Perceived and expected quantity constraints in inventory dynamics

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Perceived and expected quantity constraints in inventory dynamics

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Abstract

Inventory dynamics play a significant role in business cycles, as inventory tends to be more sensitive to excess demand fluctuations than it is to production. This study addresses the decision-making problem of firms and households under possible quantity-constrained trade using a simple model based on Keynesian unemployment in disequilibrium economics. In the presented model, production and consumption are determined by the firm and household, respectively. In particular, they solve intertemporal optimizations under expected future constraints that depend on current states such as sales and income. Their independent decisions cause disequilibrium in goods: the gaps between production and sales and between planned and actual trade. While the former gap is buffered by inventory, we use United States data to show that the existence of inventory holdings cannot moderate the disequilibrium. Our simple model contributes to the body of knowledge on this topic as it endogenizes the firm's objective value of the inventory–sales ratio in Metzlerian dynamics, reproduces the qualitative inventory dynamics, and reveals how stock disequilibrium affects flow disequilibrium dynamics.

Keywords: Inventory cycle, Metzlerian dynamics, Endogenous business cycle, Quantity-constrained trade

1 Introduction

According to Blinder and Maccini (1991), inventory dynamics play a significant role in business cycles. Specifically, inventory exhibits a persistent

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tendency to be more sensitive to excess demand fluctuations than it is to production. Figure 1, which shows the cyclical components of the business cycle in the United States (US) for 1960Q1–2020Q4, confirms that this argument is still valid.[1] The data are regulated using a Hodrick–Prescott filter with a smoothing parameter of 1600, which is suitable for quarterly periodicity. The cyclical component is calculated by dividing the difference between the actual and trend figure by the trend, so that it corresponds to the rate at which the variables deviate from the trend. The dashed line represents the cyclical component of real production. We see that sales dynamics seem slightly more sensitive than production dynamics.[2]

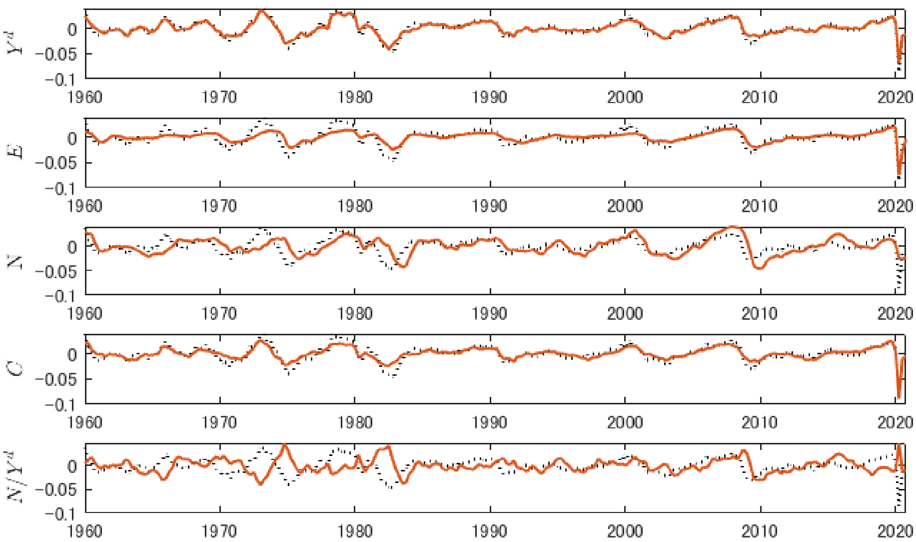

Figure 1: Cyclical components of sales (Y^d), the employment rate (E), inventory holdings (N), consumption (C), and the inventory–sales ratio (N/Y^d)

Inventory dynamics confirm this, as the cyclical component of inventory lags behind that of production (see Figure 2). This implies that during booms, production exceeds sales and thus unsold goods temporarily accumulate as inventory in the next period. Therefore, Blinder claimed that

[1]The datasets were created using the method available at `https://fairmodel.econ.yale.edu/`; see Appendix B. This method was also used by Chiarella et al. (2005). In this study, inventory is aggregated so that it includes final sales inventory, work-in-progress inventory, and raw materials. Among previous equilibrium research, Wen (2011) also aggregated multi-level inventories. We use this again in the numerical experiment.

[2]Their variances are similar: the variance of the cyclical component of production is $2.50 \cdot 10^{-04}$, whereas that of its sales counterpart is $1.80 \cdot 10^{-04}$.

the production-smoothing model is unrealistic and recommended using the so-called (S, s) model (e.g., Fisher and Hornstein (2000)).

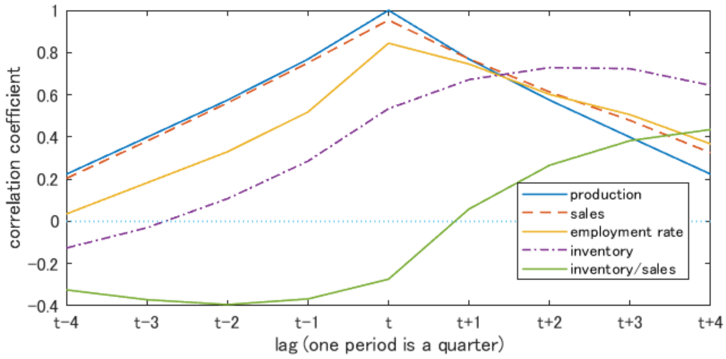

Figure 2: Correlation coefficients of the cyclical components of lagged economic variables and real production

Modern equilibrium macroeconomics has addressed this puzzle using various models. Bils and Kahn (2000) argued that the cost structure (i.e., a procyclical marginal production cost and countercyclical markup due to the costly change in factor utilization) maintains the inventory–sales ratio. As inventory appears in the goods demand function, it directly affects goods sales. Iacoviello et al. (2011) distinguished input inventory (e.g., raw materials) from output inventory. The former is used in goods production as a production factor and the latter is directly bought by households. Sarte et al. (2015) constructed a multi-sector, multi-level production model and showed that the cycle of the inventory–sales ratio has been moderated since 1984 in the US (i.e., the beginning of the Great Moderation). In these general equilibrium models, inventory is regarded as a good rather than as a buffer of the gap between demand for and the supply of goods.[3] Treating inventory as a special good in dynamic general equilibrium models allows researchers to calculate dynamic equations by solving the social planner's optimization problem. For instance, Christiano (1988) also built an equilibrium model with a total factor productivity shock, but treated inventory dynamics as the gap between supply and demand. In his model, the firm controls production under a complicated information structure with noisy observations of the shock. He used the results of a numerical experiment to argue that this noise plays an important role in procyclical inventory investment. However,

[3]Bils and Kahn (2000) and Iacoviello et al. (2011) included (output) inventory in the household's formulation. In particular, Iacoviello et al. (2011) argued for further research to examine how a firm controls its output inventory.

while this argument emphasizes the residual role of inventory investment, how inventory is controlled in decentralized markets remains unclear. Wen (2011) used an unanticipated shock on a household's preference for goods to show the procyclical inventory–sales ratio of final goods. His model emphasizes the role of inventory as an asset: holding inventory generates a type of liquidity premium in dynamic economies that face shocks. In summary, while general equilibrium models that include inventory fit the data well, how inventory, which is a residual of the gap between demand for and the supply of goods, works in such models remains underexplored. Moreover, in equilibrium models, it is difficult to motivate a firm to hold inventory without adding exogenous unanticipated shocks.

By contrast, according to Keynesian disequilibrium economists, so-called Metzlerian dynamics play a central role in the business cycle.[4] In Metzlerian dynamics, the firm forms an expectation about current sales and determines the production level to maintain the desired inventory–sales ratio, which is usually assumed to be fixed ad hoc. Production associated with this fixed ratio is a source of instability in Metzlerian dynamics, as inventory holdings are unrelated to production smoothing.[5]

Our intuition is that inventory is regarded as a residual resulting from the gap between production and actual sales. Its existence allows for the inconsistency of one period's trade in the model analysis and the inventory adjustment process may explain the fluctuations. Although Blinder and Maccini (1991) denied the stabilization effect of inventory in dynamics, the utility of holding inventory is undeniable, as inventory dynamics can offset the flow disequilibrium between production and sales. The central question in this study is whether inventory's potential production-smoothing (and offsetting disequilibrium) role and its procyclical movement can coexist in a model. Therefore, we explore the role of inventory using a disequilibrium model.

