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Abstract

We fully characterize the transition to sustained growth of resource-constrained economies using a model of industrialization that reproduces key stylized facts of resource use and prices. Natural scarcity, endogenous demography and innovations generate different growth regimes: knowledge-based innovations can potentially feed productivity growth in the long run, but exhaustible primary inputs and population pressure may halt economic development at earlier stages. Our model reproduces two well-documented empirical regularities – a U-shaped path of resource prices and a hump-shaped path of resource extraction – as *secular trends* that arise across growth regimes. Resource use and prices reach their respective turning points at different stages of development, and we may observe a peak in extraction followed by a long period where both resource use and its market price fall. The decoupling of price and quantity dynamics hinges on general-equilibrium interactions between demography and three sources of endogenous technological change, namely, increases in the mass of intermediate firms, vertical innovations within each intermediate firm, and endogenous extraction costs affected by learning-by-doing in the primary sector.

JEL Codes E10, L16, O31, O40

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1 Introduction

The sustainability of living standards in a world of finite natural resources is a research question at the roots of economics as a scientific discipline. Besides differing views about the current state of global resource scarcity, economists generally agree that technological progress plays a central role in relieving the constraints set by natural resources. As scholars in the field know, building a comprehensive theory of economic growth with finite non-renewable inputs has posed serious challenges.[1] Yet, sustainability models incorporating endogenous technological change are now a well-established framework even for applied analysis – see Acemoglu et al. (2012), Andre and Smulders (2014), Lanz et al. (2017), Hassler et al. (2021). Two prominent questions in the current agenda are (i) the impact of population on sustainability prospects and (ii) making predictions on resource use and prices that are consistent with observed stylized facts. In this paper, we propose a tractable model of growth regimes providing clear predictions on both issues.

The existing literature has often overlooked the multiple roles played by demography at different stages of economic development in environments where exhaustible resources are essential inputs in production. Besides recent contributions that successfully incorporate endogenous fertility in growth models with natural scarcity – notably, Bretschger (2020) – there is still little recognition that 'sustainable paths' comprise different phases and that each new growth regime may be triggered or impeded by exhaustible resource constraints, depending on the concurrent interactions with population and technology. Peretto (2021) makes this point in a model where population growth exherts a demand-side pressure on scarce resources before innovations take off but also induces market expansions that spur different forms of technological change. If the economy completes the transition to the last phase, where knowledge-based vertical innovations drive productivity growth, increasing shares of value added generated by reproducible inputs and intangible assets sustain economic growth in the long run. We adopt this framework with several modifications and one important difference: Peretto (2021) uses the most basic Hotelling model with zero extraction costs and thus exhibits the well-known counterfactual that the resource price always grows at the rate of interest. In this paper, we introduce fully endogenous extraction costs arising from learning-by-doing in the primary sector. This hypothesis, in conjunction with horizontal and vertical innovations in the intermediate sector, allows us to reproduce two well-documented stylized facts on (a) prices and (b) consumption/extraction of non-renewable natural resources.

The empirical literature typically interprets price dynamics by comparing observed trends in real commodity prices to the predictions of the basic Hotelling model (Hotelling, 1931), according to which the price of non-renewable resources should permanently increase at the rate of interest

[1] See Smulders (2005) for an excellent overview of the origins of the modern literature on sustainable growth. Dasgupta and Heal (1974) and Stiglitz (1974) pioneered the field by incorporating the optimal depletion of exhaustible resources into the Ramsey model of exogenous growth. A seminal extension of this framework to endogenous growth is Barbier (1999). Subsequent contributions extended this framework to vertical innovations (Grimaud and Rouge, 2003), directed technological change (Di Maria and Valente, 2008), endogenous demography (Bretschger, 2020), mixed innovations with sequential growth regimes (Peretto, 2021).

that firms use to discount future profits. Barnett and Morse (1963) first pointed out the existence of long periods of declining, or non-increasing resource prices. Subsequent work argued that for many resources, including minerals and fossil fuels, data suggest a *U-shaped* time path of resource prices over long periods.[2] Different extensions of the basic Hotelling model can reconcile optimal depletion theory with U-shaped prices (Krautkraemer, 1998). If extraction costs fall sufficiently fast to offset increased scarcity in the short run, the market price of extracted units may initially decline and then revert its trend when natural scarcity effects dominate cost-reducing effects. Specific microeconomic explanations include technical improvements in the extractive process, discovery of new resource deposits, time-varying capacity constraints. Importantly, these studies and the accompanying evidence typically refer to partial-equilibrium models applied to specific industries or sectors.[3]

With respect to resource use, the literature documenting the existence of *hump-shaped* time paths in extracted units of non-renewable resources is vast and not limited to economics. After the peak in US oil production of 1970, geophysics and engineering studies have been using bell-shaped functions to approximate the time series of oil extraction and to forecast future peaks at regional and global levels.[4] This literature does not interpret the existence of peaks in the extraction of non-renewable resources as the result of economic forces. Still, as shown by Holland (2008), optimal depletion theories do provide economic explanations: augmented Hotelling models can reproduce a hump-shaped time path of resource use if we assume technical change in extraction, increases in production capacity via site development, or new discoveries. Okullo et al. (2015) bring the discussion one step forward by showing that an augmented Hotelling model with geological constraints can generate both the peak in resource extraction and the U-shaped path of resource prices.

Our main takeaway from this literature is that partial-equilibrium models do not capture the full picture since they treat the key explanatory variable – be it changes in demand, technological progress, or input substitution – as an exogenous process. Many exhaustible resources are essential inputs on a global scale, and the time paths of resource use and prices are general-equilibrium phenomena driven by the endogenous interactions between demography, technological change and resource use. Our model captures these endogeneities and reproduces both stylized facts as *secular trends* that emerge across different growth regimes, with a decoupling of price-quantity dynamics:

[2] See, e.g., Berck and Roberts (1994), Pindyck (1999), and Slade and Thille (2009).

[3] Early contributions to this topic include Slade (1982), Pindyck (1978), and Cairns and Lasserre (1986). Krautkraemer (1998) provides a comprehensive analysis of these augmented Hotelling models and further references.

[4] This literature originates in Hubbert's (1959) conjecture that oil production in the United States would keep on increasing for a decade, reach a plateau, and reverse its trend to enter an extended phase of declining extraction rates. The predicted turning point of extraction materialized in 1970 and inspired a vast class of forecasting models that incorporate 'Hubbert peaks' in their representation of extractive industries. This literature does not provide formal explanations based on economic theory: the existence of peaks in the extraction of exhaustible resources is motivated by the joint dynamics of current reserves and new discoveries, which allow to represent resource extraction over time by means of bell-shaped, Gaussian-type functional forms. See, e.g., Brandt (2007), Smith (2012) and the references cited therein.

the time paths of resource use and resource prices reach their respective turning points at different stages of development. In particular, our model shows that we may observe a peak in resource extraction followed by a long period during which both extracted units and resource prices fall over time. The intuition for these results lies in the sequence of growth phases that characterize the transition of the economy from the pre-innovation era to sustained growth in the long run. In particular, our model highlights that demography can play a central role in triggering growth regimes via its interactions with technological change.

In the initial phase, agents build up the economy by exploiting nature to support population growth: neither horizontal nor vertical innovations are operative in the final goods' sector, whereas learning-by-doing in the extractive industry allows resource extraction to increase over time and meet growing consumption needs. Resource extraction will necessarily peak in finite time due to the physical constraint set by the resource stock, but resource prices may keep on falling as the economy enters the second phase in which output production becomes less reliant on material inputs. Since growing population leads to market expansion, entrepreneurs turn on the Schumpeterian engine of endogenous innovation: while the mass of good-producing firms increases, each firm pursues in-house R&D to increase production efficiency. The transitional decline in resource prices and its disconnection from the peak in extraction hinge on the combined effects of falling relative demand for primary inputs and cost-reducing technical change in the primary sector. Eventually, the counteracting effect of natural scarcity becomes dominant in the long run: the resource price reaches and minimum and grows thereafter. In the terminal phase, vertical innovations take the lead and economic growth exclusively relies on knowledge accumulation substituting for asymptotically exhausted natural inputs.[5] This long-run outcome is possible but not guaranteed, since completing the path to sustainability requires satisfying conditions across all phases. Unfavorable initial conditions, including low natural endowments, may disrupt the transition to sustained growth by forcing the economy into under-development traps before innovations step in – a scenario that we label as 'failure to industrialize'. Similarly, unfavorable values of technological parameters may leave the economy in post-development stagnation, that is, a second phase of 'fragile industrialization' where knowledge-driven vertical innovations are not operative and per capita incomes permanently decline.

Our paper contribution firmly stands within the agenda of the sustainable growth literature that incorporates resource scarcity in modern endogenous growth theory. The general aim is to build macroeconomic models that successfully accommodate the existence of diminishing returns – due to fixed or vanishing physical resources – as well as the potential of human knowledge accumulation to relax the contraints imposed by nature on our ability to extract economic value from those resources.[6] This provides our analysis with a distinctive content with respect to the

[5]The notions of asymptotic exhaustion and input substituion holding in models of sustained growth have been largely discussed in the literature. See Smulders (1995) and Bretschger and Smulders (2012).

[6]This general objective is easily recognizable to the foundation of economics, which Simpson et al. (2005) summarize in two core ideas: the principle of diminishing return – whereby "economies relying on fixed stocks of land and other resources are, at best, destined for stagnation" (p.5) – and the escape from it, as envisaged by Alfred Marshall

applied microeconomic literature addressing U-shaped resource prices and peaks in extraction. Our model generates these stylized facts as secular trends induced by different phases of economic development, not as sectoral phenomena driven by exogenous processes. The general-equilibrium nature of our results can be traced in just a few important contributions. Andres and Smulders (2014) is an immediate reference and source of inspiration: the authors show that directed technical change can generate peaks in extraction, and they argue that endogenous extraction costs would be necessary to obtain realistic time paths of resource prices. Differently from the analysis of Andres and Smulders (2014) – where population is fixed and the transition is driven by technological change progressively becoming less labour-augmenting and more resource-augmenting – our model considers a vertical Schumpeterian structure where *endogenous population drives the sequence of growth regimes* and the activation of horizontal and vertical innovations, while learning-by-doing in the primary sector makes extraction costs endogenous and ultimately yields a U-shaped path of the resource price. The Schumpeterian structure is laid out in the model by Peretto (2021), which however cannot not replicate our stylized facts. Other contributions that exploit the Schumpeterian structure with endogenous population include Peretto and Valente (2015) and Lanz et al. (2017), which abstract from exhaustible resources to analyze, instead, population dynamics with resource constraints arising from fixed, non-exhaustible factors like land. Our research question also differs from that in Bretschger (2020), which combines endogenous fertility and innovations to study the determinants of climate change assuming that exhaustible resource use generates stock pollution externalities.[7]

2 The model

The supply side of the economy comprises three production sectors: final producers, intermediate producers, and a primary sector that extracts a natural resource from a non-renewable stock. The economy produces the final good using labor, differentiated intermediate goods, and the natural resource input. The final good can be consumed, used to produce intermediate goods, or invested in two productivity-enhancing activities that characterize post-industrialization regimes: the creation of new intermediate goods ('horizontal innovations') and the improvement of the quality of existing intermediate goods ('vertical innovations'). A third source of technological change is learning-by-doing in the primary sector, which progressively reduces extraction costs. The supply of labour depends on demography, which is endogenously determined by households' fertility choices. The following subsections describe each of these elements in turn.

and John Stuart Mill, which hinges on "whatever adds to the general power of mankind over nature, and especially by any extension of their knowledge".

[7] Bretschger (2020) characterizes the transition to a polluted steady state with endogenous growth, and shows that climate change is independent of population growth in the long run.

2.1 The final sector

A competitive representative firm produces the final good according to the technology

$$Y = \int_0^N X_i^\theta \left(Q_i L_i^\gamma R^{1-\gamma} \right)^{1-\theta} di, \quad 0 < \theta, \gamma < 1 \tag{1}$$

where Y is final output, N is the mass of non-durable intermediate goods, X_i is the quantity of the i-th intermediate good, and L_i and R are, respectively, labor and an exhaustible natural resource (henceforth, resource). The technology features full dilution of labor across intermediate goods, reflecting the property that both labor and intermediate goods are rival inputs. The resource, instead is non-rival across intermediate goods and labor. Quality, Q_i, is the good's ability to raise the productivity of the other factors. The final good is the numeraire so its price is normalized to unity. Let P_i be the price of intermediate good i. The final producer demands intermediate goods according to

$$X_i = \left(\frac{\theta}{P_i} \right)^{\frac{1}{1-\theta}} Q_i L_i^\gamma R^{1-\gamma}. \tag{2}$$

Denoting by w the wage rate and by p the price of the resource, the final producer pays total compensation

$$\int_0^N P_i X_i di = \theta Y \quad \text{and} \quad wL = \int_0^N w L_i di = \gamma (1-\theta) Y \quad \text{and} \quad pR = (1-\gamma)(1-\theta) Y \tag{3}$$

to, respectively, suppliers of intermediate goods, labor and the resource.

2.2 The primary sector

In the primary sector, a competitive representative firm owns a finite stock S of the non-renewable natural resource. The instantaneous depletion of the stock equals uR, where R is resource use – i.e., the flow of extracted resources sold to final producers – and $u \geq 1$ is the extraction cost in terms of resource units foregone during the process.[8] Hence, the resource stock evolves over time according to

$$\dot{S}(t) = -u(t) R(t), \tag{4}$$

and, given a positive initial stock $S_0 = S(0) > 0$, the intertemporal resource constraint of the firm is

$$S_0 \geq \int_0^\infty u(t) R(t) dt, \qquad u(t) \geq 1, \qquad R(t) \geq 0. \tag{5}$$

The firm's problem is to maximize present-value profits

$$\int_0^\infty p(t) R(t) e^{-\int_0^t r(s) ds} dt,$$

[8] In models with iceberg extraction cost, u is typically defined as 1 plus the extraction cost. It makes no difference in this application. The paper's notation is simpler.

where r is the prevailing interest rate, subject to (5). Denoting the associated Hamiltonian by $\mathcal{L}^R = pR - q_S uR$, where q_S is the shadow value of the resource stock, the first-order conditions yield $p = q_S u$ and thus the augmented Hotelling rule

$$\frac{\dot{p}(t)}{p(t)} = r(t) + \frac{\dot{u}(t)}{u(t)}. \tag{6}$$

Differently from the basic Hotelling rule – according to which the price of a non-renewable resource should persistently grow over time, given a positive rate of profit discount – the augmented rule (6) implies that technological progress can induce a negative price trend over a (possibly, very long) time interval: when extraction costs fall at a rate that more than compensates for the rate of interest, $-\dot{u}/u > r$, the resource price falls, $\dot{p} < 0$. A distinctive feature of our model is that technological progress in resource extraction is endogenously determined by learning by doing.

Let K_R be the knowledge accumulated by the extraction industry, which the firm takes as given, and assume that the extraction cost is inversely related to knowledge according to $u = K_R^{-1}$. Knowledge accumulates over time as extraction activity proceeds according to $\dot{K}_R = \zeta uR$, where $\zeta > 0$ is a constant parameter that reflects learning. Denoting initial knowlegde by $\kappa = K_R(0)$, the knowledge stock at time t equals

$$K_R(t) = \kappa + \zeta \int_0^t u(s) R(s) \, ds, \tag{7}$$

and the associated growth rate of extraction costs $u = K_R^{-1}$ is

$$\frac{\dot{u}(t)}{u(t)} = -\frac{\zeta u(t) R(t)}{\kappa + \zeta \int_0^t u(s) R(s) \, ds} = -\frac{\zeta R(t)}{\left[\kappa + \zeta (S_0 - S(t))\right]^2}, \tag{8}$$

where $S_0 - S(t)$ represents cumulated extraction, the difference between the initial stock and current reserves at time t.[9] Expression (8) summarizes the supply-side forces driving the evolution of the extraction cost: the rate of decay of u is a function of current resource use as well as of cumulated extraction.

2.3 The intermediate sector

The configuration of the intermediate sector changes across regimes. In the *pre-industrial regime*, the economy starts with N_0 intermediate goods markets, each one with infinitely many atomistic producers operating a technology with constant returns to scale (henceforth, CRS) that requires $\omega > 1$ units of the final good per unit produced. Because of the competitive market structure, in the pre-industrial regime the price of each intermediate good is ω, firms make no profits and thus agents have no incentives to innovate.

[9] The central term of (8) follows from time-differentiating (7) and using the definition $\dot{u}/u = -\dot{K}_R/K_R$. Integration of (4) between time zero and t yields $S(t) = S_0 - \int_0^t u(s) R(s) \, dt$. Substituting this result in the central term of (8) yields the last term of the expression.

