Sraffa: some alternative proofs

Saccal, Alessandro

Thapar School of Liberal Arts and Sciences, TIET

4 March 2024

Sraffa: some alternative proofs

Alessandro Saccal*

Thapar School of Liberal Arts and Sciences, TIET

March 4, 2024

Abstract

Relative to the germane academic literature in this work I offer alternative and more direct proofs for (i) the existence and unicity of Sraffa's 'Standard System', (ii) the (existence and) unicity of $R = r_i$ (for $w = 0$) across the 'Real System' and the 'Standard System' and (iii) the existence (and unicity) of Sraffa's Fundamental Equation $r = R(1-w)$ across both kinds of system. While the proof for (iii) be outrightly unprecedented and that for (ii) certainly shorter, the proof for (i) is not necessarily superior to those of the germane academic literature, which judgement is left open for debate.

JEL classification codes: B24.
MSC codes: 91B55; 91B80.
Keywords: Abel Ruffini Theorem; mathematical induction; multiple and single production; Perron Frobenius Theorem; Real and Standard System; Schur decomposition; Sraffa's Fundamental Equation.

1. Introduction

Consider Sraffa's [15] 'Real System' $AP \cdot (\mathbf{1}+\mathbf{r}) = QP$ and 'Standard System' $K \cdot AP \cdot (\mathbf{1}+\mathbf{r}) = K \cdot QP$, in which industry input matrix $A \in \mathbb{R}_+^{n\times n}$, industry output matrix $Q \in \mathbb{R}_+^{n\times n}$, vectors of input prices, augmented average rate of profits and rescaling factor $\{P,\ \mathbf{1}+\mathbf{r},\ K\} \subset \mathbb{R}_{++}^n$ and average rate of profits $r_i \in \mathbb{R}_+$ such that (i) $\sum_{i=1}^n a_{ij} \leq \sum_{i=1}^n q_{ij}$ (i.e. self-reproducibility) and (ii) K for $\tilde{a}_j \equiv \sum_{i=1}^n \kappa_i a_{ij} = \sum_{i=1}^n \kappa_i q_{ij} \equiv \tilde{q}_j$ and $\frac{\tilde{q}_j}{\tilde{q}_{\neg j}} = \frac{\tilde{a}_j}{\tilde{a}_{\neg j}}$ in[1] $K \cdot A(\mathbf{1}+\mathbf{r}) = K \cdot Q$ (i.e. proportionality).

In equation form respectively consider $\sum_{i=1}^n \sum_{j=1}^n a_{ij}p_j(1+R) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}p_j(1+r_i) = \sum_{i=1}^n \sum_{j=1}^n q_{ij}p_j$ and $\sum_{i=1}^n \kappa_i \sum_{j=1}^n a_{ij}p_j(1+R) = \sum_{i=1}^n \kappa_i \sum_{j=1}^n a_{ij}p_j(1+r_i) = \sum_{i=1}^n \kappa_i \sum_{j=1}^n q_{ij}p_j$ as well, in which maximum rate of profits $R \in \mathbb{R}_+$.

Notice that $Q \in \mathbb{R}_+^{n\times n}$ models multiple production and industry output vector $Q \in \mathbb{R}_{++}^n$ models single production, for which 'Real System' $AP \cdot (\mathbf{1}+\mathbf{r}) = Q \cdot P$ and 'Standard System' $K \cdot AP \cdot (\mathbf{1}+\mathbf{r}) = K \cdot Q \cdot P$.

Relative to the germane academic literature in this work I offer alternative and more direct proofs for (i) the existence and unicity of the 'Standard System', (ii) the (existence and) unicity of $R = r_i$ (for $w = 0$) across both kinds of system and (iii) the existence (and unicity) of Sraffa's Fundamental Equation

---

*alessandro.saccal@thapar.edu. Disclaimer: the author has no declaration of interest related to this research; all views and errors in this research are the author's.

[1] $A(\mathbf{1}+\mathbf{r}) = Q$ expresses the relation between industry inputs A and output Q in real terms, whereby industry input a_{ij} is naturally priced in terms of r_i for the production of output $\sum_{j=1}^n q_{ij}$. $AP \cdot (\mathbf{1}+\mathbf{r}) = QP$ expresses the same relation in nominal terms, whereby industry input a_{ij} is priced both in terms of r_i and p_j in view of the inter-connexion between industries for the production of $\sum_{j=1}^n q_{ij}$, a_{ij} not however necessarily entering the production of all output; in other words, although the economy be not fully interconnected, inputs are outputs or finished commodities themselves, production is not static, but dynamic, and output is irreducible to capital (i.e. land) and labour alone, but is reduced to a commodity residue. The 'Standard System' is therefore paramount because it gives rise to the 'Standard Commodity' (see Proposition 3), being the said commodity residue acting as a numeraire for both kinds of system, whose intended role was to flesh out the independence of R (i) relative to the distribution of $(K\cdot)Q$ between r and wage share $w \in [0,\ 1] \subset \mathbb{R}_+$ of R and (ii) chronologically relative to P (i.e. without knowledge thereof), as per Ricardian and Marxian endeavours; it in fact turns out that Sraffa's Fundamental Equation $r = R(1-w)$ through which such an independence is fleshed out is not proper to the 'Standard System' alone but to the 'Real System' as well (see Saccal [7] and Proposition 3).

$r = R(1-w)$ across both kinds of system. While the proof for (iii) be outrightly unprecedented and that for (ii) certainly shorter, the proof for (i) is not necessarily superior to those of the germane academic literature, which judgement is left open for debate.

2. Propositions

Proposition 1 ('Standard System' existence and unicity) *All else equal, under single and multiple production the 'Standard System' exists and is unique. Formally: ceteris paribus,*

$$\exists! K \cdot AP \cdot (\mathbf{1}+\mathbf{r}) = K \cdot Q \cdot P \text{ and } K \cdot AP \cdot (\mathbf{1}+\mathbf{r}) = K \cdot QP.$$

Proof. Lemma 1.1.1 (Single production, proportionality) Existence of the 'Standard System' under single production is ensured by proportionality[2], which is achieved by rewriting $K \cdot A(\mathbf{1}+\mathbf{r}) = K \cdot Q$ as $(\mathbf{1}+\mathbf{r}) \cdot A^\top K = K \cdot Q$ and solving for a non-trivial $K \in \mathbb{R}^n_{++}$.

Notice that rewriting $K \cdot A(\mathbf{1}+\mathbf{r}) = K \cdot Q$ as $(\mathbf{1}+\mathbf{r}) \cdot A^\top K = K \cdot Q$ is possible if and only if $(\mathbf{1}+\mathbf{r}) = (1+r)$, consequently, $(\mathbf{1}+\mathbf{r}) \cdot A^\top K = K \cdot Q \longleftrightarrow (1+r)A^\top K = Q \cdot K = D(Q)K$, in which $D(Q) \in \mathbb{R}^{n\times n}_+$ is the diagonal matrix of Q. Strictly speaking, by adding normalisation equation $\sum_{i=1}^n l_i = 1$ to $(1+r)A^\top = Q$ such that

$$(1+r)\underbrace{\begin{bmatrix} a_{11} & \cdots & a_{n1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{1n} & \cdots & a_{nn} & 0 \\ l_1 & \cdots & l_n & 0 \end{bmatrix}}_{\hat{A}^\top} \underbrace{\begin{bmatrix} \kappa_1 \\ \vdots \\ \kappa_n \\ 1+r \end{bmatrix}}_{\hat{K}} = \underbrace{\begin{bmatrix} q_1 \\ \vdots \\ q_n \\ 1 \end{bmatrix}}_{\hat{Q}} \cdot \begin{bmatrix} \kappa_1 \\ \vdots \\ \kappa_n \\ 1+r \end{bmatrix}$$

one can determine both $\{\kappa_i\}_{i=1}^n \subset \mathbb{R}_{++}$ and r, whereby $(1+r) = [D(\hat{Q}) - \hat{A}^\top]\hat{K} \longrightarrow [D(\hat{Q}) - \hat{A}^\top]^{-1} = (1+r)^{-1}\hat{K}$, in which $D(\cdot)$ is a diagonal matrix and $1+r$ is the square root of the last element of $(1+r)^{-1}\hat{K}$.

Premultiplication of both sides by matrix $D(Q^{-1}) \in \mathbb{R}^{n\times n}_+$ is nonetheless such that $Bv \equiv D(Q^{-1})A^\top K = (1+r)^{-1}D(Q^{-1})D(Q)K = (1+r)^{-1}I_n K \equiv \lambda I_n v = \lambda v \longrightarrow (B - \lambda I_n)v = 0$, being the characteristic polynomial of matrix $B \in \mathbb{R}^{n\times n}_+$ and an eigenvalue problem thereby; in full:

$$(1+r)\underbrace{\begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix}}_{A^\top} \underbrace{\begin{bmatrix} \kappa_1 \\ \vdots \\ \kappa_n \end{bmatrix}}_{v \equiv K} = \underbrace{\begin{bmatrix} q_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & q_n \end{bmatrix}}_{D(Q)} \begin{bmatrix} \kappa_1 \\ \vdots \\ \kappa_n \end{bmatrix}$$

$$\longrightarrow \underbrace{\begin{bmatrix} \frac{a_{11}}{q_1} & \cdots & \frac{a_{n1}}{q_1} \\ \vdots & \ddots & \vdots \\ \frac{a_{1n}}{q_n} & \cdots & \frac{a_{nn}}{q_n} \end{bmatrix}}_{B} \begin{bmatrix} \kappa_1 \\ \vdots \\ \kappa_n \end{bmatrix} = \underbrace{\frac{1}{1+r}}_{\lambda} \underbrace{\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}}_{I_n} \begin{bmatrix} \kappa_1 \\ \vdots \\ \kappa_n \end{bmatrix},$$

in which

[2] On page 21 Sraffa [15] writes: "The possibility of speaking of a ratio between two collections of miscellaneous commodities without need of reducing them to the common measure of price arises of course from the circumstance that both collections are made up in the same proportions – from their being in fact quantities of the same composite commodity. The result would therefore not be affected by multiplying the individual component commodities by their prices. The ratio of the values of the two aggregates would inevitably be always equal to the ratio of the quantities of their several components. Nor, once the commodities had been multiplied by their prices, would the ratio be disturbed if those individual prices were to vary in all sorts of divergent ways. Thus in the Standard system the ratio of the net product to the means of production would remain the same whatever variations occurred in the division of the net product between wages and profits and whatever the consequent price changes.".

$$D(Q^{-1}) = \begin{bmatrix} q_1^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & q_n^{-1} \end{bmatrix},$$

for $q_i^{-1} \in (0,\ 1) \subset \mathbb{R}_{++}$, and $b_{ii} \equiv q_i^{-1} a_{ii},\ b_{ij} \equiv q_j^{-1} a_{ij} \in (0,\ 1] \subset \mathbb{R}_{++}$, owing to $\sum_{i=1}^{n} a_{ij} \leq q_i$. Indeed, by Cramer's Rule[3], $\exists v \neq 0$ for $B(\lambda)v \equiv (B - \lambda I_n)v = 0$ if and only if $det[B(\lambda)] = 0$, for which $B(\lambda)$ is non-invertible.

