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Abstract

I derive Sraffa's fundamental equation $r = R(1-w)$ by means of differential equations and optimisation, on which I work three remarks: (i) I analytically provide an alternative formulation of Sraffa's fundamental equation; (ii) it is analogous to the optimisation problem of a particle moving along a straight line; (iii) the optimisation problem's objective function is that of the minimisation of R. I additionally ask whether such an optimisation problem may also apply to any corresponding 'Real System' of the 'Standard System', to which it is already found to apply, and I answer positively. I ulteriorly assess the application of Heisenberg's Uncertainty Principle to the same equation and derive an equation for the momentum of the particle in terms of its momentum uncertainty and in terms of its position, in which the particle is the 'Standard Net Product' and its momentum is R. I finally appraise the brachistochrone problem from a Sraffian perspective and find that in the presence of distributional gravity, for a meaningful mass function for the 'Standard Net Product', the optimal path for the distribution of the 'Standard Net Product' between profits and wages is no longer $r = R(1-w)$, but a Sraffian cycloid with specific position coordinates w and r.

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1. Introduction

In this work I derive Sraffa's fundamental equation $r = R(1-w)$ by means of differential equations and optimisation. I specifically compute the rate of change in the average rate of profits r on invested capital with respect to time t in terms of the wage share w of R, which wage share w of R is itself measured in terms of the 'Standard Commodity' under total labour normalisation; the maximum rate of profits R is simultaneously parametrised on the real non-negative number line to obtain the differential equation

$$\dot{r} = -R\dot{w},$$

the solution to which under the initial condition $r(w) = r(1) = 0$ is $r = R(1-w)$. In view of its nature I then regard such a differential equation as the solution to a minimisation problem for a particle moving along a straight line with coordinates w and r on the Cartesian plane.

The minimisation problem specifically dictates the identification of the path traversed or travelled by the particle so that its Hamiltonian cost function be minimal, the path being $\dot{r} = -R\dot{w}$. I thus integrate $\dot{r} = -R\dot{w}$ with respect to t and obtain a Hamiltonian cost function in discretional terms of the particle's position coordinates $(w,\ r)$ and velocity coordinates $(\dot{w},\ \dot{r})$:

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$$\mathcal{H}(w,\ r,\ \dot{w},\ \dot{r}) = r + Rw + \lambda h(\dot{w},\ \dot{r}) = r + Rw + \lambda(\dot{r} + R\dot{w}) = r + Rw;$$

the discretion is both in the permission for the Hamiltonian cost function to feature velocity coordinates and for them to be expressed in terms of the differential equation, specifically being its null form $h(\dot{w},\ \dot{r}) = \dot{r} + R\dot{w} = 0$. I thereupon work three remarks whose objects are: (i) the analytical provision of an alternative formulation of Sraffa's fundamental equation; (ii) its analogy with the optimisation problem of a particle moving along a straight line; (iii) the economic rationale for the Hamiltonian cost function $r + Rw + \lambda h(\dot{w},\ \dot{r})$ as being its identification with the constrained objective function of the constrained minimisation of R in $r = R(1 - w)$, in which $\dot{r} + R\dot{w} = 0$ discretionally acts as a constant 'velocity' constraint.

In short, the particle's overall movement is the distribution of the 'Standard Net Product' between profits and wages, its position coordinates are $(w,\ r)$ and its velocity coordinates are $(\dot{w},\ \dot{r})$. The particle's position function is $r + Rw$, which equals R in $r = R(1 - w)$ and thereby acts as the objective function of a minimisation problem, in line with the remark of Sinha [8] and of Sraffa [11] himself; it is discretionally constrained by the constant velocity function $\dot{r} + R\dot{w} = 0$ and the path it traverses at minimal overall movement is its rearrangement $\dot{r} = -R\dot{w}$.

I additionally pose the following question: does such an optimisation problem also apply to any corresponding 'Real System' of the 'Standard System', to which it is already found to apply? In other words, does $r = R(1 - w)$ also apply to any 'Real System' and not to the 'Standard System' alone? Since R is equal across both kinds of system for $w = 0$ at least, as argued by Sraffa [11] and shown by Sinha [10], the answer is positive.

I ulteriorly explore the application of Heisenberg's Uncertainty Principle to $r = R(1 - w)$ and derive an equation for momentum R of the 'Standard Net Product' in terms of its momentum uncertainty dR and in terms of its position w. The reason for which R is treated as equal to both the position function of the particle and its momentum is the following: the momentum of a particle is the quantification of its movement or motion, that is, the overall movement, and since the minimisation of a particle's position function gives rise to a path of minimal overall movement, being a temporal sequence of positions of minimal overall movement, in terms of time, in the context of overall movement minimisation a particle's position function and momentum are conceptually equal.

I finally appraise the brachistochrone problem from a Sraffian perspective and find that in the presence of distributional gravity g, for a meaningful mass function $m(y)$ for the 'Standard Net Product', the optimal path for the distribution of the 'Standard Net Product' between profits and wages is no longer $r = R(1-w)$, but a Sraffian cycloid with specific position coordinates w and r.

2. Derivation

In an economy whose means of production reduces to at least one basic commodity[1], rather than labour, Sraffa [11] establishes the existence of a maximum rate of profits R (i.e. 'Standard Ratio', of the 'Standard Net Product' to the 'Standard Means of Production'), which he associates with null wages and identifies as independent of the distribution of the 'Standard Net Product' between profits and wages and as independent of prices.

He expresses such by means of his fundamental equation $r = R(1 - w)$, defined hereinbefore, being a structural relation whereby the economy is statically represented. Such an equation arises for any given Sraffian 'Real System' successfully transformed into his 'Standard System'[2], whose probability according to Dupertuis and Sinha [2], as well as Sraffa [11] himself, is ultimately one.

