Monetary-Fiscal Forward Guidance

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Abstract

When central banks announce cuts to future interest rates, the expected costs of government debt service decrease, generating additional resources in future budgets. This paper demonstrates that if the rational-expectations assumption is dropped, fiscal authority can exploit those gains by spending them on future transfers and, by announcing those transfers to households today, can enhance the output effects of forward guidance. Employing a version of the New Keynesian setup featuring bounded rationality in the form of level-k thinking, I derive an analytical expression capturing the output effects of that additional fiscal announcement. Subsequently, a similar formula is derived in a tractable heterogeneous agent New Keynesian model with bounded rationality, uninsured idiosyncratic risk, and redistributive effects of transfers. Finally, I use these analytical insights to explore the effects of the forward-guidance-induced fiscal announcement in a fully-blown heterogeneous agent New Keynesian framework with level-k thinking calibrated to match US data. The findings suggest that fiscal communication can amplify the output effects of standard forward guidance by 66%. Moreover, those gains can reach 120% when the debt-to-GDP ratio doubles. This suggests that forward guidance enriched with fiscal announcements about future transfers can be considered an effective policy option when both monetary and fiscal policies are constrained, e.g., during liquidity trap episodes accompanied by high levels of public debt.

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1 Introduction

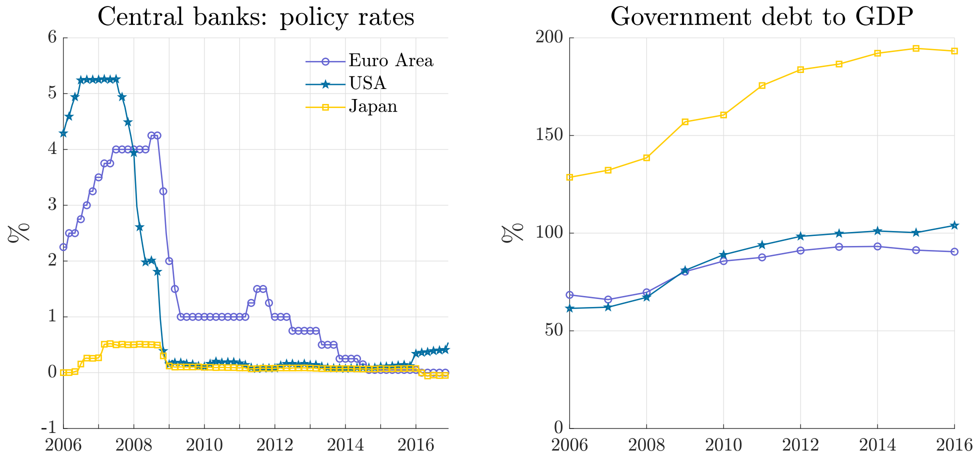
Forward guidance (FG) - a type of unconventional monetary policy based on promises about future interest rates - was initially considered useful for mitigating recessions in liquidity traps (see, e.g., Eggertsson and Woodford (2003)). The Great Recession of 2008-2009 and the Global Financial Crisis accompanied by nominal interest rates reaching the zero lower bound made central bankers turn their attention to FG which, as a consequence, became a standard element of the monetary toolkit (Blinder et al. (2017)).

When the central bank cuts future interest rates, future borrowers - including fiscal authority - gain additional resources in their budgets. I use this elementary monetary-fiscal interaction to construct a simple fiscal policy aimed at boosting the FG effects. In particular, I analyze the scenario under which fiscal authority exploits those resources on future transfers and, symmetrically to the central bank, it announces them to households today to stimulate their current consumption spending, which gives rise to the so-called fiscal forward guidance.[1] The FG extended by the fiscal announcement described above is henceforth referred to as monetary-fiscal forward guidance (MFFG).

Augmenting the standard FG with fiscal announcement has several desirable properties. First, the additional fiscal stimulus requires neither a rise in the level of government debt nor increase in taxes (either current or future). As such, it overcomes the problems associated with standard fiscal stimuli that are related to tax-adjustment costs, tight borrowing constraints or high sovereign spreads driven by a rise in debt. Second, the amount of additional fiscal resources generated by FG (and thus the aggregate value of transfers to households) increases in the amount of the maturing public debt. Therefore, MFFG can be viewed as a useful stabilization tool when both monetary and fiscal policies are constrained (by the ZLB and high

[1] I restrict the timing of the fiscal stimulus - it is deployed exactly in the same period as the future monetary shock. Naturally, one could think of alternative stimuli that are implemented sooner or earlier than the monetary policy shock and are still financed with additional resources generated by monetary easing (this is possible because if, e.g., the alternative stimulus occurs earlier, the government can issue additional debt to finance it and this debt is then repaid in the period of the monetary shock's arrival using the resources saved on debt service costs). Unlike the fiscal forward guidance, however, this would involve changes to the path of government debt. In particular, in the case of the current stimulus financed with a rise in debt, constraints imposed on the government by financial markets (e.g., debt crisis) may exlcude such a policy option.

Figure 1: Policy rates and debt to GDP ratios in the largest advanced economies in 2006-2016



Notes: the left panel displays policy rates (Fed funds rate, the BOJ call rate, and the rate of main refinancing operations of ECB).

government debt, respectively), which was the case during the Great Recession in the largest advanced economies (see Figure 1). Third, fiscal transfers directly affect household budget constraints and therefore this fiscal measure is well-understood by households.[2]

In this paper, I argue that dropping the rational-expectations assumption in favor of bounded rationality is key for fiscal forward guidance to affect economic allocation and, in particular, to have an impact on current output.[3] To see this, note that rational agents are aware of the mechanisms governing the economy and, in particular, they recognize that the FG announcement (about lower interest rates) relaxes fiscal constraints in the future. Thus, as it is shown formally in my article, the value of an additional information announced by the fiscal authority to rational households is zero and thus their consumption behavior remains unchanged. As a consequence, there is no difference between the effects of MFFG and FG when

[2]Note that it is not the case if an announcement about an alternative fiscal policy measure, i.e., government spending, is considered. It is because the effectiveness of the latter requires that households recognize the dependence of their incomes on government expenditures, which is not true if their rationality is bounded. More specifically, it can be inferred from the proof of Theorem 1 that when bounded rationality takes the form of level-k thinking, there is no difference between MFFG and FG if fiscal authority announces changes to future government spending.

[3]I consider transfers instead of government spending because the latter may feature zero effects if agents' rationality is severly limited (see Bianchi-Vimercati et al. (2021)).

agents are rational.[4] By contrast, if households feature bounded rationality, they are not (fully) aware of the impact of changes to policy rates on the fiscal budget and thus fiscal forward guidance is able to influence their choices by enriching their current information sets.

Importantly, bounded rationality assumed in this paper is a widely-accepted ingredient in macroeconomic models studying the effects of announcements about future policy actions. In particular, the departure from rational expectations allows for obtaining plausible predictions about the FG effectiveness in models, that may otheriwse exhibit explosive dynamics in response to future interest rates (the so-called "FG puzzle").[5]

The second premise is that

The main goals of this paper are the theoretical analysis of the MFFG effects and their quantitative assessment. To this end, I follow the exposition strategy applied in a related paper by Farhi and Werning (2019) and use a sequence of tightly-related models of ascending complexity. Starting with the standard representative agent New Keynesian model (RANK) extended with level-k thinking, I derive a closed-form expression for the MFFG effectiveness measured with the elasticity of output in period 0 with respect to changes to future interest rates (i.e., at time $\tau > 0$). Moreover, to isolate the effects of fiscal forward guidance, I present an analytical formula describing the difference between output elasticities under MFFG and FG. Most importantly, I find that it if the rationality of agents is sufficiently low ($k \leq \tau$), the fiscal announcement induced by FG becomes non-neutral, i.e., it affects current output. Moreover, while the effectiveness of MFFG relies solely on the mechanism of intertemporal substitution scaled by a factor related to cognitive constraints of agents, the difference between output elasticities under MFFG and FG depends on the severity of bounded rationality, the amount of maturing government debt, and

[4]Note that the rational-expectations assumption implies a deep asymmetry in the abilities of the central bank and fiscal authority to communicate future policies when the fiscal guidance (coordinated with the standard FG) is considered.

[5]Numerous papers sought for estimating the FG effects and, starting from Del Negro et al. (2023), found that a workhorse model for monetary policy analysis - the standard New Keynesian setup - generates explosive dynamics of output and inflation in response to the central bank's promise to cut policy rates below the natural interest rate in the future. This unrealistic feature of the representative agent New Keynesian model was often attributed to the rational-expectations assumption (see seminal contributions by Angeletos and Lian (2018), Farhi and Werning (2019), Garcia-Schmidt and Woodford (2019), and Gabaix (2020), among others).

marginal propensisties to consume.

In general, RANK turns out to be a useful starting point for the analysis of MFFG and the effectivenes of the fiscal announcement. By definition, however, it is unable to capture the redistributive effects of transfers. To address this point and to preserve the tractability of the analysis, I extend RANK with level-k thinking by introducing incomplete insurance markets and idiosyncratic income risk. I then use the resulting framework (the so-called tractable heterogeneous agent New Keynesian model - THANK, see Bilbiie (2019)) for deriving analytical formulas describing the analogous objects as those obtained in RANK. The transmission of MFFG is considerably more complex than in RANK: in addition to the intertemporal substitution channel, the elasticity of output in period 0 with respect to interest rates at τ depends on changes to interest earnings in τ, the redistributive effects of transfers in τ and their impact on aggregate demand that influences output in τ. As in RANK, all those terms are scaled with terms describing the bounded rationality friction. By contrast, the term describing the difference between MFFG and FG effectiveness is very similar to its RANK counterpart.[6]

Equipped with an organizing framework provided by RANK and THANK, I use a tightly related heterogeneous agent New Keynesian (HANK) model, calibrated to match the moments characterizing the US economy, to quantify the output elasticities with respect to future interest rate shocks under MFFG and calculate the effectiveness of fiscal forward guidance (given by the difference in output elasticities between MFFG and FG). In the benchmark simulation featuring debt to GDP equal to 55%, I find that the additional fiscal announcement raises the FG effectiveness measured with the interest rate elasticity of output by up to 66% (for transfers targeted towards low-income earners). Importantly, I find that the additional fiscal announcement is most effective when agents' rationality is severely bounded, which are exactly the circumstances for which the standard FG is least effective and thus improving its effectiveness is most desired (see Farhi and Werning (2019)). Moreover, I consider an additional scenario under which debt to GDP doubles when compared to the benchmark and I find that fiscal announcement raises the FG effectiveness by 120%. This corroborates the intuition described when discussing the properties of fiscal forward guidance above: the effectiveness of the fiscal announcement increases

[6] It depends on bounded rationality, the amount of maturing government debt, and marginal propensisties to consume. Additionally, it is non-zero only if $k \leq \tau$.

in public debt. Therefore, the MFFG can be viewed as a policy option for economies facing the ZLB and high public debt.

The rest of the paper is organized as follows. Section 2 discusses the related literature. In Section 3, I derive the interest rate elasticity of output related to MFFG in the RANK model. Section 4 displays an analogous outcome for THANK. Section 5 quantifies MFFG in the calibrated HANK model. Section 6 concludes.

2 Literature

Eggertsson and Woodford (2003) use the RANK model and analyze the role of FG in mitigating recessions in the economy facing a liquidity trap. They find that the monetary authority is able to eliminate the recession if it commits to holding nominal interest rates equal to zero for some additional period of time after the economic crisis. As shown by Del Negro et al. (2023), however, such extensions of low interest rates may give rise to explosive dynamics of output and inflation - the so-called "FG puzzle". Numerous researchers searched for realistic extensions of the RANK model able to generate more realistic predictions of the FG effects. In the seminal paper, McKay et al. (2016) extend RANK by incorporating uninsured idiosyncratic income risk (incomplete markets) and find that this additional ingredient resolves the "FG puzzle", i.e., the possibility of occasionally binding borrowing constraints significantly lower the elasticity of aggregate demand to future interest rates. This conclusion was undermined by Werning (2015) and Hagedorn et al. (2019), who showed that the "FG puzzle" may dissappear, persist or even get aggrevated under incomplete markets, depending on the income redistribution scheme.[7] Motivated by those findings, Farhi and Werning (2019) analyze the role of the interactions between incomplete markets and bounded rationality in mitigating the responsiveness of current output to future interest rates. They find that, if both frictions are in place, the FG effectiveness is considerably constrained when compared to RANK. Moreover, Angeletos and Lian (2018), Garcia-Schmidt and Woodford (2019), and Gabaix (2020) study the model with complete markets and find that bounded ratio-

[7] Relatedly, Bilbiie (2019) and Bilbiie (2020) use THANK and argue that eliminating the "FG puzzle" requires that the income share of high MPC agents is countercyclical. Additionally, Acharya and Dogra (2020) construct the PRANK model (i.e., Pseudo RANK) and demonstrate that the cyclicality of income risk determines whether idiosyncratic income risk resolves the "FG puzzle".

nality alone helps to resolve New Keynesian anomalies, including the "FG puzzle".[8] Additionally, Iovino and Sergeyev (2023) use the model with bounded rationality to analyze the macroeconomic effects of quantitative easing.

Alternative resolutions to the "FG puzzle" include Campbell et al. (2019) and Michaillat and Saez (2021). First of them develops the model of imperfect imperfect central bank communications. By contrast, I follow Farhi and Werning (2019) by analyzing credible monetary and fiscal policy announcements where the first of them is fully incorporated into the expected yield curve. Moreover, I assume that households and firms have to form indirect expectations about other forward-looking variables crucial for agents' decisions (e.g., future income levels). Similarly to Farhi and Werning (2019), I use level-k thinking to model this dichotomy between directly and indirectly observed future variables. To resolve the New Keynesian anomalies, Michaillat and Saez (2021) incorporate wealth into the utility function. Such social-status considerations are absent in my work.[9]

Auclert et al. (2020) and Dobrew et al. (2023a) use models with bounded rationality and incomplete markets to analyze the effects of conventional monetary policy. In particular, Auclert et al. (2020) use the HANK model with sticky household expectations and highlight a central role of investment for the propagation of monetary policy shocks. Dobrew et al. (2023a) find that market incompleteness is not an important determinant of the effectiveness of make-up strategies. They draw opposite conclusions for the significance of bounded rationality.

Angeletos and Sastry (2021) discuss the dilemma whether central banks should use forward guidance about the policy instrument (nominal interest rates) or announce targets for economic outcomes (e.g., unemployment) and show that the latter communication becomes more desirable if general equilibrium (GE) feedbacks are powerful enough. By contrast, my paper concentrates on the coordination of announcements rather than on the choice between them. Woodford and Xie (2019) and Woodford and Xie (2022) study the role of transfers in a liquidity trap using

[8]Dobrew et al. (2023b) use the framework developed by Gabaix (2020) to compare the effectiveness of inflation targeting and history-dependent monetary rules under ZLB. Similar problems are studied by Dupraz et al. (2022) in the model developed by Garcia-Schmidt and Woodford (2019). Additionally, Dupraz and Marx (2023) apply the framework by Garcia-Schmidt and Woodford (2019) to study the anchoring of inflation expectations.

[9]Note that models with wealth in utility function feature rational expectations and, by construction, the effects of the fiscal forward guidance discussed in this paper are zero.

a representative model with bounded rationality (captured with a limited foresight of agents) and argue that the resulting failure of the Ricardian equivalence implies that the analyzed fiscal policy can stimulate output. They abstract, however, from the additional fiscal policy announcement induced by FG and from household heterogeneity that are investigated in my paper.

Intuitively, bounded rationality influences the extent to which agents understand the equilibrium effects of policy interventions. In that context, Angeletos and Lian (2018) and Farhi et al. (2020) study the size of the government-spending multiplier in economies with cognitive limitations and Bianchi-Vimercati et al. (2021) compare the effectiveness of stimuli based on either higher government spending or tax cuts and conclude that the latter exhibit larger output effects when the level of cognitive sophistication is low.

The impact of incomplete markets on the conduct of fiscal policy has been recently studied by Angeletos et al. (2023) who explore the possibility of self-financing deficits. Additionally, Wolf (2021) uses an analytical model with occasionally-binding borrowing constraints to establish an equivalence between interest rate cuts and stimulus payments (that take a form of uniform transfers).

This work is related to papers studying the role of fiscal policy at the ZLB. Seminal contributions by Woodford (2011), Eggertsson (2011), Christiano et al. (2011), and Rendahl (2016) use the standard RANK model and find that government spending multipliers are potentially large in liquidity traps. This policy, however, requires a rise in taxes or government debt to finance additional government expenditures. By contrast, the policy studied in my paper does not increase the levels of these fiscal instruments (current or future), which is potentially desirable when fiscal policy is constrained by either tax-adjustment costs or borrowing constraints. A related strand of literature studies the role of the interactions between monetary policy and public debt. Rachel and Summers (2019) argue that higher levels of public debt help to avoid ZLB episodes and Bhattarai et al. (2023) find that QE, which lowers the maturity of government bonds, is effective in liquidity traps because it generates expectations about low interest rates in the future. Both mechanisms are different from the one analyzed in my article. Billi and Walsh (2022) evaluate the role of super-active fiscal policy rules (involving substantial rises in government debt in adverse economic conditions) in the standard New Keynesian model with occasionally-binding ZLB. By contrast, the MFFG discussed in this paper keeps the path of public debt unchanged. Gali (2020) studies a money-financed fiscal stimulus as a policy option in a depressed economy that relies neither on lower nominal interest rates (which is constrained at the ZLB) nor on rises in government debt (which is not feasible when its level is high or when the economy is facing a debt crisis). Thus, MFFG studied in this paper can be viewed as an alternative to the stimulus analyzed in Gali (2020).

There are empirical works corroborating the impact of policy announcements on current household spending. As for monetary policy, Coibion et al. (2023) study the impact of FG communication on households' expectations about future inflation, mortgage rates and unemployment rate. They find that changes to perceived real interest rates driven by FG lead to shifts in household spending on durable goods and document horizon effects (i.e. lower responsiveness of households to policy announcments about more distant interest rate changes). Agarwal and Qian (2014) find that household consumption rises considerably after fiscal policy announcements about transfers and document significant anticipation effects captured by the consumer response financed via credit cards.

It seems that the closest paper to mine is Farhi and Werning (2019). My work can be viewed as a modification of their analysis that adds positive public debt and explores the resulting monetary-fiscal interactions. I also follow their exposition strategy that is based on a set of models starting from those allowing for analytical characterizations and leading to more realistic quantitative setups that can be solved only numerically.[10]

3 Monetary-fiscal forward guidance in RANK

I start by analyzing the effects of MFFG in the standard New Keynesian setup. Despite its simplicity, this parsimonious model offers important insights into the determinants of fiscal forward guidance efficacy that remain valid in the model with incomplete insurance markets.

[10] In the analytical part, I develop the THANK model (instead of the continuous-time OLG analyzed in Farhi and Werning (2019)) featuring level-k thinking to give rise to the aggregate demand channel of transfers (see Oh and Reis (2012)) that would be otherwise absent in the continuous-time OLG model (because, by construction, the measure of the constrained, high-MPC agents is zero).

3.1 Environment

Time is infinite and divided into discrete subperiods indexed with $t \in \{0, 1, 2, ...\}$. The demand block consists of forward-looking identical households (consumers) of measure one. It is assumed that prices are perfectly rigid and thus ouptut is demand-driven (see, e.g., Angeletos and Lian (2018), Bilbiie (2019), Farhi and Werning (2019), Auclert et al. (2023b)). From the Fisher equation, this assumption can be alternatively interpreted as a situation when the central bank is able to control real interest rates. The government consists of two branches: monetary and fiscal authority. There are two markets: for consumption goods and assets (government bonds). I introduce bounded rationality by assuming that consumers form expectations using level-k thinking.

3.2 Households

Consumers discount future utility streams with discount factor $\beta \in (0, 1)$ and it is assumed that the instantaneous utility function u takes the following form:

$$u(c) = \frac{c^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}}$$

where $\sigma > 0$ is the intertemporal elasticity of substitution.

Household maximizes lifetime utility given by:

$$\max_{\{c_t, b_{t+1}\}_{t=0}^{+\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \cdot u(c_t) \tag{1}$$

subject to a sequence of budget constraints:

$$\forall_{t \geq 0} \; c_t + b_{t+1} - (1-\theta) \cdot b_t = R_t \cdot \theta \cdot b_t - T_t + Tr_t + Y_t$$

where c_t is consumption in period t, b_t are asset (bond) holdings accumulated in $t-1$, R_t is the nominal return on bonds (that equals to the real return under perfectly rigid prices), θ denotes the maturity of bonds (where $\theta \in [0, 1]$ and where $\theta = 1$ corresponds to one-period bonds). Moreover, to highlight the role of the FG-induced fiscal transfer, I introduce a distinction between taxes T_t and transfers Tr_t: the latter are financed by additional fiscal gains generated by the monetary shock while the former finance the usual (i.e., steady-state) debt service costs of the government. Additionally, by Y_t I denote the pre-tax income and by Υ_t I denote the after-tax income, i.e.:

$$\Upsilon_t \equiv -T_t + Tr_t + Y_t.$$

3.3 Monetary authority

I consider monetary policy scenarios analyzed in Farhi and Werning (2019): monetary authority keeps R_t at the constant level that equals to its steady state value R (steady-state value of nominal interest rate) and it deviates from this strategy only once, i.e., in period τ.[11] This period can be thought of as the time when the policy change (announced in period 0) materializes. More formally, the monetary rule is described by:

$$R_t = \begin{cases} R & \text{if } t \neq \tau, \\ R + dR & \text{if } t = \tau. \end{cases} \tag{2}$$

In what follows, I calculate the elasticities of output in period 0 with respect to the announced monetary policy changes for $\tau \in \{1, 2, ...\}$ where $\tau = 1$ corresponds to the standard monetary policy shock.

3.4 Fiscal authority

Fiscal budget constraint is:

$$\forall_{t \geq 0} \, T_t + B_{t+1} - (1 - \theta) \cdot B_t = Tr_t + R_t \cdot \theta \cdot B_t$$

where B_t is the level of aggregate government debt.

I assume the following fiscal rules: first, fiscal authority keeps the level of public debt unchanged:

$$\forall_{t \geq 0} \, B_t = \bar{B}$$

where $\bar{B} > 0$ is a parameter, which implies that MFFG is neutral for the aggregate stock of government bonds. Transfers are financed with additional resources

[11] I follow the convention that variables without time subscripts denote their values in the stationary equilibrium/steady state.

generated by a change to the monetary policy rate:

$$\forall_{t\geq 0}\, Tr_t = -\left(R_t - R\right)\cdot\theta\cdot\bar{B}$$

and taxes cover debt-service costs in the steady state:

$$\forall_{t\geq 0}\, T_t = T = \left(R - 1\right)\cdot\theta\cdot\bar{B}.$$

3.5 Market clearing

The resource constraint for the market of goods is:

$$c_t = Y_t$$

and the market clearing condition for assets is:

$$b_t = B_t.$$

Both of them are satisfied for all $t \geq 0$.

3.6 Consumption function

Before formulating the equilibrium defimition, it is useful to define the consumption function (see Kaplan et al. (2018), Farhi and Werning (2019), Auclert et al. (2023b), among others).

In particular, in the forward-looking RANK model, the solution to the household problem (1) takes the following form (note that T_t is constant over time and thus can be omitted as an argument of the consumption function):

$$\forall_{t\geq 0}\, c_t = C\left(R_t, Y_t, Tr_t, \left\{R_{t+m}, Y_{t+m}, Tr_{t+m}\right\}_{m>0}\right).$$

Note that under MFFG, the paths of transfers and interest rates are directly observed by households. By contrast, households have to form expectations about future values of output. As discussed, I use level-k thinking when modeling that process. By contrast, under the standard FG, only interest rates are observed directly and future values of both output and transfers are determined by level-k thinking.

3.7 Equilibrium

We are in a position to define the equilibrium in the RANK model with level-k thinking:

Definition. The equilibrium in RANK with level-k thinking under MFFG is $\left\{Y_t^k\right\}_{t\geq 0}$ such that for each $t \geq 0$, given $\{R_{t+m}, Tr_{t+m}\}_{m\geq 0}$ and for all $k \geq 1$:

$$\forall_{t\geq 0}\, Y_t^k = C\left(R_t, Y_t^k, Tr_t, \left\{R_{t+m}, Y_{t+m}^{k-1}, Tr_{t+m}\right\}_{m>0}\right),$$

where $\{Y_t^0\}_{t\geq 0} = \{Y\}_{t\geq 0}$ (i.e. level-0 expectations correspond to the steady state equilibrium) such that market clearing conditions, monetary and fiscal rules hold. Under the FG, for each $t \geq 0$ in the equation above: $\{Tr_{t+m}\}_{m>0}$ is replaced with $\left\{Tr_{t+m}^{k-1}\right\}_{m>0}$, Tr_t is replaced with Tr_t^k and, additionally, it is assumed that $\left\{Tr_{t+m}^0\right\}_{m>0} = \{0\}_{m>0}$.

3.8 Interest rate output elasticities in general equilibrium

This subsection presents the main results for the RANK model: the output elasticity in period 0 with respect to the MFFG communication about monetary shock and fiscal transfer in period τ and the difference between elasticities under MFFG and FG that captures the impact of fiscal forward guidance (see Theorem 1).

Before presenting those outcomes, it is useful to characterize the optimal response of household consumption in period 0 (in partial equilibirum) to sequences of infinitesimal deviations of interest rates and after-tax income levels $\{dR_t, d\Upsilon_t\}_{t\geq 0}$ from their steady-state values (for analogous characterizations in the RANK model see Angeletos and Lian (2018) and Angeletos and Sastry (2021)):

Lemma 1. *Under the optimal behavior of households, dc_0 implied by sequences $\{dR_t, d\Upsilon_t\}_{t\geq 0}$ satisfies:*

$$dc_0 = \sum_{t=1}^{\infty} \beta^t \cdot \left[-\beta \cdot \theta \cdot c \cdot \sigma \cdot dR_t + \frac{1-\beta}{\beta} \cdot \left(\theta \cdot \bar{B} \cdot dR_{t-1} + d\Upsilon_{t-1}\right)\right].$$

All proofs are delegated to the Appendix. Note that the equation in Lemma 1 is a linearized version of the non-linear equation that characterizes aggregate consumption function in Farhi and Werning (2019).[12] I use the linearized formulation because it allows for obtaining analytical results in general equilibrium not only in RANK but also when analyzing the THANK model in Section 4.

Note that Lemma 1 implies that the marginal propensity to consume (MPC) in RANK reads:

$$MPC \equiv \frac{dc_0}{dY_0} = 1 - \beta \tag{3}$$

Let us denote the intertemporal marginal propensity to consume (iMPC, see Auclert et al. (2023b)) with respect to a one-period-ahead income shock as $iMPC$. From Lemma 1:

$$iMPC \equiv \frac{dc_0}{dY_1} = \beta \cdot (1 - \beta)\,. \tag{4}$$

Before moving to Theorem 1, let us introduce several useful definitions.

By $\bar{R}$, I denote the effective rate on assets in the steady state:

$$\bar{R} \equiv R \cdot \theta + 1 - \theta$$

where $R = \bar{R}$ in the economy with one-period debt (i.e., when $\theta = 1$).

