How Information Design Shapes Optimal Selling Mechanisms

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Abstract

A monopolistic seller jointly designs trading trules and (new) information about a pay-off relevant state to a buyer with private types. When the new information flips the ranking of willingness to pay across types, a *screening* menu of prices and threshold disclosures is optimal. Conversely, when its impact is marginal, *bunching* via a single posted price and threshold disclosure is (approximately) optimal, as in standard mechanism design. While information design expands the scope for random mechanisms to outperform their deterministic counterparts, its presence leads to an equivalence result regarding sequential versus. static screening.

1 Introduction

The evolution of informational technology has significantly broadened sellers' ways of selling their products. They can design not only *trading trules* which specify how to allocate products and charge payments to buyers, but also *information policies* which control how much buyers learn about the products, thereby refining their willingness to pay. For instance, they may offer a posted price, associated with full information, to everyone. Alternatively, they could propose a rich menu of trading trules and information policies.

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As an example, many software such as McAfee and various (mobile) apps like Spotify provide users with a *single* free trial version, followed by a *single* subscription fee schedule. The trial version is, therefore, *merely* a learning opportunity for potential buyers to make well-informed purchasing decisions. An opposite example is travel agency platforms such as Priceline and Hotwire practice so-called "opaque pricing" by which, buyers either book hotels with detailed information at standard prices or opt for limited details at discounted prices. Thus, these travel agencies *screen* their buyers via a menu of prices and information policies.

Price and information discrimination is also in the form of pre-order offers for buyers of not-yet-released products, as exemplified by Google's recent pre-order bonus for the Pixel 8. By contrast, well-known products are typically sold via a *single* posted price, coupled with a *single* timeframe for free return to all buyers.

What leads to these diverse selling strategies? In particular, when is a single posted price and disclosure policy optimal and conversely, when is it necessary to provide a screening menu of prices and information? In addition, is there any benefit from offering random mechanisms? Given that classical mechanism design results (Myerson (1981)) predict that a posted price is optimal when the informational environment is *fixed*, answering these questions explains how information design shapes optimal selling mechanisms. Regarding the timing, can the seller's revenue be improved by contracting with the buyer at the "*interim*" stage where he knows his type but before the seller's information disclosure? Or equivalently, should she allow the buyer to walk away at the "posterior" stage where he observes both his type and the information provided? Answering this question helps understand the impact of consumer protection regulations that grant the consumer a withdrawal right such as the European directive 2011/83/EU.[1] Finally, if the buyer privately observes the information disclosed by the seller, can the buyer enjoy any rent induced from such an *endogenously private* information?

This paper aims to answer these questions. The model, as formally described in Section 2, features a seller (she) who sells an object to a buyer (he) with a privately known initial valuation (*initial type*). The seller controls how much the buyer learns about an *additional component* in his valuation. For example, this additional component represents what the buyer learns via product trials. The seller designs a menu of *information policies* for different types of the buyer, and *trading trules* for different types and signals. Therefore, she solves a joint mechanism and information design problem in which information plays

[1]For a detailed discussion on such policies, see Krähmer and Strausz (2015*b*).

a dual role. First, it allows the seller to screen the buyer's type through discriminatory disclosure policies. Second, disclosed information serves as input for designing trading trules. We focus on the case where the buyer privately observes the new information (private signals) and investigate the case with public signals as a benchmark.

1.1 Summary of results

First, we establish a revenue-equivalence result regarding sequential vs. static screening. Specifically, we show that for any *feasible* and *deterministic* mechanism, there exists a mechanism that generates the same revenue for the seller and non-negative payoff for the buyer at any type and signal realization. As a consequence, there is no revenue loss if contracting at the posterior stage when the buyer knows both his type and signal. This result counters the well-established idea in sequential screening suggesting that the seller's revenue is strictly higher if contracting with the buyer before, rather than after, he learns additional information.[2] The basic intuition is that the seller's ability to flexibly design information can crowd out the advantages of sequential over static screening. A practical implication is that afore-mentioned consumer protections do not necessarily harm the seller, rationalizing the prevalence of free information in many markets.

