The Market-Based Probability of Stock Returns

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25 November 2023

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Version: 10.Feb.2024

Abstract

Markets possess all available information on stock returns. The randomness of market trade determines the statistics of stock returns. This paper describes the dependence of the first four market-based statistical moments of stock returns on statistical moments and correlations of current and past trade values. The mean return of trades during the averaging period coincides with Markowitz's definition of portfolio value weighted return. We derive the market-based volatility of return and return-value correlations. We present approximations of the characteristic functions and probability measures of stock return by a finite number of market-based statistical moments. To forecast market-based average return or volatility of return, one should predict the statistical moments and correlations of current and past market trade values at the same time horizon.

Keywords : stock returns, volatility, correlations, probability, market trades

JEL: C0, E4, F3, G1, G12

This research received no support, specific grant or financial assistance from funding agencies in the public, commercial or nonprofit sectors. We welcome valuable offers of grants, support and positions.

1. Introduction

The irregular results of market trades cause fluctuating stock returns during almost any period. That makes the use of probabilistic methods almost inevitable. The correct choice of return probability is valuable for all investors. The conventional approach considers return probability to be proportional to the frequency of the return's events. However, top funds, banks, and large investors should take into account the impact market deal's size statistics and consider the probability of return as a result of market trade randomness. In this paper, we describe the dependence of statistical moments of market trade values and volumes on statistical moments of return.

Studies of stock returns are endless (Ferreira and Santa-Clara, 2008; Diebold and Yilmaz, 2009; Kelly et al., 2022). The assessments of the factors that impact the expected return play a central role (Fisher and Lorie, 1964; Mandelbrot, Fisher, and Calvet, 1997; Campbell, 1985; Brown, 1989; Fama, 1990; Fama and French, 1992; Lettau and Ludvigson, 2003; Greenwood and Shleifer, 2013; van Binsbergen and Koijen, 2015; Martin and Wagner, 2019). The irregular behavior of stock prices and returns makes probability theory a major tool for modeling returns. The probability distributions and correlation laws that can match the return change are studied by Kon (1984) and (Campbell, Grossman, and Wang, 1993; Davis, Fama, and French, 2000; Llorente et al., 2001; Dorn, Huberman, and Sengmueller, 2008; Lochstoer and Muir, 2022). The description of the expected return is complemented by research on the realized return and volatility (Schlarbaum, Lewellen, and Lease, 1978; Andersen et al., 2001; Andersen and Bollerslev, 2006; McAleer and Medeiros, 2008; Andersen and Benzoni, 2009). The probability distributions of the realized and expected return are studied by Amaral et al. (2000), Knight and Satchell (2001), and Tsay (2005). That is only a tiny fraction of the studies on stock returns.

Since Bachelier (1900), who outlined the probabilistic character of the price change, it has become routine to consider the frequency of price and return values as the basis for their probabilistic description. In our paper, we discuss reasons to consider the randomness of the size of market trade values and volumes as the basis for the introduction of a market-based probabilistic description of return.

We consider the return $r(t_i,\tau)$ (1.1) as a simple price ratio of price $p(t_i)$ at time t_i to price $p(t_i-\tau)$ in the past with time shift τ:

$$r(t_i,\tau) = \frac{p(t_i)}{p(t_i-\tau)} \qquad (1.1)$$

The number m_r of the trades that result in a particular return's value $r(t_i,\tau)=r$ (1.1) or inside a small interval near r at times t_i, $i=1,..N$ during the averaging interval Δ gives the assessment of a regular frequency-based probability $P(r)$ (1.2) of return:

$$P(r) \sim \frac{m_r}{N} \tag{1.2}$$

The conventional frequency-based mathematical expectations $E[r^n(t_i,\tau)]$ of the *n-th* power of return $r^n(t_i,\tau)$ or the *n-th* statistical moments of return are assessed by a finite number N of market trades during the averaging interval Δ as:

$$E[\, r^n(t_i,\tau)] \sim \frac{1}{N} \sum_{i=1}^{N} r^n(t_i,\tau) \tag{1.3}$$

We use the symbol *"~"* to underline that (1.2; 1.3) should be treated as the assessments of the corresponding mathematical expectations by a finite number N of the terms. The frequency-based assessments of probability $P(r)$ (1.2; 1.3) of return $r(t_i,\tau)$ serve as a basis for almost all probabilistic models. That is the correct and verified approach based on probability theory (Shephard, 1991; Shiryaev, 1999; Shreve, 2004). We note it as the frequency-based probability of a stock return.

Meanwhile, any particular finite sample of N terms of the irregular time series itself doesn't determine the averaging procedure or the correct probability distribution. The same samples of irregular time series can be the result of different random variables. The choice of an adequate averaging procedure that highlights the economic sense of the problem is the challenge for the researchers.

Actually, the description of a highly irregular time series of stock returns as a standing-alone, independent problem leaves no chance except using frequency-based probability (1.2; 1.3). However, the time series of stock returns $r(t_i,\tau)$ are completely determined by the time series of stock prices, and price random properties impact the random properties of returns. In turn, the stochasticity of market trade completely determines the randomness of stock prices. To describe the random properties of stock returns, one should take into account the stochastic properties of market trades. Indeed, at least since Markowitz (1952), the return of the portfolio is determined as the value weighted return of the securities that compose the portfolio: return "weighted with weights equal the relative amount invested in security." Markowitz (1952). That definition almost coincides with the well-known definition of volume weighted average price (VWAP) that was proposed 36 years later (Berkowitz et al., 1988; Duffie and Dworczak, 2018). In this paper, we use Markowitz's definition of a portfolio's return to define the market-based average return of trades during the averaging period, and that is almost similar to the definition of VWAP. We use

Markowitz's definition of a portfolio's return as the average return of sale trades and derive the dependence of the statistical moments of return on the statistical moments and correlations of current and past market trade values. Our theoretical paper reveals that the properties of returns' stochasticity depend on the randomness of the size of the current and past trade values.

