Market-Based Asset Price Probability

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14 August 2022

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Abstract

We consider the randomness of market trade values and volumes as the origin of asset price stochasticity. We define the first four market-based price statistical moments that depend on statistical moments and correlations of market trade values and volumes. Market-based price statistical moments coincide with conventional frequency-based ones if all trade volumes are constant during the time averaging interval. We present approximations of market-based price probability by a finite number of price statistical moments. We consider the consequences of the use of market-based price statistical moments for asset-pricing models and Value-at-Risk. We show that the use of volume weighted average price results in zero price-volume correlations. We derive market-based correlations between price and squares of volume and between squares of price and volume. To forecast market-based price volatility at horizon T one should predict the first two statistical moments of market trade values and volumes and their correlations at the same horizon T.

Keywords : asset price, price probability, returns, inflation, market trade

JEL: G12

This research received no support, specific grant or financial assistance from funding agencies in the public, commercial or nonprofit sectors. We welcome valuable offers of grants, support and positions.

1. Introduction

Investors and traders need reliable price forecasts, but exact price projections as well as tomorrow's fortune forecasts are too variable and almost stochastic. The history of asset pricing research (Dimson and Mussavian, 1999) tracked price probability up to Bernoulli's studies in 1738, but possibly Bachelier (1900) was the first to really highlight the probabilistic character of price behavior and forecasting. "The probabilistic description of financial prices, pioneered by Bachelier." (Mandelbrot et al., 1997). "in fact, the first author to put forward the idea to use a random walk to describe the evolution of prices was Bachelier." (Shiryaev, 1999). During the last century, countless studies discussed asset pricing models and different hypotheses, laws, and properties of a random asset price (Kendall and Hill, 1953; Muth, 1961; Sharpe, 1964; Fama, 1965; Stigler and Kindahl, 1970; Black and Scholes, 1973; Merton, 1973; Tauchen and Pitts, 1983; Mackey, 1989; Friedman, 1990; Cochrane and Hansen, 1992; Campbell, 2000; Heaton and Lucas, 2000; Cochrane, 2001; Poon and Granger, 2003; Andersen et al., 2005; 2006; Cochrane, 2005; Wolfers and Zitzewitz, 2006; DeFusco et al., 2017; Weyl, 2019; Cochrane, 2022). Rigorous mathematical treatments of stochastic description and probabilistic modeling of asset prices can be found in Shiryaev (1999) and Shreve (2004). We refer only to a negligible part of the asset pricing studies.

Asset price dynamics are under the impact of multiple factors that cause irregular or random price changes during almost any time interval. That generates studies of price variations and price dependence on the market (Fama, 1965; Tauchen and Pitts, 1983; Odean, 1998; Poon and Granger, 2003; DeFusco et al., 2017), on macroeconomics (Cochrane and Hansen, 1992; Heaton and Lucas, 2000; Diebold and Yilmaz, 2008), on business cycles (Mills, 1946; Campbell, 1998), on expectations (Muth, 1961; Malkiel and Cragg, 1980; Campbell and Shiller, 1988; Greenwood and Shleifer, 2014), on trading volumes (Karpoff, 1987; Campbell et al., 1993; Gallant et al., 1992; Brock and LeBaron, 1995; Llorente et al., 2001), and on many other factors that for sure impact price change and fluctuations. The line of factors and references can be continued (Goldsmith and Lipsey, 1963; Andersen et al., 2001; Hördahl and Packer, 2007; Fama and French, 2015).

The conventional treatment (Shiryaev, 1999) of a random price $p(t_i)$ times series during the averaging interval Δ is based on the frequency of trades at a price p. If m_p is the number of trades at a price p and N is the total number of trades during Δ, then the probability $P(p)$ of a price p is assessed as:

$$P(p) \sim \frac{m_p}{N} \qquad (1.1)$$

We denote mathematical expectation as *E[..]*. The finite number *N* of trades for *n=1,2,..* estimates the *n-th* statistical moments of price $E[p^n(t_i)]$ as:

$$E[\,p^n(t_i)] \sim \frac{1}{N}\sum_{i=1}^{N} p^n(t_i) \qquad (1.2)$$

The set of the *n-th* statistical moments (1.2) provides a conventional frequency-based description of a random variable (Shiryaev, 1999; Shreve, 2004).

However, asset prices are not independent issues of economics and finance. Asset pricing is woven deeply into market trade, financial laws, and economic properties. We consider market trade randomness as the economic origin of price stochasticity and show that market-based price statistical moments determined by market stochasticity could take a form that is different from (1.2). That results in significant distinctions, which are valuable for investors and financial markets. We consider the description of random asset prices and price statistical moments as a problem of market trade stochasticity.

Indeed, each market deal at time t_i is described by trade value $C(t_i)$, trade volume $U(t_i)$, and trade price $p(t_i)$, which match simple relations (1.3):

$$C(t_i) = p(t_i)U(t_i) \qquad (1.3)$$

However, the trivial equation (1.3) states that given the probabilities of the trade value and volume (1.3), determine the market price probability. Given statistical moments of market trade value and volume, determine statistical moments of market price.

The market-based approach to asset price probability highlights the direct dependence of price *n-th* statistical moments on the statistical moments of market trade values and volumes. However, market trade records allow us to estimate only a finite number of trade statistical moments. Hence, one can derive only a finite number of price statistical moments, which describe a finite approximation of price probability. Actually, we replace the problem of what the "correct" form of the price probability is with a different one. We consider how an approximate description of the random market trade value and volume by a finite number of statistical moments describes a finite number of price statistical moments.