Second, we investigate the (ir)relevance of signal privacy. In the benchmark problem with public signals, only *expected* allocations and payments (over signals) matter. Hence, this benchmark admits multiple solutions, including $\mathbf{M}^\star$, a *screening* menu of threshold disclosures $\pi^\star$ and prices paid conditional on trade.[3] We provide a simple way to verify the (ir)relevance of signal privacy, which is to check if, under $\mathbf{M}^\star$, the highest type pays the lowest price. If this is true, privacy of signals is irrelevant and $\mathbf{M}^\star$ solves the seller's original problem. We find that this is not always the case and consequently, not observing signals generally hurts the seller. Moreover, *per-signal* allocations and payments matter, which significantly complicates the characterization of optimal mechanisms In particular, it is not *a priori* clear how many signals are needed and which incentive compatibility (IC) constraints are relevant. The seller must also handle double deviations when the buyer lies about both his type and observed signal. Leveraging techniques for mechanisms with non-convex type spaces, we make it *always* possible for the buyer to "correct his lie," facilitating the characterization of optimal double deviations and thereby, optimal mechanisms.

[2]See Courty and Li (2000) and Krähmer and Strausz (2015*b*).

[3]See Definition 4 for a formal description of $\mathbf{M}^\star$.

Our main result characterizes optimal mechanisms, starting with binary types. The seller faces a trade-off between maximizing virtual surplus and minimizing the posterior rent. A threshold disclosure rule, under which signal realization is either "*good news*" if the state is above some cutoff or "*bad news*" otherwise, is optimal in both targets.[4] Under the optimal mechanism, the seller either *screens* the buyer's types (via a menu of threshold disclosures and posted prices) or *bunches* them (via a single posted price and threshold disclosure), depending on whether the threshold disclosure $\pi^\star$ induces a *threshold flip* of type order: the high type's value after "*bad news*" is *lower* than the low type's after "*good news*." Specifically, screening is optimal when this flip of type order occurs, and bunching otherwise.

To grasp the intuition, note that such a flip of type order occurs when the variation of valuations is mainly driven by the unknown component, leaving some room for the threshold disclosure $\pi^\star$ to reverse the ranking of valuation. Information (about the unknown component) matters, serving as a screening tool. Conversely, if the buyer's type is the main driver, which prevents $\pi^\star$ from flipping the type order, information is not crucial and screening disappears. The optimal mechanism echoes its counterpart in standard mechanism design where the buyer's valuation is his type: a posted price (but associated with threshold disclosure) is optimal.

The significance of this bunching vs. screening result is two-fold. First, it implies that in the above-mentioned scenarios, eliciting signals and random mechanisms are worthless. Second, it rationalizes observed mechanisms in practice. For coming-soon items, the unknown component's impact on the variation of valuations is large and a screening menu is employed. By contrast, its impact is marginal for well-known products where bunching comes into play. The significance of the unknown component also varies across different industries. In the realm of hotels, it matters much more than in software or mobile apps, leading to screening for the former and bunching for the latter.

Having characterized the optimal mechanism for the binary-type setting, we consider larger type spaces. With more than two types, there are also cases where an information policy reverses the ranking of valuations within a group of types but fails to do so for another. Consequently, not only information but also trading probabilities are needed to screen the buyer, leading to a *random* solution. However, the two scenarios of bunching/screening extend to the case with finitely many types, under stronger notions of flip (no flip) of type order. Specifically, a screening menu is optimal under a *partition* flip by

[4] See Definition 1 for our formal definition of a threshold disclosure.

$\pi^{\star}$ of type order - which generalizes the threshold flip of type order by $\pi^{\star}$, taking into account medium types and their associated cut-off states. Instead, bunching via a fixed price and threshold disclosure maximizes the seller's revenue when there is *uniformly* threshold preservation of type order under which, the type order is to be preserved between *any* pair of types and after *any* threshold disclosure. This strong requirement of type order preservation helps deal with the challenge of determining the lowest type being served in a rich type space.

