# Inspecting a seasonal ARIMA model with a random period

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# Inspecting a seasonal ARIMA model with a random period

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**Abstract**

This work proposes a class of seasonal autoregressive integrated moving average models whose period is an independent and identically distributed random process valued in a finite set. The causality, invertibility, and autocovariance shape of the model are first revealed. Then, the estimation of the parameters which are the model coefficients, the innovation variance, the probability distribution of the period, and the (unobserved) sample-path of the period, is carried out using the expectation-maximization algorithm. In particular, a procedure for random elimination of seasonality is proposed. An application of the methodology to the annual Wolfer sunspot numbers is provided.

**Keywords:** Seasonal ARIMA models, irregular seasonality, random period, non-integer period, SARIMAR model, EM algorithm.

## 1 Introduction

Numerous time series observed in empirical studies (e.g. physics, environmental sciences, economics, and finance) are characterized by similar patterns recurring at regular time-periods. This feature, called conventionally *seasonality*, or more generally *periodicity* is mainly caused by natural factors, institutional and organizational measures, cultural traditions, or religious

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rituals (Granger, 1979; Hylleberg, 1992). There is currently a wide range of models dedicated to the representation of seasonality whose main objectives are forecasting, control, deseasonalization, smoothing, and filtering (see e.g. Brockwell and Davis, 1991; McLeod and Hipel, 2005; Box et *al*, 2008; Ghysels and Osborn, 2001; Franses and Paap, 2004, Hurd and Miamee, 2008; Bittanti and Colaneri, 2009 and the references therein). These models, also called seasonal or periodic, can be classified into two general categories: *deterministic seasonality* models and *stochastic seasonality* models (see e.g. Ghysels and Osborn, 2001). Models in the first category are expressed through deterministic periodic functions of time perturbed by innovation processes, implying that any current shock in the data does not affect future evolutions. In the second category, however, the models consist of parametric stochastic difference equations involving lagged values of the underlying process and/or the innovation. Thus, the model consists at any time of an accumulation of past shocks, so present shocks affect more or less the future evolution, depending on the position of the parameters inside the stability/instability domain.

Among stochastic seasonality models, there are in particular i) linear seasonal models with time-invariant coefficients (Box-Jenkins SARIMA model, Box et *al*, 2008), ii) linear periodic models with periodically time-varying coefficients (periodic ARMA models, e.g. Tiao and Grupe, 1980; McLeod and Hipel, 2005; Hurd and Miamee, 2007; Bittanti and Colaneri, 2009), and iii) non-linear seasonal/periodic models such as seasonal GARCH, periodic GARCH, periodic stochastic volatility, periodic Markov switching autoregression (see Ghysels and Osborn, 2001; Franses and Paap, 2004; Aknouche and Guerbyenne, 2009; Bibi and Aknouche, 2010; Aknouche et *al*, 2022). Periodic B-spline models, Fourier approximation-based models, and wavelet-based periodic models are typical examples of deterministic seasonality models (e.g. Tsiakas, 2006; Tesfaye et *al*, 2011; Franses and Paap, 2011; Rossi and Fantazani, 2015; Ziel et *al*, 2015; Ziel et *al*, 2016; Ambach and Croonenbroeck, 2015; Ambach and Schmid, 2015).

For all these models, a central role is played by the *period* which represents the portion of time after which the pattern of the phenomenon is more or less repeated. Almost invariably, the period of a model is supposed constant, reflecting the well-established finding that the causes of seasonality occur regularly over time, even though their effects may be delayed and hence occur irregularly. Thus, the assumption of period constancy might be restrictive for quasi-periodic phenomena whose pattern is certainly repeated, but after non-equidistant time intervals. Notable examples are climate changes, business cycles, recurrent strikes, moving holidays, seasonal breaks, calendar variations, etc. The combination of these factors might cause a phenomenon to repeat multiple patterns over different time-intervals, leading to irregular but frequency stable cyclical fluctuations. For constant period seasonality models, this "period shift" is only supported by the model innovation and therefore remains unexplained.

The best-known example of a "visually" periodic time series with a varying period is the annual Wolfer sunspot numbers (Waldmeier, 1961), better known as "sunspot data" (Box et al, 2008, series E). The seasonal fluctuation of sunspot data has been the subject of significant study since the work of Box and Jenkins (1976, series E) and continues to attract interest. It has been recognized that this series can be affected by a varying periodicity (Hipel and McLeod, 2005, Chapter 5). Examining the series of sunspots (see Figure 5.1), it turns out that the most frequent intervals between three successive turning points are 11 and 12 years. This irregular cyclicity is often caused by so many unidentifiable factors that the best way to explain it is to consider it as generated by a random mechanism having a certain probabilistic law.

Thus, giving the period a random aspect, that is to say, the period is itself a stochastic process, allows great flexibility in modeling both regular and irregular cyclicity. This significantly eases the burden on model innovation and reduces it of a part considered unexplained in traditional seasonal models.

Several structures can be assigned to the probabilistic evolution of the period. In particular, three usual dependence mechanisms can be considered. The first one is an endogenous threshold structure where the period transition depends on the past of the observed process. The second one is an unobservable Markov switching evolution in which the period follows an unobserved Markov chain. The third structure is a special case of the second one and consists of an independent and identically distributed switching (iid), which implies a purely random character of the period. The first structure is quite arbitrary and it seems difficult to find a suitable relationship between the period and the past of the observed process. The second structure can be useful, but since the Markov chain is generally assumed to be stationary and ergodic, this assumption only seems appropriate for long time series. The third structure is consistent with the hypothesis of a purely random period.

This paper proposes inspecting a new class of time series models with purely random period. We restrict ourself to the class of seasonal ARMA models with time-invariant coefficients. Specifically, we propose a seasonal autoregressive moving average model in which the period is an unobserved iid process $\{S_t, t \in \mathbb{Z}\}$ valued in a finite set of possible periods $\left\{S^{(1)}, ..., S^{(K)}\right\}$. We call the model SARMAR. For example, $\{11, 12, 13\}$ could be suitable for monthly series, and for quarterly series, the set $\{3, 4, 5\}$ could be considered. Of course, a period identification procedure based on the frequency of three successive turning points could be introduced.

Section 2 defines the SARMAR model and the corresponding assumptions to which it is subject. Special cases concerning the seasonal random period autoregressive (SARR) model, the seasonal random period moving average (SMAR) model, and the first-order SARMAR are studied in some detail. In Section 3, the model parameters are estimated using the EM algorithm. Specific examples are provided. In addition, a procedure for random elimination of seasonality is provided and the SARIMAR (SARIMA with a random period) model is introduced. A simulation study is conducted in Section 4 while Section 5 presents an empirical example regarding the Wolfer number of sunspots. The forecasting ability of the proposed model is compared to a known benchmark. Additional comments and perspectives are presented in the concluding Section.

# 2 Seasonal random period autorgressive moving average model

## 2.1 General presentation

Before defining the model in its general form, we need first to consider a *random backward shift operator*. Let $\{S_t, t \in \mathbb{Z}\}$ be an integer-valued random sequence and let the operator $g : \mathbb{Z} \to \mathbb{Z}$ be given by

$$g(t) = t - S_t. \tag{2.1a}$$

Define the sequence $\left(g^{(l)}(t)\right)_l$ by

$$g^{(l)}(t) = \begin{cases} t & \text{if } l = 0 \\ g\left(g^{(l-1)}(t)\right) & \text{if } l \geq 1. \end{cases} \tag{2.1b}$$

For example, the first four terms of the sequence $\left(g^{(l)}(t)\right)$ are respectively: $g^{(0)}(t) = t$, $g^{(1)}(t) = t - S_t$, $g^{(2)}(t) = t - S_{t-S_t}$, and $g^{(3)}(t) = t - S_{t-S_{t-S_t}}$. Now define a second sequence of random backward shift operators $\left(h^{(l)}(t)\right)_l : \mathbb{Z} \to \mathbb{Z}$ by $h^{(0)}(t) = g^{(0)}(t)$ and

$$h^{(l)}(t) = g^{(0)}(t) + \sum_{j=1}^{l} \left(g^{(j)}(t) - t\right), \ l \geq 1. \tag{2.1c}$$

The first four terms of the sequence $\left(h^{(l)}(t)\right)_l$ are: $h^{(0)}(t) = t$, $h^{(1)}(t) = t - S_t$, $h^{(2)}(t) = t - S_t - S_{t-S_t}$, and $h^{(3)}(t) = t - S_t - S_{t-S_t} - S_{t-S_{t-S_t}}$.

When the sequence $\{S_t, t \in \mathbb{Z}\}$ is non-random and stationary, say $S_t = S$ for all $t$ (for some positive integer $S$), then

$$h^{(l)}(t) = t - lS,$$

is nothing but the standard seasonal backward shift operator

$$L^{lS} Y_t := Y_{t-lS} = Y_{h^{(l)}(t)},$$

where $L$ is the backward shift operator given by $LY_t = Y_{t-1}$ (e.g. Box et al, 2008).

Having defined the random backward shift $h^{(l)}(t)$, we can now consider our random period model. A seasonal autoregressive moving average model with a random period ($\mathrm{SARMAR}_{S_t}(p,q)$) satisfies the following recursion

$$Y_t = \phi_1 Y_{h^{(1)}(t)} + ... + \phi_p Y_{h^{(p)}(t)} + \varepsilon_t - \theta_1 \varepsilon_{h^{(1)}(t)} - ... - \theta_1 \varepsilon_{h^{(q)}(t)},\ t \in \mathbb{Z}, \tag{2.2}$$

where the following assumptions are considered.