In Section 2, we describe how the statistical moments of market trade values and volumes determine the statistical moments of the asset price. In Sections 3 and 4, we briefly consider the consequences of our results on description of the random properties of returns and inflation, asset pricing models, and Value-at-Risk (VaR) as most conventional risk management tool. Section 5: Conclusion. We assume that our readers are familiar with standard issues of asset pricing theory, probability theory, statistical moments, characteristic functions, etc. This paper is not for novices, and we propose that readers already know or can find on their own the notions, terms, and models that are not given in the text.

2. Market-based price statistical moments

Properties of a random variable can equally be described by probability measure, characteristic function, and a set of the *n-th* statistical moments (Shephard, 1991; Shiryaev, 1999; Shreve, 2004). We describe a market price as a random variable using market-based *n-th* statistical moments. The finite set of *n-th* statistical moments of price, $n=1,2,..m,$ describes the *m-th* approximation of the price characteristic function and price probability measure (App.A). Each additional *n-th* statistical moment adds additional accuracy to the approximation of market-based price probability.

Let us consider time series records that describe the transaction value $C(t_i)$, volume $U(t_i)$, and price $p(t_i)$ at a time t_i. The times t_i of market deals introduce the initial division of the time axis. We describe random properties of market trade using time series records of performed transactions only. Thus, all possible factors that impact asset pricing are already imprinted into the time series records of the market trade value $C(t_i)$ and volume $U(t_i)$. For simplicity, we assume that transactions are performed with a constant, small interval ε that can be equal to 1 second or even a fraction of a second:

$$t_i - t_{i-1} = \varepsilon \tag{2.1}$$

The time series (2.1) establishes a time axis division multiple of ε. To describe market trade and price at a time horizon $T >> \varepsilon$, the precise time division ε is of little help. To describe the market price at a horizon T, which can be equal to weeks, months, or years, one should aggregate or average the initial trade time series during some reasonable averaging interval Δ:

$$\varepsilon < \Delta < T \quad ; \quad \Delta = \left[t - \frac{\Delta}{2}; t + \frac{\Delta}{2}\right] \quad ; \quad t_i \in \Delta \quad ; \quad i = 1,2,..N \tag{2.2}$$

We assume that each interval Δ contains the same number N of terms of the trade time series and take time t as the current time. We assume that all prices are adjusted to the current time t. N terms of time series of the market trade value $C(t_i)$, volume $U(t_i)$, and price $p(t_i)$ can behave irregularly during averaging interval Δ, and we consider them as random variables during Δ. The averaging of market trade value $C(t_i)$, volume $U(t_i)$, and price $p(t_i)$ during Δ estimates their statistical moments and assesses their properties as random variables. We consider averaging during Δ (2.2) as a tool for estimating statistical moments of market trade value and volume and, through them, deriving statistical moments of market price. The finite number N of market trades estimates only a finite number of trade statistical moments and describes a finite number of price statistical moments. A finite set of statistical moments describes a finite approximation of a characteristic function and probability measure of a random variable (App. A).

The choice of the averaging interval Δ (2.2) is a particular problem. Too long Δ can have more terms of time series and thus permit the estimation of more trade statistical moments. However, long Δ establishes time axis division, which reduces the abilities of traders and investors to take "this hour" or "this day" trade decisions. Short Δ give opportunity to take immediate market decisions, but statistical moments may be irregular on long horizon T. We highlight that the choice of different averaging intervals Δ could result in different forecasts on long horizons.

We start the definition of the market-based statistical moments of price with the choice of the average price or the 1-st statistical moment $a(t;1)$. To highlight the distinction between the market-based price probability and the conventional frequency-based probability, we shall note market-based mathematical expectation as $E_m[..]$ and due to (2.2):

$$a(t;1) = E_m[p(t_i)] \quad ; \quad i = 1,..N \ ; \quad t_i \in \Delta \tag{2.3}$$

The reasons for justification of the choice of the market-based average price $a(t;1)$ are as follows: As usual, investors estimate the average price of the shares in their portfolio as a simple ratio of the total value to the total number of shares in the portfolio. The same meaning has the well-known definition of volume weighted average price (VWAP) (Berkowitz et al., 1988; Buryak and Guo, 2014; Busseti and Boyd, 2015; CME Group, 2020; Duffie and Dworczak, 2021). Taking into account the trade price equation (1.3), one can present VWAP $p(t;1,1)$ as:

$$p(t;1,1) = \frac{\sum_{i=1}^{N} p(t_i)U(t_i)}{\sum_{i=1}^{N} U(t_i)} = \frac{C_\Sigma(t;1)}{U_\Sigma(t;1)} = \frac{C(t;1)}{U(t;1)} \tag{2.4}$$

$$C_\Sigma(t;1) = \sum_{i=1}^{N} C(t_i) \quad ; \quad U_\Sigma(t;1) = \sum_{i=1}^{N} U(t_i) \tag{2.5}$$

$$C(t;1) = \frac{1}{N}\sum_{i=1}^{N} C(t_i) \quad ; \quad U(t;1) = \frac{1}{N}\sum_{i=1}^{N} U(t_i) \tag{2.6}$$

