\{U_{CA}^{(-,+)}, U_{BC}^{(+,-)}\right\}$. Notice that each of these edges intersect with one of the edges from the case $\overrightarrow{01}$. Moreover, the edge compensating the index in the case $\overrightarrow{01}$ determines the one in the case $\overrightarrow{12}$: for instance.

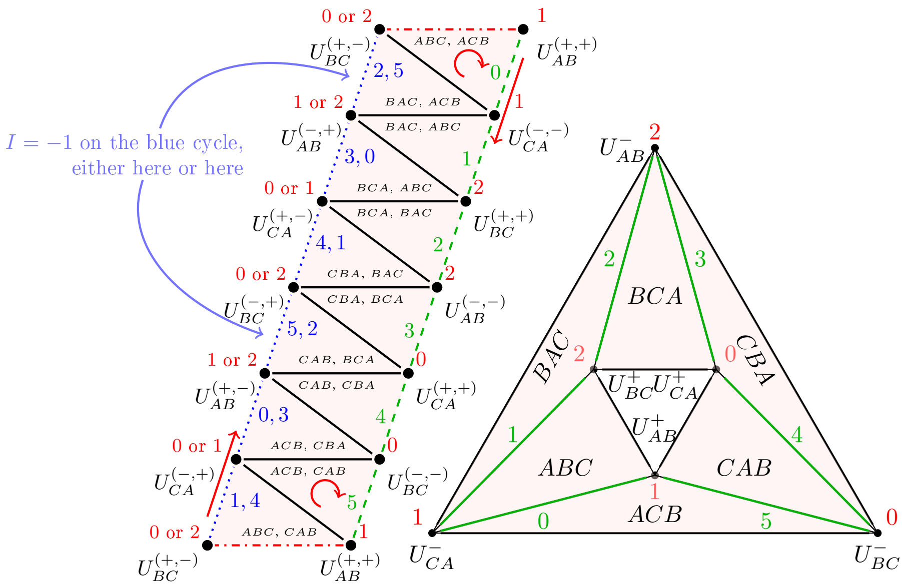


Fig. 11: The index lemma on the cylinder C_1. The red arrows represents the orientation on the triangles and its induced orientation on the boundary circles. Moreover, the red numbers associated to each vertex is its coloring. Whereas the vertices on the green-dashed circle are determined by the unanimity, the ones in the blue-dotted circle have two possibilities.

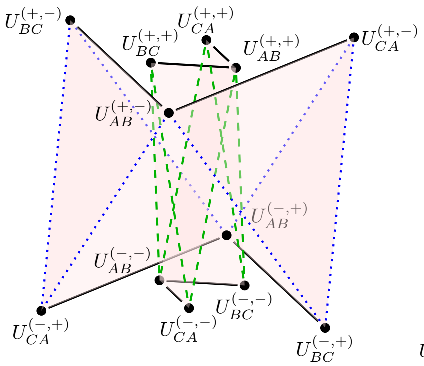
if $\left\{U_{AB}^{(-,+)}, U_{BC}^{(+,-)}\right\}$ compensate the index in the $\overrightarrow{01}$ case, $U_{BC}^{(+,-)}$ must be coloured with 0. Then, $U_{CA}^{(+,-)}$ must compensate index in the case $\overrightarrow{12}$ because $\left\{U_{CA}^{(-,+)}, U_{BC}^{(+,-)}\right\}$ do not (because $U_{BC}^{(+,-)}$ is coloured with 0 and not 2). Following an equivalent argument, if $\left\{U_{AB}^{(+,-)}, U_{BC}^{(-,+)}\right\}$ compensate the index in the case $\overrightarrow{01}$, $\left\{U_{CA}^{(-,+)}, U_{BC}^{(+,-)}\right\}$ must compensate it in the case $\overrightarrow{01}$.

Notice that the same argument can be applied into the case $\overrightarrow{20}$. So, chosing the top edge $\left\{U_{AB}^{(-,+)}, U_{BC}^{(+,-)}\right\}$ is the one colored $\overrightarrow{10}$ or if it is the $\left\{U_{AB}^{(+,-)}, U_{BC}^{(-,+)}\right\}$, determines the colouring of all vertices. Moreover, it can be easily computed that if the top edge $\left\{U_{AB}^{(-,+)}, U_{BC}^{(+,-)}\right\}$ is the one colored $\overrightarrow{10}$, then the 2nd voter is the dictator, else if the edge $\left\{U_{AB}^{(+,-)}, U_{BC}^{(-,+)}\right\}$ is the one colored $\overrightarrow{10}$, then the 1st voter is the dictator.

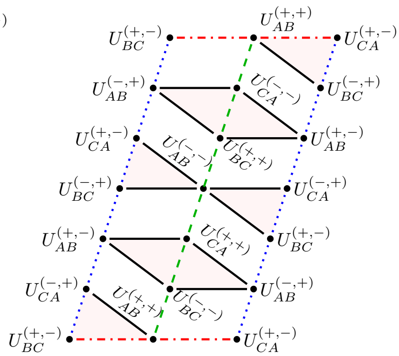
4.2 Eluding Arrow's impossibility while preserving non-contractibility

It has been argued that the existence of a rule that permits aggregation is related to contractibility of a topological space. For the existence case in the continuous setting (which is different from our Arrovian setting), Chichilnisky and Heal (1983), and a 1954 topology theorem by Eckmann (2004) show that, for a general class of domains, contractibility is necessary and sufficient. Building on this result and on Baryshnikov (1993), for weak orders, Tanaka (2009) shows a connection with Brower's fixed point theorem, in the case of $n = 2$ and $|X| = 3$. Baryshnikov (1993) and other authors such as Lauwers (2000) and Baigent (2011) conjectured in subsequent publications that the aggregation on non-universal domains could be equivalent to the contractibility of the induced input simplicial complex. That is, the aggregation à la Arrow on a domain $D \subseteq W^n$ would be possible iff the induced complex N_I' is contractible. Moreover, they added that in the well-known case of single-peaked preferences (in which aggregation is possible) contractibility is satisfied.

Next, we present a domain of preferences that proves that Baryshnikov's conjecture above is not true. That is, the domain N_I'' represented in Figure 12 is not contractible and it allows non-dictatorial aggregation maps.



(a)



(b)

Fig. 12: The restricted domain N_I'' is the union of the simplicial complexes represented in (a) and (b) according the identifications defined by vertices' labeling and colours.

This restricted domain N_I'' corresponds to a polarised society where political parties are classified as left-wing and right-wing parties. Assume that every left-wing voter will prefer all left-wing parties over all right-wing parties (vice-versa for right-wing voters). *A priori* we do not know if a voter is right-wing

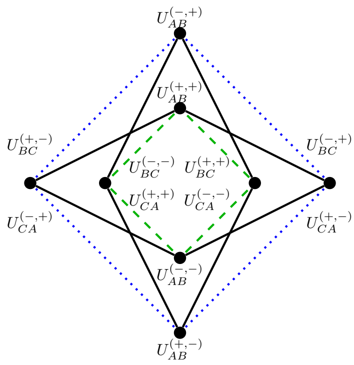


Fig. 13: The simplicial complex N^*. The colours and patterns of the edges (resp. the labellings of the vertices) show where the edges (resp. the vertices) of N_I'' have been compressed in N^*.

or left-wing. The polarized preferences in this section are a particular case of group-separable preferences (see. e.g. Elkind et al. (2022)).

We focus on the case in which there are two right-wing parties $\{A, B\}$ and one left-wing party C and two voters ($n = 2$). This way, N_I'' can be compared with the previous examples and proofs on this article.

The polarised domain restriction deletes the profiles in which a voter has C as the middle preferred party[5]. For example, no voter will have the preference ACB because it prefer the right-wing party A over the left-wing party C and C over the right-wing party B. Formally, applying this restriction means deleting from Figure 6 the edges of the form $\left\{U_{CA}^{(+,\cdot)}, U_{BC}^{(+,\cdot)}\right\}$, $\left\{U_{CA}^{(-,\cdot)}, U_{BC}^{(-,\cdot)}\right\}$, $\left\{U_{CA}^{(\cdot,+)}, U_{BC}^{(\cdot,+)}\right\}$ and $\left\{U_{CA}^{(\cdot,-)}, U_{BC}^{(\cdot,-)}\right\}$ and all triangles containing them, and we obtain the simplicial complex N_I'' represented in Figure 12.

