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Abstract

In this study, the author demonstrates that the selection of an appropriate money-demand function is crucial to ascertain the relationship between fiscal deficits and inflation. To do so, the author incorporates a Selden-Latané money-demand function into a micro-founded extension of the model introduced by Sargent, Williams, and Zha (2009). The use of this particular function results in a model that more accurately replicates Mexican money supply's past history, and furthermore, establishes a stronger historical association between fiscal and monetary policy, namely, between fiscal deficits and seigniorage. As a result, the author is able to provide more compelling evidence for the dominance of fiscal policy as the major cause of high inflation in Mexico during the last three decades of the twentieth century.

1 Introduction

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Theories of inflation have been primarily built upon Cagan's money-demand function (1956). Nevertheless, recent empirical research conducted by Benati, Lucas, Nicolini, and Weber (2021) suggests that, in numerous countries, alternative specifications of the money-demand function provide a more accurate characterization of the long-run relation between money demand and inflation expectations embedded in nominal interest rates.

The principal objective of this study is to investigate whether the selection of a suitable money-demand function matters to study the interaction between fiscal and monetary policies. In light of this, an alternative to Cagan's money-demand function is proposed, and it is embedded into a micro-founded extension of the model proposed in Sargent, Williams, and Zha (2009). Throughout the remainder of this paper, the latter model shall be referred to as SWZ. Consequently, the proposed model is estimated and its projections are contrasted with respect to Mexican inflation, inflation expectations, money supply, and seigniorage-financed deficits, as well as those implied by the SWZ model.

Before proceeding, it is important to clarify that the model presented in this study does not directly address fiscal deficits. Instead, the focus is on the seigniorage generated by the monetary authority. When a country is in a fiscal dominance regime, seigniorage's magnitude is anticipated to have a strong connection with fiscal deficits. If the connection between fiscal deficits and seigniorage is robust, then the case for fiscal dominance as the underlying cause of inflation becomes more compelling. This study's findings reveal that this was indeed the case for the Mexican economy.

To demonstrate the suitability of the alternative money-demand functional proposed in this study, a monetary equilibrium inflation model is utilized. The model enables the computation of historical model-implied money supply and seigniorage time series, which align with the patterns observed in the historical Mexican monetary base and fiscal deficits, respectively.

It is worth noting that the selection of the alternative money-demand specification not only leads to a stronger correlation between fiscal deficits and seigniorage but also has a significant impact on the equilibrium behavior and dynamics of inflationary expectations.

Indeed, the interaction between inflation dynamics and seigniorage regimes has noteworthy implications for the stability of inflation expectations. In our model, if inflation beliefs surpass a certain threshold, expectations may rise significantly without necessarily becoming unstable or explosive. Instead, we observe stable equilibria at extremely high levels of inflation, and convergence towards these equilibrium levels is how the model represents hyper-inflationary processes. This stands in contrast to other models, where hyper-inflation is the outcome of unstable, explosive dynamics. Thus, this study's model offers an alternative framework for comprehending economic behavior in high-inflation environments. The presence of stable hyper-inflationary equilibria enhances the availability of quantitative tools to analyze the model's projections, which is a significant advantage compared to models that exhibit explosive behavior. However, the use of an alternative money-demand specification to depart from Cagan's approach has its drawbacks, notably an increased complexity in terms of estimation and the computation of equilibria. This study outlines how to navigate these challenges.

To understand why the choice of money-demand function specification matters in examining the relationship between fiscal deficits, inflation, and seigniorage, we must consider the relationship between money supply and seigniorage. For example, Cagan (1956) examined dual inflation equilibria in a model that featured a semi-logarithmic money-demand function and an explicit inflation expectations mechanism. There was no budget constraint or money-supply function. Cagan focused on the problem of maximizing seigniorage. For any feasible seigniorage level, there existed a low inflation equilibrium and a high inflation equilibrium. At the maximum feasible seigniorage level, there was only one inflation equilibrium, which represented the optimal seigniorage level. If a monetary authority increased the money supply beyond this level, high inflation would result, and the government would no longer be able to increase its seigniorage.

This suggests that there isn't a simple relationship between money supply and seigniorage, because we can theoretically find a situation when expanding the money supply could cause a decrease, rather than an increase, in seigniorage. This situation is called Cagan's paradox and occurs because Cagan's money-demand function predicts that money demand decreases rapidly as expected inflation increases. But Benati (2018) finds that there is little empirical evidence to support the latter prediction.

It is clear that Cagan's paradox significantly affects the relationship between seigniorage and money supply and thus, it may obfuscate the real relationship between fiscal deficits and inflation. Under Cagan's hypothesis, if fiscal deficits cannot be financed by the monetary authority beyond some maximum seigniorage level, then it is hard to explain the persistence of fiscal dominance regimes or the insistence of monetary authorities on expanding the money supply beyond Cagan's optimal seigniorage level during historically high inflation periods. The lack of empirical evidence supporting Cagan's money-demand functional predictions can be explained by these issues.

In conclusion, the choice of the money-demand function specification is important because it implicitly places a theoretical limit on the amount of seigniorage a government can raise. A money-demand function that does not display Cagan's paradox, such as the alternative specification used in this work, theoretically allows the government to raise more seigniorage, and it makes possible a better fit between historical deficits and model-implied seigniorage-financed deficits.

The micro-economic model presented below, underlying the proposed money-demand function, provides a clear economic rationale for its specific form. The model shows that households face increasing costs when trying to substitute their consumption payments away from the official medium of exchange, even during high inflation episodes. This, in turn, makes it very difficult for households to set their demand for real balances to zero. The reason for this is that arranging alternative payment arrangements for an increasing number of household expenses takes additional time away from work, which represents an opportunity cost to households. Since real balances cannot be set to zero, the government can always keep collecting some seigniorage.

The model also has other important empirical features. For instance, it captures better the trend and volatility of Mexican's monetary base growth. Additionally, the model's findings are consistent with the work of Sargent, Williams, and Zha (2009) regarding the occurrence of fundamental reforms in South America in response to hyper-inflationary periods. In particular, the model suggests that similar reforms were introduced in Mexico when the economy found itself in the domain of attraction of a hyper-inflationary "high" inflation equilibrium. This is a significant confirmation of SWZ's findings, even though the model presented in this work uses different data, a different model, and different dynamics.

The model also has important implications for the interpretation of past economic events in Mexico. It suggests that gradual fiscal adjustments were not enough to stabilize and decrease inflation, and that other measures such as external debt renegotiation were crucial because of their implications for government interest expenditures and its access to international capital markets. Solving the debt crisis was essential for bringing down the deficit sufficiently to stabilize the economy. In fact, the magnitude of the fiscal retrenchment is critical in the model, and in some cases, seigniorage-financed deficits may need to be reduced to levels lower than those observed at the beginning of the inflationary episode just to restore inflation to its previous levels.

The paper proceeds by first presenting the proposed money-demand function. The monetary market is then described, and several definitions of equilibrium are provided. Next, the model is estimated using Mexican monthly inflation data, and the dynamics of the estimated model are analyzed using Monte Carlo simulations and the ordinary differential equation (ODE) method of Kushner and Yin (2003). Finally, the model predictions are compared with Mexico's fiscal deficit and money supply data, as well as with its history of stabilization programs.

2 A Transactional Model Of Money Demand

In inflationary economies, money loses its function as a store of value, and it becomes a costly transactional instrument. To reduce the costs of employing the default medium of exchange in an inflationary economy, households spend time securing alternative arrangements to preserve the value of their purchasing power. This is because, inflation depreciates any nominal money balances M_t the household is holding to pay for its desired consumption later C_{t+1}.

If inflation is a concern, these balances can be reduced by spending the money just after income is received, which takes more time away from work, or equivalently, by increasing the number of trips to the shop n_t. These trips take time that could be spent working so their opportunity cost is $zH(C_{t+1}, n_t)$ where z is the real wage per unit of labor and H gives shopping time as a function of the number of trips to the shop n_t and desired consumption.

The solution to this problem, detailed in the online appendix A, yields the key result

$$zH_{n,t} = \frac{1}{n_t}\frac{M_t}{P_t}\frac{P^e_{t+1}}{P_t}, \tag{1}$$

which says that the marginal opportunity cost of increasing trading trips in consumption units, must equal the marginal benefit of saving on inflationary costs by reducing the demand for real balances. Here nominal money demand is denoted as M_t which is a percentage of the output at period t; P_t is the price level; and P^e_{t+1} is next period's expected price level.

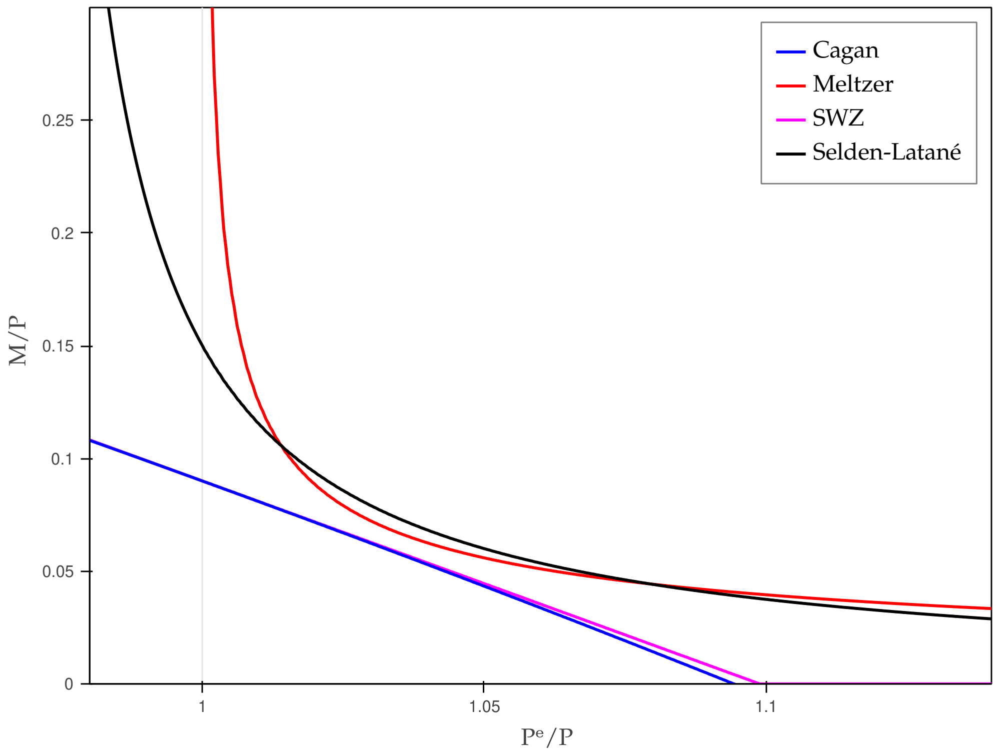

Figure 1: **Money-Demand Functions**. M/P is money demand and P^e/P denotes expected inflation. Note: The linear money-demand function in SWZ is shown here with $\lambda = 0.91$, which is close to Cagan's semi log money-demand function, displayed here with the same parameter value. Meltzer's log-log money-demand function is shown with an elasticity of 0.5, and re-scaled to show that, for high levels of inflationary expectations, it can be very close to our version of Selden-Latané's money-demand function, here shown with $\lambda_0 = 0.15$ and $\lambda_1 = 30$.

With the functional specification for H given in the online appendix A, the last equation gives the money-demand of Selden (1956) and Latané (1960):

$$\frac{M_t}{P_t} = \frac{1}{\gamma}\lambda\left(\frac{P^e_{t+1}}{P_t}\right), \quad \lambda\left(\frac{P^e_{t+1}}{P_t}\right) = \frac{\lambda_0}{1+\lambda_1(P^e_{t+1}/P_t - 1)}, \tag{2}$$

where λ_0 and λ_1 depend on deep model parameters. The function $\lambda(\cdot)$ captures the sensitivity of real money demand to changes in expected inflation. We are adding here a scale parameter γ, with $\gamma > 0$.

In Sargent, Williams, and Zha (2009), the money-demand function is a linear approximation of Cagan's semi logarithmic money-demand function, which is (up to scale) $M_t/P_t = \exp(-\lambda P^e_{t+1}/P_t)$. SWZ employ an approximation to this demand based on the Taylor expansion of the exponential function: $M_t/P_t \approx 1 - \lambda P^e_{t+1}/P_t$. In this money-demand function, λ is a scalar interpreted as the semi-elasticity of the money-demand with respect to the expected inflation. Due to the linearity of this functional form, the demand for real balances can take negative values for very high levels of expected inflation. We summarize the behavior of these money-demand specifications in figure 1.

We now describe the formation mechanism of inflation expectations. We assume here a mechanism with constant-gain learning. Defining

$$\beta_t \equiv \frac{P^e_{t+1}}{P_t}, \quad \pi_{t+1} \equiv \frac{P_{t+1}}{P_t} \tag{3}$$

we then have the following adaptive expectations mechanism:

$$\beta_t = \beta_{t-1} + \nu\left(\pi_{t-1} - \beta_{t-1}\right), \tag{4}$$

where $0 < \nu < 1$. Constant-gain learning means that ν is constant. π_t denotes gross inflation. In what follows we use interchangeably the terms "inflation expectations" and "inflation beliefs".

3 The Monetary Market

The model is completed by specifying the money supply which is given by:

$$M_t = \theta M_{t-1} + d_t(m_t, \varsigma_t, d_{t-1})P_t. \tag{5}$$

The parameter $\theta < 1$, adjusts the money supply for growth in real output and cash taxes. A lower bound on θ is given by $1 - 1/\lambda_1 < \theta$, and it is sometimes a condition for the existence of an equilibrium, as explained in online appendix C.

The government raises seigniorage in the amount of d_t. It is assumed that it will be used to finance fiscal deficits. Seigniorage has the following dynamics:

$$d_t(m_t, \varsigma_t, d_{t-1}) = \bar{d}(m_t) + \varepsilon_d(\varsigma_t, d_{t-1}),$$

in which we assume that seigniorage has an average level $\bar{d}$ which depends on a regime m_t. The regime m_t follows a Markov chain that captures the monetary authority expansionary stance. Regimes m_t with a very low $\bar{d}$ imply that the central bank is raising little revenues through monetary expansion, and on the other hand, regimes with a high $\bar{d}$ imply that the central bank is being very proactive at raising revenue through seigniorage.

Shocks to seigniorage are captured by $\varepsilon_d(\varsigma_t, d_{t-1})$, similarly depending on a volatility regime and on the deficit during period $t-1$. This shock induces a log-normal conditional distribution for the seigniorage-financed deficit with a mean equal to $\log \bar{d}(m_t)$ and variance $\sigma_d^2(\varsigma_t, d_{t-1}) = \sigma_d^2(\varsigma_t) d_{t-1}^{\vartheta}$ whenever d_t is positive. The seigniorage volatility also depends on the parameter σ_d^2, which, in turn, is a function of a state that changes according to the regime ς_t.

In our results, the volatility state spends most of the time in the lower regime and only briefly increases to the high level. The high-volatility regime may be capturing external shocks to the economy, by temporarily allowing seigniorage shocks to take higher values than usual. Defining the joint seigniorage state as $s_t \equiv (m_t, \varsigma_t)$, we can write the seigniorage as depending on just two arguments: $d_t(s_t, d_{t-1}) = d_t(m_t, \varsigma_t, d_{t-1})$.

Seigniorage's distribution has a density function $p_d(\varepsilon_d | s_t, d_{t-1})$:

$$p_d(\varepsilon_d | s_t, d_{t-1}) = \frac{\exp\left(\frac{-\left[\log(\bar{d}(m_t)+\varepsilon_d) - \log(\bar{d}(m_t))\right]^2}{2\sigma_d^2(\varsigma_t, d_{t-1})} \right)}{\sqrt{2\pi}\sigma_d(\varsigma_t, d_{t-1})(\log(\bar{d}(m_t)) + \varepsilon_d)}. \tag{6}$$

The elements of the joint seigniorage state (m_t, ς_t) follow independent Markov chains, respectively with transition probabilities $Q_m = \{p_{i,j}\}_{i,j=1,\ldots,m_h}$ and $Q_\varsigma = \{q_{i,j}\}_{i,j=1,\ldots,\varsigma_h}$, with a total of $m_h \times \varsigma_h$ possible states. The transition probability matrix of the joint state $s_t \equiv (m_t, \varsigma_t)$ is Q_s, with $Q_s = Q_m \otimes Q_\varsigma$, where $\otimes$ denotes the Kronecker product.

Imposing monetary equilibrium, the demand for money (2), the dynamics of inflationary expectations (4), and money supply (5) together imply equilibrium inflation:

$$\pi_t = \frac{\theta\lambda\left(\beta_{t-1}\right)}{\lambda\left(\beta_t\right) - \gamma d_t\left(s_t, d_{t-1}\right)}, \tag{7}$$

for all t, provided that the numerator and denominator are positive.

We need to set some additional restrictions, first a lower bound on inflation expectations β_t, and an upper bound on inflation $\pi_t < \delta^{-1}$ given by δ:

$$\begin{aligned} \beta_t &> 1 - \frac{1}{\lambda_1} && (8) \\ \lambda(\beta_t) - \gamma d_t(s_t, d_{t-1}) &> \delta\theta\lambda(\beta_{t-1}), && (9) \end{aligned}$$

almost surely for all $t > 0$. Restriction (8) sets a lower bound for inflation expectations and, together with equation (2), implies that the real money stocks are positive and finite for all $t > 0$. Restriction (9) sets δ^{-1} as an upper bound for gross inflation. The first bound is a necessary condition for the existence of a self-confirming equilibrium (SCE), which we will define later. The second bound is enforced through a cosmetic reform, to be defined below as an inflation shock with variance σ_π, to be applied instantaneously whenever inflation surpasses the bound, and it ensures that inequality (9) holds, preventing that $\pi_t \to \infty$.

Restriction (8) ensures that gross inflation is always positive, but inflation can still be negative. In equation (7), if d_t is small enough and $\lambda(\beta_{t-1})/(\lambda(\beta_t) - \gamma d_t)$ is close or equal to one then the equilibrium inflation will be around θ, which is less than one. As equation (9) ensures that the denominator of equation (7) is always positive, expectations may not satisfy equations (8) and (9) at the same time unless $1 - 1/\lambda_1 < \theta$. The model allows net equilibrium inflation and expectations to be slightly negative, as long as equation (8) continues to hold.