The (general) disequilibrium model constructed by Barro and Grossman (1971) was inspired by the quantity adjustment theory described by Clower (1965) and Leijonhufvud (1968). Bénassy (1975) also built a model to define the K-equilibrium, an interpretation of the effective demand principle

[4]See Metzler (1941). Metzlerian dynamics were examined by Franke (1996), Wegener et al. (2009), and Grasselli and Nguyen-Huu (2018). Chiarella et al. (2005) introduced the Keynes–Metzler–Goodwin model to explain the mechanism behind the growth cycle.

[5]Sarte et al. (2015) argued that the inventory–sales ratio, which was countercyclical before 1984, became acyclical thereafter. Our data also confirm this: the correlation between the simultaneous cyclical components of output and the inventory–sales ratio after 1984 is 0.018. However, this does not imply that the ratio is independent of the cycle; the correlation between the ratio after three quarters and output is 0.60. This means that an increase in output increases the ratio with a lag.

proposed by Keynes (1936).[6] The disequilibrium model allows for transactions under excess demand or supply, and the gap between planned and actual transactions (i.e., quantity constraints) affects individuals' decisions. This spillover effect, termed the *dual-decision hypothesis* by Clower (1965), induces a persistent demand shortage; low demand for goods induces correspondingly low labor demand and vice versa. In disequilibrium economics, the economy is characterized by excess demand or supply in goods and labor markets. Specifically, the Keynesian unemployment regime (i.e., excess supply in both goods and labor markets) has been the focus of macroeconomic studies such as Chiarella and Flaschel (2000) and Chiarella et al. (2000). Despite few analyses since the 1990s (Backhouse and Boianovsky, 2012), disequilibrium economics has increasingly attracted attention in macroeconomic research in the context of secular stagnation (Mankiw and Weinzierl (2011), Michaillat and Saez (2015), Eggertsson et al. (2019), Dupor et al. (2019), and Ogawa (2022)).

In disequilibrium economics, Blinder (1980, 1981) argued that the existence of a buffer stock of goods (inventory) moderates the spillover effect in the near absence of regime switching. Existing disequilibrium analyses such as those of Honkapohja and Ito (1980), Simonovits (1982), and Eckalbar (1985) show that regime switching occurs only when inventory stockouts occur because regimes are defined as the relationships among traded quantities. However, the disequilibrium analyses in these models are incomplete because they do not account for individuals' decision making; the disequilibrium between planned and actual transactions is relevant for dynamic economies. Thus, decision-making should be discussed based on the perception of quantity constraints rather than stockouts.

In this study, we formulate agents' behaviors to solve intertemporal optimization problems. Agents perceive quantity constraints, as shown by Barro and Grossman (1976), Yoshikawa (1984), and Murakami (2015). In this respect, the present model is based on Keynesian unemployment in disequilibrium economics. In static disequilibrium models, agents perceive quantity constraints on their current trades. In this dynamic model, expected future constraints are used to assess the current constraint. Despite adaptive expectations, the firm decides on employment and the household plans consumption. This premise provides scope for future research on intertemporal spillover effects.

In the model analysis, we find that the "intertemporal optimization"

[6]Korliras (1975) and Malinvaud (1977, 1980) emphasized the persistence of the Keynesian situation. For a review of these economic concepts, see Backhouse and Boianovsky (2012).

framework reveals some aspects of disequilibrium dynamics. Stock disequilibrium accelerates these dynamics by adjusting the optimal solutions. Maintaining the objective value of the sales–inventory ratio destabilizes the dynamics in ordinal Metzlerian models. Although the present model with optimization does not assume this feature, Metzlerian dynamics partially survive. The firm chooses production to ride on the saddle path toward objective values and keeps the objective value of the sales–inventory ratio stable relative to the actual ratio (i.e.,, the firm aims to maintain a constant ratio). However, actual sales are not determined by the firm; therefore, it must revise its expectations. This process results in cyclical dynamics. Furthermore, we find that this model reproduces the lagged dynamics of inventory. In a numerical experiment using US data, we show that inventory follows production with a two-quarter lag (see Figure 2).

This study does not use price dynamics (a speculative motivation for holding inventory). Instead, the firm's perception of quantity constraints (sales) and its decision to forecast future sales reproduce Metzlerian dynamics. On this point, our model can be interpreted as a "micro-founded" Metzlerian model.[7]

The remainder of this paper is organized as follows. In Section 2, we build a canonical economic model and use intertemporal optimization for the decision making by firms and households. In Section 3, we analyze the dynamics of the economic model. We find that a persistent cycle can occur by Hopf bifurcation, which corresponds to Metzlerian dynamics. In Section 4, we simulate the model estimated with US data. The persistent regular cycle itself is unrealistic; however, the results show that the relationship between economic variables such as lagged inventory dynamics is reproduced in our simple model. In Section 5, we discuss the simulation results. The simulation evaluated the accuracy of the model by approximating the data. However, a more interesting aspect of this study is that the simulation reveals the firm's objective values; the dynamics of the difference between the planned inventory and its current level show how the firm reacts to existing disequilibria. In Section 6, we conclude our analysis.

2 The model

In our model, households and firms trade consumption goods and labor.

[7]For the micro-foundation of inventory control in evolutionary dynamics, see Shiozawa et al. (2019). These authors also emphasized Metzlerian dynamics by constructing a large and sophisticated model with intertemporal optimization. Meanwhile, our model focuses on the endogenous objective inventory–sales ratio using a simple one-sector model.

1. The firm determines current employment $E(t)$ depending on current expected sales and future sales
2. The firm pays wages to the employed
3. The households determine consumption demand depending on the wages
4. The goods market opens and the gap between goods demand and supply determines the inventory dynamics
5. The firm revises its expected sales

In this study, we use intertemporal optimization *as the tool*. Sometimes, this is treated as a "sound micro-foundation." However, we do not use this to justify our model; it can only approximate the behavior formulated above as a mathematical tool. However, this does not directly illustrate the real behaviors of humans. The optimization approximates the decision in the long run: the firm attempts to predict future sales and believes that current sales will converge to a normal level in the future. Our perspective is harmless unless expectations are forced to be consistent with the actual economy, which is exogenous to the firms.

2.1 Firm's optimization model

The firm produces consumption good Y using E. Production technology is described by the following smooth production function F:

$$Y = F(E), \quad F'(E) > 0, \quad F''(E) < 0, \quad F(0) = 0, \quad \lim_{E \to 0} F'(E) = \infty, \quad \lim_{E \to \infty} F'(E) = 0. \tag{1}$$

The firm generates revenue by selling goods at price P and pays nominal wage W to employed labor E. Sales are determined by demand for good Y^d such that production and sales are usually different. This gap affects the firm's inventory by holding N constant:

$$\dot{N} = Y - Y^d - \delta N = F(E) - Y^d - \delta N, \tag{2}$$

where $\delta > 0$ is the constant inventory depreciation rate. Although holding inventory incurs a cost, we assume that the firm avoids inventory stockouts.[8]

[8] Lovell (1962) and Wen (2011) also made this stockout avoidance assumption. Bo (2001) empirically showed that Dutch firms aim to hold excess inventory to avoid stockouts.

For simplicity, we assume that the subjective inventory-holding cost is a function of N and sales, Y^d:

$$\Phi(N, Y^d), \quad \Phi > 0, \quad \Phi_{NN} > 0. \tag{3}$$

For convenience, we use the following function:

$$\Phi(N, Y^d) = \phi\left(\frac{N}{Y^d}\right) N, \quad \lim_{x\to 0} \phi'(x) = -\infty, \quad \lim_{x\to\infty} \phi'(x) \geq 0. \tag{4}$$

The unit cost of holding inventory ϕ is a function of the expected inventory–sales ratio. The conditions for the derivatives eliminate the possibility of stockouts and infinite inventory holdings (see Figure 3). For N, this function is U-shaped.[9] The condition $\Phi_{NN} > 0$ implies that $\phi''(x)x + 2\phi'(x) > 0$ holds true for all $x > 0$. This unit cost form is convenient because we can apply the following conversion equation:

$$\Phi_{NY} = -\left(\frac{N}{Y^d}\right)\Phi_{NN} < 0. \tag{5}$$

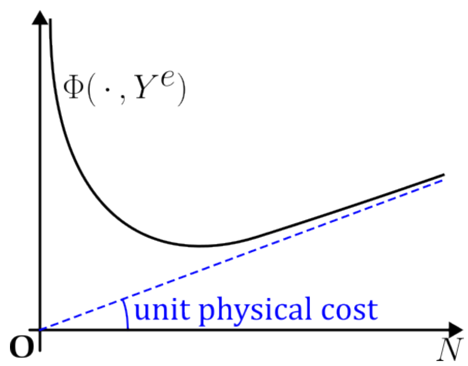

Figure 3: Canonical inventory-holding cost function $\Phi(N, Y^d)$

The firm maximizes its current value by controlling its production. Let t denote the current period in which the firm plans its production and $\tau \geq t$ be a future period. The firm's current value is calculated by aggregating its expected profits discounted by the expected real interest rate. Its nominal value in the current period, $V(t)$, is defined as follows:

$$V(t)/P(t) = \int_{\tau=t}^{\infty} [Y^e(\tau) - wE(\tau) - \Phi(N^e(\tau), Y^e(\tau))] e^{-\int_{s=0}^{\tau}(r(s)-\pi(s))\,ds}\, d\tau \tag{6}$$

[9]The utility of the U-shaped inventory-holding cost is explained by Otani (1983).

where $r(\tau)$ and $\pi(\tau)$ are the nominal interest and inflation rates, respectively. As the firm calculates this notional value using its information, it is natural to describe future sales and inventory holdings in the form of expectations. Therefore, we use expected sales $Y^e(\tau)$ and inventory holdings $N^e(\tau)$ instead of actual sales and inventory holdings.