The elasticity of output in period 0 with respect to a monetary policy shock in period τ under level-k thinking of order k and under MFFG is denoted by:

$$\epsilon(\tau, k) \equiv -\frac{\bar{R}}{Y} \cdot \frac{dY(\tau, k)}{dR} \tag{5}$$

where $dY(\tau, k)$ is the deviation of output in period 0 from its steady-state value resulting from a one-time monetary shock (and the induced fiscal transfer) in period τ. The analogous object for the equilibrium with FG is denoted by $\hat{\epsilon}(\tau, k)$. Moreover, the difference between output elasticities under MFFG and FG is denoted by $\Delta\epsilon(\tau, k)$:

$$\Delta\epsilon(\tau, k) \equiv \epsilon(\tau, k) - \hat{\epsilon}(\tau, k)\,.$$

Finally, let us denote by F the cumulative distribution function of the binominal dis-

[12]See p. 3905 in their paper.

tribution (describing the probability of obtaining $k-1$ successes in $\tau-1$ independent trials with success probability $1-\beta$):

$$F\left(k-1|\tau-1,1-\beta\right)=\sum_{l=0}^{k-1}\left(\begin{array}{c}\tau-1\\ l\end{array}\right)\cdot\beta^{\tau-l-1}\cdot\left(1-\beta\right)^{l}$$

for $k\in\{1,2,...,\tau\}$. Its value is equal to 0 for $k<1$ and 1 for $k>\tau$.

The corresponding probability mass function is:

$$f\left(k-1|\tau-1,1-\beta\right)=\begin{cases}\left(\begin{array}{c}\tau-1\\ k-1\end{array}\right)\cdot\beta^{\tau-k}\cdot\left(1-\beta\right)^{k-1} & \text{for } k\in\{1,2,...,\tau\}\\ 0 & \text{otherwise}\end{cases}$$

We can now formulate the main result for the RANK economy:

Theorem 1. *Consider a monetary policy shock in period $\tau>0$ in the RANK model under level-k thinking of order k and under the MFFG. We have:*

$$\epsilon\left(\tau,k\right)=F\left(k-1|\tau-1,1-\beta\right)\cdot\theta\cdot\sigma$$

and the impact of fiscal forward guidance is:

$$\Delta\epsilon\left(\tau,k\right)=f\left(k-1|\tau-1,1-\beta\right)\cdot\frac{iMPC}{1-MPC}\cdot\frac{\bar{R}\cdot\bar{B}\cdot\theta}{c}.$$

First, note that as Theorem 1 shows, bounded rationality is crucial for the additional fiscal announcement to have real effects. Indeed, if $k>\tau$ then $f=0$ and thus $\Delta\epsilon\left(\tau,k\right)=0$, i.e., there is no difference in the output reaction between MFFG and FG. By contrast, $\Delta\epsilon\left(\tau,k\right)$ is positive for $k\leq\tau$, i.e., if agents' cognitive frictions are sufficiently severe. As we shall see, the neutrality of fiscal forward guidance for $k>\tau$ continues to hold in THANK and it is almost exactly satisfied in the quantitative HANK model. The neutrality of the additional fiscal announcement in RANK for $k>\tau$ follows from the fact that sufficiently rational agents recognize that the additional fiscal transfer in period τ exactly offsets the change to interest earnings in that period and thus fiscal forward guidance is neutral for their consumption behavior.

Second, notice that $k > \tau$ implies $F = 1$, which corresponds to the Rational Expectations Equilibrium (REE), i.e.:

$$\epsilon(\tau, k) = \theta \cdot \sigma.$$

This means that the effectiveness of policy MFFG does not depend on horizon τ (i.e., the model features the "FG puzzle").

Third, observe that the effectiveness of MFFG (measured with $\epsilon(\tau, k)$) is an increasing function of k for $k \geq 1$ (F is a cdf and, as such, grows with k). To put it differently, MFFG is more effective when agents become more rational. By contrast, the impact of k on the effectivenes of fiscal forward guidance $\Delta\epsilon(\tau, k)$ is ambiguous. The reasons underlying this fact have an economic interpretation that can be illustrated using the equation that follows from the proof of Theorem 1:

$$\Delta\epsilon(\tau, k) = \sum_{t=1}^{\tau-k+1} \beta^t \cdot \frac{1-\beta}{\beta} \cdot \Delta\epsilon(\tau - t, k - 1). \tag{6}$$

The interpretation of the equation above follows from Lemma 1: the output effects of fiscal forward guidance of horizon τ under bounded rationality featuring level k equal the impact of changes to income levels between periods 1 and $\tau - k + 1$ on aggregate demand in period 0. Those changes to income levels, in turn, are driven by fiscal forward guidance and the expectations about them are formulated by agents using level-k thinking and thus they are equal to output effects of fiscal forward guidance computed in the $k-1$-th iteration of the equilibrium computation process. This, coupled with the fact that the model is forward-looking and features no aggregate state variables, means that these income changes can be expressed as: $\Delta\epsilon(\tau - 1, k - 1)$ for period 1, $\Delta\epsilon(\tau - 2, k - 1)$ for period 2, etc. The upper bound on the sum in equation (6) follows from the fact that:

$$\Delta\epsilon(k-2, k-1) = \Delta\epsilon(k-3, k-1) = ... = 0$$

as shown in Theorem 1. We have now the following trade-off affecting the monotonicity of $f(k-1|\tau-1, 1-\beta)$ in k (and thus the monotonicity of $\Delta\epsilon(\tau, k)$). On the one hand, as agents become more rational, they value the information conveyed by fiscal forward guidance less - they are sufficiently smart and realize that monetary policy shock at τ affects the constraint of the fiscal authority in that period, which induces transfers to households. This effect is captured with the number of elements of sum in equation (6) that decreases with k. On the other hand, however, more rational agents recognize that any change to their income at τ affects the economy in all previous periods down to period 0 which, in turn, magnifies the reaction of aggregate demand (and output due to the assumed perfectly rigid prices) in period 0. Thus the cost of the missing information (about fiscal transfers that stimulate the economy between 0 and τ) tends to increase with k. For instance, consider a rise of k from 1 to 2. For $k = 1$ we have:

$$\Delta\epsilon\left(\tau,1\right) = \sum_{t=1}^{\tau} \beta^t \cdot \frac{1-\beta}{\beta} \cdot \Delta\epsilon\left(\tau - t, 0\right) = \beta^{\tau-1} \cdot \theta \cdot \sigma \tag{7}$$

i.e., the sum collecting the income effects consists of one element only - i.e., the impact of fiscal forward guidance on the expectations about receiving additional income (i.e., fiscal transfer) in period τ. For $k = 2$ we have:

$$\Delta\epsilon\left(\tau,2\right) = \sum_{t=1}^{\tau-1} \beta^t \cdot \frac{1-\beta}{\beta} \cdot \Delta\epsilon\left(\tau - t, 1\right)$$

$$= \sum_{t=1}^{\tau-1} \beta^t \cdot \frac{1-\beta}{\beta} \cdot \beta^{\tau-t-1} \cdot \theta \cdot \sigma$$

where I used equation (7). In other words, if $k = 2$ then the sum in condition (6) consists of $\tau - 1$ elements which captures the fact that, in contrast to $k = 1$, the agents who feature $k = 2$ start recognizing the impact of general equilibrium effects of fiscal forward guidance on the path of their future incomes which, in turn, affects consumption and output at $t = 0$.

Intuitively, as Theorem 1 shows, both $\Delta\epsilon$ and ϵ increase with the amount $\frac{\bar{R}\cdot\bar{B}\cdot\theta}{c}$ of standardized maturing debt, which indicates that MFFG can be useful in liquidity traps in economies facing large amount of public debt.

Note that Theorem 1 indicates that the relative rise in the effectiveness of MFFG when compared to FG, which is given by:

$$\frac{\Delta\epsilon\left(\tau,k\right)}{\hat{\epsilon}\left(\tau,k\right)} = \frac{\Delta\epsilon\left(\tau,k\right)}{\epsilon\left(\tau,k\right) - \Delta\epsilon\left(\tau,k\right)},$$

is independent of debt maturity θ, which occurs because θ affects the effectiveness of both FG (denominator) and fiscal forward guidance (numerator).

Finally, let us relate Theorem 1 to Proposition 2 in Farhi and Werning (2019). Note that under the MFFG and $\theta = 1$ the first formula in Theorem 1 boils down to the one obtained in their paper because:

$$F\left(k-1|\tau^{FW}, 1-\beta\right) \cdot \sigma = \frac{\sum_{m=0}^{k-1}\left(\bar{R}-1\right) \cdot \sum_{s_0=0}^{\tau^{FW}-1} \cdot \sum_{s_1=0}^{\tau^{FW}-1-s_0} \cdot \ldots \cdot \sum_{s_{m-1}=0}^{\tau^{FW}-1-s_{m-2}} 1}{\bar{R}^{\tau^{FW}}}$$

where $\tau^{FW} = \tau - 1$ is the timing convention followed in their paper. This implies that MFFG (and not FG) in the model with government debt is a direct counterpart of FG in the model without public debt (analyzed in Farhi and Werning (2019)).

4 Monetary-fiscal forward guidance in THANK

This section repeats the exercises conducted in the RANK model in a framework extended to capture household heterogeneity and precautionary motives. This extension is aimed at studying the dependence of the MFFG effects on various types of transfers (e.g., uniform, targeted) announced by fiscal authority together with FG communicated by the central bank. By construction, such an analysis is impossible in RANK. More specifically, I extend the heterogeneous agent setup introduced by Bilbiie (2019) by introducting level-k thinking. I use his framework as a starting point for my investigation because it allows for preserving analytical tractability in a model featuring income heterogeneity, uninsured idiosyncratic income risk and positive government liquidity.[13] As such, it also serves as a useful device that bridges the analyses of MFFG in RANK and HANK (conducted in Sections 3 and 5, respectively).

[13]The last feature becomes possible because of the 'island' structure of the Bilbiie's model. All this means that I diverge from the exposition strategy by Farhi and Werning (2019): i.e., instead of analyzing a tractable OLG in continuous time to give rise to the notion of incomplete markets, I use the framework by Bilbiie (2019) which, additionally, allows for capturing precautionary motives. An additional motivation for this deviation is that in the continuous-time OLG model the mass of constrained agents is zero which annihilates the effects of transfers targeted to those agents.

4.1 Environment

The main difference when compared to the RANK model is the demand block that is taken from the paper by Bilbiie (2019). Supply side and monetary policy is the same as in RANK. The conduct of fiscal policy is analogous to the RANK but is adjusted to address the presence of household heterogeneity in the model. As in RANK, there are two markets: for consumption goods and for assets. Moreover, as in Section 3, I depart from the assumption about rational expectations by assuming that consumers form expectations using level-k thinking.

4.2 Households

It is assumed that a unit mass of households lives on two islands: the first is populated by agents of type S (agents having access to financial markets) and the second by agents of type H (constrained, i.e., not having access to financial markets). In every period, agents face the risk of being reallocated to another island. More specifically, the probability of staying on island S and H is s and h, respectively, where $s,\ h \in (0,1)$. The measures of ergodic populations on islands H and S (see Bilbiie (2019)) are:

$$\lambda = \frac{1-s}{2-s-h},\ 1-\lambda = \frac{1-h}{2-s-h},$$

respectively.

All households constitute a family governed by a family head that makes consumption and savings decisions on behalf of agents of type S and H. Additionally, to give rise to uninsured income risk, it is assumed that the family head is unable to move resources across both islands within a period. It is assumed that pre-tax incomes of agent H and S (denoted by Y_t^H and Y_t^S, respectively) satisfy:

$$\forall_{t\geq 0}\ \frac{Y_t^H}{Y_t^S} = \omega \in (0,1) \tag{8}$$

which is the first assumption needed for the incomplete-markets irrelevance (see Werning (2015) and Farhi and Werning (2019)) to hold in the THANK model with positive government liquidity. The incomplete-markets irrelevance is defined as a situation when there are no differences in the effects of policy announcements on current output between RANK and the model with incomplete markets. As such, the incomplete-markets irrelevance serves as an important benchmark. Note, however, that equation (8) is insufficient for the incomplete-markets irrelevance to hold in THANK and has to be complemented with additional conditions (discussed below). Observe, that THANK nests RANK as a special case. Specifically, for $s = 1$, $h = 0$, and $\omega = 1$ THANK is equivalent to RANK analyzed in the previous section.

The end-of-period-t real asset values per capita on islands H and S are Z^H_{t+1} and Z^S_{t+1}, respectively and the beginning-of-period-$t+1$ real asset value per capita on island H and S are B^H_{t+1} and B^S_{t+1}, respectively. As in Bilbiie (2019), this implies the following laws of motion:

$$\begin{cases} B^S_{t+1} = s \cdot Z^S_{t+1} + (1-s) \cdot Z^H_{t+1}, \\ B^H_{t+1} = (1-h) \cdot Z^S_{t+1} + h \cdot Z^H_{t+1}. \end{cases}$$

I denote the taxes levied on households H and S by T^H_t and T^S_t, respectively and the corresponding transfers are Tr^H_t and Tr^S_t. The Bellman equation describing the maximization problem of the family head is:

$$V_t\left(B^S_t, B^H_t\right) = \max_{\left\{c^H_t, c^S_t, Z^H_{t+1}, Z^S_{t+1}\right\}} \left\{(1-\lambda) \cdot u\left(c^S_t\right) + \lambda \cdot u\left(c^H_t\right) + \beta \cdot V_{t+1}\left(B^S_{t+1}, B^H_{t+1}\right)\right\} \tag{9}$$

$$\begin{cases} c^S_t + Z^S_{t+1} - (1-\theta) \cdot B^S_t = R_t \cdot \theta \cdot B^S_t - T^S_t + Tr^S_t + Y^S_t \\ c^H_t + Z^H_{t+1} - (1-\theta) \cdot B^H_t = R_t \cdot \theta \cdot B^H_t - T^H_t + Tr^H_t + Y^H_t \\ Z^S_{t+1},\ Z^H_{t+1} \geq 0 \end{cases}$$

where c^S_t / c^H_t is consumption of a household on island S/H and V_t is the value function in period t. Analogously to RANK, I denote the after-tax incomes by:

$$\begin{cases} \Upsilon^S_t \equiv -T^S_t + Tr^S_t + Y^S_t, \\ \Upsilon^H_t \equiv -T^H_t + Tr^H_t + Y^H_t. \end{cases}$$

As in Bilbiie (2019), I consider the equilibrium with $\forall_{t \geq 0}\ Z^H_{t+1} = 0$, i.e., when households of type H are constrained (hand-to-mouth).

4.3 Monetary authority

The conduct of monetary policy is the same as in RANK (see subsection 3.3), i.e., R_t is given by:

$$R_t = \begin{cases} R & \text{if } \; t \neq \tau, \\ R + dR & \text{if } \; t = \tau. \end{cases}$$

4.4 Fiscal authority

Both the government budget constraint and fiscal rules are simple generalizations of those analyzed in RANK. In particular, the budget constraint is:

$$\forall_{t \geq 0} \, (1-\lambda) \cdot T_t^S + \lambda \cdot T_t^H + B_{t+1} - (1-\theta) \cdot B_t$$

$$= (1-\lambda) \cdot Tr_t^S + \lambda \cdot Tr_t^H + R_t \cdot \theta \cdot B_t.$$

As in RANK, government debt is constant over time:

$$\forall_{t \geq 0} B_t = \bar{B} > 0$$

Analogously to RANK, taxes are constant and finance steady-state debt service costs:

$$T_t^S = \frac{\bar{B}}{1-\lambda} \cdot \left(\bar{R} \cdot s - 1\right), \; T_t^H = \frac{\bar{B} \cdot \bar{R}}{1-\lambda} \cdot (1-h) \,. \tag{10}$$

Those conditions are the second assumption guaranteeing that the incomplete-markets irrelevance holds in THANK.

As before, transfers are financed with a windfall resulting from the monetary shock:

$$Tr_t^S = -\frac{1-\delta}{1-\lambda} \cdot (R_t - R) \cdot \theta \cdot \bar{B}, \; Tr_t^H = -\frac{\delta}{\lambda} \cdot (R_t - R) \cdot \theta \cdot \bar{B} \tag{11}$$

where $\delta \in [0,1]$ governs the way the windfall is redistributed. In particular, I consider:

$$\delta = \begin{cases} 1-s & \text{neutral transfers,} \\ \lambda & \text{uniform transfers,} \\ 1 & \text{targeted transfers.} \end{cases}$$

The value of parameter δ that corresponds to neutral transfers is the last requirement that has to be satisfied (together with conditions (8) and (10)) to give rise to the incomplete-markets irrelevance in the model with public debt. Uniform transfers are equal for both H and S households. Targeted transfers, in turn, are directed solely towards high MPC (i.e. households of type H).

4.5 Market clearing

The sequence of the resource constraints is:

$$\forall_{t\geq 0}\ \lambda \cdot c_t^H + (1-\lambda)\cdot c_t^S = Y_t \tag{12}$$

and the set of market clearing condition for assets is:

$$\forall_{t\geq 0}\ (1-\lambda)\cdot Z_t^S = B_t \tag{13}$$

note that it follows because $Z_t^H = 0$ for H households.

4.6 Consumption function

In the THANK model, consumption functions characterizing the solution to the maximization problem (9) are:

$$\forall_{t\geq 0}\ c_t^S = C^S\left(R_t, Y_t^S, Tr_t^S, \left\{R_{t+m}, Y_{t+m}^H, Y_{t+m}^S, Tr_{t+m}^H, Tr_{t+m}^S\right\}_{m>0}\right),$$

$$\forall_{t\geq 0}\ c_t^H = \bar{R}\cdot(1-h)\cdot\bar{B} - T_t^H + Tr_t^H + Y_t^H.$$

Observe that that T_t^H and T_t^S are omitted as arguments of function C^S because taxes are constant over time. Unconstrained households receive a direct announcement $\left\{R_{t+m}, Tr_{t+m}^H, Tr_{t+m}^S\right\}_{m>0}$ under MFFG and have to form expectations about $\left\{Y_{t+m}^H, Y_{t+m}^S\right\}_{m>0}$. As in RANK, I use level-k thinking to model that process. Additionally, under the standard FG, households S have to form expectations about $\left\{Tr_{t+m}^H, Tr_{t+m}^S\right\}_{m>0}$ using level-k thinking. Households H do not need to formulate expectations because they are hand-to-mouth.

4.7 Equilibrium

Equilibrium in THANK with level-k thinking is defined as follows:

Definition. The equilibrium in THANK under the MFFG is $\left\{Y_t^{H,k}, Y_t^{S,k}, Y_t^k\right\}_{t\geq 0}$ such that given $\left\{R_{t+m}, Tr_{t+m}^H, Tr_{t+m}^S\right\}_{m\geq 0}$ for each $k \geq 1$:

$$\forall_{t\geq 0}\, Y_t^k = C^S\left(R_t, Y_t^{S,k}, Tr_t^S, \left\{R_{t+m}, Y_{t+m}^{H,k-1}, Y_{t+m}^{S,k-1}, Tr_{t+m}^H, Tr_{t+m}^S\right\}_{m>0}\right)$$

$$+\lambda \cdot \left(\bar{R} \cdot (1-h) \cdot \bar{B} - T_t^H + Tr_t^{H,k} + Y_t^{H,k}\right)$$

where $\left\{Y_t^{H,0}, Y_t^{S,0}, Y_t^0\right\}_{t\geq 0} = \left\{Y^H, Y^S, Y\right\}_{t\geq 0}$ such that market clearing conditions, monetary and fiscal rules hold. Under FG, $\left\{Tr_{t+m}^H, Tr_{t+m}^S\right\}_{m>0}$ is replaced with $\left\{Tr_{t+m}^{H,k-1}, Tr_{t+m}^{S,k-1}\right\}_{m>0}$ and $\left\{Tr_t^S, Tr_t^H\right\}$ with $\left\{Tr_t^{S,k}, Tr_t^{H,k}\right\}$ in the formula above and, additionally, $\left\{Tr_{t+m}^{H,0}, Tr_{t+m}^{S,0}\right\}_{m>0} = \{0,0\}_{m>0}$.

4.8 Interest rate output elasticities in general equilibrium

In contrast to RANK, the intra-period reallocation of resources across households in the model with uninsured idiosyncratic risk may affect aggregate demand of households and, as such, may have an impact on aggregate output (see Oh and Reis (2012)). In the analyzed THANK model, this reallocation is driven by heterogeneous changes to interest earnings driven by a monetary policy shock and fiscal transfers. Therefore, before moving to the analysis of interest rate output elasticities in THANK, it is useful to formulate the following object:

Lemma 2. *The output elasticity with respect to a 0-horizon monetary shock (i.e., for $\tau = 0$) is:*

$$\epsilon\left(0|\delta\right) = -\frac{\bar{B} \cdot \bar{R} \cdot \theta}{Y} \cdot \frac{1-\lambda+\lambda \cdot \omega}{1-\lambda} \cdot (1-\delta-s)$$

Note that $\epsilon\left(0|\delta\right)$ is encapsulating the impact of a monetary shock on output that works through heterogeneous interest earnings and transfers related to that shock (the latter are determined with parameter δ). Moreover, the output elasticity in Lemma 2 is proportional to the standardized real value of the maturing government debt $\frac{\bar{B}\cdot\bar{R}\cdot\theta}{Y}$. Observe that $\epsilon\left(0|\delta\right)$ is increasing in δ for all possible parametrizations and maximized for $\delta = 1$ (i.e., for transfers targeted towards constrained agents) and its value equals to 0 for neutral transfers (i.e., when the changes to interest earnings are exactly offset by the transfer received for each agent).

Let us now formulate the counterpart of Lemma 1 in the THANK model (which extends the results in Angeletos and Lian (2018) and Angeletos and Sastry (2021) to the model with incomplete markets). To guarantee the uniqueness of the consumption response characterization, it is assumed that:

$$s \cdot \omega + h < 1. \tag{14}$$

We have:

Lemma 3. *Under the optimal behavior of households, dc_0^S implied by sequences $\left\{dR_t, d\Upsilon_t^H, d\Upsilon_t^S\right\}_{t\geq 0}$ satisfies:*

$$dc_0^S = \frac{1}{\beta \cdot \bar{R} \cdot s} \cdot \sum_{t=1}^{\infty} \mathcal{M}^t \cdot \left[-\frac{\theta \cdot c^S \cdot \sigma}{\bar{R}} \cdot dR_t + \frac{\beta \cdot \bar{R} \cdot (1-s)}{\omega^{\frac{1}{\sigma}+1}} \cdot \left(\theta \cdot B^H \cdot dR_t + d\Upsilon_t^H\right) \right.$$
$$\left. + \frac{\beta \cdot \bar{R} \cdot s - \mathcal{M}}{\mathcal{M}} \cdot \left(\theta \cdot B^S \cdot dR_{t-1} + d\Upsilon_{t-1}^S\right) \right]$$

where $\mathcal{M}$ is the larger root of $\Psi\left(\frac{1}{\mathcal{M}}\right) = 0$ and where Ψ is a quadratic polynominal with coefficients expressed as functions of the model's parameters.

The closed-form expression for $\mathcal{M}$ is shown in the Appendix. The comparison of Lemmas 1 and 3 implies that $\mathcal{M}$ in THANK plays an analogous role to β in RANK - it discounts the impact of future changes to income levels and interest rates on current consumption of household S.

Lemma 3 implies that MPC of household S is:

$$MPC^S \equiv \frac{dc_0^S}{dY_0^S} = 1 - \frac{\mathcal{M}}{\beta \cdot \bar{R} \cdot s} \tag{15}$$

and the iMPC of the S household with respect to a one-period-ahead income shock (a unit of consumption good out of which δ is received in state H and $1-\delta$ in state

S):[14]

$$iMPC^S(\delta) \equiv \frac{\mathcal{M}}{\beta \cdot \bar{R} \cdot s} \cdot \left[\frac{\beta \cdot \bar{R} \cdot (1-s)}{\omega^{\frac{1}{\sigma}+1}} \cdot \frac{\delta}{\lambda} + \frac{\beta \cdot \bar{R} \cdot s - \mathcal{M}}{\mathcal{M}} \cdot \frac{1-\delta}{1-\lambda} \right]. \tag{16}$$

Note that, unlike $\epsilon(0|\delta)$, $iMPC^S(\delta)$ is not an increasing function of δ for all possible parameter values (i.e. indipendently of the model parametrization). This occurs because there is a trade-off between transfering income to state H and S in the future: on the one hand, increasing δ (i.e., the proportion of resources directed to state H in the next period) boosts c_0^S because higher future income in the state in which consumer is constrained reduces current precautionary motives. On the other hand, if s is sufficiently high, state H materializes with a relatively low probability and thus additional resources transferred to that state are valued less.

The elasticity of output in period 0 with respect to a monetary policy shock in period $\tau > 0$ under level-k thinking of order k under the MFFG and transfers governed by parameter δ is denoted by:

$$\epsilon(\tau, k|\delta) \equiv -\frac{\bar{R}}{Y} \cdot \frac{dY(\tau, k|\delta)}{dR} \tag{17}$$

where $dY(\tau, k|\delta)$ is the reaction of output in period 0 to a shock in period τ. The FG counterpart of $\epsilon(\tau, k|\delta)$ is denoted by $\hat{\epsilon}(\tau, k|\delta)$ and the difference between interest rate elasticities of output under MFFG and FG is denoted by $\Delta\epsilon(\tau, k|\delta)$:

$$\Delta\epsilon(\tau, k|\delta) \equiv \epsilon(\tau, k|\delta) - \hat{\epsilon}(\tau, k|\delta). \tag{18}$$

We now turn to the main result formulated for THANK:

Theorem 2. *Consider a monetary policy shock featuring horizon $\tau > 0$ in the THANK model under level-k thinking of order k and under the MFFG. We have:*

$$\epsilon(\tau, k|\delta) = \underbrace{F(k-1|\tau-1, 1-\mathcal{M}) \cdot \theta \cdot \sigma}_{\text{intertemporal substitution}}$$

$$\underbrace{-F(k-1|\tau-1, 1-\mathcal{M}) \cdot \frac{iMPC^S(1-s)}{1-MPC^S} \cdot \frac{\bar{R} \cdot \bar{B} \cdot \theta}{c^S}}_{\text{interest earnings}}$$

[14]Naturally, $MPC^H = 1$ and $iMPC^H = 0$.

$$+\underbrace{F\left(k-1|\tau-1,1-\mathcal{M}\right)\cdot\frac{iMPC^{S}\left(\delta\right)}{1-MPC^{S}}\cdot\frac{\bar{R}\cdot\bar{B}\cdot\theta}{c^{S}}}_{\text{transfers (restribution)}}$$

$$+\underbrace{F\left(k-2|\tau-1,1-\mathcal{M}\right)\cdot\left(1-\mathcal{M}\right)\cdot\epsilon\left(0|\delta\right)}_{\text{transfers (GE effects)}}$$

and the impact of fiscal forward guidance is:

$$\Delta\epsilon\left(\tau,k|\delta\right)=f\left(k-1|\tau-1,1-\mathcal{M}\right)\cdot\frac{iMPC^{S}\left(\delta\right)}{1-MPC^{S}}\cdot\frac{\bar{R}\cdot\bar{B}\cdot\theta}{c^{S}}$$

Comparison of Theorems 1 and 2 shows that although the MFFG transmission is considerably more sophisticated in THANK than in RANK (i.e., $\epsilon\left(\tau,k|\delta\right)$ is substantially more complex than $\epsilon\left(\tau,k\right)$), the isolated effects of fiscal forward guidance captured by $\Delta\epsilon\left(\tau,k|\delta\right)$ in THANK have a very similar structure to $\Delta\epsilon\left(\tau,k\right)$ in RANK.

