On identifying efficient, fair and stable allocations in "generalized" sequencing games.

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Abstract

We model sequencing problems as coalitional games and study the Shapley value and the non-emptiness of the core. The ”optimistic” cost of a coalition is its minimum waiting cost when the members are served first in an order. The ”pessimistic” cost of a coalition is its minimum waiting cost when the members are served last. We take the weighted average of the two extremes and define the class of ”weighted optimistic pessimistic (WOP)” cost games. If the weight is zero, we get the optimistic scenario and if it is one, we get the pessimistic scenario. We find a necessary and sufficient condition on the associated weights for the core to be non-empty. We also find a necessary and sufficient condition on these weights for the Shapley value to be an allocation in the core. We impose ”upper bounds” to protect agents against arbitrarily high disuilities from waiting. If an agent's disutility level is his Shapley payoff from the WOP cost game, we find necessary and sufficient conditions on the upper bounds for the Shapley value to conform to them.

Keywords: Sequencing, disutility upper bounds, core, cooperative game, Shapley value

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1 Introduction

Waiting lines are pervasive in our daily lives. Several activities involve waiting, such as, standing in line at a grocery store checkout, waiting for public transportation, or lining up for a popular coffee shop. In the digital realm, online customer support and streaming services often entail waiting for assistance or content delivery. Healthcare systems witness waiting lines in clinics and hospitals, emphasizing the importance of efficient patient flow. Waiting is costly. Long waiting lines result in time loss, fostering customer dissatisfaction and subsequently contributing to business losses. Moreover, increased waiting times indicate inefficient resource utilization, resulting in heightened operational costs, particularly within the manufacturing, transportation, and health sectors. Economists have studied the class of sequencing problems with an objective to design waiting lines (the order in which agents wait for a service) that satisfy certain desirable properties. Efficiency is one such fundamental property that minimizes the total cost of waiting. We now introduce the premise of sequencing problems more formally.

There is a finite set of agents in need of a service. Each agent has one job to process and the service provider can only serve one agent at a time. An agent is identified by two parameters, his per unit time waiting cost and his job processing time. We allow both parameters to differ across agents. We also allow for monetary compensations that can be positive (an agent receives money) or negative (an agent pays money). An **allocation** in a sequencing problem specifies the order in which agents are served and the monetary compensations they pay or receive. Preferences are defined over pairs specifying - 1) the position of an agent in an order, which in turn determines the amount of time an agent waits to get served and, 2) their consumption of money. Preferences are continuous and quasi linear. The disutility of an agent is equal to their waiting cost to get served minus their consumption of money. The literature has extensively studied sequencing problems from both incentive and normative point of views [1].

We solve this class of problems using tools from cooperative game theory. In our day-to-day lives, cooperation among individuals waiting for a service can be motivated through several practical and ethical considerations. For instance,

1. During times of high demand for medical services or vaccination clinics, individuals often form universal alliances while waiting in an organized queue. Cooperation is essential to ensure that those in need receive services in a fair manner.

2. During emergencies or evacuations, it is essential for people to cooperate in larger groups for the safety and well-being of everyone involved.

3. Large-scale events often involve queueing for entry. Attendees typically cooperate by identifying themselves as a part of a comprehensive group to ensure a smooth and safe entry process. This cooperation helps prevent congestion and maintains a positive experience for all participants.

Such instances highlight the importance of cooperation to facilitate smooth functioning of queues. It enhances overall customer satisfaction and instills a sense of solidarity among the agents awaiting the service.

This paper addresses the following concerns:

1. How to divide the (minimum) aggregate cost across all agents in a ”fair” and ”stable” manner?

    - We first define the “urgency” of an agent as the ratio of their per unit time waiting cost to their job processing time. Minimizing the aggregate waiting cost requires serving agents in a non-increasing order of their urgency indices.

[1] For sequencing problems with incentives, see Dolan [12], Mendelson and Whang [19], Suijs [28], Mitra ([21], [22]), De [10], Banerjee et al. [1]. For the normative studies, see Chun [3], Chun, Mitra and Mutuswami ([5], [6], [7]), Chun and Yengin [**?**], and De [11]

- For fairness, we use the widely recognized notion of Shapley value. It is the unique division of payoffs which satisfies the axioms of efficiency, symmetry, linearity and dummy player ([26]). An agent's Shapley payoff is the expected value of their contribution to each coalition when all permutations of agents are equally likely.
- For stability, we use the concept of core. A payoff vector is in the core if no subgroup of agents can benefit by deviating and forming their own coalition. The division of payoff is such, that it is possible to sustain cooperation among all agents.

We begin by associating a sequencing problem to a cost sharing game. There are multiple ways of calculating the cost of serving the members of a coalition. The literature studies two such extreme definitions that depend on when the coalitional members are served relative to the non-coalitional members in a queue. In the **optimistic** definition, the members of a coalition are served first (before the non-coalitional members). Under an optimistic assessment, the cost of a coalition is the minimum waiting cost of serving its members when they are served first. In the **pessimistic** definition, the members of a coalition are served last (after the non-coalitional members). Under a pessimistic assessment, the cost of a coalition is the minimum waiting cost of its members when they are served last. In this paper, we propose a more general way of defining the cost of serving coalitional members. We refer to this class of cost sharing games as "weighted optimistic-pessimistic (WOP) games". As the name suggests, each game is defined as a weighted average of the aforementioned optimistic and pessimistic cost games. The weight can be interpreted as the priority received by the non-coalitional members while calculating the cost of serving the coalitional members. Specifically, in the optimistic definition, this weight is zero while in the pessimistic definition, it is one. We then calculate the Shapley value of this weighted optimistic-pessimistic game and ask the following:

- Under what condition on the associated weights does the Shapley value of the weighted optimistic-pessimistic game qualify as a member of the core of this game?
- Under what condition on the associated weights is core of this game non-empty?

We get the following results.

- For a given weight, the Shapley value of the weighted optimistic-pessimistic game belongs to the core of this game if and only if the value of the associated weight is at least one-half.
- For a given weight, the core of the weighted optimistic pessimistic game is non-empty if and only if the value of the associated weight is at least one-half.

When the associated weight is exactly one-half, it means that the coalitional members have an equal chance of being served consecutively anywhere in the queue.

2. Unpredictability of waiting time in certain situations might lead to agents suffering arbitrarily high levels of disutility. To safeguard them against such adversities, we impose an upper bound on each agent's disutility level, which in turn acts as a safety net for that agent. We ask the following question: if each agent's disutility is equal to his Shapley payoff from the associated WOP cost game, then under what necessary and sufficient condition on the upper bounds will the Shapley payoffs conform to them?

    - We begin by introducing the concept of *"disutility upper bounds"*. Alternatively, this has also been addressed as the *"generalized welfare lower bounds (GWLB)"* and extensively analysed by Banerjee et. al [1]. It collectively represents a family of several bounds (for instance, identical costs bound, expected costs bound, bound with respect to a status quo, such as, initial arrival order, etc) that have been previously studied in the literature [2].

- Given a sequencing problem, each agent's upper bound is represented as a product of two components: 1) his per unit time waiting cost, and 2) a "benchmark" function which can be any strictly positive function of the vector of job processing times. This is a more general way of defining an upper bound since it encompasses several bounds of such multiplicative forms within itself (see Banerjee et. al [1] for a more detailed discussion). It means that the functional form of a benchmark will vary depending on the particular bound under consideration. We also assume that there is no consumption of money while computing an upper bound.

- Finally, we introduce the *"disutility upper bound relative to a benchmark"* property which ensures that each agent's disutility from waiting does not exceed his upper bound. The "Shapley rule" is an allocation rule [3] that maps each sequencing problem to a set of allocations, such that, every agent's disutility at those allocations coincides with his Shapley payoff from the WOP cost sharing game. The Shapley rule satisfies *disutility upper bound relative to a benchmark* if and only if each agent's upper bound on his disutility is at least as high as his expected cost of waiting when all orders are equally likely.