To address the quantity adjustment process, we omit price dynamics in this study.

Assumption 1. The nominal wage rate and price of goods increase at the same constant rate $\pi > 0$. The nominal interest rate $r > \pi$ remains constant.

This assumption implies that real wage $w = W/P$ is constant.

Therefore, the firm's optimization problem is as follows:

$$\max_{\{E(\tau)\}} \int_t^\infty [Y^e(\tau) - wE(\tau) - \Phi(N^e(\tau), Y^e(\tau))]e^{-(r-\pi)(\tau-t)}\, d\tau, \tag{7}$$

$$\text{subject to } \dot{N}^e(\tau) = F(E(\tau)) - Y^e(\tau) - \delta N^e(\tau), \tag{8}$$

$$\dot{Y}^e(\tau)_{\tau \geq t} \text{ is given.} \tag{9}$$

Inventory N^e is a state variable and employment E is a control variable. The firm decides on its flow of employment $\{E(\tau)\}_{\tau=t}$ to maximize the current value. The flow of employment and sales determines inventory dynamics and inventory affects value as a cost function.

The final condition for expected sales dynamics (9) is undefined. Of course, we can imagine various characteristics in the formation of expected sales. However, only the simplest settings are used.

Assumption 2. The firm expects that future sales in its notional optimization are equal to current expected sales: $Y^e(\tau) = Y^e(t)$ for all $\tau \geq t$. Expected sales at each time point $Y^e(t)$ are adjusted to current actual sales.

There are two reasons for making this assumption. First, this intuitive simplification reproduces the qualitative features of inventory dynamics. The numerical experiment shows that the production dynamics in our model are more aggressive than sales; therefore, inventory fluctuates with a lag, as shown in the previous section. The second reason is strategic: distinguishing the sales dynamics in notional intertemporal optimization and the reference point of expectation is a good baseline for future extensions.

Using this assumption, we can solve the optimization problem. The current Hamiltonian value for optimization is

$$\mathcal{H}(E(\tau), N(\tau)) = [Y^e - wE(\tau) - \Phi(N^e(\tau), Y^e)] + \lambda(\tau)[F(E(\tau)) - Y^e - \delta N^e(\tau)]. \tag{10}$$

In the following discussion, we omit the description of τ. The first-order and transversality conditions are as follows:

$$\mathcal{H}_E = -w + \lambda F'(E) = 0, \tag{11}$$

$$\mathcal{H}_{N^e} = -\Phi_N(N^e, Y^e) - \delta\lambda = -\dot{\lambda} + (r - \pi)\lambda, \tag{12}$$

$$\mathcal{H}_\lambda = F(E) - Y^e - \delta N = \dot{N}^e, \tag{13}$$

$$\lim_{\tau\to\infty} \lambda N^e e^{(r-\pi)\tau} = 0. \tag{14}$$

The first two conditions induce employment controls. By unifying the control for employment and inventory dynamics, we obtain the firm's (notional) optimal control:

$$\begin{cases} \dot{E} & = -\frac{(F'(E))^2}{wF''(E)} \left[\Phi_N(N^e, Y^e) + (r - \pi + \delta)\frac{w}{F'(E)}\right] \\ \dot{N}^e & = F(E) - Y^e - \delta N^e \end{cases} \tag{15}$$

The solution of the firm's optimization problem is illustrated in Figure 4. When the cost function Φ is not excessive, the two nullclines cross uniquely at (E^*, N^{e*}). This is the steady state of *the firm's optimization problem* and not the steady state of the economy. In this study, we call it "objective values," since the firm plans its production so that (E, N^e) converges to that level. By linearizing $(\dot{E}, \dot{N}^e)$ around the objective values, we obtain the Jacobian matrix $\mathbf{J}_2$ as follows:

$$\mathbf{J}_2 = \begin{pmatrix} r - \pi + \delta & -\frac{(F'(E))^2}{wF''(E)}\Phi_{NN}(N^{e*}, Y^e) \\ F'(E^*) & -\delta \end{pmatrix} = \begin{pmatrix} \oplus & \oplus \\ \oplus & \ominus \end{pmatrix} \tag{16}$$

As 1-D stable manifold exists and there is only one control variable (employment E), the firm chooses $E(t)$ such that $(E(t), N(t))$ lies along the saddle path. Notably, the given $Y^e(t)$ affects objective values (E^*, N^{e*}) (see Appendix A). As the firm sets current employment depending on the objective values and current inventory holdings, the employment function is as follows:

$$E = E(N, E^*(Y^e), N^{e*}(Y^e)) = E(N, Y^e) \tag{17}$$

The difference between the current and objective values $N^{e*} - N(t)$ can be interpreted as stock disequilibrium. The phase plane in Figure 4 illustrates how firms react to stock disequilibrium.

Unfortunately, it is difficult to implement comparative statics on the employment function except for the steady state of dynamic economies (see Appendix A). This is because current employment $E(t)$ is an optimum intertemporal adjustment and not the objective value E^*.

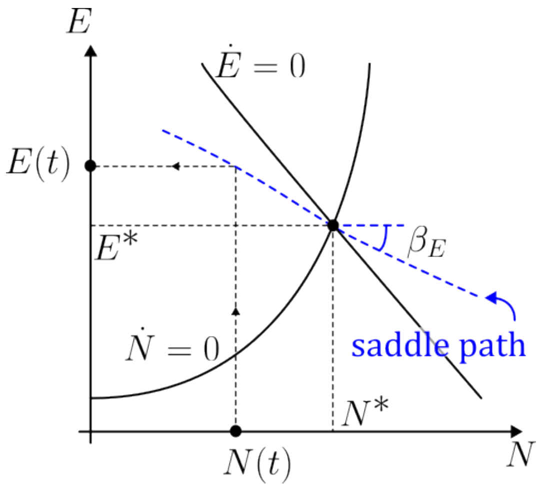

Figure 4: Firm's optimization under constant sales

2.2 Goods demand

We also use an intertemporal optimization for the households. A representative household is expected to solve the standard intertemporal utility maximization problem. The aggregate household size is L, which is a constant supply of efficient labor (a composite of the pure workforce and labor productivity). The household derives utility from its consumption per unit of effective labor at time τ, $c(\tau)$. The objective utility function at time τ is described using $u(C(\tau))$:

$$\int_t^\infty u(C(\tau))e^{-\sigma(\tau-t)}\,d\tau \text{ where} u' > 0,\ u'' < 0, \tag{18}$$

where $\sigma > 0$ is the constant rate of the time preference. The household rations its net real income into consumption and savings to maximize Equation (18). The perceived intertemporal budget constraint is expressed as

$$\dot{Y}_I(\tau) = (r - \pi)(Y_I N(\tau) - c(\tau)) \tag{19}$$

where $Y_I(\tau)$ is the net real income at time τ. In this study, we assume $Y_I(t) = F(E(t))$. This budget constraint implies that the return on savings grows at the rate $r - \pi$.[10]

Following Murakami (2015), we assume that the utility function is in the constant relative risk aversion form and includes the "social status" term

[10] This budget constraint is drawn from Uzawa (1969) and Murakami (2015).