4 Deterministic Stationary-State Equilibrium

We now turn to solution methods for the model just described. In this section, a simple deterministic version of the model is obtained by fixing the seigniorage-mean state m, and setting shocks ε_d equal to zero for all t. These assumptions imply that the volatility and the state ς_t are inconsequential in computing the deterministic steady-state and, thus, we have $d_t = \bar{d}(m)$ for all t. Likewise, inflation expectations are already settled at this steady-state, so even for adaptive expectations $\beta_t = \pi_{t+1}$ for all t. Next, consider the money-demand function, equation (2), and the money supply, equation (5), which yield, under these conditions:

$$\frac{M_t}{P_t} = \frac{1}{\gamma}\lambda(\pi_{t+1}), \tag{10}$$
$$\frac{M_t}{P_t} = \theta\frac{M_{t-1}}{P_{t-1}}\frac{1}{\pi_t} + \bar{d}(m). \tag{11}$$

The above imply the following equation:

$$\pi_t = \frac{\theta\lambda(\pi_t)}{\lambda(\pi_{t+1}) - \gamma\bar{d}(m)}. \tag{12}$$

If $\pi_{t+1} = \pi_t = \pi(m)$, then we obtain the following nonlinear equation with inflation as the unknown variable:

$$\pi = \frac{\theta\lambda(\pi)}{\lambda(\pi) - \gamma\bar{d}(m)}. \tag{13}$$

This equation might have zero, one, or two solutions. We define a stationary state equilibrium (SSE), denoted as π^*, as a zero of equation (13) This equation does not have a closed-form solution, but its zeros can be readily calculated numerically. By equation (8), first, we know that a solution must satisfy $\pi^* > 1 - 1/\lambda_1$, and by equation (9), we can also determine that a solution exists if

$$\gamma\bar{d}(m) < \lambda(\pi). \tag{14}$$

These are necessary but not sufficient conditions. A tighter upper bound for seigniorage can be found numerically: there is a maximum seigniorage level d_{max}, such that a deterministic steady-state exists. This will imply a maximum level of (low) steady-state inflation, denoted by π^*_{max}. For seigniorage levels lower than d_{max}, two solutions will exist, which we denote $\pi^*_1(m) < \pi^*_2(m)$. The first solution is the low-inflation SSE, and the second is the high-inflation SSE. On the other hand, when $\bar{d}(m) = d_{max}$, there will be only one SSE, denoted as $\pi^*_1(m) = \pi^*_2(m) = \pi^*_{max}$. When the mean seigniorage approaches zero, the solution of equation (13) becomes unique again. In

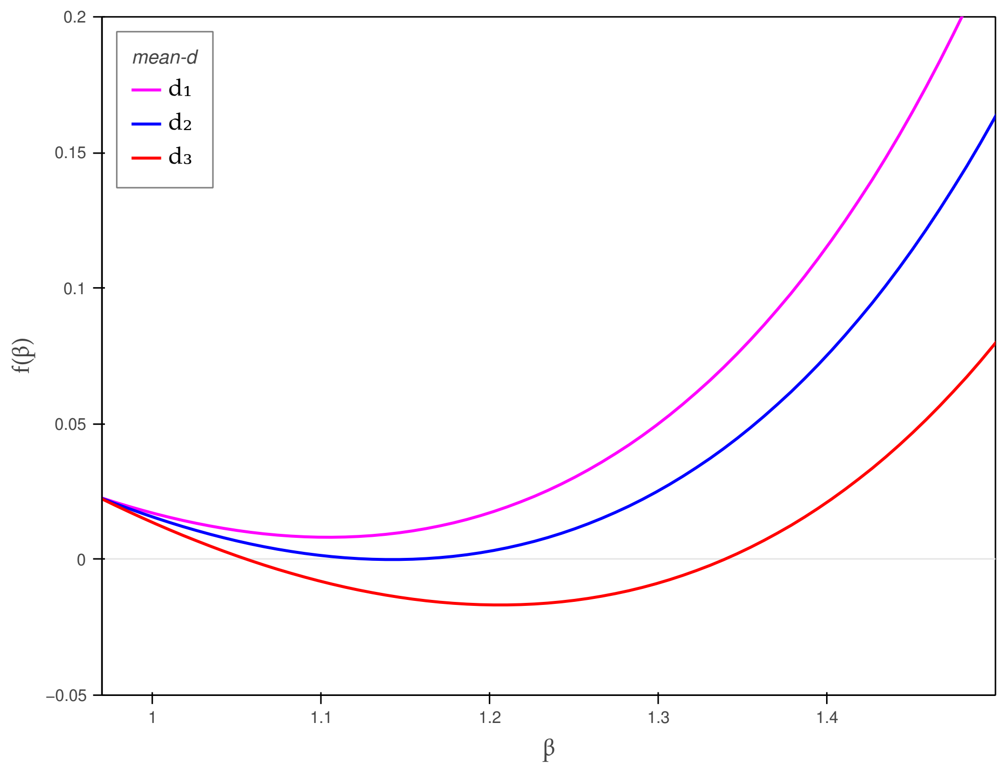

Figure 2: **Seigniorage means and Deterministic Stationary State Equilibria**. Note: For regimes with a low seigniorage-mean such as d_3, the model has two SSEs, defined as the zeros of the function in equation (13). When the mean seigniorage increases to d_2, the number of SSEs is reduced to one, while for values of the mean seigniorage greater than d_2, such as d_1, the model will have no SSEs.

particular, when the mean seigniorage is zero, then the unique solution is $\pi^* = \theta$, which can only be an equilibrium if equation (8) is satisfied–that is, if $1 - 1/\lambda_1 < \theta$.

When the seigniorage-mean is close to zero but not zero, the second (high) solution $\pi_2^*(m)$ to equation (13) can be greater than $1/\delta$, exceeding inflation's upper bound given by equation (9); thus, this solution is not an admissible SSE, leaving $\pi_1^*(m)$ as the unique equilibrium. In figure 2, we show that for a low seigniorage-mean level such as d_3, the model has two SSEs. As the seigniorage-mean increases to $d_2 = d_{max}$, the two SSEs become close to each other and eventually become the same. For any seigniorage-mean level greater than d_2, such as d_1, the model does not have an SSE.

5 Self-Confirming Equilibrium and Its Related Dynamics

More realistic equilibria are obtained holding the state m as fixed and allowing the shocks ε_d to impact seigniorage.

Definition 1. Fixed-m self-confirming equilibrium (SCE). For each m-state, a fixed-m SCE is a probability distribution over inflation histories $\pi^T \equiv \{\pi_1, \pi_2, ..., \pi_T\}$, and $\beta(m)$, possibly non-unique, such that

$$E\left[\pi_t | m_t = m \ \forall t\right] - \beta(m) = 0. \tag{15}$$

A SCE represents a good approximation to steady-state expectations when the seigniorage regime process is highly persistent and stays in a fixed mean seigniorage state for a long time. This definition of equilibrium has the advantage that the dynamical behavior of inflation expectations around the SCEs can be summarized by means of a Kushner-Yin ordinary differential equation (ODE): when agents are confident about their last-period's beliefs, i.e., when ν is close to zero ($\nu \to 0$), and when the seigniorage regime becomes persistent, the sequence of inflation beliefs $\{\beta_t\}$ converge in distribution to a random variable which is the solution to the following ODE:

$$\dot{\beta} = \hat{G}(\beta, m), \tag{16}$$

for a broad class of probability distributions of $\varepsilon_d(s_t, \bar{d}(m))$, including the one we considered here. Kushner and Yin (2003) contains further technical details. A fixed-m SCE is a fixed point of β, that is $\dot{\beta} = 0$, or $\hat{G}(\beta(m), m) = 0$, where

$$\begin{aligned}\hat{G}(\beta, m) \equiv& E[\tilde{g}(\pi_t^*, \beta, d_t(m_t, \varsigma_t, d_{t-1})))|m_t = m \ \forall t] \\ =& \sum_{k=1}^{\varsigma_h} \left[\theta\lambda(\beta)\Psi_{[m,k]}(\beta, \tilde{\omega}(\beta))\right] \bar{q}_{\varsigma,k} \\ &+ \sum_{k=1}^{\varsigma_h} \bar{\pi}_1^*(k) \left[1 - \Phi_{[m,k]}(\beta, \tilde{\omega}(\beta))\right] \bar{q}_{\varsigma,k} - \beta. \end{aligned} \tag{17}$$

This particular ODE implies that for each m based on equation (16), there exists at least one conditional SCE.

Proposition 1. *If $1 - 1/\lambda_1 < \theta$, there exists at least one conditional fixed-m SCE for every m.*

The proof of this proposition is in online appendix C. In general, the model can have up to three SCEs. We denote these equilibria as follows: first, a low-inflation stable SCE denoted by $\beta_1^*(m)$, which is typically very close to the low-inflation deterministic SSE $\pi_1^*(m)$; a high-inflation unstable SCE denoted by $\beta_2^*(m)$; and a very high-inflation stable SCE denoted by $\beta_3^*(m)$.

We depict the typical situations that can arise depending on the state m (i.e., the level of the seigniorage-mean) in figures 3a and 3b. These figures have been constructed with $\lambda_0 = 0.30$, $\lambda_1 = 30$, $\vartheta = 2$, $\delta = 0.01$, $\theta = 0.99$, and $\gamma = 1$. The seigniorage levels d_1 to d_5 are respectively 0.0080, 0.0075, 0.0070, 0.0055, and 0.0053. We see that for the highest seigniorage-mean, there is only one equilibrium $\beta_3^*(m_1)$, and it is stable, i.e. $\dot{\beta} > 0$ for $\beta < \beta_3^*(m_1)$, and $\dot{\beta} < 0$ for $\beta > \beta_3^*(m_1)$. The level d_2 has two equilibria: The first one, called $\beta_2^*(m_2)$, is unstable if $\beta > \beta_2^*(m_2)$, and the second one is $\beta_3^*(m_2)$ and it is stable. For the level d_3, the model has three equilibria, $\beta_1^*(m_3)$ and $\beta_3^*(m_3)$ are stable, but $\beta_2^*(m_3)$ is unstable, i.e. $\dot{\beta} < 0$ for $\beta < \beta_2^*(m_3)$ and $\dot{\beta} > 0$ for $\beta > \beta_2^*(m_3)$. The fourth level d_4 has two equilibria: A stable low equilibrium $\beta_1^*(m_4)$, and a very high equilibrium $\beta_3^*(m_4)$ which becomes unstable as soon as $\beta < \beta_3^*(m_4)$.

The fifth state m_5 has only a stable low equilibrium $\beta_1^*(m_5)$. It is also noteworthy that, when the economy is in the three-equilibria situation, (i.e.

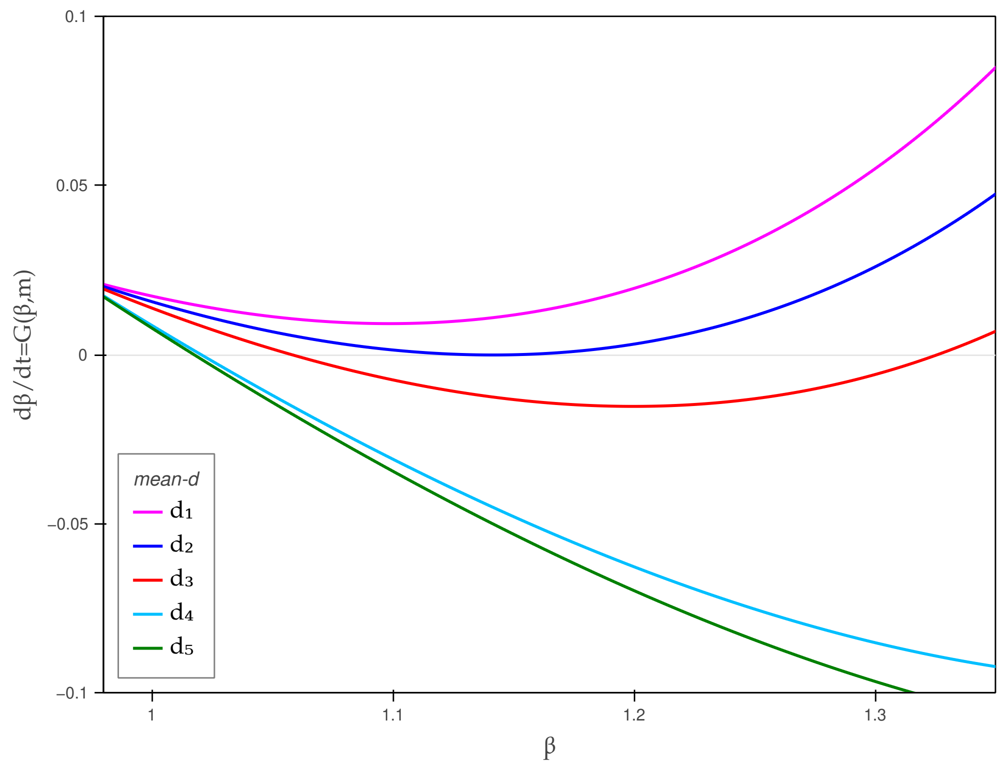

Figure 3a: **Seigniorage Means and Conditional Self-Confirming Equilibria**. Note: For intermediate levels of the mean seigniorage, such as d_3, the model has three conditional SCEs: Two of them are visible in this figure and the third one is visible in figure 3b. d_2 and d_4 have two equilibria, whereas d_1 and d_5 have one equilibrium each. Levels d_1 to d_4 have an equilibrium with very high inflationary expectations; they are visible in figure 3b. The lowest mean seigniorage level, d_5, only has one SCE, shown here at a very low level of inflationary expectations.

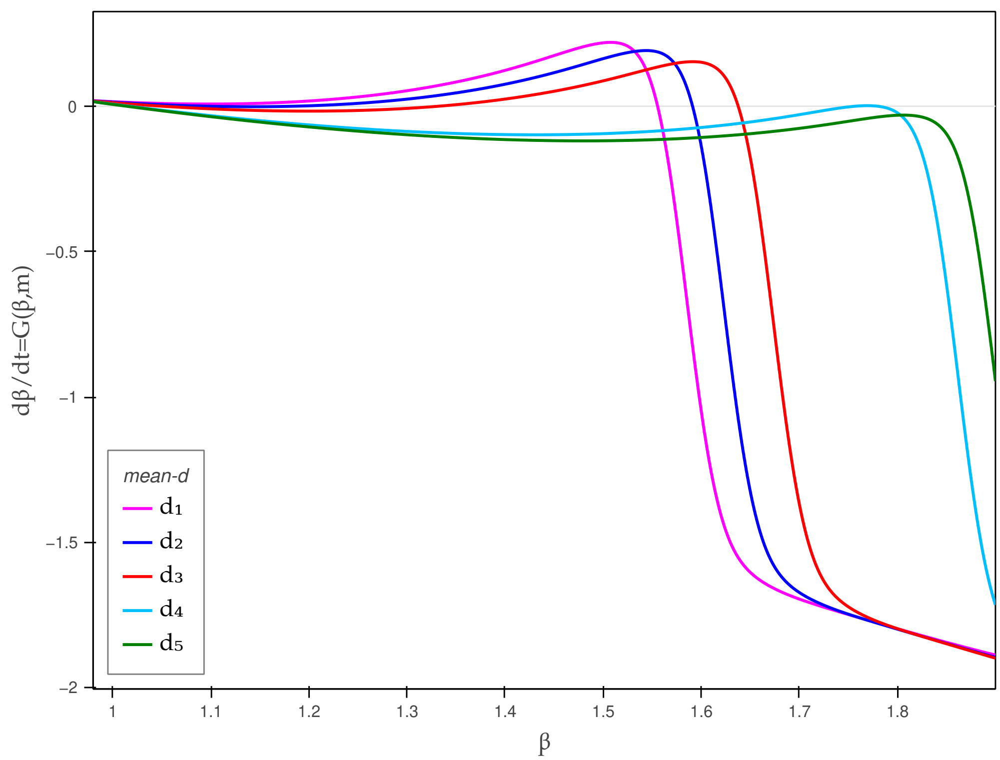

Figure 3b: **Seigniorage Means and Conditional Self-Confirming Equilibria**. Note: This figure shows that, for mean seigniorage levels on or above a threshold, here d_4, the model has a stable conditional SCE at a very high inflation level. Counterintuitively, for all mean seigniorage levels above d_4, this high equilibrium increases, not decreases, as the seigniorage-mean decreases. See the main text for the parameter values employed to elaborate figures 3a and 3b.

$d_4 < d < d_2$), and if the model switches to a higher mean-seigniorage regime inside this interval, $\beta_1^*(m)$ increases, but $\beta_2^*(m)$ decreases. Accordingly, the domain of attraction of $\beta_1^*(m)$ shrinks, whereas the one belonging to $\beta_3^*(m)$ expands, increasing the probability of jumping to the domain of attraction of the latter. Such a jump is defined below as an escape event. Analytically, we have $\beta_2^*(m') > \beta_2^*(m'')$, whenever $d_4 < d(m') < d(m'') < d_2$. A higher seigniorage-mean regime may not only lead to a greater level of inflation but to a smaller (bigger) interval in which inflation expectations are in the domain of attraction of the low- (very high-) level SCE, and thus, a higher probability of an escape event.

When there is a single equilibrium–that is, when $\beta_2^*(m)$ does not exist– inflation expectations are always stable. This is obvious when the unique equilibrium is $\beta_1^*(m)$. But even For high seigniorage-mean levels, the stable level of expected inflation may be $\beta_3^*(m)$, and it still implies stable hyper-inflationary equilibrium inflationary expectations.

It is a curious feature of the model that, whenever the economy is in the domain of attraction of the high inflation equilibrium, a higher mean seigniorage regime may lead to a decrease, not an increase in equilibrium inflation. In other words, we will get $\beta_3^*(m') > \beta_3^*(m'')$, whenever $d_4 < d(m') < d(m'')$. This is a result of the interaction of the slow increase in equilibrium inflation as mean seigniorage increases, and the bound on inflation given by δ. To see why, let us examine equation (17). The first two terms give the "physical" expected value of inflation –not the households' inflation beliefs–, roughly, as a weighted average of inflation when there isn't a reset shock, and inflation when there is a reset shock, with the weights given by the probability mass of each of these two cases. Now, when the mean seigniorage is higher, the weight for the first term, the probability mass of inflation below the bound, will decrease given each value of β, whenever inflation is close enough to the bound. Further, the weight for the second term, the probability mass of reset shocks, will increase, as inflation approaches the bound, given each value of β. As inflation approaches the bound more often when the mean seigniorage is higher, cosmetic reforms are triggered more often, driving down the expected value of inflation.

A necessary condition to maximize the chance of the success of a stabilization program is to ensure that the economy finds itself in the domain of attraction of a low-inflation equilibrium, after the program, and this is achieved by reducing the seigniorage level from d_1 to a level $d < d_4$.

Inflation cannot be lowered by gradually reducing the seigniorage from

d_1 to levels higher than d_4, unless the economy experiments a shock that accidentally takes inflation expectations to the domain of attraction of the low-level SCE. At the beginning of such a program, inflation will not fall, and it may even become higher. The necessary reduction in the seigniorage-mean level has to be done at once, not progressively.

Thus, this model proposes that a fiscal reform drastically cutting seigniorage finance is a necessary condition to reduce inflation. Further, we find that a reform must be strong enough if the economy starts its stabilization program when it is inside the domain of attraction of a high-inflation equilibrium, as it often has happened in Latin American monetary history.

As mentioned, SCEs are defined for each m-state and they determine the stability regions of inflation expectations. While economic agents form their inflation beliefs following the rule in equation (3), the SCEs represent the average dynamics of such expectations as $\nu \to 0$. They will be close to the actual dynamics when the seigniorage state m is highly persistent. Following SWZ, we also define an unconditional SCE.

Definition 2. An unconditional self-confirming equilibrium (SCE) is a probability distribution over inflation histories $\beta^T \equiv \{\pi_1, \pi_2, ..., \pi_T\}$ and a β such that

$$E[\pi_t] - \beta = 0. \tag{18}$$

This equilibrium is found by finding the zero(s) of the following function:

$$\begin{aligned}\hat{G}(\beta) \equiv & E[\tilde{g}(\pi_t^*, \beta, d_t(m_t, \varsigma_t, d_{t-1}))] \\ = & \sum_{k=1}^{h} \left[\theta\lambda(\beta)\Psi_{[k]}(\beta, \tilde{\omega}(\beta))\right] \bar{q}_k \\ & + \sum_{k=1}^{h} \bar{\pi}_1^*(k) \left[1 - \Phi\left(\frac{\log(\tilde{\omega}(\beta) - \log(\bar{d}(k))}{\sigma_d(k, \bar{d}(k))}\right)\right] \bar{q}_k - \beta,\end{aligned}$$

where $\bar{q}_k$ is the ergodic distribution of the joint state k. Note that $\hat{G}(\beta)$ is an expectation over m of $\hat{G}(\beta, m)$. Thus, using arguments similar to those in Proposition 1, it can be shown that there exist up to three unconditional SCEs and at least one unconditional SCE.

6 Escapes and Reforms

In this section we introduce some additional definitions that will be helpful to tie model dynamics with monetary and fiscal policy discussions. They will be helpful first, to describe when an economy switches into a hyper-inflationary regime, and second, to classify monetary and fiscal reforms in such a way that we can diagnose why they were successful or unsuccessful. We will proceed using the definition of SCE that fixes m, i.e., with a fixed seigniorage-mean level. This will usually be the most useful case for practical purposes because we want to evaluate the most likely path of the economy given a policy stance.

Definition 3. A *fundamental reform* takes place when there is a switch from a high mean-seigniorage state m to a lower one.

While a regime change towards a lower mean-seigniorage state is a necessary condition to stabilize high inflation, it is not a sufficient condition, since it might be required additionally to switch to a regime with a low enough $\bar{d}$ such that inflation expectations find themselves into the domain of attraction of a low inflation equilibrium, as explained in the previous section.

Definition 4. A *cosmetic reform* occurs when there is a large negative shock in inflation, but the current state m remains the same. Such a shock is constructed by setting inflation to the inflation's low deterministic SSE value $\pi_1^*(m_t)$ plus some noise:

$$\pi_t^* = \pi_1^*(m_t) + \varepsilon_\pi, \tag{19}$$

where ε_π has the probability density:

$$p_\pi(\varepsilon_\pi|\tilde{m}_t) = \frac{\exp\{-\log[\pi_1^*(m_t) + \varepsilon_\pi] - \log\pi_1^*(m_t)]^2/2\sigma_\pi^2\}}{\sqrt{2\pi}\sigma_\pi[\pi_1^*(m_t) + \varepsilon_\pi]\Phi[(-\log\delta - \log[\pi_1^*(m_t)])/\sigma_\pi}, \tag{20}$$

if $-\pi_1^*(m_t) < \varepsilon_\pi < 1/\delta - \pi_1^*(m_t)$, and $p_\pi(\varepsilon_\pi|\tilde{m}_t) = 0$ otherwise[1].

This definition captures those unsuccessful reforms that lowered inflation temporarily, but without affecting inflationary expectations, and consequently, high inflation resumed shortly thereafter.