As binding (IC) constraints can involve local, global, and upward ones, characterizing optimal random mechanisms becomes difficult. We thus focus on shedding light on how random mechanisms outperform their deterministic counterparts.[5] We first establish the "*no randomization at the top*" result, extending the well-known "no distortion at the top" to a setting with information design: the highest type receives an efficient (and hence, deterministic) allocation. In turn, this implies an optimal contract for this type, featuring a posted price and no disclosure. While randomization is not needed for the highest type, it can be helpful for the lower types, leading to a better balance of the efficiency vs. rent trade-off.[6] We analyze, by examples, how random mechanisms facilitate screening distant types as well as screening signals.

Finally, we consider a setting with a continuum of types. In this case, the optimality of a screening menu of posted prices and threshold disclosures under a partition flip of type order extends readily. Particularly, in a "continuous" model when valuation shifts smoothly across types and states, this notion corresponds to the ranking of valuations at the zero-virtual-value states by types being reversed. On the other hand, the fact that there are always types whose valuations are close to others' makes it impossible to flip the ranking of willingness to pay across *all* types. We show that when the type order is *almost* preserved, bunching via a fixed price-threshold disclosure bundle is *approximately* optimal. If there is only two states, we establish the "exact" optimality of bunching within the class of deterministic mechanisms..

[5] In the Online Appendix, we solve for the optimal random mechanism in several examples.

[6] While it is natural to expect the two-dimensionality feature of the buyer's valuation to lead to random mechanisms, the seller has another tool for randomization: the distribution of signals, which potentially makes random mechanisms redundant. However, signal misreporting off-path shuts off this additional instrument. Thus, random mechanisms arise to deter double deviations, minimizing the posterior rent.

1.2 Related literature

We contribute to the literature on joint mechanism and information design, comprising two main strands. The first, more related, strand endows the buyer with a private type, initiated by Eső and Szentes (2007) who focus on full disclosure. Most other papers focus on *posted-price* mechanisms,[7] which in turn, makes it without loss of generality to focus on *binary-signal* information structures (Li and Shi (2017), Guo et al. (2022), Wei and Green (2023), Smolin (2023)).[8] Our findings imply that these restrictions are not innocuous in general.

Our model builds on Eső and Szentes (2007) who focus on full disclosure and an environment with (i) the above-mentioned "continuous" model and (ii) certain assumptions on the valuation function. Under such an environment, they show that the upper bound of revenue with public signals can be achieved via full disclosure, associated with a screening menu of prices (for the good) and information fees. However, their optimal mechanism is not incentive compatible and moreover, privacy of signals generally matters outside their environment.[9] Not only do we allow for general information structures, we also characterize a *joint* design of information and trading trules in a more general environment of type space and valuation functions. This allows us to uncover how information design reshapes the optimal selling mechanism which features not just screening, but also bunching and a random mechanism. At the same time, we strengthen Eső and Szentes (2007)'s finding by showing that the irrelevance of signals extends to other (but not all) environments, with appropriate information design.

Bergemann and Wambach (2015) and Wei and Green (2023) revisit Eső and Szentes (2007)'s continuous model, showing that the latter's *optimal* allocation can be implemented under stronger participation constraints. We show that with deterministic allocations (including Eső and Szentes (2007)'s), this is true for any *feasible* allocations, not just optimal. In turn, this provides an alternative proof for Wei and Green (2023).[10]

In the second, less related, strand of this literature, the buyer's valuation is the unknown

[7]In posted-price mechanisms, each type receives a posted price for the good and in some cases, a posted fee for information.

[8]Exceptions include Zhu (2023) and Krähmer (2020) who establish full surplus extraction results when the seller can correlate information disclosed to multiple buyers, and when randomizing over information structures is allowed and the buyer's type correlates with the unknown component, respectively.

