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Abstract

We study cost-sharing rules in network problems where agents seek to ship quantities of some good to their respective locations, and the cost on each arc is linear in the flow crossing it. In this context, Core Selection requires that each subgroup of agents pay a joint cost share that is not higher than its stand-alone cost. We prove that the demander rule, under which each agent pays the cost of her shortest path for each unit she demands, is the unique cost-sharing rule satisfying both Core Selection and Merge Proofness. The Merge Proofness axiom prevents distinct nodes from reducing their joint cost share by merging into a single node. An alternative characterization of the demander rule is obtained by combining Core Selection and Cost Solidarity. The Cost Solidarity axiom says that each agent's cost share should be weakly increasing in the cost matrix.

JEL Classification: C71, D85.

Keywords: Shortest path games, cost sharing, core, merge proofness, solidarity.

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1 Introduction

Shortest path games (Rosenthal, 2013; Bahel and Trudeau, 2014) are network games where agents must ship their demands of some homogeneous good from a source point to their respective locations. The cost on each arc of the network is linear in the total demand crossing it, which gives rise to a cooperative game (with transferable cost) where each coalition seeks to minimize the total cost of shipping the demands of its members. An optimal network configuration is obtained when each agent i's demand is shipped through a *shortest path* to agent i, that is a path that achieves minimum cost among all paths from the source to agent i. A natural way to share the cost is the *demander rule*, which requires each agent i to pay the cost of her shortest path for each unit she demands. It is known from Rosenthal (2013) that this cost allocation belongs to the *core*, i.e., no subgroup of players can ship their respective demands (using only the connections available in the subgroup) at a cost lower than their joint share under the demander rule.

The present work focuses on two consistency properties: Merge Proofness and Cost Solidarity. The axiom *Merge Proofness* prevents any subgroup S of distinct agents from reducing their joint share by merging. Such a merging manipulation typically results in a shortest path problem with a lower total shipping cost (since it does not include the internal shipping cost for agents in S), and it is profitable to the subgroup S if the sum of their share in the resulting problem and their internal connection is lower than their joint share in the original problem. Merge Proofness has been studied in many different settings, such as bilateral monopolies (Horn and Wolinsky, 1988), bankruptcy problems (de Frutos, 1999), cost-sharing problems (?), scheduling problems (Moulin, 2008), bipartite matching games (Hezarkhani, 2016), or minimum cost spanning tree problems (Gómez-Rúa and Vidal-Puga, 2011, 2017). We also examine a stronger version of Merge Proofness, which prevents any merging manipulation from negatively affecting any agent not in S.

The axiom *Cost Solidarity* states that an agent's cost share should be weakly increasing in the cost matrix. Note that, following an increase in the unit cost for some arcs (all else unchanged), Cost Solidarity says that no agent should have a lower cost share. If this property is not satisfied, some agents may be incentivized to sabotage the shipping operation and report higher costs to pay lower-cost shares. Cost solidarity generalizes the well-known idea of cost monotonicity, and it has also been studied in minimum cost spanning tree problems (Bergantiños and Vidal-Puga, 2007; Gómez-Rúa and Vidal-Puga, 2017).

We offer two characterizations of the demander rule. The first one, stated in Theorem

5.1, asserts that the demander rule is the only cost allocation scheme satisfying Core Selection and Merge Proofness. The second one, stated in Theorem 5.2, asserts that the demander rule is the only cost allocation scheme satisfying Core Selection and Cost Solidarity.

The paper is organized as follows. In Section 2, we set up the framework by defining shortest-path problems and cost-sharing rules. In Section 3, we introduce some basic properties of cost-sharing rules in shortest-path problems. In Section 4, we define the demander rule and show that it satisfies the basic properties. In Section 5, we provide our characterization results.