Incomplete insurance markets give rise to three additional transmission channels of the MFFG when compared to RANK (in the latter model it is driven solely by intertemporal substitution). First of them is associated with interest earnings: changes to interest rates in period τ have a differentiated impact on households incomes due to heterogeneous asset holdings. Note that the impact of that channel is negative for an expansionary monetary policy shock (i.e., for $dR<0$) because it lowers interest earnings. Second, monetary shock in period τ induces transfers in that period. Households in period 0 expect to receive them at time τ, which boosts aggregate demand and output in period 0. Both the interest earnings and transfers channels are proportional to the standardized value of public debt $\frac{\bar{R}\cdot\bar{B}\cdot\theta}{C^{S}}$. Note that neutral transfers with $\delta=1-s$ offset the impact of the interest earnings channel and thus the net effect of both channels on $\epsilon\left(\tau,k|\delta\right)$ is zero. Third, transfers in period τ reallocate resources across households featuring different MPC levels affecting aggregate demand and thus income at time τ, which, in turn, influences private spending at time 0.

Analogously to RANK, the transmission of MFFG is more powerful when agents become more rational - all transmission channels are multiplied by term F which increases in k. Note that, as in Farhi and Werning (2019), the impact of MFFG on current output is influenced by the interaction of incomplete markets and bounded rationality: $1-\beta$ that enters F in RANK is replaced with $1-\mathcal{M}$ in THANK.

Moreover, as in RANK, there exists a threshold value for k above which $\epsilon(\tau, k|\delta)$ is a constant function of k and corresponds to the REE. More specifically, the level-k and REE equilibria feature identical responses to monetary policy shocks if $k \geq \tau + 1$. If additionally $\delta = 1 - s$ then $\epsilon(\tau, k|\delta) = \epsilon(\tau, k)$, i.e., the incomplete markets irrelevance holds.

Note that, unlike $\epsilon(\tau, k|\delta)$, the effects of fiscal forward guidance captured by $\Delta\epsilon(\tau, k|\delta)$ are very similar to their RANK counterpart. The crucial difference is that $\Delta\epsilon(\tau, k|\delta)$ depends on δ in THANK, which results from the impact of the transfer type δ on $iMPC(\delta)$ - i.e., the way a unit rise in disposable income is redistributed between two possible states (in which households are either constrained or unconstrained) in the future. As the monotonicity of $iMPC(\delta)$ is parameter-dependent (see the discussion that follows formula (16)), a quantitative model is needed to evaluate the type of transfer maximizing the MFFG transmission. This exercise is conducted in Section 5 using the fully-blown HANK model.

Similarly to RANK, the impact of cognitive constraints on $\Delta\epsilon(\tau, k|\delta)$ is ambiguous and a more formal discussion of that issue follows the line of reasoning presented in Section 3 - it suffieces to replace β with $\mathcal{M}$ in the formal analysis of formulas (6) and (7).

Finally, as in RANK, the relative impact of the fiscal announcement when compared to FG effectiveness:

$$\frac{\Delta\epsilon(\tau, k|\delta)}{\epsilon(\tau, k|\delta) - \Delta\epsilon(\tau, k|\delta)} \tag{19}$$

is independent of θ.

5 Monetary-fiscal forward guidance in HANK

This section lays out a quantitative model with uninsured idiosyncratic risk that is based on Auclert et al. (2023a). The departure from the two-island structure analyzed in the previous section implies that wealth distribution becomes a relevant state variable which, coupled with precautionary motives that give rise to concave consumption functions, implies that linear aggregation does not obtain and therefore the model ceases to be analytically tractable. I consider three versions of the model in the quantitative exercise. The first differs from THANK only in that it features a non-degenerate wealth distribution and allows for bridging the THANK model with the analysis based on HANK. In the second variant of HANK (which is referred to as the benchmark), perfectly rigid prices are replaced with the standard New Keynesian Phillips Curve (NKPC). There is only one difference between the second HANK and the third model: the latter features higher value of the government debt. I use those three calibrations of the same HANK model to highlight the role of interactions between demand and supply in the propagation of MFFG and to study the impact of debt level on the effectiveness of MFFG. The former is achieved by comparing the first and the second variants of HANK and the latter is attained by analyzing the differences between the second and third version of the model.

5.1 Environment

The demand block is based on the canonincal Bewley-Huggett-Aiyagari model of incomplete markets. More specifically, I use a version of the model analyzed by Auclert et al. (2023a) (extended to capture bounded rationality) because it guarantees the similarity of the problems solved by households in HANK and in THANK/RANK (in particular, the assumptions made by Auclert et al. (2023a) imply zero profits and inelastic labor supply as in RANK and THANK). Moreover, it allows for dealing with the so-called MPC-MPE-Multipliers trilemma discussed in their paper. The supply side is represented by the standard NKPC but, as mentioned, I consider also a variant of HANK with perfectly rigid prices to bridge the quantitative analysis with exercises in Sections 3 and 4. The conduct of fiscal policy is analogous to RANK and THANK. The conduct of monetary policy generalizes the monetary rule (2) to the model with NKPC. There are three markets in the model: for consumption goods, assets and, labor. Moreover, as in previous sections, I depart from the assumption about rational expectations and use level-k thinking, instead.

5.2 Households

There is mass one of households facing changes to idiosyncratic labor productivity levels y_t governed by a Markovian process featuring transition probability $\mathbb{P}\left(y_{t+1}|y_t\right)$. The instaneous utility function is:

$$u\left(c\right) - v\left(n\right) = \frac{c^{1-\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}} - \gamma \cdot \frac{n^{1+\frac{1}{\phi}}}{1+\frac{1}{\phi}}$$

where $\gamma > 0$ is a parameter and $\phi > 0$ is Frisch elasticity of labor supply. The Bellman equation associated with the maximization problem of a household that enters period t with asset holdings b and productivity level y is:

$$V_t(b, y) = \max_{\{c_t, b_{t+1}\}} \{u(c_t) - v(n_t) + \beta \cdot \mathbb{E}_t V_{t+1}(b_{t+1}, y_{t+1})\} \tag{20}$$

$$\begin{cases} c_t + b_{t+1} - (1-\theta) \cdot b = \frac{R_t \cdot \theta}{\Pi_t} \cdot b - T_t + Tr_t(y|\delta) + y \cdot \frac{W_t}{P_t} \cdot n_t \\ b_{t+1} \geq 0 \end{cases}$$

where V_t is value function, W_t is nominal wage, P_t denotes the price of consumption goods, Π_t is gross inflation rate:

$$\Pi_t \equiv \frac{P_t}{P_{t-1}},$$

$Tr_t(y|\delta)$ is the transfer received by household with productivity y under transfer policy δ. Due to the assumed labor market structure (see subsection 5.3), n_t is taken as given by households. Thus, the solution to the maximization problem (20) are two policy functions: $c_t(b, y)$ and $b_{t+1}(b, y)$. As we shall see, $W_t/P_t = 1$ and firm profits are zero, which coupled with the fact that n_t is exogenous for households, implies that the budget constraint of the problem (20) is a direct counterpart of budget constraints in RANK/THANK.

5.3 Labor unions and producers of consumption goods

As in Auclert et al. (2023a), nominal wages are negotiated by labor unions. Each union offers a different labor variety $N_{j,t}$ (with $j \in [0, 1]$) and nominal wage $W_{j,t}$ to producers of consumption goods and maximizes welfare of its members (i.e. households) together with the quadratic disutility from wage adjustment, subject to labor demand. In particular, labor union j solves the following problem:

$$F_t(W_{j,t-1}) = \max_{W_{j,t},\, N_{j,t}} \left\{ \int (u(c_t) - v(n_t))\, d\mu_t(b, y) \right. \tag{21}$$

$$\left. - \frac{\psi}{2} \cdot \int \left(\frac{W_{j,t}}{W_{j,t-1}} - 1 \right)^2 + \beta \cdot \mathbb{E}_t F_{t+1}(W_{j,t}) \right\}$$

subject to:

$$N_{j,t} = \left(\frac{W_{j,t}}{W_t}\right)^{-\xi} \cdot N_t$$

which is a generalization of the problem considered by Erceg et al. (2000) in RANK, developed by Auclert et al. (2023b) and Auclert et al. (2023a), where $\mu_t\,(b,y)$ is the measure of households with assets b and productivity level y, ψ is the parameter of wage-adjustment, and N_t is aggregate labor. The solution to problem (21) is the standard NKPC (see Auclert et al. (2023b) for derivation):

$$\left(\Pi_t^W - 1\right) \cdot \Pi_t^W = \frac{\xi}{\psi} \cdot N_t \cdot \left(v'\left(N_t\right) - \frac{\xi - 1}{\xi} \cdot \int y \cdot u'\left(c_t\left(b,y\right)\right) d\mu_t\left(b,y\right)\right)$$
$$+\beta \cdot \left(\Pi_{t+1}^W - 1\right) \cdot \Pi_{t+1}^W \tag{22}$$

where $\Pi_t^W \equiv W_t/W_{t-1}$ is the gross inflation rate of nominal wages.

Moreover, W_t and the union-specific wages $W_{j,t}$ satisfy the following relationship:

$$W_t = \left(\int_0^1 W_{j,t}^{1-\xi} dj\right)^{\frac{1}{1-\xi}}$$

where parameter $\xi > 0$ governs the substitutability between labor varieties that are packed by competitive labor packers using technology:

$$N_t = \left(\int_0^1 N_{j,t}^{\frac{\epsilon-1}{\epsilon}} dj\right)^{\frac{\epsilon}{\epsilon-1}}.$$

Packed labor services are then sold to producers of consumtpion goods who operate a linear technology:

$$Y_t = N_t$$

and maximize profits:

$$P_t \cdot Y_t - W_t \cdot N_t.$$

It is assumed that they are perfectly competitive, which implies that:

$$P_t = W_t$$

and thus:

$$\Pi_t^W = \Pi_t.$$

This means that producers generate zero profits which implies that household maximization problems in RANK, THANK and HANK are analogous.

5.4 Monetary authority

I follow Farhi and Werning (2019) and generalize the monetary rule from THANK and RANK to account for the presence of the NKPC:

$$R_t = \begin{cases} R & \text{if } t < \tau \\ R + dR & \text{if } t = \tau \\ R \cdot \left(\frac{\Pi_t}{\Pi}\right)^{\phi_\Pi} & \text{if } t > \tau \end{cases}$$

where $\phi_\Pi > 0$ is the Taylor rule parameter.

5.5 Fiscal authority

Analogously to RANK and THANK, the government budget constraint reads:

$$\forall_{t\geq 0}\, T_t + B_{t+1} - (1-\theta) \cdot B_t = Tr_t + \frac{R_t}{\Pi_t} \cdot \theta \cdot B_t.$$

The fiscal policy rules are as follows. First, the real value of government debt is constant over time:

$$\forall_{t\geq 0}\, B_t = \bar{B} > 0.$$

Taxes finance the steady state level of debt service costs:

$$T_t = \left(\frac{R}{\Pi} - 1\right) \cdot \theta \cdot \bar{B}.$$

Aggregate transfers equal to:

$$Tr_t = -\left(\frac{R_t}{\Pi_t} - \frac{R}{\Pi}\right) \cdot \theta \cdot \bar{B}$$

i.e., Tr_t is financed with a windfall resulting from the monetary shock at $t = \tau$.

As in THANK, I consider three redistribution schemes:

$$Tr_t(y|\delta) = \begin{cases} \frac{b}{B} \cdot Tr_t & \text{neutral transfers} \\ Tr_t & \text{uniform transfers} \\ \frac{1}{\int_{y \in \mathcal{Y}} d\mu_t(b,y)} \cdot Tr_t & \text{targeted transfers} \end{cases}$$

where $\mathcal{Y}$ is the set of income levels of households that are entitled to targeted transfers.

5.6 Market clearing

Let us specify the time-evolution of the household distribution. The internal consistency of the model requires that the Markovian changes to labor productivity and optimal saving policies induce the following law of motion of $\mu_t(b,y)$:

$$\forall_{t \geq 0}\, \mu_{t+1}(\mathcal{B},\mathcal{Y}) = \int \left[\mathbb{I}_{\{b_{t+1}(b,y) \in \mathcal{B}\}} \cdot \mathbb{P}\left(y_{t+1} \in \mathcal{Y}|y\right) \right] d\mu_t(b,y)$$

where $\mathcal{B}$ and $\mathcal{Y}$ are Borel subsets of spaces of assets holdings and labor productivity levels, respectively and $\mathbb{I}$ is the indicator function.

Additionally, I standardize the aggregate labor productivity and the population size:

$$\forall_{t \geq 0} \int y d\mu_t(b,y) = \int d\mu_t(b,y) = 1.$$

Market clearing condition for labor is:

$$\forall_{t \geq 0, j \in [0,1]}\, n_t = N_t = N_{j,t},$$

and for consumption goods it reads:

$$\forall_{t \geq 0} \int c_t(b,y)\, d\mu_t(b,y) = Y_t,$$

The market clearing condition for assets it is given by:

$$\forall_{t \geq 0} \int b_{t+1}(b,y)\, d\mu_t(b,y) = B_{t+1}$$

5.7 Consumption and wage-setting functions

Given that $W_t = P_t$, $Y_t = n_t$, $\Pi_t = \Pi_t^W$, and T_t is constant over time, the aggregate consumption can be formulated as:

$$C_t \equiv \int c\left(b, y | R_t, \Pi_t, Tr_t\left(\cdot|\delta\right), Y_t, \left\{R_{t+m}, \Pi_{t+m}, Tr_{t+m}\left(\cdot|\delta\right), Y_{t+m}\right\}_{m>0}\right) d\mu_t\left(b, y\right).$$

Now, adopting this formulation to level-k and the MFFG I define:

$$C_t^k \equiv \int c\left(b, y | R_t, \Pi_t^k, Tr_t\left(\cdot|\delta\right), Y_t^k, \left\{R_{t+m}, \Pi_{t+m}^{k-1}, Tr_{t+m}\left(\cdot|\delta\right), Y_{t+m}^{k-1}\right\}_{m>0}\right) d\mu_t^k\left(b, y\right) \tag{23}$$

and, moreover, I denote:

$$c_t^k\left(b, y\right) \equiv c\left(b, y | R_t, \Pi_t^k, Tr_t\left(\cdot|\delta\right), Y_t^k, \left\{R_{t+m}, \Pi_{t+m}^{k-1}, Tr_{t+m}\left(\cdot|\delta\right), Y_{t+m}^{k-1}\right\}_{m>0}\right). \tag{24}$$

Notice that, under the standard FG, Tr_{t+m} is replaced with Tr_{t+m}^{k-1} and Tr_t is replaced with Tr_t^k in formulas (23) and (24). The saving policy under level-k is defined as:

$$b_{t+1}^k\left(b, y\right) \equiv \left(1-\theta\right) \cdot b + \frac{R_t \cdot \theta}{\Pi_t} \cdot b_t - T_t + Tr_t\left(y|\delta\right) + y \cdot \frac{W_t}{P_t} \cdot n_t - c_t^k\left(b, y\right).$$

It is assumed that, symmetrically to households, the forward-looking labor unions are subject to bounded rationality and use level-k thinking to forecast the values of future aggregate variables that are not announced by the government. To this end, let us first define:

$$\Omega_t \equiv \frac{\xi}{\psi} \cdot N_t \cdot \left(v'\left(N_t\right) - \frac{\xi - 1}{\xi} \cdot \int y \cdot u'\left(c_t\left(b, y\right)\right) d\mu_t\left(b, y\right)\right).$$

Using this definition and $\Pi_t^W = \Pi_t$ let us rewrite the NKPC as:

$$\left(\Pi_t - 1\right) \cdot \Pi_t = \Omega_t + \sum_{m=1}^{+\infty} \beta^m \cdot \Omega_{t+m}$$

Thus, under level-k thinking, we have the following condition summarizing the optimal price-setting behavior of labor unions:

$$\left(\Pi_t^k - 1\right) \cdot \Pi_t^k = \Omega_t^k + \sum_{m=1}^{+\infty} \beta^m \cdot \Omega_{t+m}^{k-1}. \tag{25}$$

5.8 Equilibrium

We can now define the equilibrium in HANK:

Definition. The equilibrium in HANK under MFFG is: $\left\{\Pi_t^k, Y_t^k, \Omega_t^k\right\}_{t\geq 0}$, $\left\{\mu_t^k\right\}_{t\geq 0}$, $\left\{c_t^k\left(b,y\right)\right\}_{t\geq 0}$, $\left\{b_{t+1}^k\left(b,y\right)\right\}_{t\geq 0}$ such that given $\left\{R_t, Tr_t\left(y|\delta\right)\right\}_{t\geq 0}$ and given $\mu_0^k = \mu_0$ for each $k \geq 1$: given $\left\{R_{t+m}\right\}_{m\geq 0}$, $\left\{Tr_{t+m}\left(y|\delta\right)\right\}_{m\geq 0}$, $\left\{\Pi_{t+m}^{k-1}, Y_{t+m}^{k-1}\right\}_{m>0}$, and $\left\{\Pi_t^k, Y_t^k\right\}$ functions $c_t^k\left(b,y\right)$, $b_{t+1}^k\left(b,y\right)$ solve household problem (20) for each $t \geq 0$, given $\left\{\Omega_{t+m}^{k-1}\right\}_{m>0}$ and Ω_t^k inflation Π_t^k solves (25), the government budget constraint holds and the monetary policy rule is satisfied, the law of motion of measure μ_t^k is induced by the Markovian process $\mathbb{P}\left(y_{t+1}|y_t\right)$ and policy function $b_{t+1}^k\left(b,y\right)$, market clearing conditions are satisfied. Additionally, $\left\{\Pi_{t+m}^0, Y_{t+m}^0\right\}_{m>0} = \left\{\Pi, Y\right\}_{m>0}$ and $\left\{\Omega_{t+m}^0\right\}_{m>0} = \left\{\Omega\right\}_{m>0}$. Moreover, under FG, $\left\{Tr_{t+m}\left(y|\delta\right)\right\}_{m>0}$ is replaced with $\left\{Tr_{t+m}^{k-1}\left(y|\delta\right)\right\}_{m>0}$ and $Tr_t\left(y|\delta\right)$ is replaced with $Tr_t^k\left(y|\delta\right)$ in the specification of variables taken as given by households, where $\left\{Tr_{t+m}^0\left(y|\delta\right)\right\}_{m>0} = \left\{0\right\}_{m>0}$.

5.9 Calibration

Steady state of the model is calibrated to match the moments characterizing the US economy. Time period is a quarter.

First, it is assumed that the idiosyncratic labor productivity process is specified as in Krueger et al. (2016) (i.e., labor productivity is affected by persistent and transitory shocks denoted by $\epsilon_{\hat{y}}$ and ϵ_y, respectively):

$$\begin{cases} \log y_{t+1} = \log \hat{y}_t + \epsilon_{y,t+1} \\ \log \hat{y}_{t+1} = \rho \cdot \log \hat{y}_t + \epsilon_{\hat{y},t+1} \end{cases}$$

where $\hat{y}_t$ is the persistent component of the process where $\epsilon_y \sim N\left(0, \sigma_y^2\right)$, $\epsilon_{\hat{y}} \sim N\left(0, \sigma_{\hat{y}}^2\right)$, and $\rho \in (0,1)$. I discretize the persistent component of the process using the Rouwenhorst algorithm and I use the Gauss-Hermite quadrature to approximate the transitory component.

To match the average MPC in the US economy, β is assumed to be uniformly distributed:

$$\beta \sim U\left[\underline{\beta}, \overline{\beta}\right].$$

Finally, it is assumed that targeted transfers are received by the bottom 25% of labor income earners (which pins down set $\mathcal{Y}$).

The values of calibrated parameters and the associated calibration targets are displayed in Table 1. I consider $\theta = 1$ (i.e., one-period debt), which is standard in the literature and, as argued in Sections 3 and 4, this choice is innocuous for the main quantitative result (i.e. the relative increase in the effectiveness of FG that can be attributed to fiscal forward guidance). I set $\psi = 700$ to match the NKPC slope in Auclert et al. (2023a). I set $\sigma = 1$ (log-utility from consumption) as in McKay and Reis (2016). The substitution between labor varieties ξ is equal to 7 as in Auclert et al. (2023a) and labor disutility γ is set to standardize $Y = 1$. Frisch elasticity ϕ is equal to 0.5 as in McKay and Reis (2016). Parameters σ_y, $\sigma_{\hat{y}}$, ρ are quarterly counterparts of the values reported by Krueger et al. (2016). The responsiveness of monetary policy to inflation (i.e., ϕ_Π) is equal to 1.5, which is a standard value in the literature. The value of government debt $\bar{B}$ is set to match the ratio between debt and annual GDP equal to 55% as in Auclert et al. (2023a). The upper bound of the distribution of β (i.e., $\bar{\beta}$) equals 0.988 to match the annual gross real interest rate of 1.02. The lower bound for β (i.e., $\underline{\beta}$) is set to match the quarterly MPC of 0.25 as in Auclert et al. (2023a). Finally, the steady-state value of gross nominal interest rate is consistent with the zero net inflation and the targeted real interest rate (i.e., it satisfies the Fisher equation).

5.10 Simulations

I now report the quantitative results for three variants of the HANK model. As in Sections 3 and 4, first of them assumes perfectly rigid prices/wages ($\psi \to +\infty$, with other parameters equal to their calibrated values) and is intended to bridge the quantitative analysis with the analytical part. Second variant of the model features $\psi < +\infty$, with all parameters equal to their values in Table 1. This is the so-called benchmark (baseline simulation). This model features a non-degenerate NKPC and

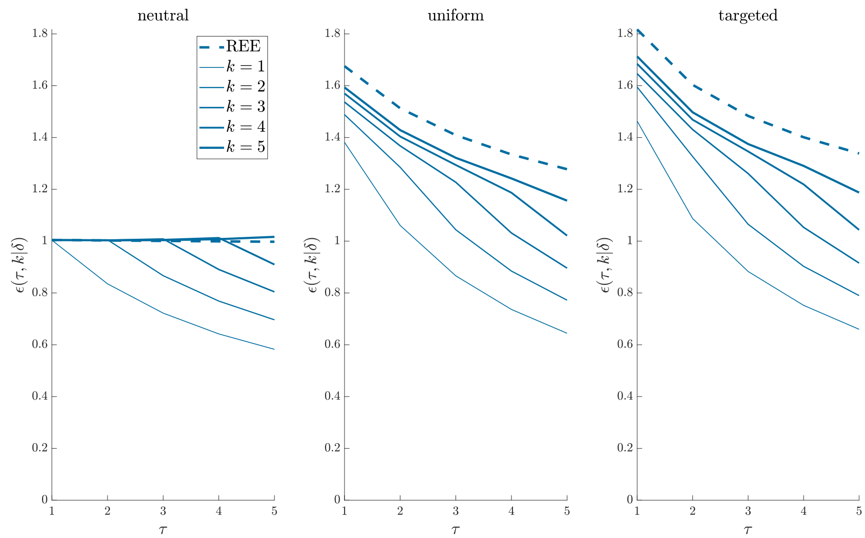
Table 1: Parameter values

Parameter	Description	Value	Target/Source
θ	debt maturity	1	one-period debt
ψ	wage-adjustment cost	700	NKPC slope (Auclert et al. (2023a))
σ	intertemporal substitution	1	McKay and Reis (2016)
ξ	substitution between labor varieties	7	Auclert et al. (2023a)
γ	labor disutility parameter	0.86	$Y = N = 1$
ϕ	Frisch elasticity	0.5	McKay and Reis (2016)
ρ	autocorrelation (persistent component)	0.99	Krueger et al. (2016)
$\sigma_{\hat{y}}$	standard error (persistent component)	0.10	Krueger et al. (2016)
σ_y	standard error (transitory component)	0.11	Krueger et al. (2016)
ϕ_Π	Taylor rule parameter	1.5	standard value
$\bar{B}$	government debt	2.2	Auclert et al. (2023a)
$\overline{\beta}$	discount factor (patient households)	0.988	annual R/Π of 2%
$\underline{\beta}$	discount factor (impatient households)	0.970	MPC (Auclert et al. (2023a))
R	steady state nominal interest rate	1.005	$\Pi = 1$

its comparison to the first version of HANK allows for isolating the impact of the interactions between the demand and supply blocks on the propagation of MFFG. Finally, I consider the MFFG effects in a variant of HANK in which the value of $\bar{B}$ is 100% higher than in the baseline. The comparison of the second and third versions of HANK allows for quantifying the influence of public debt on the effectiveness of MFFG and fiscal forward guidance.

HANK with $\psi \to +\infty$. Let us start with the HANK model featuring perfectly rigid prices. Figure 2 displays the interest rate elasticities of output corresponding to a monetary policy shock at horizon τ accompanied by the fiscal announcement about transfers. I consider both REE and level-k equilibria and three types of transfers: neutral, uniform, and targeted. The left panel shows that the interest rate elasticity corresponding to MFFG with neutral transfers and REE satisfies the incomplete market irrelevance, i.e., it is a constant function of τ that equals $\sigma = 1$, which coincides with the value of output elasticity in RANK under REE (see Theorem 1 for $k > \tau$). This echoes the conclusion by Werning (2015): the assumption of incomplete markets is insufficient to resolve the FG puzzle in the standard New Keynesian model. As the solid lines (representing elasticities under level-k thinking) show, bounded rationality fixes that problem because lower values of k lead to significant horizon effects (i.e. a decreasing relationship between τ and

Figure 2: Interest rate elasticities of output in HANK with perfectly rigid prices under MFFG for neutral, uniform and targeted transfers.



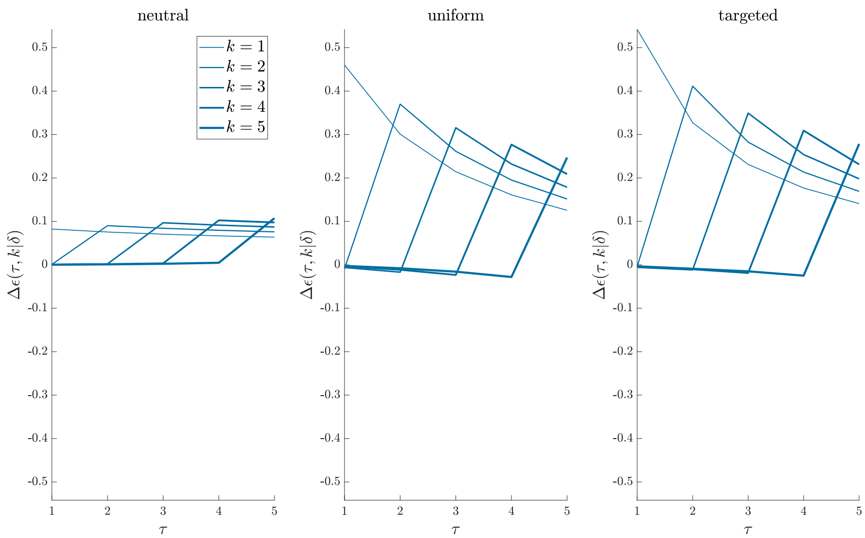
Notes: Interest rate elasticities of output in period 0 computed for a one-time drop in interest rates equal to $dR = -0.0025$ (i.e. a one-percentage-point decrease in nominal rates in annual terms) that occurs in period τ - see formula (17) for three types of transfers in period τ (induced by monetary shock): neutral, uniform, and targeted. Dashed lines correspond to the rational expectations equilibria and solid lines denote output elasticities under level-k thinking. Different thickness of solid lines represent different values of k.