[9]See Krähmer and Strausz (2015*a*) for a detailed discussion

[10]Wei and Green (2023) also shows that information disclosure triggers reverse price discrimination. We show that this can also be derived from the properties of Eső and Szentes (2007)'s optimal mechanism.

component itself. See, for example, Lewis and Sappington (1994), Bergemann and Pesendorfer (2007), Bergemann et al. (2022). Without the buyer's private types, information cannot serve as a screening tool. Moreover, the buyer's private information (about his valuation) arrives only once, making the seller's problem static.[11]

We also contribute to the literature on dynamic mechanism design in which handling off-path misreporting is a notable issue. Eső and Szentes (2007) explicitly characterize an agent's optimal double deviation, which is to "correct the lie". However, such a lie correction is feasible only if the agent's payoff shares a *common support* across types, which is rather restrictive. We show that by leveraging mechanism design techniques for a non-convex type space, lie correction is feasible even with non-common supports. Moreover, the existing literature (for instance, Battaglini (2005), Eső and Szentes (2007), Pavan et al. (2014)) extensively relies on the first-order approach considering only local incentive compatibility constraints.[12] Instead, we characterize different scenarios of binding constraints, showing that global deviations (associated with double deviation off-path) lead to bunching and random solutions.[13]

Finally, we contribute to the recent literature on Bayesian persuasion following Kamenica and Gentzkow (2011), where a sender designs *only* information disclosure to affect a receiver's action. When the latter has a private type, Kolotilin et al. (2017) show that with binary actions and linear valuation functions, non-discriminatory disclosure is optimal. In our *joint* design problem, the buyer's action space (which is the menu of allocations and payments) is endogenous and can consist of more than two options. We show that the optimality of non-discriminatory disclosure, while not being true in general, holds in some environments even if the seller also designs trading trules and the valuation function is non-linear.

[11] If the buyer in our model has no private type, the seller fully extracts the surplus by offering no disclosure and a posted price for the good, which is equal to the expected valuation.

[12] The validity of this approach usually requires certain regularity conditions, which are not easy to satisfy, see Battaglini and Lamba (2019).

[13] Even with full disclosure, which makes our problem become a standard dynamic screening problem, random mechanisms can outperform their deterministic counterparts. See Example 3(b).

2 Model

2.1 Environment

The principal, a seller (she) sells an object to an agent, the buyer (he). The buyer's valuation for the object, $v(\theta, x) \in \mathbb{R}_+$, depends on two components: (i) the buyer's type $\theta \in \Theta \subset \mathbb{R}$ and (ii) an unknown state $x \in X \subset \mathbb{R}$. There are a N possible types and M possible states: $|\Theta| = \{\theta_1, \theta_2, \cdots, \theta_N\}$ and $|X| = \{x_1, x_2, \cdots, x_M\}$.[14] Random variables θ and x are independent.[15] Let $f(\theta)$ be the probability of each type θ and $\mu(x)$ of each state x. Without loss of generality, assume $f(\theta) > 0$ and $\mu(x) > 0$ for all θ and x.

The realization of $\theta \in \Theta$ is privately known by the buyer. Neither the seller nor the buyer knows the state $x \in X$. The seller commits to a policy of information disclosure about the state, formally defined in Section 2.2.

To define payoffs, let $q \in [0, 1]$ be the trading probability and $p \in \mathbb{R}$ the expected transfer from the buyer to the seller. The seller's ex-post payoff is then p and the buyer's is $v(\theta, x)q - p$.

Let

$$\phi(\theta_n, x) \equiv v(\theta, x) - [v(\theta_{n+1}, x) - v(\theta_n, x)] \frac{\sum_{\theta' > \theta_n} f(\theta_n)}{f(\theta_n)}$$

denote the buyer's virtual value. Throughout, assume that both the valuation and virtual valuation increase in the buyer's type and the state.

Assumption 1 (Monotone value). *$v(\theta, x)$ increases in θ and x.*

Assumption 2 (Monotone virtual value). *$\phi(\theta, x)$ increases in θ and x.*

Discussion of modeling assumptions: We study the infinite type and state spaces in Section 4.4.

2.2 Selling mechanism

The seller designs, and *ex ante* commits to a *grand* mechanism or a menu of (i) information policies for different types of the buyer and (ii) trading trules for different types and information received by the buyer.

[14] See Section 4.4 for our results in a continuous type space.
[15] See Section 5.1 for our results when types and states are correlated.