In the *industrial regime*, instead, agents engage in costly innovations whereby the technology features increasing returns to scale (henceforth, IRS). This characterization captures the idea that incentives to innovation in the pre-industrial regime are weaker than in the industrial regime, so much so that the pre-industrial regime cannot deliver sustainable growth while the industrial regime can. Still, our analysis fully recognizes that industrialization is an endogenous process and, at the same time, an equilibrium outcome that should not be taken for granted: we will determine explicit conditions under which the pre-industrial economy makes, or does not make, a successful transition to modern growth.

In general, whether the economy enters the industrial regime depends on whether innovations are feasible and profitable, that is, firms can cover the costs of R&D and obtain a rate of return that matches the interest rate making the firm willing to invest. A more specific aspect is that our model includes two different types of innovations: R&D activities may be directed at creating new intermediate goods ('horizontal innovations') or at improving the quality of existing intermediate goods ('vertical innovations'). While both types of innovations are possible, their operativeness is determined by separate conditions so that we may observe post-industrial equilibria where only one type of innovations takes place. Therefore, *different post-industrialization regimes* can arise, and as we will show, industrialization per se *does not* guarantee sustained growth in the long run. The specific characteristics of horizontal and vertical innovations in our model are described below.

2.3.1 Vertical innovations: quality improvements

The typical intermediate firm develops and applies specialized knowledge. We model quality as

$$Q_i = Z_i^{\alpha} Z^{1-\alpha}, \quad 0 < \alpha < 1. \tag{9}$$

In words, the contribution of good i to factor productivity downstream depends on the knowledge of firm i, Z_i, and on average knowledge, $Z = \int_0^N (Z_j/N)\, dj$. Moreover, the firm's own production requires one unit of final output per unit of intermediate good and a fixed operating cost $\phi Z_i^{\alpha} Z^{1-\alpha}$, also in units of final output. The firm accumulates knowledge according to the technology

$$\dot{Z}_i = I_i, \tag{10}$$

where I_i is R&D in units of final output. Using (2), the firm's gross profit is

$$\Pi_i = \left[(P_i - 1) \left(\frac{\theta}{P_i} \right)^{\frac{1}{1-\theta}} Q_i L_i^{\gamma} R^{1-\gamma} - \phi \right] Z_i^{\alpha} Z^{1-\alpha}. \tag{11}$$

The firm chooses the time path of its price, P_i, and R&D, I_i, to maximize

$$V_i(0) = \int_0^{\infty} e^{-\int_0^t r(s)ds} \left[\Pi_i(t) - I_i(t) \right] dt \tag{12}$$

subject to (10) and (11), where r is the interest rate and 0 is the point in time when the firm makes decisions. The firm takes average knowledge, Z, in (11) as given. The firm's decision yields

$$r = r_{Zi} \equiv \alpha \frac{\Pi_i}{Z_i}, \tag{13}$$

which says that the return to in-house innovation delivered by the firm must equal the interest rate for the firm to be willing to invest. See the Appendix for the derivation and details on the interpretation.

2.3.2 Horizontal innovations: entry

Creation of a new firm requires βX units of final output, where $X = \int_0^N (X_i/N)\, di$ is average intermediate output. Because of this sunk cost, the new firm cannot supply an existing good in Bertrand competition with the incumbent monopolist but must introduce a new good that expands product variety. New firms enter at the average knowledge level and finance entry by issuing equity. Entry is positive if the value of the new firm, denoted i without loss of generality, is equal to its setup cost, i.e., if the free-entry condition $V_i = \beta X$ holds. Taking logs and time derivatives of the free-entry condition and of the value of the firm in (12) yields

$$r = r_{Ni} \equiv \frac{\Pi_i - I_i}{\beta X} + \frac{\dot{X}}{X}. \tag{14}$$

This equation says that the return to creating a new firm must equal the interest rate for agents to be willing to undertake the project.

2.4 Households

Demography and endogenous reproduction choices play a pivotal role in our model. We model consumption and fertility decisions as in Peretto and Valente (2015). Adult population evolves according to

$$\dot{M} = B - \delta M, \qquad \delta > 0 \tag{15}$$

where M is the mass of adults, B is the mass of children and δ is the exogenous death rate. This law of motion postulates that childhood lasts for one instant, after which children become adult workers. Children consume the same homogeneous good as adults but do not work. Consumption and reproduction decisions are endogenous and reflect the intertemporal choices of a single representative household who maximizes

$$U = \int_0^\infty e^{-\rho t} \left[\mu \log \left(C_M(t)\, M(t)^\eta\right) + (1-\mu) \log \left(C_B(t)\, B(t)^\eta\right)\right] dt, \quad 0 < \mu, \eta < 1, \tag{16}$$

where $\rho > 0$ is the discount rate, C_M is consumption per adult, M is the mass of adults, C_B is consumption per child, B is the mass of children. In this structure, the decision maker cares about utility of adults and utility of children with weights μ and $1-\mu$. Adults and children derive utility from their individual consumption and from the masses of adults and children, with parameter η regulating the trade-off between consumption per adult (child) and the mass of adults (children).[10]

[10]See Peretto and Valente (2015) and Barro and Sala-i-Martin (2004, pp.408-412) for a detailed discussion of this preference structure.

Each adult is endowed with one unit of labor. Since children do not work, the household supplies inelastically M units of labor and faces the flow budget constraint

$$\dot{A} = rA + wM + \Pi_R - C_M M - C_B B, \tag{17}$$

where A is assets holding, r is the rate of return on assets, and $\Pi_R = pR$ are profits of resource-extracting firms distributed to households as dividends. Both types of consumption expenditures, for adults and for children, are part of the decision problem: the household maximizes utility (16) subject to (15) and (17) using C_M, C_B and B as control variables.

3 The economy's general equilibrium

In this section we characterize the behavior of the household. We then impose general equilibrium conditions and characterize how market interactions determine the dynamics of resource supply and use. Finally, we characterize how these dynamics drive the evolution of the economy.

3.1 Household behavior

The following exposition focuses on intuition, see the appendix for the detailed derivation. Let $C \equiv C_M M + C_B B$ be total household consumption. Combining the first-order conditions for consumption per adult, C_M, and consumption per child, C_B, and denoting by v_A the dynamic multiplier attached to the wealth constraint (17), we obtain

$$C = C_M M + C_B B = 1/v_A \tag{18}$$

which says that consumption equals the inverse of the shadow value of financial wealth at any point in time. Denote the ratio of consumption to final output (henceforth, *consumption ratio*) as $c \equiv C/Y$.[11] Manipulation of the first-order conditions for consumption and financial wealth, A, yields

$$r = \rho + \frac{\dot{C}}{C} = \rho + \frac{\dot{c}}{c} + \frac{\dot{Y}}{Y}. \tag{19}$$

The first-order conditions for fertility, B, financial wealth, A, and adult population, M, yield

$$\frac{(1-\mu)\,\eta}{B} + v_M = v_A C_B, \tag{20}$$

$$\frac{\frac{\mu\eta}{M} + v_A\,(w - C_M)}{v_M} + \frac{\dot{v}_M}{v_M} - \delta = \rho, \tag{21}$$

where v_M is the dynamic multiplier attached to the demographic law (15). Equation (20) says that the household equates the marginal benefit of a child to the marginal cost. The former is the child's contribution to current utility, the term, $(1-\mu)\,\eta/B$, plus his shadow value as a future working

[11] The empirical counterpart of c is not the consumption-to-GDP ratio – that is, one minus the saving rate – because Y is not GDP in our model (see section 3.3). Nonetheless, the consumption ratio $c \equiv C/Y$ plays the same role as the average propensity to consume in the characterization of the consumption-saving path.

adult, v_M. The marginal cost is the child's consumption, C_B, evaluated at the marginal cost of spending on consumption, the term v_A. Equation (21) says that the household views fertility as investment in an asset, a working adult, that pays a stream of dividends in the future. Along the utility-maximizing path, the household equates the return generated by this asset to the discount rate, ρ. The return has a dividend-price ratio component and a capital gain-loss component. The former, consists of the contribution of the adult to current utility, the term $\mu\eta/M$, plus his net contribution to financial wealth accumulation, the term $v_A(w - C_M)$.

Denoting births per adult as $b \equiv B/M$, conditions (20)-(21) collapse to

$$\frac{\dot{b}}{b} = \left[\frac{\gamma(1-\theta)}{c(1-\eta)} - 1\right]\frac{b}{1-\mu} - \rho. \tag{22}$$

This simple expression describes the dynamics of the *crude birth rate* along the utility-maximizing path.[12] Equation (22) is conceptually analogous to Euler conditions for consumption growth: births per adult grow (decline) over time if the private 'rate of return' to fertility – i.e., the first term in the right hand side – exceeds (falls short of) the utility discount rate. Such rate of return to fertility depends positively on the labor share of final production, $\gamma(1-\theta)$, and negatively on the consumption ratio, c.

3.2 Resource extraction

The resource market clears when the flow supplied by the resource firm equals demand, i.e., when $R = (1-\gamma)(1-\theta)Y/p$. Time-differentiating this expression and using the Hotelling rule (6) yields

$$\frac{\dot{R}}{R} = \frac{\dot{Y}}{Y} - \frac{\dot{p}}{p} = \frac{\dot{Y}}{Y} - r - \frac{\dot{u}}{u} = -\left(\frac{\dot{c}}{c} + \rho\right) - \frac{\dot{u}}{u}. \tag{23}$$

Expression (23) shows that resource use R may increase over time during periods in which the rate of technological progress in extraction, $-\dot{u}/u > 0$, is sufficiently large in magnitude to compensate for the desired consumption rate of return. From (23), we can characterize optimal resource depletion in terms of both extraction flows and 'reserves' – i.e., the residual stock of natural resource – as follows:

Lemma 1 *Along the optimal depletion path, resource extraction equals*

$$u(t)R(t) = \frac{e^{-\varepsilon(t)t}}{\int_0^\infty e^{-\varepsilon(t)t}dt} \cdot S_0 \tag{24}$$

and the resource stock equals

$$S(t) = S_0 \cdot \left[1 - \frac{\int_0^t e^{-\varepsilon(s)s}ds}{\int_0^\infty e^{-\varepsilon(t)t}dt}\right], \tag{25}$$

where $\varepsilon(t) \equiv \frac{1}{t}\int_0^t [\dot{c}(s)/c(s) + \rho]\,ds$.

[12] The crude birth rate, often indicated as CBR in demography studies, is the empirical counterpart of $b \equiv B/M$ in our model.

Result (24) shows that the intial extraction flow, $u(0)\,R(0)$, is proportional to the endowment S_0 and thereafter the path of the extraction flow is pinned down by the path of the growth rate of the consumption ratio c. In other words, the market interaction between forward-looking households and forward-looking extraction firms make the economy behaves *as if* there were a single agent with three forms of wealth at their disposal – financial, human, and natural wealth – who chooses the extraction path jointly with the consumption-saving path. Such agent takes into account that in order to sustain faster consumption growth he needs to extract more aggressively. Result (25) confirms that available reserves $S(t)$ converge to zero as $t \to \infty$, that is, optimal depletion requires the natural resource stock to be exhausted asymptotically.

3.3 Productivity, GDP and sustained growth

Labor market clearing yields $\int_0^N L_i di = M$. The equilibrium of the intermediate sector is symmetric because firms make identical decisions: each firm i sells $X_i = X$ units of intermediate good at price $P_i = P$ to the final sector. Using the demand schedule (2) to eliminate X from the final sector's technology (1), we obtain the reduced-form production function

$$Y = \left(\frac{\theta}{P}\right)^{\frac{\theta}{1-\theta}} N^{1-\gamma} Z M^{\gamma} R^{1-\gamma}, \quad P = \begin{cases} \omega & \text{pre-industrial} \\ \min\left\{\omega, \frac{1}{\theta}\right\} & \text{industrial} \end{cases} \tag{26}$$

which emphasizes that the equilibrium price P depends on the intermediate technology in use. In the pre-industrial state, the economy has N_0 intermediate-good markets, each populated by a representative competitive firm that exploits the CRS technology and sells the good at price $P = \omega$. If the adoption of the industrial technology were not a drastic innovation, the monopolization of an intermediate-good market would occur at the limit price $P = \omega$. In our analysis, the industrial technology is a drastic process innovation that suddenly reduces the price of intermediates to $P = 1/\theta < \omega$.[13] In other words, we characterize industrialization as a large-scale, one-shot, process-innovation event: when entrepreneurs find profitable to pay the sunk entry cost to set up new firms, they adopt the IRS technology and wipe out the representative firms in the existing N_0 markets, establishing themselves as uncontested local monopolists.[14] After this change in the market structure of the intermediate sector, the new firms behave as described in section 2.3.

We can further distinguish the two regimes in terms of gross domestic product (GDP) per worker. Denote by $x_i \equiv X_i/Z_i$ the quality-adjusted size of intermediate firm i (henceforth, *firm size*). Since the final producer pays $PX = \theta Y/N$ to each intermediate producer, the definition of x and the reduced-form production function (26) yield

$$x = \frac{X}{Z} = \frac{\theta}{P}\frac{Y}{NZ} = \left(\frac{\theta}{P}\right)^{\frac{1}{1-\theta}} \cdot \left(\frac{M}{N}\right)^{\gamma} R^{1-\gamma}. \tag{27}$$

[13] This assumption also accounts for the fact that in the industrial regime firms eventually engage in quality innovation and thus can put the competitive fringe out of business due to their falling quality-adjusted price.

[14] This feature keeps the analysis simple because it avoids the co-existence of two classes of markets. Technically, it requires that at the point in time when it becomes profitable to enter with the IRS technology the economy supports a flow of entrants at least as large as N_0.

In the industrial regime, firm size x directly affects the efficiency of intermediate firms. In fact, combining (26) with (27) and using the definition of intermediate profits (11) for the industrial regime, we can express *GDP per worker* as

$$\frac{G}{M}=\begin{cases} \left(\frac{\theta}{P}\right)^{\frac{\theta}{1-\theta}}\left(1-\frac{\theta}{P}\right)N_0^{1-\gamma}Z_0\cdot\left(\frac{R}{M}\right)^{1-\gamma} & \text{pre-industrial} \\ \left(\frac{\theta}{P}\right)^{\frac{\theta}{1-\theta}}\left[1-\left(\frac{\theta}{P}\right)\left(1+\frac{\phi}{x}\right)\right]N^{1-\gamma}Z\cdot\left(\frac{R}{M}\right)^{1-\gamma} & \text{industrial} \end{cases} \tag{28}$$

where $G\equiv Y-PNX$ is the economy's gross domestic product. Expression (28) shows that GDP per worker depends on *total factor productivity* and on *resource use per worker*, R/M. The TFP component depends on static efficiency in intermediate production – which, in the industrial regime, increases with firm size x – and on technology – which enhances productivity via expansions of product variety, N, and improvements in product quality, Z. In the pre-industrial regime, firm size does not yield static efficiency gains and intermediate firms do not innovate so that product variety and product quality remain at their initial levels, N_0 and Z_0. In the industrial regime, instead, TFP may grow as a result of static efficiency gains from firm size, horizontal innovations that increase N, and vertical innovations that raise Z.

The central message of (28) is that industrialization fundamentally changes prospects for sustainability – i.e., the economy's ability to generate non-declining levels of GDP per worker over time. Since resource extraction follows the optimal depletion path (24), there is relentless downward pressure on resource use per worker, R/M, and such downward pressure becomes more intense with population growth. Consequently, *the pre-industrial regime cannot sustain GDP per worker in the long run.* Sustainability requires a successful transition to the industrial regime. This conclusion is formally restated in the next Lemma.

Lemma 2 *(Interest rate and GDP per worker growth) Denote the vertical innovation rate by $z\equiv\dot{Z}/Z$, the horizontal innovation rate by $n\equiv\dot{N}/N$, the growth rate of adult population by $m\equiv\dot{M}/M$, and the growth rate of GDP per worker by $g\equiv\dot{G}/G-m$. Let*

$$\xi(x)\equiv\frac{\theta^2\phi/x}{1-\theta^2(1+\phi/x)}$$

be the elasticity of GDP with respect to firm size. At any point in time, the interest rate and the growth rate of GDP per worker are, respectively:

$$r=\rho+(1-\gamma)n+z+\gamma m-(1-\gamma)\left(\rho+\frac{\dot{u}}{u}\right); \tag{29}$$

$$g=\underbrace{\xi(x)\frac{\dot{x}}{x}+(1-\gamma)n+z}_{TFP\ growth}-\underbrace{(1-\gamma)\left(m+\rho+\frac{\dot{u}}{u}\right)}_{growth\ drag}. \tag{30}$$

In the pre-industrial regime, these expression hold with $n=z=\xi(x)=0$.