$\epsilon(\tau, k|\delta)$), as discussed in Farhi and Werning (2019). As the middle panel shows, the effectiveness of MFFG is larger when transfers become uniform - the increase for $\tau = 1$ ranges from 38% to 66% for level-k equilibrium featuring $k = 1$ and REE, respectively. The analogous numbers for the MFFG with targeted transfers amount to 45% and 83%. In other words, the increasing progressivity of transfers raises the effectiveness of the MFFG.

Let us turn to the effectiveness of the fiscal forward guidance measured with $\Delta\epsilon(\tau, k|\delta)$, i.e., the difference in the interest rate output elasticities between MFFG and FG (see formula (18)). As shown in Figure 3, the effectiveness of the fiscal announcement is the largest in the case of targeted transfers for both bounded rationality and under REE. As explained by formula $\Delta\epsilon(\tau, k|\delta)$ in THANK (see Theorem 2), this result hinges on the value of $iMPC(\delta)$ that is affected by two opposite effects.[15] On the one hand, transfers that are directed to future states where the income is low (and where the chances of becoming constrained is substantially higher) curb precautionary motives in period 0 and stimulate aggregate demand

[15]See the discussion that follows formula (16).

Figure 3: Difference between interest rate elasticities of output under MFFG and FG in HANK with perfectly rigid prices for neutral, uniform and targeted transfers.



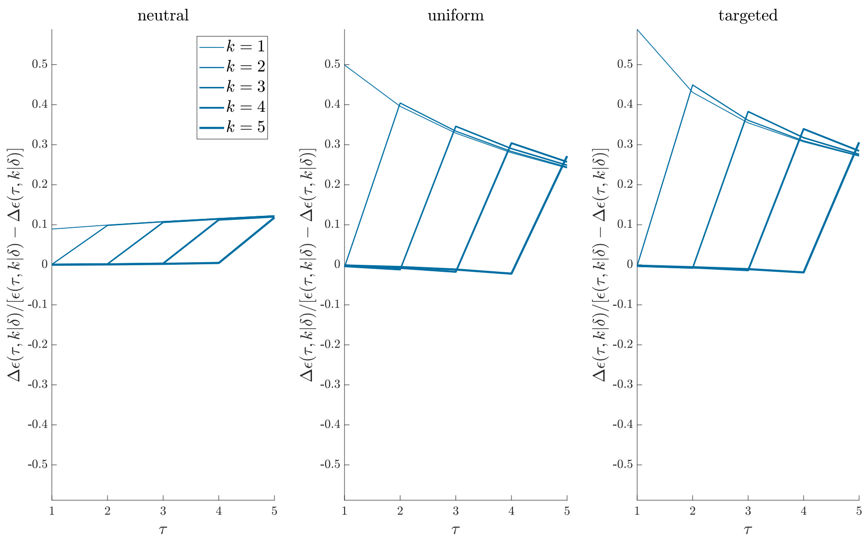
Notes: Difference (between MFFG and FG) in interest rate elasticities of output in period 0 computed for a one-time drop in interest rates equal to $dR = -0.0025$ (i.e. a one-percentage-point decrease in nominal rates in annual terms) that occurs in period τ. This difference is specified by formula (18) for three types of transfers in period τ (induced by monetary shock): neutral, uniform, and targeted. Solid lines denote output elasticities under level-k thinking. Different thickness of solid lines represent different values of k.

which, given perfectly rigid prices, translates into a rise in output. On the other hand, the chance of the materialization of low-income states for high-income households in period 0 (who feature higher responsiveness to signals about future income shocks than the poor and constrained) is small given the substantial persistence of labor productivity process ρ. As Figure 3 displays, the former force dominates the latter and thus fiscal forward guidance is most effective for targeted transfers.

Importantly, note that the difference in output elasticities between MFFG and FG is not monotonic in k (as predicted by the RANK and THANK): it first increases in k and then, when agents are sufficiently rational (i.e. when $k > \tau$), it drops to values that are close to zero. The intuition behind the increasing relationship between k and $\Delta\epsilon\,(\tau, k|\delta)$ is analogous to RANK and THANK: when k is low and increases to $k + 1$, more rational agents realize that the fiscal announcement stimulates output in periods 1, 2, ..., $\tau - 1$ which boosts aggregate demand in period 0.

To interpret those results quantitatively, let us normalize $\Delta\epsilon\,(\tau, k|\delta)$ and divide its value by the interest rate output elasticity corresponding to the standard forward guidance (see formula (19)). Figure 4 shows the values of the corresponding ratios.

Figure 4: The ratio between fiscal and monetary forward guidance effectiveness in HANK with perfectly rigid prices for neutral, uniform and targeted transfers.

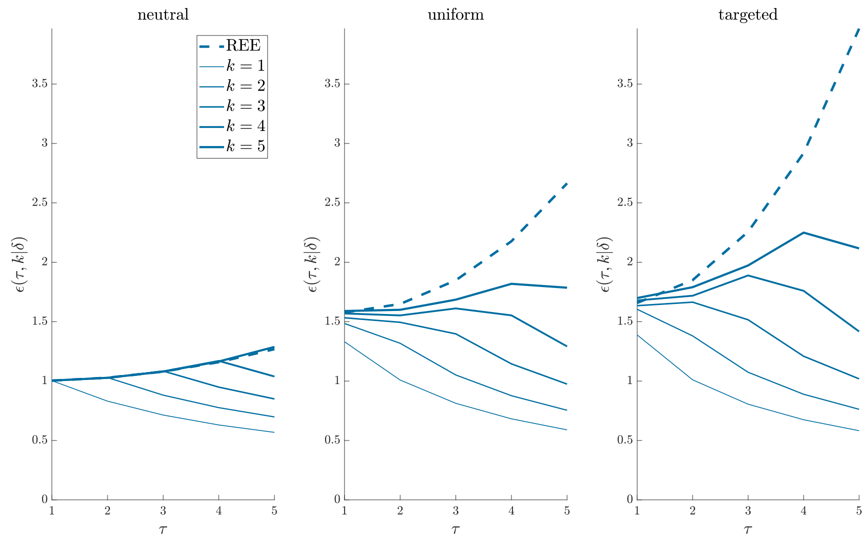


Notes: The difference (between MFFG and FG) in interest rate elasticities of output in period 0 computed for a one-time drop in interest rates equal to $dR = -0.0025$ (i.e. a one-percentage-point decrease in nominal rates in annual terms) that occurs in period τ, standardized by the interest rate elasticity corresponding to the standard FG (see formula 19) for three types of transfers in period τ (induced by monetary shock): neutral, uniform, and targeted. Solid lines denote output elasticities under level-k thinking. Different thickness of solid lines represent different values of k.

Note that the standardized effectiveness of fiscal forward guidance is "polarized": it is almost constant for $k \leq \tau$ and then it drops to values close to zero for $k > \tau$ and it is almost constant, too. The mid and right panels show that the relative change to the FG effectiveness driven by the additional fiscal announcement may reach 50% under uniform transfers and up to 56% under targeted transfers. Moreover, these relative gains from the fiscal announcement are substantial even for horizons exceeding a year (about 30% for both uniform and targeted transfers) and are present for $k \leq \tau$. I.e., they are most powerful when agents are least rational and when the standard FG is least effective (see Farhi and Werning (2019)). Recall that, as argued in Sections 3 and 4, those quantitative assessments is robust to changes to debt maturity.

HANK with $\psi < +\infty$ - baseline simulation. Let us now allow for interactions between aggregate demand and the supply side by analyzing the model with standard nominal rigidities, i.e., the supply side ceases to be passive and it is governed by the NKPC instead. Figure 5 shows interest rate elasticities of output in this variant of the HANK model. By comparing it to Figure 2, it can be

Figure 5: Interest rate elasticities of output in HANK (benchmark with the NKPC, $\psi < +\infty$) under MFFG for neutral, uniform and targeted transfers.

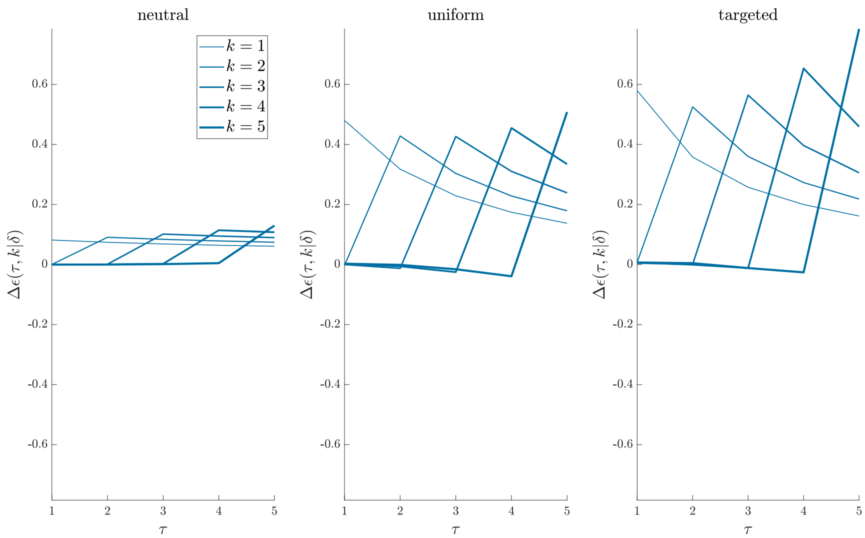


Notes: Interest rate elasticities of output in period 0 computed for a one-time drop in interest rates equal to $dR = -0.0025$ (i.e. a one-percentage-point decrease in nominal rates in annual terms) that occurs in period τ - see formula (17) for three types of transfers in period τ (induced by monetary shock): neutral, uniform, and targeted. Dashed lines correspond to the rational expectations equilibria and solid lines denote output elasticities under level-k thinking. Different thickness of solid lines represent different values of k.

concluded that relaxing the assumption about constant prices leads to an increasing relationship between interest rate elasticity of output and the MFFG horizon under REE. This becomes particularly apparent if left panels of both figures (i.e., the panels corresponding to neutral transfers) are compared: while the output elasticity is a constant function of τ when $\psi = +\infty$, its dynamics becomes explosive if $\psi < +\infty$ reflecting mechanisms analogous to a dynamic beauty contest between firms and households analyzed by Angeletos and Lian (2018). As the rationality of both households and firms becomes more constrained, however, the reaction of current output exhibits horizon effects, as predicted in Farhi and Werning (2019). Additionally, observe that similarly to the case when $\psi \to +\infty$, the MFFG is most effective under targeted transfers.

Let us turn to the difference between output elasticities under MFFG and FG displayed in Figure 6. Similarly to the case with perfectly rigid prices, the announcement of additional fiscal transfer is most effective when it is targeted. There is, however, an important difference between the values of $\Delta\epsilon\left(\tau, k|\delta\right)$ in the HANK featuring $\psi = +\infty$ and the baseline (compare Figures 3 and 6): the interactions

Figure 6: Difference between interest rate elasticities of output in HANK (benchmark with the NKPC, $\psi < +\infty$) between MFFG and FG for neutral, uniform and targeted transfers.

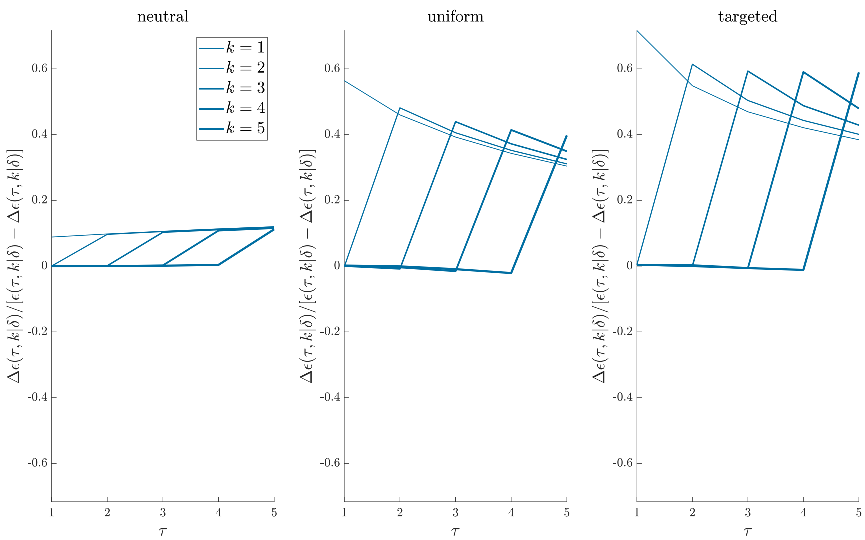


Notes: Difference (between MFFG and FG) in interest rate elasticities of output in period 0 computed for a one-time drop in interest rates equal to $dR = -0.0025$ (i.e. a one-percentage-point decrease in nominal rates in annual terms) that occurs in period τ. This difference is specified by formula (18) for three types of transfers in period τ (induced by monetary shock): neutral, uniform, and targeted. Solid lines denote output elasticities under level-k thinking. Different thickness of solid lines represent different values of k.

between the supply side and the demand block (absent in the former model) imply that $\Delta\epsilon\,(\tau, k|\delta)$ increases in k more dynamically for the lowest values of k, so that the absolute effectiveness of fiscal forward guidance peaks at $k = \tau$ and is significantly higher than for $k = 1$. This more pronounced reaction of output to fiscal communication in the economy with more rational agents when $k \leq \tau$ can be attributed to the fact that they recognize the presence of the dynamic beauty contest between the demand and the supply blocks that additionally propagates the effects of fiscal forward guidance.

The standardized effectiveness of fiscal guidance (see Figure 7) is less "polarized" than under $\psi \to +\infty$, i.e., its value features more variation for $k \leq \tau$ than in the model with perfectly rigid prices. This can be explained by the impact of the agents' awareness of the dynamic beauty contest between firms and households which increases in k and leads to more pronounced output effects of fiscal forward guidance. Note that if the interactions between aggregate demand and the supply side are taken into account, the value added of the fiscal announcement to the output effects of FG may reach 58% under uniform transfers and 66% under targeted transfers. As

Figure 7: The ratio between fiscal and monetary forward guidance effectiveness in HANK (benchmark with the NKPC, $\psi < +\infty$) for neutral, uniform and targeted transfers.



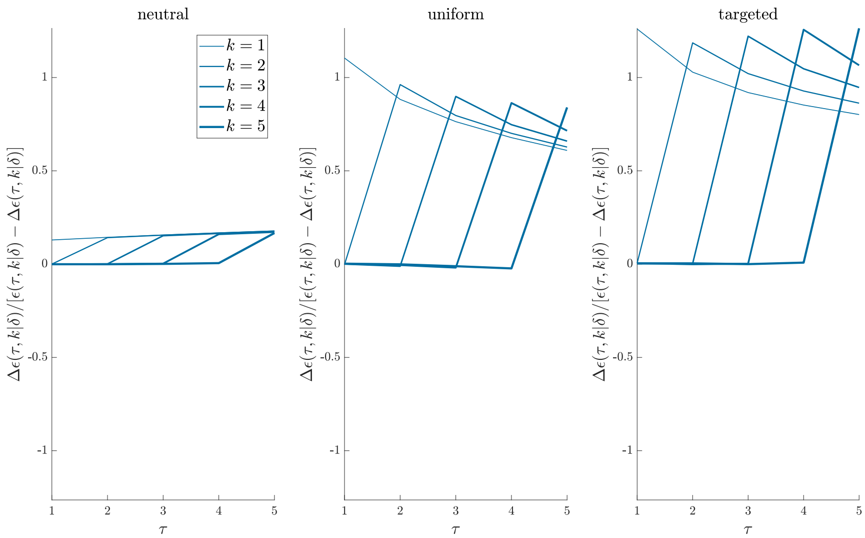
Notes: The difference (between MFFG and FG) in interest rate elasticities of output in period 0 computed for a one-time drop in interest rates equal to $dR = -0.0025$ (i.e. a one-percentage-point decrease in nominal rates in annual terms) that occurs in period τ, standardized by the interest rate elasticity corresponding to the standard FG (see formula 19) for three types of transfers in period τ (induced by monetary shock): neutral, uniform, and targeted. Solid lines denote output elasticities under level-k thinking. Different thickness of solid lines represent different values of k.

in the model with perfectly rigid prices, the gains from the fiscal forward guidance are present for $k \leq \tau$, i.e., they are most powerful when agents are least rational and when the standard FG is least effective.

HANK with $\psi < +\infty$ and high debt. Finally, let us discuss the role of public debt in boosting the effects of the standard FG through a coordinated fiscal communication. To this end, consider the model, in which the real value of government debt is 100% higher than in baseline and compute the FG and fiscal forward guidance in the economy featuring the new stationary equilibrium corresponding to higher aggregate level of liquid assets. Figure 8 shows the ratio between current output effects of the fiscal announcement and the effectiveness of the standard FG (see formula (19)).[16] Comparison with Figure 7 shows that higher public debt substantially amplifies the positive impact of the additional fiscal announcement on the effectiveness of the standard FG. Specifically, it may increase the FG effectiveness by up to 108% under uniform transfers and up to 120% under targeted transfers,

[16]Figures displaying the total effectiveness of MFFG and the absolute impact of fiscal forward guidance are delegated to the Appendix.

Figure 8: The ratio between fiscal and monetary forward guidance effectiveness in HANK (model with the NKPC and high debt) for neutral, uniform and targeted transfers.



Notes: The difference (between MFFG and FG) in interest rate elasticities of output in period 0 computed for a one-time drop in interest rates equal to $dR = -0.0025$ (i.e. a one-percentage-point decrease in nominal rates in annual terms) that occurs in period τ, standardized by the interest rate elasticity corresponding to the standard FG (see formula 19) for three types of transfers in period τ (induced by monetary shock): neutral, uniform, and targeted. Solid lines denote output elasticities under level-k thinking. Different thickness of solid lines represent different values of k.

i.e., the standardized fiscal forward guidance effects are almost two times larger than in the baseline.

6 Conclusions

In this paper, I discussed the coordination of fiscal policy announcements with FG aimed at boosting the effects of the latter. To this end, I considered a simple fiscal policy under which future transfers (financed with the expected budget gains resulting from FG) are announced to agents today to stimulate their current consumption. Using the RANK model with level-k thinking, I showed analytically that the impact of this fiscal communication becomes positive when agents feature bounded rationality. I then extended that framework by incorporating uninsured idiosyncratic income risk and derived the closed-form expressions characterizing the transmission of coordinated monetary-fiscal announcements and the isolated impact of fiscal forward guidance on current output in THANK. Using these analytical insights, I explored a related fully-blown HANK model with bounded rationality to quantify the output effects of FG enriched with fiscal communication. I found that fiscal forward guidance about transfers targeted towards 25% of the poorest (in terms of labor income) is able to raise the FG effects by 66% when debt-to-GDP ratio equals to 55%. This positive impact of fiscal forward guidance may reach 120% in the economy with debt-to-GDP ratio equal to 110%. Given this and the fact that the proposed policy does not increase neither taxes nor debt (along their transition paths) the MFFG can be viewed as a potentially effective stabilization tool in the economy where both the standard fiscal and monetary policies are constrained by high public debt levels and the ZLB, respectively.

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Appendix

Proofs

Before going to the proofs of statements presented in the main text, let us proof two helpful lemmas. First of them is a simple consequence of the Pascal identity:

Lemma 4. *Let a and b be positive integers and $a > b$. We have:*

$$\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-2}{b-1} + ... + \binom{b-1}{b-1}.$$

Moreover, if $a > c > b$:

$$\binom{a}{b} - \binom{c}{b} = \binom{a-1}{b-1} + \binom{a-2}{b-1} + ... + \binom{c}{b-1}.$$

Proof. To prove the first equation, we use the Pascal identity in a repetitive way:

$$\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b} = \binom{a-1}{b-1} + \binom{a-2}{b-1} + \binom{a-2}{b}$$

$$= ... = \binom{a-1}{b-1} + \binom{a-2}{b-1} + ... + \binom{b}{b-1} + \binom{b}{b}$$

$$= \binom{a-1}{b-1} + \binom{a-2}{b-1} + ... + \binom{b}{b-1} + \binom{b-1}{b-1}$$

where I used the fact that $\binom{b-1}{b-1} = \binom{b}{b}$. Second equation holds because:

$$\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b} = \binom{a-1}{b-1} + \binom{a-2}{b-1} + \binom{a-2}{b}$$

$$= ... = \binom{a-1}{b-1} + \binom{a-2}{b-1} + ... + \binom{c}{b-1} + \binom{c}{b}$$

which is equivalent to:

$$\begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} c \\ b \end{pmatrix} = \begin{pmatrix} a-1 \\ b-1 \end{pmatrix} + \begin{pmatrix} a-2 \\ b-1 \end{pmatrix} + ... + \begin{pmatrix} c \\ b-1 \end{pmatrix}$$

which I wanted to show. □

Second lemma plays an important role in the proofs related to THANK:

Lemma 5. *In the THANK model with the monetary policy shock at horizon $\tau > 0$ and monetary-fiscal rules specified as in Section 4 we have:*

$$\omega \cdot \left(dY_t^S - dT_t^S\right) = dY_t^H - dT_t^H \textit{ for } t \geq 0, \tag{26}$$

$$\frac{c^H}{c^S} = \omega \textit{ and } c^S = \frac{Y}{1-\lambda+\lambda\cdot\omega} \textit{ in the steady state,} \tag{27}$$

$$dY_0^S = dc_0^S = \frac{dY_0}{1-\lambda+\lambda\cdot\omega}. \tag{28}$$

Proof. Identity (26) follows immediately from the fact that, by condition (10), taxes are constant over time and thus:

$$dT_t^S = dT_t^H = 0$$

Then, from equation (8) we obtain condition (26).

From $Z^H = 0$ (H households are constrained), from (13) in the steady state, from (10), households' budget constraints in the stationary equilibrium can be rewritten as:

$$c^S + \frac{\bar{B}}{1-\lambda} + \frac{\bar{B}}{1-\lambda} \cdot \left(\bar{R}\cdot s - 1\right) = \bar{R}\cdot s \cdot \frac{\bar{B}}{1-\lambda} + Y^S$$

$$c^H + 0 + \frac{\bar{B}}{1-\lambda} \cdot \bar{R} \cdot (1-h) = \bar{R}\cdot(1-h)\cdot\frac{\bar{B}}{1-\lambda} + Y^H$$

which boils down to:

$$c^S = Y^S,\ c^H = Y^H.$$

Now, from (8):

$$\frac{c^H}{c^S} = \omega$$

and, from the resource constraint (12):

$$c^S = \frac{Y}{1-\lambda+\lambda\cdot\omega}.$$

Finally, let us prove condition (28). From the budget constraint of household S, from $Z_0^H = 0$, $R_0 = R$ (the last follows because $\tau > 0$ and hence interest rate is equal to its steady state value in period 0):

$$c_0^S + Z_1^S = \bar{R}\cdot B_0^S + T_0^S + 0 + Y_0^S.$$

Now, given (10) and from the market clearing for assets (13):

$$c_0^S + \frac{\bar{B}}{1-\lambda} = \bar{R}\cdot s\cdot\frac{\bar{B}}{1-\lambda} + Y_0^S$$

where I used the fact that $B = s\cdot\frac{\bar{B}}{1-\lambda}$ which implies:

$$c_0^S = Y_0^S.$$

Analogously, in period 0, the budget constraint of household H satisfies:

$$c_0^H + 0 = \bar{R}\cdot(1-h)\cdot\frac{\bar{B}}{1-\lambda} - \bar{R}\cdot(1-h)\cdot\frac{\bar{B}}{1-\lambda} + Y_0^H$$

and so:

$$c_0^H = Y_0^H.$$

We now plug $c_0^S = Y_0^S$ and $c_0^H = Y_0^H$ into the resource constraint (12) in period 0:

$$\lambda\cdot Y_0^H + (1-\lambda)\cdot Y_0^S = Y_0$$

which together with condition (8) gives:

$$Y_0^S = \frac{Y_0}{1-\lambda+\lambda\cdot\omega}$$

which, together with $c_0^S = Y_0^S$ and, after taking differences, yields equation (28). □

We are now in a position to prove statements from Sections 3 and 4.