$\tilde{C}(w)$:

$$u(C(\tau)) = \begin{cases} ([C(\tau) - \tilde{C}(w(t))]^{1-\theta} - 1)/(1-\theta) & \text{if } \theta \neq 1, \\ \ln(C(\tau) - \tilde{C}(w(t))) & \text{if } \theta = 1, \end{cases} \tag{20}$$

where $\tilde{C}'(w(t)) > 0$. Furthermore, the transversality condition $\lim_{\tau\to\infty} u'(C(\tau))Y_N(\tau)e^{-\sigma(\tau-t)} = 0$ holds. The following consumption function with a Keynesian flavor is obtained:

$$C^d(t) = c_1(r)Y_I(t) + (1 - c_1(r))\hat{c}(w(t)) = c_1(r)F(E(t)) + c_2(r, w), \tag{21}$$

where $c_1 = 1 - \frac{r-\sigma}{\theta r} \in (0, 1)$.[11] Consumption demand is affected by current net income because households perceive that the flow of future income depends on current income (see Equation (19).

3 Dynamics of the economy

We formulate the firms' and households' behavior. Let us describe the transactions in each period for the dynamic analysis. First, goods demand Y^d consists of autonomous demand G (constant) and the consumption demand function C^d:

$$Y^d(E) = C^d(E) + G, \tag{22}$$

We assume that expected sales Y^e are adaptively adjusted as follows:

$$\dot{Y}^e = \beta_e(Y^d - Y^e), \tag{23}$$

where β_e denotes a positive constant. This adjustment is related to the flow disequilibrium. Disequilibrium typically refers to the gap between supply and demand and $Y^d - Y^e$.

We now describe the dynamic system of the economy:

$$\dot{N} = F(E(N, Y^e)) - Y^d(E(N, Y^e)) - \delta N, \tag{24}$$

$$\dot{Y}^e = \beta_e(Y^d(E(N, Y^e)) - Y^e), \tag{25}$$

which is a 2-D autonomous system. Let subscript 0 denote the steady-state value of the system. The variable N is actual inventory holdings, which differ from the firm's expected inventory holdings N^e. This implies that the dynamics of actual inventory holdings may differ from the firm's expectations.

[11] For the proof, see Murakami (2015, Proposition 2.3).

This difference is expressed as the path of N in the following numerical experiment. The shape of the path of N differs from the saddle path illustrated in Figure 4.

However, the consistency between N^e and N is ensured in the steady state:

Proposition 1.

$$N_0 = N^{e*}(Y_0^e), \quad E_0 = E^*(N_0, Y_0^e)$$

Proof. By unifying the steady-state conditions in (24) and (25), $F(E_0) - Y_0^e - \delta N_0 = 0$ holds for (N_0, Y_0^e). This condition states that (N_0, E_0) is on the curve $\dot{N}^e = 0$ of the firm's optimization (Figure 4). As the firm chooses E on the saddle path in Figure 4, this implies that $N_0 = N^{e*}$ and $E_0 = E^*$. $\square$

This proposition characterizes the steady state corresponding to the firm's objective value. The firm's expectation is realized such that notional optimization is also realized in the steady state. In other words, there is no unexpected gap between supply and demand, which is suitable for equilibrium economics. This property partially ensures the uniqueness of the steady state.

Proposition 2. If the objective value (E^*, N^{e*}) is uniquely determined for any $Y^e > 0$, then the steady state (N_0, Y_0^e) exists uniquely.

This proposition arises from $N_0 = N^{e*}(Y_0^e)$.

The Jacobian matrix around the steady state is[12]

$$\mathbf{J} = \begin{pmatrix} \left(F'(E_0) - \frac{\partial Y^d}{\partial E}\right)\frac{\partial E}{\partial N} - \delta & \left(F'(E_0) - \frac{\partial Y^d}{\partial E}\right)\frac{\partial E}{\partial Y^e} \\ \beta_e \frac{\partial Y^d}{\partial E}\frac{\partial E}{\partial N} & \beta_e \left(\frac{\partial Y^d}{\partial E}\frac{\partial E}{\partial Y^e} - 1\right) \end{pmatrix} = \begin{pmatrix} \ominus & \oplus \\ \ominus & ? \end{pmatrix} \tag{26}$$

Although we have not formulated particular functions, we can derive the instability condition for Metzlerian dynamics: the steady state is unstable when employment around the firm's objective value reacts excessively to the change in expected sales Y^e. By observing the $(2,2)$ element, we find that

$$\frac{\partial Y^d}{\partial E}\frac{\partial E}{\partial Y^e} - 1 = \frac{dY^d}{dY^e} - 1, \tag{27}$$

This term is positive when Metzlerian instability is emphasized. In our model, this situation can occur when objective employment $E^* = E_0$ reacts excessively to changes in Y^e.

[12]For the comparative statics, see Appendix A.

3.1 Metzlerian instability and Hopf bifurcation

Here, we consider the situation in which a closed orbit appears in our dynamic economic system. This situation occurs when Metzlerian instability is effective. First, we examine the conditions of the cyclical dynamics. We assume the following conditions.

Assumption 3. Around the steady state (N_0, Y_0^e), an increase in Y^e increases objective inventory holdings N^{e*}.

An increase in expected sales would increase inventory holdings because the firm would like to avoid stockouts, as the unit cost of inventory holdings ϕ increases as N/Y^e approaches 0. Meanwhile, the firm increases employment to increase expected sales, which is shown as $\frac{dE^*}{dY^e} > 0$ in Appendix A. This increase in employment partially eliminates the possibility of stockouts, which decreases objective inventory holdings; however, the assumption above excludes this secondary effect from dominating the primary effect.

In fact, Assumption 3 implies that the change in production becomes larger than the change in expected sales.

Lemma 1. Under Assumption 3, $\frac{dF(E)}{dY^e} > 1$ remains around the steady state.

Proof. The calculations in Appendix A indicate that

$$\begin{aligned}\frac{dF(E)}{dY^e} - 1 &= \frac{dF(E)}{E}\frac{dE}{dY^e} - 1 \\ &= \frac{F'}{Q}\left[\beta_E\left(\Phi_{NY}(F')^2 + \Phi_N F''\right) + \Phi_{NN}F' - \delta\Phi_{NY}F'\right] - 1,\end{aligned}$$

where $Q = \Phi_{NN}(F')^2 + \delta\Phi_N F'' > 0$ remains around the steady state. The sign of the term is as follows:

$$\begin{aligned}F'\left[\beta_E\left(\Phi_{NY}(F')^2 + \Phi_N F''\right) + \Phi_{NN}F' - \delta\Phi_{NY}F'\right] - \Phi_{NN}(F')^2 - \delta\Phi_N F'' \\ = (\beta_E F' - \delta)(\Phi_N F'' + \Phi_{NY}(F')^2).\end{aligned}$$

This result implies that $\Phi_N F'' + \Phi_{NY}(F')^2 < 0$. Appendix A shows that

$$\frac{\partial N^{e*}}{\partial Y^e} = -\frac{\Phi_{NY}(F')^2 + \Phi_N F''}{Q}$$

Therefore, Assumption 3 is equivalent to $\frac{dF(E)}{dY^e} > 1$. □

This lemma states that the firm's production control reacts excessively to the change in expected sales when objective inventory holdings react positively to expected sales. This mechanism corresponds to Metzlerian dynamics: the firm would like to keep the ratio N/Y^e constant (this is shown as condition $\frac{\partial N^{e*}}{\partial Y^e} > 0$) so that production reacts excessively to the change in expected sales. Because $\frac{dY^d}{dY^e} = c_1\frac{dF(E)}{dY^e}$, the above equation is a necessary condition for Metzlerian instability.

In this study, we present the conditions for cyclical dynamics using the parameters c_1 (propensity for consumption) and β_e (adjustment speed of expected sales).

Proposition 3. Suppose the propensity for consumption is bounded within the range $c_1 \in (\underline{c}_1, \bar{c}_1)$. Then, the dynamic system in Equations (24) and (25) goes through Hopf bifurcation as the adjustment speed of expected sales β_e changes.

The boundaries are defined as follows:

$$\text{Metzlerian instability}: \ c_1 > \underline{c}_1 = \frac{1}{F'(E^*)E_{Ye}},$$
$$\text{Excluding saddle path}: \ c_1 < \bar{c}_1 = \frac{\frac{\delta}{F'(E^*)} - E_N}{\delta E_{Ye} - E_N},$$

where $E_{Ye} = \frac{\partial E^*}{\partial Y^e}$ and $E_N = \frac{\partial E^*}{\partial N}$.