2 The model

Our framework is close to that of Bahel and Trudeau (2014) and Bahel et al. (2024). The main differences stem from the fact that we consider here a variable set of agents —whereas the set of agents in these previous papers is fixed. Let $U = \{1, 2, \dots\}$ denote the (infinite) set of potential agents. We consider problems where a (finite) number of agents in U need to ship units of some commodity from a fixed point $\mathbf{0}$ to their respective locations ($\mathbf{0}$ is called the *source*). Let $\mathcal{N}$ denote the set of nonempty, finite subsets of U. We use the symbol N to refer to a generic element of U, that is to say, $N \in \mathcal{N}$.

To ease on notation, we will often write i instead of $\{i\}$ and $S \setminus i$ instead of $S \setminus \{i\}$, for any $i \in S \subseteq N$. Moreover, for any vector $x \in \mathbb{R}^N$ and any subset $S \subseteq N$, we write

$$x_S := \sum_{i \in S} x_i.$$

A *Shortest Path Problem* (SPP) is a tuple $P = (N, c, x)$, where:

- $N \in \mathcal{N}$ is the set of agents who need to connect to the source $\mathbf{0}$;
- $c = \{c(i, j) : i \in N \cup \mathbf{0}, j \in N, i \neq j\}$ is a collection of non-negative numbers (often referred to as a "cost matrix") giving the unit cost of shipping demands through each arc (i, j);
- $x \in \mathbb{R}_+^N$ is the demand vector: each agent $i \in N$ has a demand $x_i \in \mathbb{R}_+$ (of the commodity) to ship from the source to her location.

The set of SPP with agent set $N \in \mathcal{N}$ will be denoted by $\mathbb{P}_N$, and $\mathbb{P} := \bigcup_{N \in \mathcal{N}} \mathbb{P}_N$. Note that the source $\mathbf{0}$ is not an agent and that the unit costs $c(i, j)$ need not be symmetric —we may well have $c(i, j) \neq c(j, i)$ for some $i, j \in N$. Whenever $c(i, j) = c(j, i)$ holds for all $i, j \in N$ s.t. $i \neq j$, we say that the SPP has *symmetric arcs*.

Definition 2.1 *Given $N \in \mathcal{N}$ and $i \in N$, we call* ***path*** *(of* length *K)* ***to*** *agent i any sequence $p := (p_k)_{k=0,\ldots,K}$ such that:*

1. *$p_k \in N$, for $k = 1, 2, \ldots, K$;*

2. *$p_0 = \mathbf{0}$ and $p_K = i$;*

3. *$p_k \notin \{p_1, \ldots, p_{k-1}\}$ whenever $2 \leq k \leq K$.*

Note that all paths p originate from the source $\mathbf{0}$ and cross any location p_k only once. Thus, the length of each path and the number of paths to any given $i \in N$ are both finite. We denote by $\mathcal{P}_N(i, c)$ the set containing all paths to i. For simplicity, and as long as the player set is clear, we write $\mathcal{P}(i, c)$ instead of $\mathcal{P}_N(i, c)$. For any path p of length K, let $[p]$ refer to the set of players in the range of p, that is:

$$[p] := \{j \in N : p_k = j \text{ for some } k = 1, \ldots, K\}.$$

Given $P = (N, c, x)$, one can extend the cost function c to paths as follows: for any path p (of length K) to agent $i \in N$, $c(p)$ represents the cost of shipping one unit from the source to agent i via the path p, i.e.,

$$c(p) := \sum_{k=1}^{K} c(p_{k-1}, p_k).$$

For any $i \in N$, we call *shortest path to i* any path $\bar{p}^i \in \mathcal{P}(i, c)$ that solves the problem $\min_{p \in \mathcal{P}(i,c)} c(p)$. Note that there exists a shortest path to any $i \in N$ — since the set $\mathcal{P}(i, c)$ is nonempty and finite — but it may not be unique.