2 Literature

One of the very first study of cooperation in sequencing situations is by Curiel et al. [9]. They show that the equal gain splitting rule belongs to the core of a cost saving coali-

[2]See Moulin [24] and Yengin [29] (study identical costs bound (ICB)), Maniquet [18], Chun [4], Banerjee et. al [1], Kayi and Ramaekars [16], and Mitra [23] (study both identical costs bound (ICB) and expected costs bounds (ECB)), Chun and Yengin [8] (study the k-welfare bounds), Gershkov and Schweinzer [13](study individual rationality constraints with respect to an initial order of arrival)

[3]An allocation rule maps each sequencing problem to a non-empty subset of allocations (defined earlier).

tional game. The worth of a coalition is its maximal cost saving when members rearrange themselves into an efficient order given a status quo order of arrival .

The special case of a sequencing problem when agents have equal job processing times is referred to as a "queueing problem". Queueing problems have been studied as coalitional games when the worth of a coalition is defined in an optimistic manner [Maniquet [18]], alternatively, when it is defined in a pessimistic manner [Chun [4]]. Both papers axiomatically characterize the Shapley values of the resulting coalitional games.

An immediate extension characterizes the Shapley value (in the optimistic scenario) for sequencing games [Mishra and Rangarajan [20]]. The corresponding monetary compensation has been referred to as "minimal transfer rule" in the literature. The minimal transfer rule is the only rule satisfying *Pareto indifference*, *individual rationality from random arrival*, *consistency*, and *cost monotonicity* [Chun[2]]. Consider a scenario where individual preferences are known to the server, and job size is observable. A new kind of cooperative manipulation stems from the inability of the server to detect the true identity of users, and the users' ability to request a job without revealing its true beneficiary. The optimistic game is *mergeproof* (coalitions merging to form a single entity) and the pessimistic game is *splitproof* (a single agent splitting thier job) [Moulin [25]].

3 Model

A finite set of agents $N = \{1, 2, \ldots, n\}$ are in need of a service. A facility provider processes their jobs but can only do so one job at a time. For each $i \in N$, agent i is identified by a pair of parameters $(\theta_i, l_i) \in \mathbb{R}^2_{++}$ where θ_i is their per unit time waiting cost and l_i is their job processing time. Let $L_i \in \mathbb{R}_+$ be the amount of time an agent has to wait to get served. Agents also consume money. Let $\tau_i \in \mathbb{R}$ be the amount of money agent i consumes. Preferences are defined on $\mathbb{R}_{++} \times \mathbb{R}$ and are both continuous and quasi linear: agent i's **disutility** at (L_i, τ_i) is $\pi_i(L_i, \tau_i) = \theta_i L_i - \tau_i$ where $\theta_i L_i$ is their cost of waiting L_i

units of time and τ_i is their consumption of money. A **sequencing problem** with agent set N is a list $(\theta, l) \in \mathbb{R}_n^{++} \times \mathbb{R}_n^{++}$ where $\theta = (\theta_1, \ldots, \theta_n)$ is the vector of per unit waiting costs and $l = (l_1, \ldots, l_n)$ is the vector of job processing times. The set of all sequencing problems is denoted by $\mathcal{S}^N$.

An order on N is a bijection $\sigma : N \to N$ that assigns a position to each agent $i \in N$. For instance, if $\sigma(i) = 3$ then i occupies the third position. Let Σ^N be the set of all serving orders on N. For an order $\sigma \in \Sigma^N$, let $P_i(\sigma) = \{j \in N \setminus \{i\} \mid \sigma(j) < \sigma(i)\}$ be the set of predecessors of i and $F_i(\sigma) = \{j \in N \setminus \{i\} \mid \sigma(j) > \sigma(i)\}$ the set of his successors. Given $\sigma \in \Sigma^N$, the waiting time of agent i is $L_i(\sigma) = \sum_{j \in P_i(\sigma)} l_j$. For each $(\theta, l) \in \mathcal{S}^N$ and $\sigma \in \Sigma^N$, agent i's cost of waiting is $C_i(\sigma) = \theta_i L_i(\sigma)$. Let $\tau \in \mathbb{R}^N$ be the vector of agents' money consumptions. An **allocation** is a pair $(\sigma, \tau) \in \Sigma^N \times \mathbb{R}^N$. Let $\boldsymbol{X}^N$ be the set of allocations. For each $(\theta, l) \in \mathcal{S}^N$ and each $(\sigma, \tau) \in \boldsymbol{X}^N$, agent i's disutility is $\pi_i(\sigma, \tau_i) = \theta_i L_i(\sigma) - \tau_i = \theta_i \sum_{j \in P_i(\sigma)} l_j - \tau_i$.

Let $F_N = \{(\sigma, \tau) \in \boldsymbol{X}^N : \sum_{i \in N} \tau_i \leq 0\}$ be the set of **feasible** allocations. An allocation $(\sigma, \tau) \in F_N$ is **efficient at $(\boldsymbol{\theta}, \boldsymbol{l})$** if there is no other allocation $(\sigma', \tau') \in F_N$ such that:

1. for each $i \in N, \pi(\sigma', \tau') \leq \pi(\sigma, \tau)$,

2. there exists at least one $k \in N$ such that $\pi_k(\sigma', \tau') < \pi_k(\sigma, \tau)$ and,

3. $\sum_{i \in N} \tau_i = 0$.

Let $E(\theta, l)$ be the set of all efficient allocations for $(\theta, l) \in \mathcal{S}^N$.

Lemma 1. If $(\sigma, \tau) \in E(\theta, l)$ then for any arbitrary redistribution of money $\tau' \in \mathbb{R}^n$ such that, $\sum_{i \in N} \tau_i' = \sum_{i \in N} \tau_i$, we have $(\sigma, \tau') \in E(\theta, l)$.

Lemma 1 is a direct implication of quasi linearity. Thanks to Lemma 1, it is meaningful to speak of the efficiency of an order. Given $(\theta, l) \in \mathcal{S}^N$, the aggregate cost of waiting at $\sigma \in \Sigma^N$ is given by $C(\sigma) = \sum_{i \in N} \theta_i L_i(\sigma)$. An order $\sigma \in \Sigma^N$ is **efficient for $(\boldsymbol{\theta}, \boldsymbol{l})$** if $\sigma \in \operatorname{argmin}_{\sigma'} C(\sigma')$. Let $\Sigma_*^N(\theta, l)$ be the set of efficient orders on N. Let the ratio of the waiting cost to the processing time of agent i, given by θ_i/l_i, be agent i's **urgency index**. It is well known that $\sigma \in \Sigma^N_*(\theta, l)$ if and only if for each pair $i, j \in N$, such that $\theta_i/l_i > \theta_j/l_j$ we have $\sigma(i) < \sigma(j)$ (Smith [27]). Given $S \subset N$, the restriction of an order $\sigma \in \Sigma^N$ to $S \subset N$ is denoted by $\sigma_S \in \Sigma^S$. For $(\theta, l) \in \mathcal{S}^N$ and $S \subset N$, let $(\theta_S, l_S) \equiv (\theta_i, l_i)_{i \in S}$. Let $\Sigma^S_*(\theta_S, l_S)$ be the set of all efficient orders on $S \subset N$.

An **allocation rule** ψ associates to each $(\theta, l) \in \mathcal{S}^N$ a non-empty subset of allocations in $\boldsymbol{X}^N(\theta, l)$.

4 Coalitional games

In order to identify solutions to sequencing problems that satisfy fairness properties, the tools of cooperative game theory can be invoked. This requires that sequencing problems be modeled as coalitional games. We propose here to apply to these games the solution concepts of Shapley value (for fairness) and core (for stability). Let $\mathcal{G}^N$ be the set of all coalitional games with agent set N. First, to each $(\theta, l) \in \mathcal{S}^N$ should be associated a cost sharing game $c(\theta, l) \in \mathcal{G}^N$. There are multiple ways of defining such a game. The literature discusses two extreme scenarios based on whether the coalitional members are served first or last. For each $S \subseteq N$, in the **optimistic scenario**, the cost of coalition S, denoted by $\boldsymbol{c^{Opt}(S)(\theta, l)}$, is the minimum waiting cost of S when its members are served first. This scenario has been studied by Maniquet [18] for queueing problems [4]. Formally, for each $(\theta, l) \in \mathcal{S}^N$ and each $S \subseteq N$,

$$c^{Opt}(S)(\theta, l) = \sum_{i \in S} \theta_i \Big(\sum_{j \in P_i(\sigma_S)} l_j \Big) = \sum_{i \in S} \theta_i L_i(\sigma_S). \tag{1}$$

where $c^{Opt}(\theta, l) \in \mathcal{G}^N$ and $\sigma_S \in \Sigma^S_*(\theta_S, l_S)$.