There are non-dictatorial aggregation rules for N_I''. One of these rules is defined by two local dictators. The first voter is a local dictator between the right-wing parties A and B, whereas the second voter is a local dictator between a right-wing party and the left wing-party C. Formally, this aggregation map F is defined for every profile $\mathbf{R}$ in the domain as:

$$AF(\mathbf{R})B \Leftrightarrow AR_1B, \qquad AF(\mathbf{R})C \Leftrightarrow AR_2C, \qquad BF(\mathbf{R})C \Leftrightarrow BR_2C.$$

Using the fact that $AF(\mathbf{R})C \Leftrightarrow BF(\mathbf{R})C$, it is straightforward to check that F is well-defined (i.e. $F(\mathbf{R})$ is transitive and complete for every $\mathbf{R}$). Additionally, F is unanimous, non-dictatorial and satisfies the independence of irrelevant alternatives.

[5]This condition is a particular case of the *triplewise value-restriction* introduced in Sen (1966). However, polarized domains with at least four parties do not satisfy the triplewise value-restriction.

It remains to check that N_I'' is not contractible. In Figure 12, N_I'' has been drawn deleting a triangle on each of the concentric cylinders of N_I, and from the torus they only remain four pairs of triangles that join both cylinders. To see that N_I'' is not contractible, we apply contractions to N_I'' obtaining a new topological space N^* (that is non-contractible). This contractions consist on contracting first the eight triangles placed in the former torus (Figure 12b) to eight edges (black edges in Figure 13). Second, we contract both cylinders (Figure 12a) into two concentric circles (green-dashed and blue-dotted edges in Figure 13).

4.3 The non-dictatorial domain restrictions

The profiles on the cylinders C_1 and C_2 are the basis of the characterization of subdomains of N_I allowing unanimous and non-dictatorial aggregation maps when $|X| = 3$ and $n = 2$.

We are interested in triangles that contain an edge in the blue-dotted cycle, and a vertex in the green-dashed cycle. Consider a profile $\mathbf{R}$ that corresponds to such a triangle, that we call it a *critical profile*. An example of a critical profile is the top one on the left, (BAC, ACB) (see Figure 10 in bold type). Notice that the two voters disagree on their preferences of the pair AB and the pair BC, but they agree on the pair CA. In general, for each critical profile, $\mathbf{R}$, there exists an edge defined by two pairs of alternatives xy and $x'y'$, such that the two voters disagree on them, but agree on the third pair of alternatives, $x''y''$. Namely, $\mathbf{R}$ is defined by the edge $\left\{U_{xy}^{(+,-)}, U_{x'y'}^{(-,+)}\right\}$, together with the vertex $U_{x''y''}^{(s,s)}$, $s \in \{+,-\}$.

We now define the main notion of a *critical pair of profiles*. It is a pair of critical profiles, $(\mathbf{R_1}, \mathbf{R_2})$, $\mathbf{R_1}$ on C_1 and $\mathbf{R_2}$ on C_2, such that their blue-dotted edges do not intersect to each other. That is, if the blue-dotted edge of $\mathbf{R_1}$ is $\left\{U_{xy}^{(+,-)}, U_{x'y'}^{(-,+)}\right\}$, then $U_{xy}^{(+,-)}$ and $U_{x'y'}^{(-,+)}$ do not belong to $\mathbf{R_2}$. Moreover, a *critical triple of profiles* is a triple of profiles $(\mathbf{R_1}, \mathbf{R_2}, \mathbf{R_3})$, $\mathbf{R_1}$ in C_1, $\mathbf{R_2}$ in C_2 and $\mathbf{R_3}$ a antiunanimity profile such that the blue-dotted edges of $\mathbf{R_1}$ and $\mathbf{R_2}$ intersect in a single vertex and they are the blue-dotted edges of $\mathbf{R_3}$. For instance, $\mathbf{R_1} = (BAC, ACB)$, $\mathbf{R_2} = (BCA, ABC)$ and $\mathbf{R_3} = (BCA, ACB)$ is a critical triple.

We are interested in characterizing domain restrictions $D \subseteq N_I$ that contain all vertices of N_I, for two voters and three alternatives. This assumption is not new in the literature: it is equivalent to requiring that every pair of alternatives is free (see Le Breton and Weymark (2011) and Gaertner (2002)). A pair $\{x, y\}$ is free on a domain D if, for every profile $\mathbf{R}$ on $\{x, y\}$, there is a profile $\overline{\mathbf{R}}$ on D whose restriction $\overline{\mathbf{R}}_{|\{x,y\}}$ on the pair $\{x, y\}$ is equal to $\mathbf{R}$[6]. The theorem below characterizes such domains.

Theorem 6 (Domain Restriction Characterization) *A domain restriction D that contains all vertices of N_I allows for a unanimous, non-dictatorial aggregation map if and only if D does not intersect at least the interior*[7] *of one critical pair or one critical triple of profiles.*

Proof To prove the "$\Rightarrow$" direction of the theorem, assume there is a unanimous, non-dictatorial aggregation map $f : D \to N_O$. We show that if D intersect every critical pair and critical triple, then f must be dictatorial. Two scenarios may occur: one of the cylinders restricted to D (i.e. $C_1 \cap D$ or $C_2 \cap D$) contains all of its triangles with blue-dotted edges, or in both of the cylinders lack at least one triangle with a blue-dotted edge and all of these triangles intersect, at least, in a single vertex. We will see that in both cases, f can be extended to a simplicial map on one of the cylinders (and we are back in the situation of Section 4.1).

We start with the first case. Suppose without loss of generality that $C_1 \cap D$ contains all the critical triangles with blue-dotted edges from C_1. We denote $C_1 \cap D$ as D_- and f_- as the restriction of f in D_-. In case D_- is not C_1, we can extend f_- to C_1 because the image of the green-dashed edges of C_1 are determined by unanimity. Since they are not mapped to the boundary of N_O, the images of the triangles with a green-dashed edge are well-defined by the image of their vertices. That is, f_- has been extended to a unanimous simplicial map f_+ defined on C_1. Using the argument in Section 4.1, we conclude that f_+ must be dictatorial. Then f must also be dictatorial because f and f_+ have the same twelve vertices with the same images.

In the second case, we denote the intersection vertex as $U_{xy}^{\overline{\sigma}}$. Notice that there are at most four critical profiles in D intersecting $U_{xy}^{\overline{\sigma}}$, and their blue-dotted edges are $\left\{U_{xy}^{\overline{\sigma}}, U_{yz}^{-\overline{\sigma}}\right\}$ and $\left\{U_{zx}^{-\overline{\sigma}}, U_{xy}^{\overline{\sigma}}\right\}$ (with $z \neq x, y$). Without loss of generality, we suppose that both edges are in D.

If both edges are mapped to inner edges of N_O, then f can be extended to the missing critical profiles using the image of the vertices and we are in the Section 4.1 situation. If both blue-dotted edges are mapped on the boundary of N_O, then we consider the antiunanimity profile corresponding to the triangle $\left\{U_{xy}^{\overline{\sigma}}, U_{yz}^{-\overline{\sigma}}, U_{zx}^{-\overline{\sigma}}\right\}$ generated by both blue-dotted edges. It does not belong to D because it would be mapped into a hole of N_O. Then, since by hypothesis D intersects any critical triple and $\left\{U_{xy}^{\overline{\sigma}}, U_{yz}^{-\overline{\sigma}}, U_{zx}^{-\overline{\sigma}}\right\}$ is not in D, the unique critical profiles not belonging to D must share a blue-dotted edge. Without loss of generality, we suppose that it is $\left\{U_{xy}^{\overline{\sigma}}, U_{yz}^{-\overline{\sigma}}\right\}$.

Then, since the blue-dotted edge is mapped on the boundary of N_O (w.l.o.g. $f\left(\left\{U_{xy}^{\overline{\sigma}}, U_{yz}^{-\overline{\sigma}}\right\}\right) = \{U_{xy}^{+}, U_{yz}^{+}\}$), f can be extended to the triangle $\left\{U_{xy}^{\overline{\sigma}}, U_{yz}^{-\overline{\sigma}}, U_{zx}^{(-,-)}\right\}$ (the triangle $\left\{U_{xy}^{\overline{\sigma}}, U_{yz}^{-\overline{\sigma}}, U_{zx}^{(+,+)}\right\}$ if $f\left(\left\{U_{xy}^{\overline{\sigma}}, U_{yz}^{-\overline{\sigma}}\right\}\right) =$

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[6]There are numerous works in social choice that escape from this framework and assume that there is some structural incapacity to compare some alternatives (Fishburn, 1976) or only allowing non-complete social rankings, but complete individual preferences (Fishburn, 1974; Weymark, 1984).