Proof. First, we consider what the boundaries mean. Inequality $c_1 > \underline{c}_1$ implies that the term in (27) is positive. Therefore, $c_1 > \underline{c}_1$ guarantees that demand for goods fluctuates more aggressively than expected sales. $c_1 < \bar{c}_1$ ensures $\det\mathbf{J} > 0$. In a 2-D system, the negative determinant of $\mathbf{J}$ involves a saddle path. Excluding the saddle path is required for cyclical dynamics. Lemma 1 is a necessary condition for Metzlerian instability and ensures $\underline{c}_1 < \bar{c}_1$.

Subsequently, there exists a value for the adjustment speed of expected sales $\tilde{\beta}_e$, where the eigenvalues of $\mathbf{J}$ consist of a pair of purely imaginary numbers. The trace of $\mathbf{J}$ is a continuous function of β_e (linear with a negative constant):

$$\mathrm{tr}\mathbf{J}(\beta_e) = \gamma_1 + \beta_e(\gamma_2 - 1), \ \text{ where } \gamma_1 = (1-c)F'E_N - \delta < 0, \ \gamma_2 = cF'E_{Ye} > 1.$$

Clearly, $\tilde{\beta}_e = \frac{-\gamma_1}{\gamma_2 - 1}$ satisfies $\mathrm{tr}\mathbf{J}(\tilde{\beta}_e) = 0$. As $\det\mathbf{J} > 0$ for all positive β_e, $\mathbf{J}$ has a pair of pure imaginary eigenvalues at $\beta_e = \tilde{\beta}_e$. Let $\mathrm{Re}\lambda(\beta_e)$ denote the real part of eigenvalue λ, which depends on β_e. $\mathrm{Re}\lambda(\beta_e) = \frac{1}{2}\mathrm{tr}\mathbf{J}$ in the neighborhood of $\tilde{\beta}_e$ such that

$$\left.\frac{\partial}{\partial \beta_e}\right|_{\beta_e = \tilde{\beta}_e} \mathrm{Re}\lambda(\beta_e) = \frac{\gamma_2 - 1}{2} > 0.$$

$\square$

As the real parts of the eigenvalues increase in β_e, an immediate change in expected sales induces instability in our dynamics. Figure 5 shows the phase plane when the aforementioned conditions for c_1 hold. The two nullclines slope upward and the Y^e nullcline is steeper. An increase in β_e increases the vertical arrows.

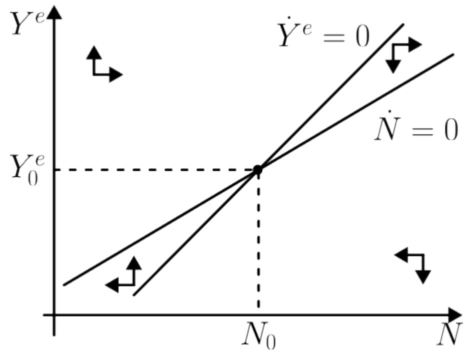

Figure 5: Phase plane of the cyclical dynamics

4 Numerical experiment with US data

In this section, we present a numerical experiment using a dynamic model. We use Ray Fair's US model to set the parameters. We find that the estimated parameters cause the cyclical dynamics described above.

First, we set steady-state values for the economy. We interpret E_0 as the average employment capacity 0.94. As we focus on short-run dynamics, we normalize production to unity: $Y_0^d = 1$. We set $N_0 = 0.64$ so that the inventory–sales ratio in the steady state is equal to the value in Q4 of 2019. We use this value instead of the average because the ratio continuously declines. The remaining fixed variables are as follows:

$$r - \pi = 0.02, \quad \delta = 0.015. \tag{28}$$

We adopt the calibrated value of δ used by Wen (2011).[13] These values are used to calculate the remaining values. For the production function, we assume $F(E) = AE^{\alpha}$, where $0 < \alpha < 1$. Because the steady state can be regarded as a notional equilibrium, E_0 maximizes profit and $F'(E_0) = w$. Furthermore, inventory dynamics cease in the steady state $F(E_0) - Y_0^d - \delta N_0 = 0$.

[13] Of course, it is unlikely that the real interest rate and expected inflation rate would be fixed—even in a short-run model. However, the values of these rates are not crucial for the results. We only need the condition $r - \pi + \delta > 0$ for the model analysis, and varying $r - \pi$ from 1% to 5% hardly affects the results of the numerical simulation.

In addition, the average labor share is 0.6768, meaning that $0.6768 = \frac{wE_0}{F(E_0)}$. Unifying the three conditions, we obtain $A = 1.0528$ and $\alpha = 0.6768 =$ labor share.

For the inventory-holding cost function Φ, we arbitrarily use

$$\Phi = (a_1(N/Y^d)^{-2} + a_2)N,$$

, where a_1 and a_2 are positive. This polynomial function is the easiest one that satisfies Equation (4. In the steady state, we assume that the non-arbitrary condition $Y_0^d - wE_0 - \Phi(N_0, Y_0^d) = r - \pi + \delta$ holds. Adding the consistency condition $(N^{e*}, Y^e) = (N_0, Y_0^d)$ implies that $\Phi_N(N_0, Y_0^d) + r - \pi + \delta = 0$[14] and we attain $a_1 = 0.0975$ and $a_2 = 0.2022$.

In the simulation, we must repeatedly calculate the saddle path of the intertemporal optimization of the firm. In this study, we carefully approximate the actual path using the reverse shooting method (see Appendix C).

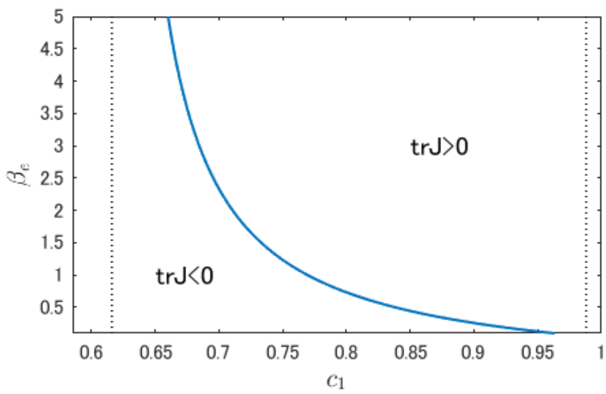

Figure 6: Hopf bifurcation conditions of the linearized model estimated with US data

Figure 6 shows $\underline{c}_1$, $\bar{c}_1$ and the locus tr$\mathbf{J} = 0$ calculated from the estimated parameters. The locus corresponds to the linearized system described in the next section. The linearized system emphasizes stability; in other words, the real locus tr$\mathbf{J} = 0$ of our simulation is located toward the right-hand side. According to the numerical experiment, $\underline{c}_1 = 0.6133$ and $\bar{c}_1 = 0.9033$ in our model. The value $\underline{c}_1 = 0.6133$ could fall if we introduced investment to the model; *animal spirit* would induce more aggressive fluctuations in production due to the change in expected sales.

[14]This condition is not needed from the theory in this study. If we add monetary features, this condition would be needed in the equilibrium.

For the value of c_1 in the simulation, we use the ordinary least squares regression $C = c_1 F(E) + \text{const}$. The results show $c_1 = 0.73$ with a standard error of $9.37{\cdot}10^{-4}$ and the revised $R^2 = 0.995$.[15] The remaining $Y^d - c_1 F(E) = \text{const}$ is calculated as $Y_0^d - c_1 F(E_0) = 0.2279$.

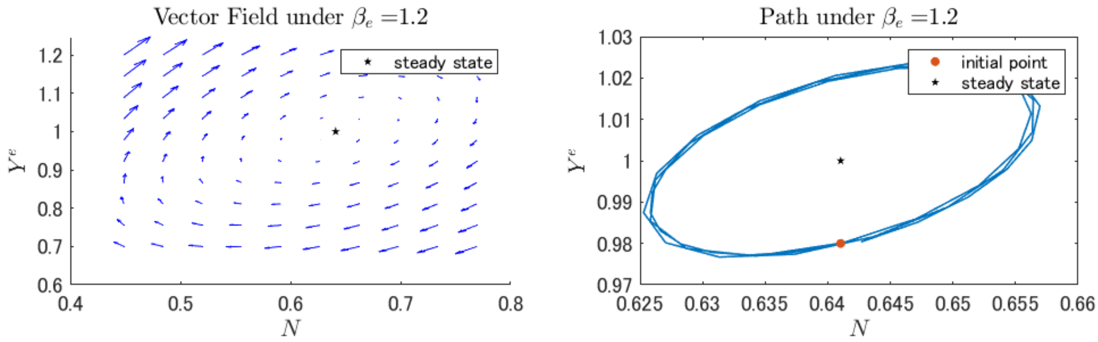

Figure 7: Simulation of the closed orbit: vector field (right) and simulated path (right)

Figure 7 illustrates the closed orbit. We set the initial value to $(N, Y^e) = (N_0, 0.98Y_0^e)$ and the adjustment speed of expected sales to $\beta_e = 1.20$. The initial values correspond to a -2% shock to expected sales (i.e., the firm becomes pessimistic). The value of β_e is equal to the bifurcation value. As described in the model analysis above, a clockwise cycle occurs and inventory holdings follow the production dynamics.