Information policies: We model information policies as information structures (experiments) $\Pi \equiv (S, \pi)$, which consists of a countable set of signals $S \subset \mathbb{R}$,[16] and a mapping π, which associates to each state θ a distribution over signals $\pi(\cdot \mid x) \in \Delta(S)$. Given a mapping π and a signal realization $s \in S$, the corresponding posterior belief $\Psi(\cdot|s) \in \Delta(X)$ is obtained by Bayes' rule whenever possible, and is given by

$$\mu_{s,\pi}(x) = \frac{\mu(x)\pi(s \mid x)}{\sum_{x' \in x} g\left(x'\right) \pi\left(s \mid x'\right)}$$

An example of information structures is the threshold rule, defined as follows.

Definition 1 (Threshold diclosure)**.** *If the information policy follows a threshold rule, each signal realization is classified as either "good news" or "bad news". Moreover,*

$$\pi(\text{“good news”}|x) = \begin{cases} 1 & \text{if} \quad x > \hat{x}, \\ \lambda & \text{if} \quad x < \hat{x}, \\ \lambda & \text{if} \quad x = \hat{x}, \end{cases} \quad \text{for some } \hat{x} \in X \text{ and } \lambda \in [0,1].$$

Thus, a threshold disclosure is represented by a pair $(\hat{x}, \lambda)$ where $\hat{x}$ is the cut-off state and λ the probability with which "*good news*" is sent at the cut-off state. It informs the buyer whether the state is (weakly) higher or lower than $\hat{x}$. To simplify notations, throughout the paper, we use "s^g" to represent "*good news*" and "s^b" for "*bad news*".

A menu of experiments is a set $\{\pi_\theta\}_{\theta \in \Theta}$. The paper focuses on the case in which the buyer privately observes the signal. The benchmark case with public signals is examined in Section 3.3.

Without loss of generality, assume that signals are ordered such that upon observing a higher signal, the buyer's posterior valuation is higher, as follows.

Assumption 3 (Ranking of signals)**.**

$$s > s' \Leftrightarrow \sum_x v(\theta, x)\mu_{s,\pi_\theta}(x) \geq \sum_x v(\theta, x)\mu_{s',\pi_\theta}(x)$$

Trading trules: An trading trule specifies the trading probability, q, and the expected transfer from the buyer to the seller, p. Given the information structure, by the revelation principle (see, for example, Myerson (1986)), we focus on direct trading trules $\{q(\theta, s), p(\theta, s)\}_{\theta,s}$.

[16] Assuming S is a countable set of $\mathbb{R}$ is without loss.

Thus, a selling mechanism is a tuple $\mathbf{M} \equiv \left\{ \pi_\theta, \left(q(\theta,s), p(\theta,s)\right) \right\}_{\theta,s}$. The formal definitions of a deterministic mechanism and its random counterpart are as follows.

Definition 2. *An mechanism **M** is deterministic if under **M**, $q(\theta,s) \in \{0,1\}$ for all $\theta \in \Theta$ and $s \in S$. **M** is random otherwise.*

Timing: The timing of interactions is as follows:

1. The seller offers a selling mechanism **M**.
2. The buyer learns his type θ and decides to accept or reject the offer. In case of acceptance, he reports a type $\hat{\theta}$ to receive information generated from $\pi_{\hat{\theta}}$.
3. The buyer *privately* observes a signal s and reports a signal $\hat{s}$.
4. The allocation $(q(\hat{\theta},\hat{s}), p(\hat{\theta},\hat{s}))$ is implemented.

According to this timing, the buyer's participation is decided at the *interim* state, as commonly assumed in the mechanism design literature. See our discussion on the timing structure in Section 3.2.