**A1**: $\{\varepsilon_t, t \in \mathbb{Z}\}$ is iid, unobserved, with zero mean and variance $\sigma^2$.

**A2**: $\{S_t, t \in \mathbb{Z}\}$ is iid, unobserved, and valued in $\mathcal{S} = \left\{S^{(1)}, ..., S^{(K)}\right\} \subset \mathbb{N}^* := \{1, 2, ...\}$ with a probability distribution $\{\pi_1, ..., \pi_K\}$ such that $P\left(S_t = S^{(k)}\right) = \pi_k$ where $\sum_{k=1}^{K} \pi_k = 1$ and $\pi_k \geq 0$, $1 \leq k \leq K$.

**A3:** $\{\varepsilon_t, t \in \mathbb{Z}\}$ and $\{S_t, t \in \mathbb{Z}\}$ are mutually independent.

Thanks to the iid property of $\{S_t, t \in \mathbb{Z}\}$ in **A2**, we have

$$\begin{aligned} P\left(S_t = S^{(k)}, S_{t-S_t} = S^{(j)}\right) &= P\left(S_t = S^{(k)}\right) P\left(S_{t-S_t} = S^{(j)} | S_t = S^{(k)}\right) \\ &= P\left(S_t = S^{(k)}\right) P\left(S_{t-S^{(k)}} = S^{(j)} | S_t = S^{(k)}\right) \\ &= P\left(S_t = S^{(k)}\right) P\left(S_{t-S^{(k)}} = S^{(j)}\right) = \pi_k \pi_j \\ &= P\left(S_t = S^{(k)}\right) P\left(S_{t-S_t} = S^{(j)}\right), \end{aligned}$$

so that $S_t$ and $S_{t-S_t}$ are independent. More generally,

$$P\left(S_t = S^{(k_1)}, S_{t-S_t} = S^{(k_2)}, ..., g^{(p)}(t) = S^{(k_p)}\right) = \pi_{k_1} \pi_{k_2} \cdots \pi_{k_p}.$$

The following subsections study in details three specific instances of the $\mathrm{SARMAR}_{S_t}(p,q)$ model (2.2), namely the seasonal random period autoregressive model ($\mathrm{SARR}_{S_t}(p)$), the seasonal random period moving average model $\mathrm{SMAR}_{S_t}(q)$, and the first-order $\mathrm{SARMAR}_{S_t}(1,1)$ model.

## 2.2 Seasonal random period autorgressive model

The $\mathrm{SARMAR}_{S_t}(p,q)$ model (2.2) reduces when $q=0$ to the following seasonal random period AR model ($\mathrm{SARR}_{S_t}(p)$ in short)

$$Y_t=\phi_1 Y_{h^{(1)}(t)}+...+\phi_p Y_{h^{(p)}(t)}+\varepsilon_t,\ t\in\mathbb{Z}, \tag{2.3}$$

Under the same assumptions **A1-A3**, the conditional probability density function (pdf) of the $\mathrm{SARR}_{S_t}(p)$ model (2.3) is given by

$$f_{Y_t}(y_t|\mathcal{F}_{t-1})=\sum_{k_1=1}^{K}\cdots\sum_{k_p=1}^{K}\frac{\pi_{k_1}\cdots\pi_{k_p}}{\sigma}f_\varepsilon\left(\frac{y_t-\phi_1 y_{h(t)}-\phi_2 y_{h(2)(t)}-...-\phi_p y_{h(p)(t)}}{\sigma}\right),\ t\in\mathbb{Z}, \tag{2.4}$$

where $f_\varepsilon$ is the pdf of $\varepsilon_t$ and $\mathcal{F}_t$ denotes the $\sigma$-algebra generated by $\{Y_{t-u},u\geq 0\}$.

For instance, the first-order $\mathrm{SARR}_{S_t}(1)$ (we take $\phi_1=\phi$) and the second-order $\mathrm{SARR}_{S_t}(2)$ are, respectively, as follows

$$\begin{aligned}Y_t&=\phi Y_{t-S_t}+\varepsilon_t\\ Y_t&=\phi_1 Y_{t-S_t}+\phi_2 Y_{t-S_t-S_{t-S_t}}+\varepsilon_t.\end{aligned}$$

To study the existence of a stationary solution to equation (2.3), define

$$\begin{aligned}Z_t&=(Y_{h^{(0)}(t)},Y_{h^{(1)}(t)},...,Y_{h^{(p-1)}(t)})',\\ B_t&=(\varepsilon_t,0,...,0)',\end{aligned}$$

and

$$A=\begin{pmatrix}\phi_1&\phi_2&\cdots&\phi_{p-1}&\phi_p\\ 1&0&\vdots&0&0\\ 0&1&\vdots&0&0\\ 0&0&\ddots&0&0\\ 0&0&\cdots&1&0\end{pmatrix}.$$

Then, model (2.3) can be cast in the following stochastic recurrence equation (SRE)

$$Z_t=AZ_{t-1}+B_t,\quad t\in\mathbb{Z}. \tag{2.5}$$

Let us denote by $\rho(A)$ the spectral radius of the matrix $A$, i.e. the maximum eigenvalue of $A$ in modulus.

**Proposition 2.1** *Equation* (2.3) *admits a unique strictly stationary solution satisfying* $E\left(Y_t^2\right) < \infty$ *if and only if*

$$\rho(A) < 1. \tag{2.6}$$

*The unique stationary solution is ergodic and is given by*

$$Z_t = \sum_{j=0}^{\infty} A^j B_{h^{(j)}(t)}, \ t \in \mathbb{Z} \tag{2.7}$$

*where the series in* (2.7) *converges absolutely almost surely (a.s.) and in mean square.*

**Proof** Standard SRE theory (e.g. Bougerol and Picard, 1992) shows that a necessary and sufficient condition for equation (2.5) to have a unique (causal) strictly stationary solution given by (2.7) is that (2.6) is satisfied. Since $E\left(\varepsilon_t^2\right)$ is by the model's definition finite then the unique stationary solution (2.7) has a finite second moment, where the series in (2.7) also converges in mean square. $\square$

Condition (2.6) is therefore the same as the stationarity condition of a standard SARMA model with a constant period. It is also the same condition for the general SARMAR model (2.1). Furthermore, (2.6) is also necessary and sufficient for the existence of a finite moment of order $r$ provided that $E\left(|\varepsilon_t|^r\right) < \infty$.

Under (2.6), the mean of the process is

$$E(Y_t) = 0,$$

while the autocovariance function, $\gamma(l) = E\left(Y_t Y_{t-l}\right)$, satisfies the following recurrence

$$\gamma(l) = \sum_{j=1}^{p} \phi_j E\left(Y_{h^{(j)}(t)} Y_{t-l}\right) \tag{2.8a}$$

where

$$E\left(Y_{h^{(j)}(t)} Y_{t-l}\right) = \sum_{k_1=1}^{K} \sum_{k_2=1}^{K} \dots \sum_{k_j=1}^{K} \pi_{k_1} \pi_{k_2} \dots \pi_{k_j} \gamma\left(l - S^{(k_1)} - S^{(k_2)} - \dots - S^{(k_j)}\right). \tag{2.8b}$$

Combining (2.8$a$) and (2.8$b$), we obtain the following *extended Yule-Walker* equations

$$\gamma(l)=\sum_{j=1}^{p}\phi_j\sum_{k_1=1}^{K}\sum_{k_2=1}^{K}\dots\sum_{k_j=1}^{K}\pi_{k_1}\pi_{k_2}\dots\pi_{k_j}\gamma\left(l-\sum_{i=1}^{j}S^{(k_i)}\right)+\delta_{l0}\sigma^2, \tag{2.9}$$

where

$$\delta_{l0}=\begin{cases}1 \text{ if } l=0\\ 0 \text{ if } l\neq 0.\end{cases}$$

In particular, the variance of $\{Y_t, t\in\mathbb{Z}\}$ has the following expression

$$\gamma(0)=\sum_{i=1}^{p}\sum_{j=1}^{p}\phi_i\phi_j E\left(Y_{h^{(i)}(t)}Y_{h^{(j)}(t)}\right)+\sigma^2.$$

It is well known that the sample autocorrelation of a seasonal ARMA model is characterized by significant peaks at lags multiple of $S$. In contrast, the sample autocorrelation function of the $\text{SARR}_{S_t}(p)$ model shows significant peaks at the lags $\sum_{j=1}^{k}S^{(i_j)}$ for all $k\in\{1,...,K\}$, $S^{(i_j)}\in\mathcal{S}$. For example when $K=2$, significant peaks are observed at the lags: $S^{(1)}$, $S^{(2)}$, $2S^{(1)}$, $2S^{(2)}$, $S^{(1)}+S^{(2)}$, $2S^{(1)}+S^{(2)}$...