Since the model has up to two stable equilibria, it is an important event when the economy switches from the domain of attraction of the low to that of the high SCE. Based on this observation, we define an escape as follows:

[1] Note that $\pi_t^* = \pi_1^*(m_t) + \varepsilon_\pi > 0$ if and only if $\varepsilon_\pi > -\pi_1^*(m_t)$. Moreover, if $\varepsilon_\pi < 1/\delta - \pi_1^*(m_t)$, then $\pi_t^* = \varepsilon_\pi + \pi_1^*(m_t) < \delta^{-1}$.

Definition 5. An escape takes place when inflation beliefs fall outside the domain of attraction of the low and stable SCE, $\beta_1^*(m)$, and inside the domain of attraction of $\beta_3^*(m)$, the high SCE. This is highly likely if $\beta_t > \beta_2^*(m)$ whenever $\beta_2^*(m)$ exists.

If there is just one SCE, being either $\beta_1^*(m)$ or $\beta_3^*(m)$, there are no escapes, but we can still compute escape probabilities depending on the relevant domain of attraction, according to Definition 5; thus, we can define escape probabilities by taking these situations as special cases, as we will do below. For the general case, with more than one SCE, online appendix D obtains the following escape probabilities:

$$\begin{aligned} Pr\{\underline{\omega}_t(m_0,\varsigma_0) < \varepsilon_d(s_0,\pi^{t-1},\phi) < \overline{\omega}_t(m_t,\varsigma)|s_0,\pi^{t-1},\phi\} \\ = \sum_{s_0=1}^{h} Pr(s_t = s_0|\pi^{t-1},\phi)\Big[F_d(\overline{\omega}_t(m_0,\varsigma_0)|s_0,d_0) \\ - F_d(\underline{\omega}_t(m_0,\varsigma_0)|s_0,d_0)\Big]. \end{aligned} \tag{21}$$

7 Data Sources

Inflation data was obtained by calculating the percentage changes in the seasonally adjusted monthly consumer price index series from the National Statistics Institute (Instituto Nacional de Estadística y Geografía, INEGI). Our dataset comprises the period starting in February 1969 and ending in July 2019 (figures 7 and 8). This was the only series used for model estimation.

For contrasting purposes we also obtained monthly and quarterly data for the money supply, figures 9 and 10, fiscal deficits, external and internal debt, figure 11, and gross domestic product. The sources are the International Financial Statistics from the International Monetary Fund (IFS-IMF), the Bank of Mexico and INEGI, again. All are seasonally unadjusted and monthly data series, with exception of the GDP data which is quarterly. To obtain monthly GDP data we interpolated the series. We used the GDP data to re-scale the fiscal deficit and the public sector debt monthly data. Since neither debt nor the deficit are corrected for seasonality, sometimes it is convenient to calculate a trailing 12-month sum, resulting in a monthly series for fiscal deficit as a percent of GDP starting in March 1980 (figure 11).

Money supply data is available from the IFS-IMF starting in 1957, and also from the Bank of Mexico, but only the most basic aggregates e.g. the monetary base, are reliable. The other aggregates M1, M2, M3 and M4, have been computed using inconsistent methodologies over time, and do not seem reliable, overall, with exception of the data from the last two decades. Therefore we use mainly M0 data, but also review M1 and M2 data from the Bank of Mexico in figure 10. These are series that have not been corrected for seasonality and we either smooth them out with a 12-month moving average or directly adjust the monthly changes using the seasonally adjusted annual rate method. The Bank of Mexico makes available data using two methodologies to calculate monetary aggregates: the 1999 and the 2018 methodology, respectively. Their data series spanning December 1985 to December 2017 uses the first methodology and another one starting in December 2000 uses to second methodology.

To help bring some context to our discussion of Mexican stabilization programs we also obtained monthly data for interest rates, figure 12, and exchange rates, figure 13, from the Bank of Mexico and the IFS-IMF. Exchange rates are available for our whole period of the study. Mexican T-bill rates are the average annualized rate in the primary market of the 28-day Certificados de Tesorería de la Federación (CETES), available starting in January 1978. Finally, the Bank of Mexico policy rate series starts in December 2001.

8 Model Estimation

The model has three key variables: inflation, inflation expectations, and seigniorage. Inflation is the only input variable used to estimate the model. The mean and variance regime states are unobservable to the econometrician, but we can estimate the probability of being in a certain regime state in a given period t. We interpret the state with the highest probability in a given period as indicative of the regime state prevalent in the economy in that period.

In comparison with SWZ's model, whose likelihood is given in closed form, estimating our model involves evaluating a likelihood in implicit form. Since the likelihood evaluation is now more susceptible to numerical errors, the optimization is much more difficult, and the likelihood evaluation code must be written carefully. Furthermore, our Kushner-Yin ODE must be evaluated with a Monte Carlo simulation method, while SWZ's Kushner-

Yin ODE is available in closed form. We also compare our model-predicted seigniorage and money supply with those implied by SWZ's model. The model in Sargent, Williams and Zha (2009) has been estimated preliminarily in Ramos-Francia, García-Verdú and Sánchez-Martínez (2018)[2], but the results in this paper use different estimates, resulting from a more accurate likelihood optimization algorithm and using an explicit model selection step.

We estimate the parameters ν, λ_0, λ_1, σ_π and ϑ from the data, along with the regime transition probabilities and the seigniorage mean and volatility in each state. The parameters γ, δ, and θ are set by calibration. The parameter γ is invariant to a re-normalization of $\bar{d}(m_t)$ and $\sigma_d^2(\varsigma_t)$ by some constant.It determines a standardization of the price level and the nominal money stocks. Without loss of generality, we set $\gamma = 1$ and $\delta = 0.01$. We assume that $\theta = 0.99$. Finally, we set $\beta_0 = \pi_1$. These values are in line with those used in SWZ.

One key aspect of the model is the number of seigniorage and volatility regime states which are quantities that are fixed before the estimation. To determine them, we use the Schwarz Criterion, a.k.a. Bayesian Information Criterion (BIC), defined as BIC= $L(\hat{\phi}) - \log(T)k/2$, where $L(\hat{\phi})$ is the maximized log-likelihood, T is the number of observations, and k is the number of estimated parameters. The results are reported in table 1. We found that the best model, according to the BIC, has six regimes for m and two for ς.

In the case of the 6×2 model, we impose the following restrictions on the transition matrices, following Sims, Waggoner, and Zha (2008) and SWZ:

$$Q_m = \begin{pmatrix} p_{11} & 1-p_{11} & 0 & 0 & 0 & 0 \\ \frac{1-p_{22}}{2} & p_{22} & \frac{1-p_{22}}{2} & 0 & 0 & 0 \\ 0 & \frac{1-p_{33}}{2} & p_{33} & \frac{1-p_{33}}{2} & 0 & 0 \\ 0 & 0 & \frac{1-p_{44}}{2} & p_{44} & \frac{1-p_{44}}{2} & 0 \\ 0 & 0 & 0 & \frac{1-p_{33}}{2} & p_{55} & \frac{1-p_{55}}{2} \\ 0 & 0 & 0 & 0 & 1-p_{66} & p_{66} \end{pmatrix},$$

$$Q_\varsigma = \begin{pmatrix} q_{11} & 1-q_{11} \\ 1-q_{22} & q_{22} \end{pmatrix}.$$

We estimate the model by maximizing the likelihood

$$\max_\phi p(\pi^T|\phi),$$

[2] I thank CEMLA for providing access to this paper's code.

		$\text{BIC}= L(\hat{\phi}) - \log(T)k/2$
$n_m = 6$	$n_\varsigma = 2$	2498.42
$n_m = 6$	$n_\varsigma = 1$	2439.85
$n_m = 6$	$n_\varsigma = 3$	2449.32
$n_m = 5$	$n_\varsigma = 2$	2494.97
$n_m = 7$	$n_\varsigma = 2$	2497.80

Table 1: **Model Selection**. We used the BIC to choose the optimal number of states. $L(\hat{\phi})$ is the maximized log-likelihood, T is the number of observations, k is the number of estimated parameters, n_m is the number of states for the mean seigniorage, and n_ς is the number of states for the seigniorage volatility. On the top row, we display the optimal number of states. Then, we show the BIC calculated for small changes in the number of states. We calculated the BIC for several other combinations of the number of states, always obtaining lower values.

where $p(\pi^T|\phi)$ is the inflation's likelihood function implied by the model. Online appendix F contains its derivation. Specifically, the parameter vector is $\phi = (\nu, \lambda, \vartheta, \bar{d}_{\{i\}}), \sigma_{\{j\}}, p_{\{i,i\}}, q_{\{j,j\}}, \sigma_\pi)$, where $i = 1, ..., 6$ and $j = 1, 2$.

To estimate the model's parameters, we used the maximum likelihood method and verified that the likelihood function is concave in the vicinity of the estimated parameters. Furthermore, the estimation algorithm is repeated many times with different initial parameter values and it never finds any sign of existence of a different set of parameters that could improve the likelihood value. We conclude that there is enough evidence to assert that the estimation exercise was successful and that the reported set of estimated parameters is indeed the solution of the maximum likelihood problem.

However, online appendix E sets necessary conditions for existence of a solution for this estimation exercise: instead of attempting to find an estimator with the best statistical properties, we examine there what information and what features of the model are important to identify the parameters. The usefulness of this exercise lies in terms of illustrating the limitations of the model to match the data and the feasibility of potential model extensions and new estimation methods.

Estimated parameter standard errors are obtained with the Hessian matrix of the likelihood function and the delta method. Table 2 presents our

Parameter	Estimate	Standard Errors
ν	0.014	0.001
λ_0	0.178	0.105
λ_1	29.27	2.071
ϑ	0.702	0.288
$\bar{d}_1$	0.0062	0.003
$\bar{d}_2$	0.0044	0.004
$\bar{d}_3$	0.0035	0.002
$\bar{d}_4$	0.0028	0.002
$\bar{d}_5$	0.0023	0.001
$\bar{d}_6$	0.0021	0.001
σ_1	1.904	1.680
σ_2	0.666	0.580
$p_{1,1}$	0.87	0.065
$p_{2,2}$	0.90	0.062
$p_{3,3}$	0.84	0.056
$p_{4,4}$	0.87	0.044
$p_{5,5}$	0.88	0.045
$p_{6,6}$	0.97	0.019
$q_{1,1}$	0.71	0.070
$q_{2,2}$	0.90	0.029
σ_π	0.03	3.812

Table 2: **Parameter Estimates**. Note: The maximized log-likelihood is 2565.688. The estimation sample comprises February 1969 to July 2019.

estimates $\hat{\phi}$ and their corresponding standard errors. We obtain a small ν implying that agents assign more weight to their previous inflation beliefs than to their past errors to form their inflation expectations.

The transition probability estimates ($p_{i,i}$) show that seigniorage-mean regimes are quite persistent, and for volatility regimes both states are persistent as well (i.e., estimates of $p_{i,i}$ and $q_{j,j}$ are close to one). Nonetheless, the high variance regime state (σ_1) is not as persistent as the low variance one, as $q_{2,2} > q_{1,1}$. A small ν and persistent mean seigniorage regimes confirm that SCEs are a good approximation to the true stochastic equilibrium.

Deficit-Mean Regime (Unconditional Probabilities)	
$m = 1$	0.15
$m = 2$	0.10
$m = 3$	0.13
$m = 4$	0.15
$m = 5$	0.17
$m = 6$	0.29
Deficit-Variance Regime (Unconditional Probabilities)	
High ($\varsigma = 1$)	0.25
Low ($\varsigma = 2$)	0.75

Table 3: **Stationary Markov Regimes Probability Estimates**. Note: By independence between the mean and variance regime states, the joint states' stationary probabilities are just the product of the seigniorage-mean and variance marginal probabilities. Probabilities may not sum to 1 due to rounding.

Table 3 presents the stationary probabilities for all regime states. Roughly, they capture the fraction of time the economy spends in each regime during the sample period. The unconditional probability of the low-variance regime is relatively high. This is because the economy seems to switch to the high-variance regime only occasionally, when there is a shock too high to be explained either by the seigniorage-mean state alone or by the low regime seigniorage volatility. As the seigniorage and the volatility states are assumed to be independent, the joint states' unconditional probabilities are just the product of the probabilities of the mean and volatility states.

We estimate the probabilities of the seigniorage-mean states conditioning

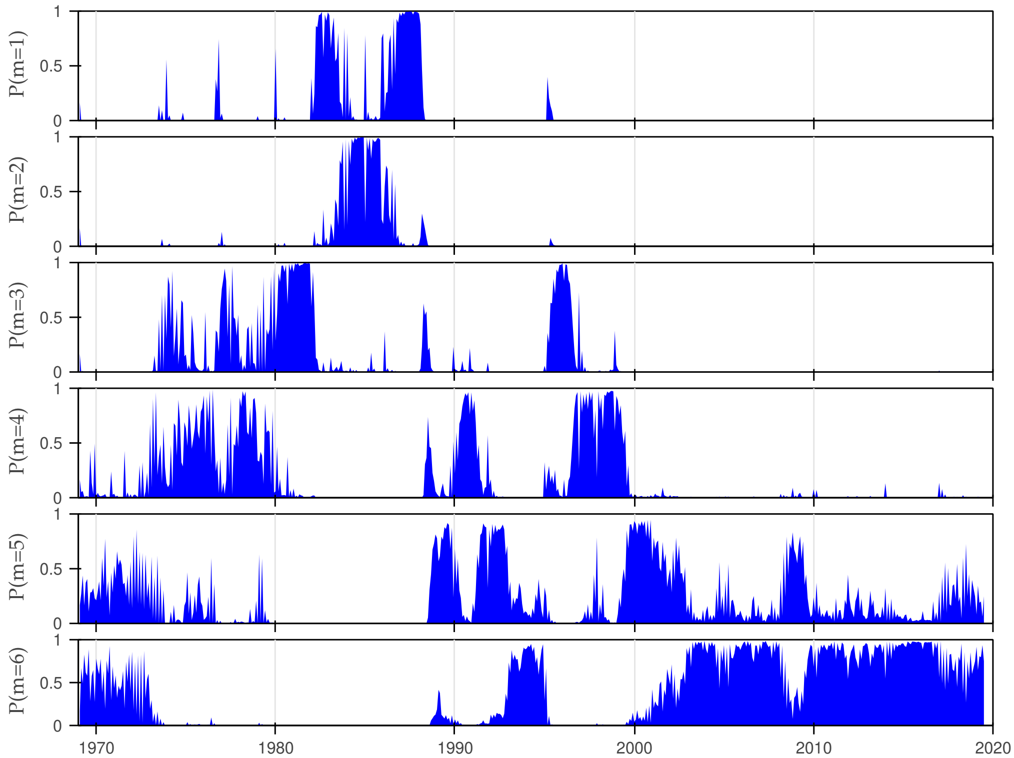

Figure 4: **Probabilities of being in each seigniorage-mean regime, conditional on the information in period *t-1***. Note: $m_t = 1$ denotes the highest seigniorage-mean state, and $m_t = 6$ the lowest seigniorage-mean state, respectively. As the seigniorage-mean state is not observable, this figure depicts the estimated probabilities of being in each state at each period.

Deterministic Equilibria (SSE)		
$\pi_1^*(1), \pi_2^*(1)$		π_{max}^*, n.a.
$\pi_1^*(2), \pi_2^*(2)$		$1.0803, 1.2550$
$\pi_1^*(3), \pi_2^*(3)$		$1.0258, 1.6558$
$\pi_1^*(4), \pi_2^*(4)$		$1.0108, 2.1405$
$\pi_1^*(5), \pi_2^*(5)$		$1.0049, 2.5914$
$\pi_1^*(6), \pi_2^*(6)$		$1.0029, 2.8380$
π_{max}^*		1.1447
Unconditional SCE		K-Y ODE
$\pi_1^*, \pi_2^*, \pi_3^*$		1.0214, n.a., n.a.
Conditional fixed-m, SCE	K-Y ODE	Monte Carlo
$\pi_1^*(1), \pi_2^*(1), \pi_3^*(1)$	n.a., n.a., 1.4236	n.a., n.a., 2.3248
$\pi_1^*(2), \pi_2^*(2), \pi_3^*(2)$	1.0962, 1.2055, 1.5899	1.0921, 1.2190, 2.7824
$\pi_1^*(3), \pi_2^*(3), \pi_3^*(3)$	1.0271, n.a., n.a.	1.0266, n.a., n.a.
$\pi_1^*(4), \pi_2^*(4), \pi_3^*(4)$	1.0112, n.a., n.a.	1.0112, n.a., n.a.
$\pi_1^*(5), \pi_2^*(5), \pi_3^*(5)$	1.0052, n.a., n.a.	1.0050, n.a., n.a.
$\pi_1^*(6), \pi_2^*(6), \pi_3^*(6)$	1.0031, n.a., n.a.	1.0030, n.a., n.a.

Table 4: **Deterministic and Self-Confirming Equilibria**. Note: SSE's are computed by solving numerically an implicit nonlinear equation. The value π_{max}^* is imputed when a low SSE does not exist. K-Y ODE SCEs are the zeros of the respective Kushner-Yin (K-Y) ordinary differential equations. Monte Carlo SCE's are obtained by simulating the monetary market with the estimated parameters. Non-existent equilibria are denoted as "n.a.".

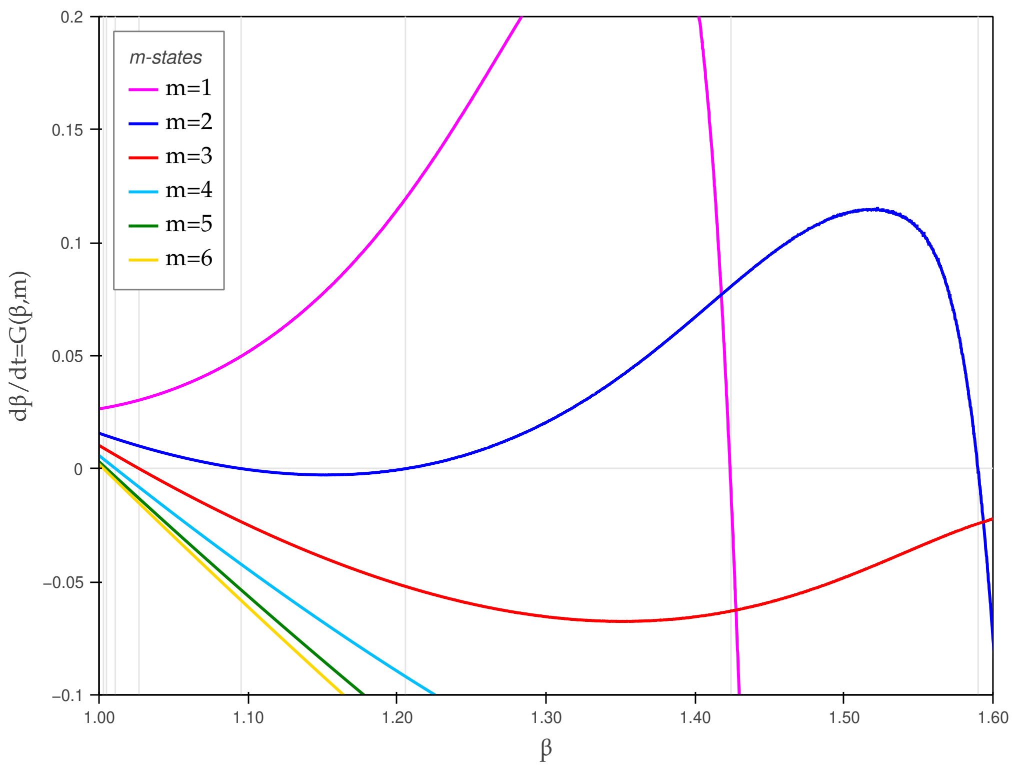

Figure 5: **Conditional fixed-$\boldsymbol{m}$ SCEs**. Note: Each conditional SCE is determined when the function $G(\beta, m)$ crosses the value of 0 for each seigniorage-mean state m. Because the equation is in continuous time, SCEs are determined when expectations do not change–i.e. $\dot{\beta} = 0$, or $d\beta/dt = 0$. The highest and lowest seigniorage-mean regimes are $m = 1$, and $m = 6$, respectively. For the estimated model, each state has only one conditional SCE, except state $m = 2$, which has three conditional SCEs.

on inflation history up to period $t-1$ and report the results in figure 4.

We now present the equilibrium dynamics implied by the estimated parameter values. First, we implement the Kushner and Yin ODE method, by computing the function $\hat{G}(\beta, m)$ with Monte Carlo integration and then obtained the SCEs by locating the zeros of that function. In our model, we have up to three SCEs, and the first two have similar properties to those in SWZ. In addition, we sometimes obtain a stable SCE with very high inflation. As explained above, we define an escape event as the probability of falling into the domain of attraction of the SCE with very high inflation. We present our estimations of deterministic and self-confirming equilibria in table 4 and figure 5 for the six m-states.