In Section 2, we introduce the trade return equation and derive the *n-th* statistical moments of return determined by the *n-th* degree of the trade return equation. In Section 3, we introduce the first four ***market-based*** statistical moments of return and derive their dependence on statistical moments and correlations of current and past trade values. In Section 4, we show how the use of market-based statistical moments of return helps derive return-value and return-price correlations. Conclusion in Section 5. In Appendix A, we describe approximations of the characteristic functions and probability measures of return by a finite set of statistical moments. In Appendix B, we consider conditions that guarantee the non-negativity of kurtosis $Ku(t,\tau)$ and volatility $\Theta^2(t,\tau)$ of squares of return.

We assume that readers are familiar with conventional models of asset prices and stock returns and have skills in probability theory, statistical moments, characteristic functions, etc. We propose that readers know or can find on their own the definitions and terms that are not given in the text.

2. Preliminary considerations

Conventional frequency-based statistics of the return time series (1.1) do not describe the impact of the randomness of market trade values on stock returns. However, the return as a result of a single trade with a large value should contribute much more to the average return during the averaging interval Δ than the returns as a result of several deals with small values.

Actually, there is almost no difference between the assessment of the return of the portfolio composed of N securities and the assessment of the average return of N market deals during the averaging period Δ. Indeed, the return of the portfolio, which is composed of N securities, is determined as an average weighted by the "relative amount X_i invested in security i" (Markowitz, 1952). One can consider market sales during Δ as a "portfolio" and each particular trade as a "security" of the portfolio. Then, to assess the average return of the trades during Δ, one should use Markowitz's definition and weight the time series of returns by their "relative value" in the past.

However, the frequency-based probability of return time series assumes that 1000 market sales with the same return r are much more probable than one deal with a return R.

Meanwhile, if 1000 sales were made with an initial value of \$1 each, then the impact of these 1000 trades on the mean return during Δ should be much less than the impact of a single deal with the initial value of \$1 billion. These reasons are similar to those that justify the use of VWAP (Berkowitz et al., 1988; Duffie and Dworczak, 2018) vs. the frequency-based average price. Indeed, a single trade of 100 million shares at a price of p_1 impacts the average price much more than 100 trades of 1 share each at a price of p_2.

There are strong financial parallels between the definition of the value weighed return of the portfolio (Markowitz, 1952) and the definition of VWAP (Berkowitz et al., 1988; Duffie and Dworczak, 2018). We use VWAP to introduce the market-based statistical moments of prices (Olkhov, 2021; 2022). In this paper, we use Markowitz's definition of portfolio return to introduce the market-based statistical moments of stock returns.

Let us consider the time series of market trade value $C(t_i)$, volume $U(t_i)$, and price $p(t_i)$ during the averaging interval Δ (2.2) as random variables. For simplicity, we take the shift ε between trades at t_i and t_{i-1} to be constant.

$$t_i - t_{i-1} = \varepsilon \tag{2.1}$$

We denote Δ the averaging interval "today" at time t:

$$t - \frac{\Delta}{2} < t_i < t + \frac{\Delta}{2} \quad ; \quad i = 1, \ldots N \tag{2.2}$$

We assume that the number $N>>1$ of the terms of the trade time series during Δ (2.2) is sufficiently large to assess the statistical moments of trade value and volume using regular frequency-based probability (1.2; 1.3). We assume that all prices are adjusted to the present time t and take the return $r(t_i,\tau)$ (1.1) with the time shift τ as a simple ratio of the price $p(t_i)$ at time t_i to the price $p(t_i\text{-}\tau)$ at time $t_i\text{-}\tau$. To consider the return of market sales, we start with the trivial trade price equations (2.3) that tie up the trade value $C(t_i)$, trade volume $U(t_i)$, and price $p(t_i)$ of a trade at time t_i during the averaging interval Δ (1.2):

$$C(t_i) = p(t_i)U(t_i) \tag{2.3}$$

As we show in Olkhov (2021; 2022), (2.3) plays the key role in the derivation of market-based price statistical moments. We believe that due to (2.3) the randomness of the trade value $C(t_i)$ and volume $U(t_i)$ completely determine properties of price $p(t_i)$ as a random variable during Δ (2.2). We use the frequency-based probability (1.2; 1.3) to determine the conventional *n-th* statistical moments of trade value $C(t;n)$ and volume $U(t;n)$ (2.4) during the interval Δ (2.2):

$$C(t;n) = E[C^n(t_i)] \sim \frac{1}{N}\sum_{i=1}^{N} C^n(t_i) \quad ; \quad U(t;n) = E[U^n(t_i)] \sim \frac{1}{N}\sum_{i=1}^{N} U^n(t_i) \tag{2.4}$$

The finite number N of trades during Δ (2.2) means that one can assess only a finite number of the *n-th* statistical moments of trade value $C(t;n)$ and volume $U(t;n)$ (2.4). Hence, due to equation (2.3), one can derive only a finite number of market-based price *n-th* statistical moments that depend on statistical moments of trade value and volume (2.4). A finite number of statistical moments of price as a random variable can describe only approximations of price characteristic functions and probability measures.