Finally, the one-step ahead forecast of $Y_t$ given $\mathcal{F}_{t-1}$ is given by

$$\begin{aligned}\mu_t \ :&=E\left(Y_t|\mathcal{F}_{t-1}\right) \\ &=\sum_{k_1=1}^{K}\dots\sum_{k_p=1}^{K}\pi_{k_1}\dots\pi_{k_p}\left(\phi_1Y_{t-S^{(k_1)}}+\phi_2Y_{t-S^{(k_1)}-S^{(i_{k2})}}+....+\phi_pY_{t-S^{(k_1)}-...-S^{(k_p)}}\right).\end{aligned} \tag{2.10}$$

### 2.2.1 The first-order $\text{SARR}_{S_t}(1)$

When $p=1$, the $\text{SARR}_{S_t}(1)$ model (2.3) becomes

$$\begin{aligned}Y_t &= \phi Y_{t-S_t}+\varepsilon_t \\ &=\begin{cases}\phi Y_{t-S^{(1)}}+\varepsilon_t \text{ with probability } \pi_1\\ \phi Y_{t-S^{(2)}}+\varepsilon_t \text{ with probability } \pi_2\\ \vdots\\ \phi Y_{t-S^{(K)}}+\varepsilon_t \text{ with probability } \pi_K\end{cases}\end{aligned} \tag{2.11}$$

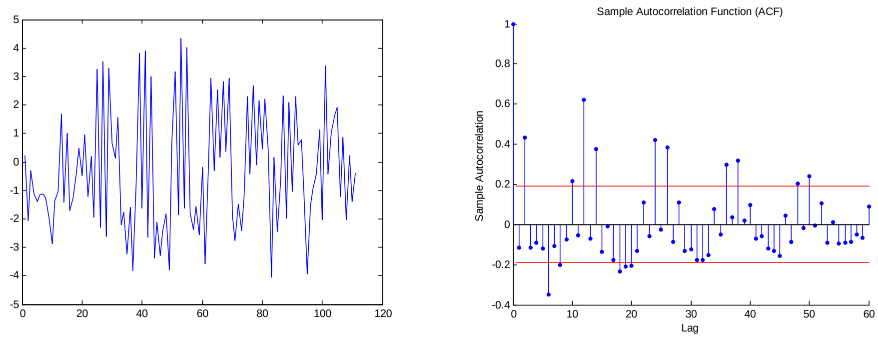
with a conditional pdf given by

$$f_{Y_t}\left(y_t|\mathcal{F}_{t-1}\right)=\sum_{k=1}^{K}\tfrac{\pi_k}{\sigma}f_\varepsilon\left(\tfrac{y_t-\phi Y_{t-S^{(k)}}}{\sigma}\right),\ t\in\mathbb{Z}. \tag{2.12}$$

We thus recognize the conditional density of a mixture of $K$ seasonal autoregressive processes. Each $Y_t$ is computed from one of the $K$ possible SAR(1) models of period $S^{(k)}$ with probability $\pi_k$ $(1 \leq k \leq K)$. Model (2.11) resembles the standard first-order seasonal SAR(1). However, here the period is no longer constant, but is a random process $\{S_t, t \in \mathbb{Z}\}$ taking values in the set $\mathcal{S}$. A each time $t$, the term $Y_t$ equals $\phi Y_{t-S^{(k)}} + \varepsilon_t$ with a probability $\pi_k$.

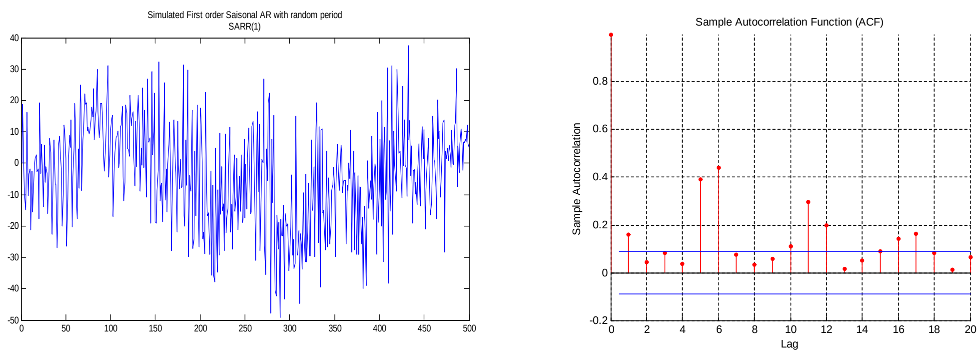
Figure 2.1 (panels $(a)$ and $(b)$) shows the time plot and the sample autocorrelation function of a simulated $\text{SARR}_{S_t}(1)$ series with sample size $n = 110$. The period $S_t$ varies across the set $\mathcal{S} = \{2, 12\}$ with probabilities $\pi_1 = P(S_t = 2) = 0.3$ and $\pi_2 = P(S_t = 12) = 0.7$. The autoregressive coefficient is $\phi = 0.8$ and the innovation $\varepsilon_t$ has a standard Gaussian distribution. For a standard seasonal $\text{AR}_S(1)$ model with a constant period, it is well-known that the sample autocorrelations is only significant at multiple values of the period $S$. In contrast, the sample autocorrelation (Figure 2.1, panels (b)) of the $\text{SARR}_{S_t}(1)$ shows significant peaks at the lags 2, 4, 12, 14, 24, 26, etc. This autocorrelation feature could be used from real series to identify possible values of the random period $S_t$.

Figure 2.2 (panels $(a)$ and $(b)$) shows the same elements as Figure 2.1 for another simulated $\text{SARR}_{S_t}(1)$ series with sample size $n = 500$ and a random period $S_t \in \mathcal{S} = \{5, 6\}$ with a probability $\pi_1 = P(S_t = 5) = 0.5$. The remaining parameters are the same as the generating model in Figure 2.1. It can be seen that the sample autocorrelation function is significantly non-zero at the lags: $l \in \{1, 5, 6, 10, 11, 12, 16, 17, 18, ...\}$, which is consistent with the shape of autocorrelation function given by (2.14).




$(a)$ $(b)$

Figure 2.1. $(a)$ Simulated $\mathrm{SARR}_{S_t}(1)$ series $n = 120$. $(b)$ Sample autocorrelation.




$(a)$ $(b)$

Figure 2.2. $(a)$ Simulated $\mathrm{SARR}_{S_t}(1)$ series with $n = 500$. $(b)$ Sample autocorrelation.

As seen above, the stationarity condition for the model (2.11) is analogous to that of a seasonal autoregressive with a constant period. Model (2.11) is strictly stationary (and also causal) if and only if $|\phi| < 1$. In this case, the solution of (2.11) admits the following causal representation

$$Y_t = \sum_{j=0}^{\infty} \phi^j \varepsilon_{h^{(j)}(t)}. \tag{2.13}$$

Under the stationarity condition $|\phi| < 1$, still $E(Y_t) = 0$ while the variance $\gamma(0)$ and the autocovariance $\gamma(l)$ of order $l > 0$ take the form

$$\gamma(0) = \tfrac{\sigma^2}{1-\phi^2} \tag{2.14a}$$

$$\gamma(l) = \sum_{k=1}^{K} \pi_k \phi \gamma\left(l - S^{(k)}\right), \quad l > 0. \tag{2.14b}$$

so that the autocorrelation function is given by

$$\rho(l) = \sum_{k=1}^{K} \phi \pi_k \rho\left(l - S^{(k)}\right).$$

Furthermore, the one-step ahead forecast of $Y_t$ given by (2.10) becomes

$$\mu_t := E\left(Y_t | \mathcal{F}_{t-1}\right) = \sum_{k=1}^{K} \pi_k \phi_1 y_{t-S^{(k)}}.$$

### 2.2.2 The second-order $\mathbf{SARR}_{S_t}(2)$

When $p = 2$ in (2.3), the $\mathrm{SARR}_{S_t}(2)$ is given by

$$Y_t = \phi_1 Y_{t-S_t} + \phi_2 Y_{t-S_t-S_{t-S_t}} + \varepsilon_t \qquad t \in \mathbb{Z}, \tag{2.15}$$

where $\{\varepsilon_t, t \in \mathbb{Z}\}$ and $\{S_t, t \in \mathbb{Z}\}$ still satisfy the assumptions **A1-A3**. The distribution of $\{S_t, t \in \mathbb{Z}\}$ in **A2** means that

$$P\left(S_t = S^{(k)}\right) = \pi_k \text{ and } P\left(S_{t-S^{(k)}} = S^{(j)}\right) = \pi_j,$$

where $\sum_{k=1}^{K} \pi_k = 1$, and $\pi_k \geq 0$ $(1 \leq k \leq K)$.

Thus, the conditional pdf given by (2.2) becomes for the $\mathrm{SARR}_{S_t}(2)$ model as follows

$$f_{Y_t}\left(y_t | \mathcal{F}_{t-1}\right) = \sum_{k=1}^{K} \sum_{j=1}^{K} \tfrac{\pi_k \pi_j}{\sigma} f_\varepsilon\left(\tfrac{y_t - \phi_1 y_{t-S^{(k)}} - \phi_2 y_{t-S^{(k)}-S^{(j)}}}{\sigma}\right), \; t \in \mathbb{Z}. \tag{2.16}$$

The stationarity condition and the autocovariance function do not take a simplified expression as in the case $\mathrm{SARR}_{S_t}(1)$ and have the same form as the general case.

Finally, the one-step ahead forecast is given by

$$\mu_t = \sum_{k=1}^{K} \sum_{j=1}^{K} \pi_k \pi_j \left(\phi_1 y_{t-S^{(k)}} + \phi_2 y_{t-S^{(k)}-S^{(j)}}\right). \tag{2.17}$$

## 2.3 Seasonal random period moving average model

When $p = 0$, the $\text{SARMAR}_{S_t}(0,q)$ model, called random period seasonal moving average ($\text{SMAR}_{S_t}(q)$), is given by

$$Y_t = \varepsilon_t - \beta_1 \varepsilon_{h^{(1)}(t)} - ... - \beta_q \varepsilon_{h^{(q)}(t)},\ t \in \mathbb{Z}, \tag{2.18}$$

and is subject to the assumptions **A1-A3**. Under these assumptions, the conditional distribution of model (2.18) has the form

$$f_{Y_t}\left(y_t|\mathcal{F}_{t-1}\right) = \sum_{j_1=1}^{K} \cdots \sum_{j_q=1}^{K} \frac{\pi_{j_1}\pi_{j_2}\cdots\pi_{j_q}}{\sigma} f_\varepsilon\left(\frac{y_t - \beta_1\varepsilon_{h^{(1)}(t)}(k) - ... - \beta_q\varepsilon_{h^{(q)}(t)}(k)}{\sigma}\right),\ t \in \mathbb{Z}, \tag{2.19}$$

where

$$\varepsilon_{h^{(j)}(t)}\left(k\right) = y_t + \beta_1 \varepsilon_{h^{(j-1)}(t)}\left(k\right) + ... + \beta_q \varepsilon_{h^{(j-q)}(t)}\left(k\right).$$

Clearly, (2.19) is a mixture of seasonal moving average processes (see also Aknouche and Rabehi, 2010), where each $y_t$ is chosen from the $K$ seasonal models of period $S^{(k)}$ $(1 \leq k \leq K)$ with probability $\pi_k$.