Example 2.1 *(Bahel et al. (2024)) Consider the SPP with symmetric arcs given by $P = (N, c, x)$, where $N = \{1, 2, 3\}$, $x = (2, 0, 1)$ and the cost structure is depicted by Figure 1. Hence, we have $c(\boldsymbol{0}, 1) = 200$, $c(3, 1) = c(1, 3) = 10$, $c(2, 1) = c(1, 2) = 70$, and so on.*

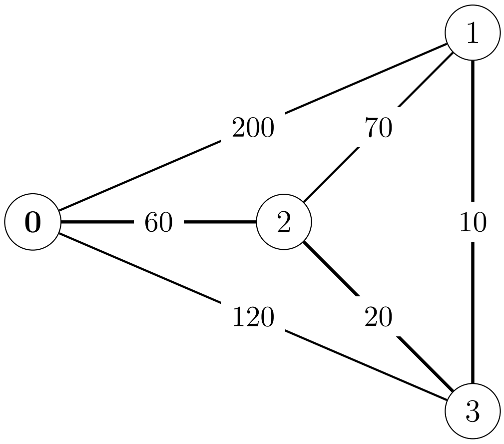

Figure 1: SPP with three agents.

One can see that there are 5 paths to agent 1, $(\boldsymbol{0},1),(\boldsymbol{0},2,1)$, $(\boldsymbol{0},3,1),(\boldsymbol{0},2,3,1)$, $(\boldsymbol{0},3,2,1)$*; and the shortest path to* 1 *is* $(\boldsymbol{0},2,3,1)$*, with cost* $c(\boldsymbol{0},2,3,1)=60+20+10=90$*. For agents* 2 *and* 3*, the costs of their respective shortest paths are* $c(\boldsymbol{0},2)=60$ *and* $c(\boldsymbol{0},2,3)=60+20=80$*.*

The cooperative game (with transferable cost) associated with P can be formulated by defining the cost of any nonempty coalition $S\subseteq N$ as

$$C_P(S):=\min\left\{\sum_{j\in S}x_j\cdot c(p^j):p^j\in\mathcal{P}(j,c)\text{ and }[p^j]\subseteq S,\forall j\in S\right\}. \tag{1}$$

That is the members of S pay the lowest possible cost of shipping their respective demands when using only the connections available in S. In particular, $C_P(S)=0$ whenever there is no demand to ship, i.e., $x_S=0$. We also adopt the convention that $C_P(\emptyset)=0$. For the problem P depicted in Example 2.1, $C_P(N)=2\cdot c(\mathbf{0},2,3,1)+0\cdot c(\mathbf{0},2)+1\cdot c(\mathbf{0},2,3)=180+80=260$.

Definition 2.2 *Given a shortest path problem* $P=(N,c,x)$*, we have the following.*
(i) An ***allocation*** *is a profile of cost shares,* $y\in\mathbb{R}^N$*, such that* $y_N=C_P(N)$*. Let* $\mathcal{A}(P)$ *be the set containing all cost allocations in* P*.*
(ii) The ***core*** *of* P *is the set*

$$Core(P):=\{y\in\mathcal{A}(P):y_S\leq C_P(S),\forall S\subsetneq N\}.$$

An allocation y *is called* ***stable*** *if* $y\in Core(P)$*.*

The above definition says that a cost allocation splits the total cost of shipping the respective demands between all agents in N. Remark that we allow for negative cost shares, which may be desirable in particular if we have agents who demand zero while providing others with cheaper access to the source. Definition 2.2-(ii) is the standard notion of stability: every coalition S should jointly pay at most its stand-alone cost $C_P(S)$. Let us now define the solution concepts studied in this work.

Definition 2.3 *A **Cost Sharing Rule** (CSR) is a mapping $y\colon \mathbb{P} \to \bigcup_{N\in\mathcal{N}} \mathbb{R}^N$ that assigns to each $P \in \mathbb{P}_N$ a cost allocation $y(P) \in \mathbb{R}^N$ such that $y_N(P) = C_P(N)$.*

In words, a CSR is a mechanism which, for any given problem P, allows to divide between agents the total cost $C_P(N)$ of satisfying the respective demands (we refer to this property as *efficiency*). In the following sections, we introduce a number of desirable properties for a CSR.

3 Anonymous Demand Additive Core Selectors

Bahel et al. (2024) introduce some basic properties for cost sharing rules, which we present in this section. Since SPP do not exhibit congestion externalities, each problem $P = (N, c, x)$ yields elementary problems of the form $P^j = (N, c, e^j)$ (for every $j \in N$), where $e^j \in \mathbb{R}^N$ is the vector of demands characterized by $e^j_j = 1$ and $e^j_i = 0$, if $i \in N \setminus j$.