All m-states have at least one fixed-m SCE, as predicted by Proposition 1, and in the case of the second-largest mean seigniorage level, there are three equilibria. Note that, for the high seigniorage state, the stable equilibrium is the very high-inflation SCE and implies annualized rates of more than $6,000\%$ (Table 4 and figure 5). Thus, unless a reform takes place, inflation expectations will significantly and unceasingly amplify until the SCE is reached. It is also surprising that a conditional SCE exists even when its corresponding deterministic steady-state equilibrium does not. The reason is that the definition of SSE effectively caps steady-state equilibria at the level $\pi^*_{\max}$, thereby ruling out "high inflation" steady-states.

We also report fixed-m SCE's obtained with Monte Carlo simulation in table 4. Since their algorithm is designed for dynamical systems with small step sizes, the Kushner-Yin ODE approach may give estimates of SCEs that depart significantly from their true values as inflation and inflationary expectations become sizable and more volatile. Under these circumstances, the Monte Carlo method becomes useful. According to Santos and Peralta-Alva (2005) the law of large numbers implies that the unconditional moments of stable dynamical systems can be computed via Monte Carlo simulation with an error that can be made arbitrarily small by increasing the simulation sample size, even in a situation when there are multiple equilibria, like ours. We used a sample size of $10,000$ plus $1,000$ initial burn-in simulations to allow for convergence to the stable equilibrium regardless of β_t's initial value. The unstable equilibrium was computed by replicating the simulations 500 times and finding the initial value β_0 where the system became more likely to converge to a different stable equilibrium. We can see that the Kushner-Yin approximations work quite well for equilibria with small inflation levels, but the numerical error is quite large for high inflation equilibria.

We conclude that the Kushner-Yin approach is very useful to characterize the dynamical properties of the model, but long-run high inflation equilibrium estimates should be verified using another approach. In what follows, we use the Monte Carlo results as the "true" SCE estimates, while we resort to the Kushner-Yin ODE to explain the model dynamics around those SCEs, particularly when there is a regime change.

9 Fitting The Fiscal Deficit And The Money Supply

Here we examine closely the *ex-post* fit between model-implied seigniorage and Mexican fiscal deficit and between model-implied money supply growth and Mexican monetary base growth. To assess quantitatively the first relationship we attempt to reconstruct the financing of past deficits, allowing for a scaling correction, unobserved variables and observational errors, estimating the linear regression equations

$$\begin{aligned} \text{fiscal deficit}_t &= \alpha_0 + \alpha_1 \Delta\text{external debt}_t + \alpha_2 \Delta\text{internal debt}_t \\ &\quad + \gamma^{\text{SWZ}-1} d_t^{\text{SWZ}} + \text{error}_t \qquad (22) \\ \text{fiscal deficit}_t &= \alpha_0' + \alpha_1' \Delta\text{external debt}_t + \alpha_2' \Delta\text{internal debt}_t \\ &\quad + \gamma^{-1} d_t + \text{error}_t' \qquad (23) \end{aligned}$$

where Δexternal debt$_t$ is external debt in t minus external debt in $t-1$ as a portion of GDP, Δinternal debt$_t$ is the same for internal debt, d_t^{SWZ} and d_t are, respectively, model-implied seigniorage from SWZ's model[3] and ours, between March 1980 and July 2019. The terms α_0 and error$_t$ capture omitted variables and measurement errors. The 12-month cumulative actual fiscal deficits and debt changes as well as model-implied seigniorage are depicted in figure 15. To perform the statistical tests we use non-smoothed monthly increments for the deficit, debt and seigniorage; since both smoothing and aggregation discard possibly relevant variation along with the seasonal effects, their use can cause a loss of statistical power.

The second relationship is assessed by simply measuring how well model-implied money supply reconstructs the actual monetary base, estimating the

[3]Our estimation of SWZ's model with Mexican inflation yielded an optimal number of seigniorage-mean and volatility states of 5 and 3, respectively.

linear regression equations

$$\Delta \mathrm{M0}_t = \alpha_0'' + \gamma'^{\mathrm{SWZ}} \Delta M_t^{\mathrm{SWZ}} + \mathrm{error}_t'' \tag{24}$$

$$\Delta \mathrm{M0}_t = \alpha_0''' + \gamma' \Delta M_t + \mathrm{error}_t''' \tag{25}$$

where $\Delta \mathrm{M0}_t$ is $M0_t/M0_{t-1}$, $\Delta M_t^{\mathrm{SWZ}}$ and ΔM_t are, respectively, model-implied money supply changes from SWZ's model and ours, between February 1969 and July 2019, obtained from equation (5)

$$\frac{M_t}{M_{t-1}} = \theta + d_t \pi_t \frac{1}{M_{t-1}/P_{t-1}}. \tag{26}$$

and substituting model-implied money demand from equation (2) into the last term. Actual and model-implied money supply growth are shown in figures 9 and 14. To compute the tests we use the non-smoothed, seasonally adjusted base money of figure 9.

Monetary base vs. model-implied money supply			
Model Contribution	J-test	d.f.	p-value
S-L model	2.0330	603	0.02
SWZ model	1.1381	603	0.13
Public sector deficit vs. model-implied seigniorage			
Model Contribution	J-test	d.f.	p-value
S-L model	4.2811	468	1.13×10^{-5}
SWZ model	1.3241	468	0.09

Table 5: **Selden-Latané and SWZ Model Comparison**. Note: The J-tests imply that the S-L model significantly improves upon SWZ's model predictions, but when the S-L model is adopted, SWZ's model fail to improve significantly the existing results. Parameter standard errors are heteroskedasticy-consistent as in MacKinnon and White (1985).

Choosing between (22) and (23) and between (24) and (25) involves testing (twice) two non-nested hypothesis, which we accomplish by means of the J-test of Davidson and MacKinnon (1981). The results are in table 5: the J-tests for the null that our model equations (23) and (25) are not significantly better than SWZ's (22) and (24) give very small p-values implying a rejection of both null hypotheses. On the other hand, the J-tests for the null hypothesis that SWZ's equations (22) and (24) are not significantly better than ours (23) and (25) fail to reject the null at the 95 percent confidence level. In conclusion, the J-tests detect a significantly statistical improvement of our model with respect to the SWZ model.

10 Regime Switching, Fiscal and Monetary Policy in Mexico: 1969-2019

In this section, we analyze the case of Mexico with a model that uses a Selden-Latané money-demand function. There are three key themes that underlie our examination of the historical events in Mexican monetary and fiscal policy during the last five decades. First, the confirmation of escape probabilities as a predictor of fundamental reforms. Second, that our choice of money-demand specification yields a model with different equilibrium dynamics, in the sense that our model always has an equilibrium, even under hyper-inflation, regardless of the prevailing seigniorage-mean regime. Third, that a model's predicted relationship between fiscal deficits and seigniorage depend on the money-demand functional form. We introduce the reader to these themes before a more detailed analysis.

Regarding the first theme, SWZ show in their study of five South American countries: Argentina, Bolivia, Brazil, Chile, and Perú, that monetary and fiscal authorities usually implemented fundamental reforms only when their economies have either fallen, or confront a high probability of falling, into the domain of attraction of unstable inflation expectations.

Our model cannot display "explosive" behavior and our analysis cannot be directly compared with SWZ's. However, in the case of Mexico, the mean seigniorage states with stable "high" inflation equilibria imply, respectively, equilibrium monthly inflation of 132.48 and 178.24 percent, implying in both cases equilibrium inflation at hyper-inflationary levels.

Our findings are consistent with SWZ in the sense that serious reforms were only introduced in Mexico when our model escape probabilities approached one, in other words, when the economy found itself in the domain of attraction of a nearly hyper-inflationary "high" inflation equilibrium. In particular, Mexico in the eighties reached a mean-seigniorage regime $m = 1$ which has only one equilibrium at 132.48 percent monthly inflation. Thus, during this time, Mexican inflationary expectations rose continuously attempting to achieve this equilibrium level. On the other hand, the regime $m = 2$ could not have brought about the needed reforms, because it has three equilibria, with the domain of attraction of "high" inflation located above the 21.90 percent monthly inflation level. Inflationary expectations at such high levels were not reached during the study period. Thus, even in this regime, Mexico was still in the domain of attraction of a "low" inflation equilibrium. All other seigniorage-mean regimes observed in Mexico have only one stable, "low" inflation equilibrium.

Our escape probability, defined as the probability of falling into the domain of attraction of the "high" inflation equilibrium, reached levels close to one first during 1982 and 1983 and then, more intensely, between 1986 and 1988, see figure 8. In both cases, fundamental reforms were implemented but only the second one was successful. That serious reforms came only after the economy fell into the domain of attraction of nearly hyper-inflationary equilibrium inflation levels, is a remarkable confirmation of SWZ's findings, notwithstanding employing here different data and a different model with different dynamics.

The third theme is that our model generates a closer relationship between fiscal deficits, money supply, seigniorage, and inflation, than alternative models, as established in the previous section. To obtain this result, it has been useful to use a money-demand function free of Cagan's paradox. This close association is important because, when a country finds itself in a fiscal dominance regime, seigniorage should closely follow fiscal deficits, because there could not be any association between inflation and fiscal deficits, unless it occurs through seigniorage. The stronger the relation between fiscal deficits and seigniorage, the stronger is the case for fiscal dominance as the explanation of a country's inflation.

It is important to note at this point that Mexico has been in a fiscal dominance regime only up to 1995. In figure 15, the model shows a clear association between the government fiscal deficit and the model-implied seigniorage-financed deficit during the first stage, thus, during this stage unconstrained fiscal largesse inevitably resulted in more inflation. After that year, as expected by an independent central bank, mean seigniorage has stayed at its minimum levels: the central bank no longer needed to raise seigniorage to finance government expenses. Because the predominant mean seigniorage states have been the lowest after 1995 in our model, it is capturing well the independence of the central bank with respect to the fiscal authority.

As explained in the last section, the simulated seigniorage-financed deficit produced by the SWZ model does not have such a clear association with Mexican fiscal deficit as our model-implied seigniorage (figure 15). This result is driven by our choice of Selden-Latané's money-demand function, and not by the number of states or by our volatility function. Since the choice of money-demand function is instrumental to obtain a better fit between seigniorage and fiscal deficits, it is also essential to establish an even stronger case for fiscal dominance as the main determinant of Mexican inflation during the seventies and eighties..

In what follows, we relate in detail past historical episodes in Mexican fiscal policy with the model's regime changes in seigniorage and equilibrium behavior. We use the behavior of the highest probability regimes (figure 6) to organize our exposition into seven periods: a first period with low inflation, then three periods depicting the rise to the highest seigniorage-mean regime, then two periods containing the two successful stabilization reforms, the latter characterized by the presence of reforms preventing fiscal dominance, and a stable last period.

1970–1976	Luis Echeverría Álvarez
1976–1982	José López Portillo
1982–1988	Miguel de la Madrid Hurtado
1988–1994	Carlos Salinas de Gortari
1994–2000	Ernesto Zedillo Ponce de León
2000–2006	Vicente Fox Quezada
2006–2012	Felipe Calderón Hinojosa
2012-2018	Enrique Peña Nieto
2018-2024	Andrés Manuel López Obrador

Table 6: **Presidents of Mexico**.

We simplify some of our statements by assigning certainty to probabilistic situations. For example, sometimes we assert that the economy switches to a low seigniorage-mean state, meaning that the probability of being in such a state is higher now than of being in any other states. Second, model-implied seigniorage refers to the seigniorage-financed portion of the deficit and it should not be confused with actual fiscal deficits.

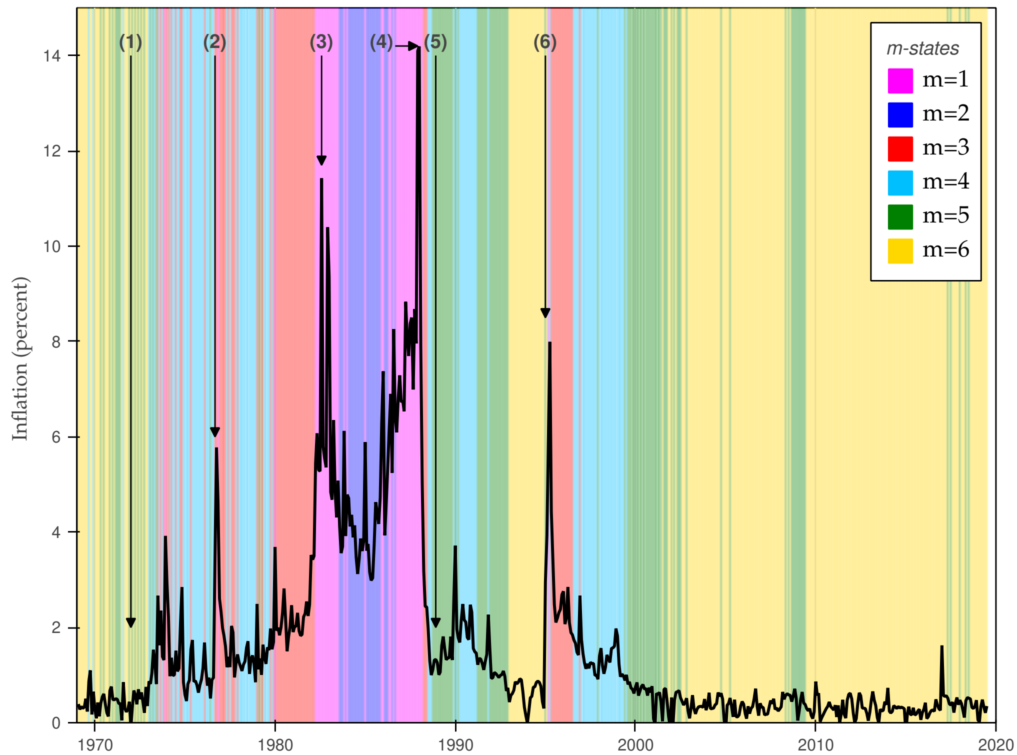

Figure 6: **Highest-Probability Deficit-Mean Regimes, Conditional on the Information up to the Previous Period, and Inflation**. Note: We show the most likely seigniorage-mean state, conditional on the information in period $t-1$. Monthly inflation is depicted on the right-hand scale. Some significant events: (1) Jan. 1972: Echeverría decides to increase public spending in 1972. (2) Sep. 1976: Echeverría's Peso devaluation. (3) Aug. 1982: López-Portillo's debt moratorium and IMF-BIS bailout, followed by De la Madrid's first stabilization program in Dec. 1982. (4) Dec. 1987: De la Madrid's second stabilization program. (5) Dec. 1988: Salinas's Stability and Economic Growth Pact. (6) Jan. 1995: The Tequila crisis started in Dec. 1994 and peaked in Jan. 1995; it ended with a U.S. bailout in Apr. 1995.

10.1 Price Stability: March 1969 to December 1972

Before 1970, inflation stayed low. The government maintained a mostly balanced budget while the monetary base growth was kept under check. Increases in government expenses depended on the expansion of public revenues, while recurring sometimes to commercial banks' reserves as a source of financing. Accordingly, the model-predicted most probable mean-seigniorage regime was a low one, and inflation expectations remained close to the predicted SCE level.

10.2 The beginning of inflationary financing: January 1973 to December 1973

This state of affairs changed during Echeverría's administration. Money-supply expansion due to seigniorage-financed expenses became more common since 1972. In figure 6, we can see that during this administration, the seigniorage-mean level progressively increased starting from year 1973 from the lowest state $m = 6$, to state $m = 3$, albeit with occasional and temporary intermediate switches to state $m = 4$. There was also a brief spike to the highest seigniorage-mean regime (figure 6).

During this time, inflation expectations were clearly lower than their equilibrium SCE levels and started a continuous increase (figure 7). Additionally, bank reserves progressively declined and for this reason, their use as a source of public financing also decreased.

10.3 Sustained fiscal expansion: January 1974 to March 1982

Consistent with developments in fiscal policy, starting in 1974, the Mexican economy settled in the area of seigniorage-mean regimes $m = 3$ and $m = 4$. In 1976, the government devalued the peso, ending the fixed exchange-rate regime in place for more than 20 years, as a consequence of the balance-of-payments crisis of that year.

Early on, López-Portillo's administration implemented a stabilization program backed by the IMF. It was considered a success initially, but it didn't last long because public expenditures started to expand again, this time financed through external debt. The wealth effect produced by the discovery of a super-giant oil field, along with the improved credit-worthiness arising from the recent stabilization reform, enabled the government to obtain financing from commercial banks, neglecting the available credit line from the IMF (IMF, 2001).

Mexican external debt grew substantially, together with fiscal and current account deficits. Indebtedness moderated temporarily the demands for seigniorage financing. Nevertheless, inflation expectations increased continuously.

10.4 The road to out-of-control inflation: April 1982 to March 1988

In April 1982, the Mexican economy reached the highest seigniorage-mean regime $m = 1$ (figure 6). The spike to the highest seigniorage-mean level was accompanied by a spike in the escape probability. In 1982, higher global interest rates and falling oil prices, along with a deteriorating balance of payments accompanied by capital outflows, resulted in increasing inflation and the depreciation of the peso. External debt due payments increased. while losing access to new financing. In this scenario, the government entered a debt moratorium in August. All these problems were compounded by the nationalization of commercial banks by López-Portillo.

De la Madrid's presidential term began with a stabilization plan that included a substantial fiscal retrenchment. This program induced a switch to a lower seigniorage-mean regime ($m = 2$) in 1983 that lasted until 1986 (figure 6), when inflation began escalating again. The temporary switch to a lower seigniorage-mean state did not prevent inflation expectations from continuing to increase, precisely because the SCE of the mean seigniorage regime $m = 2$ was still above current expectations.

According to our model predictions, equilibrium inflation expectations as estimated by the SCE were close to 10% monthly even after switching to a lower mean-seigniorage regime, and this is why expectations kept increasing, although there was a temporary decline in inflation following the stabilization program. A stronger fiscal retrenchment than implemented was required to stabilize inflation. In the end, fiscal accounts could not be balanced, in part, due to two unexpected shocks: the catastrophic 1985 Mexico City earthquake, and the 1986 oil price collapse affecting the terms of trade and the fiscal accounts. Toward the end of the De la Madrid administration, the economy returned to the highest mean seigniorage state due to fiscal

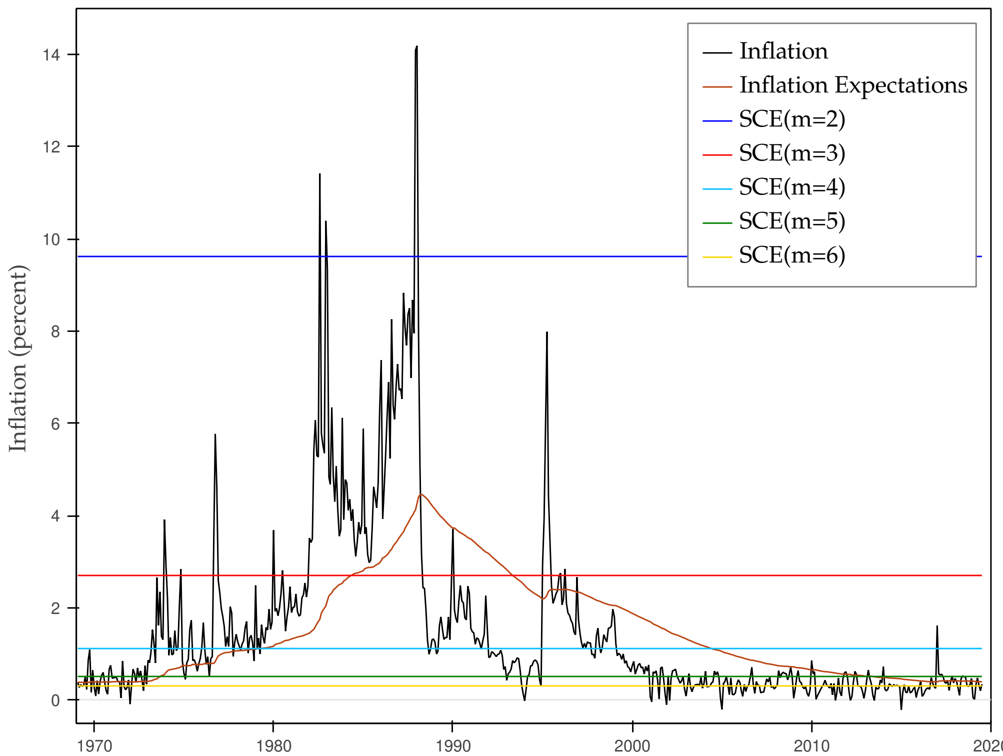

Figure 7: **Monthly Inflation, Inflation Expectations, and Conditional SCEs**. Note: Monthly inflation is the percentage change of the seasonally adjusted monthly CPI. Expectations are defined in equation (4) and shown here using the parameters in table 2. Conditional SCEs are as in definition 1. Their computed values are in table 4 and depicted here by horizontal colored lines. The lowest and highest mean-seigniorage regimes are $m = 6$ and $m = 1$, respectively. There are some conditional SCEs outside this figure. The relevant SCE for most of the 1970s was the corresponding to $m = 4$ (see figure 6) and expectations increased toward this level. After the first De la Madrid's reform in 1982, inflation expectations kept increasing because they were below the equilibrium SCE: the one corresponding to $m = 2$. However, after his second reform in 1987, expectations started to fall toward the SCE corresponding to $m = 4$ almost immediately after switching to this regime.