Whatever the value of the coefficients $\left(\beta_j\right)_j$, the unique solution of (2.18) is strictly stationary and also second-order stationary due to the iid assumptions on $\{\varepsilon_t, t \in \mathbb{Z}\}$ and $\{S_t, t \in \mathbb{Z}\}$ which are of finite variance. The stability condition to be studied for model (2.18) is rather the invertibility.

Put $W_t = (\varepsilon_{h^{(0)}(t)}, \varepsilon_{h^{(1)}(t)}, ..., \varepsilon_{h^{(q-1)}(t)})'$,

$$C = \begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_{q-1} & \beta_q \\ 1 & 0 & \vdots & 0 & 0 \\ 0 & 1 & \vdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

and $D_t = (Y_t, 0, ..., 0)'$. Then model (2.18) can be rewritten in following SRE

$$W_t = CW_{t-1} + D_t, \quad t \in \mathbb{Z}. \tag{2.20}$$

The invertibility condition for model (2.18) is now obtained.

**Proposition 2.2** *Equation* (2.18) *is invertible in the sense*

$$W_t = \sum_{j=0}^{\infty} C^j D_{h^{(j)}(t)}, \tag{2.21}$$

*where the latter series converges a.s., if and only if*

$$\rho(C) < 1. \tag{2.22}$$

*In addition, the series in* (2.21) *also converges in mean square.*

**Proof** Using standard SRE theory again, a necessary and sufficient condition for equation (2.20) to have a solution given by (2.21) is that exponential top Lyapunov exponent $\rho(C)$ satisfies (2.22) (Bougerol and Picard, 1992). Since $E(\varepsilon_t^2) < \infty$, the unique stationary solution (2.21) has a finite second moment, where the series in (2.21) converges in mean square. □

The invertibility condition (2.22) is also the same as the invertibility condition of a constant period SARMA$(p, q)$ model. It is also the same invertibility condition for the general SARMAR$_{S_t}(p, q)$ model (2.2).

### 2.3.1 First-order SMAR$_{S_t}$(1)

When $q = 1$, the SMAR$_{S_t}(1)$ satisfies the following equation

$$Y_t = \varepsilon_t - \beta \varepsilon_{t-S_t} \qquad t \in \mathbb{Z}, \tag{2.23$a$}$$

with a conditional pdf $f_{Y_t}(y_t|\mathcal{F}_{t-1})$ having the form

$$f_{Y_t}(y_t|\mathcal{F}_{t-1}) = \begin{cases} \sum_{k=1}^{K} \frac{\pi_k}{\sigma} f_\varepsilon\left(\frac{\varepsilon_t(k)}{\sigma}\right) \\ \varepsilon_t(k) = y_t + \beta \varepsilon_{t-S^{(k)}}(k), \quad 1 \leq k \leq K, \end{cases} \qquad t \in \mathbb{Z}. \tag{2.23$b$}$$

Model (2.23) is invertible if and only if $|\beta| < 1$. In this case, $\varepsilon_t$ admits the following invertible representation

$$\varepsilon_t = \sum_{j=0}^{\infty} \beta^j Y_{h^{(j)}(t)}.$$

The $\mathrm{SMAR}_{S_t}(1)$ has zero mean and an autocovariance function given by

$$\gamma(l) = \begin{cases} \sigma^2 + \beta^2\sigma^2 & \text{if } \ l = 0 \\ -\beta\pi_k\sigma^2 & \text{if } \ l = S^{(k)},\ 1 \leq k \leq K \\ 0 & \text{otherwise.} \end{cases} \tag{2.24}$$

The autocorrelation function therefore becomes

$$\rho(l) = \begin{cases} 1 & \text{if } \ l = 0 \\ \frac{-\beta\pi_k}{1+\beta} & \text{if } \ l = S^{(k)},\ 1 \leq k \leq K \\ 0 & \text{otherwise.} \end{cases}$$

Note that the only non-zero values of the $\mathrm{SMAR}_{S_t}(1)$ autocorrelations are found at lags $S^{(1)}, ..., S^{(K)}$.

### 2.3.2 Second-order $\mathbf{SMAR}_{S_t}(2)$

The $\mathrm{SMAR}_{S_t}(2)$ model satisfies the following equation

$$Y_t = \varepsilon_t - \beta_1\varepsilon_{t-S_t} - \beta_2\varepsilon_{t-S_t-S_{t-S_t}}, \qquad t \in \mathbb{Z}, \tag{2.25$a$}$$

with a conditional pdf

$$f_{Y_t}(y_t|\mathcal{F}_{t-1}) = \begin{cases} \sum_{k=1}^{K} \frac{\pi_k}{\sigma} f_\varepsilon\left(\frac{\varepsilon_t(k)}{\sigma}\right) \\ \varepsilon_t(k) = y_t + \theta_1\varepsilon_{h(t)}(k) + \theta_2\varepsilon_{h^{(2)}(t)}(k),\ 1 \leq k \leq K, \end{cases} \tag{2.25$b$}$$

It is invertible if and only if

$$\rho\begin{pmatrix} \beta_1 & \beta_2 \\ 1 & 0 \end{pmatrix} < 1,$$

and admits the following invertible representation

$$W_t = \sum_{j=0}^{\infty} C^j D_{h^{(j)}(t)}.$$

The mean of $Y_t$ is zero and the autocorrelation function is

$$\rho(l) = \begin{cases} 1 & \text{if } l = 0 \\ \frac{\pi_k(-\beta_1+\beta_1\beta_2)}{1+\beta_1^2+\beta_2^2} & \text{if } \ l = S^{(k)},\ 1 \leq k \leq K \\ 0 & \text{if } l = S^{(i_k)} + S^{(i_j)},\ \ i_j, i_k \in \{1, ..., K\}. \end{cases} \tag{2.26}$$

According to (2.26), it turns out that the only non-zero autocorrelation values are foud at the lags which are the realizations taken by the random variable

$$S_t + S_{t-S_t} \in \left\{ S^{(1)}, ..., S^{(K)}, S^{(k_1)} + S^{(k_2)},\ 1 \leq k_1, k_2 \leq K \right\}.$$

For instance when $K = 2$,

$$S_t + S_{t-S_t} \in \left\{ S^{(1)}, S^{(2)}, 2S^{(1)}, 2S^{(2)}, S^{(1)} + S^{(2)} \right\}.$$

## 2.4 First-order SARMAR$_{S_t}(1,1)$

A special case of (2.2) is the first-order SARMAR$_{S_t}(1,1)$ process given by the following representation

$$Y_t = \phi Y_{t-S_t} + \varepsilon_t - \beta \varepsilon_{t-S_t}, \quad t \in \mathbb{Z},$$

with a conditional pdf

$$f_{Y_t}\left(y_t | \mathcal{F}_{t-1}\right) = \begin{cases} \sum\limits_{k=1}^{K} \frac{\pi_k}{\sigma} f_\varepsilon \left( \frac{\varepsilon_t(k)}{\sigma} \right) \\ \varepsilon_t\left(k\right) = Y_t - \phi Y_{t-S^{(k)}} + \theta \varepsilon_{t-S^{(k)}}\left(k\right), \, 1 \leq k \leq K. \end{cases}$$

The $SARMAR(1,1)$ model admits a stationary solution given by

$$Y_t = \sum_{j=0}^{\infty} \left( \phi^j + \beta \phi^{j-1} \right) \varepsilon_{h^{(j)}(t)} + \varepsilon_t$$

if and only if $|\phi| < 1$. It invertible if and only if $|\theta| < 1$ with the invertible representation

$$\varepsilon_t = Y_t + \sum_{j=1}^{+\infty} \left( \beta^j - \phi \beta^{j-1} \right) Y_{h^{(j)}(t)}.$$

The model has a mean zero, a variance $\gamma\left(0\right) = \frac{\left(1-2\phi\beta+\beta^2\right)\sigma^2}{\left(1-\phi^2\right)}$, and an autocovariance function

$$\gamma\left(l\right) = \begin{cases} \sum\limits_{k=1}^{K} \pi_k \phi \gamma \left( l - S^{(k)} \right) - \pi_k \theta \sigma^2 & \text{if } l = S^{(k)} \\ \sum\limits_{k=1}^{K} \pi_k \phi \gamma \left( l - S^{(k)} \right) & \text{otherwise.} \end{cases}$$

# 3 Parameter estimation via the EM algorithm

Based on a series $\mathbf{y} := (y_1, y_2, ..., y_n)$ generated from (2.2), we consider estimating the $\mathrm{SARMAR}_{S_t}(p, q)$ model using the (Gaussian) quasi-maximum likelihood method through the EM (expectation-maximization) algorithm (Dempster et *al*, 1977; McLaclahan and Peel, 2000; Wong and Li, 2000-2001). The model parameters to be estimated are the autoregressive coefficients $\phi_1, ..., \phi_p$, the innovation variance $\sigma^2$, the distribution $\{\pi_1, ..., \pi_{K-1}\}$ of $\{S_t, t \in \mathbb{Z}\}$ as well as the unobservable sample-path $S_1, S_2, ..., S_n$ of the period process. The estimation is considered under the stability conditions (2.6) and (2.22). As for the mixture autoregression (MAR, cf. Wong and Li, 2000; Aknouche and Rabehi, 2010; Aknouche, 2013), it will be seen that the estimators in the M-step can be obtained explicitly in the pure SARR model, due to the linearity in parameters of the maximization criterion. In the case of a moving average component, the M-step estimates do not have a closed form and are obtained iteratively.