In detail, while the existence of a 'Standard System' for single production systems may be overall accepted[3], Dupertuis and Sinha [2] offer a circumvention to the 'Manara Problem' in the case of multiple

[1] Under single production a basic commodity is one which is directly or indirectly used for the production of all output; under single production a super basic commodity, which I term as such for clarity and completeness, is one which is directly used for the production of all output.

[2] Sraffa [11] characterises it for the 'Standard System'; I hereinafter characterise it as equivalent across both the 'Standard System' and the 'Real System'.

[3] Manara [4] bluntly concedes it as an application of the Perron Frobenius Theorem, thereby superficially suggesting universal applicability. Lippi [3], Salvadori [7] and Miyamoto [5] argue to the contrary, suggesting suitable corrections, albeit rather insufficiently (see Miyamoto [5]).

production systems, suitably compiled by Pasinetti [6]. In point of fact, for the most recent controversy on the 'Manara Problem' one can consult Chapter 6 in Sinha [9].

I presently derive such an equation by means of differential equations and optimisation, thereby providing both an alternative formulation and an alternative rationale therefor, whose economic application is the minimisation of R in $r = R(1 - w)$, confirming the remark of Sinha [8] and of Sraffa [11] himself.

Since Sraffa [11] studies the distribution of the 'Standard Net Product' between profits and wages, whereby $w(t)$ lies in a real open interval between zero and one, I express $r = R(1 - w)$ in terms of non-negative time t while treating R as a real non-negative parameter; the said distribution can then be specifically studied by computing the rate of change in $r(t)$ given that in $w(t)$ with respect to t, thereby obtaining an Ordinary Differential Equation (ODE): $\forall t \in \mathbb{R}_+,\ R \in \mathbb{R}_+,\ w(t) \in [0,\ 1] \subset \mathbb{R}_+,$

$$\begin{aligned} &r(t) = R[1 - w(t)] \\ &\longrightarrow \dot{r}(t) = -R\dot{w}(t); \end{aligned}$$

in more detail,

$$\begin{aligned} &\frac{dr(t)}{dt} = \frac{dr(t)}{dw(t)}\frac{dw(t)}{dt} \\ &\longrightarrow \frac{dr(t)}{dt} = \frac{dR[1 - w(t)]}{dw(t)}\frac{dw(t)}{dt} \\ &\longrightarrow \frac{dr(t)}{dt} = -R\left(\frac{dw(t)}{dt}\right). \end{aligned}$$

The solution to such an ODE under the initial condition $r[w(t)] = r(1) = 0$, which models the full distribution of the 'Standard Net Product' to wages rather than profits, is $r(t) = R[1 - w(t)]:$

$$\begin{aligned} &\int \frac{dr(t)}{dt}dt = -R\int \frac{dw(t)}{dt}dt \\ &\longrightarrow \int dr(t) = -R\int dw(t) \\ &\longrightarrow r(t) = -Rw(t) + C; \\ &0 = -R + C,\ \text{for } r[w(t)] = r(1) = 0, \\ &\longrightarrow C = R,\ \text{thus}, \\ &r(t) = -Rw(t) + R = R[1 - w(t)]. \end{aligned}$$

I now regard $\dot{r}(t) = -R\dot{w}(t)$ as the solution to a minimisation problem for a particle moving along a straight line with horizontal position coordinates $w(t)$ and vertical position coordinates $r(t)$. This is because the rates of change in $w(t)$ and $r(t)$ with respect to t are inversely and linearly related by proportionality factor $-R$, whereby functional inversion alludes to minimisation and functional linearity alludes to the straight line. The minimisation problem specifically identifies the path traversed by the particle so that its Hamiltonian cost function be minimal (i.e. minimal overall movement), the path being $\dot{r}(t) = -R\dot{w}(t):$

$$\mathcal{H}[w(t),\ r(t)] = \int [\dot{r}(t) + R\dot{w}(t)]dt = \int 0dt = r(t) + Rw(t) + C = K.$$

While the Hamiltonian cost function may resemble the general solution of the ODE, that is, $\mathcal{H}[w(t),\ r(t)] = r(t) + Rw(t) + C$ and $r(t) = -Rw(t) + C$, the constant of integration C is to be especially different in the minimisation problem. In an effort to characterise the Hamiltonian cost function in terms of both position and velocity, with horizontal velocity coordinates $\dot{w}(t)$ and vertical velocity coordinates $\dot{r}(t)$, C can thus specifically equal the real Lagrange multiplier λ multiplied by a constant 'velocity' constraint, discretionally chosen to equal none other than the null form $\dot{r}(t) + R\dot{w}(t) = 0$ of the ODE: $\forall \lambda \in \mathbb{R},\ C = \lambda h[\dot{w}(t),\ \dot{r}(t)] = \lambda[\dot{r}(t) + R\dot{w}(t)]$,

$$\begin{aligned}
&\mathcal{H}[w(t),\ r(t),\ \dot{w}(t),\ \dot{r}(t)] = r(t) + Rw(t) + \lambda h[\dot{w}(t),\ \dot{r}(t)] \\
&= r(t) + Rw(t) + \lambda[\dot{r}(t) + R\dot{w}(t)] = r(t) + Rw(t) + \lambda(0) \\
&= \tilde{K}.
\end{aligned}$$

The constrained minimisation problem and the constrained objective function (i.e. constrained Hamiltonian cost function) are thus effectively unconstrained:

$$\underset{w(t),\ r(t)}{argmin}\ \mathcal{H}[w(t),\ r(t)] = \underset{w(t),\ r(t) \in \mathbb{R}_+}{min}\ r(t) + Rw(t) = \underset{w(t),\ r(t) \in \mathbb{R}_+}{min}\ \tilde{K}[w(t),\ r(t)],$$

in which the derivatives $\mathcal{H}_{w(t)w(t)} = \mathcal{H}_{r(t)r(t)} = \mathcal{H}_{w(t)r(t)} = 0$, satisfying convexity as a necessary and sufficient condition for minimisation together with the invertibility of $w(t)$ and $r(t)$. First Order Conditions (FOCs) and the optimal path are consequently the following:

$$\left.\begin{aligned}
\tfrac{\partial \mathcal{H}[w(t),\ r(t)]}{\partial w(t)} &= R\dot{w}(t) = 0 \\
\tfrac{\partial \mathcal{H}[w(t),\ r(t)]}{\partial r(t)} &= \dot{r}(t) = 0
\end{aligned}\right\} \dot{r}(t) = R\dot{w}(t) = 0.$$