The same considerations are valid for the description of return (1.1) as a random variable during Δ (2.2). We consider the return $r(t_i,\tau)$ (1.1) as a random variable during the averaging interval Δ (2.2). One can describe the properties of a random variable by a probability measure, a characteristic function, or a set of the *n-th* statistical moments (Shephard, 1991; Shiryaev, 1999; Shreve, 2004). We use (2.3) to derive the dependence of market-based statistical moments of return on the statistical moments of market trade values. From (1.1) and (2.3), we obtain the trade return equation (2.5):

$$C(t_i) = \frac{p(t_i)}{p(t_i-\tau)}\, p(t_i-\tau)U(t_i) = r(t_i,\tau)\; C_o(t_i,\tau) \tag{2.5}$$

As $C_o(t_i,\tau)$ (2.6) we denote the past value of the trade volume $U(t_i)$ at time t_i that is determined by the market price $p(t_i\text{-}\tau)$ in the past at $t_i\text{-}\tau$:

$$C_o(t_i,\tau) = p(t_i-\tau)U(t_i) \tag{2.6}$$

The trade return equation (2.5) takes the same form as the trade price equation (2.3). The equation (2.5) has a simple interpretation in terms of portfolio theory (Markowitz, 1952). For $i=1,2,..\ N$, one can consider (2.5), as the relation between the past value of the "security" $C_o(t_i,\tau)$, the return $r(t_i,\tau)$, and the current value $C(t_i)$. The *n-th* degree of (2.5) gives:

$$C^n(t_i) = r^n(t_i,\tau)\; C_o^n(t_i,\tau) \quad ; \quad n = 1,2,\ldots \tag{2.7}$$

One can state that the equations (2.7) for $n=1,2,..$ prohibit the independent description of the random properties of the *n-th* power of the current trade value $C^n(t_i)$, past value $C_o^n(t_i,\tau)$ (2.6), and return $r^n(t_i,\tau)$. That justifies the intention to describe the dependence of market-based statistical moments of return on statistical moments of current and past trade values. We use (1.3) and similar to (2.4) to determine the conventional *n-th* statistical moments $C_o(t,\tau;n)$ of the past values $C_o(t_i,\tau)$:

$$C_o(t,\tau;n) = E[C_o^n(t_i,\tau)] \sim \frac{1}{N}\sum_{i=1}^{N} C_o^n(t_i,\tau) \tag{2.8}$$

We highlight that relations (2.4; 2.8) give the assessments of the *n-th* statistical moments by a finite number N of trades during Δ (2.2). Let us consider each trade $i=1,..N$ during Δ (2.2) as a security i of the portfolio and use Markowitz's definition of portfolio return. Then the return $r(t,\tau;1,1)$ of the portfolio or average return of N trade during Δ takes the form:

$$r(t,\tau;1,1) = \frac{1}{C_{\Sigma o}(t,\tau;1)} \sum_{i=1}^{N} r(t_i,\tau) C_o(t_i,\tau) = \frac{C(t;1)}{C_o(t,\tau;1)} = \frac{C_\Sigma(t;1)}{C_{\Sigma o}(t,\tau;1)} \quad (2.9)$$

$$C_\Sigma(t;1) = \sum_{i=1}^{N} C(t_i) \quad ; \quad U_\Sigma(t;1) = \sum_{i=1}^{N} U(t_i) \quad ; \quad C_{\Sigma o}(t,\tau;1) = \sum_{i=1}^{N} C_o(t_i,\tau) \quad (2.10)$$

$$C(t;1) = r(t,\tau;1,1)\, C_o(t,\tau;1) \quad ; \quad C_\Sigma(t;1) = r(t,\tau;1,1)\, C_{\Sigma o}(t,\tau;1) \quad (2.11)$$

One can present (2.9) as sum of returns $r(t_i,\tau)$ that are weighted by $z(t_i,\tau;1)$

$$z(t_i,\tau;1) = \frac{C_o(t_i,\tau)}{\sum_{i=1}^{N} C_o(t_i,\tau)} \quad ; \quad \sum_{i=1}^{N} z(t_i,\tau;1) = 1 \quad (2.12)$$

$$r(t,\tau;1,1) = \sum_{i=1}^{N} r(t_i,\tau) z(t_i,\tau) \quad (2.13)$$

Relations (2.12; 2.13) completely reproduce Markowitz's definition of a portfolio return. If one substitutes return $r(t_i,\tau)$ by price $p(t_i)$ and past trade values $C_o(t_i,\tau)$ by trade volumes $U(t_i)$, then relations (2.9-2.13) take the form of VWAP (Berkowitz et al., 1988; Duffie and Dworczak, 2018; Olkhov, 2021; 2022). To highlight the parallels with VWAP, we note Markowitz's definition of portfolio return $r(t,\tau;1,1)$ (2.9-2.13) as value weighted average return (VaWAR). The economic meaning of VWAP and VaWAR is almost the same. Indeed, VaWAR $r(t,\tau;1,1)$ (2.9) has the form of a simple ratio of a total sum of current value $C_\Sigma(t,1)$ to a total sum of past value $C_{\Sigma o}(t,\tau;1)$ (2.10). That is similar to the definition of VWAP $p(t;1,1)$ (2.14) that is equal to the ratio of the total value $C_\Sigma(t,1)$ (2.10) to the total volume $U_\Sigma(t,1)$ (2.10) during Δ (2.2):

$$p(t;1,1) = \frac{1}{U_\Sigma(t;1)} \sum_{i=1}^{N} p(t_i) U(t_i) = \frac{C(t;1)}{U(t;1)} = \frac{C_\Sigma(t;1)}{U_\Sigma(t;1)} \quad (2.14)$$

From (2.9; 2.14) obtain:

$$r(t,\tau;1,1) = \frac{C(t;1)}{C_o(t,\tau;1)} = \frac{C(t;1)}{U(t;1)} \frac{U(t;1)}{C_o(t,\tau;1)} = \frac{p(t;1,1)}{p(t-\tau;1,1)} \quad (2.15)$$

We define volume weighted average past price $p(t\text{-}\tau;1,1)$ in (2.15) as:

$$p(t-\tau;1,1) = \frac{1}{U_\Sigma(t;1)} \sum_{i=1}^{N} p(t_i-\tau) U(t_i) = \frac{C_o(t,\tau;1)}{U(t;1)}$$

Relations (2.15) ties up the average return $r(t,\tau;1,1)$ with VWAP current price $p(t;1,1)$ (2.14) and past price $p(t\text{-}\tau;1,1)$.