For each $S \subseteq N$, according to the **pessimistic scenario**, the cost of a coalition S, de-

[4] In queueing, the job processing time of each agent is identical and normalized to 1. The vector of job processing times is $l = (1, 1, \ldots, 1)$

noted by $\boldsymbol{c^{Pes}(S)(\theta, l)}$, as the minimum waiting cost of S when its members are served last. Formally, for each $(\theta, l) \in \mathcal{S}^N$ and each $S \subseteq N$,

$$c^{Pes}(S)(\theta, l) = \sum_{i \in S} \theta_i (\sum_{j \in N \setminus S} l_j + \sum_{j \in P_i(\sigma_S)} l_j) = c^{Opt}(S)(\theta, l) + \sum_{i \in S} \theta_i (\sum_{j \in N \setminus S} l_j) \qquad (2)$$

where $c^{Pes}(\theta, l) \in \mathcal{G}^N$ and $\sigma_S \in \Sigma_*^S(\theta_S, l_S)$.

4.1 Generalized sequencing games

The optimistic and pessimistic scenarios are somewhat extreme ways to compute the cost of serving the members of a coalition. We thus propose a more general way of associating a sequencing problem to a sequencing game. It includes the optimistic and pessimistic scenarios as special cases. For each $S \subseteq N$, let $|S| = s$ where $s = \{1, 2, \ldots, n\}$. Define $\mathcal{I} = \{(s, k) \mid 0 < k \leq s \leq |N|\}$. Given $(s, k) \in \mathcal{I}$, the variable s specifies the size of coalition S and k specifies the position of agent $i \in S$ when its members are ordered efficiently. A **weighting function** $a : \mathcal{I} \to \mathbb{R}$ associates a real number to each pair $(s, k) \in \mathcal{I}$. We now associate with a sequencing problem $(\theta, l) \in \mathcal{S}^N$ and a weighting function a, a **generalized sequencing game** $c^a(\theta, l) \in \mathcal{G}^N$. For each $(\theta, l) \in \mathcal{S}^N$ and each $S \subseteq N$, $c^a(S)(\theta, l)$ is the minimum aggregate "weighted" waiting cost of its members. Then, the cost of serving the members of S will depend on two things. First is how a coalition S is treated in comparison to its complement $N \setminus S$. This determines which positions in the order are allotted to the members of S. Second, how positions are treated within S. We allow for the possibility of weighing positions differently instead of assigning equal weights to all positions. Generalized queueing games have been studied by Kar et al. [15]. We extend the notion to define generalized sequencing games. Let the weight function satisfy the following. For each $i \in N$,

1. $a(|N|, \sigma(i)) = \sum_{j \in P_i(\sigma)} l_j$ where $\sigma \in \Sigma_*^N(\theta, l)$ and,

2. $a(1,1) \in [0, \sum_{j \neq i} l_j]$.

Then for each $(\theta, l) \in \mathcal{S}^N$ and each $\phi \neq S \subseteq N$ let,

$$c^a(S)(\theta, l) = \sum_{i \in S} a(|S|, \sigma_S(i))\theta_i \tag{3}$$

where $\sigma_S \in \Sigma_*^S(\theta_S, l_S)$.

If for each $S \subseteq N$, $a(|S|, \sigma_S(i)) = \sum_{j \in P_i(\sigma_S)} l_j$, we get the optimistic scenario according to which S is served before $N \backslash S$. If $a(|S|, \sigma_S(i)) = \sum_{k \in N \backslash S} l_k + \sum_{j \in P_i(\sigma_S)} l_j$, we get the pessimistic scenario according to which S is served after $N \backslash S$. The restriction $a(1,1) \in [0, \sum_{j \neq i} l_j]$ is a natural one. The left endpoint of the interval $[0, \sum_{j \neq i} l_j]$ corresponds to the best-case scenario when an agent is served first in the queue. The right endpoint corresponds to the worst-case when he is served last (all other agents being served before him).

Let $\delta(a) = \frac{a(1,1)}{\sum_{j \neq i} l_j} \in [0,1]$. The **"weighted optimistic pessimistic"** (WOP) game relative to $\delta(a)$, denoted by $c^{\delta(a)} \in \mathcal{G}^N$, is the generalized sequencing game associated with the weighting function a defined as follows: for each $(\theta, l) \in \mathcal{S}^N$ and each $S \subseteq N$ let,

$$c^{\delta(a)}(S)(\theta, l) = \sum_{i \in S} \left[\delta(a) \sum_{j \in N \backslash S} l_j + \sum_{j \in P_i(\sigma_S)} l_j \right] \theta_i. \tag{4}$$

where $\sigma_S \in \Sigma_*^S(\theta_S, l_S)$.

For each $S \subseteq N$, $a(|S|, \sigma_S(i))$ has two components. The component $\delta(a) \sum_{j \in N \backslash S} l_j$ is the additional waiting time imposed on i by the members of $N \backslash S$. The component $\sum_{j \in P_i(\sigma_S)} l_j$ is the additional waiting time incurred by an agent $i \in S$ due to members of S.

5 The Shapley value

The Shapley value is a solution concept from cooperative game theory that assigns a unique distribution of a total payoff among a group of agents. For a game $c \in \mathcal{G}^N$, the

burden imposed by agent $i \in N$ on each $S \subseteq N\backslash\{i\}$ is defined by $c(S \cup \{i\}) - c(S)$. For each $(\theta, l) \in \mathcal{S}^N$ and each $i \in N$, the Shapley value of i in c is the expected value of the burden imposed by i on each coalition when all orders are equally likely (Shapley [17]), namely,

$$Sh_i(c) = \sum_{S \subseteq N \backslash \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!}[c(S \cup \{i\}) - c(S)]. \tag{5}$$

The following lemma gives an explicit expression the Shapley value of $c^{\delta(a)} \in \mathcal{G}^N$.

Proposition 1. Let $(\theta, l) \in \mathcal{S}^N$ and $\sigma \in \Sigma_*^N(\theta, l)$. For each $i \in N$, the Shapley payoff of i in $c^{\delta(a)} \in \mathcal{G}^N$ is

$$\begin{aligned} Sh_i(c^{\delta(a)}) =&(1 - \delta(a))(\sum_{j \in P_i(\sigma)} \theta_i l_j/2 + \sum_{j \in F_i(\sigma)} \theta_j l_i/2) \\ &+ \delta(a)(\theta_i \sum_{j \neq i} l_j - \sum_{j \in P_i(\sigma)} \theta_j l_i/2 - \sum_{j \in F_i(\sigma)} \theta_i l_j/2). \end{aligned} \tag{6}$$

For a set of agents N, we first define a game $u_T \in \mathcal{G}^N$ on a coalition $T \subseteq N$ before proving the lemma.

Definition 1. Let $T \subseteq N$. The **unanimity** game on T is the game (N, u_T) defined by setting for each $T \subseteq N$, $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise [Shapley[26]].

Remark 1. A coalitional game $c \in \mathcal{G}^N$ can be uniquely expressed as a linear combination of unanimity games, i.e., for each $c \in \mathcal{G}^N$ and each $S \subseteq N$ there exists a unique vector of real numbers $(\Delta_S)_{S \subseteq N}$, such that, $c = \sum_{S \subseteq N} \Delta_S u_S$. For each $S \subseteq N$, the **dividend** of S in c, the number Δ_S, is defined as follows: if $|S| = 1$, $\Delta_S = c(S)$ and if $|S| > 1, \Delta_S = c(S) - \sum_{\substack{T \subseteq S \\ T \neq S}} \Delta_T$ [Shapley[26]].