5 Discussion

5.1 What the simulation reveals about the model

Figure 8 shows the time series of the economic variables. The dashed curves in the dynamics of N, E, and N/Y^d correspond to N^{e*}, E^*, and N^{e*}/Y^e (objective values), respectively. The simulation confirms the theoretical analysis of the Metzlerian cycle above. As sales Y^d increase, the firm increases its objective inventory holdings N^* and employment E^* to avoid stockouts. It also increases its production $F(E)$ because it becomes optimistic about expected sales Y^e. After a while, inventory N actually increases. The figure also confirms the mechanism of the Metzlerian cycle because the firm would like to maintain the inventory–sales ratio at the objective value, but actual

[15] Using this simplified estimation of the consumption function is controversial. However, a detailed discussion of the estimation of the consumption demand function is beyond the scope of this study.

sales fluctuate more. Our model endogenizes the objective value N^{e*}/Y^e and finds that its dynamics are below the actual value.

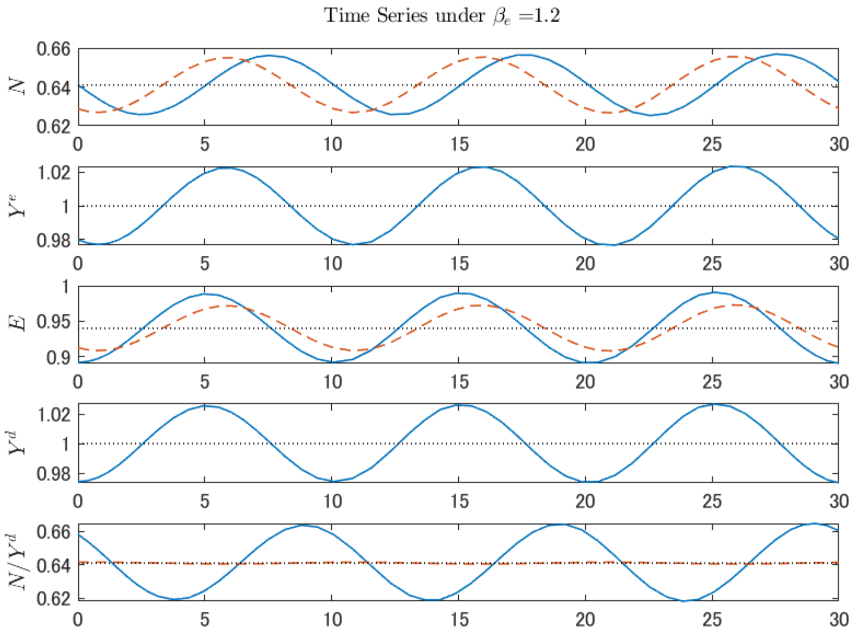

Figure 8: Time series of the variables under a closed orbit

A novel result of our model is the mechanism of cyclical dynamics. The figure shows that employment E and sales Y^d, which are directly connected by the consumption function, decrease, while expected sales Y^e increase around the top of the time series of Y^e. The firm decreases production, while expected sales increase. Is this contradiction true?

Let us consider what happens in the situation (see Figure 9). First, the firm's objective value is at P_1: a shortage of inventory holdings (N_1) promotes employment to avoid stockouts. The firm's reaction is excessive in the sense that actual employment is above the objective value. This reaction promotes goods sales since goods demand depends on wage payments, and then the firm revises its expected sales so that Y^e increases. This change in expected sales moves the firm's objective value from P_1 to P_2 and inventory holdings rise during a boom because the firm optimistically increases production. Therefore, stock disequilibrium $N^* - N_2$ becomes relatively small and the firm's control of employment decreases.

During a boom, as the firm's inventory holdings are close to the objective value, its employment adjustment is moderate. The firm's change of attitude toward its employment adjustment on the saddle path causes a decrease in employment, and this change is a trigger for the end of the boom. A decrease in production lowers sales and the firm ends the boom. This tells us how stock disequilibrium affects flow disequilibrium. As agents perceive stock disequilibrium, decisions under this disequilibrium can enhance and moderate flow disequilibrium.[16]

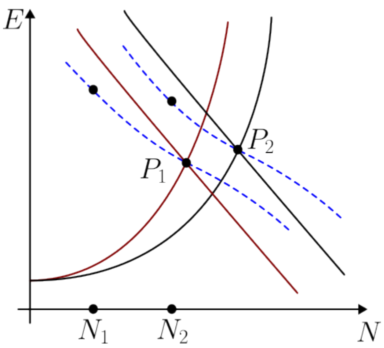

Figure 9: How stock disequilibrium $N - N^{e*}$ affects employment

This result emphasizes the inconsistency in decentralized markets. The firm does not know the consumption demand function and determines production using a simple rule of expected sales dynamics. In the optimization problem, the firm simply determines employment to ride on the saddle path. However, in dynamic economies, actual inventory holdings, which are a state variable for intertemporal optimization, change exogenously. Hence, the firm's aim (i.e., convergence to the steady state) fails and cyclical dynamics continue.

5.2 Quantitative evaluation of the simulation

We set the benchmark of the simulation at the bifurcation point with a -2% shock to Y^e. At this bifurcation point $\beta_e = 1.2$, we can generate any sized cycle of N by varying the initial point. Hence, it makes no sense to compare the result with a certain range of data. Instead, to evaluate the simulation results, it is constructive to examine the relationship between the change in the economic variables and the lag of inventory to production.

Our benchmark simulation is set to reproduce the average lag of inventory reported in the Introduction. The simulated lag is approximately 2.4301 (i.e.,

[16]Of course, this does not imply that flow disequilibrium fluctuates passively. The dynamics of Y^e directly depend on flow disequilibrium, and these affect future flow disequilibrium and inventory holdings. The spillover of stock and flow disequilibria connect complicatedly.

approximately two or three quarters); this is similar to the US data. The lag of the inventory–sales ratio is 4.3283, which is also similar to the US data (see Figure 2). The estimated lags are thus good approximations.

Next, we examine the dynamics of the economic variables. We define the cycle size as the maximum deviation from the steady-state value. For example, the simulated dynamics of inventory N have the range $[0.6253, 0.6569]$. The maximum deviation is $(0.6569/0.6410) - 1 = 0.0248$, or 2.48 percentage points. The remaining variables are set as Y^d : 2.67%, E : 5.41%, $F(E)$: 4.62%, and N/Y^d : 3.68%. Figure 10 compares the cyclical component of the US economy with our simulation. The dashed lines correspond to our simulation (upper and lower deviations).

The simulated size seems a good approximation for the US business cycle before 1984. After 1984, however, quantitative evaluation is difficult. During the period of the Great Moderation (1984–2009), as the real cycle contracts, the simulated cycle exaggerates fluctuations. By contrast, the fluctuations in inventory holdings around 2009 and 2010 as well as in production and sales in 2020 are more serious than those in the simulation.

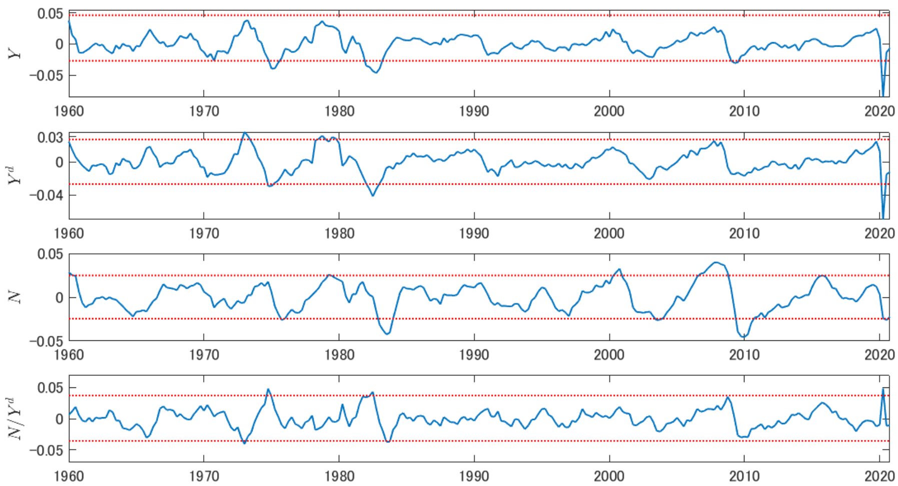

Figure 10: Cyclical components of the US business dynamics (solid) and simulated cycles (dashed)

5.3 Future issues to address to simulate various sized cycles

We have shown that our simple one-sector model can quantitatively reproduce both (1) the lagged dynamics of inventory holdings and the inventory–sales ratio and (2) the cyclical components of production, sales, inventory holdings, and the inventory–sales ratio before 1984 *on average*. However, our model cannot show (1) the various sizes and frequencies of cycles or (2) the smaller cycle during the period of the Great Moderation and larger fluctuations in 2008 and 2020. Hence, while our model captures the average characteristics of the cycle of inventory holdings, it does not simulate any one cycle.