Lemma 1.1.2 (Single production, Perron Frobenius Theorem) The Perron Frobenius Theorem[4] dictates that:

(i) $\forall B = (b_{ij}) > 0$, whereby B is positive irreducible, $\exists \bar{\lambda} \equiv \lambda_i \in \{\lambda_i\}_{i=1}^{n} \subset \mathbb{C}$ such that $\bar{\lambda} \in (0,\ \infty) = \mathbb{R}_{++}$ and $\bar{\lambda} > |\lambda_{\neg i}|$ with algebraic multiplicity $\mu_B(\bar{\lambda}) = 1$, in other words, eigenvalue $\bar{\lambda}$ is positive real, dominant and simple; for $\bar{\lambda}$, $\exists! \bar{v} \equiv v_i \in \mathbb{R}_{++}^{n}$ such that $B\bar{v} = \bar{\lambda} I_n \bar{v} = \bar{\lambda}\bar{v}$, in other words, $\bar{\lambda}$'s eigenvector $\bar{v}$ is positive and simple, whereby $\bar{\lambda}$'s geometric multiplicity $\gamma_B(\bar{\lambda}) = 1$;

(ii) $\forall B = (b_{ij}) \geq 0$, whereby B is primitive and thus non-negative irreducible[5], $\exists \bar{\lambda} \equiv \lambda_i \in \{\lambda_i\}_{i=1}^{n} \subset \mathbb{C}$ such that $\bar{\lambda} \in (0,\ \infty) = \mathbb{R}_{++}$ and $\bar{\lambda} > |\lambda_{\neg i}|$ with algebraic multiplicity $\mu_B(\bar{\lambda}) = 1$, in other words, eigenvalue $\bar{\lambda}$ is positive real, dominant and simple; for $\bar{\lambda}$, $\exists! \bar{v} \equiv v_i \in \mathbb{R}_{++}^{n}$ such that $B\bar{v} = \bar{\lambda} I_n \bar{v} = \bar{\lambda}\bar{v}$, in other words, $\bar{\lambda}$'s eigenvector $\bar{v}$ is positive and simple, whereby $\bar{\lambda}$'s geometric multiplicity $\gamma_B(\bar{\lambda}) = 1$;

(iii) $\forall B = (b_{ij}) \geq 0$, whereby B is non-negative irreducible and imprimitive, $\exists \bar{\lambda} \equiv \lambda_i \in \{\lambda_i\}_{i=1}^{n} \subset \mathbb{C}$ such that $\bar{\lambda} \in (0,\ \infty) = \mathbb{R}_{++}$ and $\bar{\lambda} \geq |\lambda_{\neg i}|$ with algebraic multiplicity $\mu_B(\bar{\lambda}) = 1$, in other words, eigenvalue $\bar{\lambda}$ is positive real, dominant and simple; for $\bar{\lambda}$, $\exists! \bar{v} \equiv v_i \in \mathbb{R}_{++}^{n}$ such that $B\bar{v} = \bar{\lambda} I_n \bar{v} = \bar{\lambda}\bar{v}$, in other words, $\bar{\lambda}$'s eigenvector $\bar{v}$ is positive and simple, whereby $\bar{\lambda}$'s geometric multiplicity $\gamma_B(\bar{\lambda}) = 1$;

(iv) $\forall B = (b_{ij}) \geq 0$, imprimitive and (thus non-negative) reducible, $\exists \bar{\lambda} \equiv \lambda_i \in \{\lambda_i\}_{i=1}^{n} \subset \mathbb{C}$ such that $\bar{\lambda} \in [0,\ \infty) = \mathbb{R}_{+}$ and $\bar{\lambda} \geq |\lambda_{\neg i}|$ with algebraic multiplicity $\mu_B(\bar{\lambda}) \geq 1$, in other words, eigenvalue $\bar{\lambda}$ is non-negative real, dominant and non-simple; for $\bar{\lambda}$, $\exists \bar{v} \equiv v_i \in \mathbb{R}_{+}^{n}$ for $v_{i_i} \in \mathbb{R}_{++}$ such that $B\bar{v} = \bar{\lambda} I_n \bar{v} = \bar{\lambda}\bar{v}$, in other words, $\bar{\lambda}$'s eigenvector $\bar{v}$ is non-negative and non-simple, whereby $\bar{\lambda}$'s geometric multiplicity $\gamma_B(\bar{\lambda}) \geq 1$.

Since reducible matrices are excluded on account of model parsimony, case (iv) is not contemplated. The application of the Perron Frobenius Theorem to a non-negative irreducible B is nonetheless insufficient[6] for the existence and unicity of the 'Standard System' under single production. This is because $\bar{\lambda} = (1+r)^{-1} \in (0,\ \infty) = \mathbb{R}_{++}$ while by definition[7] $r = \bar{\lambda}^{-1} - 1 \in [0,\ \infty) = \mathbb{R}_{+} \longrightarrow \bar{\lambda} \in (0,\ 1] \subset \mathbb{R}_{++}$; in other words, the Perron Frobenius Theorem only ensures that $\bar{\lambda} \in (0,\ \infty) = \mathbb{R}_{++}$, rather than the demanded $\bar{\lambda} \in (0,\ 1] \subset \mathbb{R}_{++}$, thereby failing to exclude $\bar{\lambda} \in (1,\ \infty) \subset \mathbb{R}_{++}$.

It additionally emerges that as $\lim_{\bar{\lambda} \to 1} r_{\bar{\lambda}} = 0$; in other words, the higher may the dominant eigenvalue be the lower is the average rate of profits to result, being relatively minimal, although neither confirming the optimality minimality of $R = r_i$ for $w = 0$ (see Proposition 2) nor suggesting that of $r = R(1-w)$ as well, shown by Sraffa [15].

Indeed, non-negative profits violate the necessary, but insufficient, requirement of $r \in \mathbb{R}_{++}$ in order for Sraffa's Fundamental Equation $r = R(1-w)$ to be the optimal path for the distribution of income

[3] https://en.wikipedia.org

[4] https://en.wikipedia.org

[5] Notice that primitive matrices are non-negative irreducible, but irreducibile matrices need not be primitive, but can be either negative or imprimitive, which are always non-negative in turn. A primitive matrix $M \in \mathbb{R}_{+}^{n \times n}$ is such that $f : M \in \mathbb{R}_{+}^{n \times n} \to M^m \in \mathbb{R}_{++}^{n \times n}$, in which $m \in [1,\ \infty) = \mathbb{N}_{+}$; an imprimitive matrix is a negation thereof. An irreducible matrix $M \in \mathbb{R}^{n \times n}$ is such that $PMP^{-1} \neq \begin{bmatrix} E & F \\ 0 & G \end{bmatrix}^{\top}$, in which $E,\ F$ are positive square matrices, G is a square matrix and P is a permutation matrix (https://en.wikipedia.org).

[6] As a matter of fact it is neither necessary, for the positive real and simple eigenvalue could fail to be dominant, thereby rendering the application of the Perron Frobenius Theorem to a non-negative irreducible B unnecessary. In fact, the consideration of the application of the Perron Frobenius Theorem to a non-negative irreducible B as a sufficient condition for the existence and unicity of the 'Standard System' under single production could be argued by claiming that $r \in [0,\ \infty) = \mathbb{R}_{+}$ already excludes all cases whereby $\bar{\lambda} \in (1,\ \infty) \subset \mathbb{R}_{++}$ in the presence of $A \in \mathbb{R}_{+}^{n \times n}$ and $Q \in \mathbb{R}_{++}^{n}$, all of which Sraffa [15] no less than admits; such an interpretation would however admit the utilisation of such algorithms as Sraffa's [15] own in Chapter 5, formalised by Lippi [3], Salvadori [10] and Miyamoto [6], and require them to be always corrective.

[7] In the fractional case of $r = \bar{\lambda}^{-1} - 1 \in (0,\ 1) = \mathbb{R}_{++}$, for instance, one observes that $\bar{\lambda} \in \left(\frac{1}{2},\ 1\right) \subset \mathbb{R}_{++}$.

between profits and wages incarnated by the Sraffian brachistochrone (see Saccal [7, 8, 9]); consequently, it is demanded that $\bar{\lambda} \in (0,\ 1) \subset \mathbb{R}_{++}$ instead (i.e. positive profits).

Lemma 1.2.1 (Joint production, proportionality) Existence of the 'Standard System' under joint production is ensured by proportionality, which is achieved by rewriting $K \cdot A(\mathbf{1}+\mathbf{r}) = QK$ as $(\mathbf{1}+\mathbf{r}) \cdot A^\top K = QK$ and solving for a non-trivial $K \in \mathbb{R}^n_{++}$.

Notice that rewriting $K \cdot A(\mathbf{1}+\mathbf{r}) = QK$ as $(\mathbf{1}+\mathbf{r}) \cdot A^\top K = QK$ is possible if and only if $(\mathbf{1}+\mathbf{r}) = (1+r)$, consequently, $(\mathbf{1}+\mathbf{r}) \cdot A^\top K = QK \longleftrightarrow (1+r)A^\top K = QK \longrightarrow QK - (1+r)A^\top K = [Q-(1+r)A^\top]K = 0$. Solving $[Q-(1+r)A^\top]K = 0$ for a non-trivial $K \in \mathbb{R}^n_{++}$ is necessary and sufficient for $det[Q-(1+r)A^\top] = 0$ (i.e. non-invertible); in fact, $Qv \equiv QK = (1+r)A^\top K \equiv \lambda A^\top v$ is a generalised eigenvalue problem:

$$\underbrace{\begin{bmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \kappa_1 \\ \vdots \\ \kappa_n \end{bmatrix}}_{v \equiv K} = \underbrace{(1+r)}_{\lambda} \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix}}_{A^\top} \begin{bmatrix} \kappa_1 \\ \vdots \\ \kappa_n \end{bmatrix}.$$

The application of a generalised Schur decomposition[8] to $Qv = \lambda A^\top v$ for $Q,\ A \in \mathbb{C}^{n\times n}\backslash\{0\}$ is such that $SJ_QT^{\mathsf{H}}v = \lambda SJ_{A^\top}T^{\mathsf{H}}v$, in which $SS^{\mathsf{H}} = SS^{-1} = TT^{\mathsf{H}} = TT^{-1} = I_n$ (i.e. unitary[9]) and J_Q and $J_{A^\top}$ are complex upper triangular[10], whereby $\lambda_i = J^{-1}_{A^\top_{ii}} J_{Q_{ii}} \in \mathbb{C}$ such that $J_{A^\top_{ii}} \neq 0$; for $Q,\ A \in \mathbb{R}^{n\times n}\backslash\{0\}$ and $\lambda_i = J^{-1}_{A^\top_{ii}} J_{Q_{ii}} \in \mathbb{R}$ such that $J_{A^\top_{ii}} \neq 0$, rather than $\lambda_i \in \mathbb{C}$, it such that $SJ_QT^\top v = \lambda SJ_{A^\top}T^\top v$, in which $SS^\top = SS^{-1} = TT^\top = TT^{-1} = I_n$ (i.e. orthogonal[11]) and J_Q and $J_{A^\top}$ are real upper triangular;

Lemma 1.2.2.1 (Joint production, Perron Frobenius Theorem) Although Mangasarian [5] may extend the Perron Frobenius Theorem to $Qv = \lambda A^\top v$, in order to obtain $\bar{\lambda} = (1+r) \in (1,\ \infty) \subset \mathbb{R}_{++}$ and $\bar{v} \in \mathbb{R}^n_{++}$ I propose the following algorithm exploiting the LAPACK routine[12]: $\forall Q,\ A \in \mathbb{R}^{n\times n}\backslash\{0\}$, whereby neither need be irreducible,