[7]Here we use interior in the topological sense (see e.g. (Henle, 1994, Section 4)).

$\{U_{xy}^{-}, U_{yz}^{-}\}$). Once f has been extended, we are in the first case where one of the cylinders contains all of their critical profiles.

If only one of the blue-dotted edges is mapped to the boundary of N_O (w.l.o.g. $f\left(\left\{U_{xy}^{\boldsymbol{\sigma}}, U_{yz}^{-\boldsymbol{\sigma}}\right\}\right) = \{U_{xy}^{+}, U_{yz}^{+}\}$), then f can be extended to one of the triangles containing $\left\{U_{zx}^{-\boldsymbol{\sigma}}, U_{xy}^{\boldsymbol{\sigma}}\right\}$. Using the same argument as before, we can conclude that f is dictatorial.

To prove the "$\Leftarrow$" direction, assume a domain not containing the critical pair $(\mathbf{R_1}, \mathbf{R_2})$ or the critical triple $(\mathbf{R_1}, \mathbf{R_2}, \mathbf{R_3})$. Without loss of generality, we can suppose that the first profile is $\mathbf{R_1} = (BAC, ACB) \in C_1$ and the second one, $\mathbf{R_2}$, can be any triangle in C_2 except (BCA, CAB) (see Figure 10). We define the following aggregation maps for the five cases (two critical triples and three critical pairs) in Table 1. It can be checked that they are all well-defined and non-dictatorial. The algorithm used to find these maps is in Appendix C. □

v	$f(v)$
$U_{AB}^{(+,-)}$	U_{AB}^{+}
$U_{AB}^{(-,+)}$	U_{AB}^{-}
$U_{BC}^{(+,-)}$	U_{BC}^{-}
$U_{BC}^{(-,+)}$	U_{BC}^{-}
$U_{CA}^{(+,-)}$	U_{CA}^{+}
$U_{CA}^{(-,+)}$	U_{CA}^{-}

(a)

v	$f(v)$
$U_{AB}^{(+,-)}$	U_{AB}^{+}
$U_{AB}^{(-,+)}$	U_{AB}^{-}
$U_{BC}^{(+,-)}$	U_{BC}^{-}
$U_{BC}^{(-,+)}$	U_{BC}^{-}
$U_{CA}^{(+,-)}$	U_{CA}^{+}
$U_{CA}^{(-,+)}$	U_{CA}^{+}

(b)

v	$f(v)$
$U_{AB}^{(+,-)}$	U_{AB}^{-}
$U_{AB}^{(-,+)}$	U_{AB}^{-}
$U_{BC}^{(+,-)}$	U_{BC}^{-}
$U_{BC}^{(-,+)}$	U_{BC}^{-}
$U_{CA}^{(+,-)}$	U_{CA}^{+}
$U_{CA}^{(-,+)}$	U_{CA}^{+}

(c)

v	$f(v)$
$U_{AB}^{(+,-)}$	U_{AB}^{-}
$U_{AB}^{(-,+)}$	U_{AB}^{-}
$U_{BC}^{(+,-)}$	U_{BC}^{-}
$U_{BC}^{(-,+)}$	U_{BC}^{+}
$U_{CA}^{(+,-)}$	U_{CA}^{+}
$U_{CA}^{(-,+)}$	U_{CA}^{+}

(d)

v	$f(v)$
$U_{AB}^{(+,-)}$	U_{AB}^{-}
$U_{AB}^{(-,+)}$	U_{AB}^{-}
$U_{BC}^{(+,-)}$	U_{BC}^{-}
$U_{BC}^{(-,+)}$	U_{BC}^{+}
$U_{CA}^{(+,-)}$	U_{CA}^{-}
$U_{CA}^{(-,+)}$	U_{CA}^{+}

(e)

Table 1: This figure contains the definition of the five aggregations maps depending on $\mathbf{R_2}$ and $\mathbf{R_3}$. Their definition relies on the image of the vertices. We do not include the images of the unanimous vertices since they are determined by the unanimity axiom. (a) $\mathbf{R_2} = (BAC, CBA)$ and $\mathbf{R_3} = (BAC, CAB)$, (b) $\mathbf{R_2} = (ABC, BCA)$, (c) $\mathbf{R_2} = (ACB, BAC)$, (d) $\mathbf{R_2} = (CAB, ABC)$, (f) $\mathbf{R_2} = (CBA, ACB)$ and $\mathbf{R_3} = (BCA, ACB)$.

The maps in Table 1 may seem somewhat opaque. However, for example, the aggregation map for $\mathbf{R_2} = (CAB, ABC)$ (Figure 1d) can be expressed as:

$$AF(\mathbf{S})B \Leftrightarrow AS_1B \text{ and } AS_2B, \quad BF(\mathbf{S})C \Leftrightarrow BS_2C \quad \text{and}$$

$$AF(\mathbf{S})C \Leftrightarrow AS_1C \text{ and } AS_2C \qquad \text{where } \mathbf{S} \in W^2 \setminus \{\mathbf{R_1}, \mathbf{R_2}\}$$

Using the expression above, we can see that the map is composed of a local dictator (the social choice between B and C) and two almost constant decisions (the social choice between A and B and between A and C).

This simplicity is mainly due to two factors: First, we are working with the simplest basis case (three alternatives and two voters). Second, as it is explained in Appendix C, these maps are deduced from the domains in which the unique removed profiles is a single critical pair. Moreover, in such domains, these maps are the unique ones that are not dictatorial. But the more profiles are removed, the more aggregation maps are compatible with the axioms. The next Section 4.2 is devoted to a domain restriction with a political interpretation, that allows more sophisticated aggregation maps.

Section 4.2 was devoted to expose a straightforward counterexample to Baryshnikov's conjecture. However, using Theorem 6 we can obtain a deeper conclusion than using this counterexample. The theorem guarantees that, for example, in $W^2 \setminus \{(BAC, ACB), (ABC, BCA)\}$ non-dictatorial aggregation is possible (see Figure 1b), on the contrary, in $W^2 \setminus \{(BAC, ACB), (BAC, BCA)\}$ it is not possible. However, both induced simplicial complexes are homeomorphic as topological spaces (and, as a consequence, homotopically equivalent).

The consequence of this fact is that neither pure topological nor homotopical (and homological) invariants characterize aggregation in restricted domains. We conclude that, at least for the Baryshnikov's construction, a pure topological study is not enough for such characterization and the labels of the vertices are required to characterize the domains in terms of aggregation.

5 Impossibility proof with pivotal voters

The goal here is to understand the geometry behind the combinatorial proofs by Geanakoplos (2005) and Yu (2012), using pivotal voters, that have received much attention e.g. Wikipedia contributors (2021). In doing so, we get another proof of Theorem 5. In Section 5.1 we introduce some concepts which will be used to give an alternative proof of Theorem 5 in Section 5.2.

5.1 Paths and pivotal voters

We say that a sequence of triangles in either N_I or N_O is a *path*, if each two consecutive triangles are adjacent (share an edge). Let $R = (R_0, \dots, R_m)$ be a sequence of preferences in W such that every R_i can be obtained from R_{i-1} by a permutation of the preference of two consecutive alternatives (see Remark 2). This sequence induces a path in N_O.

Similarly, a sequence of profiles $\mathbf{R} = (\mathbf{R}_0, \dots, \mathbf{R}_m)$ in W^2 defines a path in N_I, if $\mathbf{R}_i$ can be obtained from $\mathbf{R}_{i-1}$ by a permutation of the preference of two alternatives of at least one of the voters (see Remark 3). We will consider here only paths in N_I where $\mathbf{R}_i$ is obtained from $\mathbf{R}_{i-1}$ by a permutation of the preference of two consecutive alternatives of exactly one of the voters.

Notice that since the aggregation map f is a simplicial map, it sends triangles to triangles, and the image of a path in N_I is a path in N_O.

We will consider paths in N_I starting and ending in unanimous profiles. Additionally, such that all triangles in the path share a vertex $U_{xy}^{\overline{\boldsymbol{\sigma}}}$, $x, y \in X$, for $\overline{\boldsymbol{\sigma}}$ consisting of the same sign, either $+$ or $-$. Notice that since all the triangles share vertex $U_{xy}^{\overline{\boldsymbol{\sigma}}}$, then all the triangles of the path in N_O of the image under f share the vertex U_{xy}^{σ}, where σ is equal to the single sign in $\overline{\boldsymbol{\sigma}}$.