Let $k, k' \in U$ and $N \in \mathcal{N}$ with $N \subset U \setminus \{k, k'\}$. We define the bijection $\sigma_{kk'}$: $N \cup \{\mathbf{0}, k\} \to N \cup \{\mathbf{0}, k'\}$ by $\sigma_{kk'}(k) = k'$ and $\sigma_{kk'}(i) = i$ otherwise. Given $P = (N \cup k, c, x)$, $P' = (N \cup k', c', x') \in \mathbb{P}$, we say that P and P' are kk'-*equivalent* if $x'_i = x_{\sigma_{kk'}(i)}$ for all $i \in N \cup k$ and $c(i, j) = c'(\sigma_{kk'}(i), \sigma_{kk'}(j))$ for all $i \in N \cup k \cup \mathbf{0}$, $j \in N \cup k$ such that $i \neq j$.

Definition 3.1 *A CSR y satisfies:*

1. ***Core Selection*** *if $y(P) \in Core(P)$ for all $P \in \mathbb{P}$.*

2. ***Demand Additivity*** *if $y(P) = \sum_{j\in N} x_j \cdot y(P^j)$ for all $P \in \mathbb{P}_N$.*

3. ***Anonymity*** *if, for all $k, k' \in U$, $N \in \mathcal{N}$ with $N \subset U \setminus \{k, k'\}$, $P = (N \cup k, c, x)$, $P' = (N \cup k', c', x') \in \mathbb{P}$ such that P and P' are kk'-equivalent, we have $y_i(P) = y_{\sigma_{kk'}(i)}(P')$ for all $i \in N \cup k$.*

Bahel et al. (2024) define the family of CSR called *Anonymous Demand Additive Core Selectors* ($ADACS$, for short).

4 The demander rule

Following Bahel et al. (2024), we define the demander rule as follows:

Definition 4.1 *The* ***demander rule*** y^d *is the* CSR *defined as:*

$$y_i^d(N, c, x) = x_i \cdot \min_{p^i \in \mathcal{P}(i,c)} c(p^i)$$

for all $(N, c, x) \in \mathbb{P}$ *and* $i \in N$*.*

In other words, the demander rule requires each agent to pay the cost of her shortest path for each unit demanded. Agents who demand zero pay nothing. As an illustration, for the SPP depicted in Example 2.1, the demander rule yields the cost share $y^d = (180, 0, 80)$.

The following result extends Theorem 4.1 in Bahel et al. (2024) —which is stated for a fixed agent set– to the class of problems with a variable agent set.

Theorem 4.1 *The demander rule* y^d *is an* $ADACS$*.*

Proof. It is straightforward to see that y^d satisfies Anonymity. Since Demand Additivity and Core Selection are defined for a fixed $N \in \mathcal{N}$, these properties follow from Theorem 4.1 in Bahel et al. (2024). ■

5 Characterization results

In this section, we propose several characterization results for the demander rule. We first introduce new axioms involved in these characterizations.

5.1 Merge Proofness

The first of these new axioms is Merge Proofness. This is an incentive compatibility requirement. In essence, it says that agents (seeking to reduce their joint cost share) should not benefit from merging and acting as a single agent. If a group of agents S merge into the single agent $s \in S$ (while every agent not in S remains the same) then we have a new problem $P^{s,S} := \left(N^{s,S}, c^{s,S}, x^{s,S}\right)$, which is defined as follows. First, the new set of agents and their demands are respectively:

$$N^{s,S} := (N \setminus S) \cup \{s\} \tag{2}$$

and

$$x_i^{s,S} := \begin{cases} x_i & \text{if } i \in N \setminus S \\ x(S) & \text{if } i = s. \end{cases} \tag{3}$$