Proof. See the Appendix. ■

Expression (30) shows that the growth rate of GDP per worker is the growth rate of total factor productivity minus the *growth drag* due to the use of the non-renewable natural resource. The drag is equal to the elasticity of final output with respect to the resource, $1-\gamma$, times the sum of the population growth rate, m, and the rate of exhaustion of the resource, $\rho+\dot{u}/u$. In the pre-industrial regime with no innovation, the growth drag dominates and the economy's prospects look seriously bleak. In the industrial regime, in contrast, endogenous technological change may sustain GDP per worker – potentially, even in the long run.

According to Lemma 2, whether firms' innovation decisions support positive growth in output per capita depends on whether the resulting TFP growth rate is larger than the growth drag; this is the classic condition for sustainability derived by Stiglitz (1974; see also Brock and Taylor 2005), with the difference that (i) TFP growth in our model is endogenous and not necessarily positive and (ii) fertility is endogenous. Note that TFP growth is not necessarily positive because from the perspective of the firm innovation entails a sunk cost that is economically justified only when the anticipated revenue flow is sufficiently large. The anticipated revenue flow, in turn, depends on the anticipated size of the market. There is thus a positive feedback running from the household's fertility behavior to the firms' decisions to invest in innovation. Such feedback is the key to the transition from the pre-industrial to the industrial regime, as we show below.

3.4 Market structure and firm size

We now develop the main building blocks for the characterization of the economy's equilibrium dynamics. We first look at the returns to innovation and the evolution of market structure in the intermediate sector.

Lemma 3 *(Rates of return and firm size dynamics) Denote the innovation rates $z \equiv \dot{Z}/Z = I/Z$ and $n \equiv \dot{N}/N$, and the growth rate of adult population $m \equiv \dot{M}/M$. Using the definition of firm size, x, in (27), the returns to innovation in quality improvement (13) and in variety creation (14) become:*

$$r = \alpha\left[\left(\frac{1}{\theta}-1\right)x-\phi\right] \equiv r^Z; \tag{31}$$

$$r = \frac{1}{\beta}\left(\frac{1}{\theta}-1-\frac{\phi+z}{x}\right)+\frac{\dot{x}}{x}+z \equiv r^N. \tag{32}$$

Firm size obeys the differential equation

$$\frac{\dot{x}}{x} = \underbrace{\gamma m-(1-\gamma)\left(\frac{\dot{c}}{c}+\rho+\frac{\dot{u}}{u}\right)}_{\textit{market growth}} - \underbrace{\gamma n}_{\textit{market fragmentation}}. \tag{33}$$

Proof. See the Appendix. ■

Expressions (31) and (32) show that decisions to invest in quality and in variety innovation depend on firm size, x. The evolution of firm size, in turn, depends on market growth and market fragmentation: in (33), the term $\gamma m - (1-\gamma)\left(\frac{\dot{c}}{c} + \rho + \frac{\dot{u}}{u}\right)$ shows that the size of the market for intermediate goods depends positively on adult population growth and negatively on resource depletion per worker, whereas the last term, γn, captures how product proliferation fragments the market for intermediate goods in smaller sub-markets and thus reduces the profitability of the individual firm.

In the pre-industrial regime, we have $n = 0$ so that firm size x grows as long as the *market growth* factor $\gamma m - (1-\gamma)\left(\frac{\dot{c}}{c} + \rho + \frac{\dot{u}}{u}\right)$ is positive. This is a necessary condition for industrialization. The specifics of the mechanism are as follows. The non-negativity constraint on in-house investment, $\dot{Z} = I \geq 0$, implies that there is a threshold of firm size, x_Z, below which incumbents do not invest because the return is too low. The value of the threshold x_Z depends on whether entry is positive or zero because active entrants compete with incumbents for the household's saving pool. Similarly, the non-negativity constraint on entry, $\dot{N} \geq 0$, implies that there is a threshold of firm, x_N, size below which entry is zero because the return is too low. The value of the threshold x_N depends on whether entrants anticipate that in the equilibrium with $I > 0$ or with $I = 0$ since in-house investment affects the net cash flow that they expect to earn as dividend. We focus on the case where the threshold for entry is smaller than the threshold for in-house investment, $x_N < x_Z$. This implies that, starting from the pre-industrial regime where $x < x_N$, positive growth in firm size will activate entry in the first place: as long as $x_N < x < x_Z$, variety expansion is the only source of TFP growth. If firm size dynamics subsequently imply $x_N < x_Z < x$, both horizonal and vertical innovations will be operative. Since the threshold x_N is the value of firm size that triggers *industrialization*, our model admits different types of post-industrialization phases, i.e., with or without vertical innovations being operative alongside variety expansion.

3.5 The dynamic system: demography and firm size

Despite the seeming complexity of the model, we can reduce the system characterizing the economy's dynamics to just two dimensions by exploiting the following

Lemma 4 *(Consumption ratio) The pre-industrial regime holds for* $0 < x \leq x_N$ *and the industrial regime holds for* $x > x_N$*. The consumption behavior of the household in the two regimes is*

$$c = \begin{cases} 1 - \theta/\omega & 0 < x \leq x_N \\ 1 - \theta + \rho\beta\theta^2 & x > x_N \end{cases}. \tag{34}$$

Proof. See the Appendix. ■

Lemma 4 shows that industrialization brings about a change in the consumption ratio with respect to the pre-industrial regime, reflecting the response of private expenditures to endogenous TFP growth. By substituting result (34) in equations (22) and (33) we obtain

Lemma 5 *(Equilibrium dynamical system) Given the growth rate of the extraction cost* $\dot{u}/u$*, the equilibrium dynamics of the economy consist of two subsystems of differential equations in* (x, b) *space describing two regimes.*

- *Pre-industrial regime:*
$$\frac{\dot{b}}{b} = \left[\frac{\gamma(1-\theta)}{(1-\eta)(1-\theta/\omega)} - 1\right]\frac{b}{1-\mu} - \rho;$$
$$\frac{\dot{x}}{x} = \gamma(b-\delta) - (1-\gamma)\left(\rho + \frac{\dot{u}}{u}\right).$$

- *Industrial regime:*
$$\frac{\dot{b}}{b} = \left[\frac{\gamma(1-\theta)}{(1-\eta)(1-\theta+\rho\beta\theta^2)} - 1\right]\frac{b}{1-\mu} - \rho;$$
$$\frac{\dot{x}}{x} = \gamma m^* - (1-\gamma)\left(\rho + \frac{\dot{u}}{u}\right) - \gamma n(x).$$

In the second equation the function $n(x)$ is the equilibrum rate of entry.

Proof. See the Appendix. ■

In the industrial regime the unstable differential equation for b does not depend on x and, consequently, b jumps to its own steady-state value b^* and determines $m^* = b^* - \delta$ at all times. In particular, we shall see that in the phase diagram in (x, b) space, the $\dot{b} = 0$ locus is the saddle path that gives the unique equilibrium trajectory. In principle, similar dynamics apply in the pre-industrial regime. Indeed, if the industrial regime were not possible this is exactly what would happen: because the unstable differential equation for b does not depend on x, b would jump to its own steady-state value, $\bar{b}$, and determine $\bar{m} = \bar{b} - \delta$ at all times. In the phase diagram in (x, b) space the $\dot{b} = 0$ locus would be the saddle path giving the unique equilibrium trajectory. However, the presence of the industrial regime implies that different dynamics take place because forward-looking agents take into account the possibility of moving to that regime.

To summarize, the two key questions are whether the economy moves to the industrial regime and whether the industrial regime delivers sustainable growth. The following characterization of innovation behavior, that exploits the features of fertility and extraction behavior just established, aids in answering these questions.

Lemma 6 *(Innovation rates) Let x_N denote the threshold of firm size that triggers entry (i.e., industrialization) and x_Z the threshold of firm size that triggers in-house innovation. Assume*

$$\frac{\rho\beta\alpha\phi}{\frac{1}{\theta} - 1 - \rho\beta} < \gamma(m^* + \rho),$$

where $m^* = b^* - \delta$ is the constant, endogenous, growth rate of population in the industrial regime. Then, the ordering of the thresholds is $x_N < x_Z$, where:

$$x_N = \frac{\phi}{\frac{1}{\theta} - 1 - \rho\beta};$$

$$x_Z = \arg\text{solve}\left\{\left[\left(\frac{1}{\theta} - 1\right)x - \phi\right]\left(\alpha - \frac{1-\gamma}{\beta x}\right) < \gamma(m^* + \rho) - (1-\gamma)\left(\rho + \frac{\dot{u}}{u}\right)\right\}.$$

Assume also $\beta x > 1-\gamma\ \forall x > \phi$, i.e., $\beta\phi > 1-\gamma$. Then, for $x > x_N$ the equilibrium rates of entry and in-house innovation are:

$$n(x) = \begin{cases} \frac{1}{\beta}\left(\frac{1}{\theta} - 1 - \frac{\phi}{x}\right) - \rho & x_N < x \leq x_Z \\ \frac{(1-\alpha)\left[\left(\frac{1}{\theta}-1\right)x-\phi\right]-\rho\beta x+\gamma(m^*+\rho)-(1-\gamma)\left(\rho+\frac{\dot{u}}{u}\right)}{\beta x-(1-\gamma)} & x > x_Z \end{cases}; \tag{35}$$

$$z(x) = \begin{cases} 0 & x_N < x \leq x_Z \\ \frac{\left(\alpha-\frac{1-\gamma}{\beta x}\right)\left[\left(\frac{1}{\theta}-1\right)x-\phi\right]-\gamma(\rho+m)+(1-\gamma)\left(\rho+\frac{\dot{u}}{u}\right)}{1-\frac{1-\gamma}{\beta x}} & x > x_Z \end{cases}. \tag{36}$$

Proof. See the Appendix. ■

We now have all the ingredients needed to study the model's dynamics. The equilibrium path in (x, b) space is a trajectory $(x(t), b(t))$ that: (i) starts from the initial condition

$$x_0 = \theta^{\frac{2}{1-\theta}}\left(\frac{M_0}{N_0}\right)^{\gamma}\left(\rho\frac{S_0}{u_0}\right)^{1-\gamma} < x_N;$$

(ii) satisfies the piece-wise differential equations characterized in Lemma 5; (iii) satisfies the transversality conditions of the agents' (household and firms) intertemporal problems. The next two sections show how the dynamic system can generate equilibrium paths that under alternative parameter constellations feature radically different outcomes in terms of long-run sustainability.

4 Path to sustainable growth and paths to collapse

In this section, we temporarily ignore equation (8) and treat $\dot{u}/u$ as an exogenous constant to discuss three scenarios produced by the model: *path to sustainability*, where the economy makes the full transition to sustainable growth; *failure to industrialize*, where the economy remains trapped in a downward spiral of no innovation, resource exhaustion and falling population; *fragile industralization*, where the economy turns on the engine of innovation only partially and converges to a steady state in which population grows but income per capita falls over time. With respect to Peretto (2020), the present model delivers analytical solutions for all the variables in each growth regime, which allows us to extend the analysis to include endogenous technological change in extraction in Section 5.

4.1 Path to sustainability

The path to sustainability consists of three phases: (i) an initial phase where agents build up the economy by exploiting nature to support population growth; (ii) an intermediate phase where agents turn on the Schumpeterian engine of endogenous innovation in response to population-led market expansion; (iii) a terminal phase where growth becomes fully driven by knowledge accumulation and no longer requires growth of physical inputs. Let $\dot{u}/u$ be an exogenous constant. With this restriction the resource input, R, follows an exponential process with constant rate of change $\rho + \dot{u}/u$, i.e., $R(t) = (\rho S_0/u_0)\, e^{-(\rho+\dot{u}/u)t}$.[15] The following proposition states the main result formally, Figure 1 illustrates the dynamics.

Proposition 7 *(Pathway to sustainability) Assume:*

$$b^* \equiv \frac{\rho(1-\mu)}{\frac{\gamma}{1-\eta}\frac{1-\theta}{1-\theta+\rho\beta\theta^2} - 1} > \bar{b} \equiv \frac{\rho(1-\mu)}{\frac{\gamma}{1-\eta}\frac{1-\theta}{1-\frac{\theta}{\omega}} - 1} \geq \tilde{b} \equiv \delta + \frac{1-\gamma}{\gamma}\left(\rho + \frac{\dot{u}}{u}\right); \tag{C1}$$

$$\gamma(b^* - \delta + \rho) - (1-\gamma)\rho - \frac{\gamma}{\beta}\left(\frac{1}{\theta} - 1 - \frac{\phi}{x_Z}\right) > 0; \tag{C2}$$

$$\frac{1-\alpha}{(b^* - \delta + \rho) - \frac{1-\gamma}{\gamma}\left(\rho + \frac{\dot{u}}{u}\right)} > \frac{\beta}{\frac{1}{\theta} - 1} > \frac{1}{\phi}; \tag{C3}$$

$$\frac{\alpha\phi - \left(\frac{1}{\theta} - 1\right)\frac{1}{\beta} + \left[(b^* - \delta + \rho) - \frac{1-\gamma}{\gamma}\left(\rho + \frac{\dot{u}}{u}\right)\right]}{(1-\alpha)\left(\frac{1}{\theta} - 1\right)\frac{1}{\beta} - \left[(b^* - \delta + \rho) - \frac{1-\gamma}{\gamma}\left(\rho + \frac{\dot{u}}{u}\right)\right]} > \frac{\frac{1-\gamma}{\gamma}\left(\rho + \frac{\dot{u}}{u}\right)}{(b^* - \delta + \rho) - \frac{1-\gamma}{\gamma}\left(\rho + \frac{\dot{u}}{u}\right)}. \tag{C4}$$

Then, there is a unique equilibrium path: given initial condition x_0, the economy jumps on the saddle path

$$b(x) = \begin{cases} \arg\text{solve}\left\{ \frac{x}{x_N} = \left(\frac{b-\bar{b}}{b^*-b}\right)^{\frac{\bar{b}-\tilde{b}}{\rho}} \left(\frac{b}{b^*}\right)^{\frac{\tilde{b}}{\rho}} \right\} & 0 < x \leq x_N \\ b^* & x > x_N \end{cases} \tag{37}$$

and converges to (x^*, b^*), where, noting that $m^* = b^* - \delta$:

$$x^* = \frac{(1-\alpha)\phi - (m^* + \rho) + \frac{1-\gamma}{\gamma}\left(\rho + \frac{\dot{u}}{u}\right)}{(1-\alpha)\left(\frac{1}{\theta} - 1\right) - \left[(m^* + \rho) - \frac{1-\gamma}{\gamma}\left(\rho + \frac{\dot{u}}{u}\right)\right]\beta} > x_Z; \tag{38}$$

$$z^* = \frac{\alpha\phi - \left(\frac{1}{\theta} - 1\right)\frac{1}{\beta} + \left[(m^* + \rho) - \frac{1-\gamma}{\gamma}\left(\rho + \frac{\dot{u}}{u}\right)\right]}{(1-\alpha)\left(\frac{1}{\theta} - 1\right)\frac{1}{\beta} - \left[(m^* + \rho) - \frac{1-\gamma}{\gamma}\left(\rho + \frac{\dot{u}}{u}\right)\right]}\left[(m^* + \rho) - \frac{1-\gamma}{\gamma}\left(\rho + \frac{\dot{u}}{u}\right)\right] > 0; \tag{39}$$

$$g^* = z^* - \frac{1-\gamma}{\gamma}\left(\rho + \frac{\dot{u}}{u}\right) > 0; \tag{40}$$

$$n^* = m^* - \frac{1-\gamma}{\gamma}\left(\rho + \frac{\dot{u}}{u}\right). \tag{41}$$

Proof. See the Appendix. ■

[15] Setting $\dot{u}/u = 0$, as in Peretto (2020), casts the economy in the harshest possible conditions, namely, the supply of the resource is always falling. An interesting property of the model's extension studied in Section 5 is that endogenous technological change in extraction can deliver $\rho + \dot{u}/u < 0$ for a finite period of time, while realistic assumptions on the process imply that eventually $\dot{u}/u \to 0$. Accordingly, one can say that Peretto (2020) studies the asymptotic limit of the economy considered here.