(i-a) generally Schur decompose $Q = \mathcal{C}(SJ_QT^{\mathsf{H}}) = SJ_QT^{\mathsf{H}}$ and $A^\top = \mathcal{C}(SJ_{A^\top}T^{\mathsf{H}}) = SJ_{A^\top}T^{\mathsf{H}}$ for $Qv = \lambda A^\top v \longrightarrow J_Qv = \lambda J_{A^\top}v$, in which $S,\ T$ are unitary and $J_Q,\ J_{A^\top}$ are complex upper triangular, whereby $\lambda_i = J^{-1}_{A^\top_{ii}} J_{Q_{ii}} \in \mathbb{C}$ and $J_{A^\top_{ii}} \neq 0$;

(i-b) if $\lambda_i \in \mathbb{R}$ then $Q = \mathcal{C}(SJ_QT^\top) = SJ_QT^\top$ and $A^\top = \mathcal{C}(SJ_{A^\top}T^\top) = SJ_{A^\top}T^\top$ for $Qv = \lambda A^\top v \longrightarrow J_Qv = \lambda J_{A^\top}v \longrightarrow Uv \equiv J^{-1}_{A^\top} J_Qv = \lambda v \longrightarrow U(\lambda) = (U - \lambda I_n)v = 0$, in which $S,\ T$ are orthogonal, $J_Q,\ J_{A^\top}$ are real upper triangular and $J^{-1}_{A^\top}$ if and only if $J_{A^\top_{ii}} \neq 0$, whereby $U_{ii} = \lambda_{U(\lambda)_i} = \lambda_i = J^{-1}_{A^\top_{ii}} J_{Q_{ii}} \in \mathbb{R},\ J_{A^\top_{ii}} \neq 0$ and $v \in \mathbb{R}^n\backslash\{0\}$; go to step (iv-a);

(i-c) if $\lambda_i \in \mathbb{C}$ then generally Schur decompose $Q = \mathcal{R}(SJ_QT^\top) = \hat{S}\hat{J}_Q\hat{T}^\top$ and $A^\top = \mathcal{R}(SJ_{A^\top}T^\top) = \hat{S}\hat{J}_{A^\top}\hat{T}^\top$ for $Qv = \lambda A^\top v \longrightarrow J_Qv = \lambda J_{A^\top}v$, in which $\hat{S},\ \hat{T}$ are orthogonal, $\hat{J}_Q$ is complex upper quasi-triangular and $\hat{J}_{A^\top}$ is complex upper triangular, whereby, $\not\forall i \in \mathbb{N}_+,\ \lambda_i = \hat{J}^{-1}_{A^\top_{ii}} \hat{J}_{Q_{ii}} \in \mathbb{R}$ and $\hat{J}_{A^\top_{ii}} \neq 0$;

(ii-a) construct $\tilde{Q} = \Re(S\hat{J}_QT^{\mathsf{H}}) \in \mathbb{R}^{n\times n}\backslash\{0\}$ and $\tilde{A}^\top = \Re(S\hat{J}_{A^\top}T^{\mathsf{H}}) \in \mathbb{R}^{n\times n}\backslash\{0\}$;

(ii-b) generally Schur decompose $\tilde{Q} = \mathcal{C}(\tilde{S}J_{\tilde{Q}}\tilde{T}^{\mathsf{H}}) = \tilde{S}J_{\tilde{Q}}\tilde{T}^{\mathsf{H}}$ and $\tilde{A}^\top = \mathcal{C}(\tilde{S}J_{\tilde{A}^\top}\tilde{T}^{\mathsf{H}}) = \tilde{S}J_{\tilde{A}^\top}\tilde{T}^{\mathsf{H}}$ for $\tilde{Q}v = \lambda\tilde{A}^\top v \longrightarrow J_{\tilde{Q}}v = \lambda J_{\tilde{A}^\top}v$, in which $\tilde{S},\ \tilde{T}$ are unitary and $J_{\tilde{Q}},\ J_{\tilde{A}^\top}$ are complex upper triangular, whereby $\lambda_i = J^{-1}_{\tilde{A}^\top_{ii}} J_{\tilde{Q}_{ii}} \in \mathbb{C}$ and $J_{\tilde{A}^\top_{ii}} \neq 0$;

(iii-a) if $\lambda_i \in \mathbb{C}$ then by imposing $Q \equiv \tilde{Q}$ and $A^\top \equiv \tilde{A}^\top$ repeat steps (i-c)-(ii-b) until $\tilde{Q} = \tilde{S}J_{\tilde{Q}}\tilde{T}^\top$ and $\tilde{A}^\top = \tilde{S}J_{\tilde{A}^\top}\tilde{T}^\top$ for $\tilde{Q}v = \lambda\tilde{A}^\top v \longrightarrow J_{\tilde{Q}}v = \lambda J_{\tilde{A}^\top}v \longrightarrow Uv \equiv J^{-1}_{\tilde{A}^\top} J_{\tilde{Q}}v = \lambda v \longrightarrow U(\lambda) = (U - \lambda I_n)v = 0$, in which $\tilde{S},\ \tilde{T}$ are orthogonal, $J_{\tilde{Q}},\ J_{\tilde{A}^\top}$ are real upper triangular and $J^{-1}_{\tilde{A}^\top}$ if and only if $J_{\tilde{A}^\top_{ii}} \neq 0$, whereby $U_{ii} = \lambda_{U(\lambda)_i} = \lambda_i = J^{-1}_{\tilde{A}^\top_{ii}} J_{\tilde{Q}_{ii}} \in \mathbb{R},\ J_{A^\top_{ii}} \neq 0$ and $v \in \mathbb{R}^n\backslash\{0\}$; go to step (iv-a);

(iii-b) if $\lambda_i \in \mathbb{R}$ then $\tilde{Q} = \mathcal{C}(\tilde{S}J_{\tilde{Q}}\tilde{T}^\top) = \tilde{S}J_{\tilde{Q}}\tilde{T}^\top$ and $\tilde{A}^\top = \mathcal{C}(\tilde{S}J_{\tilde{A}^\top}\tilde{T}^\top) = \tilde{S}J_{\tilde{A}^\top}\tilde{T}^\top$ for $\tilde{Q}v = \lambda\tilde{A}^\top v \longrightarrow J_{\tilde{Q}}v = \lambda J_{\tilde{A}^\top}v \longrightarrow Uv \equiv J^{-1}_{\tilde{A}^\top} J_{\tilde{Q}}v = \lambda v \longrightarrow U(\lambda) = (U - \lambda I_n)v = 0$, in which $\tilde{S},\ \tilde{T}$ are orthogonal,

[8] https://en.wikipedia.org
[9] https://en.wikipedia.org
[10] https://en.wikipedia.org
[11] https://en.wikipedia.org
[12] https://netlib.org

$J_{\tilde{Q}},\ J_{\tilde{A}^\top}$ are real upper triangular and $J_{\tilde{A}^\top}^{-1}$ if and only if $J_{\tilde{A}_{ii}^\top} \neq 0$, whereby $U_{ii} = \lambda_{U(\lambda)_i} = \lambda_i = J_{\tilde{A}_{ii}^\top}^{-1} J_{\tilde{Q}_{ii}} \in \mathbb{R},\ J_{A_{ii}^\top} \neq 0$ and $v \in \mathbb{R}^n \backslash \{0\}$; go to step (iv-a);

(iv-a) [Auxiliary 'Real System'] since for positive profits $r = \bar{\lambda} - 1 \in (0,\ \infty) = \mathbb{R}_{++} \longrightarrow \bar{\lambda}_{U(\lambda)} = \bar{\lambda} = (1+r) \in (1,\ \infty) \subset \mathbb{R}_{++}$, whereby $\lim_{\bar{\lambda}\to\infty} r_{\bar{\lambda}} = \infty$ (i.e. r is relatively maximal), if $\lambda_{U(\lambda)_i} \in (-\infty,\ 1) \subset \mathbb{R}$ then select $\epsilon_{U_{ii}} \in (1 - \lambda_{U(\lambda)_i},\ \infty) \subset \mathbb{R}_{++}$ and construct $\bar{\lambda}_{\tilde{U}(\lambda)} = \lambda_{U(\lambda)_i} + \epsilon_{U_{ii}} \in (1,\ \infty) \subset \mathbb{R}_{++}$, ensuring that $\tilde{U}_{\neg i \neg i} = U_{\neg i \neg i} + \epsilon_{U_{\neg i \neg i}} \geq 0$ for $\epsilon_{U_{\neg i \neg i}} \in [-U_{\neg i \neg i},\ \infty) \subset \mathbb{R}$, whereby $\bar{v} \in \mathbb{R}^n_+$ and $\mu_{\tilde{U}}(\bar{\lambda}) = \gamma_{\tilde{U}}(\bar{\lambda}) = 1$ by the application of the Perron Frobenius Theorem to $\tilde{U}v = \lambda v$, in which $\tilde{U} \in \mathbb{R}^{n\times n}_+$ need not be irreducible, nor $(\tilde{U},\ I_n)$ self-reproducible, for a more efficient r : for example,

$$\underbrace{\begin{bmatrix} \underbrace{U_{11} + \epsilon_{U_{11}}}_{\tilde{U}_{11}} & \cdots & \underbrace{U_{1n} + \epsilon_{U_{1n}}}_{\tilde{U}_{1n}} \\ \vdots & \ddots & \vdots \\ \underbrace{U_{n1} + \epsilon_{U_{n1}}}_{\tilde{U}_{n1}} & \cdots & \underbrace{U_{nn} + \epsilon_{U_{nn}}}_{\tilde{U}_{nn}} \end{bmatrix}}_{\tilde{U}}.$$

(iv-b) if $\bar{\lambda}_{\tilde{U}(\lambda)} \equiv \lambda_{\tilde{U}(\lambda)_i} \equiv \lambda_{U(\lambda)_i} \in (1,\ \infty) \subset \mathbb{R}_{++}$ and $\tilde{U}_{\neg i \neg i} \equiv U_{\neg i \neg i} \geq 0$ then by the application of the Perron Frobenius Theorem to $\tilde{U}v \equiv Uv = \lambda v$, in which $\bar{v} \in \mathbb{R}^n_+$ and $\mu_{\tilde{U}}(\bar{\lambda}),\ \gamma_{\tilde{U}}(\bar{\lambda}) \geq 1$.

Lemma 1.2.2.2 (Joint production, Perron Frobenius Theorem) Convergence to $\lambda_i \in \mathbb{R}$ in step (iii-a) and the direct obtainment of $\lambda_i \in \mathbb{R}$ in step (iii-b), which are the linchpin of the algorithm and the proof, are ensured through mathematical induction in $\mathbb{R}$, which dictates that, $\forall x_0,\ x \in \mathbb{R},\ \{\mathcal{P}(x_0) \wedge [\mathcal{P}(x) \vdash \mathcal{P}(x'),\ \forall x' \in B[x] := \{x,\ y \in \mathbb{R} :\ ||y - x|| \leq \varepsilon > 0\}]\} \vdash \mathcal{P}(\mathbb{R})$, in which (i) $\mathcal{P}(x_0)$ ensures $\mathcal{P}(\cdot)$ for some real number x_0 and (ii) $[\mathcal{P}(x) \vdash \mathcal{P}(x'),\ \forall x' \in B[x] := \{x,\ y \in \mathbb{R} :\ ||y - x|| \leq \varepsilon > 0\}]$ ensures $\mathcal{P}(\cdot)$ for all other real numbers x and x', mimicking $\mathcal{P}(n) \vdash \mathcal{P}(n+1)$ for natural number line $\mathbb{N}$ in the absence of a good order and thereby spanning the entire real number line $\mathbb{R}$ by means of closed balls $B[x]$.