Proof of Lemma 1

The Euler equation associated with the households problem is:

$$u'\left(R_t \cdot \theta \cdot b_t - T_t + Tr_t + Y_t - b_{t+1} + (1-\theta) \cdot b_t\right)$$

$$= \beta \cdot (R_{t+1} \cdot \theta + 1 - \theta) \cdot u'\left(R_{t+1} \cdot \theta \cdot b_{t+1} - T_{t+1} + Tr_{t+1} + Y_{t+1} - b_{t+2} + (1-\theta) \cdot b_{t+1}\right)$$

where I have used budget constraint to substitute for consumption. Note that in the Lemma we consider infinitesimal deviations of variables from their steady-state values and thus all variables in the Euler equation can be rewritten as $X_t = X + dX_t$ where dX_t is an infinitesimal deviation of variable X_t from its steady state value X. Using the Taylor approximation and subtracting $u'(c) = \beta \cdot (R \cdot \theta + 1 - \theta) \cdot u'(c)$ from both sides yields:

$$u''(c) \cdot \left[R \cdot \theta \cdot db_t + dR_t \cdot \theta \cdot b - dT_t + dTr_t + dY_t - db_{t+1} + (1-\theta) \cdot db_t\right] \tag{29}$$

$$= \beta \cdot dR_{t+1} \cdot \theta \cdot u'(c)$$

$$+\beta \cdot (R \cdot \theta + 1 - \theta) \cdot u''(c)$$

$$\cdot \left[R \cdot \theta \cdot db_{t+1} + dR_{t+1} \cdot \theta \cdot b - dT_{t+1} + dTr_{t+1} + dY_{t+1} - db_{t+2} + (1-\theta) \cdot db_{t+1}\right].$$

Note that from the Euler equation in the steady state we have: $\beta \cdot (R \cdot \theta + 1 - \theta) = 1$ or, equivalently $\beta \cdot \bar{R} = 1$. Moreover, using the assumed functional form of u:

$$\frac{u'(c)}{u''(c)} = -c \cdot \sigma.$$

Thus, we can rewrite equation (29) can be rewritten as:

$$db_{t+2} + db_{t+1} \cdot \left(-1 - \bar{R}\right) + db_t \cdot \bar{R}$$

$$= -\left(dR_t \cdot \theta \cdot b - dT_t + dTr_t + dY_t\right)$$

$$-\beta \cdot \theta \cdot c \cdot \sigma \cdot dR_{t+1} + \left(dR_{t+1} \cdot \theta \cdot b - dT_{t+1} + dTr_{t+1} + dY_{t+1}\right).$$

By denoting the RHS of the equation by D_{t+1} and shifting the indices backwards we obtain:

$$db_{t+1} + db_t \cdot \left(-1 - \bar{R}\right) + db_{t-1} \cdot \bar{R} = D_t.$$

Using lag operator $\mathcal{L}$ we get:

$$\left(1+\left(-1-\bar{R}\right)\cdot\mathcal{L}+\bar{R}\cdot\mathcal{L}^2\right)\cdot db_{t+1}=D_t \tag{30}$$

Factorizing:

$$1+\left(-1-\bar{R}\right)\cdot\mathcal{L}+\bar{R}\cdot\mathcal{L}^2=\left(1-\lambda_1\cdot\mathcal{L}\right)\cdot\left(1-\lambda_2\cdot\mathcal{L}\right)$$

$$=1-\left(\lambda_1+\lambda_2\right)\cdot\mathcal{L}+\lambda_1\cdot\lambda_2\cdot\mathcal{L}^2$$

which implies that:

$$\begin{cases}1+\bar{R}=\lambda_1+\lambda_2\\ \bar{R}=\lambda_1\cdot\lambda_2\end{cases}$$

and gives the following quadratic equation:

$$\psi\left(\lambda_1\right)=\lambda_1^2+\left(-1-\bar{R}\right)\cdot\lambda_1+\bar{R}$$

with the following roots:

$$\begin{cases}\lambda_1=1\\ \lambda_2=\bar{R}>1\end{cases}$$

which allows for rewriting equation (30) as:

$$\left(1-\mathcal{L}\right)\cdot\left(1-\bar{R}\cdot\mathcal{L}\right)\cdot db_{t+1}=D_t.$$

The root exceeding unity - $\bar{R}$ - is used for solving the equation forwards:

$$\left(1-\mathcal{L}\right)\cdot db_{t+1}=\frac{D_t}{1-\bar{R}\cdot\mathcal{L}}+\tilde{c}\cdot\bar{R}^t$$

To obtain a bounded solution constanc $\tilde{c}$ is set to zero and we then rewrite:

$$\left(1-\mathcal{L}\right)\cdot db_{t+1}=\frac{D_t}{1-\bar{R}\cdot\mathcal{L}}$$

$$\Leftrightarrow\left(1-\mathcal{L}\right)\cdot db_{t+1}=\frac{-\left(\bar{R}\cdot\mathcal{L}\right)^{-1}\cdot D_t}{1-\left(\bar{R}\cdot\mathcal{L}\right)^{-1}}$$

$$\Leftrightarrow (1-\mathcal{L})\cdot db_{t+1} = -\sum_{m=1}^{+\infty}\frac{D_{t+m}}{\bar{R}^m}.$$

Note that in the Lemma we consider period $t=0$ for which $db_0 = 0$ (because it is a pre-determined state variable). So:

$$db_1 = -\sum_{m=1}^{+\infty}\frac{D_m}{\bar{R}^m}. \tag{31}$$

Now, from the budget constraint in period 0 we have (we use $db_0 = 0$ again):

$$dc_0 + db_1 = dR_0\cdot\theta\cdot b - dT_0 + dTr_0 + dY_0.$$

We use it to substitute for db_1 in equation (31):

$$dc_0 = -\sum_{m=1}^{+\infty}\frac{D_m}{\bar{R}^m} + dR_0\cdot\theta\cdot b - dT_0 + dTr_0 + dY_0$$

Using the definitions of D_t and Υ_t:

$$dc_0 = \sum_{m=1}^{+\infty}\frac{-\beta\cdot\theta\cdot c\cdot\sigma\cdot dR_m + (dR_m\cdot\theta\cdot b + d\Upsilon_m) - (dR_{m-1}\cdot\theta\cdot b + d\Upsilon_{m-1})}{\bar{R}^m}$$

$$+dR_0\cdot\theta\cdot b + d\Upsilon_0.$$

Reordering terms:

$$dc_0 = \sum_{m=1}^{+\infty}\frac{-\beta\cdot\theta\cdot c\cdot\sigma\cdot dR_m - (dR_{m-1}\cdot\theta\cdot b + d\Upsilon_{m-1})}{\bar{R}^m}$$

$$+\sum_{m=0}^{+\infty}\frac{dR_m\cdot\theta\cdot b + d\Upsilon_m}{\bar{R}^m}.$$

Regrouping further:

$$dc_0 = \sum_{m=1}^{+\infty}\frac{-\beta\cdot\theta\cdot c\cdot\sigma\cdot dR_m}{\bar{R}^m} + \sum_{m=0}^{+\infty}\frac{dR_m\cdot\theta\cdot b + d\Upsilon_m}{\bar{R}^m}\cdot\left(1-\frac{1}{\bar{R}}\right).$$

using $\beta\cdot\bar{R} = 1$ (steady state version of the Euler equation) and $\bar{B} = b$ (market clearing condition for assets) yields:

$$dc_0 = \sum_{m=1}^{\infty} \beta^m \cdot \left[-\beta \cdot \theta \cdot C \cdot \sigma \cdot dR_m + \frac{1-\beta}{\beta} \cdot \left(\theta \cdot \bar{B} \cdot dR_{m-1} + d\Upsilon_{m-1} \right) \right]$$

which we wanted to show. QED.

Proof of Theorem 1

It is convenient to first prove the statement in the Theorem re-expressed in terms of $dY(\tau, k)$ (the deviation of output in period 0 from its steady-state value resulting from a one-time monetary shock, see equation (5)). I.e., we want to show that under MFFG: if $k > \tau$ then, we have

$$dY(\tau, k) = -\theta \cdot \sigma \cdot \beta \cdot c \cdot dR \tag{32}$$

if $k \leq \tau$ then we have

$$dY(\tau, k) = F(k-1|\tau - 1, 1-\beta) \cdot (-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR) \tag{33}$$

and the difference in output response between MFFG and FG (where the latter is denoted by $d\hat{Y}(\tau, k)$) is:

$$\Delta dY(\tau, k) = dY(\tau, k) - d\hat{Y}(\tau, k)$$

$$= f(k-1|\tau - 1, 1-\beta) \cdot (1-\beta) \cdot \left[-\theta \cdot \bar{B} \cdot dR \right]. \tag{34}$$

Before going to the main proof of (32)-(34), let us make several useful remarks.

Remarks. Note that the model has no time-varying state variables and thus it is purely forward-looking. Therefore, for $m > \tau$ we have $Y_m = Y$ and thus $dY_m = 0$. Moreover, given the assumed fiscal rule for taxes, we have $dT_m = 0$ for all $m \geq 0$. Additionally, $dR_m = dTr_m = 0$ for all $m \geq 0$ but for $m = \tau$. Moreover, note that if $k \geq 1$ and $\tau = 0$ then the equilibrium in period 0 satisfies (see the definition of the equilibrium in RANK):

$$Y_0^k = C\left(R + dR, Y_0^k, Tr_0, \{R, Y, 0\}_{m>0}\right)$$

$$Tr_0 = -dR \cdot \theta \cdot \bar{B}$$

which is, in fact equivalent, to the Rational Expectations Equilibrium, which (in RANK) implies that $Y_0^k = Y$, as transfers have no real effects. In other words, $dY(\tau = 0, k) = 0$. Additionally, note that for $\tau > 0$, the agents featuring $k \geq 2$ expect that the equilibrium described by those two equations materializes in period τ.

$dY(\tau, k)$ **in the case when** $k > \tau$**.** Let us start with the proof of expression (32), i.e., $dY(\tau, k)$ in the case when $k > \tau$. I will use the induction method. I.e., I first prove the statement for $\tau = 1$ and then, by assuming that it holds for $\{1, 2, ..., \tau - 1\}$ I show that it is true for any $\tau > 1$.

Let us start with $\tau = 1$. The characterization of dc_0 in that case is (see Lemma 1):

$$dc_0 = \beta \cdot \left[-\beta \cdot \theta \cdot \sigma \cdot c \cdot dR_1 + \frac{1-\beta}{\beta} \cdot (\theta \cdot b \cdot dR_0 - dT_0 + dTr_0 + dY(\tau, k)) \right]$$

$$+\beta^2 \cdot \left[0 + \frac{1-\beta}{\beta} \cdot (\theta \cdot b \cdot dR_1 - dT_1 + dTr_1 + dY_1) \right].$$

Let us modify the equation above by: replacing dc_0 with $dY(\tau, k)$ (equilibrium condition), plugging:

$$dT_0 = dTr_0 = dR_0 = dY_1 = dT_1 = 0$$

(see "Remarks"), replacing $dR_1 = dR$ and $Tr_1 = -dR \cdot \theta \cdot \bar{B}$. This gives:

$$dY(\tau, k) = -\theta \cdot \sigma \cdot \beta \cdot c \cdot dR$$

which is identical to (32).

I now show that the result holds for any $\tau > 1$ if it holds for $\{1, 2, ..., \tau - 1\}$. Substitute the equilibrium condition $dc_0 = dY(\tau, k)$ into the characterization from Lemma 1 and use "Remarks" to get:

$$dY(\tau, k) = \beta \cdot \left[0 + \frac{1-\beta}{\beta} \cdot (\theta \cdot b \cdot 0 - 0 + 0 + dY(\tau, k)) \right]$$

$$+\beta^2 \cdot \left[0 + \frac{1-\beta}{\beta} \cdot (\theta \cdot b \cdot 0 - 0 + 0 + dY(\tau - 1, k - 1))\right]$$

$$+...$$

$$+\beta^{\tau-1} \cdot \left[0 + \frac{1-\beta}{\beta} \cdot (\theta \cdot b \cdot 0 - 0 + 0 + dY(2, k - 1))\right]$$

$$+\beta^{\tau} \cdot \left[-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR + \frac{1-\beta}{\beta} \cdot (\theta \cdot b \cdot 0 - 0 + 0 + dY(1, k - 1))\right]$$

$$+\beta^{\tau+1} \cdot \left[0 + \frac{1-\beta}{\beta} \cdot \left(\theta \cdot b \cdot dR - 0 - dR \cdot \theta \cdot \bar{B} + 0\right)\right].$$

Now, by the principle of induction we assume that statement (32) holds for 1, 2, ..., $\tau - 1$ and thus (note that for all terms below we have $k - 1 > \tau - 1$ because we consider the case $k > \tau$ so we can use induction)

$$dY(\tau - 1, k - 1) = ... = dY(2, k - 1) = dY(1, k - 1)$$

$$= -\theta \cdot \sigma \cdot \beta \cdot c \cdot dR$$

and the market clearing for assets implies $b = \bar{B}$. All this means implies that:

$$dY(\tau, k) = \beta \cdot \frac{1-\beta}{\beta} \cdot [-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR]$$

$$+...$$

$$+\beta^{\tau-2} \cdot \frac{1-\beta}{\beta} \cdot [-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR]$$

$$+\beta^{\tau-1} \cdot \left[-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR + \frac{1-\beta}{\beta} \cdot [-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR]\right]$$

Reformulating:

$$dY(\tau, k) = (1-\beta) \cdot [-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR]$$

$$+\beta \cdot (1-\beta) \cdot [-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR]$$

$$+\beta^2 \cdot (1-\beta) \cdot [-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR]$$

$$+...$$

$$+\beta^{\tau-3} \cdot (1-\beta) \cdot [-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR]$$

$$+\beta^{\tau-2}\cdot(1-\beta)\cdot[-\theta\cdot\sigma\cdot\beta\cdot c\cdot dR]+\beta^{\tau-1}\cdot[-\theta\cdot\sigma\cdot\beta\cdot c\cdot dR]$$

which after canceling terms gives:

$$dY(\tau,k)=-\theta\cdot\sigma\cdot\beta\cdot c\cdot dR.$$

$dY(\tau,k)$ **in the case when** $k\leq\tau$**.** Let us turn to the case when $k\leq\tau$ now. By contrast to the proof of the case when $k>\tau$, the induction method is now applied to index k (instead of τ). In what follows, I first prove formula (33) for $k=1$ (and for all $\tau>0$). Next, I prove it for any k assuming that formula (33) holds for $k-1$.

Let us start with $k=1$. Observe that, using Lemma 1 and "Remarks", for all $\tau>0$:

$$\begin{aligned}dc_0=\beta\cdot&\left[0+\frac{1-\beta}{\beta}\cdot dY(\tau,1)\right]\\+\beta^2\cdot&\left[0+\frac{1-\beta}{\beta}\cdot dY(\tau-1,0)\right]\\&+...\\+\beta^{\tau-1}\cdot&\left[0+\frac{1-\beta}{\beta}\cdot dY(2,0)\right]\\+\beta^{\tau}\cdot&\left[-\theta\cdot\sigma\cdot\beta\cdot c\cdot dR+\frac{1-\beta}{\beta}\cdot dY(1,0)\right]\\+\beta^{\tau+1}\cdot&\left[0+\frac{1-\beta}{\beta}\cdot\left(\theta\cdot b\cdot dR-0-dR\cdot\theta\cdot\bar{B}+0\right)\right].\end{aligned}\tag{35}$$

We now use the fact that $b=\bar{B}$ and the fact that for $k=0$ agents expect the steady state to hold in the future. The latter implies that:

$$dY(1,0)=dY(2,0)=...=dY(\tau-1,0)=0.$$

Imposing the equilibrium condition (i.e., $dc_0=dY(\tau,1)$), yields:

$$dY(\tau,1)=\beta^{\tau-1}\cdot(-\theta\cdot\sigma\cdot\beta\cdot c\cdot dR)$$

which completes the proof for $k = 1$ because:

$$F(0|\tau-1, 1-\beta) = \sum_{l=0}^{0} \begin{pmatrix} \tau-1 \\ l \end{pmatrix} \cdot \beta^{\tau-l-1} \cdot (1-\beta)^l = \beta^{\tau-1}.$$

Let us now show that formula (33) holds for $1 < k \leq \tau$. Again, using Lemma 1, formula for $dY(\tau, k)$ for $k > \tau$, and "Remarks":

$$dc_0 = \beta \cdot \left[0 + \frac{1-\beta}{\beta} \cdot dY(\tau, k)\right]$$

$$+\beta^2 \cdot \left[0 + \frac{1-\beta}{\beta} \cdot dY(\tau-1, k-1)\right]$$

$$+...$$

$$+\beta^{\tau-k+2} \cdot \left[0 + \frac{1-\beta}{\beta} \cdot dY(k-1, k-1)\right]$$

$$+\beta^{\tau-k+3} \cdot \left[0 + \frac{1-\beta}{\beta} \cdot (-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR)\right]$$

$$+...$$

$$+\beta^{\tau-1} \cdot \left[0 + \frac{1-\beta}{\beta} \cdot (-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR)\right]$$

$$+\beta^{\tau} \cdot \left[-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR + \frac{1-\beta}{\beta} \cdot (-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR)\right]$$

$$+\beta^{\tau+1} \cdot \left[0 + \frac{1-\beta}{\beta} \cdot \left(\theta \cdot b \cdot dR - 0 - dR \cdot \theta \cdot \bar{B} + 0\right)\right].$$

Using the induction method, I assume that formula (33) holds for $k-1$. Given the equilibrium condition $dc_0 = dY(\tau, k)$, we get:

$$dY(\tau, k) = \tag{36}$$

$$= \beta \cdot \frac{1-\beta}{\beta} \cdot \sum_{l=0}^{k-2} \begin{pmatrix} \tau-2 \\ l \end{pmatrix} \cdot \beta^{\tau-l-2} \cdot (1-\beta)^l \cdot (-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR)$$

$$+\beta^2 \cdot \frac{1-\beta}{\beta} \cdot \sum_{l=0}^{k-2} \begin{pmatrix} \tau-3 \\ l \end{pmatrix} \cdot \beta^{\tau-l-3} \cdot (1-\beta)^l \cdot (-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR)$$

$$+...$$

$$+\beta^{\tau-k+1}\cdot\frac{1-\beta}{\beta}\cdot\sum_{l=0}^{k-2}\left(\begin{array}{c}k-2\\ l\end{array}\right)\cdot\beta^{k-l-2}\cdot(1-\beta)^{l}\cdot(-\theta\cdot\sigma\cdot\beta\cdot c\cdot dR)$$

$$+\beta^{\tau-k+2}\cdot\frac{1-\beta}{\beta}\cdot(-\theta\cdot\sigma\cdot\beta\cdot c\cdot dR)$$

$$+...$$

$$+\beta^{\tau-2}\cdot\frac{1-\beta}{\beta}\cdot(-\theta\cdot\sigma\cdot\beta\cdot c\cdot dR)$$

$$+\beta^{\tau-1}\cdot\left[-\theta\cdot\sigma\cdot\beta\cdot c\cdot dR+\frac{1-\beta}{\beta}\cdot(-\theta\cdot\sigma\cdot\beta\cdot c\cdot dR)\right]$$

$$+\beta^{\tau}\cdot\left[0+\frac{1-\beta}{\beta}\cdot\left(\theta\cdot b\cdot dR-0-dR\cdot\theta\cdot\bar{B}+0\right)\right].$$

I now use the market clearing condition for assets $b=\bar{B}$ and regroup/cancel terms to obtain:

$$dY(\tau,k)=(-\theta\cdot\sigma\cdot\beta\cdot c\cdot dR)$$

$$\times\Bigg\{(1-\beta)\cdot\left[\left(\begin{array}{c}\tau-2\\ 0\end{array}\right)\cdot\beta^{\tau-2}\cdot(1-\beta)^{0}+...+\left(\begin{array}{c}\tau-2\\ k-2\end{array}\right)\cdot\beta^{\tau-k}\cdot(1-\beta)^{k-2}\right]$$

$$+(1-\beta)\cdot\left[\left(\begin{array}{c}\tau-3\\ 0\end{array}\right)\cdot\beta^{\tau-2}\cdot(1-\beta)^{0}+...+\left(\begin{array}{c}\tau-3\\ k-2\end{array}\right)\cdot\beta^{\tau-k}\cdot(1-\beta)^{k-2}\right]$$

$$+...$$

$$+(1-\beta)\cdot\left[\left(\begin{array}{c}k-2\\ 0\end{array}\right)\cdot\beta^{\tau-2}\cdot(1-\beta)^{0}+...+\left(\begin{array}{c}k-2\\ k-2\end{array}\right)\cdot\beta^{\tau-k}\cdot(1-\beta)^{k-2}\right]$$

$$+\beta^{\tau-k+1}\Big\}$$

Rewriting:

$$dY(\tau,k)=(-\theta\cdot\sigma\cdot\beta\cdot c\cdot dR)$$

$$\times\Bigg\{(1-\beta)\cdot\left[\left(\begin{array}{c}\tau-2\\ 0\end{array}\right)+\left(\begin{array}{c}\tau-3\\ 0\end{array}\right)+...+\left(\begin{array}{c}k-2\\ 0\end{array}\right)\right]\cdot\beta^{\tau-2}\cdot(1-\beta)^{0}$$

$$+(1-\beta)\cdot\left[\left(\begin{array}{c}\tau-2\\ 1\end{array}\right)+\left(\begin{array}{c}\tau-3\\ 1\end{array}\right)+...+\left(\begin{array}{c}k-2\\ 1\end{array}\right)\right]\cdot\beta^{\tau-3}\cdot(1-\beta)^{1}$$

$$+...$$

$$+(1-\beta)\cdot\left[\begin{pmatrix}\tau-2\\k-3\end{pmatrix}+\begin{pmatrix}\tau-3\\k-3\end{pmatrix}+...+\begin{pmatrix}k-2\\k-3\end{pmatrix}\right]\cdot\beta^{\tau-(k-1)}\cdot(1-\beta)^{k-3}$$

$$+(1-\beta)\cdot\left[\begin{pmatrix}\tau-2\\k-2\end{pmatrix}+\begin{pmatrix}\tau-3\\k-2\end{pmatrix}+...+\begin{pmatrix}k-2\\k-2\end{pmatrix}\right]\cdot\beta^{\tau-k}\cdot(1-\beta)^{k-2}$$

$$\left.+\beta^{\tau-k+1}\right\}.$$

I use Lemma 4:

$$dY(\tau,k)=(-\theta\cdot\sigma\cdot\beta\cdot c\cdot dR)$$

$$\times\left\{(1-\beta)\cdot\left[\begin{pmatrix}\tau-1\\1\end{pmatrix}-\begin{pmatrix}k-2\\1\end{pmatrix}\right]\cdot\beta^{\tau-2}\cdot(1-\beta)^{0}\right.$$

$$+(1-\beta)\cdot\left[\begin{pmatrix}\tau-1\\2\end{pmatrix}-\begin{pmatrix}k-2\\2\end{pmatrix}\right]\cdot\beta^{\tau-3}\cdot(1-\beta)^{1}$$

$$+...$$

$$+(1-\beta)\cdot\left[\begin{pmatrix}\tau-1\\k-2\end{pmatrix}-\begin{pmatrix}k-2\\k-2\end{pmatrix}\right]\cdot\beta^{\tau-(k-1)}\cdot(1-\beta)^{k-3}$$

$$+(1-\beta)\cdot\begin{pmatrix}\tau-1\\k-1\end{pmatrix}\cdot\beta^{\tau-k}\cdot(1-\beta)^{k-2}$$

$$\left.+\beta^{\tau-k+1}\right\}.$$

I now group together all positive terms with binominal coefficients (and do the same for negative terms):

$$dY(\tau,k)=(-\theta\cdot\sigma\cdot\beta\cdot c\cdot dR)$$

$$\times\left\{\left[\begin{pmatrix}\tau-1\\1\end{pmatrix}\cdot\beta^{\tau-2}\cdot(1-\beta)+...+\begin{pmatrix}\tau-1\\k-1\end{pmatrix}\cdot\beta^{\tau-k}\cdot(1-\beta)^{k-1}\right]\right.$$

$$-\left[\begin{pmatrix}k-2\\1\end{pmatrix}\cdot\beta^{\tau-2}\cdot(1-\beta)+...+\begin{pmatrix}k-2\\k-2\end{pmatrix}\cdot\beta^{\tau-k-1}\cdot(1-\beta)^{k-2}\right]$$

$$\left.+\beta^{\tau-k+1}\right\}. \tag{37}$$

Now, note that we have:

$$\binom{k-2}{1} \cdot \beta^{\tau-2} \cdot (1-\beta) + ... + \binom{k-2}{k-2} \cdot \beta^{\tau-k-1} \cdot (1-\beta)^{k-2}$$

$$= \beta^{\tau-1-(k-2)} \cdot \left[\binom{k-2}{1} \cdot \beta^{k-3} \cdot (1-\beta) + ... + \binom{k-2}{k-2} \cdot (1-\beta)^{k-2}\right]$$

$$= \beta^{\tau-k-1} \cdot \left[1 - \underbrace{\binom{k-2}{0}}_{=1} \cdot \beta^{k-2}\right] \tag{38}$$

which follows from the fact that the probability masses of the binominal distribution add up to unity:

$$\binom{k-2}{0} \cdot \beta^{k-2} + \binom{k-2}{1} \cdot \beta^{k-3} \cdot (1-\beta) + ... + \binom{k-2}{k-2} \cdot (1-\beta)^{k-2} = 1.$$

Applying formula (38) to equation (37) yields:

$$dY(\tau,k) = (-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR)$$

$$\times \left\{ \binom{\tau-1}{1} \cdot \beta^{\tau-2} \cdot (1-\beta) + ... + \binom{\tau-1}{k-1} \cdot \beta^{\tau-k} \cdot (1-\beta)^{k-1} \right.$$

$$\left. + \binom{k-2}{0} \cdot \beta^{k-2} \cdot \beta^{\tau-k+1} \right\}.$$

Given that $\binom{k-2}{0} \cdot \beta^{k-2} \cdot \beta^{\tau-k+1} = \binom{\tau-1}{0} \cdot \beta^{\tau-1}$ (as $\binom{k-2}{0} = \binom{\tau-1}{0}$), we get:

$$dY(\tau,k) = (-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR) \cdot \sum_{l=0}^{k-1} \binom{\tau-1}{l} \cdot \beta^{\tau-1-l} \cdot (1-\beta)^{l},$$

so:

$$dY(\tau,k) = (-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR) \cdot F(k-1|\tau-1, 1-\beta),$$

which proves formula (33).

$\Delta dY(\tau,k)$ **in the case when** $k>\tau$**.** Given that we have computed $dY(\tau,k)$ corresponding to MFFG above, it is sufficient to compute $d\hat{Y}(\tau,k)$ and take the difference:

$$dY(\tau,k)-d\hat{Y}(\tau,k)$$

to get $\Delta dY(\tau,k)$.

Note that as $\tau\geq 1$ is considered, $k>\tau$ implies that $k\geq 2$. This, given "Remarks", means that agents anticipate fiscal transfer (induced by monetary shock at horizon τ) even if it is not announced by fiscal authority. This, in turn, implies that the line of reasoning applied when calculating $dY(\tau,k)$ (for $k>\tau$) can be applied here, too. Thus: $dY(\tau,k)=d\hat{Y}(\tau,k)$ and therefore $\Delta dY(\tau,k)$.

$\Delta dY(\tau,k)$ **in the case when** $k\leq\tau$**.** Given formulas (33) and (34), proving formula (34) is equivalent to showing that:

$$d\hat{Y}(\tau,k)=(-\theta\cdot\sigma\cdot\beta\cdot c\cdot dR)\cdot F(k-1|\tau-1,1-\beta) \tag{39}$$

$$+f(k-1|\tau-1,1-\beta)\cdot(1-\beta)\cdot\theta\cdot\bar{B}\cdot dR$$

To obtain this result for $d\hat{Y}(\tau,k)$, I use the induction method: I start by proving it for $k=1$ and then I show that it holds for $k>1$ if it holds for $k-1$.

Let us begin with $k=1$. Observe that, using Lemma 1 and "Remarks", for all $\tau>0$:

$$dc_0=\beta\cdot\left[0+\frac{1-\beta}{\beta}\cdot d\hat{Y}(\tau,1)\right]$$

$$+\beta^2\cdot\left[0+\frac{1-\beta}{\beta}\cdot d\hat{Y}(\tau-1,0)\right]$$

$$+...$$

$$+\beta^{\tau-1}\cdot\left[0+\frac{1-\beta}{\beta}\cdot d\hat{Y}(2,0)\right]$$

$$+\beta^{\tau}\cdot\left[-\theta\cdot\sigma\cdot\beta\cdot c\cdot dR+\frac{1-\beta}{\beta}\cdot d\hat{Y}(1,0)\right]$$

$$+\beta^{\tau+1}\cdot\left[0+\frac{1-\beta}{\beta}\cdot(\theta\cdot b\cdot dR-0-0+0)\right].$$

Note that, in contrast to dc_0 related to MFFG (see formula (35)), the impact of the fiscal announcement $-dR \cdot \theta \cdot \bar{B}$ is absent in the formula above. By the same token as in the case of MFFG:

$$d\hat{Y}(1,0) = d\hat{Y}(2,0) = ... = d\hat{Y}(\tau-1,0) = 0$$

and, after imposing equilibrium conditions $dc_0 = d\hat{Y}(\tau,1)$, $b = \bar{B}$ we obtain:

$$d\hat{Y}(\tau,1) = \beta^{\tau-1} \cdot (-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR) + \beta^{\tau-1} \cdot (1-\beta) \cdot \left(\theta \cdot \bar{B} \cdot dR\right)$$

which is consistent with formula (39).