Lemma 2. Let $c^{\delta(a)} \in \mathcal{G}^N$. For each $S \subseteq N$, the dividend Δ_S is

$$\Delta_S = \begin{cases} \delta(a)\theta_i(\sum_{j\neq i} l_j) & \text{if } |S| = 1 \\ (1-\delta(a))\min_{i,j\in S}\{\theta_i/l_i, \theta_j/l_j\}l_i l_j - \delta(a)\max_{i,j\in S}\{\theta_i/l_i, \theta_j/l_j\}l_i l_j & \text{if } |S| = 2 \\ 0 & \text{if } |S| \geq 3 \end{cases} \tag{7}$$

Proof.

- Case 1: $|S| = 1$. Let $S = \{i\}$. Then $\Delta_{\{i\}} = c^{\delta(a)}(i) = \delta\theta_i(\sum_{j\neq i} l_j)$.
- Case 2: $|S| = 2$. Let $S = \{i, j\}$ and without loss of generality suppose that $\theta_i/l_i \geq \theta_j/l_j$. We have $\Delta_{\{i,j\}} = c^{\delta(a)}\{i,j\} - \Delta_{\{i\}} - \Delta_{\{j\}} = (1-\delta)\theta_j l_i - \delta\theta_i l_j = (1-\delta)\min\{\theta_i/l_i, \theta_j/l_j\}l_i l_j - \delta\max_{i,j\in S}\{\theta_i/l_i, \theta_j/l_j\}l_i l_j$.
- Case 3: $|S| = 3$. Let $S = \{i, j, k\}$ and without loss of generality suppose that $\theta_i/l_i \geq \theta_j/l_j \geq \theta_k/l_k$. Then $\Delta_{\{i,j,k\}} = c^{\delta(a)}\{i,j,k\} - \Delta_{\{i,j\}} - \Delta_{\{j,k\}} - \Delta_{\{i,k\}} - \Delta_{\{i\}} - \Delta_{\{j\}} - \Delta_{\{k\}} = \delta\theta_i(\sum_{m\in N\setminus\{i,j,k\}} l_m) + \delta\theta_j(\sum_{m\in N\setminus\{i,j,k\}} l_m) + \theta_j l_i + \delta\theta_k(\sum_{m\in N\setminus\{i,j,k\}} l_m) + \theta_k(l_i + l_j) - [(1-\delta)$
  $\theta_j l_i - \delta\theta_i l_j] - [(1-\delta)\theta_k l_j - \delta\theta_j l_k] - [(1-\delta)\theta_k l_i - \delta\theta_i l_k] - \delta\theta_i(\sum_{m\neq i} l_m) - \delta\theta_j(\sum_{m\neq j} l_m) - \delta\theta_k(\sum_{m\neq k} l_m) = 0$ (all the terms cancel out).

We proceed by induction on the size of the coalition S. Let us assume $\Delta_{S'} = 0$ for all S' such that $3 \leq |S'| \leq |S|$. Without loss of generality, let $S = \{1, 2, \ldots k\}$ be such that

$\theta_1/l_1 \geq \theta_2/l_2 \geq \ldots \geq \theta_k/l_k$. Using the induction hypothesis,

$$\begin{aligned}
\Delta_S =& c^{\delta(a)}(S) - \sum_{T\subset S;|T|=2} \Delta_T - \sum_{T\subset S;|T|=1} \Delta_T \\
=& \sum_{i\in S} \theta_i(\delta(\sum_{j\in N\setminus S} l_j) + \sum_{j\in P_i(\sigma_S)} l_j) - [(1-\delta)\sum_{i\in S}\theta_i(\sum_{j\in P_i(\sigma_S)} l_j) - \delta\sum_{i\in S}\theta_i(\sum_{j\in F_i(\sigma_S)} l_j)] - \delta\sum_{i\in S}\theta_i(\sum_{j\neq i} l_j) \\
=& \delta\sum_{i\in S}\theta_i(\sum_{j\in N\setminus S} l_j) + \delta\sum_{i\in S}\theta_i(\sum_{j\in S\setminus\{i\}} l_j) - \delta\sum_{i\in S}\theta_i(\sum_{j\neq i} l_j) \\
=& 0
\end{aligned}$$

This proves the claim and we can now show Lemma 2.

Proof. The Shapley value is obtained by distributing the dividend of each coalition among its members. The Shapley payoff of agent $i \in N$ in the game c is given by $Sh_i(c) = \sum_{\substack{S\subseteq N \\ i\in S}} \frac{\Delta_S}{|S|}$ [Harsanyi [14]. By substituting Eq. (7) in this expression, we obtain

$$\begin{aligned}
Sh_i(c^{\delta(a)}) =& \delta\theta_i(\sum_{j\neq i} l_j) + \frac{(1-\delta)}{2}\sum_{j\in N\setminus\{i\}} \min\{\theta_i/l_i, \theta_j/l_j\} l_i l_j - \frac{\delta}{2}\sum_{j\in N\setminus\{i\}} \max\{\theta_i/l_i, \theta_j/l_j\} l_i l_j \\
=& \delta\theta_i\sum_{j\neq i} l_j + \frac{(1-\delta)}{2}(\theta_i\sum_{j\in P_i(\sigma)} l_j + l_i\sum_{j\in F_i(\sigma)}\theta_j) - \frac{\delta}{2}(\theta_i\sum_{j\in F_i(\sigma)} l_j + l_i\sum_{j\in P_i(\sigma)}\theta_j) \\
=& (1-\delta)(\sum_{j\in P_i(\sigma)}\theta_i l_j/2 + \sum_{j\in F_i(\sigma)}\theta_j l_i/2) + \delta(\theta_i\sum_{j\neq i} l_j - \sum_{j\in P_i(\sigma)}\theta_j l_i/2 - \sum_{j\in F_i(\sigma)}\theta_i l_j/2).
\end{aligned} \tag{8}$$

The desired conclusion. □

Remark 2. For each $(\theta, l) \in \mathcal{S}^N$ and each $i \in N$, if i's disutility is his Shapley payoff from the associated cost sharing game $c^{\delta(a)} \in \mathcal{G}^N$ (in Lemma 2) then his consumption of money is

$$\tau_i = (1-\delta)(\sum_{j\in P_i(\sigma)}\theta_i l_j/2 - \sum_{j\in F_i(\sigma)}\theta_j l_i/2) + \delta(\sum_{j\in P_i(\sigma)}\theta_j l_i/2 - \sum_{j\in F_i(\sigma)}\theta_i l_j/2).$$

where $\sigma \in \Sigma_*^N(\theta, l)$ is an efficient order on N.

6 Disutility upper bound (DUB)

This section imposes upper bounds on agents' disutilities from waiting. Such upper bounds act as safety nets and shield agents from arbitrarily high disutilities due to unpredictability of waiting times. The *"disutility upper bound relative to a benchmark"* has been formally defined below. We also define the Shapley rule and find a necessary and sufficient condition on the upper bounds for the Shapley rule to satisfy this property. Its counterpart, namely the "generalized welfare lower bound relative to a benchmark" guarantees every agent a minimum level of utility and has been studied by Banerjee et. al [1].

Let $O : \mathbb{R}^n_{++} \rightarrow \mathbb{R}^n_{++}$. Given a sequencing problem $(\theta, l) \in \mathcal{S}^N$, let *agent i's upper bound* be given by $\theta_i O_i(l)$. This expression encompasses all upper bounds of multiplicative form since $O_i(l)$ can be any function of the vector of job processing times l. Specifically, for identical costs bound (ICB), i's upper bound is $\theta_i(n-1)l_i/2$ where $O_i(l) = (n-1)l_i/2$. For expected costs bound (ECB), i's upper bound is $\theta_i \sum_{j \neq i} l_j/2$ where $O_i(l) = \sum_{j \neq i} l_j/2$. The functional form of $O_i(l)$ varies depending on the particular bound being considered. For ease of verbal expression, we will call $O_i(l)$ to be "*agent i's benchmark*" and $O(l) := (O_1(l), \ldots, O_n(l)) \in \mathbb{R}^n$ to be the vector of such benchmarks.