Relations (2.5) define the total value $C_\Sigma(t;1)$ and the total volume $U_\Sigma(t;1)$ of market trades, and (2.6) estimates the average trade value $C(t;1)$ and volume $U(t;1)$ by a finite number N of trades during Δ (2.2). Relations between the frequency-based average price determined by (1.2) for $n=1$ and VWAP (2.4) are trivial. VWAP $p(t;1,1)$ (2.4) coincides with the average price determined by (1.2) if all trade volumes $U(t_i)$ $=constant$ for $i=1,2,..N$ during Δ (2.2). However, financial markets demonstrate highly irregular or random behavior in trade volumes. VWAP $p(t;1,1)$ takes into account the randomness of the trade volumes $U(t_i)$, and we set the average price of the market-based price probability $a(t;1)$ to be equal to VWAP:

$$a(t;1) = p(t;1,1) \tag{2.7}$$

To derive market-based price *n-th* statistical moments *a(t;n)*:

$$a(t;n) = E_m[p^n(t_i)] \quad ; \quad n = 2,3,\ldots \tag{2.8}$$

let us take the *n-th* power of the trade equation (1.3):

$$C^n(t_i) = p^n(t_i)U^n(t_i) \quad ; \quad n = 2,3,.. \tag{2.9}$$

Equations (2.9) generate the set of price *n-th* statistical moments *p(t;n,m)* determined by weight functions $w(t_i;m)$ of the *m-th* power of trade volume $U^m(t_i)$:

$$w(t_i;m) = \frac{U^m(t_i)}{\sum_{i=1}^{N} U^m(t_i)} \quad ; \quad \sum_{i=1}^{N} w(t_i;m) = 1 \tag{2.10}$$

$$p(t;n,m) = \sum_{i=1}^{N} p^n(t_i)w(t_i;m) = \frac{1}{\sum_{i=1}^{N} U^m(t_i)} \sum_{i=1}^{N} p^n(t_i)U^m(t_i) \tag{2.11}$$

We highlight that weight functions $w(t_i;m)$ (2.10) don't define particular price probability measures. For the simplest case, $U(t_i)=constant$, all weight functions $w(t_i;m)=1/N$, and that coincides with the weight *1/N* of the frequency-based probability (1.1; 1.2). For *n=m* price *n-th* statistical moments *p(t;n,n)*:

$$p(t;n,n) = \frac{1}{\sum_{i=1}^{N} U^n(t_i)} \sum_{i=1}^{N} p^n(t_i)U^n(t_i) = \frac{C_\Sigma(t;n)}{U_\Sigma(t;n)} = \frac{C(t;n)}{U(t;n)} \tag{2.12}$$

$$C_\Sigma(t;n) = \sum_{i=1}^{N} C^n(t_i) \quad ; \quad C(t;n) = \frac{1}{N}\sum_{i=1}^{N} C^n(t_i) \tag{2.13}$$

$$U_\Sigma(t;n) = \sum_{i=1}^{N} U^n(t_i) \quad ; \quad U(t;n) = \frac{1}{N}\sum_{i=1}^{N} U^n(t_i) \tag{2.14}$$

Relations (2.13) define the total sums $C_\Sigma(t;n)$ of the *n-th* power of trade value $C^n(t_i)$ and *C(t;n)* defines the estimate of the *n-th* statistical moments or an average of the n-th power of trade value $C^n(t_i)$ and relations (2.14) define the total sums $U_\Sigma(t;n)$ of the *n-th* power of trade volume $U^n(t_i)$ and assessment *U(t;n)* of the *n-th* statistical moments of trade volume by a finite number *N* of trades during Δ (2.2). From (2.12), obtain equations (2.15) on the *n-th* statistical moments of market trade value *C(t;n)*, volume *U(t;n),* and price *p(t;n,n)*:

$$C(t;n) = p(t;n,n)U(t;n) \quad ; \quad n = 1,2,\ldots \tag{2.15}$$

Relations (2.9-2.15) establish the basis for the definition of market-based price statistical moments *a(t;n)* (2.8). Indeed, the 1-st statistical moment *a(t;1)* (2.7) is determined by the VWAP *p(t;1,1)*, weight function $w(t_i;1)$, and is based on the trade price equation (1.3). To define the 2-d price statistical moment *a(t;2),* one should take into account (2.9) for *n=2* and the statistical moments *p(t;1,2)* and *p(t;2,2)* generated by the weight function $w(t_i;2)$.

One should verify that market-based price statistical moments *a(t;1)*, *a(t;2), a(t;3),...,* which depend on statistical moments *p(t;n,m)* (2.11) generated by different weight functions (2.10), are jointly consistent. In particular, one should check that the definitions of market-based price statistical moments *a(t;1)*, *a(t;2), a(t;3)*,.. ensure that central even moments are non-negative. To define the *2-d* price statistical moment $a(t;2)$ (2.8), one should require market-based price volatility $\sigma^2(t)$ (2.16) that depends on $a(t;2)$, and the average price $a(t;1)$ (2.7) is non-negative. To obtain that, we derive the *2-d* statistical moment $a(t;2)$ as a solution of the equation (2.17) that defines market-based price volatility $\sigma^2(t)$ (2.16):

$$\sigma^2(t) = E_m\left[\left(p(t_i) - a(t;1)\right)^2\right] = a(t;2) - a^2(t;1) \tag{2.16}$$

$$\sigma^2(t) = \sum_{i=1}^{N}(p(t_i) - a(t;1))^2 w(t_i;2) = M(t;2) \tag{2.17}$$

The factor $M(t;2)$ in the right side of (2.17) is always non-negative and that approves non-negative market-based price volatility $\sigma^2(t)$ (2.16). The factor $M(t;2)$ ties up the average price $a(t;1)$ (2.7) that is determined by the weight function $w(t_i;1)$, and the price averaging by the weight function $w(t_i;2)$. From (2.10; 2.11) for $w(t_i;2)$, obtain:

$$M(t;2) = \sum_{i=1}^{N}(p(t_i) - a(t;1))^2 w(t_i;2) = p(t;2,2) - 2p(t;1,2)a(t;1) + a^2(t;1) \tag{2.18}$$