In detail, $\mathcal{P}(\tilde{U}_0) := \{\forall Q_0,\ A_0 \in \mathbb{R}^{n\times n}\backslash\{0\},\ \exists! \tilde{U}_0 = \mathcal{A}(Q_0,\ A_0) :\ Q_0 v = \lambda A_0^\top v \vdash \tilde{U}_0 v = \lambda v :\ \bar{\lambda} \in (1,\ \infty) \subset \mathbb{R}_{++},\ v \in \mathbb{R}^n_+,\ \mu_{\tilde{U}_0}(\bar{\lambda}),\ \gamma_{\tilde{U}_0}(\bar{\lambda}) \geq 1\}$ is observed for the example of Manara [4] or Schefold [12] hereunder; notice that $\mathcal{A}_0$ is bijective because by definition all $(Q_0,\ A_0)$ always (i.e. non-injectively) admit some $\tilde{U}_0$ and no other thereby[13].

Analogously, $\forall \tilde{U}' \in B[\tilde{U}] := \{\tilde{U},\ \tilde{V} \in \mathbb{R}^{n\times n}\backslash\{0\} :\ ||\tilde{U} - \tilde{V}|| \leq \varepsilon > 0\},\ \mathcal{P}(\tilde{U}) \vdash \mathcal{P}(\tilde{U}')$ is due to $\tilde{U} = \mathcal{A}(Q,\ A)$ for $\bar{\lambda} \vdash \tilde{U}' = \mathcal{A}(Q',\ A') = \mathcal{A}(Q + \delta Q',\ A + \delta A')$ for $\bar{\lambda},\ Q' \gg \delta Q'$ and $A' \gg \delta A'$ such that $||\tilde{U} - \tilde{U}'|| \leq ||\tilde{U} - \tilde{V}|| \leq \varepsilon > 0$, whereby $\mathcal{P}(\mathbb{R}^{n\times n}\backslash\{0\})$.

The reason for which tweaks in Q and A are algorithmically preserved is as follows: composite continuous functions preserve continuity; $\mathcal{A}(Q,\ A)$ is a composition of generalised Schur decompositions conditionally linked by a multiplication at step (ii-a), but while multiplication preserve continuity generalised Schur decompositions are not continuous because S and T are not unique, therefore, $||(Q,\ A) - (Q',\ A')|| \leq \delta > 0 \nvdash ||\mathcal{A}(Q,\ A) - \mathcal{A}(Q',\ A')|| = ||\tilde{U} - \tilde{U}'|| \leq ||\tilde{U} - \tilde{V}|| \leq \varepsilon > 0$; however, for $(Q'_{ij} \gg \delta Q'_{ij}) \veebar (A'_{ij} \gg \delta A'_{ij}) \longrightarrow 1 \gg \delta$ one can enforce $S' \approx S$ and $T' \approx T$ out of those available such that relative condition number[14]

$$Cond(\tilde{U}) = \lim_{\delta\to 0_+} \sup_{||(Q',\ A')||\leq\delta} \frac{||\tilde{U}'||/||\tilde{U}||}{||(Q',\ A')||/||(Q,\ A)||} = \lim_{\delta\to 0_+} \sup_{||(Q',\ A')||\leq\delta} \underbrace{\frac{\max_{i,\ j}|\tilde{U}'_{ij}|}{\max_{i,\ j}|\tilde{U}_{ij}|}}_{\approx 1} \underbrace{\frac{\max_{i,\ j}|(Q' \veebar A')_{ij}|}{\max_{i,\ j}|(Q \veebar A)_{ij}|}}_{=1} \approx 1$$

for norm $||\cdot||_{\ell_\infty}$.

In brief, the algorithm features a unique solution and, although it may not be continuous, it is well conditioned, which is the best a pseudo-composition of generalised Schur decompositions can achieve in terms of algorithmic preservation, consequently, it is quasi-well posed, thereby ensuring $\mathcal{P}(\tilde{U}) \vdash \mathcal{P}(\tilde{U}')$.

[13] $\forall Q_0,\ A_0 \in \mathbb{R}^{n\times n}\backslash\{0\},\ \tilde{U}_0 = \mathcal{A}(Q_0,\ A_0)$ such that $\exists \tilde{U}_0 \in \mathbb{R}^{n\times n}\backslash\{0\} \vdash \exists! \tilde{U}_0 \in \mathbb{R}^{n\times n}\backslash\{0\}$, lest some $\mathcal{A}(Q_0,\ A_0) = \tilde{U}_0 \neq \neg\tilde{U}_0 = \mathcal{A}(Q_0,\ A_0)$, which is a contradiction. The converse holds too: $\forall Q_0,\ A_0 \in \mathbb{R}^{n\times n}\backslash\{0\},\ \tilde{U}_0 = \mathcal{A}(Q_0,\ A_0)$ such that $\exists! \tilde{U}_0 \in \mathbb{R}^{n\times n}\backslash\{0\} \vdash \exists \tilde{U}_0 \in \mathbb{R}^{n\times n}\backslash\{0\}$, lest some $\tilde{U}_0 \neq \mathcal{A}(Q_0,\ A_0)$ in potency, which is a contradiction.

[14] https://en.wikipedia.org

Lemma 1.2.2.3 (Joint production, Perron Frobenius Theorem) Notice that the implosive application of the Perron Frobenius Theorem to $Qv = \lambda A^{\top} v$ by Bapat *et alii* [1] such that $\bar{\lambda} \in (0,\ 1) \subset \mathbb{R}_{++}$ and $v \in \mathbb{R}^n_{++}$ with $\mu_{Q,\ A^{\top}}(\lambda) = \gamma_{Q,\ A^{\top}}(\lambda) = 1$ does not impinge on the explosive formulation of my algorithm whereby $\bar{\lambda} \in (1,\ \infty) \subset \mathbb{R}_{++}$ and $v \in \mathbb{R}^n_{+}$ with $\mu_{\tilde{U}}(\bar{\lambda}),\ \gamma_{\tilde{U}}(\bar{\lambda}) \geq 1$ for $\tilde{U}v = \lambda v$.

In order to allow for the application of the Perron Frobenius Theorem to all 'Real Systems' featuring single production the same can be expressed as $Qv = \lambda A^{\top} v$ for $Q = D(Q),\ A^{\top} \in \mathbb{R}^{n \times n} \backslash \{0\}$, in which neither need be irreducible, in which $\lambda \equiv 1 + r$ and in which the addition of $\epsilon_{U_{ii}} \in (1 - \lambda_{U(\lambda)_i},\ \infty) \subset \mathbb{R}_{++}$ to $U_{ii} = \lambda_{U(\lambda)_i}$ in the auxiliary 'Real System' is not to signify the impossibility of altering the production of output without borrowing from abroad, but a more efficient r afresh. In brief, under single and multiple production there exists a unique 'Standard System' whereby r is more efficient. *QED*

Schefold [11] applies the Perron Frobenius Theorem for the construction of a unique 'Standard System' under single production, which Manara [4] had suggested as being universally applicable; not only have I shown that for $\bar{\lambda} \in (1,\ \infty) \subset \mathbb{R}_{++}$ in $Bv = \lambda v$ is the application of the Perron Frobenius Theorem unnecessary but that without correcting for $\bar{\lambda} \in (0,\ 1) \subset \mathbb{R}_{++}$ (i.e. implosive dominant eigenvalues) it is insufficient as well.

Such aligns with the criticisms of Lippi [3], Salvadori [10] and Miyamoto [6] relative to the 'imaginary experiment' for the construction of a unique 'Standard System' under single production in Sraffa's [15] Chapter 5, which apart from being incomplete, unlike my algorithm, is insufficient even upon completion (i.e. see Miyamoto [6]).

Schefold's [11] sufficient condition for the existence and unicity of the 'Standard System' under multiple production in which $n > 2$ is $(1 + r) \in \left(\frac{q_{ij}}{a_{ij}},\ \frac{q_{ii}}{a_{ii}}\right) \subseteq [0,\ \infty] = \bar{\mathbb{R}}_{++}$ and, according to both Verger [16] and Schefold [11] himself, is rather stringent[15].

By defining commodities "from the perspective of the system and not of the observer" (Abstract) Dupertuis and Sinha [2] prove the existence and unicity of the 'Standard System' under multiple production in which $n \geq 2$, thereby fully resolving the 'Manara Problem' no less than mathematically; in detail, although they and Verger [16] contend its economic resolution as well Schefold [12] dissents.

The prime merit of my algorithm for $\bar{\lambda} \in (1,\ \infty) \subset \mathbb{R}_{++}$ in $\tilde{U}v = \lambda v$, being a sufficient condition for the construction of a unique 'Standard System', is that of refraining from differentiating between basic, non-basic and atomic commodities under multiple production (see Dupertuis and Sinha [2]), but of merely requiring, by means of a more efficient r in a naively broader fashion, an *ex post* construction at worst; in detail, my algorithm is such that any 'Real System', in which $(Q,\ A)$ need be neither irreducible nor self-reproducible, can be univocally transformed into an auxiliary 'Real System' so as to obtain a 'Standard System', the economics of which algorithm are advantageously ingenuous.

The secondary merit of my algorithm for $\bar{\lambda} \in (1,\ \infty) \subset \mathbb{R}_{++}$ in $\tilde{U}v = \lambda v$, is to highlight that under multiple production for $n \geq 5$, as per Abel Ruffini Theorem[16], r cannot be analytically elaborated in general, notwithstanding the sufficient condition for the construction of a unique 'Standard System' under multiple production and the possibility of calculating it for some $n \geq 5$ cases in either fashion (i.e. mine or Dupertuis and Sinha's [2]); in fact, the generalised Schur decomposition occurs through the QZ algorithm[17], which is numerical even for $n < 5$.

Manara's [4] 2×2 example is such that $r = 0.034 \pm 0.109i$ (3 *d.p.*) because Q is reducible (i.e. weakly connected[18]), thereby failing the requirements for Bapat *et alii*'s [1] implosive application of the Perron Frobenius Theorem to $Qv = \lambda A^{\top} v$ and confirming Dupertuis and Sinha's [2] intuition of reducing 'Real Systems' to atomic ones first, whereby the blocks of Q and A are not distinct subsystems, to Schefold's [12] detriment:

[15] In Verger's [16] words, "an industry which does not use one of its outputs as input should be ruled out, and so independently of the input-output coefficients of the other industries. Hence, this condition seems to convey no real economic significance, as it eliminates a large number of normal economic systems. ... it is more a mathematical requirement than an economic one; ... the requirement for a real solution is so restrictive that the construction of a Standard product is inapplicable for a large number of normal economic systems" (Section 4.2).