To simplify the presentation, we first show the EM calculations on the simple $\mathrm{SARR}_{S_t}(1)$ model, then, on the $\mathrm{SARR}_{S_t}(2)$ model, then, on the general $\mathrm{SARR}_{S_t}(p)$ model, and finally on the $\mathrm{SARMAR}_{S_t}(1, 1)$ model. We also provides a procedure for random elimination of seasonality leading to a SARIMAR representation.

## 3.1 EM algorithm for the $\mathbf{SARR}_{S_t}(1)$

Since the $\mathrm{SARR}_{S_t}(1)$ model (2.11) is an iid mixture of $K$ seasonal autoregressions ($\mathrm{SAR}_{S^{(k)}}(1)$), the value of $S_t$ $(1 \leq t \leq n)$ in the mixture set $\mathcal{S}$ is ignored/unobservable. Let $(\mathbf{z}_t, 1 \leq t \leq n)$ be a sequence of latent variables indicating the occurrences of $S_t$ in $\mathcal{S}$ $(1 \leq t \leq n)$. It is expressed via the following binary $K$-vector

$$\mathbf{z}_t(k) = \begin{cases} 1 & \text{if } S_t = S^{(k)} \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Denote by $\theta = (\pi', \phi, \sigma^2)'$ the model (constant) parameters where $\pi = (\pi_1, ..., \pi_{K-1})'$.

Then the complete (log-) likelihood function $l_c(\theta)$ of the model is given by

$$\begin{cases} l_c(\theta) := l(\theta|\mathbf{y},\mathbf{z}) = \sum\limits_{t=1}^{n}\sum\limits_{k=1}^{K}\mathbf{z}_t(k)\left(\log\pi_k - \log\sqrt{2\pi}\sigma - \frac{\varepsilon_t^2(k)}{2\sigma^2}\right) \\ \varepsilon_t(k) = y_t - \phi y_{t-S^{(k)}}. \end{cases} \tag{3.2}$$

The EM algorithm consists of two steps: the E-step and the M-step. At the $i$th iteration of the E-step, the parameter estimate $\theta^{(i-1)} = (\pi^{(i-1)\prime}, \phi^{(i-1)}, \sigma^{(i-1)2})'$ being obtained, $\mathbf{z}_t(k)$ is estimated by its expected value $\tau_t^{(i)}(k)$ conditional on $\mathbf{y}$ as follows

$$\tau_t^{(i)}(k) := E_{\theta^{(i-1)}}(\mathbf{z}_t(k)|\mathbf{y}) = \frac{\frac{\pi_k^{(i-1)}}{\sigma^{(i-1)}} f_\varepsilon\left(\frac{\varepsilon_t^{(i-1)}(k)}{\sigma^{(i-1)}}\right)}{\sum\limits_{k=1}^{K}\frac{\pi_k^{(i-1)}}{\sigma^{(i-1)}} f_\varepsilon\left(\frac{\varepsilon_t^{(i-1)}(k)}{\sigma^{(i-1)}}\right)},\ 1\le k\le K, \tag{3.3}$$

where $\varepsilon_t^{(i-1)}(k) = y_t - \phi^{(i-1)} y_{t-S^{(k)}}$.

At the $i$th iteration of the M-step, the retained estimate $\tau_t^{(i)}(k)$ is replaced by $\mathbf{z}_t(k)$ in the complete log-likelihood, giving $l_c(\theta|\mathbf{y},\boldsymbol{\tau}^{(i)})$. The latter is maximized over $\theta$ to obtain the M-step estimate $\theta^{(i)}$. The two steps are alternatively iterated until convergence. Calculating the first derivatives of the complete log-likelihood with respect to each parameter gives the normal equations

$$\begin{cases} \frac{\partial l_c}{\partial \pi_k} = \sum\limits_{t=1}^{n}\frac{\mathbf{z}_t(k)}{\pi_k} - \frac{\mathbf{z}_t(K)}{\pi_K}, & 1\le k\le K-1 \\ \frac{\partial l_c}{\partial \sigma^2} = \sum\limits_{t=1}^{n}\sum\limits_{k=1}^{K}\mathbf{z}_t(k)\left(-\frac{1}{\sigma} + \frac{\varepsilon_t^2(k)}{\sigma^3}\right) \\ \frac{\partial l_c}{\partial \phi} = \sum\limits_{t=1}^{n}\sum\limits_{k=1}^{K}\frac{\mathbf{z}_t(k) y_{t-S^{(k)}}\varepsilon_t(k)}{\sigma^2} \end{cases} \tag{3.4}$$

whose explicit solutions in terms of $\pi_k^{(i)}$, $\sigma^{(i)2}$, and $\phi^{(i)}$ (while replacing $\mathbf{z}_t(k)$ by $\tau_t^{(i)}(k)$) are given by

$$\pi_k^{(i)} = \frac{1}{n-m}\sum_{t=1}^{n}\tau_t^{(i)}(k) \tag{3.5a}$$

$$\sigma^{(i)2} = \frac{\sum\limits_{t=1}^{n}\sum\limits_{k=1}^{K}\tau_t^{(i)}(k)\left(\varepsilon_t^{(i-1)}(k)\right)^2}{\sum\limits_{t=1}^{n}\sum\limits_{k=1}^{K}\tau_t^{(i)}(k)} \tag{3.5b}$$

$$\phi^{(i)} = \frac{\sum\limits_{t=1}^{n}\sum\limits_{k=1}^{K}\tau_t^{(i)}(k) y_t y_{t-S^{(k)}}}{\sum\limits_{t=1}^{n}\sum\limits_{k=1}^{K}\tau_t^{(i)}(k) y_{t-S^{(k)}}^2} \tag{3.5c}$$

where $m = \max_{1 \le k \le K} \left(S^{(k)}\right)$.

## 3.2 EM algorithm for the $\mathbf{SARR}_{S_t}(2)$ model

The parameter $\theta$ has now the form $\theta = (\pi', \phi_1, \phi_2, \sigma^2)'$. Similarly to (3.1), we introduce a sequence of latent binary $K \times 2$-matrices

$$\mathbf{z}_t(k,j) = \begin{cases} 1 & \text{if } S_t = S^{(k)} \text{ and } S_{t-S_{t-S^{(k)}}} = S^{(j)} \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

The complete log-likelihood now has the form

$$\begin{aligned} l_c(\theta) &= \sum_{t=1}^{n} \sum_{k=1}^{K} \sum_{j=1}^{K} \mathbf{z}_t(k,j) \left( \log \pi_k + \log \pi_j - \log \sqrt{2\pi}\sigma - \tfrac{(\varepsilon_t(k,j))^2}{2\sigma^2} \right) \\ \varepsilon_t(k,j) &= y_t - \phi_1 y_{t-S^{(k)}} - \phi_2 y_{t-S^{(k)}-S^{(j)}}. \end{aligned} \tag{3.7}$$

The $\mathbf{z}_t(k,j)$ are estimated in the E-step by $\tau_t^{(i)}(k,j)$ which are given by

$$\tau_t^{(i)}(k,j) := E_{\theta^{(i-1)}}\left(\mathbf{z}_t(k,j)\,|\mathbf{y}\right) = \frac{\frac{\pi_k^{(i-1)}\pi_j^{(i-1)}}{\sigma^{(i-1)}} f_\varepsilon\left(\frac{\varepsilon_t^{(i-1)}(k,j)}{\sigma^{(i-1)}}\right)}{\sum_{k=1}^{K}\sum_{j=1}^{K} \frac{\pi_k^{(i-1)}\pi_j^{(i-1)}}{\sigma^{(i-1)}} f_\varepsilon\left(\frac{\varepsilon_t^{(i-1)}(k,j)}{\sigma^{(i-1)}}\right)},\ 1 \le k,j \le K, \tag{3.8}$$

where

$$\varepsilon_t^{(i-1)}(k,j) = y_t - \phi_1^{(i-1)} y_{t-S^{(k)}} - \phi_2^{(i-1)} y_{t-S^{(k)}-S^{(j)}}.$$

For the M-step, the first derivatives $\frac{\partial l_c}{\partial \pi_k}$, $\frac{\partial l_c}{\partial \sigma^2}$, $\frac{\partial l_c}{\partial \phi_1}$, and $\frac{\partial l_c}{\partial \phi_2}$ are given by

$$\begin{cases} \frac{\partial l_c}{\partial \pi_k} = \sum_{t=1}^{n}\sum_{j=1}^{K} \frac{\mathbf{z}_t(k,j)}{\pi_k} - \frac{\mathbf{z}_t(K,j)}{\pi_K}, \quad 1 \le k \le K-1 \\ \frac{\partial l_c}{\partial \sigma^2} = \sum_{t=1}^{n}\sum_{k=1}^{K}\sum_{j=1}^{K} \mathbf{z}_t(k,j) \left(-\frac{1}{\sigma} + \frac{\varepsilon_t(k,j)^2}{\sigma^3}\right) \\ \frac{\partial l_c}{\partial \phi_1} = \sum_{t=1}^{n}\sum_{k=1}^{K}\sum_{j=1}^{K} \frac{\mathbf{z}_t(k,j)}{\sigma^2} y_{t-S^{(k)}} \varepsilon_t(k,j) \\ \frac{\partial l_c}{\partial \phi_2} = \sum_{t=1}^{n}\sum_{k=1}^{K}\sum_{j=1}^{K} \frac{\mathbf{z}_t(k,j)}{\sigma^2} y_{t-S^{(k)}-S^{(j)}} \varepsilon_t(k,j). \end{cases} \tag{3.9}$$