First Order Conditions (FOCs) and the optimal path in ultimate terms of time t are analogously the following:

$$\begin{aligned}
&\underset{t}{argmin}\ \mathcal{H}[w(t),\ r(t)] = \underset{t \in \mathbb{R}_+}{min}\ r(t) + Rw(t) = \underset{t \in \mathbb{R}_+}{min}\ \tilde{K}[w(t),\ r(t)]; \\
&\frac{\partial \mathcal{H}[w(t),\ r(t)]}{\partial t} = \dot{r}(t) + R\dot{w}(t) = 0 \\
&\longrightarrow \dot{r}(t) = -R\dot{w}(t),
\end{aligned}$$

in which the derivative $\mathcal{H}_{tt} = \ddot{r}(t) + R\ddot{w}(t) = -R\ddot{w}(t) + R\ddot{w}(t) = \ddot{r}(t) - \ddot{r}(t) = 0$, satisfying convexity as a necessary and sufficient condition for minimisation together with the invertibility of t. Finally, the solution to the optimal path $\dot{r}(t) = -R\dot{w}(t)$ under the initial condition $r[w(t)] = r(1) = 0$, whose nature is dictated by the problem itself, being the distribution of the 'Standard Net Product' between profits and wages, is Sraffa's fundamental equation $r(t) = R[1 - w(t)]$, as seen.

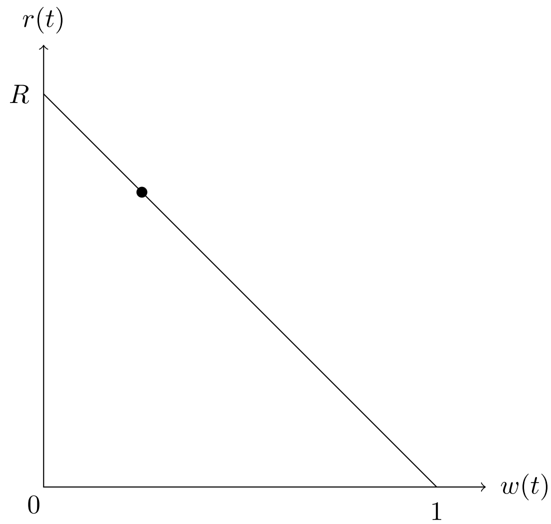
Such signifies that the minimisation problem in question is effectively the minimisation of $R = \tilde{K}$, that is, the Hamiltonian cost function of the distribution of the 'Standard Net Product' between profits and wages, which is the particle's overall movement, in hindsight is $R[w(t),\ r(t)]$:

$$\underset{r(t),\ w(t)}{argmin}\ \mathcal{H}[w(t),\ r(t)] = \underset{r(t),\ w(t)}{argmin}\ R[w(t),\ r(t)] = \underset{r(t),\ w(t)}{argmin}\ \tilde{K}[w(t),\ r(t)];$$

not for nothing does $\int [\dot{r}(t) + R\dot{w}(t)]dt = \int 0dt = r(t) + Rw(t) + C = K$ and thence $r(t) + Rw(t) = \tilde{K}$. In point of fact, $\dot{r}(t) = -R\dot{w}(t)$ is effectively Newton's Third Law of Motion (i.e. 'Action Reaction'), whereby a change in the distribution of the 'Standard Net Product' in favour of $r(t)$ is neutralised by a reciprocal one to the detriment of $w(t)$, scaled by R.

The problem of minimising the Hamiltonian cost function $\mathcal{H}[w(t),\ r(t)] = r(t) + Rw(t)$ in terms of overall physical movement while the particle moves along a straight line is therefore equivalent to the problem of minimising the maximum rate of profits R in terms of overall economic movement while the 'Standard Net Product' is distributed between profits and wages, as remarked by Sinha [8] and Sraffa [11] himself.

Indeed, while the Hamiltonian cost function $\mathcal{H}[w(t),\ r(t)] = r(t) + Rw(t)$ may appear to lack an economic rationale $r(t) + Rw(t)$ precisely captures the (cost of) distribution of the 'Standard Net Product'



Note. Sraffa's fundamental equation $r(t) = R[1 - w(t)]$, depicting the distribution of the 'Standard Net Product', that is, the black particle's overall movement, between $r(t)$ and $w(t)$.

between the average rate of profits $r(t)$ and the wage share $w(t)$ of the maximum rate of profits R, measured in terms of the 'Standard Commodity'.

Although R may be minimal the arisen minimisation problem does not allude to single production over multiple production, in which R is respectively relatively minimal and relatively maximal, but to either production mode, in which R is optimally minimal, regardless of whether it be relatively minimal or relatively maximal for the existence and unicity of a 'Standard System'.

3. Moving particle optimisation problem for 'Real Systems'

Sraffa [11] derives his fundamental equation $r(t) = R[1 - w(t)]$ by means of his 'Standard System', which originates from any given 'Real System', as mentioned hereinbefore. In fact, Dupertuis and Sinha [1, 2] and Sraffa [11] himself show that the 'Standard System' of any 'Real System' is unique. Consequently, the question one poses is whether $r(t) = R[1 - w(t)]$ may also apply to all corresponding 'Real Systems'.

To answer it, while omitting t for simplicity, I begin by denoting Sraffa's fundamental equation for the 'Standard System' as $r_S = R_S(1 - w)$ and that for the 'Real System' as $r_R = R_R(1 - w)$, whereby both equations arise[4] as properties of their respective systems.