[16] https://en.wikipedia.org

[17] https://de.wikipedia.org

[18] https://en.wikipedia.org

$$\underbrace{\begin{bmatrix} 1.09 & 1.144 \\ 1.144 & 0.99 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix}}_{v \equiv K} = \underbrace{(1+r)}_{\lambda} \underbrace{\begin{bmatrix} 1 & 1.1 \\ 1.1 & 1 \end{bmatrix}}_{A^\top} \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix}.$$

In detail, each Q element is a directed edge from row index i to column index j, whereby non-zero element q_{ij} is a directed edge with weight q_{ij} from node i to node j; Q is thus weakly connected because while node 2 may be reached by node 1 the converse does not hold: node 1 connects to itself with weight 1.09 and to node 2 with weight 1.144; node 2 connects to itself with weight 0.99 and to node 1 with no outgoing edge. The application of my algorithm to Manara's [4] 2×2 example requires no correction of $\lambda_{U(\lambda)_i}$, but ends at step (ii-b) after one iteration:

(i-a)

$$\underbrace{\begin{bmatrix} -0.689+0.158i & -0.692+0.144i \\ -0.692-0.144i & 0.689+0.158i \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 0.477+0.05i & 0.0432-2.0796i \\ 0 & 0.477-0.05i \end{bmatrix}}_{J_Q} \underbrace{\begin{bmatrix} -0.158-0.689i & -0.144+0.692i \\ 0.144+0.692i & -0.158+0.689i \end{bmatrix}}_{T^{\mathsf{H}}} v =$$

$$= \lambda \underbrace{\begin{bmatrix} -0.689+0.158i & -0.692+0.144i \\ -0.692-0.144i & 0.689+0.158i \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 0.458 & 0.0416-1.9996i \\ 0 & 0.458 \end{bmatrix}}_{J_{A^\top}} \underbrace{\begin{bmatrix} -0.158-0.689i & -0.144+0.692i \\ 0.144+0.692i & -0.158+0.689i \end{bmatrix}}_{T^{\mathsf{H}}} v,$$

whereby $\lambda_{1,\,2} = J^{-1}_{A^\top_{11,\,22}} J_{Q_{11,\,22}} = \frac{0.477 \pm 0.05}{0.458} = 1.034 \pm 0.109i$;

(i-c)

$$\underbrace{\begin{bmatrix} -0.707 & -0.707 \\ -0.707 & 0.707 \end{bmatrix}}_{\hat{S}} \underbrace{\begin{bmatrix} 2.184 & -0.05 \\ 0.05 & 0.104 \end{bmatrix}}_{\hat{J}_Q} \underbrace{\begin{bmatrix} -0.707 & -0.707 \\ 0.707 & -0.707 \end{bmatrix}}_{\hat{T}^\top} v =$$

$$= \lambda \underbrace{\begin{bmatrix} -0.707 & -0.707 \\ -0.707 & 0.707 \end{bmatrix}}_{\hat{S}} \underbrace{\begin{bmatrix} 2.1 & 0 \\ 0 & 0.1 \end{bmatrix}}_{\hat{J}_{A^\top}} \underbrace{\begin{bmatrix} -0.707 & -0.707 \\ 0.707 & -0.707 \end{bmatrix}}_{\hat{T}^\top} v,$$

whereby $\hat{J}_{Q_{11}} = |\Re(\lambda_{Q(\lambda)_1})| = 2.184$, $\hat{J}_{Q_{22}} = |\Re(\lambda_{Q(\lambda)_2})| = 0.104$, $\hat{J}_{A^\top_{11}} = |\Re(\lambda_{A^\top(\lambda)_1})| = 2.1$, $\hat{J}_{A^\top_{22}} = |\Re(\lambda_{A^\top(\lambda)_2})| = 0.1$ and $|\hat{S}_{ii}| = |\hat{S}_{ij}| = |\hat{T}_{ii}| = |\hat{T}_{ij}| = |\Re(v_{A^\top(\lambda)_{i_i}})| = 0.707$;

(ii-a)

$$\underbrace{\begin{bmatrix} 0 & 0.477 \\ 0.477 & 0 \end{bmatrix}}_{\tilde{Q} = \Re(S\hat{J}_Q T^{\mathsf{H}})}, \underbrace{\begin{bmatrix} 0 & 0.458 \\ 0.458 & 0 \end{bmatrix}}_{\tilde{A}^\top = \Re(S\hat{J}_{A^\top} T^{\mathsf{H}})},$$

whereby $\tilde{Q} = diag^{-1}[\Re(J_{Q_{ii}})] = 0.477$ and $\tilde{A}^\top = diag^{-1}[\Re(J_{A^\top_{ii}})] = 0.458$ (i.e. anti-diagonal[19]);

(ii-b)

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\tilde{S}} \underbrace{\begin{bmatrix} 0.477 & 0 \\ 0 & 0.477 \end{bmatrix}}_{J_{\tilde{Q}}} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\tilde{T}^\top} v = \lambda \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\tilde{S}} \underbrace{\begin{bmatrix} 0.458 & 0 \\ 0 & 0.458 \end{bmatrix}}_{J_{\tilde{A}^\top}} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\tilde{T}^\top} v,$$

$$\underbrace{\begin{bmatrix} 2.1 & 0 \\ 0 & 2.1 \end{bmatrix}}_{J^{-1}_{\tilde{Q}}}, \underbrace{\begin{bmatrix} 2.184 & 0 \\ 0 & 2.184 \end{bmatrix}}_{J^{-1}_{\tilde{A}^\top}},$$

[19] https://en.wikipedia.org

whereby $\lambda_{\tilde{U}(\lambda)_{1,\,2}} = \lambda_{1,\,2} = J^{-1}_{\tilde{A}^{\top}_{11,\,22}} J_{\tilde{Q}_{11,\,22}} = \Re(J_{A^{\top}_{11,\,22}})^{-1}\Re(J_{Q_{11,\,22}}) = \frac{0.477}{0.458} = 1.034 \in (1,\ \infty) \subset \mathbb{R}_{++}$, $J^{-1}_{\tilde{Q}} = diag(|\Re(\lambda_{A^{\top}(\lambda)_1})|)$, $J^{-1}_{\tilde{A}^{\top}} = diag(|\Re(\lambda_{Q(\lambda)_1})|)$, $\tilde{S} = \tilde{S}^{\top} = J_n$ (i.e. exchange[20]) and $\tilde{T} = \tilde{T}^{\top} = I_n$ for

$$\underbrace{\begin{bmatrix} 1.034 & 0 \\ 0 & 1.034 \end{bmatrix}}_{\tilde{U}\equiv U}$$

in $\tilde{U}v = \lambda v$ and $v_2 = \begin{bmatrix} 0.488 & 0.873 \end{bmatrix}^{\top}$. Notice that $\tilde{U}v = \lambda I_2 v = \lambda v$ self-reproduces:

$$\underbrace{\begin{bmatrix} 1.034 & 0 \\ 0 & 1.034 \end{bmatrix}}_{\tilde{U}} \underbrace{\begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix}}_{v\equiv K} = \underbrace{(1+r)}_{\lambda} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I_2} \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix}.$$

Even if Manara's [4] 2×2 example had featured an irreducible Q Bapat *et alii*'s [1] application of the Perron Frobenius Theorem to $Qv = \lambda A^{\top}v$ would have merely delivered $\bar{\lambda} \in (0,\ 1) \subset \mathbb{R}_{++} \longrightarrow r \in (-1,\ 0) \subset \mathbb{R}_{--}$ (i.e. implosive r); my algorithm by contrast extracts $\Re(r) = 0.034 \in (0,\ \infty) = \mathbb{R}_{++}$ through the creation of an auxiliary 'Real System', which self-reproduces.

In brief, my algorithm for the construction of a unique 'Standard System' under multiple production, which (correctively) delivers of $r \in (0,\ \infty) = \mathbb{R}_{++}$ through the creation of an auxiliary 'Real System', can be understood as a complete and sufficient counterpart of Sraffa's [15] incomplete and insufficient algorithm for the construction of a unique 'Standard System' under single production. The application of my algorithm to Schefold's [12] 2×2 example, as presented by Sinha and Verger [14], ends at step (ii-b) after two iterations:

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix}}_{v\equiv K} = \underbrace{(1+r)}_{\lambda} \underbrace{\begin{bmatrix} \frac{2}{5} & \frac{-1}{5} \\ 1 & \frac{3}{5} \end{bmatrix}}_{A^{\top}} \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix}$$

whereby $r = 0.136 \pm 0.991i$ (3 *d.p.*) and Q reducible (i.e. weakly connected);

First iteration (i-a)

$$\underbrace{\begin{bmatrix} 0.019+0.408i & -0.898+0.163i \\ -0.898-0.163i & -0.019+0.408i \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 0.754+0.657i & 0 \\ 0 & 0.754-0.657i \end{bmatrix}}_{J_Q} \underbrace{\begin{bmatrix} -0.254-0.3197i & -0.5698+0.713i \\ -0.5698-0.713i & 0.254-0.3197i \end{bmatrix}}_{T^{\mathsf{H}}} v =$$

$$= \lambda \underbrace{\begin{bmatrix} 0.019+0.408i & -0.898+0.163i \\ -0.898-0.163i & -0.019+0.408i \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 0.663 & 0.312-0.763i \\ 0 & 0.663 \end{bmatrix}}_{J_{A^{\top}}} \underbrace{\begin{bmatrix} -0.254-0.3197i & -0.5698+0.713i \\ -0.5698-0.713i & 0.254-0.3197i \end{bmatrix}}_{T^{\mathsf{H}}} v,$$

whereby $\lambda_{1,\,2} = J^{-1}_{A^{\top}_{11,\,22}} J_{Q_{11,\,22}} = \frac{0.754\pm 0.657i}{0.663} = 1.136 \pm 0.991i \in \mathbb{C}$;

(i-c)

$$\underbrace{\begin{bmatrix} -0.223 & -0.975 \\ -0.975 & 0.223 \end{bmatrix}}_{\hat{S}} \underbrace{\begin{bmatrix} 0.6402 & -0.768 \\ 0.768 & 0.6402 \end{bmatrix}}_{\hat{J}_Q} \underbrace{\begin{bmatrix} -0.892 & -0.453 \\ 0.453 & -0.892 \end{bmatrix}}_{\hat{T}^{\top}} v =$$

$$= \lambda \underbrace{\begin{bmatrix} -0.223 & -0.975 \\ -0.975 & 0.223 \end{bmatrix}}_{\hat{S}} \underbrace{\begin{bmatrix} 1.193 & 0 \\ 0 & 0.369 \end{bmatrix}}_{\hat{J}_{A^{\top}}} \underbrace{\begin{bmatrix} -0.892 & -0.453 \\ 0.453 & -0.892 \end{bmatrix}}_{\hat{T}^{\top}} v;$$

[20] https://en.wikipedia.org

(ii-a)

$$\underbrace{\begin{bmatrix} 0.483 & -0.106 \\ 0.106 & 0.483 \end{bmatrix}}_{\tilde{Q}=\Re(S\hat{J}_Q T^{\mathsf{H}})}, \underbrace{\begin{bmatrix} 0.382 & -0.425 \\ 0.321 & 0.796 \end{bmatrix}}_{\tilde{A}^\top=\Re(S\hat{J}_{A^\top} T^{\mathsf{H}})};$$

(ii-b)

$$\underbrace{\begin{bmatrix} -0.743+0.384i & 0.304+0.456i \\ 0.304-0.456i & 0.743+0.384i \end{bmatrix}}_{\tilde{S}} \underbrace{\begin{bmatrix} 0.488+0.079i & 0 \\ 0 & 0.488-0.079i \end{bmatrix}}_{J_{\tilde{Q}}} \underbrace{\begin{bmatrix} -0.696-0.169i & 0.535+0.448i \\ 0.535-0.448i & 0.696-0.169i \end{bmatrix}}_{\tilde{T}^{\mathsf{H}}} v =$$

$$= \lambda \underbrace{\begin{bmatrix} -0.743+0.384i & 0.304+0.456i \\ 0.304-0.456i & 0.743+0.384i \end{bmatrix}}_{\tilde{S}} \underbrace{\begin{bmatrix} 0.663 & 0.2003+0.377i \\ 0 & 0.663 \end{bmatrix}}_{J_{\tilde{A}^\top}} \underbrace{\begin{bmatrix} -0.696-0.169i & 0.535+0.448i \\ 0.535-0.448i & 0.696-0.169i \end{bmatrix}}_{\tilde{T}^{\mathsf{H}}} v,$$

whereby $\lambda_{1,\,2} = J^{-1}_{\tilde{A}^\top_{11,\,22}} J_{\tilde{Q}_{11,\,22}} = \frac{0.488\pm0.079i}{0.663} = 0.735 \pm 0.119i \in \mathbb{C}$, thus, $Q \equiv \tilde{Q}$ and $A^\top \equiv \tilde{A}^\top$;
Second iteration (i-c)

$$\underbrace{\begin{bmatrix} -0.394 & 0.919 \\ 0.919 & 0.394 \end{bmatrix}}_{\hat{S}} \underbrace{\begin{bmatrix} 0.464 & -0.169 \\ 0.169 & 0.464 \end{bmatrix}}_{\hat{J}_Q} \underbrace{\begin{bmatrix} 0.159 & 0.987 \\ 0.987 & -0.159 \end{bmatrix}}_{\hat{T}^\top} v =$$

$$= \lambda \underbrace{\begin{bmatrix} -0.394 & 0.919 \\ 0.919 & 0.394 \end{bmatrix}}_{\hat{S}} \underbrace{\begin{bmatrix} 0.9104 & 0 \\ 0 & 0.483 \end{bmatrix}}_{\hat{J}_{A^\top}} \underbrace{\begin{bmatrix} 0.159 & 0.987 \\ 0.987 & -0.159 \end{bmatrix}}_{\hat{T}^\top} v;$$