Solving the above normal equations (replacing $\mathbf{z}_t(k,j)$ by $\tau_t^{(i)}(k,j)$), we find the following

$i$th M-step estimate $\theta^{(i)} = (\pi^{(i)\prime}, \phi_1^{(i)}, \phi_2^{(i)}, \sigma^{(i)2})'$ such that

$$\pi_k^{(i)} = \frac{1}{n-2m} \sum_{t=1}^{n} \sum_{j=1}^{K} \tau_t^{(i)}(k,j), \quad 1 \le k \le K-1 \tag{3.10a}$$

$$\sigma^{(i)2} = \frac{\sum_{t=1}^{n}\sum_{k=1}^{K}\sum_{j=1}^{K} \tau_t^{(i)}(k,j)\left(\varepsilon_t^{(i-1)}(k,j)\right)^2}{\sum_{t=1}^{n}\sum_{k=1}^{K}\sum_{j=1}^{K} \tau_t^{(i)}(k,j)} \tag{3.10b}$$

$$\phi_1^{(i)} = \frac{\sum_{t=1}^{n}\sum_{k=1}^{K}\sum_{j=1}^{K} \tau_t^{(i)}(k,j) y_t y_{t-S^{(k)}-S^{(j)}} - \phi_2^{(i)} \sum_{t=1}^{n}\sum_{k=1}^{K}\sum_{j=1}^{K} \tau_t^{(i)}(k,j) y^2_{t-S^{(k)}-S^{(j)}}}{\sum_{t=1}^{n}\sum_{k=1}^{K}\sum_{j=1}^{K} \tau_t^{(i)}(k,j) y_{t-S^{(k)}} y_{t-S^{(k)}-S^{(j)}}} \tag{3.10c}$$

$$\phi_2^{(i)} = \frac{A-B}{\left(\sum_{t=1}^{n}\sum_{k=1}^{K}\sum_{j=1}^{K} \tau_t^{(i)}(k,j) y_{t-S^{(k)}} y_{t-S^{(k)}-S^{(j)}}\right)^2 - \sum_{t=1}^{n}\sum_{k=1}^{K}\sum_{j=1}^{K} \tau_t^{(i)}(k,j) y^2_{t-S^{(k)}} \sum_{t=1}^{n}\sum_{k=1}^{K}\sum_{j=1}^{K} \tau_t^{(i)}(k,j) y^2_{t--S^{(k)}-S^{(j)}}} \tag{3.10d}$$

where

$$A = \left(\sum_{t=1}^{n}\sum_{k=1}^{K}\sum_{j=1}^{K} \tau_t^{(i)}(k,j)\, y_{t-S^{(k)}} y_t\right)\left(\sum_{t=1}^{n}\sum_{k=1}^{K}\sum_{j=1}^{K} \tau_t^{(i)}(k,j)\, y_{t-S^{(k)}} y_{t-S^{(k)}-S^{(j)}}\right)$$

$$B = \left(\sum_{t=1}^{n}\sum_{k=1}^{K}\sum_{j=1}^{K} \tau_t^{(i)}(k,j)\, y^2_{t-S^{(k)}}\right)\left(\sum_{t=1}^{n}\sum_{k=1}^{K}\sum_{j=1t}^{K} \tau_t^{(i)}(k,j)\, y_t y_{t-S^{(k)}-S^{(j)}}\right).$$

## 3.3 EM algorithm for the general SARR$_{S_t}(p)$ model

The parameter $\theta$ to estimate is now $\theta = (\pi', \phi_1, ..., \phi_p, \sigma^2)'$. As for the SARR$_{S_t}(2)$ model, define the latent variable $\mathbf{z}_t$ $(1 \le t \le n)$, a binary $K \times p$-matrix of the form

$$\mathbf{z}_t(k_1, k_2, \ldots, k_p) = \begin{cases} 1 \text{ if} & \begin{array}{l} h^{(1)}(t) = t - S^{(k_1)} \\ h^{(2)}(t) = t - S^{(k_1)} - S^{(k_2)} \\ \vdots \\ h^{(p)}(t) = t - S^{(k_1)} - S^{(k_2)} - \cdots - S^{(k_p)} \end{array} \\ 0 \text{ otherwise.} \end{cases} \tag{3.11}$$

Thus, the complete log-likelihood becomes

$$\begin{cases} l_c(\theta) = \sum_{t=1}^{n}\sum_{k_1=1}^{K} \cdots \sum_{k_p=1}^{K} \mathbf{z}_t(k_1, k_2, \ldots, k_p)\left(\log \pi_{k_j} - \log\sqrt{2\pi}\sigma - \frac{\varepsilon_t^2(k_1,k_2,\ldots,k_p)}{2\sigma^2}\right) \\ \varepsilon_t(k_1, k_2, \ldots, k_p) = y_t - \sum_{j=1}^{p} \phi_j y_{h^{(j)}(t)}. \end{cases} \tag{3.12}$$

At the $i$th iteration of the E-step, $\mathbf{z}_t\left(k_1,k_2,\ldots,k_p\right)$ is estimated by

$$\begin{aligned}\tau_t^{(i)}\left(k_1,k_2,\ldots,k_p\right) \quad &: \quad = E_{\theta^{(i-1)}}\left(\mathbf{z}_t\left(k_1,k_2,\ldots,k_p\right)|\mathbf{y}\right) \\ &= \frac{\frac{\pi_{k_1}\pi_{k_2}\cdots\pi_{k_p}}{\sigma^{(i-1)}} f_\varepsilon\left(\frac{\varepsilon_t^{(i-1)}(k_1,k_2,\ldots,k_p)}{\sigma^{(i-1)}}\right)}{\sum_{t=1}^{n}\sum_{k_1=1}^{K}\sum_{k_2=1}^{K}\cdots\sum_{k_p=1}^{K}\frac{\pi_{k_1}\pi_{k_2}\cdots\pi_{k_p}}{\sigma^{(i-1)}} f_\varepsilon\left(\frac{\varepsilon_t^{(i-1)}(k_1,k_2,\ldots,k_p)}{\sigma^{(i-1)}}\right)}\end{aligned} \tag{3.13}$$

where $\varepsilon_t^{(i-1)}\left(k_1,k_2,\ldots,k_p\right) = y_t - \sum_{j=1}^{p}\phi_j^{(i-1)} y_{h^{(j)}(t)}$.

Likewise, the first derivatives of the complete log-likelihood, $\frac{\partial l_c}{\partial \pi_{k_j}}$, $\frac{\partial l_c}{\partial \phi_l}$, and $\frac{\partial l_c}{\partial \sigma^2}$ are given for all $1 \leq k_j \leq K-1$ and $1 \leq l \leq p$ by

$$\frac{\partial l_c}{\partial \pi_{k_j}} = \sum_{t=1}^{n}\sum_{k_1=1}^{K}\cdots\sum_{k_{j-1}=1}^{K}\sum_{k_{j+1}=1}^{K}\cdots\sum_{k_p=1}^{K}\frac{\mathbf{z}_t(k_1,\ldots,k_{j-1},k_j,k_{j+1},\ldots,k_p)}{\pi_{k_j}} - \frac{\mathbf{z}_t(k_1,\ldots,k_{j-1},K,k_{j+1},\ldots,k_p)}{\pi_K} \tag{3.14a}$$

$$\frac{\partial l_c}{\partial \sigma^2} = \sum_{t=1}^{n}\sum_{k_1=1}^{K}\cdots\sum_{k_p=1}^{K}\mathbf{z}_t\left(k_1,\ldots,k_p\right)\left(-\frac{1}{\sigma}+\frac{\varepsilon_t^2(k_1,\ldots,k_p)}{\sigma^3}\right) \tag{3.14b}$$

$$\frac{\partial l_c}{\partial \phi_l} = \sum_{t=1}^{n}\sum_{k_1=1}^{K}\cdots\sum_{k_p=1}^{K}\mathbf{z}_t\left(k_1,\ldots,k_p\right)\frac{\varepsilon_t(k_1,\ldots,k_p)}{\sigma^2} y_{h^{(l)}(t)}. \tag{3.14c}$$

Solving the normal equations corresponding to (3.14), we obtain the M-step estimates $\pi_{k_j}^{(i)}$ and $\sigma^{(i)2}$

$$\begin{aligned}\pi_{k_j}^{(i)} &= \frac{1}{n}\sum_{t=1}^{n}\sum_{k_1=1}^{K}\cdots\sum_{k_{j-1}=1}^{K}\sum_{k_{j+1}=1}^{K}\cdots\sum_{k_p=1}^{K}\tau_t^{(i)}\left(k_1,\ldots,k_p\right),\ 1 \leq k_j \leq K-1. \\ \sigma^{(i)2} &= \frac{\sum_{t=1}^{n}\sum_{k_1=1}^{K}\cdots\sum_{k_p=1}^{K}\tau_t^{(i)}(k_1,\ldots,k_p)\left(\varepsilon_t^{(i-1)}(k_1,\ldots,k_p)\right)^2}{\sum_{t=1}^{n}\sum_{k_1=1}^{K}\cdots\sum_{k_p=1}^{K}\tau_t^{(i)}(k_1,\ldots,k_p)}.\end{aligned}$$

The M-step estimates $\phi_l^{(i)}$ $(1 \leq l \leq p)$, are obtained are obtained in the same way as (3.10) via the resolution of the following system of $p$ linear equations

$$\frac{\partial l_c}{\partial \phi_l} = 0,$$

with $p$ variables $\phi_l^{(i)}$, $1 \leq l \leq p$.