An example is the path $\boldsymbol{R} = (\boldsymbol{R}_0, \ldots, \boldsymbol{R}_4)$ in N_I, defined on the left of Table 2. All the triangles in this path contain the vertex $U_{BC}^{(+,+)}$, since both voters prefer B over C. Additionally, the path starts in the unanimous profile (ABC, ABC) and ends in the unanimous profile (BCA, BCA). In the table there is another example, the path $\boldsymbol{R'} = (\boldsymbol{R'_0}, \ldots, \boldsymbol{R'_4})$ starting in the triangle (BAC, BAC), ending in the triangle (ACB, ACB), and around the vertex $U_{CA}^{(-,-)}$.

Consider the path $\mathbf{R}$ of Table 2, and its depiction in Figure 15. We call such a path *bivalent* because the social choice has to move from $f(\mathbf{R}_0) = ABC$ to $f(\mathbf{R}_4) = BCA$, by the unanimity axiom. The notion of pivotal voter arises in such bivalent paths. The social choice has to exchange the preferences of the pair A, B and also A, C, because it starts in the edge $\left\{U_{AB}^{(+,+)}, U_{CA}^{(-,-)}\right\}$ and ends in the edge $\left\{U_{AB}^{(-,-)}, U_{CA}^{(+,+)}\right\}$. It does not change preferences over B, C, since the path keeps fixed the vertex $U_{BC}^{(+,+)}$.

Consider a sequence of profiles in which the first profile unanimously prefers an alternative x over another y, we change at each step the preference of a *single* individual from x over y to y over x until we arrive at the unanimous profile in which everyone prefers y over x. By unanimity, the first profile socially prefers x over y, whereas the last one y over x. Barberá (1980) named the first voter who produces the change on the social preference from x over y to y over x, the *pivotal voter* of y over x. Denote this voter by k_{yx}.

In Section 5.2, we will use these paths to prove Theorem 5. Whereas in Section B we will compare this topological proof based on pivotal voters with the combinatorial ones by Geanakoplos (2005) and Yu (2012).

5.2 The proof based on pivotal voters

Following Geanakoplos (2005) and Yu (2012), we will first prove that all pivotal voters are the same, and then apply a simple argument to show that this pivotal voter is, in fact, a dictator.

Step 1: all pivotal voters are the same

Consider the path $\mathbf{R}$ of Figure 14 and its depiction in Figure 15. Notice that indeed all the triangles of the path share the vertex $U_{BC}^{(+,+)}$, and it starts in the edge $\left\{U_{AB}^{(+,+)}, U_{CA}^{(-,-)}\right\}$ and ends in the edge $\left\{U_{AB}^{(-,-)}, U_{CA}^{(+,+)}\right\}$. Traversing the path, we see that voter 1 changes its preferences twice, first from $\boldsymbol{R}_0$ to $\boldsymbol{R}_1$ (AB to BA) and then from $\boldsymbol{R}_1$ to $\boldsymbol{R}_2$ (AC to CA). The next two changes of preferences are by voter 2, from $\boldsymbol{R}_2$ to $\boldsymbol{R}_3$ (AB to BA) and then from $\boldsymbol{R}_3$ to $\boldsymbol{R}_4$ (AC to CA). We are interested in comparing k_{CA} with k_{BA}.

The fact that the image of this path in N_O has to exchange the preferences of the pair A, B and also A, C, means that the path in N_O has to cross the triangle BAC. The figure shows why it has to cross first the edge adjacent to U_{CA}^{-}, and then the one adjacent to the vertex U_{AB}^{-}, both of this edges incident on U^+_{BC}. The social preference has to change to B over A before it changes C over A, and given that in the path $\mathbf{R}$ the first changes are by voter 1, followed by the changes by voter 2. We conclude that either $k_{BA} = 1 = k_{CA}$, or $k_{BA} = 1 < 2 = k_{CA}$, or $k_{BA} = 2 = k_{CA}$. In any case, $k_{BA} \leq k_{CA}$.

	1	2		1	2
$\mathbf{R}_0$	A B C	A B C	$\mathbf{R}'_0$	B A C	B A C
$\mathbf{R}_1$	B A C	A B C	$\mathbf{R}'_1$	B C A	B A C
$\mathbf{R}_2$	B C A	A B C	$\mathbf{R}'_2$	C B A	B A C
$\mathbf{R}_3$	B C A	B A C	$\mathbf{R}'_3$	C B A	B C A
$\mathbf{R}_4$	B C A	B C A	$\mathbf{R}'_4$	C B A	C B A

Table 2: The sequences $\mathbf{R}$ and $\mathbf{R}'$ are defined in the table. Writing an alternative on the top on another means that the one on top is preferred to the one in the bottom.

This argument can be repeated using any path analogous to $\mathbf{R}$ around the green-dashed cycle in Figure 15. That is, taking any two of the three unanimous green-dashed triangles labeled ABC, BCA or CAB, and the corresponding bivalent path connecting them clockwisely (that preserves along the path the vertex in the intersection of the two selected triangles). This proves three inequalities $k_{yx} \leq k_{zx}$, for the corresponding $x, y, z \in X$. Conversely, taking the three unanimous blue-dotted triangles labeled BAC, CBA and ACB and the corresponding bivariant paths connecting them also clockwisely (as $\mathbf{R}'$), we obtain the three additional inequalities $k_{AB} \leq k_{AC}$, $k_{BC} \leq k_{BA}$, and $k_{CA} \leq k_{CB}$. Joining the six inequalities we obtain that $k_{BA} \leq k_{CA} \leq k_{CB} \leq k_{AB} \leq k_{AC} \leq k_{BC} \leq k_{BA}$. So, there is a unique pivotal voter.

Surprisingly, as we saw on Section 4.1, the triangles contained in these six bivariant paths constitute a minimal subsimplex N'_I of N_I (see N'_I in Figure 10) that causes an impossibility. That is, the cylinder N'_I contained in the torus is sufficient to connect the unanimity vertices and the vertices with opposite pairwise preferences leading to an impossibility result. Whereas we use here 6 paths going across the 12 triangles of N'_I, in Section 4.1 they have been joined together in a single closed path. Using this closed path we will describe a geometric argument for the impossibility. Cutting this closed path into 6

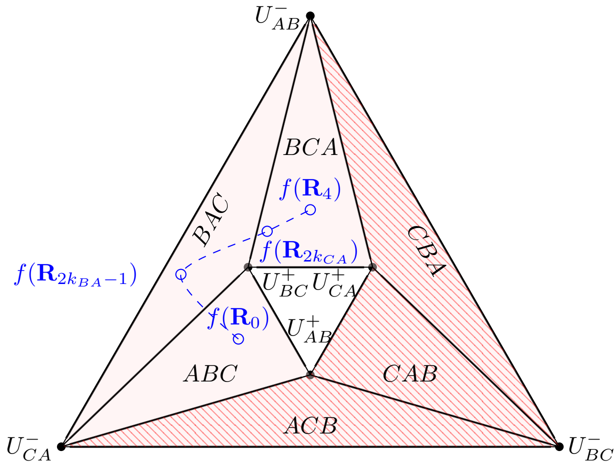

Fig. 14: A graphical representation of the paths defined by $f(\mathbf{R})$.

paths, we have connected the geometrical arguments with the classical pivotal argument.

Thus, the domain does not need to contain all preferences and, consequentially, the whole complex N_I, to apply the arguments contained in this section.

Step 2: the pivotal voter is a dictator

It remains to prove that f is a projection over the k-th component. That is, $f\left(U_{xy}^{\overline{\boldsymbol{\sigma}}}\right) = U_{xy}^{\overline{\boldsymbol{\sigma}}(k)}$. However, this is immediate to see taking the definition of pivotal voter (for $n = 2$). When there are two voters, being a pivotal voter and a dictator is equivalent. The Figure 3 shows, as an example, how to use the definition of a pivotal voter to compute $f\left(U_{xy}^{(+,-)}\right)$ when $k = 1$ and $k = 2$.