In view of the rescaling to which the 'Real System' is subjected towards the derivation of the 'Standard System', Sraffa [11] argues that the maximum rate of profits R be equal across both the 'Standard' and 'Real' system: $R_S = R_R$. In effect, Sinha [10] shows[5] the application of such an equality for $w = 0 : \ \forall w = 0, \ R \equiv R_S = R_R$.

The question could therefore be restated by asking whether equality $R_S = R_R$ (and $r_S = r_R$ thereby) may hold for $w \in (0, \ 1] \subset \mathbb{R}_{++}$ as well, giving rise to $\bar{R}_S = \frac{r_S}{1-w} \equiv \frac{r}{1-w} = \bar{R} \equiv \bar{R}_R = \frac{r_R}{1-w}$ for $w \in [0, \ 1) \subset \mathbb{R}_+$.

[4]The two equations arise as follows. The two kinds of system are respectively $AP \cdot (\mathbf{1} + \mathbf{r}) = Q \cdot P$ and $K \cdot AP \cdot (\mathbf{1} + \mathbf{r}) = K \cdot Q \cdot P$, in which industry input $(a_{ij}) = A \in \mathbb{R}^{n \times n}_+$, input price $(p_j) = P \in \mathbb{R}^n_{++}$, augmented average rate of profits $(1 + r_i) = (\mathbf{1} + \mathbf{r}) \in \mathbb{R}^n_{++}$, output $(q_i) = Q \in \mathbb{R}^n_{++}$ and rescaling factor $\kappa_i = K \in \mathbb{R}^n_{++}$. Sinha [10] shows that there exists a unique $R = r_i$ for $w = 0$ and all industries $i \in \mathbb{N}_+$ across both kinds of system, thus, $r_i = r$ throughout each one and $R = r \equiv r_S = r_R$. Now, on adding Lw to each kind of system, in which labour $(l_i) = L \in \mathbb{R}^n_{++}$, together with the equation for the 'Standard Commodity' $\sum_{i, \ j=1}^{n} \kappa_i(q_i - \sum_{i=1}^{n} a_{ij}) = 1$, for any $r \in \mathbb{R}_+$ (independent of the unique one pinning down a unique K for a unique 'Standard System'), one can determine $P \in \mathbb{R}^n_{++}$ and $w \in [0, \ 1] \subset \mathbb{R}_+$ such that $\bar{R} = \frac{r}{1-w} \in \mathbb{R}_+$ for $w \in [0, \ 1) \subset \mathbb{R}_+$ in both kinds of system, that is, $\bar{R}_S = \frac{r_S}{1-w}$ and $\bar{R}_R = \frac{r_R}{1-w}$. The reason for which the wage share w of the maximum rate of profits R is the same within each kind of system is that it is paid *post factum* and the reason for which it is the same across the two kinds of system is precisely because it is measured in terms of the 'Standard Commodity', which acts as the numeraire of the 'Real System' as well as the 'Standard System'.

[5]Afresh, a unique $R \equiv R_S = R_R = r_i$ for $w = 0$ and all industries $i \in \mathbb{N}_+$ implies $r_i \equiv r_S = r_R$ (i.e. uniquely), whereby $r_S = R_S \equiv R \equiv R_R = r_R$.

Whereupon, (i) $r_S = R_S \equiv R \equiv R_R = r_R$ for $w = 0$ and (ii) $r_S = 0 = r_R$ for $w = 1$. Consequently, the position coordinates of the 'Standard System' and 'Real System' coincide such that the affine[6] equation passing through them be univocal, that is, $(w,\ r_R) = (0,\ R) = (w,\ r_S)$ and $(w,\ r_R) = (1,\ 0) = (w,\ r_S)$ such that $r_S = R(1-w) = r_R$, whence $r \equiv r_S = r_R$ and $R \equiv R_S = R_R$ for $w \in [0,\ 1] \subset \mathbb{R}_+$.

In detail, for affine equation $y = \alpha + \beta x$, in which parameters $\alpha,\ \beta \in \mathbb{R}$, there follows that gradient $\beta = \frac{y_2 - y_1}{x_2 - x_1}$ implies $\beta = \frac{0-R}{1-0} = -R$ hereby and that intercept $\alpha = y_1 - \beta x_1$ implies $\alpha = R - \beta(0) = R$ hereby, whence $y = \alpha + \beta x$ is hereby declined as $r_S = R - Rw = R(1-w) = r_R$.

The established equality between the average rate of profits across the two kinds of system r_S and r_R is forthwith accompanied by that between the two Hamiltonian cost functions $R_S = r_S + R_R w$ and $R_R = r_R + R_R w$, confirming the application of $r = R(1-w)$, *qua* ultimate solution of the optimisation problem, to all corresponding 'Real Systems' as well:

$$\begin{aligned} r &\equiv r_S = r_R = R(1-w) \\ &\longrightarrow R = r + Rw \equiv r_S + Rw = r_R + Rw, \end{aligned}$$

whereby, $\forall w \in [0,\ 1] \subset \mathbb{R}_+,\ R \equiv R_S = R_R$ and $r \equiv r_S = r_R$. The saliency is that the optimisation problem in question also applies to any corresponding 'Real System' of the 'Standard System', to which it has already been found to apply. Such does not only signify an alternative derivation of $r = R(1-w)$ as the optimisation problem analogue of a particle moving along a straight line but an alternative derivation of the same equation as applicable to both kinds of system, that is, to the entire[7] Sraffian economy.

4. Heisenberg's Uncertainty Principle: a Sraffian application

I now ulteriorly explore the application of Heisenberg's Uncertainty Principle to Sraffa's fundamental equation $r = R(1-w)$, in which t is omitted for simplicity afresh. Heisenberg's Uncertainty Principle states that the precision of estimation of the position of a given particle is inversely proportional to that of its momentum, whereby both cannot be simultaneously estimated with the same precision.