(ii-a)

$$\underbrace{\begin{bmatrix} 0.456 & -0.061 \\ 0.061 & 0.456 \end{bmatrix}}_{\tilde{Q}=\Re(S\hat{J}_Q T^{\mathsf{H}})}, \underbrace{\begin{bmatrix} 0.707 & -0.379 \\ 0.01204 & 0.616 \end{bmatrix}}_{\tilde{A}^\top=\Re(S\hat{J}_{A^\top} T^{\mathsf{H}})};$$

(ii-b)

$$\underbrace{\begin{bmatrix} -0.956 & 0.294 \\ 0.294 & 0.956 \end{bmatrix}}_{\tilde{S}} \underbrace{\begin{bmatrix} 0.4598 & 0 \\ 0 & 0.4598 \end{bmatrix}}_{J_{\tilde{Q}}} \underbrace{\begin{bmatrix} -0.909 & 0.417 \\ 0.417 & 0.909 \end{bmatrix}}_{\tilde{T}^\top} v =$$

$$= \lambda \underbrace{\begin{bmatrix} -0.956 & 0.294 \\ 0.294 & 0.956 \end{bmatrix}}_{\tilde{S}} \underbrace{\begin{bmatrix} 0.838 & 0.212 \\ 0 & 0.525 \end{bmatrix}}_{J_{\tilde{A}^\top}} \underbrace{\begin{bmatrix} -0.909 & 0.417 \\ 0.417 & 0.909 \end{bmatrix}}_{\tilde{T}^\top} v,$$

whereby $\lambda_{1,\,2} = J^{-1}_{\tilde{A}^\top_{11,\,22}} J_{\tilde{Q}_{11,\,22}} = \frac{0.4598}{0.838,\,0.525} = 0.5489,\ 0.875 \in \mathbb{R}$ for

$$\underbrace{\begin{bmatrix} 0.5489 & -0.222 \\ 0 & 0.875 \end{bmatrix}}_{U}$$

in $Uv = \lambda v$;
(iv)

$$\underbrace{\begin{bmatrix} 1.001 & 0.001 \\ 0 & 0.875 \end{bmatrix}}_{\tilde{U}}$$

for $\lambda_{\tilde{U}(\lambda)_1} = \lambda_{U(\lambda)_1} + \epsilon_{U_{11}} = 0.5489 + (1.001 - 0.5489) = 1.001 \in (1,\ \infty) \subset \mathbb{R}_{++} \longrightarrow r = 0.001 \in (0,\ \infty) \subset \mathbb{R}_{+}$ and $\tilde{U}_{12} = U_{12} + \epsilon_{U_{12}} = -0.222 + (0.001 + 0.222) = 0.001 > 0$, whereby $v_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top$, since $\tilde{U}$ is reducible.

In fact, the reducibility of $\tilde{U}$, whereby $v_i \in \mathbb{R}^n_+$ for at least some $v_{i_i} = 0$, does not invalidate the existence of Sraffa's Fundamental Equation $r = R(1-w)$ because the derivation of the same is not affected by $v_{i_i} = 0$ (see Proposition 3). Notice that $\tilde{U}v = \lambda I_2 v = \lambda v$ does not self-reproduce:

$$\underbrace{\begin{bmatrix} 1.001 & 0.001 \\ 0 & 0.875 \end{bmatrix}}_{\tilde{U}} \underbrace{\begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix}}_{v \equiv K} = \underbrace{(1+r)}_{\lambda} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I_2} \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix}.$$

Proposition 2 [$R = r_i$ (for $w = 0$) existence and unicity] *All else equal, under single and multiple production there respectively exists a unique $R = r_i \in \mathbb{R}_+$ (for $w = 0$) for both the 'Real System' and the 'Standard System'. Formally, ceteris paribus,*

$\exists! R = r_i \in \mathbb{R}_+$ respectively for:
(i) $AP \cdot (\mathbf{1} + \mathbf{r}) = Q \cdot P$ and $K \cdot AP \cdot (\mathbf{1} + \mathbf{r}) = K \cdot Q \cdot P$;
(ii) $AP \cdot (\mathbf{1} + \mathbf{r}) = QP$ and $K \cdot AP \cdot (\mathbf{1} + \mathbf{r}) = K \cdot QP$.

Proof. Lemma 2.1 (Single production) Rewrite $AP \cdot (\mathbf{1} + \mathbf{r}) = Q \cdot P$ as $P^\top A^\top \cdot (\mathbf{1} + \mathbf{r}) = P^\top Q$. Rearrange it such that $P^\top[A^\top \cdot (\mathbf{1} + \mathbf{r}) - Q] = 0$, which holds for $P^\top \neq 0$ if and only if $A^\top \cdot (\mathbf{1} + \mathbf{r}) - Q = 0$, whence[21] $A^\top \cdot (\mathbf{1} + \mathbf{r}) = Q$, which by $A \cdot (\mathbf{1} + \mathbf{r}) = Q$ is necessary and sufficient for $\sum_{j=1}^n a_{ij}(1 + r_i) = \sum_{i=1}^n a_{ji}(1 + r_i) = q_i$ and $r = r_i = r_{\neg i}$, so that $(\mathbf{1} + \mathbf{r}) = \begin{bmatrix} (1+r) & \cdots & (1+r) \end{bmatrix}^\top$.

As a consequence, $(1 + R) = (1 + r) = (1 + r_i) \longrightarrow R = r = r_i \in \mathbb{R}_+$ for $\sum_{i=1}^n \sum_{j=1}^n a_{ij}p_j(1 + R) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}p_j(1 + r) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}p_j(1 + r_i) = \sum_{i=1}^n q_i p_j$.

The same $R = r = r_i \in \mathbb{R}_+$ also applies to $K \cdot AP \cdot (\mathbf{1} + \mathbf{r}) = K \cdot Q \cdot P$ and $\sum_{i=1}^n \kappa_i \sum_{j=1}^n a_{ij}p_j(1 + R) = \sum_{i=1}^n \kappa_i \sum_{j=1}^n a_{ij}p_j(1 + r) = \sum_{i=1}^n \kappa_i \sum_{j=1}^n a_{ij}p_j(1 + r_i) = \sum_{i=1}^n \kappa_i q_i p_j$, since K is a dot factor; it is also because the existence and unicity of $K \cdot AP \cdot (\mathbf{1} + \mathbf{r}) = K \cdot Q \cdot P$ presupposes $r = r_i$ in order to solve for a unique K, as seen. There thus exists a unique $R = r_i \in \mathbb{R}_+$ for both $AP \cdot (\mathbf{1} + \mathbf{r}) = Q \cdot P$ and $K \cdot AP \cdot (\mathbf{1} + \mathbf{r}) = K \cdot Q \cdot P$.

Lemma 2.2 (Multiple production) Rewrite $AP \cdot (\mathbf{1} + \mathbf{r}) = QP$ as $P^\top A^\top \cdot (\mathbf{1} + \mathbf{r}) = P^\top Q^\top$. Rearrange it such that $P^\top[A^\top \cdot (\mathbf{1} + \mathbf{r}) - Q^\top] = 0$, which holds for $P^\top \neq 0$ if and only if $A^\top \cdot (\mathbf{1} + \mathbf{r}) - Q^\top = 0$, whence[22] $A^\top \cdot (\mathbf{1} + \mathbf{r}) = Q^\top$, which by definition is necessary and sufficient for $I = [A^\top \cdot (\mathbf{1} + \mathbf{r})]^{-1} Q^\top$. Indeed, since A^{-1} and thereby $A^{\top^{-1}}$ exist it follows that $[A^\top \cdot (\mathbf{1} + \mathbf{r})]^{-1}$ exists if and only if $(\mathbf{1} + \mathbf{r}) = (1 + r) \in \mathbb{R}_{++}$.

As a consequence, $(1 + R) = (1 + r) = (1 + r_i) \longrightarrow R = r = r_i \in \mathbb{R}_+$ for $\sum_{i=1}^n \sum_{j=1}^n a_{ij}p_j(1 + R) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}p_j(1 + r) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}p_j(1 + r_i) = \sum_{i=1}^n \sum_{j=1}^n q_{ij}p_j$.

The same $R = r = r_i \in \mathbb{R}_+$ also applies to $K \cdot AP \cdot (\mathbf{1} + \mathbf{r}) = K \cdot QP$ and $\sum_{i=1}^n \kappa_i \sum_{j=1}^n a_{ij}p_j(1 + R) = \sum_{i=1}^n \kappa_i \sum_{j=1}^n a_{ij}p_j(1 + r) = \sum_{i=1}^n \kappa_i \sum_{j=1}^n a_{ij}p_j(1 + r_i) = \sum_{i=1}^n \kappa_i \sum_{j=1}^n q_{ij}p_j$, since K is a dot factor; it is also because the existence and unicity of $K \cdot AP \cdot (\mathbf{1} + \mathbf{r}) = K \cdot QP$ presupposes $r = r_i$ in order to solve for a unique K, as seen. There thus exists a unique $R = r_i \in \mathbb{R}_+$ for both $AP \cdot (\mathbf{1} + \mathbf{r}) = QP$ and $K \cdot AP \cdot (\mathbf{1} + \mathbf{r}) = K \cdot QP$. *QED*

Relative to Sinha [13] Proposition 2 offers an alternative and more direct derivation of the existence and unicity of $R = r_i$ (for $w = 0$) under single (and multiple) production.

Proposition 3 (Sraffa's Fundamental Equation existence and unicity) *Let expanded 'Real System' $\hat{A}\hat{P} \cdot (\mathbf{1} \hat{+} \mathbf{r}) + \hat{L}w = \hat{Q}\hat{P}$ and expanded 'Standard System' $\hat{K} \cdot \hat{A}\hat{P} \cdot (\mathbf{1} \hat{+} \mathbf{r}) + \hat{L}w = \hat{K} \cdot \hat{Q}\hat{P}$, in which the*

[21] Notice that if $P^\top[A^\top \cdot (\mathbf{1} + \mathbf{r}) - Q] = 0$ and $P^\top \neq 0$ hold then so does $A^\top \cdot (\mathbf{1} + \mathbf{r}) = Q$ (and *vice versa);* thus, because $P^\top[A^\top \cdot (\mathbf{1} + \mathbf{r}) - Q] = 0$ and $P^\top \neq 0$ (satisfied by $P \in \mathbb{R}^n_{++}$) hold by definition $A^\top \cdot (\mathbf{1} + \mathbf{r}) = Q$ must hold. Conversely, $A^\top \cdot (\mathbf{1} + \mathbf{r}) = Q$ is not required to hold in the first place, but holds as an implication of $P^\top[A^\top \cdot (\mathbf{1} + \mathbf{r}) - Q] = 0$ and $P^\top \neq 0$, semantically (i.e. soundly, apodictically) implying them in turn.