Equations (2.7) for each $m=1,2,..$ generate the weights $z(t_i,\tau;m)$ (2.16) that define the average $r(t,\tau;n,m)$ (2.17) of the n-th degree of return $r^n(t_i,\tau)$:

$$z(t,\tau;m) = \frac{C_o^m(t_i,\tau)}{\sum_{i=1}^{N} C_o^m(t_i,\tau)} \quad ; \quad \sum_{i=1}^{N} z(t_i,\tau;m) = 1 \quad (2.16)$$

$$r(t,\tau;n,m) = \sum_{i=1}^{N} r^n(t_i,\tau) z(t,\tau;m) = \frac{1}{\sum_{i=1}^{N} C_o^m(t_i,\tau)} \sum_{i=1}^{N} r^n(t_i,\tau) C_o^m(t_i,\tau) \quad (2.17)$$

If $n=m$, then the average return $r(t,\tau;n,n)$ takes the form (2.18) that is similar to (2.9):

$$r(t,\tau;n,n) = \frac{C(t;n)}{C_o(t,\tau;n)} = \frac{C_\Sigma(t;n)}{C_{\Sigma o}(t,\tau;n)} \tag{2.18}$$

Relations (2.18) present the *n-th* statistical moment of return $r(t,\tau;n,n)$ in the form (2.19) that is similar to equations (2.11):

$$C(t;n) = r(t,\tau;n,n)\, C_o(t,\tau;n) \quad ; \quad C_\Sigma(t;n) = r(t,\tau;n,n)\, C_{\Sigma o}(t,\tau;n) \tag{2.19}$$

In the case that all past values $C_o(t_i,\tau)$ (2.6) of all N trades during Δ (2.2) are equal, then the weights $z(t_i,\tau;m)$ (2.16) coincide with the conventional frequency-based assumption that all trades have equal probability $\sim 1/N$ (1.2;1.3). If so, relations (2.9-2.19) coincide with the frequency-based assessments of statistical moments of stock return (1.3). We highlight that if past values $C_o(t_i,\tau)$ (2.6) during Δ (2.2) are not equal, then (2.9-2.19) describe the impact of the randomness of the current and past trade values on statistical moments of stock return.

Now we use relations (2.9-2.19) to define market-based statistical moments of stock return.

3. Market-based statistical moments of stock return

In this paper, we describe the first four market-based statistical moments of stock returns. A finite number of statistical moments describe only approximations of the characteristic function and probability measure of a random variable. In App. A. we derive the approximations of return characteristic functions and probability measures by a finite number of market-based statistical moments.

We note $E_m[..]$ as market-based mathematical expectation to distinguish it from the conventional notion $E[..]$ (1.3). As the 1-st market-based statistical moment of return $h(t,\tau;1)$ or market-based average return we choose VaWAR $r(t,\tau;1,1)$ and set:

$$E_m[r(t_i,\tau)] = h(t,\tau;1) = r(t,\tau;1,1) \tag{3.1}$$

We denote market-based statistical moments of return $h(t,\tau;n)$ (3.1; 3.2) to emphasize the distinctions with statistical moments of return $r(t,\tau;n,m)$ (2.17) that are averaged by weight functions $z(t_i,\tau;m)$ (2.16). To justify the choice (3.1), we refer to Markowitz's definition of portfolio return and consider it one that supports the economic sense of (3.1). The second argument in favor of (3.1) is, as we discussed above, that (3.1) almost completely reproduces the economic meaning of VWAP. One can consider Markowitz's definition of portfolio return as the origin for the definition of VWAP. Both have the same economic meaning and the same structure. Now let us consider step-by-step definitions of the next three market-based statistical moments of return.

$$E_m[r^n(t_i,\tau)] = h(t,\tau;n) \quad ; \quad n = 2,3,4. \tag{3.2}$$

Each next one market-based statistical moment of return should depend on statistical moments of current and past trade values and should be consistent with the previous ones. In particular, the 2-d market-based statistical moment of return $h(t,\tau;2)$ should be consistent with the first one, $h(t,\tau;1)$. Such consistency should guarantee that the market-based volatility $\sigma^2(t,\tau)$ (3.3) of return that is determined by the first two statistical moments should be non-negative.

$$\sigma^2(t,\tau) = E_m\left[\left(r(t_i,\tau) - h(t,\tau;1)\right)^2\right] = h(t,\tau;2) - h^2(t,\tau;1) \ \geq 0 \tag{3.3}$$

Let us consider the factors $W(t,\tau;n,m,q)$ (3.4):

$$W(t,\tau;n,m,q) = \sum_{i=1}^{N}[r^q(t_i,\tau) - h(t,\tau;q)]^n z(t,\tau;m) \tag{3.4}$$

that describes the average of the *n-th* degree of variations of the *q-th* degree of return $r^q\,(t_i,\tau)$ near the average $h(t,\tau;q)$ by the weights (2.16). To obey (3.3) we set that the market-based volatility $\sigma^2(t,\tau)$ (3.3) of return should be equal to $W(t,\tau;2,2,1)$ (3.4). From (2.17; 3.4) obtain:

$$\sigma^2(t,\tau) = W(t,\tau;2{,}2{,}1) = \sum_{i=1}^{N}[r(t_i,\tau) - h(t,\tau;1)]^2 z(t,\tau;2) \geq 0 \tag{3.5}$$

$$\sigma^2(t,\tau) = r(t,\tau;2{,}2) - 2r(t,\tau;1{,}2)h(t,\tau;1) + h^2(t,\tau;1) \ \geq 0 \tag{3.6}$$

From (3.3; 3.6) obtain definition of the 2-d market-based statistical moment $h(t,\tau;2)$ of return:

$$h(t,\tau;2) = \sigma^2(t,\tau) + h^2(t,\tau;1) = r(t,\tau;2{,}2) - 2r(t,\tau;1{,}2)h(t,\tau;1) + 2h^2(t,\tau;1) \tag{3.7}$$