Formally, it states that the product of the particle's position uncertainty Δx and of its momentum uncertainty Δp is no smaller than Planck's constant h divided by 4π, thereby acting as a nether bound on uncertainty (i.e. minimum uncertainty):

$$\Delta x \Delta p \geq \frac{h}{4\pi},$$

in which Planck's constant h approximately equals 6.626×10^{-34} Joule seconds, thereby modelling the smallest possible amount of energy associated with a single quantum of an electromagnetic wave. In physical terms the Planck constant can be viewed as modelling the scarcity of energy at the smallest possible level, defining the minimum quantity of energy associated with each quantum.

In economic terms it can then be viewed as modelling the scarcity of resources at the smallest possible level, defining the minimum quantity of resources associated with each micro-transaction, which can itself be understood as a quasi-costless transaction in the broadest sense (i.e. technological, informational and institutional); its unit of measure could thus be termed "Eco-Joule seconds".

The particle's position uncertainty Δx specifically models the position distribution of the particle, often quantified as the standard deviation σ_x of the particle's position x. The particle's momentum uncertainty

[6] It behoves it to be affine, for both $r_S = R_S(1-w)$ and $r_R = R_R(1-w)$ are affine.

[7] It is crucial to note that the equivalence of $r = R(1-w)$ across both kinds of system hinges on the derivation of the 'Standard Commodity' and the existence of the 'Standard System' in turn. The discovery whereby $r_R = R_R(1-w)$ somehow arose in the absence of the 'Standard Commodity' would be a considerable result indeed, for the equality between $r_R = R_R(1-w)$ and $r_S = R_S(1-w)$ would then signify that the 'Standard System' would exist unconditionally of rescaling factors, behoving one at last to determine its construction. In fact, the alternative derivation of $r = \bar{R}(1-w)$ hereby presented can be said to apply to an economy without the consideration of a 'Real System' and the attempted construction of a 'Standard System', whereby R's independence (i) of the distribution of Q between r and w and (ii) chronologically of P is shown all the same; in other words, Sraffa's [11] endeavour to prove R's chronological independence of P is equivalent to an optimisation problem whose ultimate solution $r = R(1-w)$ emerges without his disingenuous consideration of P precisely because P does not matter therefor.

Δp accordingly models the momentum distribution of the particle, often quantified as the standard deviation σ_p of the particle's momentum p. For a constant quotient $\frac{h}{4\pi}$ as either uncertainty measure Δx or Δp increases the other must decrease.

That clarified, since the particle's Hamiltonian cost function $\mathcal{H}(w,\ r) = r + Rw$ models the (cost of) distribution of the 'Standard Net Product' between r and w and thereby equals R it follows that R models the (cost of) overall movement of the 'Standard Net Product' between r and w, that is, the momentum p of the 'Standard Net Product', which is to be minimised indeed; the particle in question is therefore the 'Standard Net Product' with position x equal to r or w.

In short, $x = w \veebar r$ and $p = R$. By assuming $x = w$ and infinitesimal quantifications of Δx and Δp I am able to express $R = r + Rw$ in terms of differentials:

$$\lim_{\Delta x \to 0} \Delta x = dx \longleftrightarrow \lim_{\Delta w \to 0} \Delta w = dw \text{ and } \lim_{\Delta p \to 0} \Delta p = dp \longleftrightarrow \lim_{\Delta R \to 0} \Delta R = dR$$

such that

$$\begin{aligned}
&\frac{dR}{dt} = \frac{dr}{dt} + \frac{Rdw}{dt}\\
&\longrightarrow dR = dr + Rdw\\
&\longrightarrow dR = dr + R\left(\frac{h}{4\pi dR}\right),
\end{aligned}$$

since

$$\begin{aligned}
&dxdp \geq \frac{h}{4\pi} \longleftrightarrow dwdR \geq \frac{h}{4\pi}\\
&\longrightarrow dw = \frac{h}{4\pi dR}.
\end{aligned}$$

I then solve the differential equation $dR = dr + R\left(\frac{h}{4\pi dR}\right)$ to obtain a solution modelling R; I specifically make use of $r = R(1-w)$:

$$\begin{aligned}
&dr = dR - R\left(\frac{h}{4\pi dR}\right)\\
&\longrightarrow \frac{dr}{dR} = 1 - w = 1 - R\left[\frac{h}{4\pi(dR)^2}\right]\\
&\longrightarrow w = R\left[\frac{h}{4\pi(dR)^2}\right]\\
&\longrightarrow R = \left[\frac{w4\pi(dR)^2}{h}\right]\\
&\longrightarrow \frac{r}{1-w} = \left[\frac{w4\pi(dR)^2}{h}\right],\ \forall w \in [0,\ 1) \subset \mathbb{R}_+.
\end{aligned}$$

Such an equation is able to quantify the momentum $R = \frac{r}{1-w}$ of the 'Standard Net Product' in terms of its momentum uncertainty dR and in terms of its position w. In short, there is relative uncertainty over the distribution of $R = \frac{r}{1-w}$ of the 'Standard Net Product', but relative certainty over its location w.

In fact, while allowing R to become a function of t, all else equal, such an equation becomes a differential equation, whose solution is computed as follows: $\forall R(t) \in \mathbb{R}_+$,

$$
\begin{aligned}
&R(t) = \left\{ \frac{w(t)4\pi[dR(t)]^2}{h} \right\}\\
&\longrightarrow [dR(t)]^2 = \frac{hR(t)}{w(t)4\pi}\\
&\longrightarrow \left[\frac{dR(t)}{dt}\right]^2 = \frac{hR(t)}{w(t)4\pi(dt)^2}\\
&\longrightarrow \dot{R}^2(t) = K_1(t)R(t),\ \forall K_1 = \frac{h}{w(t)4\pi(dt)^2},\\
&\longrightarrow \dot{R}(t) = \sqrt{K_1(t)R(t)}\\
&\longrightarrow \int \dot{R}(t)dt = \int \sqrt{K_1(t)R(t)}dt\\
&\longrightarrow R(t) = \int \sqrt{K_1(t)R(t)}dt + C\\
&\longrightarrow \frac{r(t)}{1-w(t)} = \int \sqrt{\frac{hr(t)}{w(t)4\pi(dt)^2[1-w(t)]}}dt + C,\ \forall w(t) \in [0,\ 1) \subset \mathbb{R}_+,\\
&\longrightarrow \frac{r(t)dt}{1-w(t)} = \int \sqrt{\frac{hr(t)}{w(t)4\pi(dt)^2[1-w(t)]}}dt^2 + Cdt\\
&\longrightarrow \frac{r(t)dt}{1-w(t)} = \int \sqrt{\frac{hr(t)}{w(t)4\pi[1-w(t)]}}dt + Cdt,
\end{aligned}
$$