### 3.4 EM algorithm for the SARMAR$(1,1)$ model

For the SARMAR$_{S_t}(1,1)$ model, the parameter is $\theta = (\pi', \phi, \beta, \sigma^2)'$. Let the latent variable $\mathbf{z}_t$ $(1 \le t \le n)$ be defined as in (3.1) for the SARR$_{S_t}(1)$ model. The complete log-likelihood function is

$$l_c(\theta) = \begin{cases} \sum_{t=1}^{n} \sum_{k=1}^{K} \mathbf{z}_t(k) \left( \log \pi^{(k)} - \log \sqrt{2\pi}\sigma - \frac{\varepsilon_t^2(k)}{2\sigma^2} \right), \\ \varepsilon_t(k) = y_t - \phi y_{t-S^{(k)}} + \beta \varepsilon_{t-S^{(k)}}. \end{cases}$$

At the $i$th iteration of the E-step, we obtain

$$\tau_t^{(i)}(k) := \frac{\frac{\pi_k^{(i-1)}}{\sigma^{(i-1)}} f_\varepsilon \left( \frac{\varepsilon_t^{(i-1)}(k)}{\sigma^{(i)}} \right)}{\sum_{k=1}^{K} \frac{\pi_k^{(i-1)}}{\sigma^{(i-1)}} f_\varepsilon \left( \frac{\varepsilon_t^{(i-1)}(k)}{\sigma^{(i-1)}} \right)}, \ 1 \le k \le K$$

where $\varepsilon_t^{(i-1)} = y_t - \phi^{(i-1)} y_{t-S^{(k)}} + \beta^{(i-1)} \varepsilon_{t-S^{(k)}}$.

At the M-step, solving the normal equation corresponding to $\frac{\partial l_c}{\partial \pi_k}$ and $\frac{\partial l_c}{\partial \sigma}$, we obtain the M-estimates

$$\begin{aligned} \pi_k^{(i)} &= \frac{1}{n} \sum_{t=1}^{n} \tau_t^{(i)}(k), \ 1 \le k \le K-1 \\ \sigma^{(i)2} &= \frac{\sum_{t=1}^{n} \sum_{k=1}^{K} \tau_t^{(i)}(k) \left( \varepsilon_t^{(i-1)}(k) \right)^2}{\sum_{t=1}^{n} \sum_{k=1}^{K} \tau_t^{(i)}(k)}. \end{aligned}$$

For the remaining parameters $\alpha = (\phi, \theta)'$, the scores are given by the recurrence relations

$$\begin{cases} \frac{\partial l_c}{\partial \phi} = -\sum_{t=1}^{n} \sum_{k=1}^{K} \frac{\mathbf{z}_t(k)}{\sigma^2} \frac{\partial \varepsilon_t(k)}{\partial \phi} \varepsilon_t(k) \\ \frac{\partial \varepsilon_t(k)}{\partial \phi} = -y_{t-S^k} + \beta \frac{\partial \varepsilon_{t-S^k}}{\partial \phi} \end{cases}$$

$$\begin{cases} \frac{\partial l_c}{\partial \beta} = -\sum_{t=1}^{n} \sum_{k=1}^{K} \frac{\mathbf{z}_t(k)}{\sigma^2} \frac{\partial \varepsilon_t(k)}{\partial \beta} \varepsilon_t(k) \\ \frac{\partial \varepsilon_t(k)}{\partial \beta} = \varepsilon_{t-S^k} + \beta \frac{\partial \varepsilon_{t-S^k}}{\partial \beta} \end{cases}$$

As the normal equations $\frac{\partial l_c}{\partial \phi} = 0$ and $\frac{\partial l_c}{\partial \beta} = 0$ do not have a closed-form solution, we call for a Fisher-Scoring procedure to estimate $\alpha$.

Starting from an initial value $\alpha_0^{(i)}$ (at the $i$th iteration of the EM algorithm), the Fisher-Scoring algorithm iterates $\alpha_l^{(i)}$ over $l$ as follows

$$\alpha_{l+1}^{(i)} = \alpha_l^{(i)} + \left(I_c\left(\alpha_l^{(i)}, \mathbf{y}\right)\right)^{-1} G\left(\alpha_l^{(i)}, \mathbf{y}\right)$$

where $G(\alpha, \mathbf{y}) = \frac{\partial l_c}{\partial \alpha}$ is the gradient of $l_c$ and $I_c(\alpha, \mathbf{y}) = -\frac{\partial^2 l_c}{\partial \alpha \partial \alpha'}$ is the complete Fisher information relatively to the subparameter $\alpha$. The latter recursion is repeated until convergence.

### 3.5 Random elimination of seasonality and the SARIMAR model

When a series has non-stationary seasonal behavior, it is well-known that the seasonal difference operator can eliminate the seasonality. This seasonal operator is given by

$$\begin{aligned} \Delta_S Y_t &= \left(1 - L^S\right) Y_t \\ &= Y_t - Y_{t-S}. \end{aligned}$$

Based on the random period sequence $\{S_t, t \in \mathbb{Z}\}$, we now define the following random seasonal difference operator as

$$\begin{aligned} \Delta_{S_t} Y_t &= \left(1 - L^{S_t}\right) Y_t \\ &= Y_t - Y_{t-S_t}, \end{aligned} \tag{3.15}$$

where $\{S_t, t \in \mathbb{Z}\}$ is subject to **A2**. Since $S_t \in \left\{S^{(1)}, ..., S^{(K)}\right\}$ is random and has a probability distribution $(\pi_k)_{1 \le k \le K}$, the effective random elimination of seasonality passes first by estimating that distribution and thus by using the EM algorithm. The following procedure illustrates the random elimination of seasonality on the simple case $\mathcal{S} = \left\{S^{(1)}, S^{(2)}\right\}$. The general case is made in the same lines.

**Algorithm 3.1 (Random deseasonalization)**

**(0)** Start with initial values for $\pi^{(0)} = \left(\pi_1^{(0)}, \pi_2^{(0)}\right)$ (say $\left(\pi_1^{(0)}, \pi_1^{(0)}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$) and $\sigma^{2(0)} = \left(\sigma_1^{2(0)}, \sigma_2^{(0)}\right)$.

Fix $I$ to be a large enough integer and specify $f_\varepsilon$ (e.g. $f_\varepsilon$ is the standard normal density).

**(1)** For $i = 1, ...I$, repeat:

**(1.1)** $\tau_t^{(i)}(k) = \dfrac{\frac{\pi_k}{\sigma} f_\varepsilon\left(\frac{y_t - y_{t-S^{(k)}}}{\sigma}\right)}{\sum\limits_{k=1}^{K} \frac{\pi_k}{\sigma} f_\varepsilon\left(\frac{y_t - y_{t-S^{(k)}}}{\sigma}\right)}$, $k = 1, 2$.

**(1.2)** $\widehat{\pi}_k^{(i)} = \frac{1}{n-m} \sum\limits_{t=1}^{n} \tau_t^{(i)}(k)$, $k = 1, 2$, where $m = \max\limits_{1 \leq k \leq 2}(S^{(k)})$.

**(1.3)** $\widehat{\sigma}^{2(i)} = \dfrac{\sum\limits_{t=1}^{n}\sum\limits_{k=1}^{K} \tau_t^{(i)}(k)\left(y_t - y_{t-S^{(k)}}\right)^2}{\sum\limits_{t=1}^{n}\sum\limits_{k=1}^{K} \tau_t^{(i)}(k)}$.

**(2)** For $t \in \{m, ..., n\}$,

If $\tau_t^{(I)}(1) > 0.5$ then $\Delta_{S_t} Y_t = Y_t - Y_{t-S^{(1)}}$.

If $\tau_t^{(I)}(1) \leq 0.5$ then $\Delta_{S_t} Y_t = Y_t - Y_{t-S^{(2)}}$.

**(3)** Return $\{\Delta_{S_t} Y_t,\ m \leq t \leq n\}$. $\square$

Finally, note that the case $K = 2$ with $S^{(1)} = 1$ and $S^{(2)} = S$ (e.g. $S = 12$ for monthly series) is particularly important since the random difference $\Delta_{S_t}$ will consist of a "randomization" of the (stochastic) trend operator $\nabla Y_t = Y_t - Y_{t-1}$ and of the seasonality operator $\nabla_S Y_t = Y_t - Y_{t-S}$. It therefore constitutes an alternative to the $\mathrm{SARIMA}_S(p, d, q)(P, D, Q)$ model

$$\phi(L)\,\Phi\left(L^S\right)\nabla^d \nabla_S^D Y_t = \beta(B)\,\Theta\left(L^S\right)\varepsilon_t \tag{3.16}$$

(with obvious notation; see Box et *al*, 2008) in which there is rather a "superposition" of the stochastic trend operator $\nabla$ and the seasonality operator $\nabla_S$.

Thus when $\Delta_{S_t} Y_t$ given by (3.15) has the $\mathrm{SARMAR}_{S_t}(p, q)$ representation (3.2) then analogously to the SARIMA model (3.16) (with $P = D = Q = 0$) we call the process $(Y_t)$ $\mathrm{SARIMAR}_{S_t}(p, 1, q)$.