	$\mathbf{S}_0$	$\mathbf{S}_1$	$\mathbf{S}_2$		$\mathbf{S}'_0$	$\mathbf{S}'_1$	$\mathbf{S}'_2$
Case $k = 1$	y y	x y	x x	Case $k = 2$	y y	x y	x x
	x x	y x	y y		x x	y x	y y
Social pref.	yx	xy	xy	Social pref.	yx	yx	xy

Table 3: The table on the left represents a sequence of profiles $\mathbf{S} = \mathbf{S}_0, \mathbf{S}_1, \mathbf{S}_2$ starting from unanimity profile of y over x to x over y in which the pivotal voter is $k = 1$. Since $k = 1$ is the pivotal voter, the social preference changes in the first step, so $f\left(U_{xy}^{(+,-)}\right) = U_{xy}^{+}$. The table on the right represents the converse situation, when $k = 2$.

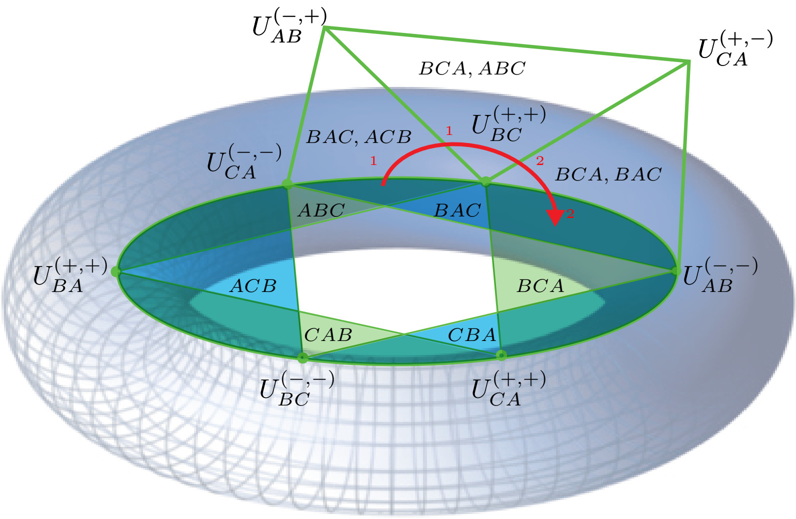

Fig. 15: The sequence $\mathbf{R} = \mathbf{R}_0, \ldots, \mathbf{R}_4$ in the complex N_I. The red curved arrow shows the order in which these triangles appear in $\mathbf{R}$, and it indicates that voter 1 changes its preference twice and then voter 2 changes its preference twice. For clarity, the triangle $\mathbf{R}_0$ is labeled with ABC, and the triangle $\mathbf{R}_4$ is labeled with BCA.

In Appendix B, we further discuss the correspondence of pivotal with the simplicial complex setting.

6 Reduction to the case of $n = 2$ and $|X| = 3$

We have presented several geometric proofs of Arrow's impossibility Theorem 5 for $|X| = 3, n = 2$. Therefore, the proof of Theorem 5 for $|X| \geq 3, n \geq 2$ follows directly from Lemma 7 and 8 below.

There are several works in which the proof of Arrow's theorem is only for $|X| = 3$ and/or $n = 2$ (e.g. Akashi (2005), Saari (2011), Baigent (2011) and Tanaka (2009)). Using Lemma 7 and 8, all these proofs are extended to $|X| \geq 3$ and/or $n \geq 2$.

A few works have used inductive arguments over the number of voters or alternatives. In the fifties, Weldon (1952) proved an impossibility theorem under a set of non-Arrovian axioms. Unlike our case, he could set the initial case of his inductive argument on the trivial case $n = 1$ (instead of $n = 2$). More recent works, Akashi (2005) and Tang and Lin (2009), use inductive arguments using the base case $|X| = 3, n = 2$, as we do. However, our proof is more general. That is, whereas the results of Akashi (Akashi, 2005, Lemma 1) and Tang and Lin (Tang and Lin, 2009, Lemma 1) are constrained to finite sets of alternatives, Lemma 7 works also for infinite X. In addition, the inductive step in (Tang and Lin, 2009, Lemma 2) is proved by contradiction using a large family of maps, while Lemma 8 uses only two, and using an explicit map that helps to understand the inductive step.

Lemma 7 *Let the number of voters be any $n \geq 2$. Arrow's impossibility theorem for $|X| = 3$ implies it for $|X| \geq 3$.*

Proof Suppose that Arrow's theorem is true when $|X| = 3$. We prove that for any X (with $|X| \geq 3$) and any $F : W^n \to W$ satisfying unanimity and independence of irrelevant alternatives, F is dictatorial.

Given F, choose three distinct alternatives $x, y, z \in X$ and denote $\overline{W}_0$ the set of all strict orders over these three alternatives. Define an aggregation map $\overline{F} : \overline{W_0}^n \to \overline{W}_0$ as follows. The image of a profile $\left(\overline{R}_1, \ldots, \overline{R}_n\right) \in \overline{W_0}^n$ by $\overline{F}$ is the restriction of the ordering $F(R_1, \ldots, R_n) \in W$ on the set $\{x, y, z\} \subseteq X$, where for each i, R_i is any extension of $\overline{R}_i$ from $\overline{W}_0$ to W. Notice that the definition of $\overline{F}$ does not depend on the chosen extension because of the independence of irrelevant alternatives of F. Moreover, it is easy to check that $\overline{F}$ satisfies unanimity as well as independence of irrelevant alternatives. So, it follows that $\overline{F}$ is dictatorial because we have supposed that Arrow's theorem is true when $|X| = 3$. It remains to prove that F is also dictatorial.

If k is the dictator of $\overline{F}$, we will prove that it is also a dictator for F. Consider a profile $\mathbf{R} = (R_1, \ldots R_n) \in W^n$ where aR_kb for some $a, b \in X$. Then take a profile $\mathbf{R}' = \left(R'_1, \ldots R'_n\right) \in W^n$ satisfying that, for every i, $xR'_ibR'_iaR'_iy$ if bR_ia, and $aR'_iyR'_ixR'_ib$ if aR_ib.

Since k is a dictator of $\overline{F}$ and yR'_kx (k prefers a over b in R_k), we know that the image by $\overline{F}$ of the restriction of $\mathbf{R}'$ over $\overline{W}_0^n$ prefers y over x, hence $F(\mathbf{R}')$ also prefers y over x. Moreover, by unanimity, it holds that $aF(\mathbf{R}')y$ and $xF(\mathbf{R}')b$. Then, we obtain that $aF(\mathbf{R}')b$ from the relations $aF(\mathbf{R}')yF(\mathbf{R}')xF(\mathbf{R}')b$ using the transitivity. Finally, using the independence of irrelevant alternatives, we obtain that $aF(\mathbf{R})b$. Since this happens for every pair $a, b \in X$, k must be the dictator of F. $\square$

The proof of the previous lemma, contrary to the ones in Akashi (2005); Tang and Lin (2009), is not inductive. This fact enables us to reduce the cases of any cardinality of X to $|X| = 3$ in a single step.

Lemma 8 *Let the number of alternatives be any $|X| \geq 3$. If Arrow's impossibility theorem is true for $n = 2$ then it is true for $n > 2$.*

Proof The proof is by induction on n. By hypothesis, the theorem is true when $n = 2$. Suppose that it is true for $n - 1$ and we will prove it for n.

Let $F^n : W^n \to W$ an aggregation map satisfying unanimity and independence of irrelevant alternatives. We will prove that F^n is dictatorial in three steps:

Step 1: We define the aggregation map on W^{n-1}, $F_1^{n-1}(R_1, \ldots, R_{n-1}) := F^n(R_1, \ldots, R_{n-1}, R_1)$. Since F_1^{n-1} satisfies unanimity and independence of irrelevant alternatives, the induction hypothesis guarantees that it has a dictator k_1. We will prove that if $k_1 \neq 1$, then k_1 is also a dictator for F^n.

Suppose $\mathbf{R} \in W^n$ and $xR_{k_1}y$. If the ordering of R_1 and R_n coincides on $\{x,y\}$, then $xF^n(\mathbf{R})y$ because F_1^{n-1} has k_1 as a dictator. Otherwise, we can suppose without loss of generality that xR_1y, yR_nx. Then, let $z \in X$ be an auxiliary alternative and let $\mathbf{R}' \in W^n$ be a profile which coincides with $\mathbf{R}$ over $\{x,y\}$, $xR'_{k_1}zR'_{k_1}y$ and z is below x and y for the remaining voters.