Note from (3) that agent s in the new problem inherits the aggregate demand of the coalition S, and the demand of every other agent $i \in N \setminus S$ remains unchanged. In addition, the cost matrix $c^{s,S}$ of the new problem is given by

$$c^{s,S}(i,j) := \begin{cases} \min_{k \in S} c(i,k) & \text{if } i \in (N \setminus S) \cup \mathbf{0} \text{ and } j = s \\ \min_{k \in S} c(k,j) & \text{if } i = s \text{ and } j \in N \setminus S \\ c(i,j) & \text{if } i \in (N \setminus S) \cup \mathbf{0} \text{ and } j \in N \setminus S. \end{cases} \tag{4}$$

In the reduced problem, any node outside S will always use the best link available in S to connect with s (and vice-versa). One can then state two versions of the property requiring that merging should not be profitable.

Definition 5.1 *A CSR y satisfies*

1. ***Merge Proofness*** *if, for all $P \in \mathbb{P}$, $S \subsetneq N$ and $s \in S$,*

$$\sum_{i \in S} y_i(P) \leq y_s\left(P^{s,S}\right) + C_P(N) - C_{P^{s,S}}\left(N^{s,S}\right);$$

2. ***Strong Merge Proofness*** *if, for all $P \in \mathbb{P}$, $S \subsetneq N$, $s \in S$ and $i \in N \setminus S$,*

$$y_i(P^{s,S}) \leq y_i(P).$$

Strong Merge Proofness ensures that no other agent $i \notin S$ will be worse off in the reduced problem if a group of agents S merge in advance to be treated as a single agent s. Given that $\sum_{i \in S} y_i(P) = C_P(N) - \sum_{i \in N \setminus S} y_i(P)$ and $\sum_{i \in N \setminus S} y_i(P^{s,S}) = C_{P^{s,S}}(N^{s,S}) - y_s(P^{s,S})$ must hold for every CSR y, it is not difficult to see that Strong Merge Proofness implies Merge Proofness, which only requires that the agents in S do not pay a lower joint share in the reduced problem (after accounting for their internal transaction costs $C_P(N) - C_{P^{s,S}}(N^{s,S})$).

Our first result of this section states that the demander rule satisfies Strong Merge Proofness (hence, it also satisfies Merge Proofness).

Proposition 5.1 *The demander rule y^d satisfies Strong Merge Proofness.*

Proof. Consider an SPP $P = (N, c, x)$ and let $\bar{p}^i = (\bar{p}_0^i = \mathbf{0}, \bar{p}_1^i, \ldots, \bar{p}_{K_i}^i = i)$ be a shortest path (for the problem P) to each $i \in N$, with K_i denoting the length of $\bar{p}^i$.

Suppose that the players of coalition $S \subsetneq N$ (with $|S| \geq 2$) decide to merge into the node $s \in S$; and consider the reduced problem $P^{s,S}$ defined by (2)-(4). Fix $j \in N \setminus S$ and, in the case where $[\bar{p}^j] \cap S \neq \emptyset$, let

$$\underline{k}^j_S := \min\{k \in \{1,\dots,K_j\} : \ p^j_k \in S\} \text{ and } \bar{k}^j_S := \max\{k \in \{1,\dots,K_j\} : \ p^j_k \in S\}.$$

Let $\mathcal{P}^{s,S}(j)$ be the set of paths to j in the problem $P^{s,S}$. We can define a reduced path $\tilde{p}^j \in \mathcal{P}^{s,S}(j)$ (which is of length $K_j - (\bar{k}^j_S - \underline{k}^j_S)$ if $[\bar{p}^j] \cap S \neq \emptyset$) as follows.

$$\begin{aligned} \tilde{p}^j &= \bar{p}^j = \left(\bar{p}^j_0, \bar{p}^j_1, \dots, \bar{p}^j_{K_j}\right) && \text{if } [\bar{p}^j] \cap S = \emptyset; \\ \tilde{p}^j &= \left(\bar{p}^j_0, \bar{p}^j_{\underline{k}^j_S - 1}, s, \bar{p}^j_{\bar{k}^j_S + 1}, \dots, \bar{p}^j_{K_j}\right) && \text{otherwise.} \end{aligned} \tag{5}$$