and by defining the t domain of the integral as $[0,\ T] \subset \mathbb{R}_+$ there arises that

$$
\begin{aligned}
&\longrightarrow \int_0^T \dot{R}(t)dt = \int_0^T \sqrt{K_1(t)R(t)}dt\\
&\longrightarrow R(T) = \int_0^T \sqrt{K_1(t)R(t)}dt\\
&\longrightarrow \frac{r(T)}{1-w(T)} = \int_0^T \sqrt{\frac{hr(t)}{w(t)4\pi(dt)^2[1-w(t)]}}dt,\ \forall w(T),\ w(t) \in [0,\ 1) \subset \mathbb{R}_+,\\
&\longrightarrow \frac{r(T)dt}{1-w(T)} = \int_0^T \sqrt{\frac{hr(t)}{w(t)4\pi[1-w(t)]}}dt,
\end{aligned}
$$

which is an equation in sole terms of $w(T),\ r(T),\ w(t)$ and $r(t)$, having solved for $\dot{R}(t)$.

5. The brachistochrone problem: a Sraffian application

The minimisation of the Hamiltonian cost function $R = r(t) + Rw(t)$ is the minimisation of the overall movement of the 'Standard Net Product', distributed between average rate of profits $r(t)$ and wage share $w(t)$ of maximum rate of profits R, measured in terms of the 'Standard Commodity'; its arguments are therefore $w(t)$ and $r(t)$, as seen.

Since such arguments are themselves functions of time t the minimisation problem is to be ultimately framed in terms of t, as its core argument, as hereinbefore, whereby the optimal path to be traversed by the moving particle, which is the 'Standard Net Product', becomes that featuring the least amount of time:

$$\underset{t}{argmin}\ \mathcal{H}[w(t),\ r(t)] = \underset{t\in\mathbb{R}_+}{min}\ \mathcal{H}[w(t),\ r(t)].$$

Consequently, if any economically gravitational force were introduced, in the invariant presence of a frictionless descent and in a reversion of the axes for reciprocity, the minimisation problem would then become that of the brachistochrone: for distributional gravity $g \in \mathbb{R}$,

$$\underset{r(t)}{argmin}\, T[r(t)] = \underset{r(t)\in\mathbb{R}_+}{min}\; T[r(t)] = \underset{r(t)\in\mathbb{R}_+}{min} \int_0^1 \sqrt{\frac{1+\dot{r}^2(t)}{2gr(t)}}dw(t),$$

with boundary conditions $r[w(t)] = r(0) = R$ and $r[w(t)] = r(1) = 0$ respectively modelling the full distribution of the 'Standard Net Product' to profits and to wages. The integral is specifically derived as follows: (i-a) the partial segment of a curve is the hypotenuse of Pythagoras' Theorem, which reads

$$\begin{aligned}
&\Delta s = \sqrt{\Delta x^2 + \Delta y^2} \\
&\longrightarrow \frac{\Delta s^2}{\Delta x^2} = \left(\frac{\Delta s}{\Delta x}\right)^2 = \frac{\Delta x^2}{\Delta x^2} + \frac{\Delta y^2}{\Delta x^2} = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2 \\
&\longrightarrow \Delta s = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\Delta x,
\end{aligned}$$

for coordinates $x,\ y \in \mathbb{R}$; (i-b) the entire segment of the curve is therefore approximated by the entire hypotenuse such that the arc length of a curve is

$$\lim_{\Delta s_i \to 0} \sum_{i=1}^{n} \Delta s_i = \int_{\mathbb{R}} ds = \int_{\mathbb{R}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{\mathbb{R}} \sqrt{1+\dot{y}^2(x)}dx = \int_{\mathbb{R}} \sqrt{1+\dot{y}^2(t)}dx(t),$$

for $x = f(t)$, in which time $t \in \mathbb{R}_+$; (ii-a) in direction terms time t is the quotient of position or space $x(t) \in \mathbb{R}_+$ divided by velocity $v(t) \in \mathbb{R}_+$, that is, $t = \frac{x(t)}{v(t)}$; (ii-b) additionally, by the conservation of energy, kinetic or actual energy gained equals potential energy lost, that is, $\frac{mv^2(t)}{2} = mgy(t)$, whence $v(t) = \sqrt{2gy(t)}$; (iii) consequently, the time employed to traverse the partial, infinitesimal segment of a curve is

$$t[y(t)] = \frac{\sqrt{1+\dot{y}^2(t)}}{\sqrt{2gy(t)}}dx(t) = \sqrt{\frac{1+\dot{y}^2(t)}{2gy(t)}}dx(t)$$

and that employed to traverse the its arc length is

$$T[y(t)] = \int_{\mathbb{R}} \sqrt{\frac{1+\dot{y}^2(t)}{2gy(t)}}dx(t).$$

Being a variational problem, for integrand $L[r(t)] = \sqrt{\frac{1+\dot{r}^2(t)}{2gr(t)}}$ the Euler Lagrange equation $L[r(t)] - \dot{r}(t)\left\{\frac{\partial L[r(t)]}{\partial \dot{r}(t)}\right\} = C$ (i.e. Beltrami Identity) is notoriously used to solve for a minimiser:

$$
\begin{aligned}
&\sqrt{\frac{1+\dot{r}^2(t)}{2gr(t)}} - \dot{r}(t)\left(\frac{\dot{r}(t)}{\sqrt{2gr(t)[1+\dot{r}^2(t)]}}\right) = C \\
&\longrightarrow \frac{1+\dot{r}^2(t)}{\sqrt{2gr(t)[1+\dot{r}^2(t)]}} - \frac{\dot{r}^2(t)}{\sqrt{2gr(t)[1+\dot{r}^2(t)]}} = \frac{1}{\sqrt{2gr(t)[1+\dot{r}^2(t)]}} = C \\
&\longrightarrow r(t)[1+\dot{r}^2(t)] = \frac{1}{2gC^2} = 2A \\
&\longrightarrow \dot{r}(t) = \sqrt{\frac{2A}{r(t)} - 1} = \sqrt{\frac{2A - r(t)}{r(t)}},
\end{aligned}
$$

being an ODE with solution

$$
r(t) = A - Acos\theta = A(1 - cos\theta), \text{ at } r(t) = 0 \text{ and } \theta = 0,
$$

in which $r(t) \in [0,\ 2A] \subset \mathbb{R}_+$, whence

$$
\begin{aligned}
&\frac{dr(t)}{d\theta} = Asin\theta = 2Acos\frac{\theta}{2}sin\frac{\theta}{2} \\
&\longrightarrow dr(t) = 2Acos\frac{\theta}{2}sin\frac{\theta}{2}d\theta
\end{aligned}
$$

such that

$$
\begin{aligned}
&\dot{r}(t) = \frac{dr(t)}{dw(t)} = \sqrt{\frac{2A - r(t)}{r(t)}} = \sqrt{\frac{A + Acos\theta}{A - Acos\theta}} = \sqrt{\frac{cos^2\frac{\theta}{2}}{sin^2\frac{\theta}{2}}} = \frac{cos\frac{\theta}{2}}{sin\frac{\theta}{2}} \\
&\longrightarrow dr(t) = 2Acos\frac{\theta}{2}sin\frac{\theta}{2}d\theta = \frac{cos\frac{\theta}{2}}{sin\frac{\theta}{2}}dw(t) \\
&\longrightarrow dw(t) = 2Asin^2\frac{\theta}{2}d\theta = A(1 - cos\theta)d\theta \\
&\longrightarrow \int dw(t) = \int A(1 - cos\theta)d\theta \\
&\longrightarrow w(t) = A(\theta - sin\theta) + D,
\end{aligned}
$$

in which $D = 0$ at $w(t) = 0$ and $\theta = 0$. The solution to the brachistochrone problem is therefore characterised by the following $w(t)$ and $r(t)$ position coordinates of a cycloid:

$$
w(t) = \frac{\theta - sin\theta}{4gC^2} \text{ and } r(t) = \frac{1 - cos\theta}{4gC^2},
$$

in which $C \in \mathbb{R}$ is a constant of integration and $\theta \in (0,\ 2\pi] \subset \mathbb{R}_{++}$ is the angle of rotation of the cycloid's circle, whereby a cycloid is the trajectory of a fixed point on a circle's circumference with a given radius. Since economic systems at large can be said to be resistant to instantaneous changes (i.e. economic inertia) the introduction of distributional gravity would model the difficulties notoriously involved in the change of income distribution, being a problem influenced by such factors as institutional frictions.

The saliency is that if g were at all present then Sraffa's fundamental equation $r(t) = R[1 - w(t)]$ would no longer delineate the optimal path in the distribution of the 'Standard Net Product' between profits and wages. In other words, the saliency lies in the potential displacement of $r(t) = R[1 - w(t)]$ from the primacy of optimality.

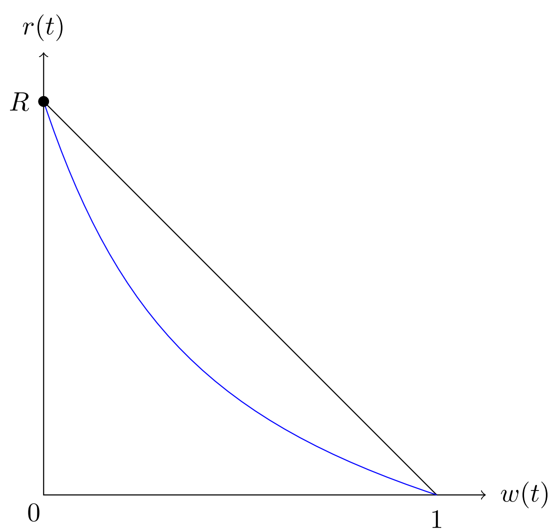
It would therefore be interesting to internalise such an eventuality and model g in terms of $r(t) =$

$R[1-w(t)]$. Since R has been established as modelling the momentum of 'Standard Net Product' and since the momentum of a particle equals the product of its mass m multiplied by its velocity $v(t)$ there results the following:

$$\begin{aligned} & R = mv(t) = m\sqrt{2gr(t)} \\ & \longrightarrow \left(\frac{R}{m}\right)^2 = 2gr(t) \\ & \longrightarrow g = \frac{\left(\frac{R}{m}\right)^2}{2r(t)} = \frac{\left(\frac{R}{m}\right)^2}{2R[1-w(t)]} = \frac{R}{2m^2[1-w(t)]}, \end{aligned}$$

in which $m = m(\overset{+}{y}) \in \mathbb{R}_{++}$, that is, mass m of the 'Standard Net Product', being the particle, is a real positive and increasing function of the selfsame 'Standard Net Product' $y \in \mathbb{R}_{++}$. It is remarkable that for a meaningful mass function $m(y)$ for the 'Standard Net Product', as is conceivable, distributional gravity g is not merely expressible in terms of R and $w(t)$ or $r(t)$ but exists altogether. In other words, Sraffa's economy realistically features distributional gravity. The solution position coordinates $w(t)$ and $r(t)$ of the Sraffian cycloid are therefore

$$\begin{aligned} & w(t) = \frac{m^2(y)[1-w(t)](\theta - sin\theta)}{2RC^2} \\ & \longrightarrow \left[1 + \frac{m^2(y)(\theta - sin\theta)}{2RC^2}\right] w(t) = \frac{m^2(y)(\theta - sin\theta)}{2RC^2} \\ & \longrightarrow w(t) = \left[1 + \frac{m^2(y)(\theta - sin\theta)}{2RC^2}\right]^{-1} \frac{m^2(y)(\theta - sin\theta)}{2RC^2} \text{ and} \\ & r(t) = \frac{m^2(y)[1-w(t)](1 - cos\theta)}{2RC^2}. \end{aligned}$$