Definition 2. *(Banerjee et.al [1])* An allocation rule ψ satisfies the *disutility upper bound relative to benchmark O* if for each $(\theta, l) \in \mathcal{S}^N$, each $(\sigma, \tau) \in X(\theta, l)$, and each $i \in N$ we have:

$$\pi_i(\sigma, \tau_i) \leq \theta_i O_i(l)$$

The definition implies that, in the absence of any consumption of money, agent i is assured that his disutility will not exceed his upper bound $\theta_i O_i(l)$.

Let us associate to each sequencing problem a weighted optimisic pessimistic cost game. The "Shapley rule" maps each sequencing problem to a set of allocations such that disutility of every agent at that allocation is equal to his Shapley payoff from the associated WOP cost game. We show that the Shapley rule satisfies disutility upper bound relative to benchmark O if and only if each agent's benchmark is greater than or equal to half the aggregate job processing times of all other agents (excluding him). The latter can be interpreted as follows: each agent's upper bound is at least as much as his expected cost of waiting when all orders are equally likely, i.e, for each $i \in N$, we have $\theta_i O_i(l) \geq \theta_i \sum_{j \neq i} l_j / 2$ where the right hand side is i's expected waiting cost when all orders are equally likely.

We formally define the Shapley rule and state the result.

Let $\psi^{Sh} : \mathcal{S}^N \to\to \Sigma \times \mathbb{R}^N$. To each $(\theta, l) \in \mathcal{S}^N$ is associated $c^{\delta(a)}(\theta, l) \in \mathcal{G}^N$ where $\delta(a) \in [0, 1]$. Define $\hat{X}(\theta, l) = \{(\sigma, \tau) \in \Sigma \times \mathbb{R}^N \mid \pi_i(\sigma, \tau_i) = Sh_i(c^{\delta(a)}(\theta, l))$ for each $i \in N\}$.

Theorem 1. The following statements are equivalent:

1. For each $(\theta, l) \in \mathcal{S}^N$, each $(\sigma, \tau) \in \hat{X}(\theta, l)$ and each $i \in N$,

$$\pi_i(\sigma, \tau_i) \leq \theta_i O_i(l)$$

   where $\pi_i(\sigma, \tau_i) = Sh_i(c^{\delta(a)}(\theta, l))$.

2. For each $(\theta, l) \in \mathcal{S}^N$ and each $i \in N$, $O_i(l) \geq \sum_{j \neq i} \frac{l_j}{2}$.

Proof.

(A) $1 \Rightarrow 2$: For $(\theta, l) \in \mathcal{S}^N$ and each $i \in N$, $\pi_i(\sigma, \tau_i)$ the disutility of $i \in N$ (corresponding to his Shapley payoff in the game $c^{\delta(a)}$) satisfies the DUB property relative to O

if,

$$
\begin{aligned}
Sh_i(c^{\delta(a)}) \leq \theta_i O_i(l) \Rightarrow & (1-\delta)(\sum_{j\in P_i(\sigma)} \theta_i l_j/2 + \sum_{j\in F_i(\sigma)} \theta_j l_i/2) + \\
& \qquad \delta(\theta_i \sum_{j\neq i} l_j - \sum_{j\in P_i(\sigma)} \theta_j l_i/2 - \sum_{j\in F_i(\sigma)} \theta_i l_j/2) \leq \theta_i O_i(l) \\
\Rightarrow & (1-\delta)(\sum_{j\in P_i(\sigma)} \theta_i l_j/2 + \sum_{j\in F_i(\sigma)} \theta_j l_i/2) + \\
& \qquad \delta(\theta_i \sum_{j\neq i} l_j - \sum_{j\in P_i(\sigma)} \theta_j l_i/2 - \sum_{j\in F_i(\sigma)} \theta_i l_j/2) - \theta_i O_i(l) \\
\leq & 0.
\end{aligned}
$$

For each $i \in N$, let $O_i(l) = \sum_{j\neq i} l_j/2 + \epsilon_i$. Adding and subtracting $\delta\theta_i(\sum_{j\neq i} l_j)/2$ we get,

$$
\frac{(1-\delta)}{2}(\sum_{j\in F_i(\sigma)} \theta_j l_i - \sum_{j\in F_i(\sigma)} \theta_i l_j) + \frac{\delta}{2}(\sum_{j\in P_i(\sigma)} \theta_i l_j - \sum_{j\in P_i(\sigma)} \theta_j l_i) - \theta_i \epsilon_i \leq 0.
$$

Since $\sigma \in \Sigma_*^N(\theta, l)$ is an efficient ordering on N, for each $i \in N$, if $j \in F_i(\sigma)$ we have $\theta_i/l_i \geq \theta_j/l_j$. This implies that the term $\sum_{j\in F_i(\sigma)} \theta_j l_i - \sum_{j\in F_i(\sigma)} \theta_i l_j \leq 0$. Further, for each $i \in N$, if $j \in P_i(\sigma)$ we have $\theta_j/l_j \geq \theta_i/l_i$. This implies that the second term $\sum_{j\in P_i(\sigma)} \theta_i l_j - \sum_{j\in P_i(\sigma)} \theta_j l_i \leq 0$. With the entire expression being non-positive and $\theta_i \in \mathbb{R}_{++}$ we have $\epsilon_i \geq 0$. This proves the necessity of $O_i(l) \geq \sum_{j\neq i} \frac{l_j}{2}$.

(B) $2 \Rightarrow 1$: For each $i \in N$, $O_i(l) \geq \sum_{j\neq i} l_j/2$. The disutility of i is given by his Shapley payoff $Sh_i(c^{\delta(a)})$. For each such agent, let $O_i(l) = \sum_{j\neq i} l_j/2$ and consider the expression,

$$
\begin{aligned}
Sh_i(c^{\delta(a)}) - \theta_i O_i(l) =&(1-\delta)(\sum_{j\in P_i(\sigma)} \theta_i l_j/2 + \sum_{j\in F_i(\sigma)} \theta_j l_i/2)+\\
&\delta(\theta_i \sum_{j\neq i} l_j - \sum_{j\in P_i(\sigma)} \theta_j l_i/2 - \sum_{j\in F_i(\sigma)} \theta_i l_j/2) - \theta_i \sum_{j\neq i} l_j/2\\
=&(1-\delta)(\sum_{j\in P_i(\sigma)} \theta_i l_j/2 + \sum_{j\in F_i(\sigma)} \theta_j l_i/2)+\\
&\delta(\sum_{j\in P_i(\sigma)} \theta_i l_j + \sum_{j\in F_i(\sigma)} \theta_i l_j/2 - \sum_{j\in P_i(\sigma)} \theta_j l_i/2) - \theta_i \sum_{j\neq i} l_j/2\\
&\text{(By adding and subtracting the term } \delta\theta_i \sum_{j\neq i} l_j/2)\\
=&\frac{1-\delta}{2}(\sum_{j\in F_i(\sigma)} \theta_j l_i - \sum_{j\in F_i(\sigma)} \theta_i l_j) + \frac{\delta}{2}(\sum_{j\in P_i(\sigma)} \theta_i l_j - \sum_{j\in P_i(\sigma)} \theta_j l_i).
\end{aligned}
$$

Since $\sigma \in \Sigma_*^N(\theta, l)$ is an efficient ordering on N, for each $i \in N$, if $j \in F_i(\sigma)$ we have $\theta_i/l_i \geq \theta_j/l_j$. This implies that the first term $\sum_{j\in F_i(\sigma)} \theta_j l_i - \sum_{j\in F_i(\sigma)} \theta_i l_j \leq 0$. Further, for each $i \in N$, if $j \in P_i(\sigma)$ we have $\theta_j/l_j \geq \theta_i/l_i$. This implies that the second term $\sum_{j\in P_i(\sigma)} \theta_i l_j - \sum_{j\in P_i(\sigma)} \theta_j l_i \leq 0$. Thus, $Sh_i(c^{\delta(a)}) - \theta_i O_i(l) = \frac{1-\delta}{2}(\sum_{j\in F_i(\sigma)} \theta_j l_i - \sum_{j\in F_i(\sigma)} \theta_i l_j) + \frac{\delta}{2}(\sum_{j\in P_i(\sigma)} \theta_i l_j - \sum_{j\in P_i(\sigma)} \theta_j l_i) \leq 0$. For each $i \in N$, we have $Sh_i(c^{\delta(a)}) \leq \theta_i O_i(l) = \theta_i \sum_{j\neq i} l_j/2 \leq \theta_i \sum_{j\neq i} l_j/2 + \epsilon_i$ where $\epsilon_i \geq 0$. Hence, proved.