Let us consider the case when $k > 1$. As discussed in "Remarks", $k > 1$ implies that households are sufficiently rational to anticipate fiscal transfers induced by the future monetary policy shock. So, using the reasoning applied in the case of MFFG, we can reformulate the characterization of dc_0 (see Lemma 1) to get:

$$dc_0 = \beta \cdot \left[0 + \frac{1-\beta}{\beta} \cdot d\hat{Y}(\tau,k)\right]$$

$$+\beta^2 \cdot \left[0 + \frac{1-\beta}{\beta} \cdot d\hat{Y}(\tau-1,k-1)\right]$$

$$+...$$

$$+\beta^{\tau-k+2} \cdot \left[0 + \frac{1-\beta}{\beta} \cdot d\hat{Y}(k-1,k-1)\right]$$

$$+\beta^{\tau-k+3} \cdot \left[0 + \frac{1-\beta}{\beta} \cdot (-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR)\right]$$

$$+...$$

$$+\beta^{\tau-1} \cdot \left[0 + \frac{1-\beta}{\beta} \cdot (-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR)\right]$$

$$+\beta^{\tau} \cdot \left[-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR + \frac{1-\beta}{\beta} \cdot (-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR)\right]$$

$$+\beta^{\tau+1} \cdot \left[0 + \frac{1-\beta}{\beta} \cdot \left(\theta \cdot b \cdot dR - 0 - dR \cdot \theta \cdot \bar{B} + 0\right)\right].$$

I now use the induction method: I assume that formula (39) holds for $k-1$. This, after using the equilibrium condition $dc_0 = d\hat{Y}(\tau, k)$, implies that:

$$d\hat{Y}(\tau,k) = \beta \cdot \frac{1-\beta}{\beta} \cdot \left[\sum_{l=0}^{k-2} \begin{pmatrix} \tau-2 \\ l \end{pmatrix} \cdot \beta^{\tau-l-2} \cdot (1-\beta)^{l} \cdot (-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR)\right.$$

$$\left. + \begin{pmatrix} \tau-2 \\ k-2 \end{pmatrix} \cdot \beta^{\tau-k} \cdot (1-\beta)^{k-2} \cdot (1-\beta) \cdot \left(\theta \cdot \bar{B} \cdot dR\right)\right]$$

$$+\beta^{2} \cdot \frac{1-\beta}{\beta} \cdot \left[\sum_{l=0}^{k-2} \begin{pmatrix} \tau-3 \\ l \end{pmatrix} \cdot \beta^{\tau-l-3} \cdot (1-\beta)^{l} \cdot (-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR)\right.$$

$$\left. + \begin{pmatrix} \tau-3 \\ k-2 \end{pmatrix} \cdot \beta^{\tau-k-1} \cdot (1-\beta)^{k-2} \cdot (1-\beta) \cdot \left(\theta \cdot \bar{B} \cdot dR\right)\right]$$

$$+...$$

$$+\beta^{\tau-k+1} \cdot \frac{1-\beta}{\beta} \cdot \left[\sum_{l=0}^{k-2} \begin{pmatrix} k-2 \\ l \end{pmatrix} \cdot \beta^{k-l-2} \cdot (1-\beta)^{l} \cdot (-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR)\right.$$

$$\left. + \begin{pmatrix} k-2 \\ k-2 \end{pmatrix} \cdot \beta^{0} \cdot (1-\beta)^{k-2} \cdot (1-\beta) \cdot \left(\theta \cdot \bar{B} \cdot dR\right)\right]$$

$$+\beta^{\tau-k+2} \cdot \frac{1-\beta}{\beta} \cdot (-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR)$$

$$+...$$

$$+\beta^{\tau-2} \cdot \frac{1-\beta}{\beta} \cdot (-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR)$$

$$+\beta^{\tau-1} \cdot \left[-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR + \frac{1-\beta}{\beta} \cdot (-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR)\right]$$

$$+\beta^{\tau} \cdot \left[0 + \frac{1-\beta}{\beta} \cdot \left(\theta \cdot b \cdot dR - 0 - dR \cdot \theta \cdot \bar{B} + 0\right)\right].$$

Now, by taking the difference between $dY(\tau, k)$ (see equation(36)) and $d\hat{Y}(\tau, k)$ above, we obtain the following formula for $\Delta dY(\tau, k)$:

$$\Delta dY(\tau,k) = \beta \cdot \frac{1-\beta}{\beta} \cdot \begin{pmatrix} \tau-2 \\ k-2 \end{pmatrix} \cdot \beta^{\tau-k} \cdot (1-\beta)^{k-2} \cdot (1-\beta) \cdot \left(-\theta \cdot \bar{B} \cdot dR\right)$$

$$+\beta^2 \cdot \frac{1-\beta}{\beta} \cdot \begin{pmatrix} \tau-3 \\ k-2 \end{pmatrix} \cdot \beta^{\tau-k-1} \cdot (1-\beta)^{k-2} \cdot (1-\beta) \cdot \left(-\theta \cdot \bar{B} \cdot dR\right)$$

$$+...$$

$$+\beta^{\tau-k+1} \cdot \frac{1-\beta}{\beta} \cdot \begin{pmatrix} k-2 \\ k-2 \end{pmatrix} \cdot \beta^0 \cdot (1-\beta)^{k-2} \cdot (1-\beta) \cdot \left(-\theta \cdot \bar{B} \cdot dR\right).$$

Which implies:

$$\Delta dY(\tau,k) = \left(-\theta \cdot \bar{B} \cdot dR\right)$$

$$\times \left[\begin{pmatrix} \tau-2 \\ k-2 \end{pmatrix} \cdot \beta^{\tau-k} \cdot (1-\beta)^k + \begin{pmatrix} \tau-3 \\ k-2 \end{pmatrix} \cdot \beta^{\tau-k} \cdot (1-\beta)^k \right.$$

$$\left. +... + \begin{pmatrix} k-2 \\ k-2 \end{pmatrix} \cdot \beta^{\tau-k} \cdot (1-\beta)^k \right].$$

Now, I use Lemma 4 to obtain:

$$\Delta dY(\tau,k) = \left(-\theta \cdot \bar{B} \cdot dR\right) \cdot \begin{pmatrix} \tau-1 \\ k-1 \end{pmatrix} \cdot \beta^{\tau-k} \cdot (1-\beta)^k$$

$$= \left(-\theta \cdot \bar{B} \cdot dR\right) \cdot \begin{pmatrix} \tau-1 \\ k-1 \end{pmatrix} \cdot \beta^{\tau-1-(k-1)} \cdot (1-\beta)^{k-1} \cdot (1-\beta)$$

$$= f(k-1|\tau-1, 1-\beta) \cdot (1-\beta) \cdot \left[-\theta \cdot \bar{B} \cdot dR\right]$$

which is identical to formula (34). This means that we proved formulas (32)-(34) and we are in a position to show that expressions in Theorem 1 hold. To this end, it suffices to standardize equations (32)-(34) and multiply them by $-\frac{1}{dR} \cdot \frac{\bar{R}}{Y}$ to re-express them as elasticities (see equation (5)).

For $k > \tau$ we have:

$$\epsilon(\tau,k) \equiv -\frac{\bar{R}}{Y} \cdot \frac{dY(\tau,k)}{dR}$$

$$= -\frac{\bar{R}}{Y} \cdot \frac{(-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR)}{dR} = \theta \cdot \sigma = F(k-1|\tau-1, 1-\beta) \cdot \theta \cdot \sigma,$$

where I used the fact that $\beta \cdot \bar{R} = 1$ (i.e., the steady-state version of the Euler equation) and that $c = Y$ (equilibrium condition in the steady state).

For $k \leq \tau$ we obtain:

$$\epsilon(\tau, k) \equiv -\frac{\bar{R}}{Y} \cdot \frac{dY(\tau, k)}{dR}$$

$$= -\frac{\bar{R}}{Y} \cdot \frac{F(k-1|\tau-1, 1-\beta) \cdot (-\theta \cdot \sigma \cdot \beta \cdot c \cdot dR)}{dR}$$

$$= F(k-1|\tau-1, 1-\beta) \cdot \theta \cdot \sigma.$$

Finally, let us turn to $\Delta\epsilon(\tau, k)$:

$$\Delta\epsilon(\tau, k) \equiv -\frac{\bar{R}}{Y} \cdot \frac{\Delta dY(\tau, k)}{dR}$$

$$= -\frac{\bar{R}}{Y} \cdot \frac{f(k-1|\tau-1, 1-\beta) \cdot (1-\beta) \cdot \left[-\theta \cdot \bar{B} \cdot dR\right]}{dR}$$

$$= f(k-1|\tau-1, 1-\beta) \cdot \frac{iMPC}{1-MPC} \cdot \frac{\theta \cdot \bar{B} \cdot \bar{R}}{c}$$

where I used formulas for MPC and $iMPC$ (see equations (3) and (4)) and the equilibrium condition $c = Y$ in the steady state. This completes the proof of Theorem 1. QED.

Proof of Lemma 2

Let us consider an arbitrary period t and the monetary policy shock dR that arrives at time $t + \tau$ where $\tau = 0$. Note that from the resource constraint (see equation (12)):

$$\lambda \cdot dc_t^H + (1-\lambda) \cdot dc_t^S = dY_t$$

which is equivalent to:

$$dY_t = (1-\lambda) \cdot MPC^S \cdot \left[dY_t^S - dT_t^S + dTr_t^S + dR \cdot \theta \cdot s \cdot Z^S\right]$$

$$+\lambda \cdot 1 \cdot \left[dY_t^H - dT_t^H + dTr_t^H + dR \cdot \theta \cdot (1-h) \cdot Z^S\right]$$

where I used the fact that $MPC^H = 1$ and that $Z^H = 0$ in the stationary equilibrium. Fiscal rule adopted in the THANK model implies that $dT_t^S = 0$ and $dT_t^H = 0$. Therefore:

$$dY_t = (1-\lambda) \cdot MPC^S \cdot \left[dY_t^S + dTr_t^S + dR \cdot \theta \cdot s \cdot Z^S\right] \tag{40}$$

$$+\lambda \cdot 1 \cdot \left[dY_t^H + dTr_t^H + dR \cdot \theta \cdot (1-h) \cdot Z^S\right].$$

Let us now turn to the aggregate budget constraint of households in period t:

$$(1-\lambda) \cdot \left[c_t^S + Z_{t+1}^S\right] + \lambda \cdot \left[c_t^H + Z_{t+1}^H\right]$$

$$= (1-\lambda) \cdot \left[R \cdot \theta \cdot B_t^S - T_t^S + Tr_t^S + Y_t^S\right]$$

$$+\lambda \cdot \left[R \cdot \theta \cdot B_t^H - T_t^H + Tr_t^H + Y_t^H\right].$$

Using the market clearing for assets (equation (13)) and $Z_t^H = 0$ (implying $B_t^S = s \cdot \frac{\bar{B}}{1-\lambda}$ and $B_t^H = (1-h) \cdot \frac{\bar{B}}{1-\lambda}$) and the resource constraint for goods (equation (12)):

$$Y_t + (1-\lambda) \cdot \frac{\bar{B}}{1-\lambda}$$

$$= (1-\lambda) \cdot \left[R \cdot \theta \cdot s \cdot \frac{\bar{B}}{1-\lambda} - T_t^S + Tr_t^S + Y_t^S\right]$$

$$+\lambda \cdot \left[R \cdot \theta \cdot (1-h) \cdot \frac{\bar{B}}{1-\lambda} - T_t^H + Tr_t^H + Y_t^H\right].$$

Taking the differences (and using $dT_t^S = 0$ and $dT_t^H = 0$ again) yields:

$$dY_t = (1-\lambda) \cdot \left[dR \cdot \theta \cdot s \cdot \frac{\bar{B}}{1-\lambda} + dTr_t^S + dY_t^S\right] \tag{41}$$

$$+\lambda \cdot \left[dR \cdot \theta \cdot (1-h) \cdot \frac{\bar{B}}{1-\lambda} + dTr_t^H + dY_t^H\right].$$

From equation (11) we have:

$$(1-\lambda) \cdot dTr_t^S + \lambda \cdot dTr_t^H = -\bar{B} \cdot dR \cdot \theta$$

and given that:

$$(1-\lambda) \cdot dR \cdot \theta \cdot s \cdot \frac{\bar{B}}{1-\lambda} + \lambda \cdot dR \cdot \theta \cdot (1-h) \cdot \frac{\bar{B}}{1-\lambda}$$

$$= \bar{B} \cdot dR \cdot \theta$$

where I used the fact that $\frac{\lambda}{1-\lambda}=\frac{1-s}{1-h}$. Thus, equation (41) becomes:

$$dY_t=(1-\lambda)\cdot dY_t^S+\lambda\cdot dY_t^H$$

which together with condition (8) implies:

$$dY_t^S=\frac{1}{1-\lambda+\lambda\cdot\omega}\cdot dY_t,\; dY_t^H=\frac{\omega}{1-\lambda+\lambda\cdot\omega}\cdot dY_t.$$

We plug those expressions for dY_t^S and dY_t^H into condition (40) and use equations (11) and (13) to get:

$$dY_t=(1-\lambda)\cdot MPC^S\cdot\left[\frac{1}{1-\lambda+\lambda\cdot\omega}\cdot dY_t-\frac{1-\delta}{1-\lambda}\cdot\bar{B}\cdot dR\cdot\theta+dR\cdot\theta\cdot s\cdot\frac{\bar{B}}{1-\lambda}\right]$$

$$+\lambda\cdot\left[\frac{\omega}{1-\lambda+\lambda\cdot\omega}\cdot dY_t-\frac{\delta}{\lambda}\cdot\bar{B}\cdot dR\cdot\theta+dR\cdot\theta\cdot(1-h)\cdot\frac{\bar{B}}{1-\lambda}\right].$$

Rearranging:

$$dY_t\cdot\frac{(1-\lambda)\cdot\left(1-MPC^S\right)}{1-\lambda+\lambda\cdot\omega}$$
$$=\left(1-MPC^S\right)\cdot(1-\delta-s)\cdot\bar{B}\cdot\theta\cdot dR$$

which gives the formula for the output reponse to monetary shock featuring horizon 0:

$$dY_t=\frac{1-\lambda+\lambda\cdot\omega}{1-\lambda}\cdot(1-\delta-s)\cdot\bar{B}\cdot\theta\cdot dR$$

which multiplied by $-\frac{1}{dR}\cdot\frac{\bar{R}}{Y}$ yields:

$$-\frac{dY_t}{dR}\cdot\frac{\bar{R}}{Y}=-\frac{\bar{B}\cdot\bar{R}\cdot\theta}{Y}\cdot\frac{1-\lambda+\lambda\cdot\omega}{1-\lambda}\cdot(1-\delta-s)$$

and therefore:

$$\epsilon\left(0|\delta\right)=-\frac{\bar{B}\cdot\bar{R}\cdot\theta}{Y}\cdot\frac{1-\lambda+\lambda\cdot\omega}{1-\lambda}\cdot\left(1-\delta-s\right).$$

This completes the proof. QED.

Proof of Lemma 3

The Euler equation of the unconstrained agent is:

$$u'\left(c_t^S\right) = \beta \cdot \bar{R}_{t+1} \cdot \left[s \cdot u'\left(c_{t+1}^S\right) + (1-s) \cdot u'\left(c_{t+1}^H\right)\right].$$

Plugging the budget constraints of both agents yields:

$$u'\left(\bar{R}_t \cdot \left[s \cdot Z_t^S + (1-s) \cdot Z_t^H\right] + \Upsilon_t^S - Z_{t+1}^S\right)$$

$$= \beta \cdot \bar{R}_{t+1} \cdot s \cdot u'\left(\bar{R}_{t+1} \cdot \left[s \cdot Z_{t+1}^S + (1-s) \cdot Z_{t+1}^H\right] + \Upsilon_{t+1}^S - Z_{t+2}^S\right)$$

$$+\beta \cdot \bar{R}_{t+1} \cdot (1-s) \cdot u'\left(\bar{R}_{t+1} \cdot \left[(1-h) \cdot Z_{t+1}^S + h \cdot Z_{t+1}^H\right] + \Upsilon_{t+1}^H - Z_{t+2}^H\right).$$

Note that we consider the situation when $Z_t^H = 0$ for all t. Rewriting the variables in terms of deviations from the steady state (i.e., as $X_t = X + dX_t$ where dX_t is an infinitesimal deviation of variable X_t from its steady state value X) and using the Taylor approximation and subtracting the steady-state version of the Euler equation of the unconstrained agent yields:

$$u''\left(c^S\right) \cdot \left(s \cdot \bar{R} \cdot dZ_t^S + s \cdot Z^S \cdot d\bar{R}_t + d\Upsilon_t^S - dZ_{t+1}^S\right) \tag{42}$$

$$= \beta \cdot \left[s \cdot u'\left(c^S\right) + (1-s) \cdot u'\left(c^H\right)\right] \cdot d\bar{R}_{t+1}$$

$$\beta \cdot \bar{R} \cdot s \cdot u''\left(c^S\right) \cdot \left(s \cdot \bar{R} \cdot dZ_{t+1}^S + s \cdot Z^S \cdot d\bar{R}_{t+1} + d\Upsilon_{t+1}^S - dZ_{t+2}^S\right)$$

$$+\beta \cdot \bar{R} \cdot (1-s) \cdot u''\left(c^H\right)\left(\bar{R} \cdot (1-h) \cdot dZ_{t+1}^S + (1-h) \cdot Z^S \cdot d\bar{R}_{t+1} + d\Upsilon_{t+1}^H\right).$$

Note that from the steady-state version of the Euler equation for the unconstrained agent:

$$\frac{\beta \cdot \left[s \cdot u'\left(c^S\right) + (1-s) \cdot u'\left(c^H\right)\right]}{u''\left(c^S\right)} = \frac{u'\left(c^S\right)}{\bar{R}} \cdot \frac{1}{u''\left(c^S\right)}$$

Rearranging further under the assumed specification of the utility function:

$$\frac{u'\left(c^S\right)}{\bar{R}} \cdot \frac{1}{u''\left(c^S\right)} = -\frac{\sigma \cdot c^S}{\bar{R}}.$$

Let us use this outcome to reformulate equation (42) divided by $\beta \cdot \bar{R} \cdot s \cdot u''\left(c^S\right)$:

$$dZ^S_{t+2} - \left(\frac{1 + \beta \cdot \bar{R}^2 \cdot \left(s^2 + (1-s) \cdot (1-h) \cdot \omega^{-\frac{1}{\sigma}-1}\right)}{\beta \cdot \bar{R} \cdot s}\right) \cdot dZ^S_{t+1} + \frac{1}{\beta} \cdot dZ^S_t \tag{43}$$

$$= \frac{1}{\beta \cdot \bar{R} \cdot s} \cdot \left(-\frac{\sigma \cdot c^S}{\bar{R}}\right) \cdot d\bar{R}_{t+1} + s \cdot Z^S \cdot d\bar{R}_{t+1} + d\Upsilon^S_{t+1}$$

$$+\frac{1-s}{s} \cdot \omega^{-\frac{1}{\sigma}-1} \cdot \left[(1-h) \cdot Z^S \cdot d\bar{R}_{t+1} + d\Upsilon^H_{t+1}\right]$$

$$-\frac{1}{\beta \cdot \bar{R} \cdot s} \cdot \left(s \cdot Z^S \cdot d\bar{R}_t + d\Upsilon^S_t\right)$$

where I used Lemma 5 when replacing $\frac{c^H}{c^S}$ with ω. I define:

$$D_{t+1} \equiv \frac{1}{\beta \cdot \bar{R} \cdot s} \cdot \left\{\left(-\frac{\sigma \cdot c^S}{\bar{R}}\right) \cdot d\bar{R}_{t+1}\right. \tag{44}$$

$$+\beta \cdot \bar{R} \cdot s \cdot \left(s \cdot Z^S \cdot d\bar{R}_{t+1} + d\Upsilon^S_{t+1}\right)$$

$$+\beta \cdot \bar{R} \cdot (1-s) \cdot \omega^{-\frac{1}{\sigma}-1} \cdot \left[(1-h) \cdot Z^S \cdot d\bar{R}_{t+1} + d\Upsilon^H_{t+1}\right]$$

$$\left. - \left(s \cdot Z^S \cdot d\bar{R}_t + d\Upsilon^S_t\right)\right\}$$

which allows for rewriting equation (43) (after shifting indices backwards) as:

$$\left(1 - \frac{1 + \beta \cdot \bar{R}^2 \cdot \left(s^2 + (1-s) \cdot (1-h) \cdot \omega^{-\frac{1}{\sigma}-1}\right)}{\beta \cdot \bar{R} \cdot s} \cdot \mathcal{L} + \frac{1}{\beta} \cdot \mathcal{L}^2\right) \cdot dZ^S_{t+1} \tag{45}$$

$$= D_t.$$

where $\mathcal{L}$ is the lag operator. I factorize the polynominal:

$$1 - \frac{1 + \beta \cdot \bar{R}^2 \cdot \left(s^2 + (1-s) \cdot (1-h) \cdot \omega^{-\frac{1}{\sigma}-1}\right)}{\beta \cdot \bar{R} \cdot s} \cdot \mathcal{L} + \frac{1}{\beta} \cdot \mathcal{L}^2$$

$$= (1 - \lambda_1 \cdot \mathcal{L}) \cdot (1 - \lambda_2 \cdot \mathcal{L})$$

$$= 1 - (\lambda_1 + \lambda_2) \cdot \mathcal{L} + \lambda_1 \cdot \lambda_2 \cdot \mathcal{L}^2$$

which implies that:

$$\begin{cases} \frac{1+\beta\cdot\bar{R}^2\cdot\left(s^2+(1-s)\cdot(1-h)\cdot\omega^{-\frac{1}{\sigma}-1}\right)}{\beta\cdot\bar{R}\cdot s} = \lambda_1+\lambda_2 \\ \frac{1}{\beta} = \lambda_1\cdot\lambda_2 \end{cases}$$

This, in turn, implies the following quadratic equation:

$$\underbrace{\lambda_1^2 - \frac{1+\beta\cdot\bar{R}^2\cdot\left(s^2+(1-s)\cdot(1-h)\cdot\omega^{-\frac{1}{\sigma}-1}\right)}{\beta\cdot\bar{R}\cdot s}\cdot\lambda_1+\frac{1}{\beta}}_{\equiv\Psi(\lambda_1)} = 0.$$

First, note that $\Psi(0) > 0$. Second, we now show that $\Psi(1) < 0$. Observe that:

$$1 - \frac{1+\beta\cdot\bar{R}^2\cdot\left(s^2+(1-s)\cdot(1-h)\cdot\omega^{-\frac{1}{\sigma}-1}\right)}{\beta\cdot\bar{R}\cdot s}+\frac{1}{\beta} < 0$$

$$\Longleftrightarrow$$

$$\beta\cdot\bar{R}\cdot s - 1 - \beta\cdot\bar{R}^2\cdot\left(s^2+(1-s)\cdot(1-h)\cdot\omega^{-\frac{1}{\sigma}-1}\right)+\bar{R}\cdot s < 0.$$

Now, $\beta\cdot\bar{R}\cdot s < 1$ so to obtain $\Psi(1) < 0$, it is sufficient to show that:

$$-\beta\cdot\bar{R}^2\cdot\left(s^2+(1-s)\cdot(1-h)\cdot\omega^{-\frac{1}{\sigma}-1}\right)+\bar{R}\cdot s < 0$$

which can be re-expressed as:

$$s < \beta\cdot\bar{R}\cdot\left(s^2+(1-s)\cdot(1-h)\cdot\omega^{-\frac{1}{\sigma}-1}\right)$$

and given that $\beta\cdot\bar{R} = \left(s+(1-s)\cdot\omega^{-\frac{1}{\sigma}}\right)^{-1}$ (from the steady-state version of the Euler equation of the unconstrained household), we get:

$$s < \frac{s^2+(1-s)\cdot(1-h)\cdot\omega^{-\frac{1}{\sigma}-1}}{s+(1-s)\cdot\omega^{-\frac{1}{\sigma}}}$$

and thus:

$$s^2+s\cdot(1-s)\cdot\omega^{-\frac{1}{\sigma}}$$

$$< s^2+(1-s)\cdot(1-h)\cdot\omega^{-\frac{1}{\sigma}-1}$$

which is equivalent to the assumed condition (14).