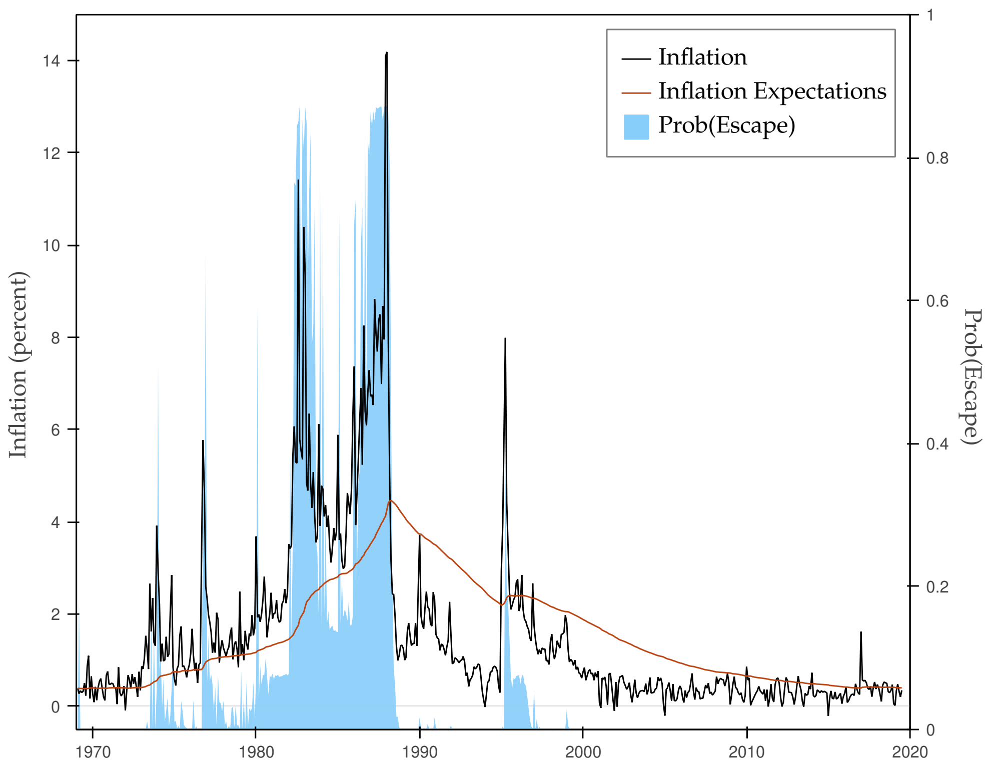

Figure 8: **Escape Probabilities and Inflation Expectations**. Note: This figure overlays monthly inflation and inflation expectations on escape probabilities–i.e., the probability of falling within the domain of attraction of the equilibrium with very high inflation during the next month. The two biggest spikes in escape probabilities occurred during De la Madrid's administration, in August 1982 and in December 1987. Both were followed by fundamental reforms. In the first case we can see a switching, in figure 6, from $m = 1$ to $m = 2$, while the second reformed involved a switching from $m = 1$ to $m = 4$ and then to $m = 5$.

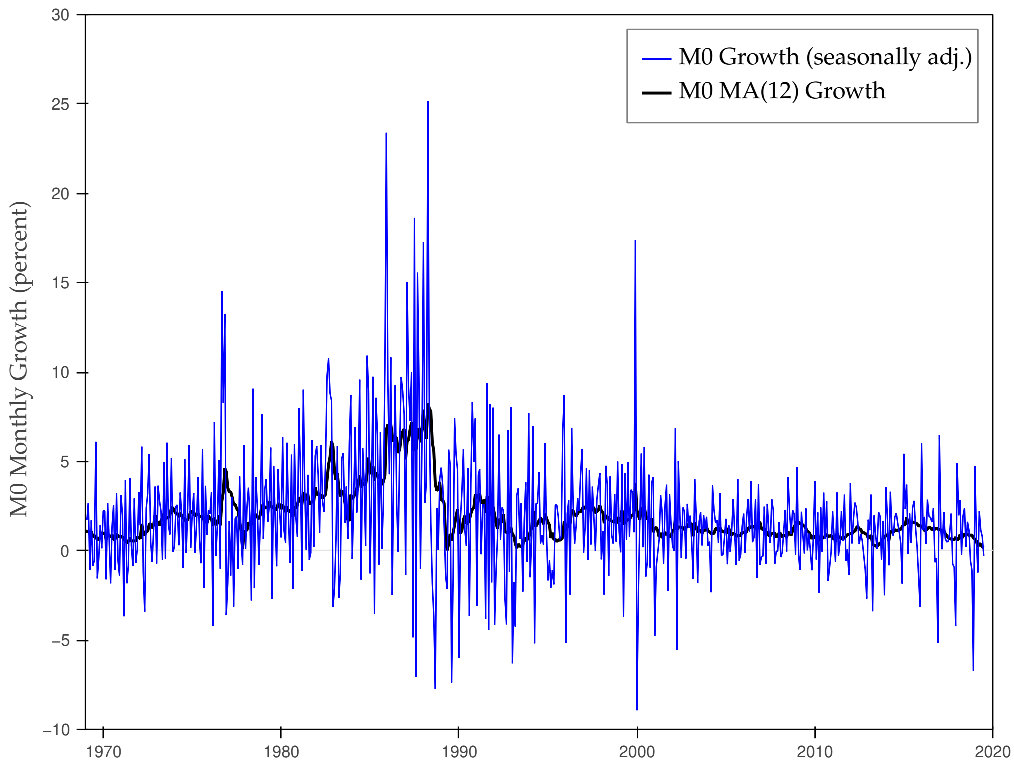

Figure 9: **Monetary Base (M0)**. Note: We show here the month-to-month growth of the 12-month moving average of the monetary base, along with a seasonally-adjusted month-to-month growth series. The adjustment was calculated with the seasonally adjusted annual rate method modified to keep yearly growth unchanged. Nonetheless, after this procedure, the last five years in our data still show some seasonality.

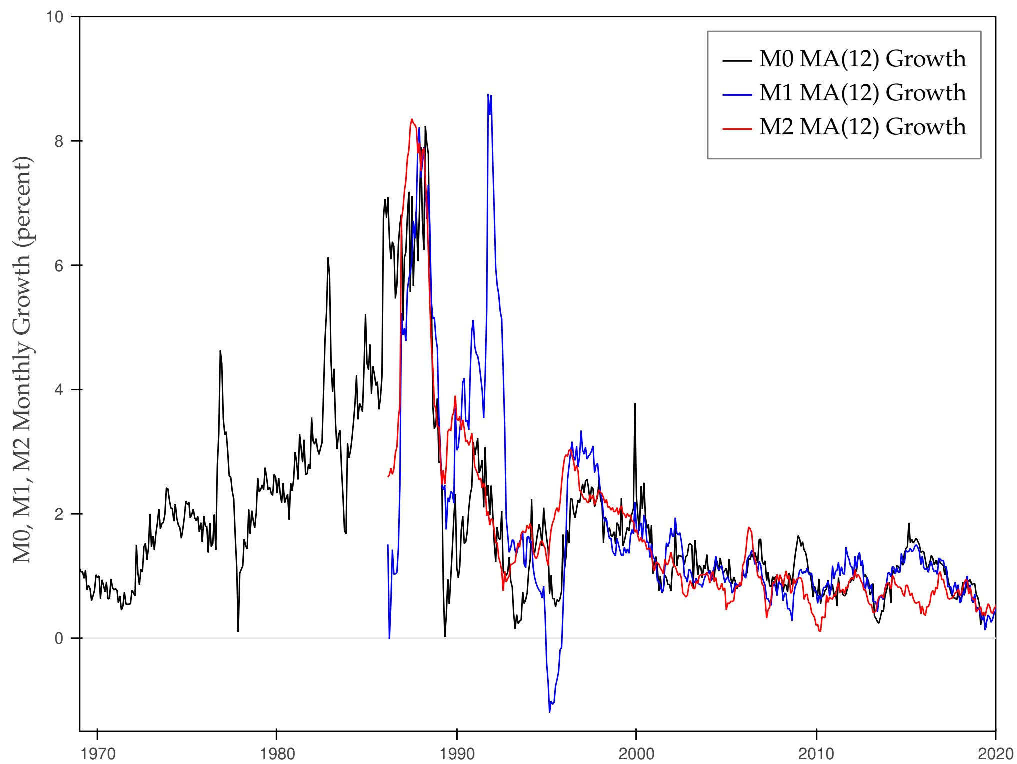

Figure 10: **Monetary Aggregates: M0, M1 and M2**. Note: Shown here are M0, M1 and M2's month-to month growth of their 12-month moving average. M0 is the more reliable aggregate and it is available for our entire period of study. Data for M1 and M2 are available starting from 1986 but do not seem to be reliable before 1995. Here we show the aggregates M1 and M2 calculated with the Bank of Mexico's 1999 methodology until the year 2004; afterwards, we use the data computed with the 1998 methodology. The initial observations seem to not change too much initially, resulting in very low growth rates which is at odds with the monetary base growth and with inflation rates. Moreover, M1 shows a noticeable peak in the early nineties which does not correspond to any event or to changes in the other monetary aggregates.

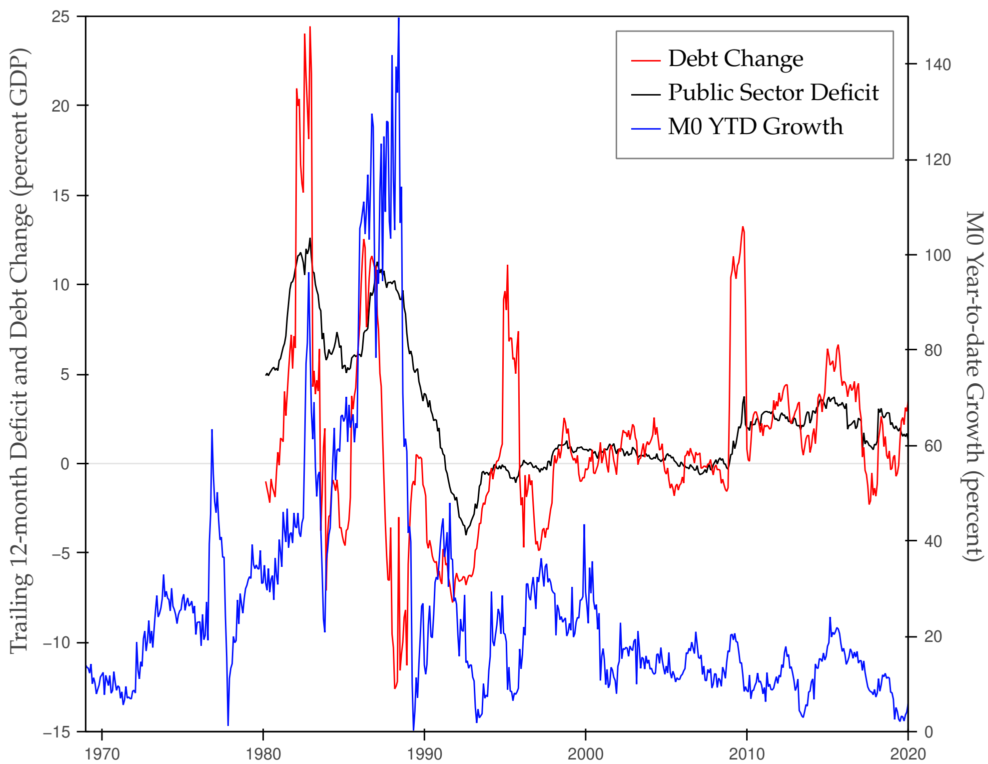

Figure 11: **Monetary Base Growth, Fiscal Deficits, And Public Sector Debt Changes**. Note: This figure shows monthly data for the 12-month cumulative public sector deficit as a fraction of GDP along with year-to-date changes in the monetary base and year-to date changes in the stock of the public sector debt, internal and external. The monetary base growth follows closely the two spikes in the deficit during the eighties and roughly tracks the deficit's inverted-u trend during the nineties and two-thousands. During the two-thousand tens the monetary base growth and the deficit show some co-movement again although neither the monetary base growth nor the deficit levels show an increasing trend.

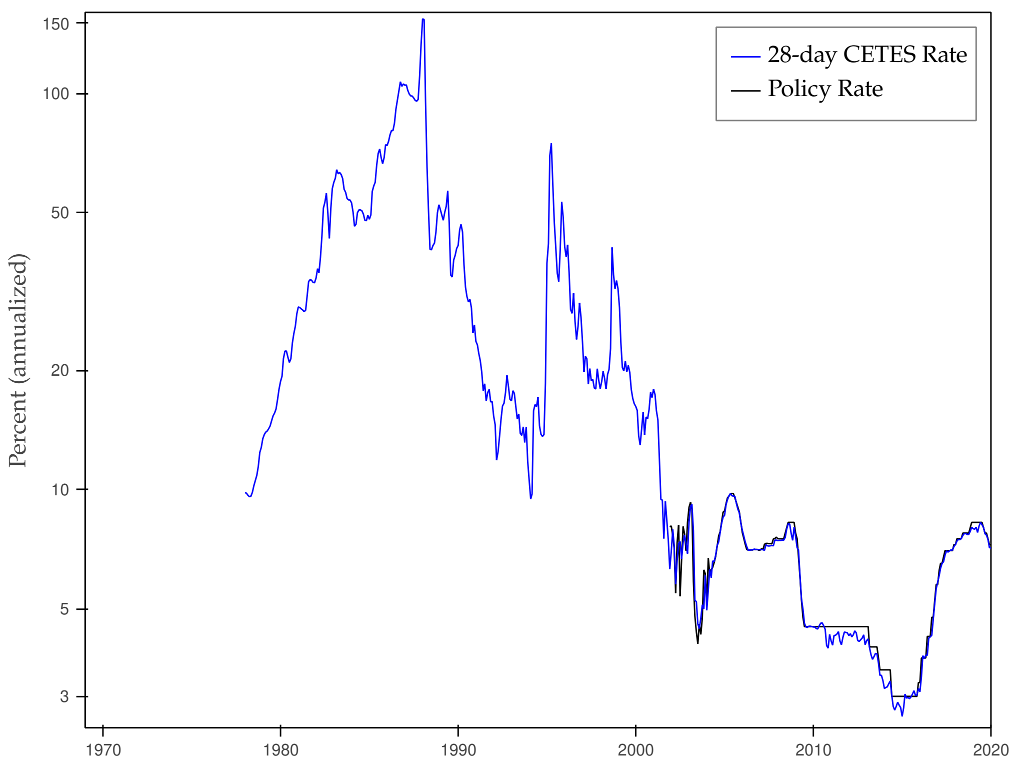

Figure 12: **Policy Rate and 28-day CETES rate**. Note: The policy rate follows closely the 28-day CETES rate.

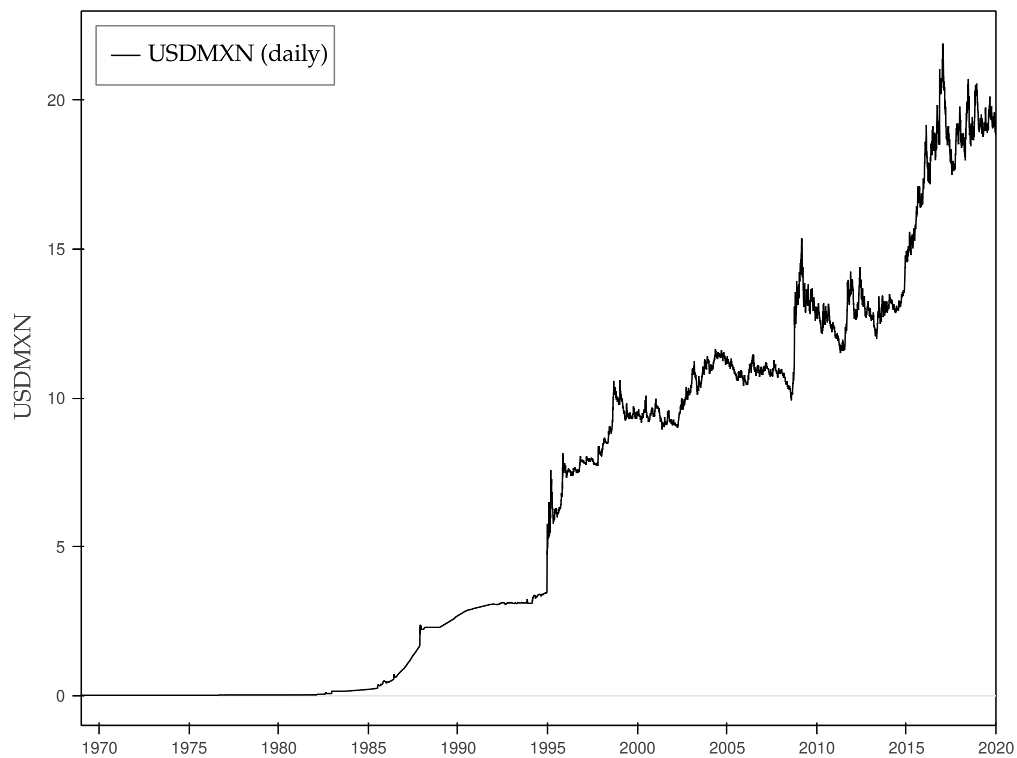

Figure 13: **USD-MXN Exchange Rate**. Note: Mexico has changed exchange rate regimes several times since 1976. Until August 31, 1976 the exchange rate was fixed at 0.0125 pesos per one dollar, or 12.50 old pesos per dollar. The new peso, introduced on January 1, 1993 is equivalent to 1000 old pesos. From September 1, 1976 until August 5, 1982 the exchange rate regime was managed floating. Starting on August 6, 1982 and until November 10, 1991, there were multiple official exchange rates; initially, until December 19, 1982, all of them were fixed. This time series includes the "general" and later the "ordinary" exchange rate. Afterwards, a "free" market exchange rate was available and reported here, although apparently it was not always updated every day. Since November 11, 1991 there is only one exchange rate. Until December 21, 1994, the exchange rate regime was floating within sliding bands and free floating after that.

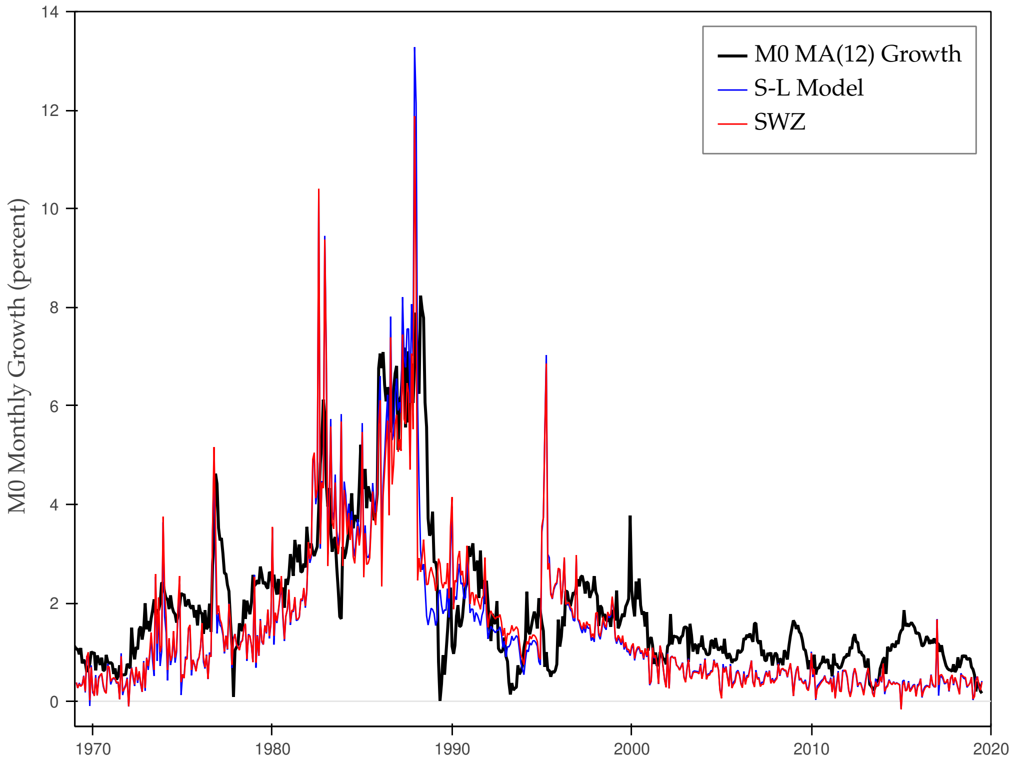

Figure 14: **Actual and Model-Implied Monetary Base Growth**. Note: Model-implied money supply growth is compared with M0's. Both the Selden-Latané and SWZ's model-implied money supply monthly changes follow the general trend shown by M0, although the S-L model results in a slightly more volatile series which is closer to the real series, see also Figure 9. It is noteworthy that the highest peak in S-L model's money supply matches better the highest peak in M0's growth shown in Figure 9 and it implies a tighter monetary policy after the 1989 policy reform, a behavior also confirmed by the actual M0.