From (2.16-2.18), obtain the 2-d market-based price statistical moment $a(t;2)$:

$$a(t;2) = E_m[p^2(t_i)] = p(t;2,2) + 2a(t;1)[a(t;1) - p(t;1,2)] \tag{2.19}$$

$$\sigma^2(t) = p(t;2,2) - 2p(t;1,2)a(t;1) + a^2(t;1) \tag{2.20}$$

We highlight that $a(t;2)$ (2.19) fits together statistical moments $p(t;1,1)$, $p(t;1,2)$, and $p(t;2,2)$ (2.11) that are determined by two different weight functions $w(t_i;1)$ and $w(t_i;2)$. The definition of $a(t;2)$ (2.19) provides correct, non-negative values (2.17; 2.18; 2.20) for market-based price volatility $\sigma^2(t)$ (2.16). We use similar recursive procedures to define higher market-based price statistical moments.

To define market-based 3-d statistical moment $a(t;3)$, we determine market-based price skewness $Sk(t)$ (2.21) via the factor $M(t;3)$ (2.22) determined by the weight function $w(t_i;3)$:

$$Sk(t)\sigma^3(t) = E_m[(p(t_i) - a(t;1))^3] = a(t;3) - 3a(t;2)a(t;1) + 2a^3(t;1) \tag{2.21}$$

$$E_m[(p - a(1))^3] = \sum_{i=1}^{N}(p(t_i) - a(t;1))^3 w(t_i;3) = M(t;3) \tag{2.22}$$

From (2.21; 2.22), obtain the 3-d statistical moment $a(t;3)$:

$$a(t;3) = p(t;3,3) - 3a(t;1)[p(t;2,3) - p(t;1,3)a(t;1)] + 3a(t;1)\sigma^2(t) \tag{2.23}$$

$$Sk(t)\sigma^3(t) = p(t;3,3) - 3p(t;2,3)a(t;1) + 3p(t;1,3)a^2(t;1) - a^3(t;1) \tag{2.24}$$

To define the 4-th statistical moment $a(t;4)$, one should check that two even market-based statistical moments: price kurtosis $Ku(t)$ (2.25) and volatility of squares of price $\theta^2(t)$ (2.27), which depend on $a(t;4)$, are non-negative.

$$Ku(t)\sigma^4(t) = E_m[(p(t_i) - a(t;1))^4] \tag{2.25}$$

$$E_m[(p(t_i) - a(t;1))^4] = a(t;4) - 4a(t;3)a(t;1) + 6a(t;2)a^2(t;1) - 3a^4(t;1) \tag{2.26}$$

$$\theta^2(t) = E_m[(p^2(t_i) - a(t;2))^2] = a(t;4) - a^2(t;2) \tag{2.27}$$

To derive that, let us consider the factor *M(t;4)* (2.28) that is similar to *Ku(t)* (2.25) and the factor *M(t;2,2)* (2.29) that is similar to $\theta^2(t)$ (2.27). The factors *M(t;4)* and *M(t;2,2)* are non-negative and determined by the weight function $w(t_i;4)$ (2.10). We derive the *4-th* market-based statistical moment *a(t;4)* as the solution of equation (2.30) that provides non-negativity of (2.25; 2.27). We set that the product of (2.25) and (2.27) should be equal to the product of *M(t;4)* (2.28) and *M(t;2,2)* (2.29):

$$M(t;4) = \sum_{i=1}^{N}(p(t_i) - a(t;1))^4 w(t_i;4) = \frac{\sum_{i=1}^{N}[p(t_i)-a(t;1)]^4 U^4(t_i)}{\sum_{i=1}^{N} U^4(t_i)} \tag{2.28}$$

$$M(t;2,2) = \sum_{i=1}^{N}(p^2(t_i) - a(t;2))^2 w(t_i;4) = \frac{\sum_{i=1}^{N}(p^2(t_i)-a(t;2))^2\, U^4(t_i)}{\sum_{i=1}^{N} U^4(t_i)} \tag{2.29}$$

$$M(t;4) = p(t;4,4) - 4p(t;3,4)a(t;1) + 6p(t;2,4)a^2(t;1) - 4p(t;1,4)a^3(t;1) + a^3(t;1)$$

$$M(t;2,2) = p(t;4,4) - 2p(t;2,4)a(t;2) + a^2(t;2)$$

One can present (2.26; 2.27), as

$$E_m[(p(t_i) - a(t;1))^4] = a(t;4) - F(t)\ ;$$

$$F(t) = 4a(t;3)a(t;1) - 6a(t;2)a^2(t;1) + 3a^4(t;1)$$

To derive the 4-th statistical moment *a(t;4),* we set the equation:

$$E_m[(p(t_i) - a(t;1))^4]E_m[(p^2(t_i) - a(t;2))^2] = M(t;4)M(t;2,2)$$

$$[a(t;4) - F(t)][a(t;4) - a^2(t;2)] = M(t;4)M(t;2,2) \tag{2.30}$$

$$a^2(t;4) - [F(t) + a^2(t;2)]a(t;4) + [F(t)a^2(t;2) - M(t;4)M(t;2,2)] = 0 \tag{2.31}$$

One can verify that (2.31) always has two roots $a_{1,2}(t;4)$ and one of the roots ensures that market-based price kurtosis *Ku(t)* (2.25; 2.32) and volatility of squares of price $\theta^2(t)$ (2.27; 2.33) are non-negative (App.B):

$$Ku(t)\sigma^4(t) = E_m[(p(t_i) - a(t;1))^4] = a(t;4) - F(t) \geq 0 \tag{2.32}$$

$$\theta^2(t) = E_m[(p^2(t_i) - a(t;2))^2] = a(t;4) - a^2(t;2) \geq\ 0 \tag{2.33}$$

That makes the first four market-based statistical moments *a(t;n)* (2.8), *n=1,2,3,4,* self-consistent; nevertheless, they are determined by different weight functions $w(t_i;n)$, *n=1,2,3,4* (2.10). Thus, the statistical moments *a(t;n)* (2.7; 2.19; 2.20; 2.23; 2.24) and the root of the equation (2.31) define a 4-finite approximation of market-based price characteristic function and probability measure (App.A). In this paper, we reduce the approximation of market-based price probability by the first four statistical moments.