Let us explain the meaning of $r(t,\tau;1,2)$ (2.17) that impacts the market-based volatility $\sigma^2(t,\tau)$ (3.6) and the 2-d statistical moment $h(t,\tau;2)$ (3.7). From the trade return equations (2.7) and (2.17), obtain:

$$r(t,\tau;1{,}2) = \frac{1}{C_\Sigma(t;n)}\sum_{i=1}^{N} r(t_i,\tau)C_o^2(t_i,\tau) = \frac{1}{C_o(t,\tau;2)}\frac{1}{N}\sum_{i=1}^{N} C(t_i)C_o(t_i,\tau) \tag{3.8}$$

Relations (3.8) highlight dependence of $r(t,\tau;1,2)$ on correlations $corr\{C(t_i)C(t_i,\tau)\}$ between the current $C(t_i)$ and past $C(t_i,\tau)$ trade values. Indeed, the frequency-based probability (1.3) determine

$$E[C(t_i)C_o(t_i,\tau)] = \frac{1}{N}\sum_{i=1}^{N} C(t_i)C_o(t_i,\tau) = C(t;1)\,C_o(t,\tau;1) + corr\{C(t_i)C_o(t_i,\tau)\} \tag{3.9}$$

From (2.18; 3.6; 3.7; 3.9), one can obtain:

$$\sigma^2(t,\tau) = \frac{\Omega^2(t) + h^2(t,\tau;1)\Phi^2(t,\tau) - 2h(t,\tau;1)corr\{C(t_i)C_o(t_i,\tau)\}}{C_o(t,\tau;2)} \tag{3.10}$$

$$h(t,\tau;2) = \frac{C(t;2) + 2h^2(t,\tau;1)\Phi^2(t,\tau) - 2h(t,\tau;1)corr\{C(t_i)C_o(t_i,\tau)\}}{C_o(t,\tau;2)} \tag{3.11}$$

In (3.10; 3.11) we denote volatility $\Omega(t)$ (3.12) of the current trade value and volatility $\Phi(t,\tau)$ (3.13) of the past trade value:

$$\Omega^2(t) = E[(C(t_i) - C(t;1))^2] = C(t;2) - C^2(t;1) \tag{3.12}$$

$$\Phi^2(t,\tau) = E[(C_o(t_i,\tau) - C_o(t,\tau;1))^2] = C_o(t,\tau;2) - C_o^2(t,\tau;1) \tag{3.13}$$

Relations (3.10; 3.11) reveal the direct dependence of the market-based volatility $\sigma^2(t,\tau)$ and the 2-d statistical moment $h(t,\tau;2)$ of stock return on the first two statistical moments, volatilities, and correlations of the current $C(t_i)$ and past $C(t_i,\tau)$ trade values.

Market-based statistical moments of stock return describe the dependence on statistical moments and correlations of current and past values of market trades. Any predictions of return volatility $\sigma^2(t,\tau)$, which ignore the dependence (3.6; 3.10) on statistical moments and correlations of the current and past trade values, could be untrustworthy. It is particularly important for the largest investment funds, banks, and traders, who perform the major market transactions and forecast macroeconomic, financial, and market trends, to take into account the impact of the random size of the market trade values on the volatility of stock returns.

Now we consider the definition of the 3-d market-based statistical moment $h(t,\tau;2)$ of return. We highlight that the 3-d central statistical moment is usually noted as skewness $Sk(t,\tau)$, and we take the conventional definition for market-based $Sk(t,\tau)$:

$$Sk(t,\tau)\sigma^3(t,\tau) = E_m\left[\left(r(t_i,\tau) - h(t,\tau;1)\right)^3\right] = h(t,\tau;3) - 3\,h(t,\tau;2)h(t,\tau;1) + 2h^3(t,\tau;1) \quad (3.14)$$

To determine the 3-d market based statistical moment $h(t,\tau;3)$ from (3.14) we use (2.17; 3.4) and set up equation (3.15):

$$E_m\left[\left(r(t_i,\tau) - h(t,\tau;1)\right)^3\right] = W(t,\tau;3,3,1) = \sum_{i=1}^{N}\left[r(t_i,\tau) - h(t,\tau;1)\right]^3 z(t,\tau;3) \quad (3.15)$$

From (2.17; 3.15) obtain:

$$Sk(t,\tau)\sigma^3(t,\tau) = r(t,\tau;3,3) - 3r(t,\tau;2,3)h(t,\tau;1) + 3r(t,\tau;1,3)h^2(t,\tau;1) - h^3(t,\tau;1) \quad (3.16)$$

From (3.14; 3.16) obtain market-based 3-d statistical moment $h(t,\tau;3)$:

$$h(t,\tau;3) = r(t,\tau;3,3) + 3h(t,\tau;1)[r(t,\tau;1,3)h(t,\tau;1) - r(t,\tau;2,3)] + 3\,h(t,\tau;1)\sigma^2(t,\tau) \quad (3.17)$$

We emphasize that the statistical moments of return $r(t,\tau;2,3)$ and $r(t,\tau;1,3)$ (2.17) depend on correlations between the current and past trade values:

$$r(t,\tau;2,3) = \frac{1}{C_o(t,\tau;3)}\frac{1}{N}\sum_{i=1}^{N} C^2(t_i)C_o(t_i,\tau) = \frac{C(t;2)C_o(t,\tau;1) + corr\{C^2(t_i)C_o(t_i,\tau)\}}{C_o(t,\tau;3)} \quad (3.18)$$

$$r(t,\tau;1,3) = \frac{1}{C_o(t,\tau;3)}\frac{1}{N}\sum_{i=1}^{N} C(t_i)C_0^2(t_i,\tau) = \frac{C(t;1)C_o(t,\tau;2) + corr\{C(t_i)C_0^2(t_i,\tau)\}}{C_o(t,\tau;3)} \quad (3.19)$$