Note. The Sraffian brachistochrone problem, whereby the fastest way for the distribution of the 'Standard Net Product' (i.e. black particle) from average rate of profits $r(t) = R$ to wage share $w(t) = 1$ of the maximum rate of profits R in the presence of distributional gravity $g = \frac{R}{2m^2(y)[1-w(t)]}$ is no longer Sraffa's fundamental equation $r(t) = R[1-w(t)]$ (i.e. black line), but the Sraffian cycloid (i.e. blue curve) with position coordinates $w(t) = \left[1 + \frac{m^2(y)(\theta - sin\theta)}{2RC^2}\right]^{-1} \frac{m^2(y)(\theta - sin\theta)}{2RC^2}$ and $r(t) = \frac{m^2(y)[1-w(t)](1-cos\theta)}{2RC^2}$.

The cycloid additionally features the remarkable property of isochronism, that is, the time employed by the moving particle to traverse the cycloid from any point until the lowest point thereon is the same. Such is particularly relevant to the Sraffian brachistochrone, for it fleshes out the Sraffian cycloid as the fastest way for the distribution of the 'Standard Net Product' from $r(t) = R$ to $w(t) = 1$ in the presence of $g = \frac{R}{2m^2(y)[1-w(t)]}$.

In short, given the presence of distributional gravity is Sraffa's fundamental equation, a straight line, the fastest way from profits to wages in the distribution of income or output? The answer is no: the Sraffian cycloid is. Consequently, insofar as they may give rise to $r(t) = R[1 - w(t)]$ the Sraffian systems are suboptimal for income distribution, thereby calling for centralised intervention towards the attainment of the Sraffian cycloid.

6. Conclusion

In this work I analytically provided an alternative formulation of Sraffa's fundamental equation $r(t) = R[1 - w(t)]$, by means of differential equations and optimisation. The optimisation problem in question is analogous to that of a particle moving along a straight line.

In effect, the momentum of the particle in question is the distribution of the 'Standard Net Product' between profits and wages; the particle's Hamiltonian cost function is precisely the maximum rate of profits R, which captures the (cost of) distribution of the 'Standard Net Product' between the average rate of profits $r(t)$ and the wage share $w(t)$ of R, measured in terms of the 'Standard Commodity'.

I then asked whether such an optimisation problem might also apply to any corresponding 'Real System' of the 'Standard System', to which it is already found to apply, and I answered positively, since R is equal across both kinds of system for $w(t) = 0$ at least.

I ulteriorly assessed the application of Heisenberg's Uncertainty Principle to the same equation and derived an equation for momentum R of the 'Standard Net Product' in terms of its momentum uncertainty dR and in terms of its position $r(t)$. I finally appraised the brachistochrone problem from a Sraffian perspective and found that in the presence of distributional gravity g, for a meaningful mass function $m(y)$ for the 'Standard Net Product', the optimal path for the distribution of the 'Standard Net Product' between profits and wages is no longer $r(t) = R[1 - w(t)]$, but the Sraffian cycloid with position coordinates $w(t) = \left[1 + \frac{m^2(y)(\theta - sin\theta)}{2RC^2}\right]^{-1} \frac{m^2(y)(\theta - sin\theta)}{2RC^2}$ and $r(t) = \frac{m^2(y)[1-w(t)](1-cos\theta)}{2RC^2}$.

References

[1] Dupertuis M.-S. and Sinha A. (2009) "A Sraffian Critique of the Classical Notion of Centre of Gravitation", Cambridge Journal of Economics 33, 1:1065-1087.

[2] Dupertuis M.-S. and Sinha A. (2009) "Existence of the Standard System in the Multiple-Production Case: a Solution to the Manara Problem", Metroeconomica 60, 3:432-454.

[3] Lippi M. (2008) "Some Observations on Sraffa and Mathematical Proofs with an Appendix on Sraffa's Convergence Algorithm", in Chiodi G. and Ditta L. (2008) "Sraffa or an Alternative Economics", Palgrave Macmillan.

[4] Manara C. F. (1968) "Il Modello di Piero Sraffa per la Produzione Congiunta di Merci a Mezzo di Merci", L'Industria 1, 1:3-18.

[5] Miyamoto J. (2008) "Correction of Sraffa's Imaginary Experiment", Kwansei Gakuin University Repository.

[6] Pasinetti L. (1980) "Essays on the Theory of Joint Production", Palgrave Macmillan.

[7] Salvadori N. (2008) "Commentary by Neri Salvadori: on a Proof of Sraffa's", in Chiodi G. and Ditta L. (2008) "Sraffa or an Alternative Economics", Palgrave Macmillan.

[8] Sinha A. (2016) "A Revolution in Economic Theory: the Economics of Piero Sraffa", Palgrave Macmillan.

[9] Sinha A. (2021) "A Reflection on Sraffa's Revolution in Economic Theory", Palgrave Macmillan.

[10] Sinha A. (2022) "A Response to Pignalosa and Trabucchi and Perri and Oro", History of Economic Ideas 30, 3:107-131.

[11] Sraffa P. (1960) "Production of Commodities by Means of Commodities: Prelude to a Critique of Economic Theory", Cambridge University Press.