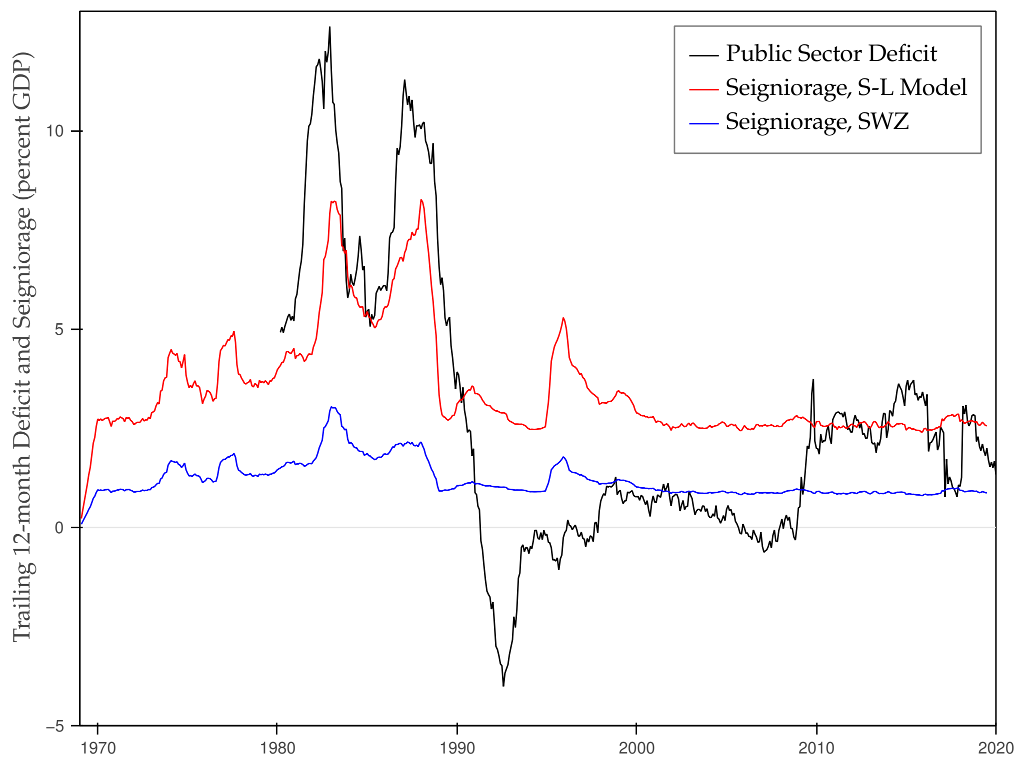

Figure 15: **Public Sector Deficit and Model-Implied Seigniorage**. Note: Monthly data is depicted here for the 12-month cumulative Mexican deficit and for model-implied seigniorage. During the Mexican fiscal dominance period, there is a closer association between the Selden-Latané money-demand model-implied seigniorage (S-L Model) and the Mexican public sector deficit since the former implies that the government can raise more seigniorage from households for similar money supply increases. After the enactment of fiscal dominance prevention reforms in 1995, seigniorage is no longer a source of deficit financing and indeed the series do not show comovements between seigniorage and the deficit.

pressures caused by external debt payments that were amplified by two large devaluations, first in 1986 and then in 1987 (figure 6).

Thus, the key problem was the ever-increasing external debt liabilities which lent minimal credibility to any fiscal or monetary policy commitment because the government budget was unsustainable. In 1986 and 1987 the country reached its highest external debt level ever as a fraction of GDP, and figure 11 shows increases in the debt stock around this time. Mexico attempted to renegotiate its external debt in 1986 and 1987, as a participant in the Baker plan, proposed at the IMF/WB 1985 meetings in South Korea, see Sachs (1989). The plan offered access to medium-term loans and the possibility of rolling over old loans, in exchange for economic reforms. The idea was that with this plan, economies would be able to grow their way out of debt. This approach was not generally successful (van Wijnbergen, King, and Portes, 1991).

The maximum inflation in our sample was reached in December 1987, and high inflation continued into January 1988, when 28-day CETES rates averaged 157.07 percent annualized (figure 12). It is worth emphasizing that inflation and the regime state probability dynamics are consistent with the economy exploding toward a very high inflation equilibrium (SCE), in other words, with an escape event. Thus, a fundamental reform became unavoidable (figure 8).

Indeed, it was around this time, in December 1987, that De la Madrid's administration enacted the Economic Solidarity Pact. This stabilization program's key element was a significant effort to balance the budget deficit by raising taxes, restricting government spending and generally reducing the size of the public sector. It also attempted to manage inflation expectations through income policies (i.e., wage and price controls) and used the exchange-rate as a nominal anchor. It included also other elements, such as trade liberalization, deregulation, and privatization of government companies. As shown in figure 6, the probability of being in a high mean seigniorage state remained the highest for a few months after the reform.

10.5 Successful stabilization: April 1988 to February 1995

During Salinas' term, the government implemented the Stability and Economic Growth Pact, which essentially continued De la Madrid's Economic

Solidarity Pact. It included a fiscal retrenchment once again. Some structural reforms, including the NAFTA, were also implemented. The most important element, however, was a successful external public debt renegotiation through the Brady plan. We can see that these measures resulted in a switch to an even lower mean seigniorage state $m = 5$ (figure 6).

Under the Brady plan, Mexican sovereign bonds could be exchanged for Brady bonds, at a discount and with longer maturities. Banks were then able to sell their Brady bonds to third parties, obtaining long-needed liquidity. The IMF, the World Bank, and the Bank of Japan acted as guarantors of the principal and the initial coupon payments, improving the bonds' credit rating and leading to lower interest rates. It was a deal that improved the status quo for all parties involved (Sanginés, 1987). Further reforms improved confidence and the return of foreign capital, such as trade liberalization, the privatization of commercial banks in 1991–1992, and the removal of most capital controls. Nevertheless, the exchange-rate was not allowed to float and in November 1991 the authorities established a target zone regime.

Overall, these measures implied a drastic decrease in monetary base growth captured well by the model (figure 14), and a much lower seigniorage-mean regime, which quickly returned to late-1960s levels (figure 6).

10.6 Another crisis and a successful new reform: March 1995 to July 1999

The 1994–1995 Mexican crisis had an underlying cause and a trigger. The underlying cause was an excessive credit expansion and an exchange rate misalignment. The financial liberalization, and its accompanying increase in confidence, along with low U.S. interest rates, attracted foreign capital inflows to Mexico. The newly privatized banks allocated a large portion of this capital to illiquid investments. Moreover, during the last years of Salinas's administration there was a considerable fiscal expansion while the real exchange-rate remained significantly misaligned.

The trigger of the crisis was the political turmoil of 1994: the Chiapas social revolt, the assassinations of a leading presidential candidate and a political leader. Furthermore, since the Federal Reserve began raising interest rates in early 1994, many did not find a reason to keep their funds in Mexico and capital outflows ensued, leading to the loss of international reserves. To prevent a devaluation the government issued dollar-indexed short-term bonds (the *Tesobonos*) (Buiter, 1987). Soon after, by November, doubts arose regarding whether the bonds had enough foreign reserves backing.

Thus, this was a balance of payments and financial crisis. Zedillo's term began in December 1994. As capital outflows continued, the Bank of Mexico announced a shift in the upper bound of the exchange-rate's target zone by 15%. But foreign capital kept fleeing. On December 22, the exchange-rate was allowed to float and it depreciated considerably (figure 13).

Meanwhile, yearly inflation increased to 52% in 1995. There was also an increase in interest rates to 74.75% (28-day CETES, figure 12), and a fall in GDP by 6.3%.

The sudden devaluation and GDP contraction was accompanied neither by an expansion in the monetary base (figure 9) nor by a larger fiscal deficit (figure 11). Nevertheless, the model displays a switch to a higher mean-seigniorage regime that lasted for most of 1995. Concomitantly, the escape-provoking probability spiked (figure 8), but by 1997 it had returned to levels close to zero.

After the initial spike in the seigniorage-mean level in 1995, it quickly switched to level $m = 3$ and then to progressively lower levels (figure 6). As can be seen in figure 5, levels $m = 3$ to 6 have a unique low-inflation SCE; thus, after reaching $m = 3$, it is feasible to gradually return to a lower seigniorage-mean without risk of staying in the domain of attraction of an SCE with very high inflation.

Rather than seigniorage-financed deficits, it seems more plausible that the 1994-1995 sudden devaluation was the trigger for the temporary bout of inflation that lasted until the end of the decade. While the data does not show fiscal deficits nor seigniorage finance during this period, it could be argued that economic agents anticipated future deficits, given the pressure to increase fiscal expenditures to fight the economic contraction and to fix the financial system, or that they still mistrusted the government and conjectured that the peso depreciation was being caused by the government financing itself with seigniorage, as in the past.

Thus, it makes sense to consider the end of fiscal dominance as crucial to regain price stability, as the reforms included the central bank's (Bank of Mexico's) independence from fiscal authorities, improving its credibility.

There were other contributing elements such as the financial support package, announced in January 1995, involving the U.S. Treasury, the IMF, the BIS, and private commercial banks, along with measures to stabilize the financial sector, provisioning dollar liquidity to banks and assuming some loans to ensure that commercial banks satisfy capital requirements, see Whitt (1996).

In 1998 the oil price dropped, an event that affected the Mexican peso, as well as fiscal revenues, but the latter still exceeded government expenditures. The model captures a shock and corresponding adjustment by a small spike in the probability of being in a higher mean-seigniorage regime. Nonetheless, the model does not show in 1998 a clear regime change regarding seigniorage finance, and indeed there was none.

That the measures taken after the Tequila Crisis avoided a return to a fiscal dominance situation, explains why there was no need for escape probabilities to rise substantially during the nineties before seeing fundamental reforms to bring inflation back to low and stable levels. The absence of fiscal dominance after 1995 is captured quite well by the dynamics of the estimated model, due to the persistence of the lowest mean seigniorage regimes (figure 6) and the disappearance of the correlation between fiscal deficits and deficit-financed seigniorage (figure 15).

10.7 Price Stability again: August 1999 to July 2019

Since 1999, the escape-provoking probability has stayed close to zero, and inflation expectations have remained between the two lowest SCEs (figure 7). The mean-seigniorage regime has been mostly in its lowest state (figure 6). While the central bank adopted an inflation-targeting regime, prudent fiscal management continued.

In 1999, there was a sudden increase in the monetary base (figures 9, 10). Curiously, it did not seem to affect inflation at all. According to the Bank of Mexico (2000) the increase in M0 obeyed to their passive provision of cash due to its increasing demand.

The probability of being in the lowest-mean seigniorage state has remained the highest among all states (figure 4). It is worth noting that there was a temporary switch to the second lowest mean regime during and after the global financial crisis (GFC), and there were as well more frequent switchings to the high volatility state. Consistently with central bank independence, fiscal deficits incurred after the GFC (figure 11) have been mostly financed with internal and external debt, not with seigniorage.

There is also an apparent co-movement between fiscal deficits and monetary base growth during the two-thousand tens (figure 11) that fails to be reflected by fluctuations in inflation or seigniorage. Depending on the deficit's specific causes, it is entirely possible that government expenditures increased the demand for cash, which was then provided by the central bank.

It is also noteworthy that since 2014 the Bank of Mexico has managed its policy rate targeting inflation, but more importantly attempting to address a persistent Mexican Peso depreciation, figures (12) and (13), and thus, since 2014 the policy rate was continuously increased even when inflation was stable. The exchange rate on the other hand, kept depreciating. The Peso's fall is commonly attributed to external factors such as the fall in oil prices, capital flight and the Federal Reserve's monetary policy. In January 2015 the 28-day CETES rate averaged 2.67 percent, and then it started to rise, anticipating an increase in the Bank of Mexico's policy rate, which came a few months later, as a response to the peso depreciation. The latter continued into the beginning of 2017.

11 Final Remarks

Latin American governments have frequently resorted to seigniorage to finance their fiscal deficits in the past, often starting a sequence of events that only gets reverted once "escape probabilities" reach levels close to one: once fiscal dominance becomes prevalent in an economy, inflation needs to get out of hand to convince policy makers to enact the necessary stabilizing reforms. In this regard, Mexico has not been an exception.

Our data depicts the self-fulfilling cycle of inflation, fiscal imbalances, and increasing external debt that comes along with fiscal dominance, and that results in economic crises almost certainly. The Mexican government implemented several adjustment programs during its fiscal dominance cycle, and many turned out to be ineffective and insufficient. One important factor to ensure a programs' success was to ensure that fiscal accounts were sustainable. This is why the renegotiation of the external debt through the Brady plan proved crucial for the success of the 1987 and 1988 stabilization programs. Another key factor was the introduction of central bank independence, which effectively put an end to the age of seigniorage finance in Mexico.

Throughout Mexican fiscal dominance cycle, the model has been able to capture the dynamics of inflation, money supplyy, seigniorage-financed fiscal deficits, and inflation expectations: its model-implied regime switching captures roughly historical developments in Mexican fiscal and monetary policy.

We show actual deficits and model-implied seigniorage in figure 15, The close correlation between these series is striking up to 1995, which depicts a typical fiscal dominance situation. We emphasize that Mexican deficit data was not used in the estimation step. This correlation disappeared in 1995 when central bank independence was implemented, bringing about continuous price stability to the economy. We also show actual and model-implied money supply growth in figure 14 which are also very close. Again, the results are surprising because historical money supply data was not used for parameter estimation.

Model-implied seigniorage fits historical fiscal deficits better than the alternative specification in two dimensions: first, its absolute levels are higher because with our proposed money-demand function, the government is allowed to raise higher levels of seigniorage, and second, seigniorage volatility depends on a volatility state as well as on the level of seigniorage. The latter allows the model to predict better short-term fluctuations in money supply. A more realistic volatility specification not only allows to identify the model parameters, it also implies that volatile money supply and inflation are key stylized facts of high-inflation episodes.

By estimating seigniorage, the model, as others in its class, contributes to understand the fiscal-monetary policy interaction since the latter occurs through seigniorage, which is usually a hidden variable. Direct comparison of monetary base growth to fiscal deficits as in figure 11 can be misleading.

It is worth reemphasizing that our main conclusion hinges on the particular money-demand function we employ. Thus, we consider that the choice of functional form for the money-demand equation is crucial to study the relationship between fiscal dominance and inflation. Cagan's paradox obfuscates the true relationship between seigniorage and inflation. We expect that this model can bring additional insight in the analysis of highly- or hyper-inflationary episodes.

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"Inflation and Seigniorage-Financed Fiscal Deficits: The Case of Mexico" Appendices

Appendix A A Transactional Model for Money Demand

In inflationary economies, money loses its function as a store of value, and it becomes a costly transactional instrument. To reduce the costs of employing the default medium of exchange in an inflationary economy, households spend time securing alternative arrangements to preserve the value of their purchasing power. We propose a model that captures the costs and benefits of holding money as a transactional instrument, and derive its implied money-demand function. To simplify the model, we focus on the spending side of the household's problem, and suppose that households receive a stable real income. In reality, nominal labor contracts are the norm, but inflationary experiences have shown that in this case, households demand labor contracts fixed in hard currency or with automatic salary increases depending on inflation. Additionally, households purchase hard currency and real assets upon payment of their salaries when inflation becomes significantly costly. All these alternative arrangements are also costly, and focusing on the spending part of the problem still captures the cost of looking for mechanisms and contracts that minimize the use of the official medium of exchange.

We envision a simple economy with an infinite-lived representative household solving the following problem at each period: in the current period, the household works and earns labor income, while at the same time, it spends time trading, which draws time away from work, thus reducing its earning potential. Trading and payment may take place in the first or the second period, but delivery and consumption occur in the second period. The household could just arrange for all of its consumption purchases in exchange for payment in cash in the second period, spending minimal trading time. In the meantime, inflation depreciates any money balances M_t the household is holding for payment. If inflation is a concern, these balances can be reduced by spending more time doing transactions in the first period, or equivalently, by increasing the number of trips to the shops n_t. Consumption requires either to keep nominal money balances on hand, or trips to the shop n_t:

$$C_{t+1} \leq \frac{M_t}{P_t} n_t, \tag{A.1}$$

where M_t/P_t is the amount of real balances held until the next period and C_{t+1} is next period's consumption. Note that when $n_t = 1$ this is a standard cash-in-advance constraint. Two trips to the shop $n_t = 2$ means that half of next period's consumption is paid upon receipt of wages, $n_t = 3$ means that payment of two-thirds of consumption is arranged in advance, and so on. Essentially, more trading time allows to secure more shopping contracts denominated in real or hard currency terms to purchase the desired level of consumption. With more time doing trading deals, the household can commit a greater portion of its wage earnings, a real asset, to guarantee future delivery of real assets, reducing the need for cash.

The household supplies labor $l_t \leq 1$ in exchange for a real wage of z per unit of labor. To turn this wage into consumption, the household can procure non-cash shopping contracts, which take time to arrange, or currency-denominated shopping contracts that can be arranged without cost. In the next period, settlement of shopping contracts takes place and the household consumes the delivered goods C_{t+1}, but in the meantime, cash has lost value due to inflation, and if the household does not plan adequately for inflation, it may have to consume less than planned. The expected utility of household is then

$$E_t u(C_{t+1}). \tag{A.2}$$

To procure non-cash shopping contracts, the household must take time away from work. The relationship between shopping trips and labor time is given by the technology

$$s_t = H\left(C_{t+1}, n_t\right) \tag{A.3}$$

where s_t is the time spent trading instead of working and earning a salary; H is continuous and twice differentiable, increasing in both C_{t+1} and n_t. The household's budget constraint is

$$z l_t + \frac{M_{t-1}}{P_t} \geq C_{t+1} + \frac{M_t}{P_t}\frac{P^e_{t+1}}{P_t}, \tag{A.4}$$

where the right hand side is next-period's consumption plus the next period's value of real money balances. The left hand side is labor income plus any remaining money balances from last period. The time constraint is given by:

$$1 \geq l_t + H\left(C_{t+1}, n_t\right). \tag{A.5}$$

The Lagrangian for the household's problem is

$$\begin{aligned} E_t u(C_{t+1}) &+ \lambda_t \left(z l_t + \frac{M_{t-1}}{P_t} - C_{t+1} - \frac{M_t}{P_t} \frac{P^e_{t+1}}{P_t} \right) \\ &+ \mu_t \left(1 - l_t - H\left(C_{t+1}, n_t\right)\right) \\ &+ \rho_t \left(\frac{M_t}{P_t} n_t - C_{t+1} \right) \end{aligned}$$

with first order conditions

$$\begin{aligned} C_{t+1} : &\quad E_t u_{C,t+1} - \lambda_t - \mu_t H_{C,t} - \rho_t = 0 \\ l_t : &\quad \lambda_t z - \mu_t = 0 \\ n_t : &\quad -\mu_t H_{n,t} + \rho_t \frac{M_t}{P_t} = 0 \\ M_t : &\quad -\lambda_t \frac{1}{P_t} \frac{P^e_{t+1}}{P_t} + \rho_t \frac{1}{P_t} n_t = 0 \end{aligned}$$

The Lagrange multipliers are:

$$\begin{aligned} \lambda_t &= \frac{E_t u_{c,t+1}}{1 + \frac{1}{n_t} \frac{P^e_{t+1}}{P_t} + z H_{C,t}} \\ \mu_t &= z \lambda_t \\ \rho_t &= \lambda_t \frac{1}{n_t} \frac{P^e_{t+1}}{P_t} \end{aligned}$$

The first order condition with respect to consumption shows that consumption is maximized by trading off holding money balances and spending time trading, while the first order condition with respect to n_t shows that households will equate the marginal cost of visiting shops with the marginal benefit of avoiding the inflation tax. Replacing the value of the Lagrange multipliers and the cash-in-advance constraint into the fist order condition with respect to n_t yields

$$z H_{n,t} = \frac{C_{t+1}}{n_t^2} \frac{P^e_{t+1}}{P_t}, \tag{A.6}$$

which implies (1); it says that the marginal cost of increasing trading trips, in consumption units, must equal the marginal benefit of reducing the demand of real balances and saving on inflationary costs.