To derive the 4-th market-based statistical moment $h(t,\tau;4)$ of return we highlight that it should be consistent with the requirement that two market-based central even statistical moments $E_m[(r(t_i,\tau)-h(t,\tau;1))^4]$ and $E_m[(r^2(t_i,\tau)-h(t,\tau;2))^2]$ should be non-negative (App.B.) and we show that the 4-th market-based statistical moment $h(t,\tau;4)$ (B.10) of return takes the form:

$$h_+(t,\tau;4) = \frac{f(t) + h^2(t,\tau;2) + \sqrt{[f(t) - h^2(t,\tau;2)]^2 + 4W(t,\tau;4,4,1)W(t,\tau;2,4,2)}}{2} \quad (3.20)$$

In this paper, we reduce our description of market-based probability of stock return by the first four statistical moments: $h(t,\tau;1)$ (3.1), $h(t,\tau;2)$ (3.7; 3.11), $h(t,\tau;3)$ (3.17), and $h(t,\tau;4)$ (3.20). The finite number of statistical moments describes the approximations of the characteristic functions and probability measures of stock return (App. A.).

4. Return-Value correlations

In this section, we use the above relations to assess correlations between stock return $r(t_i,\tau)$ and past trade values. From (2.4; 2.8; 2.9; 3.1), one obtains that correlations $corr\{r(t_i,\tau)C_o(t_i,\tau)\}$ between return $r(t_i,\tau)$ and past trade values $C_o(t_i,\tau)$ during the averaging interval Δ (2.2) equal zero:

$$corr\{r(t_i,\tau)C_o(t_i,\tau)\} = E[r(t_i,\tau)C_o(t_i,\tau)] - E_m[r(t_i,\tau)]\, E[C_o(t_i,\tau)] \tag{4.1}$$

From (2.5) obtain:

$$E[r(t_i,\tau)C_o(t_i,\tau)] = E[C(t_i)] = C(t;1)$$

Hence from (2.9; 2.11; 3.1) obtain:

$$corr\{r(t_i,\tau)C_o(t_i,\tau)\} = C(t;1) - h(t,\tau;1)\, C_o(t,\tau;1) = 0 \tag{4.2}$$

One can easy derive correlations $corr\{r(t_i,\tau)C_o^2(t_i,\tau)\}$ between return $r(t_i,\tau)$ and squares of the past trade values $C_o^2(t_i,\tau)$:

$$corr\{r(t_i,\tau)C_o^2(t_i,\tau)\} = E[r(t_i,\tau)C_o^2(t_i,\tau)] - E_m[r(t_i,\tau)]\, E[C_o^2(t_i,\tau)]$$

$$E[r(t_i,\tau)C_o^2(t_i,\tau)] = E[C(t_i)C_o(t_i,\tau)] = C(t;1)C_o(t,\tau;1) + corr\{C(t_i)C_o(t_i,\tau)\}$$

$$E[C(t_i)C_o(t_i,\tau)] = \frac{1}{N}\sum_{i=1}^{N} C(t_i)C_o(t_i,\tau)$$

Hence, correlations $corr\{r(t_i,\tau)C_o^2(t_i,\tau)\}$ during Δ (2.2) takes the form:

$$corr\{r(t_i,\tau)C_o^2(t_i,\tau)\} = C(t;1)C_o(t,\tau;1) - h(t,\tau;1)\, C_o(t,\tau;2) + corr\{C(t_i)C_o(t_i,\tau)\}$$

From (2.9; 3.13) obtain:

$$corr\{r(t_i,\tau)C_o^2(t_i,\tau)\} = corr\{C(t_i)C_o(t_i,\tau)\} - h(t,\tau;1)\, \Phi^2(t,\tau) \tag{4.3}$$

Relations (4.3) reveal direct dependence of the market-based $corr\{r(t_i,\tau)C_o^2(t_i,\tau)\}$ between return $r(t_i,\tau)$ and squares of the past trade values $C_o^2(t_i,\tau)$ on the correlations $corr\{C(t_i)C_o(t_i,\tau)\}$ between current $C(t_i)$ and past $C_o(t_i,\tau)$ trade values.

6. Conclusion

The irregular time series of stock returns themselves don't uniquely determine the choice of the averaging procedure and the probability distribution. The conventional approach considers return time series as a standing-alone sample of a random variable, and that results in a frequency-based (1.2; 1.3) assessment of the random properties of return. However, the market nature of the randomness of stock returns implies that statistical moments of return should depend on statistical moments of market trades. The market-based origin of return statistics results in the dependence of return statistical moments, return-value, and return-price correlations on statistical moments and correlations of current and past trade values, and that differs from the frequency-based assessments of random return time series.

The choice between frequency-based and market-based assessments of stock return statistical moments is determined by the habits and goals of investors. The largest investors, traders, and banks, that manage substantial portfolios and major market transactions should take into account the impact of the randomness of trade values on the statistical properties of return as we describe in our paper. To predict market-based statistical moments of stock return, one should forecast statistical moments and correlations of current and past trade values. That complicates the assessment of market-based statistical moments of return but highlights direct ties between market trade stochasticity and the randomness of stock returns.

Frequency-based assessment of return statistics is more simple, familiar, and may follow most investors' expectations. These expectations can influence investment decisions and thus impact market trade stochasticity and, hence, the randomness of stock returns. However, the forecasts of the frequency-based properties of random return have almost no ties with predictions of market trade statistical moments. Thus, frequency-based forecasts have low reliability and a poor economic basis.

Eventually, the description of the random properties of stock returns as a result of market trades requires the use of market-based statistical moments.