Since R'_1 and R'_n agrees on $\{y,z\}$ and k_1 is a dictator for F_1^{n-1}, we have that $zF^n(\mathbf{R}')y$. Moreover, $xF^n(\mathbf{R}')z$ because of the unanimity. Using the transitivity we obtain that $xF^n(\mathbf{R}')y$, and applying the independence of irrelevant alternatives we obtain that $xF^n(\mathbf{R})y$. So, k_1 is a dictator of F^n (if $k_1 \neq 1$).

Step 2: We define $F_2^{n-1}(R_1,\ldots,R_{n-1}) := F^n(R_1,\ldots,R_{n-1},R_2)$. Using the inductive hypothesis, F_2^{n-1} has a dictator k_2. If $k_2 \neq 2$, apply a symmetric reasoning to the one in step 1 to deduce that k_2 is the dictator of F^n (if $k_2 \neq 2$).

Step 3: If $k_1 = 1$ and $k_2 = 2$, we show that n is the dictator of F^n. Let $\mathbf{R} \in W^n$ be a profile with xR_ny. Consider $z \in X$, and $\mathbf{R}' \in W^n$ coinciding with $\mathbf{R}$ over $\{x,y\}$, $xR'_nzR'_ny$, xR'_1z and zR'_2y. Using that 1 (resp. 2) is the dictator of F_1^{n-1} (resp. F_2^{n-1}), we obtain that $xF^n(\mathbf{R}')z$ (resp. $zF^n(\mathbf{R}')y$). So, using the transitivity on $F^n(\mathbf{R}')$ and the independence of irrelevant alternatives, we obtain that $xF^n(\mathbf{R})y$. Finally, we have concluded that n is the dictator of F^n (if $k_1 = 1$ and $k_2 = 2$). □

The reader may wonder why Lemma 8 is inductive, instead of applying some direct argument extending from $n = 2$ to any number of voters (as we have done in Lemma 7). If such argument existed, it would allow to extend the theorem to an infinite number of voters. However, this is not possible because Arrow's impossibility is not true when n is infinite (see Fishburn (1970)).

7 Conclusions

We have given new proofs of Arrow's theorem consisting of two parts. The first part deals with the base case of two voters and three alternatives, and we presented three different versions: using the index lemma, using pivotal voters, and using domain restrictions. The second part proves the general case by a simple reduction to the base case. Both parts are new; the first one exposes the geometry behind Arrow's impossibility, while the second simplifies previous similar proofs.

The first part shows that any aggregation function is dictatorial, because in essence it is mapping a torus onto a cylinder, in a continuous way, respecting unanimity. The argument sheds light on the remarkable algebraic topology proof of Baryshnikov (1993), and makes it accessible to a wider audience. Also, it connects it to standard proofs of Arrow's theorem based on pivotal arguments, by explaining how the paths of such arguments move along the torus and the cylinder. Furthermore, it provided a guide on how to characterize the domain restrictions that allow non-dictatorial maps.

The structure of our proofs, in two parts, suggests that the interesting geometry happens in the base case. The structure of the base case has a clear geometric intuition, which led us to obtain a characterization of the non-dictatorial domain restrictions. We hope these ideas are helpful in generalizing the characterization beyond the base case. The two cylinders C_1 and C_2 are the crux of the impossibility, representing social profiles with intermediate disagreement (neither complete agreement nor complete disagreement of the preferences of the two voters). Moreover, we presented a domain restriction in Section 4.2 that answers a 20-year-old question set out by Baryshnikov in the concluding remarks of his article (Baryshnikov, 1997) about the relation between the classic and the topological social choice: we used a natural domain in political science to show that non-dictatorial rules are possible in non-contractible spaces. Furthermore, contrary to our expectations, Theorem 5 shows that we can not reduce the question of restriction domains to a pure topological problem. The labels of the vertices are not irrelevant. So, we should study the spaces of preferences, at least, as chromatic simplicial complexes.

We hope that bringing combinatorial topology to social choice problems opens interesting opportunities for future work. These tools have been encountering many applications recently. Some examples are in concurrency (Herlihy et al., 2013), logic Castañeda et al. (2023), image processing (Babaei and Hersch, 2016), political structures (Mock and Volić, 2021), data analysis (Kannan et al., 2019) and wireless networks (Ramazani et al., 2016). Such tools are at the same time more accessible to an audience unfamiliar with algebraic topology, and also more concrete, closer to computational approaches.

In particular, combinatorial topology has been very useful in distributed computing (Herlihy et al., 2013). We described some analogies that are worth exploring here and by Rajsbaum and Raventós-Pujol (2022), since computing processes that communicate with each other need to agree on one of their inputs in many applications. Remarkably, while Sperner's lemma is the key to the impossibilities of tasks where processes need to reach agreement such as *consensus*, *set agreement* (Attiya and Rajsbaum, 2002), *vector consensus* (Neves et al., 2005) and *interactive consistency* (Friedman et al., 2007) (where domain restrictions are studied), for Arrow's impossibility, the key is the index lemma, as it is for tasks related to renaming and weak symmetry breaking (Castañeda and Rajsbaum, 2010; Goubault et al., 2019). Here we studied only Arrow's setting, where the aggregation map is defined directly on the input complex; it would be interesting to explore the case where the agents can communicate with each other and subdivisions of the input complex arise. Notice that the index lemma is preserved under subdivisions e.g. (Goubault et al., 2019, Corollary 4). However, we are not aware of a distributed task where the impossibility is proved in dimension 2, and then extended easily to any dimension. Aggregation functions that elude Arrow's theorem have been considered in the distributed setting (Chauhan and Garg, 2013; Melnyk et al., 2018; Allouah et al., 2022).

We have derived the first complete characterization of domains where non-dictatorial aggregation is possible, but only for the base case. We hope our techniques can be used to obtain corresponding characterizations for the general case. We are working on taking advantage of the structure exposed by combinatorial topology to understand Gibbard-Satterthwaite theorem Gibbard (1973); Satterthwaite (1975), and following Eliaz unification program Eliaz (2004) of Arrow and Gibbard-Satterthwaite theorems. There are a few attempts to such unification projects (see Keiding (1981); Nehring and Puppe (2010)), but they use an abstract approach such as category theory. The geometric flavour and simplicity of the combinatorial topology techniques could help understanding the connection between the two theorems, as well as helping deriving new non-dictatorial domain restrictions for this case, analogous to the Arrow domain restrictions presented here.

Some of the results of this paper were presented in the 16^{th} *meeting of the Society for Social Choice and Welfare*. We believe that combinatorial topology techniques could be useful to study other aspects of social choice discussed in the conference, such as strategy-proofness (Barberà et al., 2010; Sato, 2013), fuzzy social choice (Gibilisco et al., 2014; Raventos-Pujol et al., 2020) or (social) choice functions (Aizerman and Aleskerov, 1995; Basile et al., 2022).

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A Generalized version of the index lemma

Here we present the generalizated version of the index lemma, and show that it holds on N_I.

Definition 2 Let K be a simplicial complex of dimension 2 satisfying that every simplex of dimension 1 has a single or an even number of 2-simplices containing it. An *orientation on* K is an orientation on every 2-simplex satisfying that the induced orientations on the 1-simplices by the 2-simplices have to be opposite by pairs.

As in the original framework, let K be an oriented simplicial complex of dimension 2 with each vertex labeled with a color from $\{0, 1, 2\}$. The content C of K is the number of tricoloured triangles in K counted $+1$ if the order of the labeling agrees with the orientation (see the right side of Figure 17) and -1 otherwise. The index I of K is the number of edges $\overrightarrow{01}$ on the boundary counted $+1$ if the order of the vertices agrees with the orientation and -1 otherwise. Now, we can state and proof the index lemma for oriented simplicial 2-complexes.

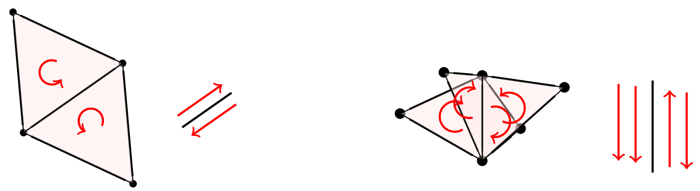


Fig. 16: The simplicial complex on the left is oriented because the induced orientations on the inner edge are opposite. However, the right one is not because it has three orientations in one direction and one on the opposite direction.