To conclude the proof, remark from (4) and (5) that $c^{s,S}(\tilde{p}^j) \leq c(\bar{p}^j)$; and therefore

$$y^d_j(P^{s,S}) = x_j \cdot \min_{p^j \in \mathcal{P}^{s,S}(j)} c^{s,S}(p^j) \leq x_j \cdot c^{s,S}(\tilde{p}^j) \leq x_j \cdot c(\bar{p}^j) = y^d_j(P).$$

■

Theorem 5.1 *The demander rule y^d is the only CSR that satisfies Merge Proofness and Core Selection.*

Proof. We have already proved that y^d satisfies both properties. Conversely, letting y be a CSR satisfying Merge Proofness and Core Selection, we must show that $y = y^d$. Let $P = (N, c, x)$ be an SPP. Since $y_N(P) = y^d_N(P) = C_P(N)$ (by efficiency of y and y^d), it is enough to prove that $y \geq y^d$, i.e., $y_i \geq y^d_i$ for all $i \in N$. Fix $i \in N$. Let $i' \in U \setminus N$ and consider a problem $P' = (N', c', x')$ such that $N' = N \cup \{i'\}$, $x'_j = x_j$ if $j \in N \setminus i$, $x_i = x'_i + x'_{i'}$ and $c'(j,k) = c(j,k)$ if $j,k \in N \setminus i$, $c'(i,i') = c'(i',i) = 0$, $c'(i,j) = c'(i',j) = c(i,j)$ for all $j \in N \setminus i$. For each $j \in N \setminus i$, let $\bar{p}^j$ be a shortest path to agent j in the problem P; and denote by $\bar{p}'^j$ the path obtained from $\bar{p}^j$ by replacing agent i with agent i' (if $i \in [\bar{p}^j]$) and keeping all agents $k \in [\bar{p}^j] \setminus i$. Then, $\bar{p}'^j$ is a shortest path to agent j in P'. Hence, by the definition of the demander rule, the minimum cost to satisfy the demands of agents in $N \setminus i$ is

$$\Upsilon = \sum_{j \in N \setminus i} y^d_j(P) = \sum_{j \in N \setminus i} c(\bar{p}^j) = \sum_{j \in N \setminus i} c(\bar{p}'^j) = \sum_{j \in N \setminus i} y^d_j(P').$$

Under Core Selection,

$$y_i(P') + \sum_{j \in N \setminus i} y_j(P') \leq y^d_i(P') + \Upsilon$$

and

$$y_{i'}(P') + \sum_{j \in N \setminus i} y_j(P') \leq y^d_{i'}(P') + \Upsilon$$

so that

$$y_i(P') + y_{i'}(P') + 2\sum_{j\in N\setminus i} y_j(P') \le y_i^d(P') + y_{i'}^d(P') + 2\Upsilon. \tag{6}$$

On the other hand, by efficiency,

$$y_i(P') + y_{i'}(P') + \sum_{j\in N\setminus i} y_j(P') = y_i^d(P') + y_{i'}^d(P') + \Upsilon. \tag{7}$$

Combining (6) and (7),

$$\sum_{j\in N\setminus i} y_j(P') \le \Upsilon. \tag{8}$$

It is not difficult to check that $C_P(N) = C_{P'}(N')$. Hence, from Definition 5.1, Merge Proofness gives

$$\begin{aligned} y_i(P) &\ge y_i(P') + y_{i'}(P') = y_i(P') + y_{i'}(P') + \sum_{j\in N\setminus i} y_j(P') - \sum_{j\in N\setminus i} y_j(P') \\ &\overset{(8)}{\ge} y_i(P') + y_{i'}(P') + \sum_{j\in N\setminus i} y_j(P') - \Upsilon \overset{(7)}{=} y_i^d(P') + y_{i'}^d(P') = y_i^d(P). \end{aligned}$$

Since this is true for each $i \in N$, we deduce $y(P) \ge y^d(P)$ which, combined with $y_N(P) = y_N^d(P)$, implies $y(P) = y^d(P)$. ■

Corollary 5.1 *An ADACS y satisfies Merge Proofness* ***if and only if*** *it is the demander rule y^d.*

The properties in Theorem 5.1 are independent.