Appendix A

Approximations of the characteristic functions and probability measures of stock return by a finite set of statistical moments

We consider stock return as a random variable during the averaging interval Δ (2.2). The random variable can equally be described by the set of statistical moments, characteristic function $F(t,\tau;x)$ (A.1), and probability measure $\mu(t,\tau;r)$ (A.2) (Shephard, 1991; Shiryaev, 1999; Shreve, 2004). The Taylor series expansion of the market-based characteristic function $F(t,\tau;x)$ presents it through the set of market-based *n-th* statistical moments $h(t,\tau;n)$:

$$F(t,\tau;x) = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} h(t,\tau;n)\, x^n \qquad \text{(A.1)}$$

$$\mu(t,\tau;r) = \frac{1}{\sqrt{2\pi}} \int F(t;x) \exp(-ixr)\, dx \qquad \text{(A.2)}$$

$$h(t,\tau;n) = \frac{d^n}{(i)^n dx^n} F(t,\tau;x)|_{x=0} = \int r^n \mu(t,\tau;r)\, dr \quad ; \quad \int \mu(t,\tau;r)\, dr = 1 \qquad \text{(A.3)}$$

In (A.1-A.3), i is the imaginary unit. For simplicity, we take the stock return as a continuous random variable during Δ (2.2). A finite number q of the statistical moments $h(t,\tau;n)$, $n=1,2,..q$, determines the q-approximation of the price characteristic function $F_q(t,\tau;x)$ (A.4):

$$F_q(t,\tau;x) = 1 + \sum_{n=1}^{q} \frac{i^n}{n!} h(t,\tau;n)\, x^n \qquad \text{(A.4)}$$

Taylor expansion (A.4) is not too useful to derive the Fourier transform (A.2), and to obtain a q-approximation of the price probability measure $\mu_q(t,\tau;r)$, we consider the approximation of price characteristic functions $G_q(t,\tau;x)$ (A.5):

$$G_q(t,\tau;x) = exp\left\{\sum_{n=1}^{q} \frac{i^n}{n!}\, b(t,\tau;n)\, x^n - B\, x^{2Q}\right\} \;\; ; \;\; q = 1,2,\dots ;\; q < 2Q \;;\; B > 0 \qquad \text{(A.5)}$$

and require that $G_q(t,\tau;x)$ (A.5) obey relations (A.3):

$$h(t,\tau;n) = \frac{d^n}{(i)^n dx^n} G_q(t,\tau;x)|_{x=0} \quad ; \quad n \le q \qquad \text{(A.6)}$$

Relations (A.6) define functions $b(t,\tau;n)$ in (A.5) through market-based statistical moments $h(t,\tau;n)$, $n \le q$. The terms Bx^{2Q}, $B>0$, and $2Q>q$ don't impact relations (A.3; A.6) but guarantees the existence of the probability measures $\mu_q(t,\tau;r)$ as Fourier transform (A.2) of the characteristic functions $G_q(t,\tau;x)$ (A.5). The uncertainty of $B>0$ and power $2Q>q$ in (A.5) highlights the well-known fact that the first q statistical moments don't explicitly determine the characteristic function and probability measure of a random variable. Relations (A.5) describe the set of characteristic functions $G_q(t,\tau;x)$ with different $B>0$ and $2Q>q$ and the corresponding set of probability measures $\mu_q(t,\tau;r)$ that match (A.2; A.5; A.6).

For $q=1$, the approximate characteristic function $G_1(t,\tau;x)$ and probability $\mu_q(t,\tau;r)$:

$$G_1(t,\tau;x) = exp\{i\, b(t,\tau;1)x\} \;\; ; \; h(t,\tau;1) = -i \frac{d}{dx} G_1(t,\tau;x)|_{x=0} = b(t,\tau;1) \qquad \text{(A.7)}$$

$$\mu_1(t,\tau;r) = \int dx\ G_1(t,\tau;x)\, exp(-irx) = \delta\big(r - b(t,\tau;1)\big) \tag{A.8}$$

For $q=2$, the approximation $G_2(t,\tau;x)$ describes the Gaussian probability measure $\mu_2(t,\tau;r)$:

$$G_2(t,\tau;x) = exp\left\{i\, b(t,\tau;1)x - \frac{b(t,\tau;2)}{2}x^2\right\} \tag{A.9}$$

It is easy to show that:

$$h(t,\tau;2) = -\frac{d^2}{dx^2}G_2(t,\tau;x)|_{x=0} = b(t,\tau;2) + b^2(t,\tau;1)$$

$$b(t,\tau;2) = h(t,\tau;2) - b^2(t,\tau;1) = \sigma^2(t,\tau) \tag{A.10}$$

The coefficient $b(t,\tau;2)$ equals the market-based return volatility $\sigma^2(t,\tau)$ (3.3; 3.6; 3.10), and the Fourier transform (A.2) for $G_2(t,\tau;x)$ (A.9) gives the Gaussian price probability $\mu_2(t,\tau;r)$:

$$\mu_2(t,\tau;r) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma^2(t,\tau)}exp\left\{-\frac{(r-b(t,\tau;1))^2}{2\sigma^2(t,\tau)}\right\} \tag{A.11}$$

One can consider also non-Gaussian approximations of the characteristic function $G_2(t,\tau;x)$:

$$G_2(t,\tau;x) = exp\left\{i\, b(t,\tau;1)x - \frac{b(t,\tau;2)}{2}x^2 - B\, x^{2Q}\right\} \tag{A.12}$$

For $q=3$, the approximation $G_3(t,\tau;x)$ has the form:

$$G_3(t,\tau;x) = exp\left\{i\, b(t,\tau;1)x - \frac{b(t,\tau;2)}{2}x^2 - i\,\frac{b(t,\tau;3)}{6}x^3 - B\, x^{2Q}\right\} \tag{A.13}$$

$$h(t,\tau;3) = i\frac{d^3}{dx^3}G_3(t;x)|_{x=0} = b(t,\tau;3) + 3b(t,\tau;2)\sigma^2(t,\tau) + b^3(t,\tau;1)$$

$$b(t,\tau;3) = E_m\left[\big(r - b(t,\tau;1)\big)^3\right] = Sk(t,\tau)\sigma^3(t,\tau) \tag{A.14}$$

The coefficient $b(t,\tau;3)$ (A.14) depends on the market-based skewness $Sk(t,\tau)$ (3.14; 3.16) of stock return and describes the asymmetry of the market-based probability of return from the normal distribution.