As the proposition states, the model is simple enough that we can solve it analytically. In the pre-industrial regime, the saddle path is the solution of the PDE problem

$$\frac{\dot{b}}{\dot{x}} = \frac{db}{dx} \Rightarrow \frac{db\left(x\right)}{dx} = \frac{b}{x} \cdot \frac{\left(\frac{\gamma}{1-\eta}\frac{1-\theta}{1-\frac{\theta}{\omega}} - 1\right)\frac{b}{1-\mu} - \rho}{\gamma\left(b-\delta\right) - \left(1-\gamma\right)\left(\rho + \frac{\dot{u}}{u}\right)},$$

with boundary conditions $x = x_0$ and $b\left(x_N\right) = b^*$. The solution is the top line in equation (37), which holds for $x \in (0, x_N]$ and $b > \bar{b}$ (see the Appendix for the details). It identifies the path in the figure connecting point $\left(0, \bar{b}\right)$ to point $\left(x_N, b^*\right)$ where the industrial regime takes hold. The kink reflects the fact that this event entails a change in the production technology and in household income. For $x \in \left(x_N, \infty\right)$ the saddle path is the flat line $b = b^*$.[16] The value of fertility $\tilde{b}$ that yields $\dot{x} = 0$ has particular meaning because it is the value below which the market growth factor in the pre-industrial regime is negative and the economy moves away from the industrial regime. The dynamics thus are: given initial condition x_0, the economy selects initial fertility b_0, the square in the figure, and rides the saddle path until convergence to the steady state $\left(x^*, b^*\right)$, the star in the figure. The conditions appearing in Proposition 7 have the following meaning. Condition C1 in Proposition 7 is the restriction on the parameters that ensures that the economy crosses the threshold x_N and activates endogenous TFP growth by innovation. When that happens, the law of motion of firm size becomes

$$\frac{\dot{x}}{x} = \begin{cases} \gamma\left(m^* + \rho\right) - \left(1-\gamma\right)\left(\rho + \frac{\dot{u}}{u}\right) - \frac{\gamma}{\beta}\left(\frac{1}{\theta} - 1 - \frac{\phi}{x}\right) & x_N < x \leq x_Z \\ \gamma m^* - \left(1-\gamma\right)\rho - \gamma\frac{(1-\alpha)\left[\left(\frac{1}{\theta}-1\right)x - \phi\right] - \rho\beta x + \gamma\left(m^*+\rho\right) - \left(1-\gamma\right)\left(\rho + \frac{\dot{u}}{u}\right)}{\beta x - (1-\gamma)} & x > x_Z \end{cases}. \tag{42}$$

Condition C2 says that the parameters are such that firm size is strictly increasing in the range $[x_N, x_Z]$ so that the economy crosses the threshold x_Z and turns on quality innovation. Thereafter the economy follows the non-linear differential equation in the second line of (42) and converges to x^*. Condition C3 ensures that this value exists because both the numerator and the denominator of (38) are positive. The model yields an analytical solution of the dynamics in the industrial regime as well. The details are in the appendix. The economy converges to the growth rate g^* in equation (40). The associated sustainability condition is condition C4, which holds for $m^* \in \left(m^*_{\min}, m^*_{\max}\right)$ with $m^*_{\min} < 0$ and $m^*_{\max} > 0$. This interval includes 0 and says that sustainabile growth is feasible for sufficiently *slow* population growth, be it positive or negative.

In the equilibrium path of Figure 1, growing births per adult in the pre-industrial regime and growing firms size through the whole transition allow the economy to achieve positive horizontal and vertical innovation rates and thereby sustained growth in the long run. This success story hinges on two key conditions, namely, the "take-off" condition C1 and the "full transition" condition C2. When the take-off condition $b^* > \bar{b} > \tilde{b}$ fails to hold, the economy may be unable to reach industrialization and two potential scenarios of collapse arise depending on the combination of

[16] Point $\left(0, \bar{b}\right)$ is in principle a steady state but is not an equilibrium because it violates the resource constraint (positive fertility with zero output is not possible).

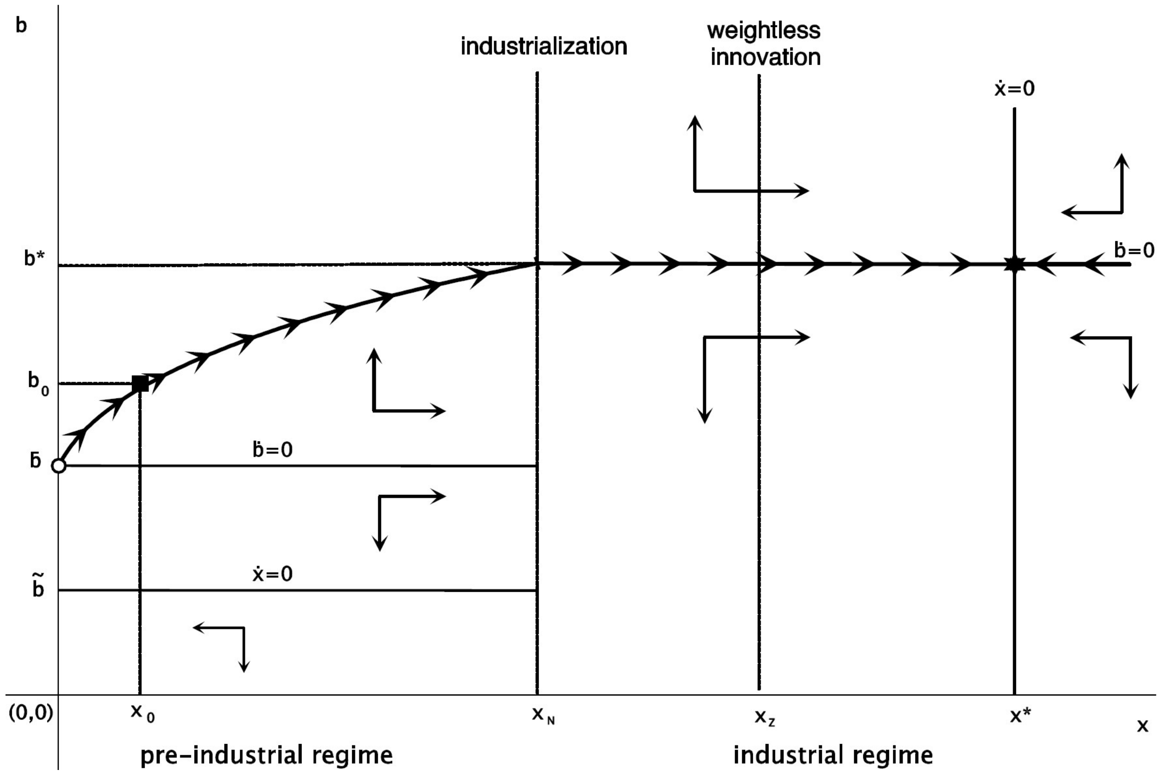

Figure 1: Path to sustainability.

parameters. We discuss these cases of *failure to industrialize* in the next subsection. A second possible obstacle to achieving sustainable growth is that the take-off condition is satisfied but condition C2 is not. In this case, the economy activates horizontal innovations but, despite the positive TFP growth yielded by net entry of firms, firm size x stops growing before reaching the threshold x_Z and vertical innovations remain inoperative because firms do not invest in quality improvements. We discuss this scenario of fragile industrialization in subsection 4.3.

4.2 Failure to industrialize: collapse scenarios

The path to industrialization described in Proposition 7 hinges on the take-off condition $b^* > \bar{b} > \tilde{b}$ which comprises two joint restrictions on critical levels of the fertility rate. The first inequality, $b^* > \bar{b}$, follows from Lemma 4 and imposes that in the industrial regime the consumption-output ratio, $1-\theta+\beta\theta^2$, is larger than in the pre-industrial regime, $1-\theta/\omega$. The second inequality, $\bar{b} > \tilde{b}$, guarantees that in the pre-industrial regime the saddle path is monotonically increasing so that a successful transition to the industrial regime features growing births per adult over time.[17] When this second

[17]The restriction $\bar{b} > \tilde{b}$ says that in the pre-industrial regime the value of the fertility rate that yields $\dot{b} = 0$ is larger than the value that yields $\dot{x} = 0$. If the industrial regime did not exist, the economy would jump on the flat saddle path featuring constant population growth $\bar{b} - \delta$ and constant firm size growth (it is immediate to see that $\bar{b} > \tilde{b}$ implies $\bar{b} > \delta$).

restriction fails to hold, the economy may be unable to reach industrialization. In particular, if the initial resource endowment is not sufficiently large, the economy undertakes a doomsday scenario where resource exhaustion drags down population and market size until economic collapse. The following proposition states the result formally.

Proposition 8 *(Failure to industrialize) Assume*

$$\bar{b} \equiv \frac{\rho(1-\mu)}{\frac{\gamma}{1-\eta}\frac{1-\theta}{1-\theta/\omega}-1} < \tilde{b} \equiv \delta + \frac{1-\gamma}{\gamma}\left(\rho + \frac{\dot{u}}{u}\right).$$

Two cases are possible:

1. *For $\tilde{b} < b^*$ there exists a threshold value $\tilde{x} \in (0, x_N)$ such that if the economy has a sufficiently large endowment S_0, so that $x_0 = \rho S_0/u_0 > \tilde{x}$, it chooses $b_0 > \bar{b}$ on the saddle path that converges to x^*. If the economy has an insufficient endowment S_0, so that $x_0 = \rho S_0/u_0 \leq \tilde{x}$, it must choose $b_0 = \bar{b}$ and is thus doomed to collapse because $\dot{x} < 0$.*

2. *For $\tilde{b} \geq b^*$ the threshold value is $\tilde{x} = x_N$ and the economy cannot make the transition to the industrial regime from the pre-industrial regime. Either it starts with endowment such that $x_0 = \rho S_0/u_0 > x_N$ or it is doomed to collapse.*

Proof. See the Appendix. ■

Figure 2 illustrates case 1 in Proposition 8. The saddle path converging to (x_N, b^*) is non-monotonic.[18] There exists a threshold value $\tilde{x}$ such that for $x_0 > \tilde{x}$, the economy may reach industrialization: growing births per adult initially reduce firm size over time, but as the gross fertility rate crosses the threshold $\tilde{b}$, firm size starts growing and eventually drives the economy to the industrialization phase. However, for $x_0 < \tilde{x}$ the economy cannot make the transition to the industrial regime and must ride the saddle path leading to economic collapse. This saddle path is the $\dot{b} = 0$ locus itself, i.e., the line $b = \bar{b}$. On this line the economy experiences falling market size, and thus falling firm size, and converges asymptotically (i.e., it does not get there in finite time) to point $(0, \bar{b})$. Notably, moreover, this path *does not* require falling population; it says that population may be rising (i.e., $\bar{b} > \delta$) but it does not compensate for resource exhaustion. Recall that $\tilde{b}$ is the value of the birth rate that delivers $\dot{x} = 0$, that is, the value that delivers population growth that exactly offsets resource exhaustion so that the size of the market, and thus firm size, remains constant.

Figure 3 illustrates case 2 in Proposition 8. It occurs for $\tilde{b} \geq b^*$, which implies that for all $x \in [0, x_N]$ the $\dot{x} = 0$ locus associated to the pre-industrial regime is above the $\dot{b} = 0$ locus associated to the industrial regime. Accordingly, there is no path connecting an initial condition $x_0 < x_N$ to

[18] Given the non-linearity in b, it is easier to think of equation (37) as a function $x(b)$ in (b, x) space. The function is U-shaped. For $\bar{b} > \tilde{b}$ it intersects the horizontal axis twice, at 0 and at $\bar{b}$. The non-negativity constraint on x then says that $x = 0$ for $b \in [0, \bar{b}]$. For $\bar{b} < \tilde{b}$, instead, the function does not intersect the horizontal axis, has a vertical asymptote at $\bar{b}$ and has a minimum at $\tilde{b}$. Rotating the axes gives the representation in (x, b) space in Figures 1-2.

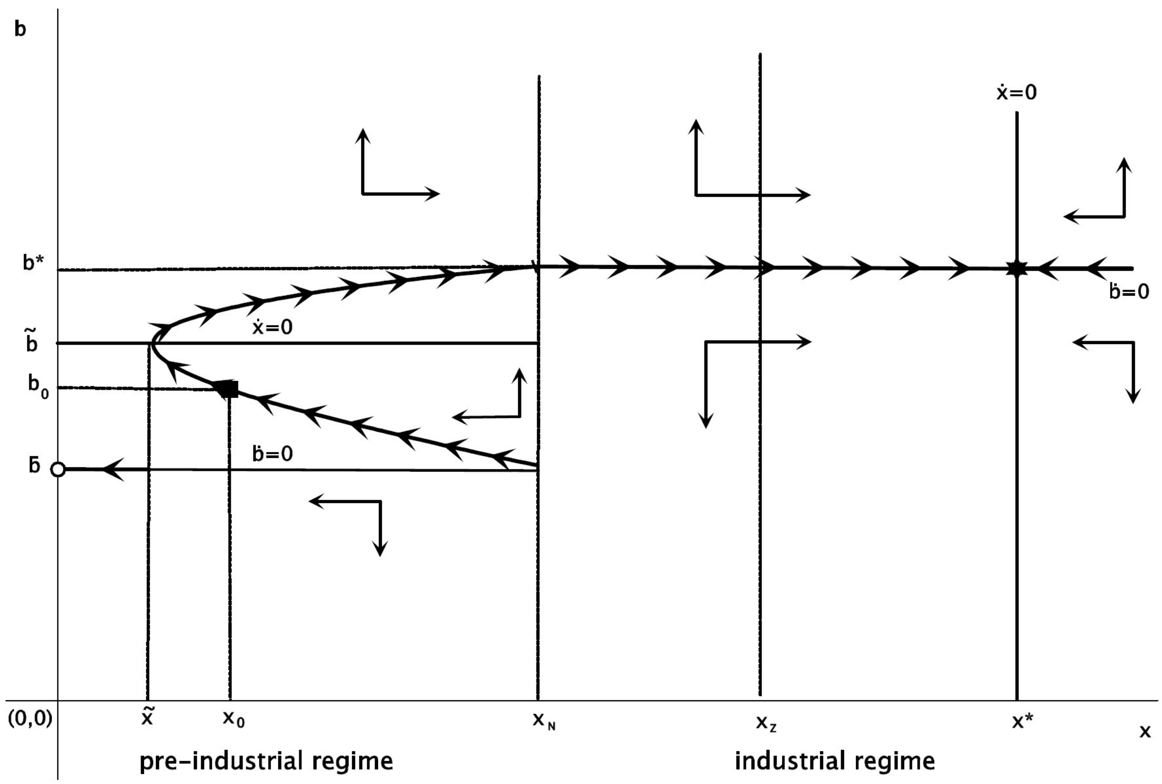

Figure 2: Failure to industrialize, case 1 in Proposition 8: with $\bar{b} < \tilde{b} < b^{*}$, the transition to industrialization requires $x_0 > \tilde{x}$, whereas $x_0 < \tilde{x}$ leads to collapse.

the saddle path of the industrial regime. The difference between cases 1 and 2 is that in the former scenario the economy may achieve industrialization starting from the pre-industrial phase, whereas avoiding collapse in case 2 requires experiencing industrialization since the beginning. But in both cases, the key to escape the doomsday scenario is the initial level of firm size, a main determinant of which is *resource abundance*: the economy avoids collapse only if the initial endowment of natural resource S_0 is sufficiently large to make the initial firm size x_0 greater than the relevant doomsday thresholds (respectively, $\tilde{x}$ in case 1 and x_N in case 2).

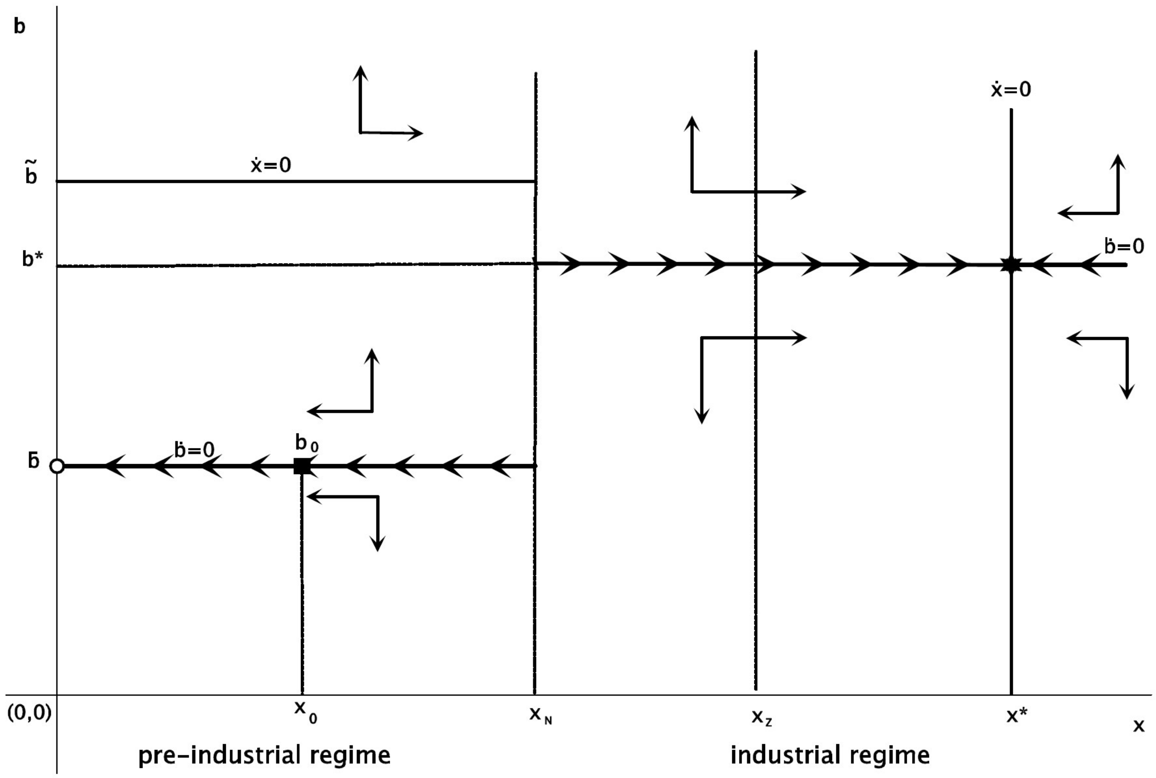

Figure 3: Failure to industrialize, case 2 in Proposition 8: with $\bar{b} < b^* \leq \tilde{b}$, avoiding collapse requires experiencing the industrialization phase since the beginning, $x_0 > x_N$. Any pre-industrial scenario $x_0 < x_N$ leads to collapse.