- The average lexicographic value defined in Tijs et al. (2011) satisfies Core Selection and fails Merge Proofness.
- For each $P = (N, c, x) \in \mathbb{P}$, let $M(P) = \{i \in N : x_i = \max_{k\in N} x_k\}$. Consider the CSR y^m defined as

$$y_i^m(P) = \begin{cases} \frac{C_P(N)}{|M(P)|} & \text{if } i \in M(P) \\ 0 & \text{if } i \notin M(P) \end{cases}$$

  for each $P \in \mathbb{P}$. Then, y^m satisfies Merge Proofness and fails Core Selection.

5.2 Cost solidarity

Definition 5.2 *A CSR y satisfies* ***Cost Solidarity*** *if, for any $P = (N, c, x)$, $P' = (N, c', x)$, we have*

$$[c(i,j) \le c'(i,j), \forall i, j \in N] \Rightarrow [y_i(P) \le y_i(P'), \forall i \in N].$$

Cost Solidarity says that no agent should be better off if the shipping costs increase on some arcs of the network (all else equal).[1] Combining Core Selection and Cost Solidarity allows to state the following result.

Theorem 5.2 *The demander rule y^d is the only CSR that satisfies Core Selection and Cost Solidarity.*

Proof. It is not difficult to check that y^d satisfies both Core Selection and Cost Solidarity. Fix a CSR y satisfying Core Selection and Cost Solidarity. We show below that $y(P) = y^d(P)$ for all $P \in \mathbb{P}$. We proceed by induction on the size of

$$\Omega(P) = \left\{ i \in N : c(\mathbf{0}, i) > \min_{p \in \mathcal{P}(i,c)} c(p) \right\}.$$

Assume first $|\Omega(P)| = 0$, i.e., $c(\mathbf{0}, i) = \min_{p \in \mathcal{P}(i,c)} c(p)$ for all $i \in N$. This means it is optimal for each agent i to ship her demand directly from the source. Hence, the associated cooperative game —defined in (1)– is additive. This implies that the core is a singleton. Under Core Selection, $y(P) = y^d(P)$. Assume now $y(P') = y^d(P')$ whenever $|\Omega(P')| < \omega$; and fix a problem P satisfying $|\Omega(P)| = \omega$. Let $i \in \Omega(P)$. This means that $c(\mathbf{0}, i) > c(\bar{p}^i) = \min_{p \in \mathcal{P}(i,c)} c(p)$. Consider the problem $P' = (N, c', x)$ defined by $c'(\mathbf{0}, i) = c(\bar{p}^i)$ and $c'(j,k) = c(j,k)$ otherwise. Under Cost Solidarity, we deduce $y_j(P) \geq y_j(P')$ and $y^d_j(P) \geq y^d_j(P')$ for all $j \in N$. Moreover, $C_P(N) = C_{P'}(N)$ because $c'(\mathbf{0}, i) = c'(\bar{p}^i) = c(\bar{p}^i) = \min_{p \in \mathcal{P}(i,c)} c(p) = \min_{p \in \mathcal{P}(i,c)} c'(p)$. Hence, $y(P) = y(P')$ and $y^d(P) = y^d(P')$. By induction hypothesis, $y(P') = y^d(P')$, and hence $y(P) = y^d(P)$. ■

Corollary 5.2 *An $ADACS$ y satisfies Cost Solidarity* ***if and only if*** *it is the demander rule y^d.*

Observe that the properties used in Theorem 5.2 are independent.

- The average lexicographic value defined in Tijs et al. (2011) satisfies Core Selection and fails Cost Solidarity.
- The equal division rule y^e, defined as $y^e_i(P) = \frac{C_P(N)}{|N|}$ for all $P \in \mathbb{P}_N$ and all $i \in N$, satisfies Cost Solidarity and fails Core Selection.

[1] Cost Solidarity has been used in the literature on minimum cost spanning trees; see, for instance, Bergantiños and Vidal-Puga (2007).

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