Figures 1 and 10 show that the real cycle is not perfectly periodic. As the cyclical dynamics of our model originate from a smooth dynamic system with complex eigenvalues, it cannot reproduce the calculated fluctuation in the figure, which occurs irregularly. On this point, it is rational to learn the mechanism of the cyclical dynamics of inventory from our results rather than regard the model as a replica of the real economy. Therefore, we do not search for more complicated functions to fine-tune our simulation.[17]

Instead, we discuss future issues to be addressed to attain more fruitful results regarding cyclical dynamics that include inventory. The following three methods are independent of each other. The first is to include price dynamics. As our model focuses on quantity adjustments during the cycle, adding price dynamics is a straightforward way to expand the analysis. As mentioned in the Introduction, some equilibrium dynamic models use the speculative motivation of holding inventory. When a firm expects price and wage inflation, it produces more. We can use sticky wages, inflation expectations, interest rate dynamics, and the discount rates of future profitability to include the price dynamic features. Our quantity dynamics model complements the price dynamics model, which corresponds to the equilibrium economics of price dynamics. Of course, this direction includes the monetary dynamics that are omitted here.

The second recommendation is to include investment. For simplicity, we omitted investment. The firm chooses its investment depending on its return, which is affected by expected price dynamics and expected future sales. For quantity dynamics, *animal spirits* play an important role: the firm would

[17]For example, we set the inventory-holding cost function and production function ad hoc. We might find better results by changing these functions. Furthermore, we need to introduce a series of exogenous shocks (e.g., productivity A, following Christiano (1988)) to trace the path of data to provide further quantitative evaluation. Since our topic is the endogenous Metzlerian cycle, this is beyond the scope of the present study.

increase its investment in a boom since it becomes optimistic.[18]

The third idea is more fundamental. In this study, we assume that the firm expects the path of sales to be constant in its optimization problem; the firm controls production to ride on the saddle path, which depends on the given standard level of sales. However, it is natural that when facing a boom or depression, a firm expects a dynamic path of sales from current sales to the standard level. In other words, we need the dynamics of $Y^e(\tau)$ in the firm's problem. If the firm is pessimistic during a depression, $Y^e(\tau)$ is underestimated. However, sales are expected to converge to the standard level in the future (precisely, $\tau \to \infty$) because such convergence would ensure that the optimization is solvable. By contrast, $Y^e(\tau)$ immediately approaches the steady state when the firm is optimistic during a depression. Adding the path of subjectively expected future sales would therefore help distinguish between ordinary cycles and serious recessions such as those in 2009 and 2020.

5.4 Approximation of the saddle path

Our simulation, in which the saddle path of the firm's optimization is precisely calculated using the reverse shooting method, supports the previous idea. For large dynamic systems, researchers typically set (log-)linear or polynomial equations as approximations. These approximations are useful as long as the actual relationships among the economic variables do not have high nonlinearity or discontinuity and the actual economic state is close to the reference point of the approximation, which is usually a steady state.

For our model, we adopt this concept using a linear approximation of the saddle path (see Figure 11). The left-hand figure compares the difference between the linearized employment function and the fully traced saddle path used in this study. The linearized employment E^l is calculated as follows:

$$E^l(t) = E^* + \beta_E(N(t) - N^{e*}), \tag{29}$$

where β_E denotes the slope of the saddle path at (N^{e*}, E^*).[19] It is clear that the linearized approximation underevaluates the adjustment to stock disequilibrium because the difference between current and objective employment is smaller. This implies that changes in employment, production, and demand

[18] For example, Howitt and McAfee (1985) and Franke and Westerhoff (2017) introduced a stochastic process to express animal spirits.

[19] We can calculate this using the Jacobian matrix. By diagonalization, we find a regular matrix that consists of eigenvectors. β_E is the ratio of the two factors of an eigenvector that corresponds to the negative eigenvalue.

for goods decrease when we used a linearized system. The figure on the right-hand side (a heatmap of the Metzlerian term) confirms this hypothesis. We calculate the following:

$$\left|\left(\frac{Y^d(E) - Y^e}{Y^e}\right)\right| - \left|\left(\frac{Y^d(E^l) - Y^e}{Y^e}\right)\right|. \tag{30}$$

The figure on the right-hand side shows that the above term is positive, except around $\dot{Y}^e = 0$, or equivalently, the theoretical region $Y^d = Y^e$. In addition, the difference continues to increase. This figure implies that Metzlerian instability is underevaluated in a linearized system.

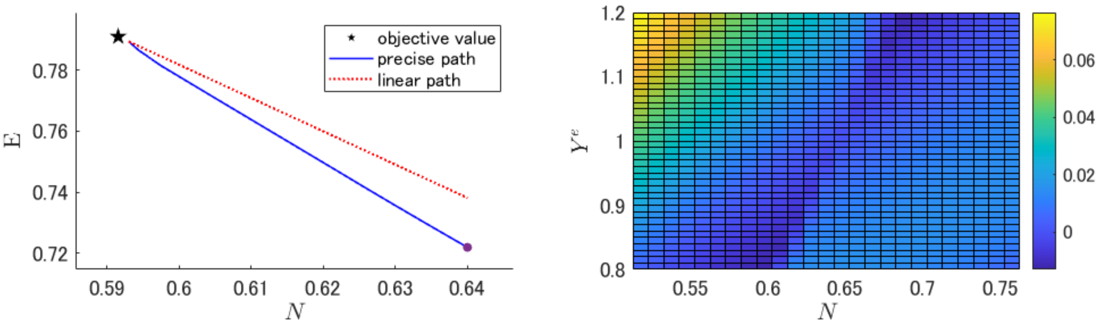

Figure 11: Difference between the linearized and precise saddle paths: employment at $(N, Y^e) = (0.64, 0.91)$ (left) and the Metzlerian instability term (right)

6 Concluding remarks

In this study, we built an inventory dynamics model driven by quantitative expectations. The firm decides current employment depending on expected sales, and employment induces goods demand so that actual sales are indirectly determined by the firm's expectations. This expectation is adjusted by actual sales; thus, the dynamics of expected (and actual) sales have a recursive structure: sales determine production and this production determines future sales.

We used intertemporal optimization modeling to express the dynamic control of employment, inventory, and consumption. This method is ordinal to equilibrium economics because the steady state of the economy is often suitable for the agent's notional optimality in the optimization problem. In our model, this consistency appears as $N^{e*} = N_0$ and $Y^e = Y_0^d$, but holds only in the steady state.

Alternatively, the model in this study addressed disequilibrium dynamics. The firm determines employment under the current stock disequilibrium $N \neq N^{e*}$ to solve this disequilibrium. However, it does not know actual future sales in a decentralized economy.[20] These disequilibrium dynamics cause the Metzlerian cyclical dynamics in the model, and the simulation partially explains the data: the existence of inventory does not mitigate the cycle; rather, its existence causes cyclical dynamics. We proved this dynamic, although inventory works as a buffer for the gap between goods demand and supply.

Among the many future issues discussed in the previous section, the most interesting one is to build a firm's expectation-path model. The firm could be both optimistic and pessimistic in a cycle, and this attitude affects how it adjusts its disequilibrium. Adding this feature would help us understand the different sizes of the cycles.

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[20] Although the main issue is not the same, our framework is similar to that of Otani (1983), who also used the independent intertemporal optimizations of firms and households and showed that inventory dynamics stop only when the market is in the (notional) equilibrium. His framework thus followed a type of neoclassical, but he emphasized the gap between the solutions (demand and supply) of each optimization to illustrate a decentralized economy.