4.3 Fragile industrialization and economic collapse

Recalling the assumptions of Proposition 7, suppose that the "take-off" condition C1 is satisfied but the "full transition" condition C2 is not. When condition C2 does not hold, the economy fails to cross the threshold x_Z and converges to the steady state $(\bar{x}^*, b^*)$ where the growth rate of GDP per worker is

$$\bar{g}^* = -\frac{1-\gamma}{\gamma}\left(m^* + \rho + \frac{\dot{u}}{u}\right), \quad m^* = b^* - \delta > 0, \quad \rho + \frac{\dot{u}}{u} > 0. \tag{43}$$

The growth rate $\bar{g}^*$ is *negative* because the economy does not possess sufficiently strong social returns to variety that compensate for resource exhaustion. To see the mechanism in detail, recall the definition of firm size, which yields $x = \theta^{\frac{2}{1-\theta}} \left(\frac{M}{N}\right)^{\gamma} R^{1-\gamma}$, and rewrite the expression for GDP per worker in (28) as

$$\frac{G}{M} = \theta^{\frac{2\theta}{1-\theta}} \left[1 - \theta^2 \left(1 + \frac{\phi}{x}\right)\right] Z \cdot \left(\frac{x}{\theta^{\frac{2}{1-\theta}}}\right)^{1-\gamma} \cdot \left(\frac{M}{N}\right)^{-(1+\gamma)(1-\gamma)}. \tag{44}$$

In the steady state that we are considering, both Z and x are constant and the dynamics of GDP per worker hinge on the dynamics of the term $(M/N)^{-1+\gamma^2}$. The restriction $\dot{x} = 0$ says that the ratio M/N must grow in steady state to offset resource exhaustion and preserve the interest rate (equivalently, incentives to invest). Therefore, output per worker shrinks.[19]

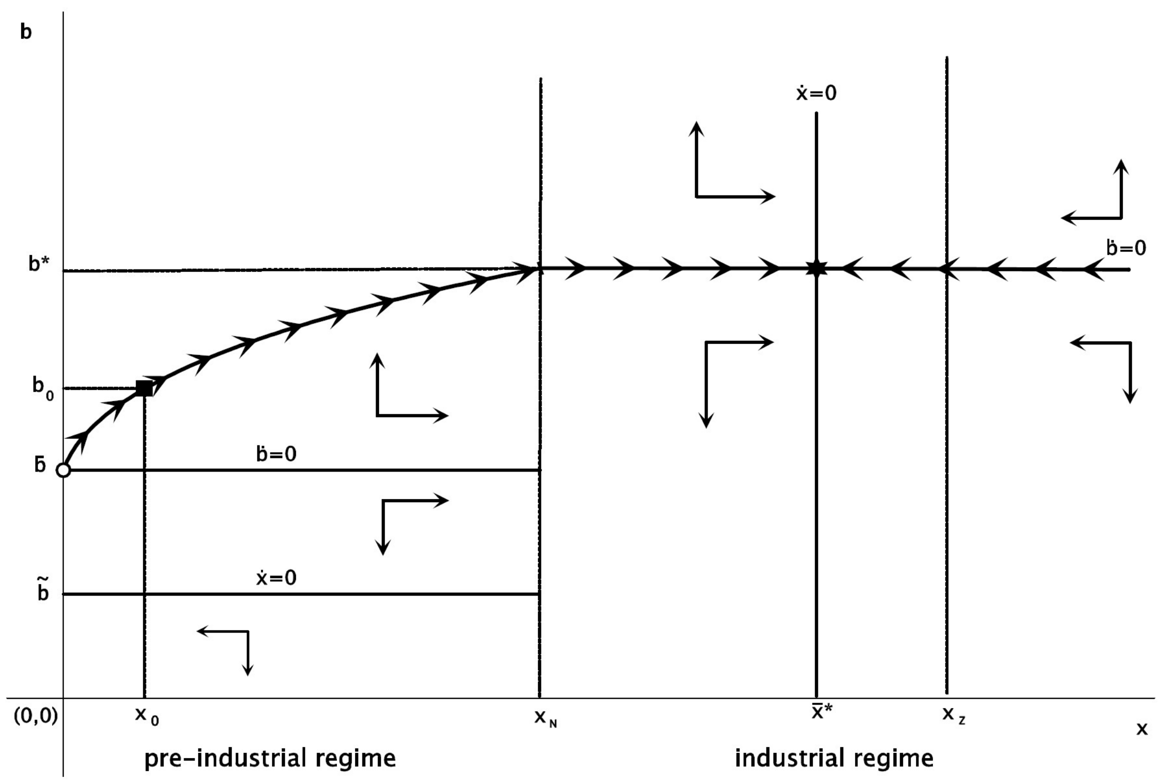

Figure 4: Fragile industrialization.

Figure 4 illustrates the scenario of fragile industrialization, which we may interpret as one of "economic collapse": the society settles into a steady state with growing population and falling living standards. This notion of long-run equilibrium is not only questionable empirically, it is also fragile analytically. In such steady state, the individual consumption of both adults and children falls at rate $\bar{g}^*$. If we then assume a subsistence level, C^{sub}, similar to that popularized

[19]The only way out of this outcome is to endow the model with an *additional* love-of-variety effect, on top of the already existing one due to the assumption that the resource is non-rival in final production, the term, $N^{1-\gamma}$. For an example of this approach and a discussion of its limitations, see Peretto (2020).

by Unified Growth Theory (see, e.g., Galor 2011), this steady state is fragile in the sense that it cannot survive extensions that incorporate realistic features that are left out of the model only to simplify the exposition. More precisely, under fragile industrialization, a persistently falling output per worker eventually becomes low enough that the subsistence constraint $C_M \geqslant C^{\text{sub}}$ becomes binding, the household is knocked off its first-order conditions and thus the birth rate obeys a different equation saying that the birth rate falls over time, eventually falling short of the death rate; consequently, the economy experiences falling population and falling market size, exactly as in the 'failure to industrialize' secenario previously discussed. We formally show in the Appendix that introducing a subsistence level of consumption in the scenario of fragile industrialization prevents the economy from settling into a stable steady state with positive and finite population: such steady state would in principle be feasible but the economy cannot reach it because the exhaustion of the resource exerts relentless downward pressure on living standards, a force that in turn pushes the birth rate below the death rate and keeps it there. Pursuing this extension further would yield a richer treatment of the global dynamics produced by this class of models but is beyond the scope of the paper. The point that matters to the present discussion is twofold. First, any path of this economy different from the 'path to sustainability' yields eventual collapse under realistic features – a conclusion that encompasses the cases of failure to industrialize and fragile industrialization. Second, the case of *fragile industralization* brings the additional insight that even though the economy initially experiences innovation, this is not a self-sustaining process and eventually must succumb to the ever-worsening scarcity of an essential factor of production.

5 Endogenous technological change in extraction

Our previous discussion of growth regimes (section 4) treated $\dot{u}/u$ as an exogenous constant. This section extends general equilibrium dynamics to include endogenous technological change in extraction, which brings about important results for the qualitative behaviour of resource use and resource price before and after industrialization. The industrialization process itself is affected since endogenous technological change partially offsets the resource drag (by partially compensating for the *exhaustion* of the resource) and interacts with population dynamics, giving rise to phases where growing resource use and falling resource prices are secular trends directly connected to demography and to endogenous TFP growth in industry. The model yields full analytical solutions for resource use and extraction costs (subsect.5.1), the pre-industrial regime (subsect.5.2) and innovation rates (subsect.5.3) that deliver clear and empirically relevant predictions on such secular trends (subsection).

5.1 Resource use and extraction cost

Recall equation (23), which describes the instantaneous market-clearing equilibrium of the resource market. Together with the law of motion of the stock, that condition yields the dynamics of resource use and resource stock,

$$\dot{R} = -R\left(\rho - \frac{\zeta R}{\left[\kappa + \zeta\left(S_0 - S\right)\right]^2}\right) \quad \text{and} \quad \dot{S} = -R, \tag{45}$$

which is a self-contained system in the jumping variable R and the state variable S that delivers well-defined saddle-path stable dynamics. Importantly, system (45) yields a solvable PDE problem:

Proposition 9 *(Extraction cost dynamics) The supply-demand interaction in the spot market for the extracted resource yields a unique equilibrium path: given initial condition S_0, the economy jumps on the saddle path*

$$R\left(S\right) = \frac{\rho}{\zeta}\left[\left(\zeta - 1\right)S_0 + S\right]\left(\kappa + S_0 - S\right)$$

and as $t \to \infty$ converges to $R\left(\infty\right) = 0$ and $S\left(\infty\right) = 0$. The resulting path of the extraction flow is

$$R\left(t\right) = \rho S_0 e^{-\rho t}\left[\kappa + \left(1 - e^{-\rho t}\right)\zeta S_0\right]. \tag{46}$$

The associated path of the extraction cost is

$$u\left(t\right) = \frac{1}{K_R\left(t\right)} = \frac{1}{\kappa + \left(1 - e^{-\rho t}\right)\zeta S_0}. \tag{47}$$

Proof. See the Appendix. ■

Proposition 9 highlights a relevant property: contrary to the basic Hotelling model of exhaustible resources, the path of resource extraction is generally hump-shaped. More precisely, time-differentiating (46) yields

$$\dot{R}\left(t\right) = \left(2e^{-\rho t}\zeta S_0 - \kappa - \zeta S_0\right) \cdot \rho S_0 \rho e^{-\rho t}$$

which for $1 > \kappa/\zeta S_0$ admits a 'peak time', a finite positive instant t_R^H before (after) which resource extraction grows (declines):

$$\dot{R}\left(t\right) \gtreqless 0 \quad \text{for} \quad t \lesseqgtr \frac{1}{\rho}\log\left(\frac{2}{1 + \kappa/\zeta S_0}\right) \equiv t_R^H. \tag{48}$$

For a sufficiently large initial stock of the resource, $S_0 > \kappa/\zeta$, the path displays a *rising* extraction flow over the interval of time $0 \leq t < t_R^H$, turns around at time $t = t_R^H$ and thereafter displays the usual falling extraction flow that converges asymptotically to zero. The peak-extraction date t_R^H is larger the smaller the ratio $\kappa/\zeta S_0$, that is, the smaller the ratio of initial knowledge κ to the knowledge accumulable over time by fully extracting the stock, ζS_0. Equivalently, we can say that the date is larger the stronger the learning potential (technological opportunity) of the extraction industry. Also, because it is proportional to $1/\rho$, it is easy to calibrate. For example, with the typical annual $\rho = 0.02$ we get $1/\rho = 200$. Letting κ be relatively very small so that we can approximate $\kappa/\zeta S_0 \simeq 0$, we get $t_R^H = 200\log 2 = 138.69$. The intuition for this result is that technological change in extraction slows down effective resource exhaustion and may imply a positive of $\dot{R}/R$ for long periods. This result is consistent with the vast empirical literature documenting the existence of hump-shaped time paths in extracted units of non-renewable resources.

The impact of technological change in extraction costs, however, is not limited to the primary sector because the time path of u also affects firms' incentives to develop productivity-enhancing innovations and thereby the rise of different growth regimes in the economy. To gain further insight from this mechanism, note that from (47) the path of the extraction cost is logistic with representation[20]

$$\frac{\dot{u}}{u} = \rho\left(1 - \frac{u}{u^*}\right), \quad u_0 = \frac{1}{\kappa} > u^* = \frac{1}{\kappa + \zeta S_0} > 1. \tag{49}$$

By combining (49) with the laws of motion of births per adult b, and firm size x, we can fully characterize general equilibrium dynamics and obtain analytical solutions for the equilibrum path of all the variables of interest. For the sake of clarity, the next subsections study the consequences of (49) for the pre-industrial and the industrial regimes separately.

5.2 Pre-industrial regime

Combining (49) with the equations derived in Lemma 5 yields the system for the pre-idustrial regime

$$\begin{cases} \frac{\dot{b}}{b} = \left(\frac{\gamma}{1-\eta}\frac{1-\theta}{1-\frac{\theta}{\omega}} - 1\right)\frac{b}{1-\mu} - \rho \\ \frac{\dot{x}}{x} = \gamma\left(b - \delta\right) - \left(1-\gamma\right)\rho\left(2 - \frac{u}{u^*}\right) \end{cases} \tag{50}$$

One difference from the case studied in section 4 is the presence of the time-varying extraction cost, which declines over time and slows down effective exhaustion. Consequently, existence of the pathway to sustainability under no technological change in extraction *implies* its existence under technological change in extraction. Technically, the most important change with respect to the previous case is that the system is now three-dimensional. However, because we added a stable differential equation for the pre-determined state variable u that operates as a forcing process on the subsystem in (x, b), we preserve the property that the economy selects the unique path that starting from (x_0, u_0, b_0) connects to the point (x_N, u_N, b^*) that constitutes the initial condition of the stable dynamics in (x, u) space that characterize the industrial regime. We can write this part of the solution as a trajectory $b_{CRS}(x, u)$. Moreover, to connect sharply this exercise to the analysis of Section 4, we posit that the value of u_0 used there is the value $1/\kappa$ obtained here from the microfoundation of the dynamics of the extraction cost. A property that aids in developing intuition is that the threshold x_N is independent of u. Thus, we can visualize the dynamics of the pre-industrial regime in a phase diagram in (x, b) space where the role of the forcing process $u(t)$ is to shift the relevant loci. On inspection, the only locus that shifts is the $\dot{x} = 0$ locus, which reads

$$b_{\dot{x}=0}(u) = \delta + \frac{1-\gamma}{\gamma}\rho\left(2 - \frac{u}{u^*}\right)$$

[20]By suitable choice of units, we can impose $\kappa + \zeta S_0 < 1$, which guarantees that the extraction cost, u, is always larger than one. It is worth stressing that this representation is a *result*, not an assumption: see the proof of Proposition 9 for the details.

and rises over time as u falls. This dynamics creates a *window of opportunity* for the economy, i.e., how long it takes for the $b_{\dot{x}=0}(u)$ locus to rise to two critical values. The first is the value $\bar{b}$ that marks the $\dot{b}=0$ locus. When this event occurs, we have the first collapse scenario illustrated in Figure 2. As the $b_{\dot{x}=0}(u)$ locus keeps rising, eventually it reaches the value b^*. When this happens, the window closes and we have the second collapse scenario illustrated in Figure 3. We can compute these times using equation (47) in Proposition 9. Starting from u_0, the windows are

$$\delta+\frac{1-\gamma}{\gamma}\rho\left(2-\frac{u(t)}{u^*}\right)=b\Rightarrow t^{\text{window}}=\frac{1}{\rho}\log\left(\frac{1}{1+\frac{\kappa}{\zeta S_0}}\frac{2-\frac{b-\delta}{(1-\gamma)\rho/\gamma}}{1-\frac{b-\delta}{(1-\gamma)\rho/\gamma}}\right)$$

for $b=\bar{b}$ and $b=b^*$. These dates are finite if the argument of the log is positive. To interpret this expression it is useful to recall that $\kappa/\zeta S_0$ is the ratio of initial knowledge, κ, to the knowledge accumulable over time by fully extracting the stock, ζS_0.

An agnostic mini-calibration. We set $\rho=2\%$ and $\kappa/\zeta S_0\simeq 0$. Then we set $\bar{b}-\delta=0.2\%$ and $b^*-\delta=1\%$, to match roughly the world averages for the the pre-industrial and industrial eras. We set $\gamma=0.77$ as in, e.g., Iacopetta and Peretto (2018). We thus obtain $\frac{\bar{b}-\delta}{(1-\gamma)\rho/\gamma}=\frac{0.2}{0.597}=0.335$ and $\frac{b^*-\delta}{(1-\gamma)\rho/\gamma}=\frac{1}{0.597}=1.675$. Therefore, we have $t_1^{\text{window}}\simeq 79.72$ and $t_2^{\text{window}}\to\infty$. The second result says that realistic values of population growth in the industrial era imply that the $b_{\dot{x}=0}(u)$ locus never reaches b^* and thus that the window for industrialization never closes, i.e., the second case in Proposition 8 never occurs. The first result says that realistic values of population growth in the pre-industrial era imply that the $b_{\dot{x}=0}(u)$ reaches $\bar{b}$ and creates a set of initial values x_0 that doom the economy to collapse in accord with the first case in Proposition 8.