We now find explicit functional forms for the money-demand by specifying the function H. Let us assume an exponential form for H

$$H\left(C_{t+1}, n_t\right) = \epsilon n_t^{\upsilon} \tag{A.7}$$

with $\upsilon > 0$, implying that H is convex or concave in its arguments depending on υ.

Then

$$H_{n,t} = \epsilon \upsilon n_t^{\upsilon - 1},$$

and replacing the latter expression jointly with the cash-in-advance constraint, (A.1) into (A.6) yields

$$\begin{aligned}
\frac{z n_t^2}{C_{t+1}} H_{n,t} &= \frac{P_{t+1}^e}{P_t} \\
\frac{z n_t^2}{C_{t+1}} \epsilon \upsilon n_t^{\upsilon-1} &= \frac{P_{t+1}^e}{P_t} \\
\frac{z \epsilon \upsilon}{C_{t+1}} n_t^{\upsilon+1} &= \frac{P_{t+1}^e}{P_t} \\
\frac{z \epsilon \upsilon C_{t+1}^{\upsilon+1}}{C_{t+1}} \frac{P_t^{\upsilon+1}}{M_t^{\upsilon+1}} &= \frac{P_{t+1}^e}{P_t}
\end{aligned}$$

and finally

$$\frac{M_t^{\upsilon+1}}{P_t^{\upsilon+1}} = z \epsilon \upsilon C_{t+1}^{\upsilon} \left(\frac{P_{t+1}^e}{P_t}\right)^{-1}.$$

Taking logs we obtain the familiar form

$$\log \frac{M_t}{P_t} = \frac{\log(z\epsilon\upsilon)}{\upsilon+1} + \frac{\upsilon}{\upsilon+1} \log(C_{t+1}) - \frac{1}{\upsilon+1} \log \frac{P_{t+1}^e}{P_t}, \tag{A.8}$$

which is the log-log money-demand of Meltzer (1963). The case $\upsilon = 1$ is the money-demand of Baumol (1952) and Tobin (1956), which features an elasticity of 1/2.

If we use the approximation $\log P_{t+1}^e / P_t \approx P_{t+1}^e / P_t - 1$ we obtain the semi-log money-demand of Cagan (1956):

$$\log \frac{M_t}{P_t} = \frac{\log(z\epsilon\upsilon)}{\upsilon+1} + \frac{\upsilon}{\upsilon+1} \log(C_{t+1}) + \frac{1}{\upsilon+1} - \frac{1}{\upsilon+1} \frac{P_{t+1}^e}{P_t}, \tag{A.9}$$

inserting into the last equation the Taylor series approximation $\exp(x) \approx 1 + x$:

$$\frac{M_t}{P_t} = \exp\left(\frac{\log(z\epsilon \upsilon)}{\upsilon+1} + \frac{\upsilon}{\upsilon+1}\log(C_{t+1})\right)\left(1 - \frac{1}{\upsilon+1}\left(\frac{P^e_{t+1}}{P_t} - 1\right)\right), \tag{A.10}$$

which is the linear money-demand used in many studies, and particularly in SWZ.

Alternatively, there exists a cost function H of the form

$$H(C_{t+1}, n_t) = C_{t+1}\left[H_0 + \epsilon_0 \log n_t - \epsilon_1 \frac{1}{n_t}\right]. \tag{A.11}$$

This cost function is linear with respect to consumption and concave with respect to n_t. Note that $H > 0$ only if n_t is above some level $\bar{n}$ that depends on parameters H_0, ϵ_0 and ϵ_1, implying an upper bound on money balances and a lower bound on the number of trips to shops. Thus, this functional form captures borrowing constraints, in the sense that money balances cannot go to infinity as in the case with an exponential cost function, because it imposes a lower bound on trading time. Likewise, this cost function is more concave than the exponential case which implies that the marginal cost of trips to the shops doesn't fall as quickly, and it is more difficult to reduce the demand for real balances. Both ϵ_0 and ϵ_1 are positive, but the parameter ϵ_1 is small and it has the restriction $\epsilon_1 < 1/z$.

If H takes the form in (A.11), then

$$H_{n,t} = C_{t+1}\left[\epsilon_0 \frac{1}{n_t} + \epsilon_1 \frac{1}{n_t^2}\right]$$

and replacing into (A.6)

$$\frac{zn_t^2}{C_{t+1}} H_{n,t} = \frac{P_{t+1}^e}{P_t}$$

$$\frac{zn_t^2}{C_{t+1}} C_{t+1} \left[\epsilon_0 \frac{1}{n_t} + \epsilon_1 \frac{1}{n_t^2} \right] = \frac{P_{t+1}^e}{P_t}$$

$$z \left[\epsilon_0 n_t + \epsilon_1 \right] = \frac{P_{t+1}^e}{P_t}$$

$$z\epsilon_0 n_t = -z\epsilon_1 + \frac{P_{t+1}^e}{P_t}$$

$$\frac{1}{n_t} = \frac{z\epsilon_0}{-z\epsilon_1 + \frac{P_{t+1}^e}{P_t}}$$

where inserting the cash-in-advance constraint (A.1) gives

$$\frac{M_t}{P_t} = \frac{C_{t+1} \frac{z\epsilon_0}{1-z\epsilon_1}}{1 + \frac{1}{1-z\epsilon_1} \left(\frac{P_{t+1}^e}{P_t} - 1 \right)}$$

which becomes the money-demand of Selden (1956) and Latané (1960), equation (2), by defining

$$\lambda_0 = \frac{z\epsilon_0 C_{t+1}}{1 - z\epsilon_1}, \text{ and} \tag{A.12}$$

$$\lambda_1 = \frac{1}{1 - z\epsilon_1}. \tag{A.13}$$

with $\lambda_1 > 1$ and $\lambda_0 > 0$, because $\epsilon_0 > 0$ and $0 < \epsilon_1 < 1/z$ imply $\lambda_1 > 1$ and $\lambda_0 > 0$. Since the model below focuses on inflation and monetary aggregates, it implicitly holds real aggregates as given, and therefore, we assume that $C_{t+1} = C$, for all t, for some constant C.

That this is a form of Selden's 1956 and Latané's 1960 money-demand function can be verified by letting $a = \gamma/\lambda_0$ and $b = \gamma\lambda_1/\lambda_0$ and then rewriting equation (1) as $M_t/P_t = 1/(a+b(P_{t+1}^e/P_t - 1))$. Under this parameterization, λ_0 determines the demand for real balances when expected inflation is zero, while λ_1 determines jointly the money-demand elasticity and the lower bound for expected inflation.

This functional form behaves similarly to the log-log money-demand function of Meltzer (1963) when the expected inflation is high, but it does not explode when the latter is low and close to zero, i.e., when P^e_{t+1}/P_t approaches one. Instead, it increases in a non-explosive way toward λ_0, as expected inflation reaches zero.

A log-log money-demand function counter-factually predicts overly high levels of real money demand when inflation expectations fall very close to zero. Intuitively, money demand cannot increase to infinity unless individuals have access to unlimited credit. Our money-demand function predicts explosive behavior for inflationary expectations below zero, that is, when expected inflation approaches the value $P^e_{t+1}/P_t \to 1 - 1/\lambda_1$. Should the need arise, this could be fixed by introducing borrowing constraints, but for our data and estimated parameters, both the realized inflation and expected inflation are well above this critical level. Benati, Lucas, Nicolini, and Weber (2021) propose setting real money demand to a constant when expected inflation is very low, in order to introduce borrowing constraints for log-log specifications to rule out explosive behavior at these expected inflation levels. Our money-demand function does not allow explosive behavior for any level of expected inflation such that $P^e_{t+1}/P_t > 1$, and thus, this specification can be thought of as an approximation of a log-log money-demand function with borrowing constraints.

In the SWZ model, the money-demand function is a linear approximation of Cagan's semi logarithmic money-demand function. Recall that Cagan's money-demand function is (up to scale) $M_t/P_t = \exp(-\lambda P^e_{t+1}/P_t)$. SWZ employ an approximation of this demand based on the Taylor expansion of the exponential function: $M_t/P_t \approx 1 - \lambda P^e_{t+1}/P_t$. In this money-demand function, λ is a scalar interpreted as the semi-elasticity of the money-demand with respect to the expected inflation. Due to the linearity of this functional form, the demand for real balances can take negative values for very high levels of expected inflation.[4] We summarize the behavior of these money-demand specifications in figure 1.

In Benati, Lucas, Nicolini, and Weber (2021), long-term estimations of money demand are performed for several countries. They find that neither the popular Cagan semi log nor the log-log specifications are appropriate for a large number of countries, including Mexico. In effect, many of these countries have experienced episodes of very high inflation, or even hyperinflation, but also episodes of low inflation, with inflation rates close to zero.

[4]Negative money-demand levels are handled in SWZ by resetting inflation and possibly expectations to a low level, as a type of cosmetic reform.

They propose either a log-log functional form with borrowing constraints, or an approximation based on the Selden-Latané money-demand function. However, the choice of money-demand function is not only an empirical fit issue, as Benati (2018) shows that replacing Cagan's semi log for a log-log specification in a standard inflation model yields different predictions regarding equilibria and dynamics, and that such a model can display explosive inflationary behavior even when steady-state equilibria are well defined.

Appendix B Derivation of $\hat{G}(\beta, m)$

To compute the functional $\hat{G}(\beta, m)$ (17), first we obtain some preliminary results. Let us start by writing down the evolution of inflation. First, since inflation is bounded by δ^{-1} (see equation (9)), we need to specify what happens when inflation reaches or exceeds such a bound. Let us rewrite the inflation bound as one on the seigniorage $d_t(s_t, d_{t-1}) < \omega(\beta_t, \beta_{t-1})$, where

$$\omega(\beta_t, \beta_{t-1}) \equiv \frac{\lambda(\beta_t) - \delta\theta\lambda(\beta_{t-1})}{\gamma}. \tag{B.1}$$

Then, when inflation is below its bound, its equilibrium level will be determined by the equilibrium condition (7); otherwise, inflation will be reset to the lowest equilibrium $\bar{\pi}_1^*(s_t)$ given the state m as follows[5]:

$$\begin{aligned} \pi_t =& \iota\big(d_t(s_t, d_{t-1}) < \omega(\beta_t, \beta_{t-1})\big) \frac{\theta\lambda(\beta_{t-1})}{\lambda(\beta_t) - \gamma d_t(s_t, d_{t-1})} \\ &+ \iota\big(d_t(s_t, d_{t-1}) \geq \omega(\beta_t, \beta_{t-1})\big) \bar{\pi}_1^*(s_t). \end{aligned} \tag{B.2}$$

Properly speaking, $\bar{\pi}_1^*(s_t)$ is here the low conditional SCEs, but, following SWZ, we instead use the low deterministic SSE to compute equation (B.2).

Defining (B.2) with the SCEs results in an implicit ODE that would require solving a computationally intractable double fixed-point problem to estimate the SCEs. Ex-post, we found the low SCEs to be very close to the low deterministic SSEs when both exist. As explained above, in those cases where there is no deterministic equilibrium, $\bar{\pi}_1^*(s_t)$ is replaced by π_{max}^*.

We now derive the Kushner-Yin (2003) ODE. Using equation (B.2), we define the inflation belief error as follows:

[5] This is a cosmetic reform to be formally defined in the main text.

$$
\begin{aligned}
g\big(\beta_t^*, \beta_t, \beta_{t-1}, &d_t(s_t, d_{t-1})\big) = \pi_t - \beta_t \\
&= \iota\left(d_t(s_t, d_{t-1}) < \omega(\beta_t, \beta_{t-1})\right) \frac{\theta\lambda(\beta_{t-1})}{\lambda(\beta_t) - \gamma d_t(s_t, d_{t-1})} \\
&\quad + \iota\left(d_t(s_t, d_{t-1}) \geq \omega(\beta_t, \beta_{t-1})\right) \bar{\pi}_1^*(s_t) - \beta_t,
\end{aligned} \tag{B.3}
$$

where ι is an indicator function. To compute the SCE, we need to define

$$
\tilde{\omega}(\beta) = \omega(\beta, \beta) = (1 - \delta\theta)\lambda(\beta)/\gamma, \tag{B.4}
$$

and note that as $\beta \to \infty$, then $\tilde{\omega}(\beta) \to 0$. We will use this result later. We can now rewrite the adaptive expectations mechanism as

$$
\beta_{t+1} - \beta_t = \nu g(\pi_t^*, \beta_t, \beta_{t-1}, d_t(s_t, d_{t-1})), \tag{B.5}
$$

or, more generally,

$$
\beta_{t+\Delta} - \beta_t = \nu g(\pi_t^*, \beta_t, \beta_{t-\Delta}, d_t(s_t, d_{t-\Delta})), \tag{B.6}
$$

which takes the form of equation (16) as $\nu \to 0$ and $\Delta \to 0$ jointly, and after taking expectations, conditioning on the state m. Then, to find the equilibrium value of β, we must evaluate the expectation

$$
Eg(\pi_t^*, \beta_t, \beta_{t-1}, d_t(s_t, d_{t-1})) = 0, \tag{B.7}
$$

conditioning on $m_t = m$, and then, solve for β. To help integrate out the seigniorage shocks and the lagged seigniorage, we define

$$
\Psi_s(\beta, b) = \int_0^\infty \int_0^{b - \bar{d}(m)} \frac{1}{\lambda(\beta) - \gamma(\bar{d}(m) + \varepsilon_d)} dF_d(\varepsilon_d | s, d') dF_d^*(d' | s), \tag{B.8}
$$

which will help compute the expectation of equilibrium inflation (8) given $d_t(s_t, d_{t-1}) < \omega(\beta_t, \beta_{t-1})$. The upper bound of the inner integral in Ψ_s is given by the conditioning seigniorage bound, and F_d^* denotes the distribution of the seigniorage, conditioning solely on the joint state $s_t = s$. Similarly, we define

$$
\Phi_s(\beta, b) = \int_0^\infty \int_0^{b - \bar{d}(m)} dF_d\left(\varepsilon_d | s, d'\right) dF_d^*\left(d' | s\right), \tag{B.9}
$$

This helps compute the expected value of post-cosmetic reform inflation, given $d_t(s_t, d_{t-1}) \geq \omega(\beta_t, \beta_{t-1})$. Φ_s is a cumulative distribution function that will be used to assess the probability of inflation reaching its upper bound. Recall that the seigniorage bound is rewritten as $\tilde{\omega}(\beta)$ when β is constant, and we evaluate these integrals from 0 to $\tilde{\omega}(\beta) - \bar{d}(m)$ to ensure they are finite. Additionally, the inflation's upper bound, together with the seigniorage bound, guarantee that $\Psi_s(\beta, b)$ is finite. As $\tilde{\omega}(\beta) \to 0$ if $\beta \to \infty$, Φ_s eventually decreases toward 0. Now, let us define

$$\tilde{g}(\pi_t^*, \beta, d_t) = g(\pi_t^*, \beta, \beta, d_t). \tag{B.10}$$

We now collect definitions to provide an expression for $\tilde{g}$. We denote $\bar{q}_{\varsigma,k}$ as the unconditional probability of the event $\varsigma_t = k$, which is an element of the ergodic distribution of Q_ς, and then we obtain equation (17). Note that the final expression indeed has the form of the ODE (16).

Appendix C Proof of Proposition 1

Proof. We need to show that equation (16) has at least one root at zero for every m. First, $\hat{G}(\beta, m)$ has a bounded support, open on the lower bound at $1 - 1/\lambda_1$ by equation (8), and open at the upper bound at $1/\delta$ by equation (9). By the properties of Ψ_s and Φ_s, $\hat{G}$ is bounded and continuous inside its support. Thus, through the intermediate value theorem, we only need to show that $\hat{G}(\beta, m)$ has at least one sign change. At the upper bound, as $\beta \to 1/\delta$ and as $\delta \to 0$, or equivalently as $\beta \to \infty$, then $\Psi_s, \Phi_s \to 0$ and $\hat{G}(\beta, m) \to -\infty$. However, at the lower bound, as $\beta \to 1 - 1/\lambda_1$, then $\tilde{\omega}(\beta) \to \infty$ and $\hat{G}(\beta, m)$ has a positive and finite limit: Expected gross inflation is finite because inflation is bounded. This expectation is always greater than β's lower bound if $1 - 1/\lambda_1 < \theta$, as θ is the lowest value an SSE can take. Note that the second term is determined by the SSE. If the bound on θ holds, the sum of the first two terms will be greater than the bound even if the first term is at the bound. On the other hand, the third term, $-\beta$, approaches its upper bound, which is smaller than expected inflation in absolute value. Intuitively, as the lowest values of β are not equilibria, they have to increase; therefore, $\hat{G}(\beta, m)$ is positive at β's lower bound. As $\hat{G}(\beta, m)$ always becomes negative at the upper bound, it is positive at the lower bound, and it is continuous, we can conclude that it crosses zero at least once by the intermediate value theorem. □

Appendix D Derivation of Escape Probabilities

We proceed to describe the computation of our escape probabilities. We find the distribution of shocks that would cause an escape event. Define $\underline{\omega}_t(m_t, \varsigma_t)$, as the value of $\varepsilon_d(\varsigma_t, d_{t-1})$, such that $\pi_t = \beta_2^*(m)$, and $\overline{\omega}_t(m_t, \varsigma_t)$ as the value of $\varepsilon_d(\varsigma_t, d_{t-1})$ such that $\pi_t = \delta^{-1}$. Realizations of ε_d between these values would push drive inflation expectations toward the domain of attraction of $\beta_3^*(m)$. Using monetary equilibrium and setting inflation to the bounds above we find that

$$\underline{\omega}_t(m_t, \varsigma_t) = \frac{1}{\gamma}(\lambda(\beta_t) - \theta\lambda(\beta_{t-1})\beta_2^*(m_t)^{-1}) - \bar{d}(m_t)$$
$$\overline{\omega}_t(m_t, \varsigma_t) = \frac{1}{\gamma}(\lambda(\beta_t) - \delta\theta\lambda(\beta_{t-1})) - \bar{d}(m_t).$$

Then, the probability of an escape-provoking event, this is, the probability of observing a seigniorage shock that would push inflation to a level greater than $\beta_2^*(m)$ but smaller than δ^{-1}, when $\beta_1^*(m)$, $\beta_2^*(m)$, and $\beta_3^*(m)$ exist, and given period t's state m is

$$Pr\{\underline{\omega}_t(m_t,\varsigma) < \varepsilon_d(\varsigma_t, d_{t-1}) < \overline{\omega}_t(m_t, \varsigma)|s_t = s, d_{t-1} = d'\}$$
$$= F_d(\overline{\omega}_t(m_t, \varsigma_t)|s_t = s, d_{t-1} = d') - F_d(\underline{\omega}_t(m_t, \varsigma_t)|s_t = s, d_{t-1} = d').$$

Under unique SCEs, there are no escapes, as mentioned, but note that if $\beta_3^*(m)$ is the unique equilibrium, then the economy is permanently in the domain of attraction of the high SCE; therefore, the probability of falling into the domain of the very high SCE, for those states m such that $\beta_3^*(m)$ is the unique equilibrium, as

$$Pr\left(\underline{\omega}_t(m_t, \varsigma_t) < \varepsilon_d(\varsigma_t, d_{t-1}) < \overline{\omega}_t(m_t, \varsigma_t)|s_t = s, d_{t-1} = d'\right) = 1,$$

likewise, if m is such that $\beta_1^*(m)$ is the unique equilibrium, the probability of falling into the domain of the very high SCE is

$$Pr\left(\underline{\omega}_t(m_t, \varsigma_t) < \varepsilon_d(\varsigma_t, d_{t-1}) < \overline{\omega}_t(m_t, \varsigma_t)|s_t = s, d_{t-1} = d'\right) = 0,$$

because the economy is permanently in the domain of attraction of the low SCE. Gathering the computations for each mean seigniorage state and since neither past seigniorage levels nor period t's m state are observed, the final computation for an escape-provoking event probability is (21).

Appendix E Identification: The Unbounded Case

To explain our procedure, let us begin by recalling that the estimation uses just inflation data to recover time series for money demand, money supply, seigniorage and inflation expectations. To achieve this, the model imposes tight restrictions on these objects and their interrelationships. These restrictions allow the identification of the model.

In particular, the monetary equilibrium equation (7) serves a dual purpose. First, it produces a seigniorage time series, which is enough to identify the parameters of its conditional distribution. Second, it determines the conditional distribution of the seigniorage implied by money demand, inflation expectations and monetary equilibrium. Parameter identification is achieved by matching these two distributions.

Since the seigniorage shock is the main element driving the dynamics of the model, the key question is if knowledge of its distribution allows the identification of the parameters of the money-demand and inflationary expectations mechanisms. In fact, starting from the seigniorage distribution, the likelihood function is constructed using a change of variables to derive the implied conditional distribution of inflation.