Theorem 9 (Index lemma) *Let K be a 3-colored oriented simplicial complex of dimension 2. Then, the index I is equal to the content C.*

Proof Let S be the number of edges $\overrightarrow{01}$ counted according to the orientation. We will prove that $I = S$ and $C = S$. First, we will see that the contribution of every interior edge $\overrightarrow{01}$ is equal to 0. Since every interior edge has an even number of incident 2-simplices, by definition of being oriented, their contribution is 0. Then $I = S$.

For every triangle in the complex, the contribution is only non-zero if the triangle is tricoloured. If it is not tricoloured, it is 0 because, in case it has at least one 0 and one 1, the third vertex has to be coloured by 0 or 1, then one edge compensates the other. Otherwise, if it is tricoloured, its contribution is the same as the content's contribution (see Figure 17).

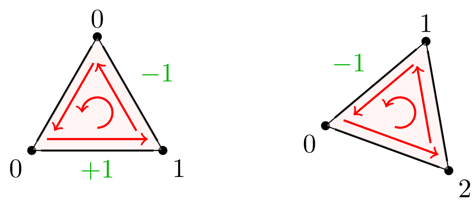


Fig. 17: On the left, the contribution of the simplex is 0 because the two edges $\overrightarrow{01}$ compensate each other. On the right, the contribution of the tricolored triangle is -1.

□

Now we provide N_I with an orientation. Recall that we assume that the number of alternatives is $|X| = 3$ and the number of voters is $n = 2$.

Proposition 10 *The complex N_I is orientable.*

Proof We will define an orientation on N_I as follows. For every 2-simplex $\Delta = \left\{U_{AB}^{\overline{\boldsymbol{\sigma}}_1}, U_{BC}^{\overline{\boldsymbol{\sigma}}_2}, U_{CA}^{\overline{\boldsymbol{\sigma}}_3}\right\}$ we define its parity as the product of all the signs of $\overline{\boldsymbol{\sigma}}_1$, $\overline{\boldsymbol{\sigma}}_2$ and $\overline{\boldsymbol{\sigma}}_3$. For instance, if $\overline{\boldsymbol{\sigma}}_1 = (+,+)$, $\overline{\boldsymbol{\sigma}}_2 = (+,-)$ and $\overline{\boldsymbol{\sigma}}_3 = (-,-)$, the parity is -1 (see Figure 18a). We define the orientation of this 2-simplex as clockwise ($AB \to CA \to BC \to AB$) if its parity is -1 and counterclockwise ($AB \to BC \to CA \to AB$) if its parity is 1.

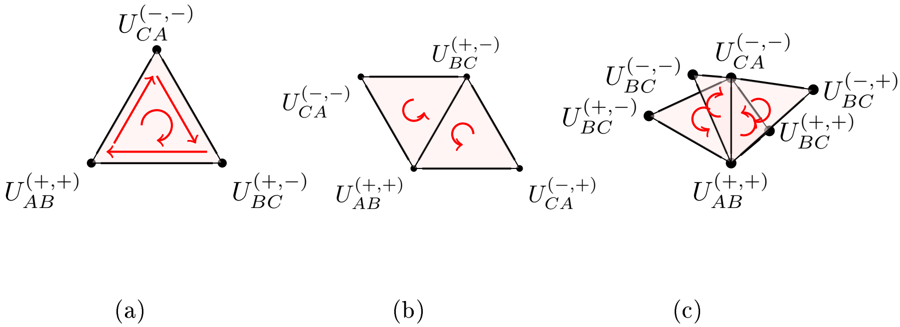


Fig. 18: (a) Since the parity of the triangle is negative, the orientation is $U_{AB}^{(+,+)} \leftarrow U_{BC}^{(+,-)} \leftarrow U_{CA}^{(-,-)}$. (b) Two triangles sharing the edge $\left\{U_{AB}^{(+,+)}, U_{BC}^{(+,-)}\right\}$. (c) Four triangles sharing the edge $\left\{U_{AB}^{(+,+)}, U_{CA}^{(-,-)}\right\}$

This is an orientation because for every non-boundary edge, there are an even number of 2-simplices containing it, and they are paired by their opposite induced orientations. For example, consider the edge $\left\{U_{AB}^{\overline{\boldsymbol{\sigma}}_1}, U_{BC}^{\overline{\boldsymbol{\sigma}}_2}\right\}$, this edge only can be completed with a vertex indexed as $U_{CA}^{\overline{\boldsymbol{\sigma}}_3}$ for some compatible $\overline{\boldsymbol{\sigma}}_3 \in \{+,-\}^n$ constrained by the transitivity property. That is, for every component $i \in \{1,\dots n\}$, if $\overline{\boldsymbol{\sigma}}_1(i) = \overline{\boldsymbol{\sigma}}_2(i) = +$ (resp. $\overline{\boldsymbol{\sigma}}_1(i) = \overline{\boldsymbol{\sigma}}_2(i) = -$, then $\overline{\boldsymbol{\sigma}}_3(i) = +$ (resp. $\overline{\boldsymbol{\sigma}}_3(i) = -$). However, if $\overline{\boldsymbol{\sigma}}_1(i)$ and $\overline{\boldsymbol{\sigma}}_2(i)$ have different signs, both signs are compatible in $\overline{\boldsymbol{\sigma}}_3(i)$. We can conclude that the admissible $\overline{\boldsymbol{\sigma}}_3$ are exactly 2^k (where k is equal to the number of voters i on the third situation). And, since by hypothesis $\left\{U_{AB}^{\overline{\boldsymbol{\sigma}}_1}, U_{BC}^{\overline{\boldsymbol{\sigma}}_2}\right\}$ is not in the boundary, $k > 0$.

Second, we can pair these 2^k 2-simplices saying that $\left\{U_{AB}^{\overline{\boldsymbol{\sigma}}_1}, U_{BC}^{\overline{\boldsymbol{\sigma}}_2}, U_{CA}^{\overline{\boldsymbol{\sigma}}_3}\right\}$ and $\left\{U_{AB}^{\overline{\boldsymbol{\sigma}}_1}, U_{BC}^{\overline{\boldsymbol{\sigma}}_2}, U_{CA}^{\overline{\boldsymbol{\sigma}}'_3}\right\}$ are paired if $\overline{\boldsymbol{\sigma}}_3$ and $\overline{\boldsymbol{\sigma}}'_3$ are equal on each component but the first one. That is, if there are $\sigma_2, \dots, \sigma_n \in \{+,-\}$ such that $\overline{\boldsymbol{\sigma}}_3 = (+, \sigma_2, \dots, \sigma_n)$ and $\overline{\boldsymbol{\sigma}}'_3 = (-, \sigma_2, \dots, \sigma_n)$. Then the parity associated to every triangle of a pair is opposite to the other member, so, their contribution on the edge $\left\{U_{AB}^{\overline{\boldsymbol{\sigma}}_1}, U_{BC}^{\overline{\boldsymbol{\sigma}}_2}\right\}$ determined by the induced orientations is also opposite. □

B Pivotal voters and paths in N_I

In this section, we further discuss the correspondence of the pivotal setting with the simplicial complex setting of Section 5.

To discuss the role of pivotal voters and the paths defined by sequences, consider as an example the path $\mathbf{R}$ defined in Table 2. This path starts and ends in the inner cylinder of N_I, that is, the unanimity simplices (see Figure 6). Obviously, this cylinder is identified with N_O because of the unanimity property of the aggregation map f. The remaining simplices $\{\mathbf{R}_1, \mathbf{R_2}, \mathbf{R}_3\}$ of the path link the inner cylinder with the outer one (see the complex at the right of Figure 19).

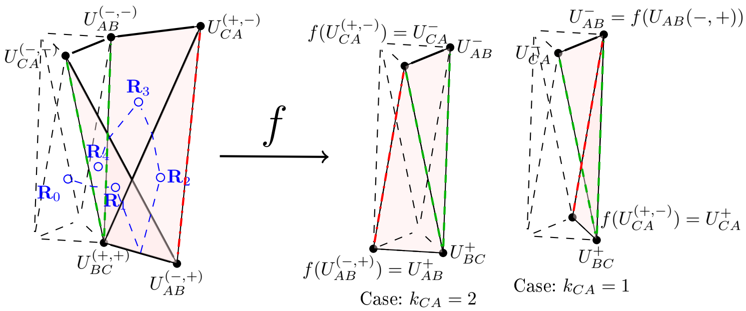


Fig. 19: The figure in the right represents the simplices $\{\mathbf{R}_1, \mathbf{R_2}, \mathbf{R}_3\}$ linking the inner cylinder of N_I (green-dashed edges) with the outer cylinder (red edge) and the path $\mathbf{R}$. The figure in the middle represents the folding of the hinges and the inner cylinder when $k_{CA} = 2$; the one on the left, when $k_{CA} = 1$.