Remark 3. The lower bound $\sum_{j\neq i} \frac{l_j}{2}$ has a very natural interpretation. It is the average waiting time of each agent when all orders are equally likely. For each $\sigma \in \Sigma(\theta, l)$ and each $i \in N$, define $\sigma_i^c = n + 1 - \sigma_i$. Let $\sigma^c \in \Sigma(\theta, l)$ be the complement order of an order $\sigma \in \Sigma(\theta, l)$. An agent $j \neq i$ precedes agent i in an order $\sigma \in \Sigma(\theta, l)$ if and only if he does not precede i in its complement order $\sigma^c \in \Sigma(\theta, l)$. When all orders are equally likely, the processing time of each $j \neq i$ appears in i's waiting time with probability $\frac{1}{2}$.

7 The core

This section explores the cost sharing concepts of core and anticore in the optimistic scenario. Let G^N be the set of all coalitional games with player set N. Let $c \in \mathcal{G}^N$ be a cost sharing game. An **allocation** for c is vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ where for each $i \in N$, x_i is the cost share of agent i. For each coalition $S \subseteq N$, $x(S)$ is the sum of individual cost shares of the members of S. An allocation $x \in \mathbb{R}^N$ is **efficient** for $c \in \mathcal{G}^N$ if $x(N) = c(N)$. The set of efficient allocations in c is denoted by $X(c)$. The core of c, denoted by $Core(c)$, is the set $\{x \in X(c) \mid x(S) \leq c(S) \text{ for each } S \subseteq N\}$.

The next theorem provides a necessary and sufficient condition on δ for the Shapley payoff vector to be a core stable allocation in a WOP sequencing game.

Theorem 2. For $c^{\delta(a)} \in \mathcal{G}^N$ relative to $\delta(a)$ we have $\{Sh_i(c^{\delta(a)})\}_{i\in N} \in Core(c^{\delta(a)})$ if and only if $\delta(a) \geq \frac{1}{2}$.

Proof.

Case A: Given $\delta(a) \geq \frac{1}{2}$, we prove that the Shapley payoff vector satisfies all the core constraints.

1. For each $i \in N$, $\sum_{i\in N} Sh_i(c^{\delta(a)}) = c^{\delta(a)}(N)$.

Proof: For each $i \in N$, using equation (6) in proposition (1) we have,

$$
\begin{aligned}
\sum_{i\in N} Sh_i(c^{\delta(a)}) &= (1-\delta)\sum_{i\in N}(\sum_{j\in P_i(\sigma)}\theta_i l_j/2 + \sum_{j\in F_i(\sigma)}\theta_j l_i/2)+ \\
&\qquad \delta\sum_{i\in N}(\theta_i\sum_{j\neq i} l_j - \sum_{j\in P_i(\sigma)}\theta_j l_i/2 - \sum_{j\in F_i(\sigma)}\theta_i l_j/2) \\
&= (1-\delta)\sum_{i\in N}(\theta_i \sum_{j\in P_i(\sigma)} l_j)+ \\
&\qquad \delta\sum_{i\in N}[\theta_i(\sum_{j\in P_i(\sigma)} l_j + \sum_{j\in F_i(\sigma)} l_j) - \sum_{j\in P_i(\sigma)}\theta_j l_i/2 - \sum_{j\in F(\sigma)}\theta_i l_j/2] \\
&= (1-\delta)\sum_{i\in N}(\theta_i \sum_{j\in P_i(\sigma)} l_j)+ \\
&\qquad \delta\sum_{i\in N}[\theta_i(\sum_{j\in P_i(\sigma)} l_j) + \sum_{j\in F_i(\sigma)}\theta_i l_j/2 - \sum_{j\in P_i(\sigma)}\theta_j l_i/2] \\
&= (1-\delta)\sum_{i\in N}(\theta_i \sum_{j\in P_i(\sigma)} l_j)+ \\
&\qquad \delta\sum_{i\in N}\theta_i(\sum_{j\in P_i(\sigma)} l_j) - \delta\sum_{i\in N}(\sum_{j\in F_i(\sigma)}\theta_i l_j)/2 + \delta\sum_{i\in N}(\sum_{j\in P_i(\sigma)}\theta_j l_i)/2 \\
&= (1-\delta)\sum_{i\in N}(\theta_i \sum_{j\in P_i(\sigma)} l_j) + \delta\sum_{i\in N}\theta_i(\sum_{j\in P_i(\sigma)} l_j) \\
&= c^{\delta(a)}(N)
\end{aligned}
\tag{9}
$$

where $\sigma \in \Sigma$, as claimed. □

2. For each $i \in N$, $Sh_i(c^{\delta(a)}) \leq c^{\delta(a)}(\{i\})$.

Proof. For $c^{\delta(a)} \in \mathcal{G}^N$, consider the following for each $i \in N$,

$$Sh_i(c^{\delta(a)}) - c^{\delta(a)}(\{i\}) = (1-\delta)(\sum_{j\in P_i(\sigma)} \theta_i l_j/2 + \sum_{j\in F_i(\sigma)} \theta_j l_i/2) + \delta(\theta_i \sum_{j\neq i} l_j - \sum_{j\in P_i(\sigma)} \theta_j l_i/2 - \sum_{j\in F_i(\sigma)} \theta_i l_j/2) - \delta(\sum_{j\neq i} \theta_i l_j)$$

$$= (1-\delta)(\sum_{j\in P_i(\sigma)} \theta_i l_j/2 + \sum_{j\in F_i(\sigma)} \theta_j l_i/2) - \delta(\sum_{j\in P_i(\sigma)} \theta_j l_i/2 + \sum_{j\in F_i(\sigma)} \theta_i l_j/2)$$

By efficiency of $\sigma \in \Sigma_*^N(\theta, l)$, for each $i \in N$ we have $\sum_{j\in P_i(\sigma)} \theta_j l_i \geq \sum_{j\in P_i(\sigma)} \theta_i l_j$ and $\sum_{j\in F_i(\sigma)} \theta_i l_j \geq \sum_{j\in F_i(\sigma)} \theta_j l_i$. Combining this with the fact that $\delta \geq \frac{1}{2}$, we get for each $i \in N$, $Sh_i(c^{\delta(a)}) - c^{\delta(a)}(\{i\}) \leq 0$.