Besides monetary supply and demand, there is a third element in the model: an inflation reset shock capturing cosmetic reforms. It is estimated by setting an upper bound on inflation. This is just a residual attempting to fit drops in inflation that are not explained either by seigniorage shocks or by seigniorage regime changes. We ignore the reset shock in this section and assume that its distribution parameters are immediately identified after setting the inflation bound and obtaining the parameters of the unbounded case.

We represent here seigniorage dynamics by its conditional distribution $P_d(d_{t+1}|d_t, s_t)$, where d_t is the information available at t, and s_t is the unobserved state.

The key identifying condition is given by (7): it implies that if seigniorage is not observed, it can be recovered with inflation data and with the true values of the parameters of $\lambda(\cdot)$: λ_0, λ_1, of those of β_t: ν, and of θ:

$$d_t = \frac{1}{\gamma}\left[\lambda\left(\beta_t\right) - \frac{\theta\lambda\left(\beta_{t-1}\right)}{\pi_t}\right], \tag{E.1}$$

We can see immediately that there is an infinite set of pairs (λ_0, γ) that satisfy this equilibrium condition. For this reason, we assume $\gamma = 1$ in the remainder of this section.

Estimation of the model with unbounded inflation can be accomplished by matching the distributions of the left hand side and of the right hand side of the equation above. Denote by $\hat{d}_t$ the right hand side. We then have the system of equations

$$E(E^d(f(d_{t+1})|d_t, s_t)) = E(E^{\hat{d}}(f(\hat{d}_{t+1})|\pi_t, \beta_t, s_t)) \tag{E.2}$$

where f is an array of measurable functions characterizing the distributions of d_t and $\hat{d}_t$, for example, containing the terms of the characteristic function, those of the moment generating functions, or a sufficient number of moment conditions. Note also that the evaluation of f in the left hand side depends on Q_s, $\bar{d}$, σ and ϑ, while the evaluation on the right hand side depends on ν, λ_0, and λ_1. Additionally, to obtain population expectations in both sides we need Q_s and its invariant distribution.

To conduct our analisis we will focus on necessary conditions and we employ the concept of global identification that says that the moment conditions (E.2) are only fulfilled by the true $\phi = \phi_0$[6].

Definition 6 (Global identification)**.** The system of equations (E.2) is not solved by any $\phi' \in \Phi$ such that $\phi' \neq \phi_0$.

On the other hand, the model is not identified whenever there is no value of ϕ that can solve (E.2). The main necessary condition for identification is then

Assumption 1. There exists $\phi_0 \in \Phi$ such that system (E.2) is solved by $\phi = \phi_0$.

We now impose some additional conditions that guarantee we can recover the parameters of P_d.

Assumption 2. The joint transition probability matrix $Q_s = [Q_s]_{i,j}$, $i, j = 1, \ldots, m_h \times \varsigma_h$ is ergodic and has full rank.

Define the distribution $P_d(\varepsilon|s, d) = \int_0^{\varepsilon} p_d(\varepsilon_d|s, d)d\varepsilon_d$, and let the ergodic distribution of Q_s be Γ. Also:

[6]See e.g. Hall (2005), Chapter 3.

Assumption 3. The seigniorage shock distributions satisfy $P_d(\varepsilon|s,d) \neq P_d(\varepsilon|s',d)$, for any pair $s \neq s'$, for any shock level ε, and for any seigniorage level d.

In other words, different states induce different conditional distributions for seigniorage. Intuitively, states can be only distinguished if they induce a unique behaviour of d_t at each state.

Assumption 4. The process $\{d_t\}$ is ergodic and it has an invariant distribution.

Neither inflation, money demand or supply need to have an invariant distribution. Instead, identification hinges on the stationarity of d_t. We now present an auxiliary result:

Proposition 2 (Alexandrovich, Holzmann, and Leister; 2016)**.** *Under assumptions 2, 3, and 4, with a fixed number of states $K = m_h \times \varsigma_h$ and a sequence $\{d_t\}_{t=1}^{T}$; then Q_s and the parameters of P_d: $\bar{d}$, σ and ϑ, are identified as $T \to \infty$. Furthermore, the parameters are unique up to label swapping.*

This is essentially Theorem 1 in Alexandrovich, Holzmann, and Leister (2016), with the difference that, here, the conditional distributions P_d depend on the state and additionally on $d_1, \ldots, d_T$ and that we do not need to initialize d_t at its ergodic distribution. Their proof needs to be modified just slightly. To accommodate the dependence of the P_d's on d_{t-1} notice that their proof relies on computing the changes of P_d at different states; since we can do these calculations keeping the data fixed, this causes no problem at all. It is still required that the block-by-block joint distributions of d_t are linearly independent among different states, whenever the data $d_{t+1}, \ldots, d_{t+K}$ is fixed within the block. Regarding the initialization at the stationary distribution, this is not needed whenever $T \to \infty$, as long as $\{d_t\}$ is ergodic. Finally, there is more than one set of parameters characterizing the distribution of d_t, with different state labellings, but they are otherwise unique.

We now present our main identification result.

Proposition 3 (Model identification)**.** *Suppose that $T \to \infty$, that assumptions 1, 2, 3, 4 hold, with f containing a number of moments sufficient to characterize the distribution of d_t. Then the model is globally identified.*

Proof. if any of ν, λ_0, λ_1 differs from its true value, it will produce a sequence for $\hat{d}_t$, with distribution parameters obtained via Proposition 2 that will not match the true distribution of d_t, and thus (E.2) will not hold. Conversely, starting from a matrix Q_s or parameters for P_d that do not match their true values, up to state relabeling, they will imply different conditional moments for d_t that will not fulfill (E.2), either. Furthermore, ϕ is possibly over-identified whenever (E.2) has more than one solution ϕ_0. $\square$

Then, provided that different parameters ϕ imply different distributions for d_t and $\hat{d}_t$, and that these distributions are characterized by a adequate and comparable number of moment conditions, then the parameters are identified. Furthermore, while the sequence $\{d_t\}$ is not observed, identification is obtained by specifying explicitly its dynamics. Note that the latter also plays a key role determining the number of moment conditions that needs to be fitted to obtain an identified system.

This proposition formalizes and extends Section VI (Identification) in SWZ. Examining (E.1), we can see that inflation levels indeed determine the values in $\bar{d}$ through inflation expectations. Moreover, inflation volatility also determines volatility states. We can also infer that more complex inflation expectation mechanisms may require more states to adequately fit the data. In addition, the parameters for the money-demand function and the inflation expectations mechanism can be recovered provided that the distribution of d_t has a sufficient number of higher moments, given here by the size of vector f. This clarifies the importance of a complex specification for the volatility of d_t: identification in this class of models relies not only on the variance and skeweness of d_t, but possibly on its kurtosis or even on higher moments, as well.

Finally, while we allow the possibility of over-identification, empirically we can still obtain a "best fit" set of parameters, such as in Hansen (1982), using alternative estimation methods. Since a large enough vector f is key to identify more complex versions of our model, this section suggests that Hansen's General Method of Moments (GMM) has substantial potential to estimate those models, while expecting an improvement in efficiency by virtue of the unrestricted number of moment conditions allowed by GMM.

Appendix F The Inflation Likelihood Function.

Inflation shocks follow an independent and identically distributed random variable with probability density

$$Pr_\pi(\varepsilon_\pi|m_t) = \frac{\exp\left(-[\log(\pi_1^*(m_t)+\varepsilon_\pi) - \log(\pi_1^*(m_t))]^2/(2\sigma_\pi^2)\right)}{\sqrt{2\pi}[\pi_1^*(m_t)+\varepsilon_\pi]\Phi[(\log(\delta) - \log(\pi_1^*(m_t)))/\sigma_\pi]},$$

if $-\pi_1^*(m_t) < \varepsilon_\pi < \frac{1}{\delta} - \pi_1^*(m_t)$ and 0 in other cases, where $\Phi[\cdot]$ denotes the normal standard cumulative function. On the one hand, the lower bound of the interval $[-\pi_1^*(m_t), 1/\delta - \pi_1^*(m_t)]$ ensures that inflation is positive after a cosmetic reform.

On the other hand, the upper one ensures that inflation is below the upper bound $\delta^(- 1)$, which we have introduced in equation (10). Thus, we denote $s^t = \{s_1, ..., s_t\}$ and $d^t = \{d_1, ..., d_t\}$ for the history of regime states up to period t, and the history of seigniorage-financed deficits up to t, respectively, we also define

$$\xi_d(s_t, d_{t-1}) = \frac{1}{\sigma_d(s_t, d_{t-1})} = \frac{1}{\sqrt{\sigma_d^2(\varsigma_t) d_{t-1}^{\vartheta}}},$$

$$\xi_\pi = \frac{1}{\sigma_\pi}.$$

Proposition B1. *The conditional likelihood is*

$$\begin{aligned}
&p\big(\pi_t|\pi_{t-1}, s^t, d^{t-1}, \phi\big) \\
&= C_{1t}\frac{|\xi_\pi|\exp\left\{\left(-\frac{\xi_\pi^2}{2}\right)\left[\log\pi_t - \log\pi_1^*\left(s_t\right)\right]^2\right\}}{\sqrt{2\pi}\Phi\left[|\xi_\pi|\left(-\log\delta - \log\left[\pi_1^*\left(s_t\right)\right]\right)\right]\pi_t} \\
&\quad + C_{2t}\ \frac{\theta|\xi_d\left(s_t, d_{t-1}\right)|\lambda\left(\beta_{t-1}\right)}{\sqrt{2\pi}\left[\lambda\left(\beta_t\right)\pi_t - \theta\lambda\left(\beta_{t-1}\right)\right]\pi_t} \\
&\quad \times \exp\Big(-\frac{\xi_d^2\left(s_t, d_{t-1}\right)}{2}\big\{\log\left[\lambda\left(\beta_t\right)\pi_t - \theta\lambda\left(\beta_{t-1}\right)\right] \\
&\quad - \log\pi_t - \log\gamma - \log\left[\bar{d}\left(m_t\right)\right]\big\}^2\Big),
\end{aligned}$$

where

$$C_{1t} = \Big(1 - \Phi\big[|\xi_d(s_t, d_{t-1})| \\ \big(\log\{\max[(\lambda(\beta_t) - \delta\theta\lambda(\beta_{t-1}))/\gamma, 0]\} - \log\left[\bar{d}(m_t)\right]\big)\,\big]\Big),$$

$$C_{2t} = \iota\left(\min\left[\frac{\theta\lambda(\beta_{t-1})}{\lambda(\beta_t)}, \frac{1}{\delta}\right] < \pi_t < \frac{1}{\delta}\right).$$

Proof: It follows SWZ closely. The likelihood describes what can happen at each t : there is a reform if $\widetilde{\varepsilon}_{dt}(s_t, d_{t-1}) \geq \overline{\omega}_t(s_t, d_{t-1})$. And if there is no reform, the dynamics is driven by the seigniorage shock jointly with the inflation equilibrium equation. We need to show that

$$\int_0^{1/\delta} p\left(\pi_t|\pi^{t-1}, d^{t-1}, s_t, \phi\right) d\pi_t = 1.$$

Rearranging the definition of $p(\pi_t|\pi^{t-1}, d^{t-1}, s_t, \phi)$, combining with the definition of $p_d(\varepsilon_d|s_t, d_{t-1})$ and $p_\pi(\varepsilon_\pi|m_t)$, and taking into account that ε_d and ε_π are independent, we get:

$$\begin{aligned}
&p\big(\pi_t|\pi_{t-1}, s^t, d^{t-1}, \phi\big) \\
&= \Big(1 - \Phi\big[\xi_d(s_t, d_{t-1}) \\
&\quad \big(\log\{\max[(\lambda(\beta_t) - \delta\theta\lambda(\beta_{t-1}))/\gamma, 0]\} - \log\left[\bar{d}(m_t)\right]\big)\,\big]\Big) \\
&\quad \times \frac{|\xi_\pi| \exp\left\{\left(-\frac{\xi_\pi^2}{2}\right)[\log\pi_t - \log\pi_1^*(s_t)]^2\right\}}{\sqrt{2\pi}\Phi\left[|\xi_\pi|\left(-\log\delta - \log\left[\pi_1^*(s_t)\right]\right)\right]\pi_t} \\
&\quad + \iota\left(\min\left[\frac{\theta\lambda(\beta_{t-1})}{\lambda(\beta_t)}, \frac{1}{\delta}\right] < \pi_t < \frac{1}{\delta}\right) \frac{\theta|\xi_d(s_t, d_{t-1})|\lambda(\beta_{t-1})}{\sqrt{2\pi}\left[\lambda(\beta_t)\pi_t - \theta\lambda(\beta_{t-1})\right]\pi_t} \\
&\quad \times \exp\Big(-\frac{\xi_d^2(s_t, d_{t-1})}{2}\Big\{\log\left[\lambda(\beta_t)\pi_t - \theta\lambda(\beta_{t-1})\right] \\
&\quad - \log\pi_t - \log\gamma - \log\left[\bar{d}(m_t)\right]\Big\}^2\Big) \\
&= \Pr\left[\widetilde{\varepsilon}_{dt}(s_t, d_{t-1}) \geq \widetilde{\omega}_t(s_t, d_{t-1})\right] p_\pi\left(\pi_t - \pi_1^*(s_t)|s_t\right) \\
&\quad + \iota\left(\min\left[\frac{\theta\lambda(\beta_{t-1})}{\lambda(\beta_t)}, \frac{1}{\delta}\right] < \pi_t < \frac{1}{\delta}\right) p_d(\varepsilon_d|s_t, d_{t-1}) \frac{d\varepsilon_{dt}(s_t, d_{t-1})}{d\pi_t},
\end{aligned}$$

where we used

$$
\begin{aligned}
& p_\pi\big(\pi_t - \pi_1^*(s_t)|s_t\big) \\
& \quad = \frac{|\xi_\pi| \exp\left\{\left(-\frac{\xi_\pi^2}{2}\right)[\log \pi_t - \log \pi_1^*(s_t)]^2\right\}}{\sqrt{2\pi}\Phi\left[|\xi_\pi|\left(-\log\delta - \log\left[\pi_1^*(s_t)\right]\right)\right]\pi_t}, \\
& \Pr\left[\widetilde{\varepsilon}_{dt}(s_t, d_{t-1}) \geq \overline{\omega}_t(s_t, d_{t-1})\right] \\
& \quad = 1 - \Phi\Big[\xi_d(s_t, d_{t-1})| \\
& \qquad \Big(\log\left\{\max\left[\left(\lambda(\beta_t) - \delta\theta\lambda(\beta_{t-1})\right)/\gamma, 0\right]\right\} - \log\left[\bar{d}(m_t)\right]\Big)\Big],
\end{aligned}
$$

since $\widetilde{\varepsilon}_{dt} \geq \overline{\omega}_t(s_t, d_{t-1})$ if and only if $\widetilde{\varepsilon}_{dt} \geq \frac{1}{\gamma}\left(\lambda(\beta_t) - \delta\theta\lambda(\beta_{t-1})\right) - \bar{d}(m_t)$ if and only if $d_{t-1} \geq \frac{1}{\gamma}\left(\lambda(\beta_t) - \delta\theta\lambda(\beta_{t-1})\right)$, and the integration by substitution

$$
\begin{aligned}
& p_d(\varepsilon_d|s_t, d_{t-1})\frac{d\varepsilon_{dt}(s_t, d_{t-1})}{d\pi_t} \\
& \quad = \frac{\theta|\xi_d(s_t, d_{t-1})|\lambda(\beta_{t-1})}{\sqrt{2\pi}\left[\lambda(\beta_t)\pi_t - \theta\lambda(\beta_{t-1})\right]\pi_t} \\
& \qquad \times \exp\Big(-\frac{\xi_d^2(s_t, d_{t-1})}{2}\Big\{\log\left[\lambda(\beta_t)\pi_t - \theta\lambda(\beta_{t-1})\right] \\
& \qquad - \log\pi_t - \log\gamma - \log\left[\bar{d}(m_t)\right]\Big\}^2\Big),
\end{aligned}
$$

to obtain an integral with respect to inflation using the equilibrium inflation equation

$$
\gamma d_t(s_t, d_{t-1}) = \frac{\lambda(\beta_t)\pi_t - \theta\lambda(\beta_{t-1})}{\pi_t}.
$$

We now recall that

$$
\int_0^{1/\delta} p_\pi\left(\pi_t - \pi_1^*(s_t)|s_t\right) d\pi_t = 1
$$

and that

$$\int_0^{1/\delta} \iota\Big(\min\left[\frac{\theta\lambda\left(\beta_{t-1}\right)}{\lambda\left(\beta_t\right)}, \frac{1}{\delta}\right] < \pi_t < \frac{1}{\delta}\Big) p_d\left(\varepsilon_d | s_t, d_{t-1}\right) \frac{d\varepsilon_{dt}\left(s_t, d_{t-1}\right)}{d\pi_t} d\pi_t$$

$$= \int_{L_t}^{1/\delta} p_d\left(\varepsilon_d | s_t, d_{t-1}\right) \frac{d\varepsilon_{dt}\left(s_t, d_{t-1}\right)}{d\pi_t} d\pi_t$$

$$= \int_{-\bar{d}(s_t)}^{\overline{\omega}_t(s_t, d_{t-1})} p_d\left(\varepsilon_d | s_t, d_{t-1}\right) d\varepsilon_{dt}\left(s_t, d_{t-1}\right)$$

$$= \Pr\left[\widetilde{\varepsilon}_{dt}\left(s_t, d_{t-1}\right) < \widetilde{\omega}_t\left(s_t, d_{t-1}\right)\right]$$

where we used the integration by substitution of seigniorage by inflation and where

$$L_t = \min\left[\frac{\theta\lambda\left(\beta_{t-1}\right)}{\lambda\left(\beta_t\right)}, \frac{1}{\delta}\right];$$

then, collecting terms:

$$\int_0^{1/\delta} p\left(\pi_t | \pi^{t-1}, d^{t-1}, s_t, \phi\right) d\pi_t$$

$$= \int_0^{1/\delta} \Bigg[\Pr\left[\widetilde{\varepsilon}_{dt}\left(s_t, d_{t-1}\right) \geq \widetilde{\omega}_t\left(s_t, d_{t-1}\right)\right] p_\pi\left(\pi_t - \pi_1^*\left(s_t\right) | s_t\right)$$

$$+ \iota\left(\min\left[\frac{\theta\lambda\left(\beta_{t-1}\right)}{\lambda\left(\beta_t\right)}, \frac{1}{\delta}\right] < \pi_t < \frac{1}{\delta}\right) p_d\left(\varepsilon_d | s_t, d_{t-1}\right) \frac{d\varepsilon_{dt}\left(s_t, d_{t-1}\right)}{d\pi_t} \Bigg] d\pi_t$$

$$= \Big[\Pr\left[\widetilde{\varepsilon}_{dt}\left(s_t, d_{t-1}\right) \geq \widetilde{\omega}_t\left(s_t, d_{t-1}\right)\right] + \Pr\left[\widetilde{\varepsilon}_{dt}\left(s_t, d_{t-1}\right) < \widetilde{\omega}_t\left(s_t, d_{t-1}\right)\right] \Big]$$

$= 1.$ *Q.E.D.*

Integrating out s^T , we have the likelihood of interest

$$p\left(\pi^T | \phi\right) = \prod_{t=1}^{T} p\left(\pi_t | \pi^{t-1}, d^{t-1}, \phi\right)$$

$$= \prod_{t=1}^{T} p\left(\pi_t | s_t, \pi^{t-1}, d^{t-1}, \phi\right) \Pr\left(s_t | \pi^{t-1}, d^{t-1}, \phi\right)$$

where

$$\Pr\left(s_t|\pi^{t-1}, d^{t-1}, \phi\right) = \sum_{s_{t-1}=1}^{h} \Pr\left(s_t|s_{t-1}, Q_s\right) \Pr\left(s_{t-1}|\pi^{t-1}, d^{t-1}, \phi\right).$$

As in Sims, Waggoner and Zha (2008) and SWZ, the probability of having observed a state, $\Pr\left(s_{t-1}|\pi^{t-1}, d^{t-1}, \phi\right)$ can be updated recursively starting with the assumption that

$$\Pr\left(s_0|\pi^0, d^0, \phi\right) = 1/h.$$

That is, at the beginning, the econometrician does not know in which state he is, so he assigns the same probability to each state. Thus, using the recursion process, we have that:

$$\Pr\left(s_t|\pi^t, d^{t-1}, \phi\right) = \frac{p\left(\pi_t|\pi^{t-1}, d^{t-1},\ s_t,\ \phi\right) \Pr\left(s_t|\pi^{t-1}, d^{t-1}, \phi\right)}{\sum_{s_t=1}^{h} \left[p\left(\pi_t|\pi^{t-1}, d^{t-1}, s_t,\ \phi\right) \Pr\left(s_t|\pi^{t-1}, d^{t-1}, \phi\right)\right]}.$$