Further execution of recursive procedures that are similar to those above will define equations on market-based price n-th statistical moments *a(t;n)* for *n=5,6,..* .

Let us highlight the important result of our consideration. We introduced the first four market-based price statistical moments *a(t;n), n=1,2,3,4* that describe the 4-approximation of market-based price characteristic function and probability measure (App.A). These approximations reveal the direct dependence of market-based price statistical moments, price characteristic function and price probability measure on statistical moments of market trade values and volumes.

Let us consider this issue in more detail. From (2.12), obtain equations (2.15) on $p(t;n,n)$ that describe dependence on the statistical moments of market trade value $C(t;n)$ and volume $U(t;n)$. However, the factors $p(t;n,m)$ for $n<m$ describe the joint statistical moments of market trade values and volumes. Indeed, from (2.11), for $n<m$, obtain:

$$p(t;n,m)U(t;m) = \frac{1}{N}\sum_{i=1}^{N} p^n(t_i)U^m(t_i) = \frac{1}{N}\sum_{i=1}^{N} C^n(t_i)U^{m-n}(t_i) = CU(t;n,m-n) \quad (2.34)$$

We denote the joint statistical moments of market trade value and volume as $CU(t;n,m-n)$:

$$CU(t;n,m-n) = E[C^n(t_i)U^{m-n}(t_i)] = \frac{1}{N}\sum_{i=1}^{N} C^n(t_i)U^{m-n}(t_i) \quad (2.35)$$

$$CU(t;n,m-n) = C(t;n)U(t;m-n) + corr\{C^n(t_i)U^{m-n}(t_i)\} \quad (2.36)$$

Relations (2.34-2.36) reveal the direct dependence of the market-based price statistical moments, characteristic function, and probability measure on statistical moments of market trade value $C(t;n)$ (2.13), volume $U(t;n)$ (2.14), and on joined statistical moments of trade value and volume $CU(t;n,m-n)$ (2.35) or on correlations (2.36) between the *n-th* power of value $C^n(t_i)$ and the *(m-n)-th* power of volume $U^{m-n}(t_i)$. Relations (2.34-2.36) allow present market-based price volatility $\sigma^2(t)$ (2.20) and the 2-d price statistical moment $a(t;2)$ (2.19) as follows:

$$\sigma_p^2(t) = \frac{\Omega_C^2(t) + a^2(t;1)\Omega_U^2(t) - 2a(t;1)corr\{C(t_i)U(t_i)\}}{U(t;2)} \quad (2.37)$$

$$a(t;2) = \frac{C(t;2) + 2a^2(t;1)\Omega_U^2(t) - 2a(t;1)corr\{C(t_i)U(t_i)\}}{U(t;2)} \quad (2.38)$$

Here we introduce trade value volatility $\Omega_C^2(t)$ and trade volume volatility $\Omega_U^2(t)$ (2.39)

$$\Omega_C^2(t) = C(t;2) - C^2(t;1) \quad ; \quad \Omega_U^2(t) = U(t;2) - U^2(t;1) \quad (2.39)$$

Relations (2,37; 2.38) highlight direct dependence of market based price $\sigma^2(t)$ (2.20) and statistical moment $a(t;2)$ on trade statistical moments $C(t;2)$ and $U(t;2)$, their volatilities $\Omega_C^2(t)$ and $\Omega_U^2(t)$ (2.39) and mutual correlation (2.36). That dependence highlights the hidden difficulties in performing any predictions of market-based price volatility $\sigma^2(t)$ (2.16). Such hidden complexity reduces the accuracy and reliability of price volatility forecasts, which neglect the above dependence.

3. Asset pricing and value-at-risk

In this section, we briefly argue some consequences of the use of market-based price statistical moments.

Asset-pricing

The introduction of market-based price statistical moments *a(t;n)* that are determined by statistical moments of the market trade value and volume (2.7; 2.19; 2.23; 2.31) makes predictions of the price probability one of the most complex problems of finance. Indeed, to forecast price probability at time horizon *T,* one should predict the statistical moments and correlations of market trade value and volume at the same time horizon *T*. That requires forecasts of economic and financial factors that impact market trade trends and fluctuations at horizon *T*. One should predict supply and demand, production function and investment, economic development and growth, etc. Forecasts of the first four market-based price statistical moments *a(t;n), n=1,2,3,4,* should match predictions of market trade statistical moments *C(t,n)* (2.13)*, U(t,n)* (2.14), and their correlations (2.34-2.36).