3. For each $S \subset N$, $\sum_{i\in S} Sh_i(c^{\delta(a)}) \leq c^{\delta(a)}(S)$.

Proof: For $c^{\delta(a)} \in \mathcal{G}^N$, using proposition 1 we can write,

$$\sum_{i\in S} Sh_i(c^{\delta(a)}) = (1-\delta)\sum_{i\in S}\left(\sum_{j\in P_i(\sigma)} \theta_i l_j/2 + \sum_{j\in F_i(\sigma)} \theta_j l_i/2\right) + \delta \sum_{i\in S}\left(\theta_i \sum_{j\neq i} l_j - \sum_{j\in P_i(\sigma)} \theta_j l_i/2 - \sum_{j\in F_i(\sigma)} \theta_i l_j/2\right)$$

$$= (1-\delta)\sum_{i\in S}\left(\sum_{\substack{j\in P_i(\sigma)\\ j\in S}} \theta_i l_j/2 + \sum_{\substack{j\in P_i(\sigma)\\ j\notin S}} \theta_i l_j/2 + \sum_{\substack{j\in F_i(\sigma)\\ j\in S}} \theta_j l_i/2 + \sum_{\substack{j\in F_i(\sigma)\\ j\notin S}} \theta_j l_i/2\right) + \delta \sum_{i\in S}\left(\theta_i \sum_{j\neq i} l_j\right) -$$

$$\delta \sum_{i\in S}\left(\sum_{\substack{j\in P_i(\sigma)\\ j\in S}} \theta_j l_i/2 + \sum_{\substack{j\in P_i(\sigma)\\ j\notin S}} \theta_j l_i/2 + \sum_{\substack{j\in F_i(\sigma)\\ j\in S}} \theta_i l_j/2 + \sum_{\substack{j\in F_i(\sigma)\\ j\notin S}} \theta_i l_j/2\right)$$

Note that $\sum_{i\in S} \theta_i \left(\sum_{\substack{j\in P_i(\sigma)\\ j\in S}} l_j\right) = \sum_{i\in S} l_i \left(\sum_{\substack{j\in F_i(\sigma)\\ j\in S}} \theta_j\right)$. Similarly, $\sum_{i\in S} l_i \left(\sum_{\substack{j\in P_i(\sigma)\\ j\in S}} \theta_j\right) = \sum_{i\in S} \theta_i \left(\sum_{\substack{j\in F_i(\sigma)\\ j\in S}} l_j\right)$.
We can write the following,

$$
\sum_{i\in S} Sh_i(c^{\delta(a)}) = \sum_{i\in S}\theta_i\Bigg[(1-\delta)\sum_{\substack{j\in P_i(\sigma)\\ j\in S}} l_j - \delta\sum_{\substack{j\in F_i(\sigma)\\ j\in S}} l_j + \delta\sum_{j\neq i} l_j\Bigg] +
$$

$$
\sum_{i\in S}\Bigg[(1-\delta)\Big(\sum_{\substack{j\in P_i(\sigma)\\ j\notin S}}\theta_i l_j/2 + \sum_{\substack{j\in F_i(\sigma)\\ j\notin S}}\theta_j l_i/2\Big) - \delta\Big(\sum_{\substack{j\in P_i(\sigma)\\ j\notin S}}\theta_j l_i/2 + \sum_{\substack{j\in F_i(\sigma)\\ j\notin S}}\theta_i l_j/2\Big)\Bigg] \tag{10}
$$

.

Claim 1. For $c^{\delta(a)} \in \mathcal{G}^N$ and each $S \subseteq N$ we have the following:

$$
\sum_{i\in S}\theta_i\Bigg[(1-\delta)\sum_{\substack{j\in P_i(\sigma)\\ j\in S}} l_j - \delta\sum_{\substack{j\in F_i(\sigma)\\ j\in S}} l_j + \delta\sum_{j\neq i} l_j\Bigg] - c^{\delta(a)}(S) = 0 \tag{11}
$$

where $\sigma \in \Sigma_*^N(\theta, l)$.

Proof. Consider the expression on the left hand side of equation 11. Using equations 3 and 4 we have, $\sum_{i\in S}\theta_i\Big[(1-\delta)\sum_{\substack{j\in P_i(\sigma)\\ j\in S}} l_j - \delta\sum_{\substack{j\in F_i(\sigma)\\ j\in S}} l_j\Big] + \delta\sum_{i\in S}\theta_i\Big(\sum_{j\in N\setminus S} l_j + \sum_{\substack{j\in P_i(\sigma)\\ j\in S}} l_j + \sum_{\substack{j\in F_i(\sigma)\\ j\in S}} l_j\Big) - \sum_{i\in S}\theta_i\Big[\delta(\sum_{j\in N\setminus S} l_j) + \sum_{j\in P_i(\sigma_S)} l_j\Big] = 0$. As claimed. □

Using claim (1) in equation (10) we have the following:

$$
\sum_{i\in S} Sh_i(c^{\delta(a)}) - c^{\delta(a)}(S) = \sum_{i\in S}\Bigg[(1-\delta)\Big(\sum_{\substack{j\in P_i(\sigma)\\ j\notin S}}\theta_i l_j/2 + \sum_{\substack{j\in F_i(\sigma)\\ j\notin S}}\theta_j l_i/2\Big) - \delta\Big(\sum_{\substack{j\in P_i(\sigma)\\ j\notin S}}\theta_j l_i/2 + \sum_{\substack{j\in F_i(\sigma)\\ j\notin S}}\theta_i l_j/2\Big)\Bigg] \tag{12}
$$

By efficiency of $\sigma \in \Sigma_*^N(\theta, l)$, for each $i \in S$, $\sum_{\substack{j\in P_i(\sigma)\\ j\notin S}}\theta_j l_i \geq \sum_{\substack{j\in P_i(\sigma)\\ j\notin S}}\theta_i l_j$ and $\sum_{\substack{j\in F_i(\sigma)\\ j\notin S}}\theta_i l_j \geq \sum_{\substack{j\in F_i(\sigma)\\ j\notin S}}\theta_j l_i$.

Combining this with the fact that $\delta(a) \geq \frac{1}{2}$, we get for each $S \subset N$, $\sum_{i\in S} Sh_i(c^{\delta(a)}) - c^{\delta(a)}(S) \leq 0$.

This proves sufficiency.

Case B: For $c^{\delta(a)} \in \mathcal{G}^N$, $\{Sh_i(c^{\delta(a)})\}_{i\in N} \in Core(c^{\delta(a)}) \Rightarrow \delta(a) \geq \frac{1}{2}$.

Proof. For $c^{\delta(a)} \in \mathcal{G}^N$, by definition of the core, $\sum_{i\in N} Sh_i(c^{\delta(a)}) = c^{\delta(a)}(N)$ and for each $S \subset N$, $\sum_{i\in S} Sh_i(c^{\delta(a)}) \leq c^{\delta(a)}(S)$. Using the expression in equation 12 we can write for each $S \subset N$,

$$\begin{aligned}
&\sum_{i\in S} Sh_i(c^{\delta(a)}) - c^{\delta(a)}(S) \leq 0 \\
&\Rightarrow \sum_{i\in S}\left[(1-\delta)\Bigg(\sum_{\substack{j\in P_i(\sigma)\\ j\notin S}} \theta_i l_j/2 + \sum_{\substack{j\in F_i(\sigma)\\ j\notin S}} \theta_j l_i/2\Bigg) - \delta\Bigg(\sum_{\substack{j\in P_i(\sigma)\\ j\notin S}} \theta_j l_i/2 + \sum_{\substack{j\in F_i(\sigma)\\ j\notin S}} \theta_i l_j/2\Bigg)\right] \leq 0
\end{aligned}$$

Let $\delta(a) = \frac{1}{2} + \epsilon$. Rewriting the above expression,

$$\begin{aligned}
&\Rightarrow \sum_{i\in S}\left[(\frac{1}{2}-\epsilon)\Bigg(\sum_{\substack{j\in P_i(\sigma)\\ j\notin S}} \theta_i l_j/2 + \sum_{\substack{j\in F_i(\sigma)\\ j\notin S}} \theta_j l_i/2\Bigg) - (\frac{1}{2}+\epsilon)\Bigg(\sum_{\substack{j\in P_i(\sigma)\\ j\notin S}} \theta_j l_i/2 + \sum_{\substack{j\in F_i(\sigma)\\ j\notin S}} \theta_i l_j/2\Bigg)\right] \leq 0 \\
&\Rightarrow \sum_{i\in S}\left[\frac{1}{2}\Bigg(\sum_{\substack{j\in P_i(\sigma)\\ j\notin S}} \theta_i l_j/2 + \sum_{\substack{j\in F_i(\sigma)\\ j\notin S}} \theta_j l_i/2\Bigg) - \frac{1}{2}\Bigg(\sum_{\substack{j\in P_i(\sigma)\\ j\notin S}} \theta_j l_i/2 + \sum_{\substack{j\in F_i(\sigma)\\ j\notin S}} \theta_i l_j/2\Bigg)\right] \\
&\qquad - \epsilon\sum_{i\in S}\left[\Bigg(\sum_{\substack{j\in P_i(\sigma)\\ j\notin S}} \theta_i l_j/2 + \sum_{\substack{j\in F_i(\sigma)\\ j\notin S}} \theta_j l_i/2\Bigg) + \Bigg(\sum_{\substack{j\in P_i(\sigma)\\ j\notin S}} \theta_j l_i/2 + \sum_{\substack{j\in F_i(\sigma)\\ j\notin S}} \theta_i l_j/2\Bigg)\right] \leq 0
\end{aligned}$$

By efficiency of $\sigma \in \Sigma_*^N(\theta, l)$, for each $i \in S$ we have $\sum_{\substack{j\in P_i(\sigma)\\ j\notin S}} \theta_j l_i \geq \sum_{\substack{j\in P_i(\sigma)\\ j\notin S}} \theta_i l_j$ and $\sum_{\substack{j\in F_i(\sigma)\\ j\notin S}} \theta_i l_j \geq \sum_{\substack{j\in F_i(\sigma)\\ j\notin S}} \theta_j l_i$.