If $q=4$, than the approximations $G_4(t;x)$ depend on the choice of $B>0$ and power $2Q>4$:

$$G_4(t,\tau;4) = exp\left\{i\, b(t,\tau;1)x - \frac{b(t,\tau;2)}{2}x^2 - i\,\frac{b(t,\tau;3)}{6}x^3 + \frac{b(t,\tau;4)}{24}x^4 - Bx^{2Q}\right\}\ ;\ 2Q > 4 \tag{A.15}$$

Simple, but long calculations give:

$$b(t,\tau;4) = h(t,\tau;4) - 4h(t,\tau;3)h(t,\tau;1) + 12h(t,\tau;2)h^2(t,\tau;1) - 6h^4(t,\tau;1) - 3h^2(t,\tau;2)$$

$$b(t,\tau;4) = E_m\left[\big(r - b(t,\tau;1)\big)^4\right] - 3E_m^2\left[\big(r - b(t,\tau;1)\big)^2\right]$$

Kurtosis $Ku(t,\tau)$ of return (B.1) describes the distinction of the tails of return probability measure $\mu_4(t,\tau;r)$ from the tails of a normal distribution.

$$Ku(t,\tau)\sigma^4(t,\tau) = E_m\left[\big(r - b(t,\tau;1)\big)^4\right]$$

$$b(t,\tau;4) = [Ku(t,\tau) - 3]\sigma^4(t,\tau)$$

Appendix B

The choice of *h(t,τ;4)* and non-negativity of kurtosis *Ku(t,τ)* and volatility *Θ²(t,τ)*

To define the 4-th market-based statistical moment $h(t,\tau;4)$ that is consistent with the first three market-based statistical moments, one should verify that two central even statistical moments (B.1) and (B.3) are non-negative. The 4-th central statistical moment is usually noted as the Kurtosis $Ku(t,\tau)$ of return:

$$Ku(t,\tau)\sigma^4(t,\tau) = E_m\left[\left(r(t_i,\tau) - h(t,\tau;1)\right)^4\right] \geq 0 \quad \text{(B.1)}$$

$$Ku(t,\tau)\sigma^4(t,\tau) = h(t,\tau;4) - 4h(t,\tau;3)h(t,\tau;1) + 6h(t,\tau;2)h^2(t,\tau;1) - 3h^4(t,\tau;1) \geq 0 \quad \text{(B.2)}$$

Let us denote the volatility $\Theta^2(t,\tau)$ of the squares of return $r^2(t_i,\tau)$:

$$\Theta^2(t,\tau) = E_m\left[\left(r^2(t_i,\tau) - h(t,\tau;2)\right)^2\right] = h(t,\tau;4) - h^2(t,\tau;2) \geq 0 \quad \text{(B.3)}$$

The 4-th statistical moment of return $h(t,\tau;4)$ must fulfill (B.1-B.3). To fulfill both conditions we take the product of (B.1; B.3) to be equal to the product of the similar central statistical moments $W(t,\tau;4,4,1)$ and $W(t,\tau;2,4,2)$ (3.4):

$$Ku(t,\tau)\sigma^4(t,\tau)\Theta^2(t,\tau) = W(t,\tau;4,4,1)W(t,\tau;2,4,2) \geq 0 \quad \text{(B.4)}$$

$$W(t,\tau;4,4,1) = r(t,\tau;4,4) - 4r(t,\tau;3,4)h(t,\tau;1) + 6r(t,\tau;2,4)h^2(t,\tau;1) - 4r(t,\tau;1,4)h^3(t,\tau;1) + h^4(t,\tau;1) \geq 0 \quad \text{(B.5)}$$

$$W(t,\tau;2,4,2) = r(t,\tau;4,4) - h^2(t,\tau;2) \geq 0 \quad \text{(B.6)}$$

One can present (B.2) as:

$$Ku(t,\tau)\sigma^4(t,\tau) = h(t,\tau;4) - f(t,\tau) \geq 0 \quad \text{(B.7)}$$

$$f(t,\tau) = 4h(t,\tau;3)h(t,\tau;1) - 6h(t,\tau;2)h^2(t,\tau;1) + 3h^4(t,\tau;1) \quad \text{(B.8)}$$

Then (B.4) takes the form:

$$[h(t,\tau;4) - f(t,\tau)][h(t,\tau;4) - h^2(t,\tau;2)] = W(t,\tau;4,4,1)W(t,\tau;2,4,2) \quad \text{(B.9)}$$

Equation (B.9) is a simple quadratic equation for $h(t,\tau;4)$:

$$h^2(t,\tau;4) - [f(t,\tau) + h^2(t,\tau;2)]h(t,\tau;4) + [f(t,\tau)h^2(t,\tau;2) - W(t,\tau;4,4,1)W(t,\tau;2,4,2)] = 0$$

The inequality (B.3) is always valid for the root $h_+(t,\tau;4)$ (B.10) of equation (B.9):

$$h_+(t,\tau;4) = \frac{f(t) + h^2(t,\tau;2) + \sqrt{[f(t) - h^2(t,\tau;2)]^2 + 4W(t,\tau;4,4,1)W(t,\tau;2,4,2)}}{2} \quad \text{(B.10)}$$

To derive it let us substitute $h_+(t,\tau;4)$ into (B.3) and for (B.5; B.6) obtain:

$$\sqrt{[F(t) - h^2(t,\tau;2)]^2 + 4W(t,\tau;4,4,1)W(t,\tau;2,4,2)} > F(t) - h^2(t,\tau;2)$$

Thus, (B.3) is always valid. Due to (B.5; B.6), obtain that (B.4) is always valid too. Hence, for the root (B.10), we obtain that (B.1) and (B.3) are valid simultaneously, and (B.10) determines the 4-th market-based statistical moment $h(t,\tau;4)$ that is consistent with the first three market-based statistical moments.

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