Value-at-risk

The approximate predictions of asset price probability determine the accuracy and reliability of Value-at-Risk (VaR) – one of the most widespread tools for hedging the risk of a random market price change. The foundation for VaR was developed more than 30 years ago (Longerstaey and Spencer, 1996; CreditMetrics™, 1997; Choudhry, 2013). "Value-at-Risk is a measure of the maximum potential change in value of a portfolio of financial instruments with a given probability over a pre-set horizon" (Longerstaey and Spencer, 1996). Despite the great progress in VaR performance since then, the core features of VaR remain the same. To assess VaR at horizon *T* one should forecast at horizon *T* the integral of the left tail of the returns or price probability density function.

Such predictions limit the possible capital loss due to market price random variations for a selected time horizon *T* with a given probability. VaR is used by the largest banks and investment funds to hedge their AUM and portfolios valued at billions of USD from the risk of a random market price change. The large size of AUM under risk requires considering the impact of large trades on market price probability. Hence, the largest banks and investment funds should take into account market-based price probability.

As we show above, the predictions of market-based price statistical moments and probability depend on the forecasts of market trade statistical moments and correlations. Hence, VaR as a method to hedge large AUM from risks of market price change at horizon *T* is based on forecasts of the market trade value and volume statistical moments at the same horizon *T.* The accuracy of VaR assessment at horizon *T* is determined by the accuracy of forecasting the market trade value and volume statistical moments. The more statistical moments of market trade are predicted, the higher the accuracy of VAR. Simply put, VaR assessment almost equals the prediction of the joint probability of market trade values and volumes. However, an imaginable exact forecast of the market trade value and volume statistical moments at horizon *T* would provide that lucky man with a unique opportunity to manage the market alone. That is much more profitable than any VaR assessments. One who succeeds in predicting market trade joint statistical moments will forget about VaR and will enjoy beating the market alone! However, there still remains a "negligible" problem – how can one *exactly* predict market trade statistical moments? It is a good issue for further research.

The accuracy of any assessments of VaR at horizon *T* is bounded by the precision of possible predictions of market trade value and volume statistical moments at horizon *T*.

4. Price-volume correlations

Price-volume correlations have been studied in numerous papers (Tauchen and Pitts, 1983; Karpoff, 1987; Campbell et al., 1993; Llorente et al., 2001; DeFusco et al., 2017). These researchers investigate the frequency-based correlations of price-volume time series.

Actually, the correlations of two random variables are determined by their joint probabilities. The choice of probabilities determines the value of mutual correlations. The choice of frequency-based price and trade volume probabilities determines the results of the above authors. However, the consideration of price-volume correlations as a result of market trade randomness and the use of market-based price statistical moments give results that differ from those presented by (Tauchen and Pitts, 1983; Karpoff, 1987; Campbell et al., 1993; Llorente et al., 2001; DeFusco et al., 2017).

The use of VWAP as the average market-based price *a(t;1)* (2.4; 2.7) immediately highlights that price-volume correlations equal zero:

$$corr\{p(t_i)U(t_i)\} = E[p(t_i)U(t_i)] - E_m[p(t_i)]E[U(t_i)] \tag{4.1}$$

Indeed, from (2.4-2.6; 2.15), obtain:

$$E[p(t_i)U(t_i)] = E[C(t_i)] = \frac{1}{N}\sum_{i=1}^{N} C(t_i) = \frac{1}{N}\sum_{i=1}^{N} p(t_i)U(t_i) =$$

$$= \frac{1}{\sum_{i=1}^{N} U(t_i)} \sum_{i=1}^{N} p(t_i)U(t_i) \cdot \frac{1}{N}\sum_{i=1}^{N} U(t_i) = E_m[p(t_i)]E[U(t_i)]$$

Hence, the correlation $corr\{p(t_i)U(t_i)\}$ (4.1) between price $p(t_i)$ and trade volume $U(t_i)$ is zero:

$$corr\{p(t_i)U(t_i)\} = 0 \tag{4.2}$$

Now let us derive correlation $corr\{p(t_i)U^2(t_i)\}$ between price $p(t_i)$ and squares of volumes $U^2(t_i)$.

$$corr\{p(t_i)U^2(t_i)\} = E[p(t_i)U^2(t_i)] - a(t;1)U(t;2) \tag{4.3}$$

$$E[p(t_i)U^2(t_i)] = E[C(t_i)U(t_i)] = C(t;1)U(t;1) + corr\{C(t_i)U(t_i)\}$$

From (2.4), obtain:

$$corr\{p(t_i)U^2(t_i)\} = corr\{C(t_i)U(t_i)\} - a(t;1)\sigma_U^2(t) \tag{4.4}$$

$$\sigma_U^2(t) = U(t;2) - U^2(t;1)$$

Let us derive correlation $corr\{p^2(t_i)U^2(t_i)\}$ between squares of price and trade volume:

$$corr\{p^2(t_i)U^2(t_i)\} = E[p^2(t_i)U^2(t_i)] - a(t;2)U(t;2)$$

$$E[p^2(t_i)U^2(t_i)] = E[C^2(t_i)] = C(t;2)$$

After simple transformations, obtain the form of $corr\{p^2(t_i)U^2(t_i)\}$:

$$corr\{p^2(t_i)U^2(t_i)\} = 2a(t;1)U(t;2)[p(t;1,2) - a(t;1)] \tag{4.5}$$

The factor $p(t;1,2)$ in (4.5) can be presented using (2.34-2.36) and we obtain:

$$corr\{p^2(t_i)U^2(t_i)\} = 2a(t;1)[corr\{C(t_i)U(t_i)\} - a(t;1)\sigma_U^2(t)] \tag{4.6}$$

From (4.4) and (4.6), obtain:

$$corr\{p^2(t_i)U^2(t_i)\} = 2a(t;1)corr\{p(t_i)U^2(t_i)\}$$

The use of different probabilities gives different expressions of correlations between the same random variables.