The first term, $\sum_{i\in S}\left[\frac{1}{2}\Bigg(\sum_{\substack{j\in P_i(\sigma)\\ j\notin S}} \theta_i l_j/2 + \sum_{\substack{j\in F_i(\sigma)\\ j\notin S}} \theta_j l_i/2\Bigg) - \frac{1}{2}\Bigg(\sum_{\substack{j\in P_i(\sigma)\\ j\notin S}} \theta_j l_i/2 + \sum_{\substack{j\in F_i(\sigma)\\ j\notin S}} \theta_i l_j/2\Bigg)\right] \leq 0$.

Since the entire expression is non-positive and for each $S \subset N$, the term, $\sum_{i\in S}\left[\Bigg(\sum_{\substack{j\in P_i(\sigma)\\ j\notin S}} \theta_i l_j/2 + \sum_{\substack{j\in F_i(\sigma)\\ j\notin S}} \theta_j l_i/2\Bigg) + \Bigg(\sum_{\substack{j\in P_i(\sigma)\\ j\notin S}} \theta_j l_i/2 + \sum_{\substack{j\in F_i(\sigma)\\ j\notin S}} \theta_i l_j/2\Bigg)\right] > 0$, we must have $\epsilon \geq 0$.

This proves the necessary condition and shows that $\delta(a) \geq \frac{1}{2}$. □

8 Player specific games

In this section, we define a class of games called ”player specific (PS) games” for which the Shapley value and the prenucleolus of the game coincide (Kar et.al [15]). We first define the PS-property which states that the sum of a player’s contribution (burden) to any coalition S and its complement $N\backslash S \cup \{i\}$ is a player specific constant. For $c \in \mathcal{G}^N$, for each $S \subseteq N\backslash\{i\}$ and each $i \in N$, let $\Delta_i c(S) = c(S \cup \{i\}) - c(S)$ be the burden imposed on S when agent i joins S.

Definition 3. A game $c \in \mathcal{G}^N$ satisfies the PS-property if for each $S \subseteq N\backslash\{i\}$ and each $i \in N$, there exists $\zeta_i \in \mathbb{R}$ such that $\Delta_i c(S) + \Delta_i c(N\backslash S \cup \{i\}) = \zeta_i$.

A game which satisfies the PS-property is called a PS-game and this class of games is denoted by $\mathcal{G}^N_{PS}$.

For $c \in \mathcal{G}^N$, the **excess of a coalition** S at an allocation $x \in \mathbb{R}^n$, denoted by $e(S, x, c)$, is defined as $c(S) - x(S)$. Construct the vector $\gamma(x)$ by arranging the $2^n - 2$ excesses corresponding to proper non-empty subsets of N in a non-decreasing order [5]. If $y, z \in X(c)$ then $y \succ_L z$ means that $\gamma(y)$ is lexicographically larger than $\gamma(z)$. We use $y \succ_L z$ to indicate that either $y \succeq_L z$ or $y = z$.

Definition 4. The **prenucleolus** of game $c \in \mathcal{G}^N$ is the set $PreN(c) = \{x \in X(c) \mid \gamma(x) \succeq_L \gamma(y)$ for each
$y \in X(c)\}$.

We state the following result from Kar et. al [15].

Theorem 3. If $c \in \mathcal{G}^N_{PS}$, then $Sh(c) = PreN(c)$.

Our next result shows that the weighted optimistic pessimistic sequencing game relative to $\delta(a)$ satisfies the PS-property.

Theorem 4. For each a, $c^{\delta(a)} \in \mathcal{G}^N$ is a PS-game.

[5] We ignore the grand coalition since for each $x \in X(c)$, $e(N, x, c) = 0$.

Proof. For $c^{\delta(a)} \in \mathcal{G}^N$, consider the following for each $i \in N$ and each $S \in N \backslash S \cup \{i\}$.

$$\begin{aligned}
&\Delta_i c^{\delta(a)}(S) + \Delta_i c^{\delta(a)}(N \backslash S \cup \{i\}) \\
&= \theta_i \Big(\delta \sum_{j \in N \backslash S \cup \{i\}} l_j + \sum_{j \in P_i(\sigma_{S \cup \{i\}})} l_j \Big) + \sum_{k \in S} \theta_k \Big[\delta \Big(\sum_{j \in N \backslash S \cup \{i\}} l_j - \sum_{j \in N \backslash S} l_j \Big) + \sum_{j \in P_k(\sigma_{S \cup \{i\}})} l_j - \sum_{j \in P_k(\sigma_S)} l_j \Big] \\
&+ \theta_i \Big(\delta \sum_{j \in S} l_j + \sum_{j \in P_i(\sigma_{N \backslash S})} l_j \Big) + \sum_{k \in N \backslash S \cup \{i\}} \theta_k \Big[\delta \Big(\sum_{j \in S} l_j - \sum_{j \in S \cup \{i\}} l_j \Big) + \sum_{j \in P_k(\sigma_{N \backslash S})} l_j - \sum_{j \in P_k(\sigma_{N \backslash S \cup \{i\}})} l_j \Big] \\
&= \theta_i \Big(\delta \sum_{j \in N \backslash \{i\}} l_j + \sum_{j \in P_i(\sigma)} l_j \Big) - \delta l_i \sum_{k \in N \backslash \{i\}} \theta_k + l_i \sum_{k \in F_i(\sigma)} \theta_k
\end{aligned} \tag{13}$$

where $\sigma \in \Sigma_*^N(\theta, l)$ and $\sigma_S \in \Sigma_*^S(\theta_S, l_S)$. The right hand side of equation (13) is independent of S. □

Corollary 1. For $c^{\delta(a)} \in \mathcal{G}^N$, $Sh(c^{\delta(a)}) = PreN(c^{\delta(a)})$.

Theorem 5. For $c^{\delta(a)} \in \mathcal{G}^N$ relative to $\delta(a)$, $Core(c^{\delta(a)}) \neq \phi$ iff $\delta(a) \geq \frac{1}{2}$.

Proof. The "if" part follows directly from theorem (2). To show the "only if" part, let us assume that $\delta(a) < \frac{1}{2}$. Using theorem (2) and corollary (1) we can say that for $c^{\delta(a)} \in \mathcal{G}^N$, $PreN(c^{\delta(a)}) \notin Core(c^{\delta(a)})$. Since it is given that $Core(c^{\delta(a)}) \neq \phi$, the prenucleolus coincides with the nucleolus of the game. This contradicts the fact that if the core is non-empty, then the nucleolus is in the core. □

9 Conclusion

This paper takes a cooperative game theoretic approach to study sequencing problems. We associate to each sequencing problem a cost sharing game and identify a necessary and sufficient condition for allocations to be efficient, fair (in the spirit of Shapley value) and stable (defined by core-stability). There are a few immediate extensions to this paper. One, what happens if we do not assume linearity of costs? Two, what are the characterizing axioms of the Shapley value from the weighted optimistic pessimistic game? Three, what happens if there are multiple machines instead of a single server? Four, can we allow for dynamic arrival of agents instead of a fixed set of agents who arrive at the same time to process their jobs? It will be interesting and worthwhile to explore each of these questions for future research.

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