2.3 Seller's problem

An optimal mechanism refers to a revenue-maximizing mechanism. By the revelation principle, it is without loss of generality to focus on direct mechanisms such that the buyer finds it optimal to (i) participate in the mechanism, (ii) truthfully report his type, and (iii) truthfully report his signal conditional on being truthful about his type. Let

$$u(\theta,\theta',s,s') \equiv \sum_x [v(\theta,x)q(\theta',s') - p(\theta',s')]\Psi_\theta(x|s)$$

denote the ex-post payoff for type-θ buyer, who reports θ', observes s, and reports s'. Note that if the buyer lies about his type, he may want to lie again about the signal. In other words, double deviations from truth-telling may be attractive. Let

$$s^\star(\theta,\theta',s) \in \operatorname*{argmax}_{s'} u(\theta,\theta',s,s')$$

be the optimal signal reporting of type-θ buyer who reports θ' and observes signal s.[17] The *ex ante* payoff for type-θ buyer, who reports θ' and then $s^\star(\theta,\theta',s)$, is then given by

$$U(\theta,\theta') \equiv \sum_x \sum_s u(\theta,\theta',s,s^\star(\theta,\theta',s))\pi(s|x).$$

[17] In case the buyer is indifferent between signals off the equilibrium path, fix arbitrarily one of the seller-preferred signals.

With abuse of notation, let $u(\theta,s) \equiv u(\theta,\theta,s,s)$, $u(\theta,s,s') \equiv u(\theta,\theta,s,s')$, and $U(\theta) \equiv U(\theta,\theta)$. For the buyer to truthfully report his signal on the equilibrium path (conditional on reporting his type truthfully), it must be that for all θ and s,

$$u(\theta,s) \geq u(\theta,s,s'). \tag{IC-signal}$$

For the buyer to truthfully report his type, it must be that for all θ and θ',

$$U(\theta) \geq U(\theta,\theta'). \tag{IC-type}$$

Finally, the buyer participates in the mechanism if and only if

$$U(\theta) \geq 0. \tag{IR}$$

Formally, the seller's maximization problem is given by

$$\sup_{\left\{\pi_\theta, q(\theta,s), p(\theta,s)\right\}_{s,\theta}} \sum_\theta \sum_x \sum_s p(\theta,s)\pi(s|x)\mu(x)f(\theta)$$
$$s.t. \quad \text{(IR), (IC-type), (IC-signal).}$$

3 Preliminary results

This section presents several preliminary results, which are helpful to characterize the optimal mechanisms. For this purpose, we first provide formal definitions of an implementable (ex-post) allocation with private and public signals.

Definition 3. *An ex-post allocation $\{\mathbb{Q}(\theta,x)\}_{\theta,x}$ is implementable with public signals if there exists a mechanism $\tilde{\boldsymbol{M}} \equiv \left\{\tilde{\pi}_\theta, \left(\tilde{q}(\theta,s), \tilde{p}(\theta,s)\right)\right\}_{\theta,s}$ under which, constraints (IR) and (IC-type) are satisfied, and*

$$\mathbb{Q}(\theta,x) \equiv \sum_s \tilde{q}(\theta,s)\tilde{\pi}_\theta(s|x)\mu(x).$$

It is implementable with private signals if $\tilde{\boldsymbol{M}}$ also satisfies (IC-signal).

Given that the main model concerns private signals, throughout the paper we say that an ex-post allocation is implementable without specifying whether signals are private or public when it is implementable with private signals.

3.1 No distortion at the top and no rent at the bottom

We first show that the solution to the seller's joint design problem bears commonly known features: the highest type receives an efficient allocation while the lowest is fully extracted.

Lemma 1. *Under any optimal mechanism,*

(a) the lowest type gets a zero payoff: $U(\theta_1) = 0$*, and*

(b) the highest type receives an efficient allocation: $q(\theta_N, x) = \begin{cases} 1 & \text{if} \quad v(\theta_N, x) > 0, \\ \in [0,1] & \text{if} \quad v(\theta_N, x) = 0. \end{cases}$

Here, we prove Lemma 1(a) or the "no rent at the bottom" feature. we first show that the buyer's rent $U(\theta)$ increases in θ under any incentive-compatible mechanism. Consider the buyer of type $\theta > \theta_1$. By (IR), type θ prefers to reveal his type than mimicking some type $\theta' < \theta$ and reporting signals truthfully. Hence,

$$\begin{aligned} U(\theta) &\geq \sum_x \left[v(\theta, x) q(\theta', s) - p(\theta', s) \right] \pi_{\theta'}(s|x) \mu(x) \\ &\geq \sum_x [v(\theta', x) q(\theta', s) - p(\theta', s)] \pi_{\theta'}(s|x) \mu(x) \\ &= U(\theta') \end{aligned}$$