A mini-calibration based on Andre' and Smulders (2014). We interpret the natural resource as the key input in the production of energy (e.g., fossil fuels). The energy share of GDP in the USA is 0.035. We take $\theta=0.77$ from Iacopetta and Peretto (2018) and $G/Y=0.5$. We thus have

$$0.035=\frac{(1-\gamma)(1-\theta)Y}{G}\Rightarrow\gamma=0.923,$$

which yields

$$\frac{(1-\gamma)\rho}{\gamma}=0.166\%.$$

This says that population growth must be faster than 0.166% to deliver industrialization. Then if we set $\bar{b}-\delta=0.2\%$ at the pre-industrial historical average from Galor (2011), we get $\frac{\bar{b}-\delta}{(1-\gamma)\rho/\gamma}=\frac{0.2}{0.10973}=1.204$ and $\frac{b^*-\delta}{(1-\gamma)\rho/\gamma}=\frac{1}{0.10973}=6.024$. Thus, both dates are infinite and the window of industrialization remains open all the time.

The key message of these mini-calibrations is that whatever the source and the precision, estimates of population growth in the industrial era are always above the threshold 0.166% and thus yield that the window remains open. Similarly, if we restrict the pre-industrial regime to 1500-1800, then estimates of population growth are above the threshold 0.166% and thus yield that the *full* window of opportunity remains open. In other words, in this extension with endogenous technological change in extraction the economy surely makes the transition to the industrial regime. One can note that these estimates, in fact, say that condition C1 of Proposition 7 holds. As argued, existence of the path to sustainability under no technological change in extraction implies its existence under technological change in extraction.

5.3 Industrial regime

Recalling result (49), the general equilibrium dynamics for the industrial regime are represented by a 3×3 system

$$\begin{cases} \frac{\dot{b}}{b} = \left[\frac{\gamma(1-\theta)}{(1-\eta)(1-\theta+\rho\beta\theta^2)} - 1\right]\frac{b}{1-\mu} - \rho \\ \frac{\dot{x}}{x} = \gamma m^* - (1-\gamma)\left(\rho + \frac{\dot{u}}{u}\right) - \gamma n(x) \\ \frac{\dot{u}}{u} = \rho\left(1 - \frac{u}{u^*}\right) \end{cases} \tag{51}$$

where the first two equations follow from Lemma 5. Once the economy reaches the industrialization stage, prospects for sustainability crucially depend on whether vertical innovations – i.e., firms developing quality improvements – become profitable an remain operative in the long run.[21] In section 4, the joint dynamics of firm size and births per adult triggered industrialization ($n > 0$) and quality improvements ($z > 0$) according to 'boundary regions' that only made reference to threshold levels of firm size, namely, x_N and x_Z. With endogenous extraction costs, the boundary for industrialization, $n > 0$, is unchanged, $\frac{1}{\beta}\left(\frac{1}{\theta} - 1 - \frac{\phi}{x}\right) > \rho \Rightarrow x > x_N$. The boundary of the region where $z > 0$, instead, is affected by the dynamics of the extraction cost: combining (49) with (36), the rate of quality improvement reads

$$z(x,u) = \frac{\left(\alpha - \frac{1-\gamma}{\beta x}\right)\left[\left(\frac{1}{\theta} - 1\right)x - \phi\right] - \gamma(\rho + m^*) + (1-\gamma)\rho\left(2 - \frac{u}{u^*}\right)}{1 - \frac{1-\gamma}{\beta x}}.$$

Therefore, the condition $z > 0$ can be re-expressed as a condition for u to fall short of a critical threshold level $u_Z(x)$ that depends on firm size, that is,

$$z > 0: \quad u < u^*\left[2 + \frac{\left(\alpha - \frac{1-\gamma}{\beta x}\right)\left[\left(\frac{1}{\theta} - 1\right)x - \phi\right] - \gamma(\rho + m^*)}{(1-\gamma)\rho}\right] \equiv u_Z(x). \tag{52}$$

Expression (52) allows us to represent graphically the dynamics of the 3×3 system (51) in two dimensions. For example, Figure 5 projects the dynamics of (u, b, x) in the (u, x) plane, where the locus $u_Z(x)$ defines the boundary for horizontal innovations to be operative, with $z > 0$ when $u < u_Z(x)$. Before explaining this diagram in detail, we state the main analytical results on the industrial regime in the following

[21] Like in section 4, our analysis maintains that the ordering of innovation thresholds is such that firm size levels trigger entry innovation before triggering quality innovation, that is, $x = x_N$ is the boundary for "industrialization" as previously defined.

Lemma 10 *(Innovation rates with endogenous extraction cost) Under the industrialization regime, $x > x_N$, the equilibrium rate of entry innovation is*

$$n = \begin{cases} \frac{\left(\frac{1}{\theta}-1\right)x-\phi}{\beta x} - \rho & \text{for } u \geq u_Z(x) \\ \frac{(1-\alpha)\left[\left(\frac{1}{\theta}-1\right)x-\phi\right]-\rho\beta x+\gamma(m^*+\rho)-(1-\gamma)\rho\left(2-\frac{u}{u^*}\right)}{\beta x-(1-\gamma)} & \text{for } u < u_Z(x) \end{cases},$$

and the growth rate of firm size is

$$\frac{\dot{x}}{x} = \begin{cases} \gamma\left(m^*+\rho-\frac{\left(\frac{1}{\theta}-1\right)x-\phi}{\beta x}\right)-(1-\gamma)\rho\left(2-\frac{u}{u^*}\right) & \text{for } u \geq u_Z(x) \\ \gamma\left(m^*-\frac{(1-\alpha)\left[\left(\frac{1}{\theta}-1\right)x-\phi\right]-\rho\beta x+\gamma(m^*+\rho)}{\beta x-(1-\gamma)}\right)-(1-\gamma)\rho\left(2-\frac{u}{u^*}\right) & \text{for } u < u_Z(x) \end{cases}, \tag{53}$$

with

$$\dot{x} \geq 0: \quad u \geq u^* \cdot \begin{cases} 2+\frac{\gamma}{(1-\gamma)\rho}\left(\frac{\left(\frac{1}{\theta}-1\right)x-\phi}{\beta x}-(m^*+\rho)\right) & \text{for } u \geq u_Z(x) \\ 2+\frac{\gamma}{(1-\gamma)\rho}\left(\frac{(1-\alpha)\left[\left(\frac{1}{\theta}-1\right)x-\phi\right]-\rho\beta x+\gamma(m^*+\rho)}{\beta x-(1-\gamma)}-m^*\right) & \text{for } u < u_Z(x) \end{cases}.$$

Proof. See the Appendix. ■

Figure 5 plots the phase diagram of the dynamic system formed by (49) and (53) in the (u, x) plane, along with the boundary locus for quality innovation $u_Z(x)$. The simultaneous steady state is

$$\dot{x} = \dot{u} = 0: \quad x^* = \frac{(1-\alpha)\phi - \left(m^*+\rho-\frac{1-\gamma}{\gamma}\rho\right)}{(1-\alpha)\left(\frac{1}{\theta}-1\right)-\left(m^*+\rho-\frac{1-\gamma}{\gamma}\rho\right)\beta}. \tag{54}$$

The equilibrium path features extraction cost u declining over time, converging to the steady state u^* from above, and firm size x growing throughout the transition and approaching x^* from below. The economy successfully completes the transition to quality innovation only if the equilibrium path crosses the boundary locus $u_Z(x)$ before reaching the steady state. From (52), the locus $u_Z(x)$ is increasing in x and asymptotically linear. The $\dot{x} = 0$ locus is also increasing, but lies above the $u_Z(x)$ locus when idustrialization starts (i.e., when $x = x_N$) and is asymptotically horizontal as $x \to \infty$, so that an intersection with $u_Z(x)$ generally exists.[22] A successful transition to sustainable growth requires that the two loci intersect to the left of x^*, which guarantees that firm size grows throughout the transition and crosses the $u_Z(x)$ locus before approaching the steady state x^*. This is the case depicted in Figure 5, where the intersection with locus $u_Z(x)$ splits the industrial regime between a first phase where the only source of TFP growth is firms' entry and a second phase where quality innovation becomes operative and guarantees sustained growth in the long run.

[22] See the Appendix for details.

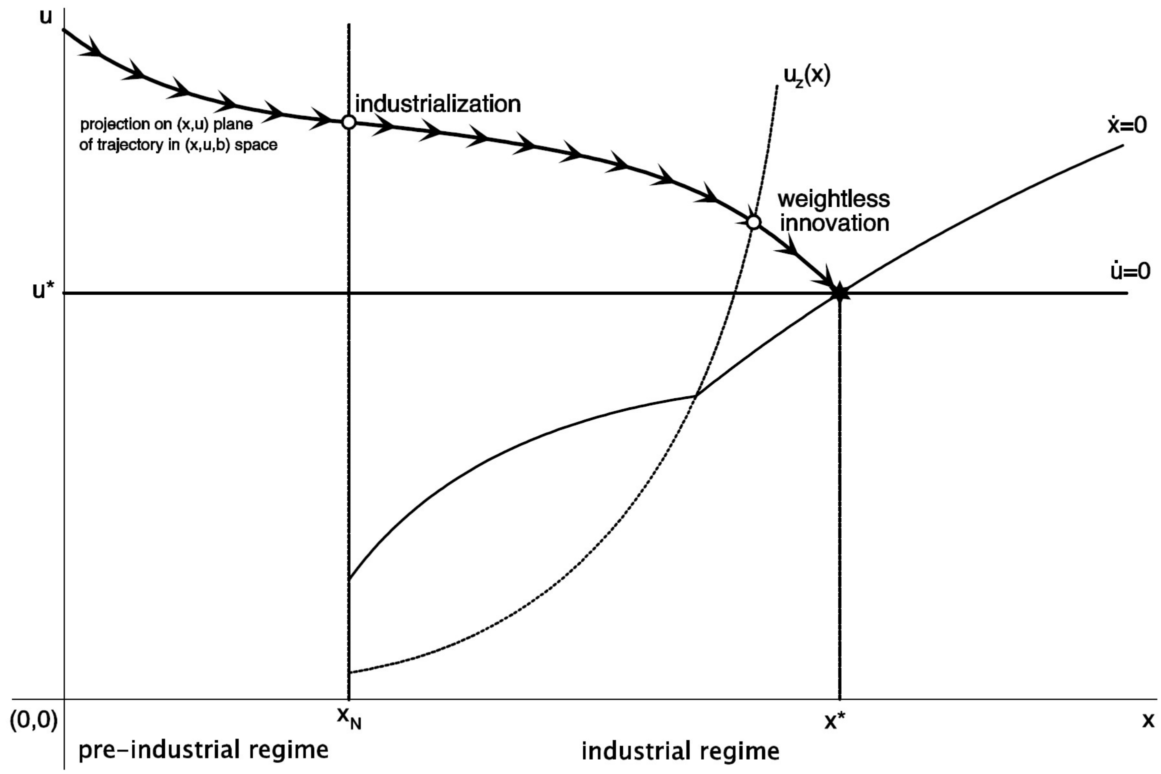

Figure 5: Firm size dynamics and extraction cost with endogenous technological change in the primary sector.

In order to gain a comprehensive view of the entire path across different regimes, suppose that in the *pre*-industrial phase the rate of technological change in extraction is sufficiently fast to deliver positive growth,

$$g = -(1-\gamma)\left[m(x,u) + \rho + \frac{\dot{u}}{u}\right] = (1-\gamma)\left[\rho\left(\frac{u}{u^*} - 2\right) - m(x,u)\right] > 0.$$

This scenario arises when the cost of extraction is initially high, i.e., $u > 2u^*$, so that in the early history of the economy the rate of learning in extraction is fast and the rate of change of the flow of extracted resource supplied to the final producer is positive and rising. This fuels the growth of the economy in aggregate and, if $\rho\left(\frac{u}{u^*} - 2\right) > m(x,u)$ holds, also in per capita terms. Eventually, however, the rate of technological change in extraction cost *must* fall below the rate of population growth, i.e., the inequality above switches direction, and result in falling income per capita. The only force that can prevent this from happening is the activation of the Schumpeterian engine of endogenous innovation in manufacturing, i.e., the sequence of, first, industrialization and, later, quality innovation – or equivalently, "weightless innovation".

Note that extraction costs affect equilibrium innovation rates in different ways. The entry rate $n(x,u)$ is increasing in firms size x but is also increasing in u, so that technological change in extraction costs provides downward pressure on the growth rate of the mass of firms: falling extraction costs tend to counteract the upward pressure that the rising firm size puts on the incentives to develop new products and set up new firms. This implies that the industrialization phase with horizontal innovation only can display a hump-shaped growth rate of output if the effect of extraction costs dominates that of firm size. For the same reason, a scenario with *fragile industrialization* similar to that studied earlier (subsect. 4.3) remains possible in the model with endogenous extraction cost. If and when the economy activates weightless innovation, the upward pressure from the rising firms size becomes much stronger, the function $z(x)$ does not contain u, and the growth rate of income per capita is more likely to accelerate.

5.4 Secular trends

A prominent question in the current agenda of growth economics is to reconcile theoretical models with observed stylized facts on resource use and resource prices, namely, a hump-shaped path of resource extraction and a U-shaped path of resource prices. We have already shown that endogenous technological change in extraction generates a peak in resource use (subsect. 5.1). The behavior of the resource price is not the mirror image of that of extracted quantities because the demand for intermediates is also affected by demography and total factor productivity, which are both endogeneous in our model. In particular, besides obtaining a U-shaped path of resource prices, we demonstrate the existence of intermediate phases during which both resource use and resource prices fall over time.

A distinctive property of our model is that different growth regimes modify the law of motion of the resource price. From (6) and (49), we have

$$\frac{\dot{p}(t)}{p(t)} = \begin{cases} \gamma\left(m(t)+\rho-\frac{e^{-\rho t}\rho\zeta S_0}{\kappa+(1-e^{-\rho t})\zeta S_0}\right) & \text{for } x(t)\le x_N \\ (1-\gamma)\,n(t)+z(t)+\gamma\left(m^*+\rho-\frac{e^{-\rho t}\rho\zeta S_0}{\kappa+(1-e^{-\rho t})\zeta S_0}\right) & \text{for } x(t)>x_N \end{cases} \tag{55}$$

Because the rate of change of the extraction cost is always decreasing, the growth rate of the resource price can start negative and turn positive when $-\dot{u}/u$ falls below $m+\rho$. Assuming $m^*+\rho<\frac{\rho\zeta S_0}{\kappa}$, we obtain such U-shaped time profile because throughout the pre-industrial phase $m(t)\in[m(0),m^*]$. The model thus predicts that there is a first phase with falling resource price whereas the conventional Hotelling result of rising resource prices takes hold in the very long run.