5. Conclusion

The price probability and its predictions are the most wanted issues for traders and investors. The conventional and generally accepted frequency-based price probability describes random market prices under the implicit assumption that all trade volumes are constant during the averaging interval. That is not the best imitation of random market trade.

As opposed to the frequency-based approach, market-based price statistical moments directly depend on the statistical moments and correlations of random market trade values and volumes. That highlights the complex impact of trade randomness on market price stochasticity and reveals the dependence of large trade deals on price variations. Market-based statistical moments help derive direct expressions for the dependence of price-volume correlations on statistical moments and correlations of random trade values and volumes. That could help investors much more than the frequency-based assessments of correlations between price-volume time series.

The complexity of predictions of market-based price statistical moments is hidden in the difficulties of forecasting market trade statistical moments. In particular, we show that predictions of price volatility depend on forecasts of the first two statistical moments and correlations of trade values and volumes. Market-based price probability reveals the economic bounds and restrictions for the reliability and accuracy of Value-at-Risk.

The development of a market-based approach to price probability can benefit market, financial, and economic modeling and management.

Appendix A.

Approximations of the price characteristic function and probability measure

We consider price as a random variable during the averaging interval *Δ* (2.2). One can equally describe a random variable by its characteristic function *F(t;x)* (A.1), probability measure *μ(t;p)* (A.2), and a set of the *n-th* statistical moments *a(t;n)* (2.8) (Shephard, 1991; Shiryaev, 1999; Shreve, 2004). The Taylor series expansion of the market-based characteristic function *F(t;x)* presents it through the set of the *n-th* statistical moments *a(t;n)*:

$$F(t;x) = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} a(t;n)\, x^n \tag{A.1}$$

$$\mu(t;p) = \frac{1}{\sqrt{2\pi}} \int F(t;x) \exp(-ixp)\, dx \tag{A.2}$$

$$a(t;n) = \frac{d^n}{(i)^n dx^n} F(t;x)|_{x=0} = \int p^n \mu(t;p)\, dp \quad ; \quad \int \mu(t;p)\, dp = 1 \tag{A.3}$$

In (A.1;A.2), i is the imaginary unit. For simplicity, we take price as a continuous random variable during *Δ* (2.2). Any predictions of the *market-based* price probability *μ(t;p)* and characteristic function *F(t;x)* at a horizon *T* should match the forecasts of price *n-th* statistical moments *a(t;n).* The direct dependence of market-based price *n-th* statistical moments *a(t;n)* on statistical moments of market trade values and volumes and their correlations highlights the dependence of forecasts of price probability *μ(t;p)* and characteristic function *F(t;x)* on predictions of market trade statistical moments at the same horizon *T*.

Finite number *q* of price statistical moments *a(t;n)*, *n=1,2,..q* determines finite q-approximation of price characteristic function $F_q(t;x)$ (A.4):

$$F_q(t;x) = 1 + \sum_{n=1}^{q} \frac{i^n}{n!} a(t;n)\, x^n \tag{A.4}$$

We present a simple example of approximation. Statistical moments determined by $F_q(t;x)$ for *n>q* can be different, but the first *q* moments are equal to *a(t;n), n=1,2,..q*. Taylor expansion (A.4) is not too useful to derive Fourier transform (A.2) and to obtain *q*-approximation of the price probability measure $\mu_q(t;p)$. Let us consider price characteristic function $G_q(t;x)$ (A.5):

$$G_q(t;x) = exp\left\{\sum_{n=1}^{q} \frac{i^n}{n!}\, b(t;n)\, x^n - B\, x^{2Q}\right\} \;\; ; \;\; q = 1, 2, \dots ; \; q < 2Q \; ; \; B > 0 \tag{A.5}$$

and require that $G_q(t;x)$ (A.5) obey relations (A.3):

$$a(t;n) = \frac{d^n}{(i)^n dx^n} G_q(t;x)|_{x=0} \quad ; \quad n \le q \tag{A.6}$$

Relations (A.6) define terms *b(t;n)* in (A.5) through price statistical moments *a(t;n)*, $n \le q$. The term Bx^{2Q}, *B>0, 2Q>q* don't impact relations (A.3; A.6) but guarantees existence of the price probability measures $\mu_q(t;p)$ as Fourier transform (A.2) of the characteristic functions $G_q(t;x)$

(A.5). The uncertainty of $B>0$ and power $2Q>q$ in (A.5) highlights the well-known fact that the first q statistical moments don't explicitly determine the characteristic function and probability measure of a random variable. Relations (A.5) describe the set of characteristic functions $G_q(t;x)$ with different $B>0$ and $2Q>q$ and the corresponding set of probability measures $\mu_q(t;p)$ that match (A.2; A.5; A.6).

For $q=1$ the approximate price characteristic function $G_1(t;x)$ and probability $\mu_q(t;p)$ are trivial:

$$G_1(t;x) = exp\{i\, b(t;1)x\} \;\; ; \; a(t;1) = -i\frac{d}{dx}G_1(t;x)|_{x=0} = b(t;1) \tag{A.7}$$

$$\mu_1(t;p) = \int dx\;\, G_1(t;x)\, exp(-ipx) = \delta\big(p - \; b(t;1)\big) \tag{A.8}$$