# 4 Simulation study

The finite-sample behavior of the EM algorithm is examined via a simulation study. The models considered are the $\mathrm{SARR}_{S_t}(1)$ and $\mathrm{SARR}_{S_t}(2)$ models with $K = 2$ and $S_t \in \mathcal{S} = \{11, 12\}$ for the $\mathrm{SARR}_{S_t}(1)$ (cf. Table 4.1) and $S_t \in \mathcal{S} = \{10, 11\}$ (cf. Table 4.2) These choices are motivated by their resemblance to the fitted model in the real application. For each model, 1000 Monte Carlo replications with a sample size $n = 100$ are generated for which the EM algorithm is run. The estimated probability $\widehat{\pi}_2$ associated to the period $S^{(2)}$ does not appear in Tables 4.1-4.2 since it is equal to $1-\widehat{\pi}_1$. In both tables (Tables 4.1-4.2) we keep the true values of parameters as initial values for the estimation of the latent variable in the E-step of the EM algorithm. Unreported simulations (available from the authors) showed that the effect of the initial values on the quality of estimates is not really significant.

| | $\pi_1$ | $\phi$ | $\sigma$ |
|---|---|---|---|
| True value | **0.6000** | **−0.9000** | **1.0000** |
| Mean of estimates | 0.6006 | −0.8781 | 0.9946 |
| Empirical standard error | 0.0689 | 0.1056 | 0.0861 |

| | $\pi_1$ | $\phi$ | $\sigma$ |
|---|---|---|---|
| True value | **0.4000** | **0.9000** | **1.0000** |
| Mean of estimates | 0.4009 | 0.8770 | 1.0001 |
| Empirical standard error | 0.0622 | 0.0912 | 0.0829 |

| | $\pi_1$ | $\phi$ | $\sigma$ |
|---|---|---|---|
| True value | **0.2000** | **0.1000** | **4.0000** |
| Mean of estimates | 0.1993 | 0.1029 | 3.9602 |
| Empirical standard error | 0.0039 | 0.1179 | 0.3001 |

| | $\pi_1$ | $\phi$ | $\sigma$ |
|---|---|---|---|
| True value | **0.4000** | **0.7000** | **1.0000** |
| Mean of estimates | 0.4041 | 0.6776 | 0.9979 |
| Empirical standard error | 0.0755 | 0.1117 | 0.0861 |

Table 4.1 Performance of $EM$ estimates for the $\mathrm{SARR}_{S_t}(1)$ model.

For the $\mathrm{SARR}_{S_t}(1)$ model, it can be seen that for all the parameters, the estimates are good in terms of bias and standard errors, despite the small sample size. For larger sample sizes, the results are more and more accurate, but they are not reported here.

Regarding the $\mathrm{SARR}_{S_t}(1)$ model, the same conclusions can be drawn. The means of estimates are near to the true values with small empirical standard deviations.

| | $\pi_1$ | $\phi_1$ | $\phi_2$ | $\sigma$ |
|---|---|---|---|---|
| True value | **0.1000** | **0.8000** | **0.2500** | **1.0000** |
| Mean of estimates | 0.1018 | 0.7921 | 0.2412 | 0.9808 |
| Empirical standard error | 0.0427 | 0.0955 | 0.1133 | 0.0897 |

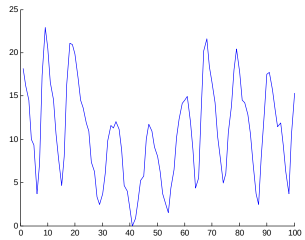
| | $\pi_1$ | $\phi_1$ | $\phi_2$ | $\sigma$ |
|---|---|---|---|---|
| True value | **0.2000** | **−0.3000** | **0.7000** | **5.0000** |
| Mean of estimates | 0.1999 | −0.2936 | 0.7807 | 4.9295 |
| Empirical standard error | 0.0372 | 0.0865 | 0.0964 | 0.4702 |

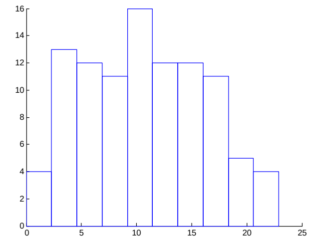
| | $\pi_1$ | $\phi_1$ | $\phi_2$ | $\sigma^2$ |
|---|---|---|---|---|
| True value | **0.2000** | **0.2500** | **0.6000** | **5.0000** |
| Mean of estimates | 0.1971 | 0.2368 | 0.5663 | 4.9514 |
| Empirical standard error | 0.0421 | 0.1153 | 0.1278 | 0.5154 |

Table 4.2 Performance of $EM$ estimates for the $\mathrm{SARR}_{S_t}(2)$ model.

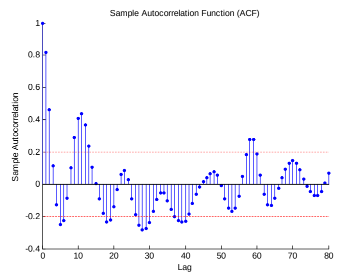
# 5 Application to the sunspot data

Observing solar activity is essential, given the consequences it could have on telecommunications and the conduct of space experiments. Solar irruptions, called sunspots, also have an impact on the variation in the global temperature of the earth, hence the need to integrate them into the study of climate change. It was Yule (1927) who first modeled the number of sunspots using an AR(2), and long before him, Schwabe (1843) noticed the existence of a cycle of 10 years. Other works gave rise to different cycle periods varying from 9 to 13 years. McLeod and Hipel (2005) argued that the annual Wolfer sunspot numbers from 1770 to 1869 (Waldmeir, 1961) could be characterized by a random periodicity; hence our choice to apply the $\mathrm{SARR}_{S_t}(2)$ model to this series (cf. Figure 5.1).





$(a)$ $(b)$




$(c)$

Figure 5.1. $(a)$ Box-Cox transformed Sunspots data from 1770 to 1869; $(b)$ histogram; $(c)$ sample autocorrelation.

Box and Jenkins (1976) proposed to model sunspot data by a (constrained) AR(9). McLeod and Hipel (2005) took up this application and presented the following model which gave better out-of-sample forecasts

$$\left(1 - 1.325L + 0.605L^2 - 0.130L^9\right)(W_t - 10.718) = \varepsilon_t \tag{5.1}$$

where $W_t = \left(\frac{1}{0.5}\right)\left[(Y_t + 1)^{0.5} - 1\right]$ is the Box-Cox transformation operated on the original series $Y_t$ of the number of sunspots. This transformation ensures homoskedasticity and confers normality to the transformed series.

We therefore propose to fit the $\mathrm{SARR}_{S_t}(2)$ model to sunspot data with $S_t \in \left\{S^{(1)}, S^{(2)}\right\}$ such that $S^{(1)} = 11$ and $S^{(2)} = 12$. The reason for this choice is that, as pointed out above, the most frequent three successive turning points of the series are 11 and 12. We first consider the first difference transformation $Z_t = \Delta Y_t$ on the Box-Cox transformed sunspot series. The we run the EM algorithm on the series $\{Z_t, t \in \mathbb{Z}\}$ giving the following estimated

SARR$_{S_t}$(2) model

$$\begin{aligned} Z_t &= 0.4442 Z_{t-S_t} + 0.1965 Z_{t-S_t-S_{t-S_t}} \qquad t \in \mathbb{Z} \\ \widehat{\sigma} &= 2.4654,\ \widehat{\pi} = (0.8944, 0.1056) \end{aligned} \tag{5.2}$$

We then compare the out-of-sample ability of the SARR$_{S_t}$(2) model to the model of Hipel and McLeod (2005). Specifically, we consider the sunspot series truncated from is last ten observations and estimate the two competing models on the truncated series with 90 observations. We compute for each model, the mean square error given by

$$MSE = \tfrac{1}{n} \sum_{t=1}^{n} \left(y_t - \widehat{y}_t\right)^2,$$

where $\widehat{y}_t$ is the estimated conditional mean computed for each model (cf. Figure 5.2).

The MSE of the SARR$_{S_t}$(2) forecasts equals 0.2636 and is significantly lower than 0.5010, the MSE obtained by the (constrained) AR(9) of Hipel and McLeod (2005).

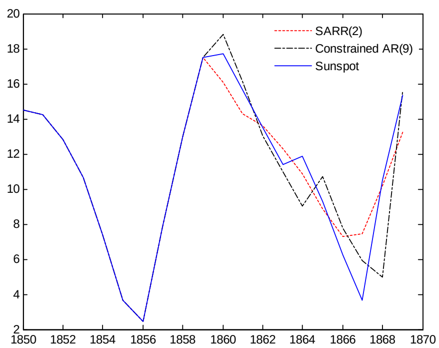



Figure 5.2. Out-of-sample forecast comparison between the SARR$_{S_t}$ (2) and the constrained AR(9).

## 5.1 Conclusion

This work proposes an extension of the Box-Jenkins SARIMA model to the case where the period is a random iid sequence. Considering the period as random gives rise to flexible modeling of seasonality, particularly for quasi-periodic phenomena whose period evolves.

The proposed SARIMAR model can also represent constant period time series with significant autocorrelations for lags that are adjacent to multiples of the period. Finally, for the SARIMAR model, the mean of the period is allowed to be non-integer so this scenario is a complementary approach to seasonal models with non-integer periods introduced by De Livera et al (2011); see also Aknouche et *al* (2018).

The SARIMAR model could be expanded in many directions. First, the parameter $\theta$ could be made dependent on $S_t$, making the model more flexible. Second, the spectral density of a $\mathrm{SARMAR}_{S_t}(p,q)$ and its estimates are useful for studying the properties of the model from the spectral domain perspective. Third, the problem of identifying the number of periods $K$ and the different periods $S^{(1)}, ..., S^{(K)}$ can be considered. In particular, we have seen how the shape of the sample autocorrelation function can be indicative of the model orders, but the classic AIC and BIC information criteria might also be adapted. Fourth, the case where the period has a dependent structure seems to be an interesting area of study. In particular, the sequence $\{S_t, t \in \mathbb{Z}\}$ might be a non-independent Markov chain (cf. Aknouche and Francq, 2022), or depend on the past of the observed process. Finally, periodic models with random periods could be introduced.

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