The reversal of the resource price trend, from negative to positive, may occur in any of the three growth regimes – i.e., pre-industrial, post-idustrial with horizontal innovations or with both horizontal and vertical innovations – depending on the underlying parameters. This is relevant for the duration of the first phase and, most importantly, for the *co-movement of resource extraction and resource price within each regime.* If the reversal takes place before the economy reaches industrialization, $x(t)=x_N$, the price keeps falling over the period

$$\frac{\dot{p}(t)}{p(t)}\le 0 \text{ for } 0\le t\le t_p^H\equiv \arg\operatorname{solve}\left\{t=\frac{1}{\rho}\log\left(\frac{2+m(t)/\rho}{1+\kappa/\zeta S_0}\right)\right\} \tag{56}$$

The reasoning is similar for the other regimes. If the price is still falling when the economy crosses the threshold x_N, we have

$$\frac{\dot{p}(t)}{p(t)} \leq 0 \text{ for } 0 \leq t \leq t_p^H \equiv \arg\text{solve}\left\{ t = \frac{1}{\rho} \log \left(\frac{2 + \frac{(1-\gamma)n(t)+\gamma(m^*+\rho)}{\gamma\rho}}{1+\kappa/\zeta S_0} \right) \right\}. \tag{57}$$

Similarly, if the price is still falling when the economy activates vertical innovation, we have

$$\frac{\dot{p}(t)}{p(t)} \leq 0 \text{ for } 0 \leq t \leq t_p^H \equiv \arg\text{solve}\left\{ t = \frac{1}{\rho} \log \left(\frac{2 + \frac{(1-\gamma)n(t)+z(t)+\gamma(m^*+\rho)}{\gamma\rho}}{1+\kappa/\zeta S_0} \right) \right\}. \tag{58}$$

Combining these results with those obtained for resource use – in particular, expression (48) – it follows that the U-shaped price path does not mirror the hump-shaped path of extracted flows, in the sense that the price starts to grow *after* resource use has already reached its peak. Expressions (56)-(58) imply $t_R^H < t_p^H$ in all cases. This means that there exists an intermediate phase lasting $t \in \left(t_R^H, t_p^H\right)$ where $\dot{R} < 0$ while $\dot{p} < 0$, that is, *resource use and prices fall together.*

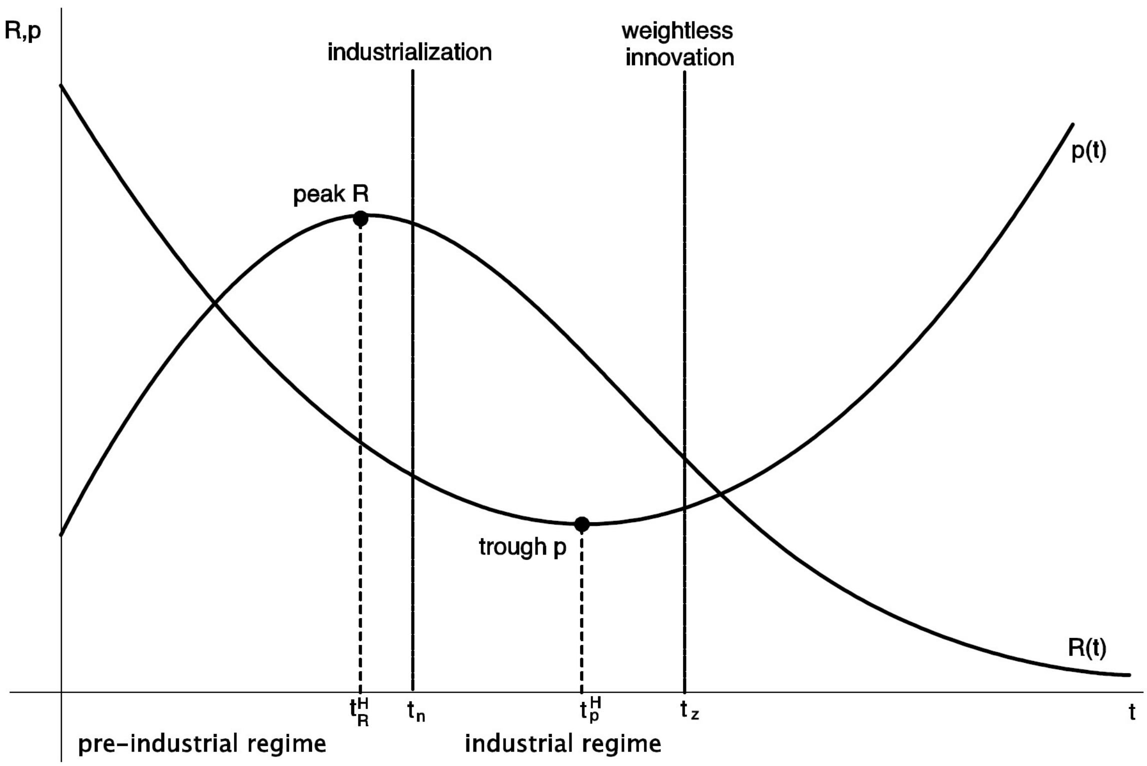

Figure 6: Dynamics of resource use and resource price with endogenous technological change in extraction.

The decoupling of price-quantity dynamics hinges on a distinctive feature of our model: the endogenous interactions between demography and technological change that generate the sequence of growth regimes. In partial-equilibrium models, the decoupling of price-quantity dynamics can be obtained by assuming ad-hoc exogenous processes that simultaneously affect resource demand and extraction. Our framework suggests, instead, that peaks in extraction followed by possibly long phases of falling prices and falling resource use can be secular trends driven by demography and industrialization. Figure 6 depicts the scenario associated with expression (57), that is, the price growth rate reverses its sign in the post-industrial phase but before quality innovation becomes operative. The intermediate phase with falling prices and falling extracted quantities can be quite long before the conventional Hotelling prediction of ever-falling resource use and ever-growing prices is realized in the long run. This conclusion, alongside the more general result that these paths are secular trends driven by demography and industrialization, are of immediate relevance to empirical work, as we argue below.

6 Conclusion

Our paper built a model of industrialization where natural scarcity, endogenous demography and technological innovations generate different stages of industrialization and economic development. The interactions between demography and incentives to innovate may lead the economy towards opposite outcomes, namely, a successful transition to sustained growth in the long run or to collapse scenarios where exhaustible primary inputs and population pressure halt economic development. The distinctive feature of our analysis is that the path to sustainability reproduces two well-documented empirical regularities – a U-shaped path of resource prices and a hump-shaped path of resource extraction – as *secular trends* that arise across growth regimes. The model also predicts long transitional phase during which both extracted quantities and their market price fall.

The nature of these secular trends differentiates our results from both the theoretical and the empirical literature. Theoretical contributions typically obtain the decoupling of price-quantity dynamics in partial equilibrium models by assuming simultaneous exogenous processes that affect resource demand and supply schedules, whereas our model suggests that (i) a U-shaped path of the resource price, (ii) a hump-shaped path of resource extraction, and (iii) an intermediate phase where prices and extracted quantities fall together, directly stem from the interaction between endogenous demography and endogenous incentives to technological innovation as the economy crosses different stages of industrialization. This view is equally absent in the empirical literature, where most contributions confirm the existence of the time-series regularities recognized by earlier contributions (Berck and Roberts, 1994; Pindyck, 1999; Slade and Thille, 2009) but do not connect these findings to general-equilibrium phenomena. Testing whether demography and stages of industrialization are the underlying causes of resource-use and price trends is our main suggestion for applied research.

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A Appendix

To facilitate the reader, all the equations from the text needed for the proofs are replicated in this document with self-contained numbering.

A.1 Derivation of the return to quality

The usual method of obtaining first-order conditions is to write the Hamiltonian for the optimal control problem of the firm. This derivation highlights the intuition. The firm undertakes R&D up to the point where the shadow value of the innovation, q_i, is equal to its cost,

$$1 = q_i \Leftrightarrow I_i > 0.$$

Since the innovation is implemented in-house, its benefits are determined by the marginal profit it generates. Thus, the return to the innovation must satisfy the arbitrage condition

$$r = \frac{\partial \Pi_i}{\partial Z_i}\frac{1}{q_i} + \frac{\dot{q}_i}{q_i} = \frac{\partial \Pi_i}{\partial Z_i}. \tag{A.1}$$

To calculate the marginal profit, observe that the firm's problem is separable in the price and investment decisions. Facing the isoelastic demand

$$X_i = \left(\frac{\theta}{P_i}\right)^{\frac{1}{1-\theta}} Z_i^{\alpha} Z^{1-\alpha} L_i^{\gamma} R^{1-\gamma}$$

and a marginal cost of production equal to one, the firm sets $P_i = 1/\theta$. Substituting this result into the expression for the cash flow,

$$\Pi_i = \left[(P_i - 1)\left(\frac{\theta}{P_i}\right)^{\frac{1}{1-\theta}} L_i^{\gamma} R^{1-\gamma} - \phi\right] Z_i^{\alpha} Z^{1-\alpha}, \tag{A.2}$$

differentiating with respect to Z_i, substituting into (A.1) yields

$$r = \alpha \frac{\Pi_i}{Z_i},$$

which is the expression in the text.

A.2 Household behavior

The household's current value Hamiltonian is

$$\begin{aligned} CVH &= \mu \log C_M + (1-\mu)\log C_B + \mu\eta \log M + (1-\mu)\,\eta \log B \\ &\quad + v_A\left(rA + wM + \Pi_R - C_M M - C_B B\right) + v_M\left(B - \delta M\right), \end{aligned}$$

where the vs denote the shadow value of respectively, financial assets and household size. The first order conditions for the control variables C_M, C_B, and B are:

$$\frac{\mu}{C_M M} = v_A = \frac{1-\mu}{C_B B}; \quad \frac{(1-\mu)\,\eta}{B} + v_M = v_A C_B.$$

The conditions for the state variables A and M are:

$$r+\frac{\dot{v}_A}{v_A}=\rho;$$

$$\frac{\frac{\mu\eta}{M}+v_A\left(w-C_M\right)}{v_M}+\frac{\dot{v}_M}{v_M}-\delta=\rho.$$

Associated to these are the transversality conditions that the value of each state variable times its shadow value converges to zero as $t\to\infty$.

Let $C=C_M M+C_B B$. The conditions for C_M, C_B yield $C=C_M M+C_B B=1/v_A$. Let $c\equiv C/Y$. The result $C=1/v_A$ and the first-order condition for financial wealth A yield the Euler equation for saving

$$r=\rho+\frac{\dot{C}}{C}=\rho+\frac{\dot{c}}{c}+\frac{\dot{Y}}{Y}. \tag{A.3}$$

The result $C=1/\lambda_A$ and the conditions for fertility B, financial wealth A, and adult population size M, yield the fertility rule

$$B=\frac{(1-\mu)(1-\eta)}{v_M} \tag{A.4}$$

and, recalling that $wM=\gamma(1-\theta)Y$, the asset-pricing equation

$$\frac{\frac{\mu\eta}{M}+v_A\left(w-C_M\right)}{v_M}+\frac{\dot{v}_M}{v_M}-\delta=\rho.$$

Using (A.4), we manipulate this expression as follows:

$$\begin{aligned}
\frac{\mu\eta+v_A\left(wM-C_M M\right)}{(1-\mu)(1-\eta)}\frac{B}{M}+\frac{\dot{v}_M}{v_M}-\delta &= \rho\\
\frac{\mu\eta+\gamma(1-\theta)\frac{Y}{C}-\mu}{(1-\mu)(1-\eta)}\frac{B}{M}-\left(\frac{\dot{B}}{B}-\frac{\dot{M}}{M}+\frac{\dot{M}}{M}\right)-\delta &= \rho\\
\left[\frac{-\mu(1-\eta)+\gamma(1-\theta)/c}{(1-\mu)(1-\eta)}-1\right]b-\frac{\dot{b}}{b} &= \rho\\
\left[\frac{-\mu(1-\eta)-(1-\mu)(1-\eta)+\gamma(1-\theta)/c}{(1-\mu)(1-\eta)}\right]b-\frac{\dot{b}}{b} &= \rho\\
\left[\frac{-(1-\eta)+\gamma(1-\theta)/c}{(1-\mu)(1-\eta)}\right]b-\frac{\dot{b}}{b} &= \rho.
\end{aligned}$$

The final result is

$$\frac{\dot{b}}{b}=\left[\frac{\gamma(1-\theta)}{c(1-\eta)}-1\right]\frac{b}{1-\mu}-\rho,$$

which is the expression in the text.

A.3 Proof of Lemma 1 (Rates of return and market structure dynamics)

Expression (A.2) for the cash flow yields the returns to quality and to variety innovation as, respectively:

$$r=\alpha\frac{\Pi}{Z}=\alpha\frac{\left(\frac{1}{\theta}-1\right)X}{Z};$$

$$r = \frac{\Pi - I}{\beta X} + \frac{\dot{X}}{X} = \frac{\left(\frac{1}{\theta} - 1\right) X - \phi Z - I}{\beta X} + \frac{\dot{X}}{X}.$$

The expression for firm size is

$$x = \frac{X}{Z} = \frac{NX}{NZ} = \frac{\theta^2 Y}{NZ} = \theta^{\frac{2}{1-\theta}} \left(\frac{M}{N}\right)^{\gamma} R^{1-\gamma}.$$

We use the first equality to rewrite the returns as in the text:

$$r = \alpha \left[\left(\frac{1}{\theta} - 1 \right) x - \phi \right];$$

$$r = \frac{1}{\beta} \left(\frac{1}{\theta} - 1 - \frac{\phi + z}{x} \right) + \frac{\dot{x}}{x} + z.$$

Next, we log-differentiate with respect to time the last equality to obtain

$$\frac{\dot{x}}{x} = \gamma (m - n) + (1 - \gamma) \frac{\dot{R}}{R} = \gamma (m - n) - (1 - \gamma) \left(\frac{\dot{c}}{c} + \rho + \frac{\dot{u}}{u} \right).$$

A.4 Proof of Lemma 2 (Consumption ratio)

Recall the household budget

$$\dot{A} = rA + wM + pR - C_M M - C_B B.$$

Use the definition $C \equiv C_M M - C_B B$ and the fact $wM + pR = (1 - \theta) Y$ to rewrite it as

$$\dot{A} = rA + (1 - \theta) Y - C.$$

When $n = 0$ assets market equilibrium requires $A = NV$ but $V < \beta Y / N$ since by definition the free-entry condition does not hold. However, the valuation equation for the firm still says that $rV = \Pi - I + \dot{V}$. Substituting this expression in the household budget derived above yields

$$0 = \Pi - I + Y - NPX - C.$$

Using the fact that in the pre-industrial (CRS) regime $\Pi = I = 0$ and rearranging terms yields

$$C = Y - NPX.$$

The definition of c and the fact that $NPX = \theta Y$ then yield

$$c = 1 - \frac{\theta}{\omega}.$$

This is the top line of the expression in the text. When $n > 0$ assets market equilibrium requires $A = NV = N\beta X = \beta\theta^2 Y$, which says that the wealth ratio A/Y is constant. Using the Euler equation $r = \rho + \dot{C}/C$ and the definition of c, rewrite the budget constraint, after rearranging terms, as

$$c = \left(\rho + \frac{\dot{c}}{c} \right) \beta\theta^2 + 1 - \theta.$$

This unstable differential equation says that c jumps to its steady-state value $\rho\beta\theta^2 + 1 - \theta$, which is the value in the bottom line of the expression in the text.

A.5 Proof of Lemma 3 (Interest Rate and GDP per worker growth)

The household intertemporal optimization problem yields the saving rule

$$r = \rho + \frac{\dot{C}}{C} = \rho + \frac{\dot{c}}{c} + \frac{\dot{Y}}{Y}.$$

Log-differentiating with respect to time the reduced-form production function,

$$Y = \left(\frac{\theta}{P}\right)^{\frac{\theta}{1-\theta}} N^{1-\gamma} Z M^{\gamma} R^{1-\gamma},$$

we have

$$\frac{\dot{Y}}{Y} = (1-\gamma)\, n + z + \gamma m - (1-\gamma)\left(\frac{\dot{c}}{c} + \rho + \frac{\dot{u}}{u}\right).$$

Therefore, we have

$$r = \rho + \frac{\dot{c}}{c} + (1-\gamma)\, n + z + \gamma m - (1-\gamma)\left(\frac{\dot{c}}{c} + \rho + \frac{\dot{u}}{u}\right),$$

where $\dot{c} = 0$ by Lemma 4. Similarly, log-differentiating with respect to time the expression for GDP per worker,

$$\frac{G}{M} = \begin{cases} \left(\frac{\theta}{P}\right)^{\frac{\theta}{1-\theta}} \left(1 - \frac{\theta}{P}\right) N_0^{1-\gamma} Z_0 \left(\frac{R}{M}\right)^{1-\gamma} & \text{pre-industrial} \\ \left(\frac{\theta}{P}\right)^{\frac{\theta}{1-\theta}} \left[1 - \left(\frac{\theta}{P}\right)\left(1 + \frac{\phi}{x}\right)\right] N^{1-\gamma} Z \left(\frac{R}{M}\right)^{1-\gamma} & \text{industrial} \end{cases},$$

we obtain

$$g = \begin{cases} -(1-\gamma)\left(m + \frac{\dot{c}}{c} + \rho + \frac{\dot{u}}{u}\right) & \text{pre-industrial} \\ \xi(x)\frac{\dot{x}}{x} + (1-\gamma)\, n + z + \gamma m - (1-\gamma)\left(m + \frac{\dot{c}}{c} + \rho + \frac{\dot{u}}{u}\right) & \text{industrial} \end{cases},$$

where $\dot{c} = 0$ by Lemma 4.