For $q=2$ approximation $G_2(t;x)$ describes Gaussian probability measure $\mu_2(t;p)$:

$$G_2(x;t) = exp\left\{i\, b(t;1)x - \frac{b(t;2)}{2}x^2\right\} \tag{A.9}$$

It is easy to show that

$$a(t;2) = -\frac{d^2}{dx^2}G_2(t;x)|_{x=0} = b(t;2) + b^2(t;1)$$

$$b(t;2) = a(t;2) - a^2(t;1) = \sigma^2(t) \tag{A.10}$$

Coefficient $b(t;2)$ equals price volatility $\sigma^2(t)$ (2.16; 2.20) and the Fourier transform (A.2) for $G_2(t;x)$ gives Gaussian price probability measure $\mu_2(t;p)$:

$$\mu_2(t;p) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma(t)} exp\left\{-\frac{(p-b(t;1))^2}{2\sigma^2(t)}\right\} \tag{A.11}$$

For $q=3$ approximation $G_3(t;x)$ has form:

$$G_3(t;x) = exp\left\{i\, b(t;1)x - \frac{\sigma^2(t)}{2}x^2 - i\,\frac{b(t;3)}{6}x^3 - B\, x^{2Q}\right\} \tag{A.12}$$

$$a(t;3) = i\frac{d^3}{dx^3}G_3(t;x)|_{x=0} = b(t;3) + 3b(t;1)\sigma^2(t) + \; b^3(t;1)$$

$$b(t;3) = E_m\left[\big(p - b(t;1)\big)^3\right] = Sk(t)\sigma^3(t) \tag{A.13}$$

Coefficient $b(t;3)$ (A.13) depends on price skewness $Sk(t)$, which describes the asymmetry of the market-based price probability from the normal distribution.

For the $q=4$ approximation $G_4(t;x)$ depends on the choice of $B>0$ and power $2Q>4$:

$$G_4(t;x) = exp\left\{i\, b(t;1)x - \frac{\sigma^2(t)}{2}x^2 - i\,\frac{b(t;3)}{6}x^3 + \frac{b(t;4)}{24}x^4 - Bx^{2Q}\right\} \;\; ; \;\; 2Q > 4 \tag{A.14}$$

Simple, but long calculations give:

$$b(t;4) = a(t;4) - 4a(t;3)a(t;1) + 12a(t;2)p^2(t;1) - 6a^4(t;1) - 3a^2(t;2)$$

$$b(t;4) = E_m\left[\big(p - b(t;1)\big)^4\right] - 3E_m^2\left[\big(p - b(t;1)\big)^2\right]$$

Price kurtosis $Ku(p)$ (B.11) describes how the tails of the price probability measure $\eta_K(t;p)$ differ from the tails of a normal distribution.

$$Ku(t)\sigma_p^4(t;p) = E_m\left[\left(p - b(t;1)\right)^4\right] \tag{A.15}$$

$$b(t;4) = [Ku(t) - 3]\sigma^4(t)$$

Even the simplest Gaussian approximation $G_2(t;x)$, $\mu_2(t;p)$ (A.9; A.11) highlights the direct dependence of the market-based price volatility $\sigma^2(t)$ (2.16; 2.20; A.10) on the first two statistical moments of trade value $C(t;1)$, $C(t;2)$ and volume $U(t;1)$, $U(t;2)$ and their correlations (2.36). Thus, prediction of price volatility $\sigma^2(t)$ for Gaussian measure $\mu_2(t;p)$ (A.9) should follow non-trivial forecasting of the first two statistical moments (2.13; 2.14) and correlations (2.34-2.36) of the market trade value and volume.

Appendix B.

The proof of non-negativity of price kurtosis *Ku(t)* and θ^2(t)

Let us consider the quadratic equation (2.31):

$$a^2(t;4) - [F(t) + a^2(t;2)]a(t;4) + [F(t)a^2(t;2) - M(t;4)M(t;2,2)] = 0 \tag{B.1}$$

The existence of two real roots valid if:

$$[F(t) + a^2\,(t;2)]^2 - 4\,[F(t)a^2(t;2) - M(t;4)M(t;2,2)] > 0 \tag{B.2}$$

The inequality (B.2) is always valid because it takes form of (B.3):

$$[F(t) - a^2(t;2)]^2 + 4M(t;4)M(t;2,2) > 0 \tag{B.3}$$

If prices $p(t_i)$ are not constant for $i=1,2,..N$ during Δ then from (2.28) and (2.29) always valid:

$$M(t;4) > 0 \quad ; \quad M(t;2,2) > 0 \tag{B.4}$$

To prove that at least one root $a(t;4)$ of equation (B.1) results in $Ku(t)$ (2.25) and $\theta^2(t)$ (2.27) are non-negative together, consider $\theta^2_+(t)$ (B.5) that is determined by the root $a_+(t;4)$ (B.6):

$$\theta_+^2(t)d = a_+(t;4) - a^2(t;2) > 0 \tag{B.5}$$

$$a_+(t;4) = \frac{F(t)+a^2(t;2)+\sqrt{[F(t)-a^2(t;2)]^2+4M(t;4)M(t;2,2)}}{2} \tag{B.6}$$

Substitute (B.6) into (B.5), and obtain:

$$\sqrt{[F(t) - a^2(t;2)]^2 + 4M(t;4)M(t;2,2)} > a^2(t;2) - F(t)$$

The square of two parts of the above inequality give:

$$[F(t) - a^2(t;2)]^2 + 4M(t;4)M(t;2,2) > [a^2(t;2) - F(t)]^2 \tag{B.7}$$

Hence, due to (B.4) inequality (B.7) is always valid and hence $\theta^2_+(t) > 0$ always. Due to (2.30), obtain $Ku(t) > 0$ always valid.

If all prices $p(t_i)$ are constant during Δ, then $\sigma^2(t)=Ku(t)=\theta^2(t)=0$.

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