All this implies that polynominal Ψ has a root between $(0,1)$ and another root is strictly larger than one. From the symmetry between λ_1 and λ_2, I denote the lower root by λ_1 and the larger one by λ_2:

$$\begin{cases} \lambda_1 = \frac{1}{2} \cdot \left(\frac{1+\beta \cdot \bar{R}^2 \cdot \left(s^2+(1-s)\cdot(1-h)\cdot\omega^{-\frac{1}{\sigma}-1}\right)}{\beta \cdot \bar{R} \cdot s} - \sqrt{\Delta} \right) \\ \lambda_2 = \frac{1}{2} \cdot \left(\frac{1+\beta \cdot \bar{R}^2 \cdot \left(s^2+(1-s)\cdot(1-h)\cdot\omega^{-\frac{1}{\sigma}-1}\right)}{\beta \cdot \bar{R} \cdot s} + \sqrt{\Delta} \right) \end{cases}$$

where:

$$\Delta = \left[\frac{1 + \beta \cdot \bar{R}^2 \cdot \left(s^2 + (1-s) \cdot (1-h) \cdot \omega^{-\frac{1}{\sigma}-1}\right)}{\beta \cdot \bar{R} \cdot s} \right]^2 - 4 \cdot \frac{1}{\beta}.$$

Now, the root that exceeds unity (i.e. λ_2) is used for solving the difference equation (45) forward:

$$(1 - \lambda_1 \cdot \mathcal{L}) \cdot dZ^S_{t+1} = \frac{D_t}{1 - \lambda_2 \cdot \mathcal{L}} + \tilde{c} \cdot \lambda_2^t$$

where $\tilde{c}$ is a constant associated with the general solution to the difference equation. To obtain bounded solution, I impose $\tilde{c} = 0$, which yields:

$$(1 - \lambda_1 \cdot \mathcal{L}) \cdot dZ^S_{t+1} = \frac{D_t}{1 - \lambda_2 \cdot \mathcal{L}}$$

$$\Leftrightarrow (1 - \lambda_1 \cdot \mathcal{L}) \cdot dZ^S_{t+1} = \frac{-\left(\lambda_2 \cdot \mathcal{L}\right)^{-1} \cdot D_t}{1 - \left(\lambda_2 \cdot \mathcal{L}\right)^{-1}}$$

$$\Leftrightarrow (1 - \lambda_1 \cdot \mathcal{L}) \cdot dZ^S_{t+1} = -\sum_{m=1}^{+\infty} \frac{D_{t+m}}{\lambda_2^m}.$$

Let us now consider period $t = 0$. From the household's S budget constraint in period 0:

$$dC^S_0 + dZ^S_1 = \bar{R} \cdot dB^S_0 + d\bar{R}_0 \cdot B^S + d\Upsilon^S_0.$$

Given that B^S_0 is pre-determined we have $dB^S_0 = 0$. By the same token, $dZ^S_0 = 0$ and therefore:

$$(1 - \lambda_1 \cdot \mathcal{L}) \cdot dZ^S_1 = -\sum_{m=1}^{+\infty} \frac{D_m}{\lambda_2^m}$$

$$\Longrightarrow$$

$$d\bar{R}_0 \cdot B^S + d\Upsilon_0^S - dC_0^S = -\sum_{m=1}^{+\infty} \frac{D_m}{\lambda_2^m}$$

$$\Longleftrightarrow$$

$$dC_0^S = \sum_{t=1}^{+\infty} \frac{D_t}{\lambda_2^t} + d\bar{R}_0 \cdot B^S + d\Upsilon_0^S.$$

I rewrite this equation using the definition of D_t (see equation (44)) and after denoting $\mathcal{M} \equiv \frac{1}{\lambda_2}$:

$$dC_0^S = \frac{1}{\beta \cdot \bar{R} \cdot s} \cdot \sum_{t=1}^{+\infty} \mathcal{M}^t \left\{ \left(-\frac{\sigma \cdot c^S}{\bar{R}} \right) \cdot d\bar{R}_t \right.$$
$$+\beta \cdot \bar{R} \cdot s \cdot \left(s \cdot Z^S \cdot d\bar{R}_t + d\Upsilon_t^S \right)$$
$$+\beta \cdot \bar{R} \cdot (1-s) \cdot \omega^{-\frac{1}{\sigma}-1} \cdot \left[(1-h) \cdot Z^S \cdot d\bar{R}_t + d\Upsilon_t^H \right]$$
$$\left. - \left(s \cdot Z^S \cdot d\bar{R}_{t-1} + d\Upsilon_{t-1}^S \right) \right\}$$
$$+d\bar{R}_0 \cdot B^S + d\Upsilon_0^S,$$

which, after observing that $d\bar{R}_t = \theta \cdot dR_t$, $B^S = s \cdot Z^S$, and $B^H = (1-h) \cdot Z^S$ (note that $Z^H = 0$), can be rewritten as:

$$dc_0^S = \frac{1}{\beta \cdot \bar{R} \cdot s} \cdot \sum_{t=1}^{\infty} \mathcal{M}^t \cdot \left[-\frac{\theta \cdot c^S \cdot \sigma}{\bar{R}} \cdot dR_t + \frac{\beta \cdot \bar{R} \cdot (1-s)}{\omega^{\frac{1}{\sigma}+1}} \cdot \left(\theta \cdot B^H \cdot dR_t + d\Upsilon_t^H \right) \right.$$
$$\left. + \frac{\beta \cdot \bar{R} \cdot s - \mathcal{M}}{\mathcal{M}} \cdot \left(\theta \cdot B^S \cdot dR_{t-1} + d\Upsilon_{t-1}^S \right) \right]$$

which we wanted to show. QED.

Proof of Theorem 2

As in the case of the proof of Theorem 1, let us first prove the statement of Theorem 2 re-expressed in terms of $dY(\tau, k|\delta)$ (the deviation of output in period 0 from its steady-state value resulting from a one-time monetary shock, see equation (17)). Therefore, I start by showing that the following formula is true (and then argue that it is equivalent to the one in Theorem 2):

$$dY^S(\tau,k|\delta) = f(k-1|\tau-1,1-\mathcal{M}) \cdot \Bigg\{ -\frac{\sigma \cdot c^S}{\bar{R}} \cdot d\bar{R} \tag{46}$$

$$+\frac{\beta \cdot \bar{R} \cdot (1-s)}{\omega^{\frac{1}{\sigma}+1}} \cdot \left(\frac{(1-h) \cdot \bar{B} \cdot d\bar{R}}{1-\lambda} - \frac{\delta}{\lambda} \cdot \bar{B} \cdot d\bar{R} \right)$$

$$+\left(\beta \cdot \bar{R} \cdot s - \mathcal{M}\right) \cdot \left(\frac{s \cdot \bar{B} \cdot d\bar{R}}{1-\lambda} - \frac{1-\delta}{1-\lambda} \cdot \bar{B} \cdot d\bar{R} \right) \Bigg\}$$

$$+F(k-2|\tau-1,1-\mathcal{M}) \cdot \Bigg\{ -\frac{\sigma \cdot c^S}{\bar{R}} \cdot d\bar{R}$$

$$+(1-\mathcal{M}) \cdot \frac{dY(0,k-1|\delta)}{1-\lambda+\lambda \cdot \omega}$$

$$+\frac{\beta \cdot \bar{R} \cdot (1-s)}{\omega^{\frac{1}{\sigma}+1}} \cdot \left(\frac{(1-h) \cdot \bar{B} \cdot d\bar{R}}{1-\lambda} - \frac{\delta}{\lambda} \cdot \bar{B} \cdot d\bar{R} \right)$$

$$+\left(\beta \cdot \bar{R} \cdot s - \mathcal{M}\right) \cdot \left(\frac{s \cdot \bar{B} \cdot d\bar{R}}{1-\lambda} - \frac{1-\delta}{1-\lambda} \cdot \bar{B} \cdot d\bar{R} \right) \Bigg\}$$

before showing this result, let us make several remarks to which I refer in the rest of the proof.

Remarks. Note that as the distribution of agents across two islands is constant over time, the model has no time-varying state variables and thus, like RANK, it is purely forward-looking. Therefore, for $m > \tau$ we have $Y_m = Y$ and thus:

$$dY_m = dY_m^H = dY_m^S = 0.$$

Moreover, given the assumed fiscal rule for taxes (see formula (10)), we have $dT_m^H = 0$ and $dT_m^S = 0$ for all $m \geq 0$. Additionally:

$$dR_m = dTr_m^H = dTr_m^S = 0$$

for all $m \geq 0$ but for $m = \tau$.

Finally, by the same token as in the "Remarks" in the proof of Theorem 1, note that for the case when $\tau = 0$, REE is equivalent to the level-k equilibrium for all $k \geq 1$ which, in turn, implies that for $\tau > 0$, agents featuring $k \geq 2$ fully anticipate the fiscal transfer induced by the monetary policy shock at horizon τ (irrespectively of the fiscal announcement).

$dY^S(\tau, k|\delta)$ **in the case when** $k > \tau$**.** Let us start with the proof of expression (46), i.e., $dY^S(\tau, k|\delta)$ in the case when $k > \tau$. I will use the induction method. I.e., I first prove the statement for $\tau = 1$ and then, by assuming that it holds for $\{1, 2, ..., \tau - 1\}$ I show that it is true for any $\tau > 1$.

Let us start with $\tau = 1$. The characterization of dc_0^S in that case is (see Lemma 3 and note that $d\bar{R} = \theta \cdot dR$, $B^S = s \cdot \frac{\bar{B}}{1-\lambda}$, and $B^H = (1-h) \cdot \frac{\bar{B}}{1-\lambda}$):

$$dc_0^S = \frac{1}{\beta \cdot \bar{R} \cdot s} \cdot \mathcal{M} \cdot \left[-\frac{c^S \cdot \sigma}{\bar{R}} \cdot d\bar{R} \right. \tag{47}$$

$$+\frac{\beta \cdot \bar{R} \cdot (1-s)}{\omega^{\frac{1}{\sigma}+1}} \cdot \left((1-h) \cdot \frac{\bar{B}}{1-\lambda} \cdot d\bar{R} + dY^H(0, k-1|\delta) + dTr_1^H \right)$$

$$\left. +\frac{\beta \cdot \bar{R} \cdot s - \mathcal{M}}{\mathcal{M}} \cdot dY^S(1, k|\delta) \right]$$

$$+\frac{1}{\beta \cdot \bar{R} \cdot s} \cdot \mathcal{M}^2 \cdot \frac{\beta \cdot \bar{R} \cdot s - \mathcal{M}}{\mathcal{M}} \cdot \left(\frac{s \cdot \bar{B}}{1-\lambda} \cdot d\bar{R} + dY^S(0, k-1|\delta) + dTr_1^S \right)$$

which follows from "Remarks". Additionally, from the proof of Lemma 2, we have:

$$dY^H(0, k-1|\delta) = \omega \cdot \frac{dY(0, k-1|\delta)}{1-\lambda+\lambda \cdot \omega},\ dY^S(0, k-1|\delta) = \frac{dY(0, k-1|\delta)}{1-\lambda+\lambda \cdot \omega},$$

which, together with: Lemma 5 (implying $dc_0^S = dY^S(1, k|\delta)$), multiplication of both sides of equation (47) by $\beta \cdot \bar{R} \cdot s$, division of both sides of equation (47) by $\mathcal{M}$, the addition of $\left(1 - \frac{\beta \cdot \bar{R} \cdot s}{\mathcal{M}}\right) \cdot dY^S(1, k|\delta)$ to both sides of (47) allows for reformulating equation (47) as follows:

$$dY^S(1, k|\delta) = -\frac{c^S \cdot \sigma}{\bar{R}} \cdot d\bar{R}$$

$$+\frac{\beta \cdot \bar{R} \cdot (1-s)}{\omega^{\frac{1}{\sigma}+1}} \cdot \left(\frac{(1-h) \cdot \bar{B}}{1-\lambda} \cdot d\bar{R} + \omega \cdot \frac{dY(0, k-1|\delta)}{1-\lambda+\lambda \cdot \omega} - \frac{\delta}{\lambda} \cdot \bar{B} \cdot d\bar{R} \right)$$

$$+\left(\beta \cdot \bar{R} \cdot s - \mathcal{M}\right) \cdot \left(\frac{s \cdot \bar{B}}{1-\lambda} \cdot d\bar{R} + \frac{dY(0, k-1|\delta)}{1-\lambda+\lambda \cdot \omega} - \frac{1-\delta}{1-\lambda} \cdot \bar{B} \cdot d\bar{R} \right)$$

We now use the steady-state version of the Euler equation: $\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}}}+\beta\cdot\bar{R}\cdot s=1$:

$$dY^S(1,k|\delta)=-\frac{c^S\cdot\sigma}{\bar{R}}\cdot d\bar{R}+(1-\mathcal{M})\cdot\frac{dY(0,k-1|\delta)}{1-\lambda+\lambda\cdot\omega}$$
$$+\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}+1}}\cdot\left(\frac{(1-h)\cdot\bar{B}}{1-\lambda}\cdot d\bar{R}-\frac{\delta}{\lambda}\cdot\bar{B}\cdot d\bar{R}\right)$$
$$+\left(\beta\cdot\bar{R}\cdot s-\mathcal{M}\right)\cdot\left(\frac{s\cdot\bar{B}}{1-\lambda}\cdot d\bar{R}-\frac{1-\delta}{1-\lambda}\cdot\bar{B}\cdot d\bar{R}\right)$$

which is equivalent to equation (46) because for $k>\tau$:

$$F(k-2|\tau-1,1-\mathcal{M})=1,\ f(k-1|\tau-1,1-\mathcal{M})=0.$$

This completes the proof for $\tau=1$.

For convenience, let us denote the formula for $dY^S(1,k|\delta)$ as:

$$\mathcal{X}\equiv-\frac{c^S\cdot\sigma}{\bar{R}}\cdot d\bar{R}+(1-\mathcal{M})\cdot\frac{dY(0,k-1|\delta)}{1-\lambda+\lambda\cdot\omega}$$
$$+\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}+1}}\cdot\left(\frac{(1-h)\cdot\bar{B}}{1-\lambda}\cdot d\bar{R}-\frac{\delta}{\lambda}\cdot\bar{B}\cdot d\bar{R}\right)$$
$$+\left(\beta\cdot\bar{R}\cdot s-\mathcal{M}\right)\cdot\left(\frac{s\cdot\bar{B}}{1-\lambda}\cdot d\bar{R}-\frac{1-\delta}{1-\lambda}\cdot\bar{B}\cdot d\bar{R}\right).$$

Let us show now that equation (46) holds for $\tau>1$. I first use Lemma 3 which, together with "Remarks" and Lemma 5 (equation (28): $dY^S(\tau,k|\delta)=dc_0^S$) yields:

$$dY^S(\tau,k|\delta)=\frac{1}{\beta\cdot\bar{R}\cdot s}\cdot\left\{\mathcal{M}\cdot\left[\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}+1}}\cdot dY^H(\tau-1,k-1|\delta)+\frac{\beta\cdot\bar{R}\cdot s-\mathcal{M}}{\mathcal{M}}\cdot dY^S(\tau,k|\delta)\right]\right.$$
$$+\mathcal{M}^2\cdot\left[\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}+1}}\cdot dY^H(\tau-2,k-1|\delta)+\frac{\beta\cdot\bar{R}\cdot s-\mathcal{M}}{\mathcal{M}}\cdot dY^S(\tau-1,k-1|\delta)\right]$$
$$+...$$
$$+\mathcal{M}^{\tau-1}\cdot\left[\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}+1}}\cdot dY^H(1,k-1|\delta)+\frac{\beta\cdot\bar{R}\cdot s-\mathcal{M}}{\mathcal{M}}\cdot dY^S(2,k-1|\delta)\right]$$
$$+\mathcal{M}^{\tau}\cdot\left[-\frac{c^S\cdot\sigma}{\bar{R}}\cdot d\bar{R}+\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}+1}}\cdot\left(\frac{(1-h)\cdot\bar{B}}{1-\lambda}\cdot d\bar{R}+dY^H(0,k-1|\delta)-\frac{\delta}{\lambda}\cdot\bar{B}\cdot d\bar{R}\right)\right.$$

$$+\frac{\beta\cdot\bar{R}\cdot s-\mathcal{M}}{\mathcal{M}}\cdot dY^{S}\left(1,k-1|\delta\right)\Bigg]$$

$$+\mathcal{M}^{\tau+1}\cdot\left[\frac{\beta\cdot\bar{R}\cdot s-\mathcal{M}}{\mathcal{M}}\cdot\left(\frac{s\cdot\bar{B}}{1-\lambda}\cdot d\bar{R}+dY^{S}\left(0,k-1|\delta\right)-\frac{1-\delta}{1-\lambda}\cdot\bar{B}\cdot d\bar{R}\right)\right]\Bigg\}.$$

Now, I use the assumption about the relationship between the income of the constrained and the unconstrained agent (see equation (8)) I reformulate the equation above further to get:

$$dY^{S}\left(\tau,k|\delta\right)=\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}}}\cdot dY^{S}\left(\tau-1,k-1|\delta\right)$$

$$+\mathcal{M}\cdot\left[\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}}}\cdot dY^{S}\left(\tau-2,k-1|\delta\right)+\frac{\beta\cdot\bar{R}\cdot s-\mathcal{M}}{\mathcal{M}}\cdot dY^{S}\left(\tau-1,k-1|\delta\right)\right]$$

$$+...$$

$$+\mathcal{M}^{\tau-2}\cdot\left[\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}}}\cdot dY^{S}\left(1,k-1|\delta\right)+\frac{\beta\cdot\bar{R}\cdot s-\mathcal{M}}{\mathcal{M}}\cdot dY^{S}\left(2,k-1|\delta\right)\right]$$

$$+\mathcal{M}^{\tau-1}\cdot\Bigg[-\frac{c^{S}\cdot\sigma}{\bar{R}}\cdot d\bar{R}+\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}+1}}\cdot\left(\frac{(1-h)\cdot\bar{B}}{1-\lambda}\cdot d\bar{R}+dY^{H}\left(0,k-1|\delta\right)-\frac{\delta}{\lambda}\cdot\bar{B}\cdot d\bar{R}\right)$$

$$+\left(\beta\cdot\bar{R}\cdot s-\mathcal{M}\right)\cdot\left(\frac{s\cdot\bar{B}}{1-\lambda}\cdot d\bar{R}+dY^{S}\left(0,k-1|\delta\right)-\frac{1-\delta}{1-\lambda}\cdot\bar{B}\cdot d\bar{R}\right)$$

$$+\frac{\beta\cdot\bar{R}\cdot s-\mathcal{M}}{\mathcal{M}}\cdot dY^{S}\left(1,k-1|\delta\right)\Bigg]$$

Now, I use the induction assumption (i.e., that formula (46) holds for 1, 2, ..., $\tau-1$, which is true because for all those terms the horizon index is lower than $k-1$ as $k>\tau$) and thus terms $dY^{S}\left(\tau-1,k-1|\delta\right)$, $dY^{S}\left(\tau-2,k-1|\delta\right)$, ..., $dY^{S}\left(1,k-1|\delta\right)$ can be replaced with formula 46, which for $k>\tau$ is equal to $\mathcal{X}$. Moreover, observe that the penultimate line and the line before are equal to $\mathcal{M}^{\tau-1}\cdot\mathcal{X}$ and therefore:

$$dY^{S}\left(\tau,k|\delta\right)=\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}}}\cdot\mathcal{X}$$

$$+\mathcal{M}\cdot\left[\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}}}\cdot\mathcal{X}+\frac{\beta\cdot\bar{R}\cdot s-\mathcal{M}}{\mathcal{M}}\cdot\mathcal{X}\right]$$

$$+...$$

$$+\mathcal{M}^{\tau-2}\cdot\left[\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}}}\cdot\mathcal{X}+\frac{\beta\cdot\bar{R}\cdot s-\mathcal{M}}{\mathcal{M}}\cdot\mathcal{X}\right]$$

$$+\mathcal{M}^{\tau-1}\cdot\left[\mathcal{X}+\frac{\beta\cdot\bar{R}\cdot s-\mathcal{M}}{\mathcal{M}}\cdot\mathcal{X}\right]$$

We now use the steady-state version of the Euler equation: $\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}}}+\beta\cdot\bar{R}\cdot s=1$, which gives (less technically speaking, terms preceeded by $\frac{\beta\cdot\bar{R}\cdot s-\mathcal{M}}{\mathcal{M}}$ in the equation above are "shifted up by one line" and then the Euler equation in the steady state is used):

$$dY^S(\tau,k|\delta)=(1-\mathcal{M})\cdot\mathcal{X}$$
$$+\mathcal{M}\cdot(1-\mathcal{M})\cdot\mathcal{X}$$
$$+...$$
$$+\mathcal{M}^{\tau-2}\cdot(1-\mathcal{M})\cdot\mathcal{X}$$
$$+\mathcal{M}^{\tau-1}\cdot\mathcal{X}.$$

Canceling terms yields:

$$dY^S(\tau,k|\delta)=\mathcal{X}$$

and using the definition of $\mathcal{X}$:

$$dY^S(\tau,k|\delta)=-\frac{c^S\cdot\sigma}{\bar{R}}\cdot d\bar{R}+(1-\mathcal{M})\cdot\frac{dY(0,k-1|\delta)}{1-\lambda+\lambda\cdot\omega}$$
$$+\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}+1}}\cdot\left(\frac{(1-h)\cdot\bar{B}}{1-\lambda}\cdot d\bar{R}-\frac{\delta}{\lambda}\cdot\bar{B}\cdot d\bar{R}\right)$$
$$+\left(\beta\cdot\bar{R}\cdot s-\mathcal{M}\right)\cdot\left(\frac{s\cdot\bar{B}}{1-\lambda}\cdot d\bar{R}-\frac{1-\delta}{1-\lambda}\cdot\bar{B}\cdot d\bar{R}\right)$$

which is equivalent to equation (46) because for $k>\tau$:

$$F(k-2|\tau-1,1-\mathcal{M})=1,\ f(k-1|\tau-1,1-\mathcal{M})=0.$$

This completes the proof of formula (46) for $k>\tau$.

$dY^S(\tau,k|\delta)$ **in the case when** $k\leq\tau$**.** By contrast to $k>\tau$, the induction method is now applied to the index k (instead of τ). In what follows, I first prove formula (46) for $k=1$ (and for all $\tau>0$). Next, I prove it for any k assuming that formula (46) holds for $k-1$.

Let us consider $k=1$. Observe that, using Lemma 3 and "Remarks", for all $\tau>0$:

$$dc_0^S=\frac{1}{\beta\cdot\bar{R}\cdot s}\cdot\left\{\mathcal{M}\cdot\left[\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}+1}}\cdot dY^H(\tau-1,k-1|\delta)+\frac{\beta\cdot\bar{R}\cdot s-\mathcal{M}}{\mathcal{M}}\cdot dY^S(\tau,k|\delta)\right]\right.$$

$$+\mathcal{M}^2\cdot\left[\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}+1}}\cdot dY^H(\tau-2,k-1|\delta)+\frac{\beta\cdot\bar{R}\cdot s-\mathcal{M}}{\mathcal{M}}\cdot dY^S(\tau-1,k-1|\delta)\right]$$

$$+...$$

$$+\mathcal{M}^{\tau-1}\cdot\left[\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}+1}}\cdot dY^H(1,k-1|\delta)+\frac{\beta\cdot\bar{R}\cdot s-\mathcal{M}}{\mathcal{M}}\cdot dY^S(2,k-1|\delta)\right]$$

$$+\mathcal{M}^{\tau}\cdot\left[-\frac{c^S\cdot\sigma}{\bar{R}}\cdot d\bar{R}+\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}+1}}\cdot\left(\frac{(1-h)\cdot\bar{B}}{1-\lambda}\cdot d\bar{R}+dY^H(0,k-1|\delta)-\frac{\delta}{\lambda}\cdot\bar{B}\cdot d\bar{R}\right)\right.$$

$$\left.+\frac{\beta\cdot\bar{R}\cdot s-\mathcal{M}}{\mathcal{M}}\cdot dY^S(1,k-1|\delta)\right]$$

$$+\mathcal{M}^{\tau+1}\cdot\left[\frac{\beta\cdot\bar{R}\cdot s-\mathcal{M}}{\mathcal{M}}\cdot\left(\frac{s\cdot\bar{B}}{1-\lambda}\cdot d\bar{R}+dY^S(0,k-1|\delta)-\frac{1-\delta}{1-\lambda}\cdot\bar{B}\cdot d\bar{R}\right)\right]\Bigg\}.$$

Note that for $k=1$ agents expect that their future income levels does not deviate from their steady-state levels and therefore:

$$dY^H(\tau-1,k-1|\delta)=dY^H(\tau-2,k-1|\delta)=...=dY^H(0,k-1|\delta)=0.$$

$$dY^S(\tau-1,k-1|\delta)=dY^S(\tau-2,k-1|\delta)=...=dY^S(0,k-1|\delta)=0.$$

Thus:

$$dc_0^S=\frac{1}{\beta\cdot\bar{R}\cdot s}\cdot\left\{\mathcal{M}\cdot\left[\frac{\beta\cdot\bar{R}\cdot s-\mathcal{M}}{\mathcal{M}}\cdot dY^S(\tau,k|\delta)\right]\right.$$

$$+\mathcal{M}^{\tau}\cdot\left[-\frac{c^S\cdot\sigma}{\bar{R}}\cdot d\bar{R}+\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}+1}}\cdot\left(\frac{(1-h)\cdot\bar{B}}{1-\lambda}\cdot d\bar{R}-\frac{\delta}{\lambda}\cdot\bar{B}\cdot d\bar{R}\right)\right]$$

$$\left.+\mathcal{M}^{\tau+1}\cdot\left[\frac{\beta\cdot\bar{R}\cdot s-\mathcal{M}}{\mathcal{M}}\cdot\left(\frac{s\cdot\bar{B}}{1-\lambda}\cdot d\bar{R}-\frac{1-\delta}{1-\lambda}\cdot\bar{B}\cdot d\bar{R}\right)\right]\right\}.$$

Note that from Lemma 5 (equation (28)) we have $dc_0^S=dY^S(\tau,k|\delta)$ and therefore:

$$dY^S(\tau,k|\delta)=$$

$$\mathcal{M}^{\tau-1}\cdot\left[-\frac{c^S\cdot\sigma}{\bar{R}}\cdot d\bar{R}+\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}+1}}\cdot\left(\frac{(1-h)\cdot\bar{B}}{1-\lambda}\cdot d\bar{R}-\frac{\delta}{\lambda}\cdot\bar{B}\cdot d\bar{R}\right)\right.$$

$$\left.+\left(\beta\cdot\bar{R}\cdot s-\mathcal{M}\right)\cdot\left(\frac{s\cdot\bar{B}}{1-\lambda}\cdot d\bar{R}-\frac{1-\delta}{1-\lambda}\cdot\bar{B}\cdot d\bar{R}\right)\right].$$

Given that for $k = 1$:

$$\mathcal{M}^{\tau-1} = \begin{pmatrix} \tau - 1 \\ k - 1 \end{pmatrix} \cdot \mathcal{M}^{\tau-k} \cdot (1 - \mathcal{M})^{k-1} = f\left(k - 1 | \tau - 1, 1 - \mathcal{M}\right)$$

and that for $k = 1$:

$$F\left(k - 2 | \tau - 1, 1 - \mathcal{M}\right) = \sum_{l=0}^{k-2} \begin{pmatrix} \tau - 1 \\ l \end{pmatrix} \cdot \mathcal{M}^{\tau-l-1} \cdot (1 - \mathcal{M})^{l} = 0$$

as the sum contains zero elements by the definition of the aggregation operator. This completes the proof for $k = 1$.