A.6 Proof of Lemma 4 (Innovation rates)

The household intertemporal optimization problem yields the saving rule

$$r = \rho + \frac{\dot{c}}{c} + \frac{\dot{Y}}{Y},$$

where $\dot{c} = 0$ by Lemma 4. Combine this expression and the return to variety innovation (32) to write

$$\rho + \frac{\dot{Y}}{Y} = \frac{1}{\beta}\left(\frac{1}{\theta} - 1 - \frac{\phi + z}{x}\right) + \frac{\dot{Y}}{Y} - n - z + z.$$

Solve this equation for

$$n = \begin{cases} \frac{1}{\beta}\left(\frac{1}{\theta} - 1 - \frac{\phi+z}{x}\right) - \rho & z > 0 \\ \frac{1}{\beta}\left(\frac{1}{\theta} - 1 - \frac{\phi}{x}\right) - \rho & z = 0 \end{cases}. \tag{A.5}$$

Combine the return to saving (29) and the return to quality innovation (31) to write

$$\alpha\left[\left(\frac{1}{\theta}-1\right)x-\phi\right]=\rho+(1-\gamma)\,n+z+\gamma m-(1-\gamma)\left(\rho+\frac{\dot{u}}{u}\right).$$

Substitute (A.5) in this expression and solve for

$$z=\frac{\left(\alpha-\frac{1-\gamma}{\beta x}\right)\left[\left(\frac{1}{\theta}-1\right)x-\phi\right]-\gamma\,(\rho+m)+(1-\gamma)\left(\rho+\frac{\dot{u}}{u}\right)}{1-\frac{1-\gamma}{\beta x}}. \tag{A.6}$$

The threshold x_N follows directly from (A.5), which says that when agents anticipate $z=0$ entry is positive for

$$x>x_N\equiv\frac{\phi}{\frac{1}{\theta}-1-\rho\beta}.$$

Solving (A.5) and (A.6) for z then yields (36), which says that in the region $x>x_N$ quality innovation is positive for

$$\left[\left(\frac{1}{\theta}-1\right)x-\phi\right]\left(\alpha-\frac{1-\gamma}{\beta x}\right)>\gamma\,(m^*+\rho)-(1-\gamma)\left(\rho+\frac{\dot{u}}{u}\right).$$

Since the left-hand side is increasing in x, we have a unique value x_Z. Finally, $x_Z>x_N$ if

$$\left[\left(\frac{1}{\theta}-1\right)x_N-\phi\right]\left(\alpha-\frac{1-\gamma}{\beta x_N}\right)<\gamma\,(m^*+\rho)-(1-\gamma)\left(\rho+\frac{\dot{u}}{u}\right),$$

which yields

$$\frac{\rho\beta\alpha\phi}{\frac{1}{\theta}-1-\rho\beta}<\gamma\,(m^*+\rho)-(1-\gamma)\,\frac{\dot{u}}{u}.$$

This condition holds for all $\frac{\dot{u}}{u}\leq 0$ under the assumption in the proposition, namely,

$$\frac{\rho\beta\alpha\phi}{\frac{1}{\theta}-1-\rho\beta}<\gamma\,(m^*+\rho)\,.$$

A.7 Proof of Proposition 1 (Path to sustainability)

We divide the proof in two parts.

A.7.1 The saddle path in the pre-industrial regime

The equation to solve is

$$\frac{db\,(x)}{dx}=\frac{b\,(x)}{x}\frac{\left(\frac{\gamma}{1-\eta}\frac{1-\theta}{1-\frac{\theta}{\omega}}-1\right)\frac{1}{1-\mu}b\,(x)-\rho}{b\,(x)-\left[\delta+\frac{(1-\gamma)\rho}{\gamma}\right]}=\frac{b\,(x)}{x}k_1\frac{b\,(x)-\frac{k_2}{k_1}}{b\,(x)-k_3},$$

where

$$k_1=\left(\frac{\gamma}{1-\eta}\frac{1-\theta}{1-\frac{\theta}{\omega}}-1\right)\frac{1}{1-\mu},\quad k_2=\rho,\quad \frac{k_2}{k_1}=\bar{b},\quad k_3=\tilde{b}\equiv\delta+\frac{(1-\gamma)\,\rho}{\gamma}.$$

The solution is

$$c_1 + \log x = \frac{(k_2 - k_3 k_1)\log(k_2 - k_1 b(x))}{k_2 k_1} + \frac{k_3 \log b(x)}{k_2}.$$

The boundary condition is $(x, b) = (x_N, b^*)$. Hence:

$$c_1 + \log x_N = \frac{(k_2 - k_3 k_1)\log(k_2 - k_1 b^*)}{k_2 k_1} + \frac{k_3 \log b^*}{k_2}.$$

Therefore, we have

$$\log\left(\frac{x}{x_N}\right) = \frac{\bar{b} - \tilde{b}}{\rho}\log\left(\frac{b - \bar{b}}{b^* - \bar{b}}\right) + \frac{\tilde{b}}{\rho}\log\left(\frac{b}{b^*}\right),$$

which gives equation (37) in the text.

A.7.2 The steady state in the industrial regime

First, we have

$$\frac{\dot{x}}{x} = \gamma m^* - (1-\gamma)\rho - \gamma n = 0 \Rightarrow n^* = m^* - \frac{1-\gamma}{\gamma}\rho.$$

Using this result, the three rates of return become:

$$r = m^* + \rho - \frac{1-\gamma}{\gamma}\rho + z;$$

$$r = \alpha\left[\left(\frac{1}{\theta} - 1\right)x - \phi\right];$$

$$r = \frac{1}{\beta}\left(\frac{1}{\theta} - 1 - \frac{\phi + z}{x}\right) + z.$$

We thus reduce them to the system:

$$z = \alpha\left[\left(\frac{1}{\theta} - 1\right)x - \phi\right] - (m^* + \rho) + \frac{1-\gamma}{\gamma}\rho;$$

$$z = \left(\frac{1}{\theta} - 1\right)x - \phi - \left[(m^* + \rho) - \frac{1-\gamma}{\gamma}\rho\right]\beta x.$$

We solve this for:

$$x^* = \frac{(1-\alpha)\phi - (m^* + \rho) + \frac{1-\gamma}{\gamma}\rho}{(1-\alpha)\left(\frac{1}{\theta} - 1\right) - \left[(m^* + \rho) - \frac{1-\gamma}{\gamma}\rho\right]\beta};$$

$$\begin{aligned} z^* &= \alpha\left[\left(\frac{1}{\theta} - 1\right)x^* - \phi\right] - (m^* + \rho) + \frac{1-\gamma}{\gamma}\rho \\ &= \frac{\alpha\phi - \left(\frac{1}{\theta} - 1\right)\frac{1}{\beta} + \left[(m^* + \rho) - \frac{1-\gamma}{\gamma}\rho\right]}{(1-\alpha)\left(\frac{1}{\theta} - 1\right)\frac{1}{\beta} - \left[(m^* + \rho) - \frac{1-\gamma}{\gamma}\rho\right]}\left[(m^* + \rho) - \frac{1-\gamma}{\gamma}\rho\right]. \end{aligned}$$

The sustainability condition is

$$\begin{aligned} g^* &= (1-\gamma)\, n^* + z^* - (1-\gamma)\,(m^* + \rho) \\ &= z^* + (1-\gamma)\left[m^* - \frac{1-\gamma}{\gamma}\rho - m^* - \rho\right] \\ &= z^* - \frac{1-\gamma}{\gamma}\rho > 0. \end{aligned}$$

This holds for

$$\frac{\alpha\phi - \left(\frac{1}{\theta} - 1\right)\frac{1}{\beta} + \left[(m^* + \rho) - \frac{1-\gamma}{\gamma}\rho\right]}{(1-\alpha)\left(\frac{1}{\theta} - 1\right)\frac{1}{\beta} - \left[(m^* + \rho) - \frac{1-\gamma}{\gamma}\rho\right]}\left[(m^* + \rho) - \frac{1-\gamma}{\gamma}\rho\right] > \frac{1-\gamma}{\gamma}\rho,$$

which is condition C4 in the text of the proposition.

A.8 Fragile industrialization and substistence consumption

Consider the scenario of fragile industrialization discussed in subsection 4.3 and assume that there is a finite, strictly positive minimum level of consumption for human subsistence, $C_M \geqslant C^{\text{sub}} > 0$. By Lemma 4 the household's consumption decisions yield $C_M = \mu\frac{C}{M} = \mu\left(1 - \theta + \rho\beta\theta^2\right)\frac{Y}{M}$ and $C_B = \frac{1-\mu}{b}\frac{C}{M} = \frac{(1-\mu)\left(1-\theta+\rho\beta\theta^2\right)}{b}\frac{Y}{M}$. Under fragile industrialization, output per worker in the long run eventually becomes low enough that the subsistence constraint $C_M \geqslant C^{\text{sub}}$ binds. Using equation (26), we have $C_M = C^{\text{sub}}$ and

$$C_B = C^{\text{sub}} \Rightarrow b = \frac{1-\mu}{C^{\text{sub}}}\left(1 - \theta + \rho\beta\theta^2\right)\theta^{\frac{2\theta}{1-\theta}} Z \cdot \left(\frac{R}{M/N}\right)^{1-\gamma}.$$

This result shows that the household is knocked off its first-order conditions and thus the birth rate obeys a different equation, an equation that says that not only the birth rate falls over time, because M/N grows and R shrinks, but that eventually it falls below the death rate. Consequently, the economy experiences falling population and falling market size — exactly as in the failure to industrialize secenario. Two things then happen. First, entry eventually stops. Second, firm size eventually falls below ϕ and firms revert to the CRS technology. When that happens, we have $C/Y = 1 - \theta/\omega$ and

$$b = \frac{(1-\mu)\left(1 - \frac{\theta}{\omega}\right)\left(\frac{\theta}{\omega}\right)^{\frac{\theta}{1-\theta}}}{C^{\text{sub}}} Z N^{1-\gamma} \cdot \left(\frac{R}{M}\right)^{1-\gamma}$$

This equation shows that while in principle the economy could settle into a stable steady state with positive and finite population size because the birth rate is decreasing in population, it cannot do so because the exhaustion of the resource exerts relentless downward pressure on living standards, a force that in turn pushes the birth rate below the death rate and keeps it there.

A.9 Proof of Proposition 7 (Extraction cost dynamics)

The system is:

$$\dot{R} = -R\left(\rho - \frac{\zeta R}{\left[\kappa + \zeta\left(S_0 - S\right)\right]^2}\right);$$

$$\dot{S} = -R.$$

It yields the PDE problem

$$\frac{dR/dt}{dS/dt} = \frac{dR}{dS} = \frac{-R\left(\rho - \frac{\zeta R}{[\kappa+\zeta(S_0-S)]^2}\right)}{-uR} = \rho\left[\kappa + \zeta\left(S_0 - S\right)\right] - \frac{\zeta R}{\kappa + \zeta\left(S_0 - S\right)}.$$

Observe that $\kappa + S_0 - S = K_R$ so that $-\zeta dS = dK_R$. With this change of variable we obtain:

$$\frac{dR}{dK_R} = \frac{R}{K_R} - \frac{\rho}{\zeta}K_R, \quad K_R \in \left[\kappa, \kappa + S_0\right], \quad R \in \left[0, S_0\right].$$

We then solve this PDE:

$$R\left(K_R\right) = k_1 K_R - \frac{\rho}{\zeta}K_R^2.$$

To determine the constant of integration k_1 we use the boundary condition $R = 0$ at $S = 0$. Hence, we want

$$R\left(\kappa + \zeta S_0\right) = k_1\left(\kappa + \zeta S_0\right) - \frac{\rho}{\zeta}\left(\kappa + \zeta S_0\right)^2 = 0 \Rightarrow k_1 = \frac{\rho}{\zeta}\left(\kappa + \zeta S_0\right).$$

The final result is

$$R\left(K_R\right) = \frac{\rho}{\zeta}\left[\left(\kappa + \zeta S_0\right) - K_R\right]K_R,$$

which we rewrite

$$R\left(S\right) = \frac{\rho}{\zeta}\left[\left(\zeta - 1\right)S_0 + S\right]\left(\kappa + S_0 - S\right).$$

Now recall that by construction $\dot{K}_R = \zeta u R = \zeta R / K_R$ so that we can write this equation as

$$\dot{K}_R = \rho\left(K_R^* - K_R\right), \quad K_R^* = \kappa + \zeta S_0.$$

This has solution

$$K_R\left(t\right) = K_R^* + e^{-\rho t}\left(K_R^* - K_R\left(0\right)\right), \quad K_R\left(0\right) = \kappa.$$

Thus, we obtain:

$$K_R\left(t\right) = \kappa + \left(1 - e^{-\rho t}\right)\zeta S_0;$$

$$u\left(t\right) = \frac{1}{K_R\left(t\right)} = \frac{1}{\kappa + \left(1 - e^{-\rho t}\right)\zeta S_0}.$$

$$\frac{\dot{u}\left(t\right)}{u\left(t\right)} = -\frac{\dot{K}_R\left(t\right)}{K_R\left(t\right)} = -\rho\left(\frac{K_R^*}{K_R\left(t\right)} - 1\right) = -\frac{e^{-\rho t}\rho\zeta S_0}{\kappa + \left(1 - e^{-\rho t}\right)\zeta S_0}.$$

Finally, we obtain:

$$R\left(t\right) = \rho S_0 e^{-\rho t}\left[\kappa + \left(1 - e^{-\rho t}\right)\zeta S_0\right].$$

A.10 Proof of Lemma ??

From (51), the growth rate of firm size is

$$\frac{\dot{x}}{x} = \gamma m^{*} - (1-\gamma)\left(\rho + \frac{\dot{u}}{u}\right) - \gamma n(x). \tag{A.7}$$

From (A.5), the growth rate of the mass of firms is

$$n = \begin{cases} \frac{1}{\beta}\left(\frac{1}{\theta} - 1 - \frac{\phi}{x}\right) - \rho & z = 0 \\ \frac{1}{\beta}\left(\frac{1}{\theta} - 1 - \frac{\phi + z}{x}\right) - \rho & z > 0 \end{cases}, \tag{A.8}$$

and, from (36), the rate of quality improvement under operative vertical innovation is

$$z = \frac{\left(\alpha - \frac{1-\gamma}{\beta x}\right)\left[\left(\frac{1}{\theta} - 1\right)x - \phi\right] - \gamma(\rho + m) + (1-\gamma)\left(\rho + \frac{\dot{u}}{u}\right)}{1 - \frac{1-\gamma}{\beta x}},$$

where we can substitute $\dot{u}/u$ by means of (49) to obtain

$$z = \frac{\left(\alpha - \frac{1-\gamma}{\beta x}\right)\left[\left(\frac{1}{\theta} - 1\right)x - \phi\right] - \gamma(\rho + m) + (1-\gamma)\rho\left(2 - \frac{u}{u^{*}}\right)}{1 - \frac{1-\gamma}{\beta x}}. \tag{A.9}$$

Using (A.9) to substitute z into (A.8), we obtain the equilibrium rate of entry innovation n in Lemma 10, which can in turn be used to substitute n in expression (A.7) to obtain the equilibrium growth rate of firm size $\dot{x}/x$ in Lemma 10.

A.11 Industrial regime with endogenous extraction cost

Further derivations of the results stated in the main text of subsection 5.3. Setting $z > 0$ in (A.9) and solving the resulting expression for u yields $z > 0: \quad u < u_Z(x)$ where we defined the locus

$$u_Z(x) \equiv u^{*}\left[2 + \frac{\left(\alpha - \frac{1-\gamma}{\beta x}\right)\left[\left(\frac{1}{\theta} - 1\right)x - \phi\right] - \gamma(\rho + m^{*})}{(1-\gamma)\rho}\right]. \tag{A.10}$$

This locus is increasing in x and eventually becomes linear. Moreover, by construction we have $z(x_N) = 0$. This implies

$$\left(\alpha - \frac{1-\gamma}{\beta x_N}\right)\left[\left(\frac{1}{\theta} - 1\right)x_N - \phi\right] - \gamma(\rho + m^{*}) + (1-\gamma)\rho\left(2 - \frac{u}{u^{*}}\right) = 0$$

so that, evaluating locus $u_Z(x)$ at the critical firm size level $x = x_N$ yields

$$u_Z(x_N) = u^{*}\left[2 + \frac{\left(\alpha - \frac{1-\gamma}{\beta x_N}\right)\left[\left(\frac{1}{\theta} - 1\right)x_N - \phi\right] - \gamma(\rho + m^{*})}{(1-\gamma)\rho}\right]. \tag{A.11}$$

Reading the above results in graphical terms in the (u, x) plane: the $u_Z(x)$ locus is increasing in x and has a "regime-boundary vertical intercept" given by (A.10). The $\dot{x} = 0$ locus, instead, has a "regime-boundary vertical intercept" given by

$$u(x_N) = u^* \left[2 - \frac{\gamma m^*}{(1-\gamma)\rho}\right]$$

and increases with x until it converges from below to

$$u_\infty = 2 + \frac{\gamma}{(1-\gamma)\rho}\left[(1-\alpha)\left(\frac{1}{\theta} - 1\right) - (m^* + \rho)\right].$$

These results guarantee the existence of an intersection between the $\dot{x} = 0$ locus and the $u_Z(x)$ locus. For simultaneous steady state (54) to exist it is sufficient to assume that $u_\infty > u^*$.