[22] Afresh, because $P^\top[A^\top \cdot (\mathbf{1} + \mathbf{r}) - Q^\top] = 0$ and $P^\top \neq 0$ hold by definition $A^\top \cdot (\mathbf{1} + \mathbf{r}) = Q^\top$ must hold.

ulterior equation added[23] *to both* $AP(1+r)+Lw=QP$ *and* $K\cdot AP(1+r)+Lw=K\cdot QP$*, respectively, for labour vector* $L\in\mathbb{R}^n_{++}$ *and wage share* $w\in[0,\ 1]\subset\mathbb{R}_+$ *of* R *such that* $\sum_{i=1}^n l_i=1$ *(i.e. normalisation) is* $\sum_{j=1}^n\sum_{i=1}^n\kappa_i(q_{ij}-a_{ij})=1$ *(i.e. 'Standard Commodity'), all else equal:*

(i) ceteris paribus, $AP(1+r)+Lw=QP$ *and* $\sum_{j=1}^n x_j=\sum_{j=1}^n\sum_{i=1}^n\kappa_i(q_{ij}-a_{ij})=1$ *give rise to*

$$\hat{A}\hat{P}\cdot(\mathbf{1}\hat{+}\mathbf{r})+\hat{L}w=\hat{Q}\hat{P}$$
$$\longleftrightarrow\begin{bmatrix}a_{11}&\cdots&a_{1n}&0\\ \vdots&\ddots&\vdots&\vdots\\ a_{n1}&\cdots&a_{nn}&0\\ x_1&\cdots&x_n&0\end{bmatrix}\begin{bmatrix}p_1\\ \vdots\\ p_n\\ 1\end{bmatrix}\cdot\begin{bmatrix}1+r\\ \vdots\\ 1+r\\ 1\end{bmatrix}+\begin{bmatrix}l_1\\ \vdots\\ l_n\\ 0\end{bmatrix}w=\begin{bmatrix}q_{11}&\cdots&q_{1n}&0\\ \vdots&\ddots&\vdots&\vdots\\ q_{n1}&\cdots&q_{nn}&0\\ 0&\cdots&0&1\end{bmatrix}\begin{bmatrix}p_1\\ \vdots\\ p_n\\ 1\end{bmatrix},$$

in which the last row of $\hat{A}$ *is* $\sum_{j=1}^n x_j$ *and the last column is* $\mathbf{0}$*, the last element of* $\hat{P}$ *is* 1*, all but the last element of* $\mathbf{1}\hat{+}\mathbf{r}$ *are* $1+r$ *and the last element thereof is* 1*, the last element of* $\hat{L}$ *is* 0*, the last row of* $\hat{Q}$ *is* $\sum_{j=1}^n 0$ *and the last column is* $\begin{bmatrix}\mathbf{0} & 1\end{bmatrix}^\top$*;*

(ii) ceteris paribus, $K\cdot AP(1+r)+Lw=K\cdot QP$ *and* $\sum_{j=1}^n x_j=\sum_{j=1}^n\sum_{i=1}^n\kappa_i(q_{ij}-a_{ij})=1$ *give rise to*

$$\hat{K}\cdot\hat{A}\hat{P}\cdot(\mathbf{1}\hat{+}\mathbf{r})+\hat{L}w=\hat{K}\cdot\hat{Q}\hat{P}$$
$$\longleftrightarrow\begin{bmatrix}\kappa_1\\ \vdots\\ \kappa_n\\ 1\end{bmatrix}\cdot\begin{bmatrix}a_{11}&\cdots&a_{1n}&0\\ \vdots&\ddots&\vdots&\vdots\\ a_{n1}&\cdots&a_{nn}&0\\ x_1&\cdots&x_n&0\end{bmatrix}\begin{bmatrix}p_1\\ \vdots\\ p_n\\ 1\end{bmatrix}\cdot\begin{bmatrix}1+r\\ \vdots\\ 1+r\\ 1\end{bmatrix}+\begin{bmatrix}l_1\\ \vdots\\ l_n\\ 0\end{bmatrix}w=\begin{bmatrix}\kappa_1\\ \vdots\\ \kappa_n\\ 1\end{bmatrix}\cdot\begin{bmatrix}q_{11}&\cdots&q_{1n}&0\\ \vdots&\ddots&\vdots&\vdots\\ q_{n1}&\cdots&q_{nn}&0\\ 0&\cdots&0&1\end{bmatrix}\begin{bmatrix}p_1\\ \vdots\\ p_n\\ 1\end{bmatrix},$$

in which the last element of $\hat{K}$ *is* 1*, the last row of* $\hat{A}$ *is* $\sum_{j=1}^n x_j$ *and the last column is* $\mathbf{0}$*, the last element of* $\hat{P}$ *is* 1*, all but the last element of* $\mathbf{1}\hat{+}\mathbf{r}$ *are* $1+r$ *and the last element thereof is* 1*, the last element of* $\hat{L}$ *is* 0*, the last row of* $\hat{Q}$ *is* $\sum_{j=1}^n 0$ *and the last column is* $\begin{bmatrix}\mathbf{0} & 1\end{bmatrix}^\top$*.*

All else equal, for any $r\in\mathbb{R}_+$*, the determination of* $P\in\mathbb{R}^n_{++}$ *and* $w\in[0,\ 1]\subset\mathbb{R}_+$ *is such that* $\bar{R}=\frac{r}{1-w}\in\mathbb{R}_+$ *for* $w\in[0,\ 1)\subset\mathbb{R}_+$ *for both* $\hat{A}\hat{P}\cdot(\mathbf{1}\hat{+}\mathbf{r})+\hat{L}w=\hat{Q}\hat{P}$ *and* $\hat{K}\cdot\hat{A}\hat{P}\cdot(\mathbf{1}\hat{+}\mathbf{r})+\hat{L}w=\hat{K}\cdot\hat{Q}\hat{P}$*. Formally: ceteris paribus,* $\forall r\in\mathbb{R}_+,\ P\in\mathbb{R}^n_{++}$ *and* $w\in[0,\ 1]\subset\mathbb{R}_+$ *are determined such that*

$$\bar{R}=\frac{r}{1-w}\in\mathbb{R}_+\text{ for }w\in[0,\ 1)\subset\mathbb{R}_+\text{ for}$$
$$\hat{A}\hat{P}\cdot(\mathbf{1}\hat{+}\mathbf{r})+\hat{L}w=\hat{Q}\hat{P}\text{ and }\hat{K}\cdot\hat{A}\hat{P}\cdot(\mathbf{1}\hat{+}\mathbf{r})+\hat{L}w=\hat{K}\cdot\hat{Q}\hat{P}.$$

Proof. Mathematical induction in $\mathbb{R}$ afresh dictates that, $\forall x_0,\ x\in X\subseteq\mathbb{R},\ \{\mathcal{P}(x_0)\wedge[\mathcal{P}(x)\vdash\mathcal{P}(x'),\ \forall x'\in B[x]:=\{x,\ y\in X:\ ||y-x||\le\varepsilon>0\}]\}\vdash\mathcal{P}(X)$, in which (i) $\mathcal{P}(x_0)$ ensures $\mathcal{P}(\cdot)$ for some real number x_0 and (ii) $[\mathcal{P}(x)\vdash\mathcal{P}(x'),\ \forall x'\in B[x]:=\{x,\ y\in X:\ ||y-x||\le\varepsilon>0\}]$ ensures $\mathcal{P}(\cdot)$ for all other real numbers x and x', mimicking $\mathcal{P}(n)\vdash\mathcal{P}(n+1)$ for $\mathbb{N}$ in the absence of a good order and thereby spanning the entire real set X or the entire real number line $\mathbb{R}$ by means of closed balls $B[x]$.

Since K is a dot factor, for both the expanded 'Real System' and expanded 'Standard System' consider $x=w$ and $\mathcal{P}(w):=\left\{w:\ \forall r\in\mathbb{R}_+,\ \bar{R}\in\mathbb{R}_{++},\ w=1-\frac{r}{\bar{R}}\in[0,\ 1]\subset\mathbb{R}_+\right\}$, which is observed for some $w\in(0,\ 1)\subset\mathbb{R}_{++}$ as a rearrangement of $\bar{R}=\frac{r}{1-w}\in\mathbb{R}_+$, whereby and R is unique and in which $r\in\mathbb{R}_+$ is

[23] The reason for which normalisation equation $\sum_{i=1}^n l_i=1$ is not added to both kinds of system *en lieu* of 'Standard Commodity' equation $\sum_{j=1}^n\sum_{i=1}^n\kappa_i(q_{ij}-a_{ij})=1$ is that L is already present in the two kinds of system, whereby $(K\cdot)AP(1+r)+Lw=(K\cdot)QP$ and w is determined through the addition of $\sum_{j=1}^n\sum_{i=1}^n\kappa_i(q_{ij}-a_{ij})=1$ to both kinds of system, that is, w is measured in terms of the 'Standard Commodity' (i.e. numeraire).

chosen[24] and $w \in (0,\ 1) \subset \mathbb{R}_{++}$ is determined[25] together with $P \in \mathbb{R}^n_{++}$ in $\hat{A}\hat{P} \cdot (\mathbf{1} \hat{+} \mathbf{r}) + \hat{L}w = \hat{Q}\hat{P}$ and $\hat{K} \cdot \hat{A}\hat{P} \cdot (\mathbf{1} \hat{+} \mathbf{r}) + \hat{L}w = \hat{K} \cdot \hat{Q}\hat{P}$; for example, see[26] page 114 in Sinha [13].

Let $x_0 = w_0 = 0$ and $x = w,\ y = v \in (0,\ 1] \subset \mathbb{R}_{++}$. It follows that $x = w = 0 \vdash \bar{R} = r$, whose truth satisfies $\mathcal{P}(x_0)$, and that $x = w = 1 - \frac{r}{\bar{R}} \vdash \forall \delta_r \in \mathbb{R} \backslash \{0\},\ x' = x + \delta = w' = w + \delta = 1 - \frac{r}{\bar{R}} + \delta = 1 - \left(\frac{r + \delta_r}{\bar{R}}\right) = 1 - \frac{r'}{\bar{R}}$ for $\delta = \frac{-r}{\bar{R}}$, which satisfies[27] $\mathcal{P}(x) \vdash \mathcal{P}(x')$ for $x' \in B[x] := \{x,\ y \in \mathbb{R} :\ ||y - x|| \leq \varepsilon > 0\}$.

Thus, $\mathcal{P}(X) = \mathcal{P}([0,\ 1])$, whereby $\bar{R} = \frac{r}{1-w} \in \mathbb{R}_+$ for $r \in \mathbb{R}_+$ and $w \in [0,\ 1) \subset \mathbb{R}_+$; accordingly, $r = \bar{R}(1 - w) \in \mathbb{R}_+$ for $\bar{R} \in \mathbb{R}_+$ and $w \in [0,\ 1] \subset \mathbb{R}_+$. *QED*

Not only does Proposition 3 derive Sraffa's Fundamental Equation by means of mathematical induction on the real number line but relative to Saccal [7] it more directly shows that it is the same across both kinds of system, under single and multiple production. The derivation of Sraffa's Fundamental Equation by means of mathematical induction on the real number line is also a deductive derivation of the same, as perhaps as yet unaccomplished (see footnote 1 on page 114 in Sinha [13]).