When the aggregation map f is applied, the inner cylinder remains invariant because we have identified it with N_O, but the outer cylinder and the links (the torus joining both cylinders) are compressed into the inner cylinder. We have to imagine the simplices between the cylinders (from Figure 6), the ones linking the cylinders, playing the role of "hinges", folding into each other so that the two cylinders fit together.

In Figure 19 we can see that the *hinge* $\{\mathbf{R}_1, \mathbf{R_2}, \mathbf{R}_3\}$ can fold two ways. It folds one way or another depending on the value of k_{CA}. Notice that its folding also determines the value of $f(U_{AB}^{(-,+)})$, and this determination of the folding is the geometrical representation of the inequality $k_{CA} \leq k_{BA}$, proved in Section 5.2. Moreover, the simplex $\mathbf{R}_3$ also belongs to another hinge, which at the same time will represent an inequality. So, all hinges are connected and they constrain each other foldings. Consequently, there are only two possible ways to fold and fit both cylinders together: the two projections.

C Schema of how obtain aggregation maps on restricted domains

Here we give an overview of the procedure we have followed to obtain the maps of Table 1 in Section 4.3.

First, we have studied the scenario in which only a critical pair or a critical triple has been removed from N_I. Notice that if a non-dictatorial map f exists in a domain D like this, then in every subdomain $D' \subseteq D$ we will have as a non-dictatorial map $f_{|D'}$. This assertion is true because D and D' have the same vertices.

As in the proof of Theorem 6, we focus on the domain D obtained by removing the critical pair $(\mathbf{R_1}, \mathbf{R_2})$ or the critical triple $(\mathbf{R_1}, \mathbf{R_2}, \mathbf{R_3})$ being $\mathbf{R_1} = (BAC, ACB)$. We will define a non-dictatorial map f. We will build f step by step imposing unanimity and the simplicial structure of D and N_O.

Our strategy will be the following: We will determine all possible images of the blue-dotted path (i.e. the antiunanimity vertices), using exclusively the simplicial properties of $C_1 \setminus \{\mathbf{R_1}\}$ and the unanimity axiom. Next, we will determine which of these images are compatible with the domain $D = N_I \setminus \{\mathbf{R_1}, \mathbf{R_2}\}$ or $D = N_I \setminus \{\mathbf{R_1}, \mathbf{R_2}, \mathbf{R_3}\}$ depending on $\mathbf{R_2}$ and $\mathbf{R_3}$.

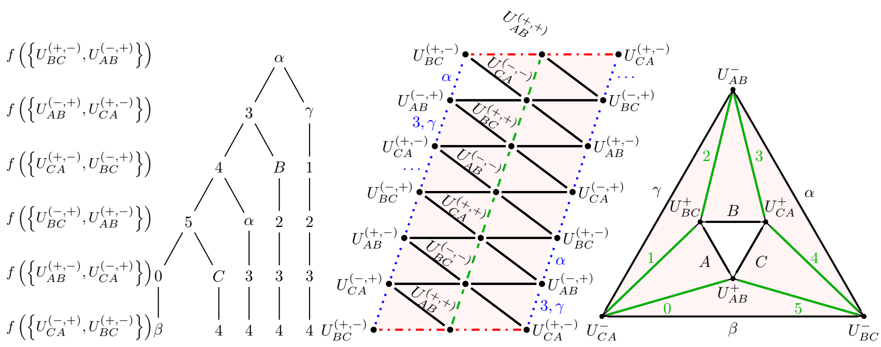
Step 1: The boundary of $\mathbf{R_1}$ must be mapped on the boundary of N_O, otherwise f can be extended to another map defined on C_1 and, using the arguments in Section 4.1, we conclude it is dictatorial. And, using that $f\left(U_{CA}^{(-,-)}\right) = U_{CA}^{-}$, we state that $f\left(U_{BC}^{(+,-)}\right) = U_{BC}^{-}$ and $f\left(U_{AB}^{(-,+)}\right) = U_{AB}^{-}$.

Step 2: We consider the next blue-dotted edge $\left\{U_{AB}^{(-,+)}, U_{CA}^{(+,-)}\}\right)$. Taking into account that $f\left(U_{AB}^{(-,+)}\right) = U_{AB}^{-}$ and $f\left(U_{BC}^{(+,+)}\right) = U_{BC}^{+}$, the image of $U_{CA}^{(+,-)}$ is not determined. In other words, the edge $\left\{U_{AB}^{(-,+)}, U_{CA}^{(+,-)}\}\right)$ can be mapped to the edge labeled with 3 or the one labeled with γ in Figure 20c.

Step 3: If $\left\{U_{AB}^{(-,+)}, U_{CA}^{(+,-)}\}\right)$ was mapped to γ, using the same reasoning, we conclude that the next edge $\left\{U_{CA}^{(+,-)}, U_{BC}^{(-,+)}\right\}$ must be mapped in 1. Otherwise, if $f(\left\{U_{AB}^{(-,+)}, U_{CA}^{(+,-)}\right\}) = 3$, then $f(\left\{U_{CA}^{(+,-)}, U_{BC}^{(-,+)}\right\})$ could be 4 of B (third row in Figure 20a).

We repeat the same types of arguments for steps 4,5 and 6, until we have mapped all possible images for the blue-dotted cycle. In Figure 20a each branch corresponds to a candidate for the mapping. Starting with α as the image of $\left\{U_{BC}^{(+,-)}, U_{AB}^{(-,+)}\right\}$ and finishing with 4 or β as the image of $\left\{U_{CA}^{(-,+)}, U_{BC}^{(+,-)}\right\}$.

We have five candidates for the image of the blue-dotted cycle, equivalently, five candidates for an aggregation map. By the definition of f, we know that these maps are simplicial in C_1, but we need to verify that these candidates



(a) (b) (c)

Fig. 20: (a) The tree representing the admissible mappings of the blue-dotted edges when $\mathbf{R_1} = \left\{U_{AB}^{(-,+)}, U_{BC}^{(+,-)}, U_{CA}^{(-,-)}\right\}$. The first row of the three represents the admissible image of the edge $\left\{U_{BC}^{(+,-)}, U_{AB}^{(-,+)}\right\}$, the second row the admissible images of $\left\{U_{AB}^{(-,+)}, U_{CA}^{(+,-)}\right\}$, and successively until the edge $\left\{U_{CA}^{(-,+)}, U_{BC}^{(+,-)}\right\}$. So, a tupple represents an admissible mapping of the blue-dotted cycle. For instance, the tuple $(\alpha, 3, B, 2, 3, 4)$ represents a map in which the first blue-dotted edge is mapped to α, the second to 3 and the sixth to 4. (b) The torus without the triangle $\mathbf{R_1} = \left\{U_{AB}^{(-,+)}, U_{BC}^{(+,-)}, U_{CA}^{(-,-)}\right\}$ and some admissible mappings of the edges represented. (c) The N_O complex with their edges labeled.

are simplicial in the whole domain $N_I \setminus \{\mathbf{R_1}, \mathbf{R_2}\}$ or $N_I \setminus \{\mathbf{R_1}, \mathbf{R_2}, \mathbf{R_3}\}$ (for a suitable $\mathbf{R_2}$ and $\mathbf{R_3}$).

It turns out that the unique obstacle for each of these five maps to be simplicial is overcomed by removing a single triangle from C_2. That is, for each critical pair $(\mathbf{R_1}, \mathbf{R_2})$ or critical triple $(\mathbf{R_1}, \mathbf{R_2}, \mathbf{R_3})$ (being $\mathbf{R_1} = (BAC, ACB)$), we obtain a unique non dictatorial aggregation map.

Given a triangle $\mathbf{R_2} \in C_2$, as we have argued before, it has to be mapped to the boundary of N_O, then the unique map compatible, is the one which maps the blue-dotted edge of $\mathbf{R_2}$ in the boundary of $\mathbf{R_2}$. For example, if $\mathbf{R_2} = (ABC, BCA)$, the unique candidate to be simplicial is the map which maps $\left\{U_{AB}^{(+,-)}, U_{CA}^{(-,+)}\right\}$ to C. That is, the map represented by the tupple $(\alpha, 3, 4, 5, C, 4)$.