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A Comparative statics of the objective values

The firm's employment control around the objective values (E^*, N^{e*}) is approximated as follows:

$$E = \beta_E(N - N^{e*}) + E^*, \tag{A.1}$$

where β_E is the slope of the saddle path stemming from the objective values.

The objective values of the system (N^{e*}, E^*) satisfy

$$\begin{aligned} \Phi_N(N^{e*}, Y^e)F'(E^*) + (r - \pi + \delta)w = 0 \\ F(E^*) - Y^e - \delta N^{e*} = 0. \end{aligned}$$

Using the total differentials, we obtain

$$\begin{aligned} \left.\frac{dE}{dN}\right|_{\dot{E}=0} &= -\frac{\Phi_{NN}F'}{\Phi_N F''} < 0, \quad \left.\frac{dE}{dN}\right|_{\dot{N}=0} = \frac{\delta}{F'} > 0, \\ \frac{\partial E^*}{\partial Y^e} &= \frac{\Phi_{NN}F' - \delta\Phi_{NY}F'}{Q}, \quad \frac{\partial N^{e*}}{\partial Y^e} = -\frac{\Phi_{NY}(F')^2 + \Phi_N F''}{Q}, \end{aligned}$$

where $Q = \Phi_{NN}(F')^2 + \delta\Phi_N F'' > 0$ and $\Phi_{NY} = -(N^e/Y^e)\Phi_{NN} < 0$. This result is intuitive. As expected sales increase, the firm becomes optimistic, and objective employment also increases. The increase in expected sales would induce the possibility of stockouts, so N^{e*} would increase; however, increased employment might compensate for the deficit in production capacity. Therefore, the increase or decrease in objective inventory holdings is indeterminate.

Let us now move toward the control for current employment. As shown in Figure 4, the slope of the curve $\dot{E} = 0$ is steeper than that of the saddle path $-\frac{\Phi_{NN}F'}{\Phi_N F''} < \beta_E < 0$. In the steady state of the economy,

$$
\begin{aligned}
\frac{dE_0}{dY^e} &= (N_0 - N^{e*})\frac{d\beta_E}{dY^e} - \beta_E \frac{dN^{e*}}{dY^e} + \frac{dE^*}{dY^e} \\
&= -\beta_E \frac{dN^{e*}}{dY^e} + \frac{dE^*}{dY^e} \\
&= Q^{-1}\left[\beta_E\left(\Phi_{NY}(F')^2 + \Phi_N F''\right) + \Phi_{NN}F' - \delta\Phi_{NY}F'\right] \\
&> Q^{-1}\left[\beta_E\Phi_{NY}(F')^2 - \Phi_{NN}F' + \Phi_{NN}F' - \delta\Phi_{NY}F'\right] \\
&= Q^{-1}\Phi_{NY}F'\left[F'\beta_E - \delta\right] > 0
\end{aligned}
$$

The second equation holds true because $N_0 = N^{e*}$. Therefore, we use the following comparative statics for the steady state:

$$\frac{dE}{dN} = \beta_E < 0, \quad \frac{dE}{dY^e} > 0$$

B On the empirical data and simulation results

The data supplied by Ray Fair (`https://fairmodel.econ.yale.edu/`) were used for the numerical experiments. Time series data for the US over 1960–2020 were used. The abbreviations below follow Fair's:

- `CD` : real consumption expenditure on durable goods
- `CN` : real consumption expenditure on nondurable goods
- `CS` : real consumption expenditure on services
- `E` : total employment
- `HN` : average number of non-overtime hours paid per job
- `HO` : average number of overtime hours paid per job
- `HF` : average number of hours paid per job
- `JF` : number of jobs
- `WF` : average hourly earnings excluding overtime of workers
- `SIFG` : employer social insurance contributions (paid to the US government)
- `SIFS` : employer social insurance contributions (paid to state and local governments)
- `U` : total unemployment
- `V` : inventory holdings
- `X` : real sales
- `Y` : real production
- `PF` : output price index for `X`

The following points on variable construction should be noted. First, labor input can be calculated as $\mathtt{JF} \cdot \mathtt{HF}$. However, this labor input index is inconvenient, and hourly input and labor productivity are excluded. As Fair (2018, Figure 4) shows, $\mathtt{Y}/(\mathtt{JF} \cdot \mathtt{HF})$ continues to grow although the present model does not include specific technical changes. As the model concerns medium-term dynamics, we exclude the continuous growth of labor productivity. Instead of various variables on labor inputs, we simply use the labor share, which does not fluctuate aggressively.

The variables in this model are specified as follows:

$$\begin{aligned} F(E) &= \mathtt{Y} \\ C &= \mathtt{CD} + \mathtt{CN} + \mathtt{CS} \\ Y^d &= \mathtt{X} \\ N &= \mathtt{V} \\ E &= \mathtt{E}/(\mathtt{E} + \mathtt{U}) \\ \text{Labor Share} &= (\mathtt{WF} \cdot \mathtt{JF} \cdot (\mathtt{HN} + 1.5\mathtt{HO}) + \mathtt{SIFG} + \mathtt{SIFS})/(\mathtt{PF} \cdot \mathtt{Y}) \end{aligned}$$

We suppose that the wage for overtime work is 1.5, similar to that paid for non-overtime work.

C Numerical solution for the employment function

In the simulation of our dynamic model, we should obtain the employment function E, which is the solution to the firm's intertemporal optimization problem. This function is usually not analytical because we cannot obtain an analytical saddle path. Therefore, a method is required to approximate the saddle path to calculate employment.

In this study, the reverse shooting (backward integration) method was used; see Judd (1998, Chapter 10) Brunner and Strulik (2002) and Stemp and Herbert (2006, 2008). This method uses the fact that the saddle path can be an attractor when time reverses (see Figure 12 (left)). The assumptions were as follows. First, the initial point was set to approximately the steady state. In particular, we use information about the values of the state variables at time t and the eigenvectors of the Jacobian matrix. Next, we simulated a dynamic system with time reversal. When the original system is autonomous, it must only be multiplied by (-1).[21] Finally, the simulation was stopped when the state variable N reached its values at t. The jump variables at this

[21] In our simulation, the firm's control system is multiplied by -0.08. As the practical

point correspond to the initially chosen values in the original optimization, or $E(t)$.

When the dimensions of the stable manifold are more than one, we face the problem of simulation feasibility (see Figure 12 (right)). Consider a 3-D system with a 2-D stable manifold and jump variable. This system corresponds to the system described as a future issue: the firm's control of E under the (E, N^e, Y^e) system with the convergence of Y^e. Reverse shooting guarantees convergence to a stable manifold (the set of saddle paths); however, finding one correct path takes time. We should find a correct saddle path that crosses the intersection of the stable manifold and set $(N^e, Y^e) = (N^e(t), Y^e(t))$; however, this takes time.

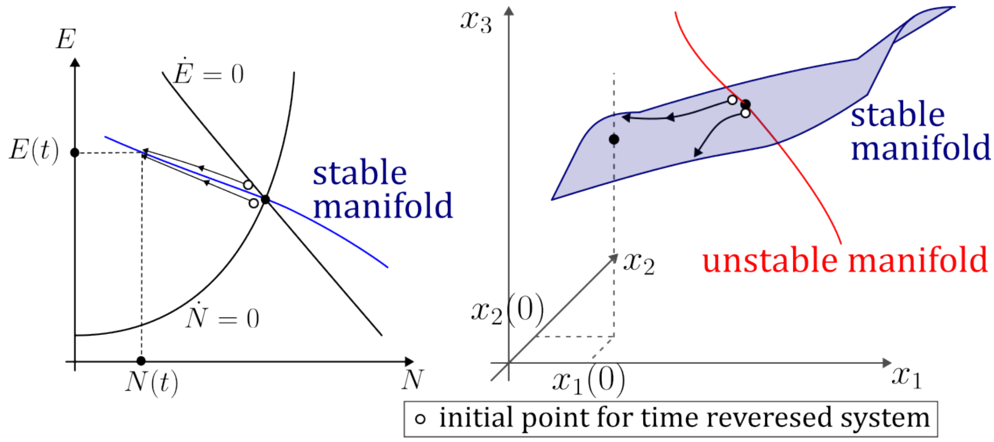

Figure 12: Images of reverse shooting to find the initial value of the jump variable: the firm's control (left) and a 3-D system with one jump variable (right)

simulation is discrete, we should make the size of the step sufficiently small that the software can find a correct path. This multiplier comes from repeated simulation so that there is no theoretical justification.