For convenience let us define:

$$\mathcal{Z} \equiv \tag{48}$$

$$\left[-\frac{c^S \cdot \sigma}{\bar{R}} \cdot d\bar{R} + \frac{\beta \cdot \bar{R} \cdot (1-s)}{\omega^{\frac{1}{\sigma}+1}} \cdot \left(\frac{(1-h) \cdot \bar{B}}{1-\lambda} \cdot d\bar{R} - \frac{\delta}{\lambda} \cdot \bar{B} \cdot d\bar{R}\right)\right.$$
$$\left. + \left(\beta \cdot \bar{R} \cdot s - \mathcal{M}\right) \cdot \left(\frac{s \cdot \bar{B}}{1-\lambda} \cdot d\bar{R} - \frac{1-\delta}{1-\lambda} \cdot \bar{B} \cdot d\bar{R}\right)\right].$$

Let us now show that formula (46) holds for $1 < k \leq \tau$. Again, using Lemma 3, formula for $dY^S\left(\tau, k|\delta\right)$ for $k > \tau$, condition $dc_0^S = dY^S\left(\tau, k|\delta\right)$ (see Lemma 5), assumption described by condition (8), and "Remarks":

$$dY^S\left(\tau, k|\delta\right) = \frac{\beta \cdot \bar{R} \cdot (1-s)}{\omega^{\frac{1}{\sigma}}} \cdot dY^S\left(\tau - 1, k - 1|\delta\right)$$

$$+\mathcal{M} \cdot \left[\frac{\beta \cdot \bar{R} \cdot (1-s)}{\omega^{\frac{1}{\sigma}}} \cdot dY^S\left(\tau - 2, k - 1|\delta\right) + \frac{\beta \cdot \bar{R} \cdot s - \mathcal{M}}{\mathcal{M}} \cdot dY^S\left(\tau - 1, k - 1|\delta\right)\right]$$

$$+ ...$$

$$+\mathcal{M}^{\tau-k} \cdot \left[\frac{\beta \cdot \bar{R} \cdot (1-s)}{\omega^{\frac{1}{\sigma}}} \cdot dY^S\left(k - 1, k - 1|\delta\right) + \frac{\beta \cdot \bar{R} \cdot s - \mathcal{M}}{\mathcal{M}} \cdot dY^S\left(k, k - 1|\delta\right)\right]$$

$$+\mathcal{M}^{\tau-k+1} \cdot \left[\frac{\beta \cdot \bar{R} \cdot (1-s)}{\omega^{\frac{1}{\sigma}}} \cdot \mathcal{X} + \frac{\beta \cdot \bar{R} \cdot s - \mathcal{M}}{\mathcal{M}} \cdot dY^S\left(k - 1, k - 1|\delta\right)\right]$$

$$+\mathcal{M}^{\tau-k+2} \cdot \left[\frac{\beta \cdot \bar{R} \cdot (1-s)}{\omega^{\frac{1}{\sigma}}} \cdot \mathcal{X} + \frac{\beta \cdot \bar{R} \cdot s - \mathcal{M}}{\mathcal{M}} \cdot \mathcal{X}\right]$$

$$+ ...$$

$$+\mathcal{M}^{\tau-2} \cdot \left[\frac{\beta \cdot \bar{R} \cdot (1-s)}{\omega^{\frac{1}{\sigma}}} \cdot \mathcal{X} + \frac{\beta \cdot \bar{R} \cdot s - \mathcal{M}}{\mathcal{M}} \cdot \mathcal{X}\right]$$

$$+\mathcal{M}^{\tau-1}\cdot\left[-\frac{c^S\cdot\sigma}{\bar{R}}\cdot d\bar{R}+\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}+1}}\cdot\left(\frac{(1-h)\cdot\bar{B}}{1-\lambda}\cdot d\bar{R}+\frac{\omega\cdot dY(0,k-1|\delta)}{1-\lambda+\lambda\cdot\omega}-\frac{\delta}{\lambda}\cdot\bar{B}\cdot d\bar{R}\right)\right.$$

$$\left.+\frac{\beta\cdot\bar{R}\cdot s-\mathcal{M}}{\mathcal{M}}\cdot\mathcal{X}\right]$$

$$+\mathcal{M}^{\tau}\cdot\frac{\beta\cdot\bar{R}\cdot s-\mathcal{M}}{\mathcal{M}}\cdot\left(\frac{s\cdot\bar{B}}{1-\lambda}\cdot d\bar{R}+\frac{dY(0,k-1|\delta)}{1-\lambda+\lambda\cdot\omega}-\frac{1-\delta}{1-\lambda}\cdot\bar{B}\cdot d\bar{R}\right)$$

where I also used:

$$dY^H(0,k-1|\delta)=\frac{\omega\cdot dY(0,k-1|\delta)}{1-\lambda+\lambda\cdot\omega}$$

$$dY^S(0,k-1|\delta)=\frac{dY(0,k-1|\delta)}{1-\lambda+\lambda\cdot\omega}$$

implied by the proof of Lemma 2. I now use: i) the induction assumption (i.e., that formula (46) holds for $k-1$), ii) aggregation of terms in the bracketts multiplied by $\mathcal{M}^{\tau-1}$ and $\mathcal{M}^{\tau}$ (all but $\frac{\beta\cdot\bar{R}\cdot s-\mathcal{M}}{\mathcal{M}}\cdot\mathcal{X}$) to get $\mathcal{X}$, iii) aggregation of terms $dY^S(\hat{\tau},k-1|\delta)$ featuring the same horizon $\hat{\tau}$, iv) using the steady-state version of the Euler equation: $\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}}}+\beta\cdot\bar{R}\cdot s=1$, v) definition of $\mathcal{Z}$ (see equation (48)), to get:

$$dY^S(\tau,k|\delta)= \tag{49}$$

$$(1-\mathcal{M})\cdot\left[\binom{\tau-2}{k-2}\cdot\mathcal{M}^{\tau-k}\cdot(1-\mathcal{M})^{k-2}\cdot\mathcal{Z}+\sum_{l=0}^{k-3}\binom{\tau-2}{l}\cdot\mathcal{M}^{\tau-2-l}\cdot(1-\mathcal{M})^{l}\cdot\mathcal{X}\right]$$

$$+\mathcal{M}\cdot(1-\mathcal{M})\cdot\left[\binom{\tau-3}{k-2}\cdot\mathcal{M}^{\tau-k-1}\cdot(1-\mathcal{M})^{k-2}\cdot\mathcal{Z}+\sum_{l=0}^{k-3}\binom{\tau-3}{l}\cdot\mathcal{M}^{\tau-3-l}\cdot(1-\mathcal{M})^{l}\cdot\mathcal{X}\right]$$

$$+...$$

$$+\mathcal{M}^{\tau-k}\cdot(1-\mathcal{M})\cdot\left[\binom{k-2}{k-2}\cdot\mathcal{M}^{0}\cdot(1-\mathcal{M})^{k-2}\cdot\mathcal{Z}+\sum_{l=0}^{k-3}\binom{k-2}{l}\cdot\mathcal{M}^{k-2-l}\cdot(1-\mathcal{M})^{l}\cdot\mathcal{X}\right]$$

$$+\mathcal{M}^{\tau-k+1}\cdot(1-\mathcal{M})\cdot\mathcal{X}$$

$$+\mathcal{M}^{\tau-k+2}\cdot(1-\mathcal{M})\cdot\mathcal{X}$$

$$+...$$

$$+\mathcal{M}^{\tau-2}\cdot(1-\mathcal{M})\cdot\mathcal{X}$$

$$+\mathcal{M}^{\tau-1}\cdot\mathcal{X}.$$

Using Lemma 4 for the aggregation of terms containing $\mathcal{Z}$:

$$\left[\binom{\tau-2}{k-2}+\binom{\tau-3}{k-2}+...+\binom{k-2}{k-2}\right]\cdot\mathcal{M}^{\tau-k}\cdot(1-\mathcal{M})^{k-1}$$

$$= \begin{pmatrix} \tau - 1 \\ k - 1 \end{pmatrix} \cdot \mathcal{M}^{\tau-1-(k-1)} \cdot (1-\mathcal{M})^{k-1} = f\left(k-1|\tau-1, 1-\mathcal{M}\right)$$

which corresponds to the coefficient of the term containing $\mathcal{Z}$ in formula (46) because it can be re-expressed as:

$$dY^S\left(\tau, k|\delta\right) = f\left(k-1|\tau-1, 1-\mathcal{M}\right) \cdot \mathcal{Z}$$

$$+F\left(k-2|\tau-1, 1-\mathcal{M}\right) \cdot \mathcal{X}. \tag{50}$$

Thus, to complete the proof that (46) holds, it suffices to show that terms containing $\mathcal{X}$ in equation (49) aggregate to $F\left(k-2|\tau-1, 1-\mathcal{M}\right) \cdot \mathcal{X}$. Let us first rewrite them from (49):

$$(1-\mathcal{M}) \cdot \left[\sum_{l=0}^{k-3} \begin{pmatrix} \tau - 2 \\ l \end{pmatrix} \cdot \mathcal{M}^{\tau-2-l} \cdot (1-\mathcal{M})^l \cdot \mathcal{X}\right]$$

$$+\mathcal{M} \cdot (1-\mathcal{M}) \cdot \left[\sum_{l=0}^{k-3} \begin{pmatrix} \tau - 3 \\ l \end{pmatrix} \cdot \mathcal{M}^{\tau-3-l} \cdot (1-\mathcal{M})^l \cdot \mathcal{X}\right]$$

$$+...$$

$$+\mathcal{M}^{\tau-k} \cdot (1-\mathcal{M}) \cdot \left[\sum_{l=0}^{k-3} \begin{pmatrix} k - 2 \\ l \end{pmatrix} \cdot \mathcal{M}^{k-2-l} \cdot (1-\mathcal{M})^l \cdot \mathcal{X}\right]$$

$$+\mathcal{M}^{\tau-k+1} \cdot (1-\mathcal{M}) \cdot \mathcal{X}$$

$$+\mathcal{M}^{\tau-k+2} \cdot (1-\mathcal{M}) \cdot \mathcal{X}$$

$$+...$$

$$+\mathcal{M}^{\tau-2} \cdot (1-\mathcal{M}) \cdot \mathcal{X}$$

$$+\mathcal{M}^{\tau-1} \cdot \mathcal{X}.$$

I now follow the ananlogous steps as in the proof of Theorem 1. I rewrite the sum above as (I cancel terms):

$$(1-\mathcal{M}) \cdot \left[\sum_{l=0}^{k-3} \begin{pmatrix} \tau - 2 \\ l \end{pmatrix} \cdot \mathcal{M}^{\tau-2-l} \cdot (1-\mathcal{M})^l \cdot \mathcal{X}\right]$$

$$+\mathcal{M}\cdot(1-\mathcal{M})\cdot\left[\sum_{l=0}^{k-3}\begin{pmatrix}\tau-3\\l\end{pmatrix}\cdot\mathcal{M}^{\tau-3-l}\cdot(1-\mathcal{M})^{l}\cdot\mathcal{X}\right]$$

$$+...$$

$$+\mathcal{M}^{\tau-k}\cdot(1-\mathcal{M})\cdot\left[\sum_{l=0}^{k-3}\begin{pmatrix}k-2\\l\end{pmatrix}\cdot\mathcal{M}^{k-2-l}\cdot(1-\mathcal{M})^{l}\cdot\mathcal{X}\right]$$

$$+\mathcal{M}^{\tau-k+1}\cdot\mathcal{X}$$

which is equivalent to:

$$\mathcal{X}\cdot\left\{\left[\begin{pmatrix}\tau-2\\0\end{pmatrix}\cdot\mathcal{M}^{\tau-2}\cdot(1-\mathcal{M})+...+\begin{pmatrix}\tau-2\\k-3\end{pmatrix}\cdot\mathcal{M}^{\tau-k+1}\cdot(1-\mathcal{M})^{k-2}\right]\right.$$

$$+\left[\begin{pmatrix}\tau-3\\0\end{pmatrix}\cdot\mathcal{M}^{\tau-2}\cdot(1-\mathcal{M})+...+\begin{pmatrix}\tau-3\\k-3\end{pmatrix}\cdot\mathcal{M}^{\tau-k+1}\cdot(1-\mathcal{M})^{k-2}\right]$$

$$+...$$

$$+\left[\begin{pmatrix}k-2\\0\end{pmatrix}\cdot\mathcal{M}^{\tau-2}\cdot(1-\mathcal{M})+...+\begin{pmatrix}k-2\\k-3\end{pmatrix}\cdot\mathcal{M}^{\tau-k+1}\cdot(1-\mathcal{M})^{k-2}\right]$$

$$\left.+\mathcal{M}^{\tau-k+1}\right\}$$

which, by Lemma 4, is equivalent to:

$$\mathcal{X}\cdot\left\{\left[\begin{pmatrix}\tau-1\\1\end{pmatrix}-\begin{pmatrix}k-2\\1\end{pmatrix}\right]\cdot\mathcal{M}^{\tau-2}\cdot(1-\mathcal{M})^{1}\right.$$

$$+\left[\begin{pmatrix}\tau-1\\2\end{pmatrix}-\begin{pmatrix}k-2\\2\end{pmatrix}\right]\cdot\mathcal{M}^{\tau-3}\cdot(1-\mathcal{M})^{2}$$

$$+...$$

$$+\left[\begin{pmatrix}\tau-1\\k-2\end{pmatrix}-\begin{pmatrix}k-2\\k-2\end{pmatrix}\right]\cdot\mathcal{M}^{\tau-(k-1)}\cdot(1-\mathcal{M})^{k-2}$$

$$\left.+\mathcal{M}^{\tau-k+1}\right\}.$$

By the same token as in the proof of Theorem 1, negative terms in the expression above aggregate to $-\mathcal{M}^{\tau-k+1}\cdot\left(1-\begin{pmatrix}k-2\\0\end{pmatrix}\cdot\mathcal{M}^{k-2}\right)$ which allows for the following reformulation of the sum above:

$$\mathcal{X}\cdot\left\{\begin{pmatrix}\tau-1\\1\end{pmatrix}\cdot\mathcal{M}^{\tau-2}\cdot(1-\mathcal{M})^{1}\right.$$

$$+\begin{pmatrix}\tau-1\\2\end{pmatrix}\cdot\mathcal{M}^{\tau-3}\cdot(1-\mathcal{M})^{2}$$

$$+...$$

$$\left.+\begin{pmatrix}\tau-1\\k-2\end{pmatrix}\cdot\mathcal{M}^{\tau-(k-1)}\cdot(1-\mathcal{M})^{k-2}+\begin{pmatrix}k-2\\0\end{pmatrix}\cdot\mathcal{M}^{\tau-1}\right\}$$

which, given that $\begin{pmatrix}k-2\\0\end{pmatrix}=\begin{pmatrix}\tau-1\\0\end{pmatrix}$, is equivalent to:

$$\mathcal{X}\cdot\sum_{l=0}^{k-2}\begin{pmatrix}\tau-1\\l\end{pmatrix}\cdot\mathcal{M}^{\tau-1-l}\cdot(1-\mathcal{M})^{l}$$

$$=\mathcal{X}\cdot F\left(k-2|\tau-1,1-\mathcal{M}\right)$$

as the term containing $\mathcal{X}$ in formula (50). Given that we have already proved the term containing $\mathcal{Z}$ in formula (50), and given that formula (50) is equivalent to formula (46), completes the proof of expression (46). Let us now standardize it to get the formulation from Theorem 2. To this end, let us multiply both sides of (46) by $-\frac{\bar{R}}{Y}\cdot\frac{1}{dR}$ and use the fact that $dY_t^S=\frac{dY_t}{1-\lambda+\lambda\cdot\omega}$ (see Lemma 5):

$$-\frac{\bar{R}}{Y}\cdot\frac{1}{dR}\cdot dY\left(\tau,k|\delta\right)= \tag{51}$$

$$-\frac{\bar{R}}{Y}\cdot\frac{1-\lambda+\lambda\cdot\omega}{dR}\cdot f\left(k-1|\tau-1,1-\mathcal{M}\right)\cdot\left\{-\frac{\sigma\cdot c^{S}}{\bar{R}}\cdot d\bar{R}\right.$$

$$+\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}+1}}\cdot\left(\frac{(1-h)\cdot\bar{B}\cdot d\bar{R}}{1-\lambda}-\frac{\delta}{\lambda}\cdot\bar{B}\cdot d\bar{R}\right)$$

$$\left.+\left(\beta\cdot\bar{R}\cdot s-\mathcal{M}\right)\cdot\left(\frac{s\cdot\bar{B}\cdot d\bar{R}}{1-\lambda}-\frac{1-\delta}{1-\lambda}\cdot\bar{B}\cdot d\bar{R}\right)\right\}$$

$$-\frac{\bar{R}}{Y}\cdot\frac{1-\lambda+\lambda\cdot\omega}{dR}\cdot F\left(k-2|\tau-1,1-\mathcal{M}\right)\cdot\left\{-\frac{\sigma\cdot c^{S}}{\bar{R}}\cdot d\bar{R}\right.$$

$$+\left(1-\mathcal{M}\right)\cdot\frac{dY\left(0,k-1|\delta\right)}{1-\lambda+\lambda\cdot\omega}$$

$$+\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}+1}}\cdot\left(\frac{(1-h)\cdot\bar{B}\cdot d\bar{R}}{1-\lambda}-\frac{\delta}{\lambda}\cdot\bar{B}\cdot d\bar{R}\right)$$

$$\left.+\left(\beta\cdot\bar{R}\cdot s-\mathcal{M}\right)\cdot\left(\frac{s\cdot\bar{B}\cdot d\bar{R}}{1-\lambda}-\frac{1-\delta}{1-\lambda}\cdot\bar{B}\cdot d\bar{R}\right)\right\}$$

which after reformulation and after using the definition of interest rate elasticity of output (see equation (17)) gives:

$$\epsilon\left(\tau,k|\delta\right)=$$

$$-\frac{\bar{R}}{Y}\cdot\frac{1-\lambda+\lambda\cdot\omega}{dR}\cdot F\left(k-1|\tau-1,1-\mathcal{M}\right)\cdot\left\{-\frac{\sigma\cdot c^{S}}{\bar{R}}\cdot\theta\cdot dR\right.$$

$$+\left(\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}+1}}\cdot\frac{1-s}{\lambda}+\left(\beta\cdot\bar{R}\cdot s-\mathcal{M}\right)\cdot\frac{s}{1-\lambda}\right)\cdot\bar{B}\cdot\theta\cdot dR$$

$$\left.-\left(\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}+1}}\cdot\frac{\delta}{\lambda}+\left(\beta\cdot\bar{R}\cdot s-\mathcal{M}\right)\cdot\frac{1-\delta}{1-\lambda}\right)\cdot\bar{B}\cdot\theta\cdot dR\right\}$$

$$-\frac{\bar{R}}{Y}\cdot\frac{1-\lambda+\lambda\cdot\omega}{dR}\cdot F\left(k-2|\tau-1,1-\mathcal{M}\right)\cdot\left(1-\mathcal{M}\right)\cdot\frac{dY\left(0,k-1|\delta\right)}{1-\lambda+\lambda\cdot\omega}$$

because $d\bar{R}=\theta\cdot dR$, $\frac{1-h}{1-\lambda}=\frac{1-s}{\lambda}$, and because:

$$F\left(k-1|\tau-1,1-\mathcal{M}\right)=f\left(k-1|\tau-1,1-\mathcal{M}\right)$$

$$+F\left(k-2|\tau-1,1-\mathcal{M}\right).$$

Using formulas for MPC^{S} and $iMPC^{S}\left(\delta\right)$ (see equations (15) and (16), respectively) and relationship $c^{S}=\frac{Y}{1-\lambda+\lambda\cdot\omega}$ (see Lemma 5):

$$\epsilon\left(\tau,k|\delta\right)=$$

$$F\left(k-1|\tau-1,1-\mathcal{M}\right)\cdot\{\sigma\cdot\theta$$

$$+\left(-\frac{\bar{R}}{c^S}\right)\cdot\frac{iMPC^S(1-s)}{1-MPC^S}\cdot\bar{B}\cdot\theta$$

$$-\left(-\frac{\bar{R}}{c^S}\right)\cdot\frac{iMPC^S(\delta)}{1-MPC^S}\cdot\bar{B}\cdot\theta\Big\}$$

$$-\frac{\bar{R}}{Y}\cdot\frac{1}{dR}\cdot F\left(k-2|\tau-1,1-\mathcal{M}\right)\cdot(1-\mathcal{M})\cdot dY\left(0,k-1|\delta\right)$$

which is equivalent to:

$$\epsilon\left(\tau,k|\delta\right)=F\left(k-1|\tau-1,1-\mathcal{M}\right)\cdot\theta\cdot\sigma$$

$$-F\left(k-1|\tau-1,1-\mathcal{M}\right)\cdot\frac{iMPC^S(1-s)}{1-MPC^S}\cdot\frac{\bar{R}\cdot\bar{B}\cdot\theta}{c^S}$$

$$+F\left(k-1|\tau-1,1-\mathcal{M}\right)\cdot\frac{iMPC^S(\delta)}{1-MPC^S}\cdot\frac{\bar{R}\cdot\bar{B}\cdot\theta}{c^S}$$

$$+F\left(k-2|\tau-1,1-\mathcal{M}\right)\cdot(1-\mathcal{M})\cdot\epsilon\left(0|\delta\right)$$

which completes the proof of formula $\epsilon\left(\tau,k|\delta\right)$ in Theorem 2.

Finally, let us prove the formula for $\Delta\epsilon\left(\tau,k|\delta\right)$ in Theorem 2. Note that the additional fiscal announcement (fiscal guidance) is captured with terms $\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}+1}}\cdot\left(-\frac{\delta}{\lambda}\cdot\bar{B}\cdot d\bar{R}\right)$ and $\left(\beta\cdot\bar{R}\cdot s-\mathcal{M}\right)\cdot\left(-\frac{1-\delta}{1-\lambda}\cdot\bar{B}\cdot d\bar{R}\right)$ in formula describing $\mathcal{Z}$ (see equation (48)). Thus, aggregating its isolated impact on the income of agent S (see equation (49) and the aggregation of terms related to $\mathcal{Z}$ following that expression):

$$\Delta Y^S\left(\tau,k|\delta\right)=f\left(k-1|\tau-1,1-\mathcal{M}\right)\cdot\Bigg[-\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}+1}}\cdot\left(\frac{\delta}{\lambda}\cdot\bar{B}\cdot d\bar{R}\right)$$

$$+\left(\beta\cdot\bar{R}\cdot s-\mathcal{M}\right)\cdot\left(\frac{1-\delta}{1-\lambda}\cdot\bar{B}\cdot d\bar{R}\right)\Bigg]$$

standardizing this expression as in (51) (i.e. multiplying by $-\frac{\bar{R}}{Y}\cdot\frac{1-\lambda+\lambda\cdot\omega}{dR}\cdot$):

$$\Delta\epsilon\left(\tau,k|\delta\right)=-\frac{\bar{R}}{Y}\cdot\frac{1-\lambda+\lambda\cdot\omega}{dR}$$

$$\cdot f\left(k-1|\tau-1,1-\mathcal{M}\right)\cdot\Bigg[-\frac{\beta\cdot\bar{R}\cdot(1-s)}{\omega^{\frac{1}{\sigma}+1}}\cdot\left(\frac{\delta}{\lambda}\cdot\bar{B}\cdot d\bar{R}\right)$$

$$+ \left(\beta \cdot \bar{R} \cdot s - \mathcal{M}\right) \cdot \left(\frac{1-\delta}{1-\lambda} \cdot \bar{B} \cdot d\bar{R}\right)\Bigg]$$

which is equivalent to:

$$\Delta\epsilon\left(\tau, k|\delta\right) =$$

$$f\left(k-1|\tau-1, 1-\mathcal{M}\right) \cdot \left[\frac{\beta \cdot \bar{R} \cdot (1-s)}{\omega^{\frac{1}{\sigma}+1}} \cdot \left(\frac{\delta}{\lambda}\right)\right.$$

$$\left. + \left(\beta \cdot \bar{R} \cdot s - \mathcal{M}\right) \cdot \left(\frac{1-\delta}{1-\lambda}\right)\right] \cdot \frac{\bar{B} \cdot \theta \cdot \bar{R}}{c^S}$$

which, by definitions of MPC^S and $iMPC^S$ (see equations (15) and (16)) is equivalent to:

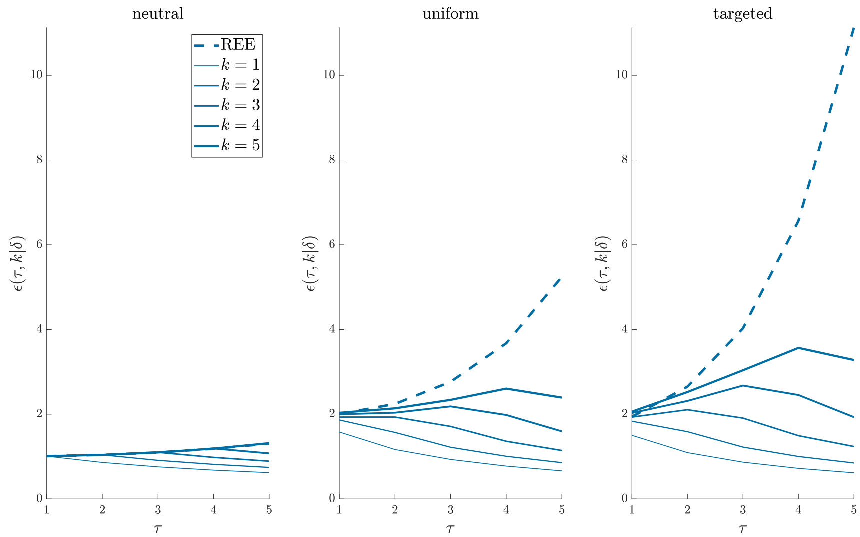
$$\Delta\epsilon\left(\tau, k|\delta\right) =$$

$$f\left(k-1|\tau-1, 1-\mathcal{M}\right) \cdot \frac{iMPC^S\left(\delta\right)}{1-MPC^S} \cdot \frac{\bar{B} \cdot \theta \cdot \bar{R}}{c^S}$$

which completes the proof. QED.

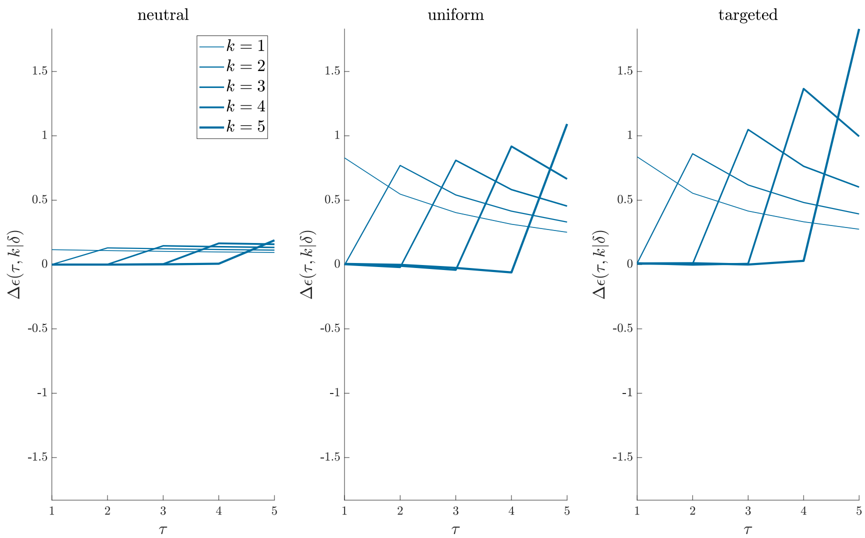
Additional figures

Figure 9: Interest rate elasticities of output in HANK (model with the NKPC and high debt) under MFFG for neutral, uniform and targeted transfers.



Notes: Interest rate elasticities of output in period 0 computed for a one-time drop in interest rates equal to $dR = -0.0025$ (i.e. a one-percentage-point decrease in nominal rates in annual terms) that occurs in period τ - see formula (17) for three types of transfers in period τ (induced by monetary shock): neutral, uniform, and targeted. Dashed lines correspond to the rational expectations equilibria and solid lines denote output elasticities under level-k thinking. Different thickness of solid lines represent different values of k.

Figure 10: Difference between interest rate elasticities of output in HANK (model with the NKPC and high debt) between MFFG and FG for neutral, uniform and targeted transfers.



Notes: Difference (between MFFG and FG) in interest rate elasticities of output in period 0 computed for a one-time drop in interest rates equal to $dR = -0.0025$ (i.e. a one-percentage-point decrease in nominal rates in annual terms) that occurs in period τ. This difference is specified by formula (18) for three types of transfers in period τ (induced by monetary shock): neutral, uniform, and targeted. Solid lines denote output elasticities under level-k thinking. Different thickness of solid lines represent different values of k.