The first fruit of the derivation Sraffa's Fundamental Equation by means of mathematical induction is its derivation without resorting to the analytical elaboration of r, which for $n \geq 5$ and undetermined coefficients is notoriously impossible (i.e. Abel Ruffini Theorem); in other words, *ex ante,* although the coefficients (i.e. rescaling factors) may be suitably determined for the existence and unicity of the 'Standard System' (see Dupertuis and Sinha [2]) and although r may be analytically elaborated for some $n \geq 5$ cases it is not in general and, *ex post,* although the coefficients may be suitably determined for the existence and unicity of the 'Standard System' by having first determined r (see Proposition 1) for cases $n \geq 5$ they are not in general, being two problems which the derivation of Sraffa's Fundamental Equation by means of mathematical induction circumvents, to the end of its very derivation.

The derivation of Sraffa's Fundamental Equation is additionally the same for single production, whereby expanded 'Real System' $\hat{A}\hat{P} \cdot (\mathbf{1} \hat{+} \mathbf{r}) + \hat{L}w = \hat{Q} \cdot \hat{P}$ and expanded 'Standard System' $K \cdot \hat{A}\hat{P} \cdot (\mathbf{1} \hat{+} \mathbf{r}) + \hat{L}w = K \cdot \hat{Q} \cdot \hat{P}$. Notice however that the linchpin of the equivalence of Sraffa's Fundamental Equation across both kinds of system is the derivation of the 'Standard Commodity', which hinges itself on the existence of the 'Standard System'.

3. Conclusion

Relative to the germane academic literature in this work I offered alternative and more direct proofs for (i) the existence and unicity of the 'Standard System', (ii) the (existence and) unicity of $R = r_i$ (for $w = 0$) across both kinds of system and (iii) the existence (and unicity) of $r = R(1 - w)$ across both kinds of system. While the proof for (iii) be outrightly unprecedented and that for (ii) certainly shorter, the proof for (i) is not necessarily superior to those of the germane academic literature, which judgement is left open for debate.

[24] $r \in \mathbb{R}_+$ is varied independently of its unique determination for the existence and unicity of the 'Standard System' (i.e. $r \in (0,\ \infty) = \mathbb{R}_{++}$ for the elaboration of a unique $K \in \mathbb{R}^n_{++}$), as in both kinds of system it is varied for their expanded counterparts.

[25] $P \in \mathbb{R}^n_{++}$ is indivisibly determined as a vector and not in terms of its individual elements p_j, for prices are determined simultaneously, whereby the loss of one commodity incapacitates all production. P and w are determined thus: $\hat{A}\hat{P} \cdot (\mathbf{1} \hat{+} \mathbf{r}) + \hat{L}w = \hat{Q}\hat{P} \longleftrightarrow \hat{A}D(\mathbf{1} \hat{+} \mathbf{r})\hat{P} + \hat{L}w = \hat{Q}\hat{P} \longrightarrow [\hat{Q} - \hat{A}D(\mathbf{1} \hat{+} \mathbf{r})]\hat{P} = \hat{L}w \longrightarrow w^{-1}\hat{P} = [\hat{Q} - \hat{A}D(\mathbf{1} \hat{+} \mathbf{r})]^{-1}\hat{L}$, in which $D(\cdot)$ is a diagonal matrix and w is the inverse of the last element of $w^{-1}\hat{P}$; a necessary and sufficient condition for a solution, indeed unique, is $det[\hat{Q} - \hat{A}D(\mathbf{1} \hat{+} \mathbf{r})] \neq 0$, which is notably different from $det[Q - (1 + r)A^\top] = 0$ as necessary, but insufficient, for the existence of the 'Standard System'.

[26] While the observation of $\bar{R} = \frac{r}{1-w} \in \mathbb{R}_+$ may occur for both kinds of system it is typically observed for the expanded 'Standard System', being that which Sraffa [15] intended, owing to the fact that "the distribution of the surplus must be determined through the same mechanism and at the same time as are the prices of commodities" *(ibidem,* page 6). Since one is nonetheless a mere rescaling of the other, Sinha [13] on page 114 observes it for the expanded 'Real System'.

[27] It is even clearer in contrapositive terms: $\forall \delta_r \in \mathbb{R} \backslash \{0\},\ x' = x + \delta = w' = w + \delta \neq 1 - \frac{r}{\bar{R}} + \delta = 1 - \left(\frac{r+\delta_r}{\bar{R}}\right) = 1 - \frac{r'}{\bar{R}}$ for $\delta = \frac{-r}{\bar{R}} \vdash x = w \neq 1 - \frac{r}{\bar{R}}$, which satisfies $\neg\mathcal{P}(x') \vdash \neg\mathcal{P}(x)$ for $x' \in B[x] := \{x,\ y \in \mathbb{R} :\ ||y - x|| \leq \varepsilon > 0\}$. Notice that ℓ_p norm $||z||_{p \in [1,\ \infty) \underline{\cup} \{\infty\} = \bar{\mathbb{N}}_+} = (\langle z,\ z \rangle)^{\frac{1}{p}} \veebar \max\{|z|\} = |z| = \begin{cases} z = y - x,\ z = y - x \in \mathbb{R}_{++} \\ -z = x - y,\ z = y - x \in \mathbb{R}_{--} \end{cases}$; for $||z||_{p \in [1,\ \infty] = \bar{\mathbb{N}}_+}$ one therefore observes $\frac{r - r'}{\bar{R}} = \left(1 - \frac{r'}{\bar{R}}\right) - \left(1 - \frac{r}{\bar{R}}\right) = w' - w = x' - x \leq y - x = v - w = v - \left(1 - \frac{r}{\bar{R}}\right) = \frac{\bar{R}(v-1)+r}{\bar{R}} \leq \varepsilon$ or $\frac{r' - r}{\bar{R}} = \left(1 - \frac{r}{\bar{R}}\right) - \left(1 - \frac{r'}{\bar{R}}\right) = w - w' = x - x' \leq x - y = w - v = \left(1 - \frac{r}{\bar{R}}\right) - v = \frac{\bar{R}(1-v)-r}{\bar{R}} \leq \varepsilon$.

References

[1] Bapat R. B., Olesky D.D. and Van Den Driessche P. (1995) "Perron-Frobenius Theory for a Generalized Eigenproblem", Linear and Multilinear Algebra 40, 2:141-152.
[2] Dupertuis M.-S. and Sinha A. (2009) "Existence of the Standard System in the Multiple-Production Case: a Solution to the Manara Problem", Metroeconomica 60, 3:432-454.
[3] Lippi M. (2008) "Some Observations on Sraffa and Mathematical Proofs with an Appendix on Sraffa's Convergence Algorithm", in Chiodi G. and Ditta L. (2008) "Sraffa or an Alternative Economics", Palgrave Macmillan.
[4] Manara C. F. (1968) "Il Modello di Piero Sraffa per la Produzione Congiunta di Merci a Mezzo di Merci", L'Industria 1, 1:3-18.
[5] Mangasarian O. L. (1971) "Perron-Frobenius Properties of $Ax - \lambda Bx$", Journal of Mathematical Analysis and Applications 36, 1:86-102.
[6] Miyamoto J. (2008) "Correction of Sraffa's Imaginary Experiment", Kwansei Gakuin University Repository.
[7] Saccal A. (2023) "An Alternative Derivation of Sraffa's Fundamental Equation with Applications", Paper.
[8] Saccal A. (2023) "From Sraffa's Fundamental Equation to the Sraffian Cycloid: a Note", Paper.
[9] Saccal A. (2023) "An Explanation of Distributional Gravity for the Sraffian Brachistochrone", Paper.
[10] Salvadori N. (2008) "Commentary by Neri Salvadori: on a Proof of Sraffa's", in Chiodi G. and Ditta L. (2008) "Sraffa or an Alternative Economics", Palgrave Macmillan.
[11] Schefold B. (1989) "Mr. Sraffa on Joint Production and Other Essays", Unwin Hyman.
[12] Schefold B. (2021) "Why Sraffa was Right not to Publish his Last Article: a Rectification of Yoann Verger", in Sinha A. (2021) "A Reflection on Sraffa's Revolution in Economic Theory", Palgrave Macmillan.
[13] Sinha A. (2022) "A Response to Pignalosa and Trabucchi and Perri and Oro", History of Economic Ideas 30, 3:107-131.
[14] Sinha A. and Verger Y. (2021) "A Response to Schefold's 'Why Sraffa was Right not to Publish his Last Article"', in Sinha A. (2021) "A Reflection on Sraffa's Revolution in Economic Theory", Palgrave Macmillan.
[15] Sraffa P. (1960) "Production of Commodities by Means of Commodities: Prelude to a Critique of Economic Theory", Cambridge University Press.
[16] Verger Y. (2021) "Sraffa and Manara: The Mystery of the Last Article of Piero Sraffa", in Sinha A. (2021) "A Reflection on Sraffa's Revolution in Economic Theory", Palgrave Macmillan.

Appendix

Julia commands for Proposition 1's algorithm (wherein # must replace %)

```
using LinearAlgebra

% A and Q are the respective input and output matrices of any Real System; they are ...
    non-zero and possibly reducible.
% They can be instantiated for the "counterexamples" of Manara (1968) and Schefold (2021).
% The upshot is that both are no longer counterexamples for the existence and unicity of ...
    the Standard System.

% 1) Manara (1968)
A=[1 1.1; 1.1 1];
Q=[1.09 1.144; 1.144 0.99];

% 2) Schefold (2021)
%A=[2/5 1; -1/5 3/5];
%Q=[1 0; 0 1];

% 3) Particular random case
%A=[0.307858 0.84353; 0.720777 0.838968];
%Q=[0.611711 0.561314; 0.849475 0.0438535];
```

```
% 4) Universal random case
%A=rand(2, 2)+rand(2, 2)*im;
%Q=rand(2, 2)+rand(2, 2)*im;

At=A'; % Compute transpose of A

F=schur(complex(Q), complex(At)); % Perform complex Schur decomposition
U=inv(F.T)*F.S; % Initialise U
counter=1;  % Initialise counter at 1

if isreal(diag(U)) % Check if diagonal elements of U are real
    println("Values of diag(U):", diag(U));  % Print diagonal elements of U
else
    G=schur(Q, At); % If not real then perform real Schur decomposition
    Qtl=real(F.Q*G.S*F.Z'); % Project Q's complex results to real numbers
    Attl=real(F.Q*G.T*F.Z'); % Project At's complex results to real numbers
    H=schur(complex(Qtl), complex(Attl)); % Perform new complex Schur decomposition on real ...
        matrices
    U=inv(H.T)*H.S; % Update U with the new complex Schur decomposition
while !isreal(diag(U)) % Iterate until diagonal elements of U be real
    counter+=1;  % Increment counter in each iteration by 1
    println("Iteration:", counter);  % Print iteration number
    println("Values of diag(U):", diag(U)); % Print diagonal elements of U
    G=schur(Qtl, Attl); % If not real then perform real Schur decomposition
    Qtl=real(H.Q*G.S*H.Z'); % Project Qtl's complex results to real numbers
    Attl=real(H.Q*G.T*H.Z'); % Project Attl's complex results to real numbers
    H=schur(complex(Qtl), complex(Attl)); % Perform new complex Schur decomposition on real ...
        matrices
    U=inv(H.T)*H.S; % Update U with the new complex Schur decomposition
end
end

return U % Display U

%Utl=[U[1]+(1.001-U[1]) U[3]+(0.001-U[3]); U[2]+(0.009-U[2]) U[4]+(0.005-U[4])]; % ...
    Construct Utl for Standard System
%E=eigen(Utl) % Display eigenvalues and eigevectors of Utl for Standard System
```