Therefore, $U(\cdot)$ is an increasing function. Toward a contradiction, suppose $U(\theta_1) = \varepsilon > 0$ under an optimal mechanism $\mathbf{M} = \{\pi_\theta, q(\theta, s), p(\theta, s)\}$. Then, by increasing $p(\theta, s)$ by ε for all θ and s, the seller's revenue strictly increases while no constraints are violated. This contradicts with $\mathbf{M}$ being optimal. Thus, $U(\theta_1) = 0$.

We leave the proof of Part (b) or the "no distortion at the bottom" result in Appendix A.1. The idea is that whenever this type does not trade with probability 1 (at some state), it is possible to improve the seller's revenue by letting him always trade under no disclosure and a posted price which is equal to his original expected payment adding the new surplus. An implicaition of this result is that random allocations are not needed for the highest type. This is, however, not necessarily true for the lower types to which, offering efficient allocations is generally sub-optimal. See Section 4.3.1 for a detailed discussion.

3.2 Sequential vs. static screening

We now establish an irrelevance result regarding the timing structure of interactions. Specifically, within the class of deterministic mechanisms, there is no revenue loss from allowing the buyer to walk away after information disclosure.

Proposition 1.

(a) If an ex-post allocation $\{q(\theta,x)\}_{\theta,x}$ is implementable by a deterministic mechanism $\mathbf{M}^d$, then it is implementable by a mechanism $\tilde{\mathbf{M}}$ which generates the same revenue for the seller and a positive pay-off for the buyer at any signal realization.

(b) Moreover, if $\mathbf{M}^d$ is optimal, then $\tilde{\mathbf{M}}$ is a menu of posted prices and binary-signal experiments under which

1. *each signal realization is either "good news" or "bad news", and*
2. *the buyer finds it optimal to buy the good if and only if he observes "good news."*

We call such a menu of experiments and posted prices that satisfies the two conditions in Proposition 1(b) a persuasive posted-price mechanism (PPM).

The proof of Part (a) is as follows. Consider an ex-post allocation $\{q(\theta,x)\}_{\theta,x}$, which is implemented by a deterministic mechanism $\mathbf{M}^d \equiv \{q(\theta,s), p(\theta,s), \pi_\theta\}$. Hence, under $\mathbf{M}^d$, $q(\theta,s) \in \{0,1\}$ for any θ and s. Fix $\theta \in \Theta$. Let $s_\theta^g \equiv \{s \mid q(\theta,s) = 1\}$ and $s_\theta^b \equiv \{s \mid q(\theta,s) = 0\}$. To induce signal truth-telling by θ, $p(\theta,s) = p(\theta,s') = \underline{p}(\theta)$ if $s \in s_\theta^g$; and $p(\theta,s) = p(\theta,s') \equiv \overline{p}(\theta)$ and if $s \in s_\theta^b$. Let

$$Q(\theta) \equiv \sum_x q(\theta,x)\mu(x)$$

represent type θ's trade probability. Consider the following two cases:

Case 1: $\underline{p}(\theta) < 0$. Then for all $s \in s_\theta^b$, $u(\theta,s) = -\underline{p}(\theta) < 0$. Suppose there exists $s' \in s_\theta^g$, $\pi(\theta,s') < 0$.Then type θ who observes s' (strictly) misreports a signal $s \in s_\theta^b$ to receive a negative transfer without buying the good. This contradicts with the incentive compatibility of $\mathbf{M}^d$. Hence, $u(\theta,s) \geq 0$ for all s and we can choose $\tilde{\mathbf{M}} = \mathbf{M}^d$.

Case 2: $\underline{p}(\theta) \geq 0$. Revise $\mathbf{M}^d$ to $\tilde{\mathbf{M}} \equiv \{\tilde{\pi}_\theta, \tilde{q}(\theta,s), \tilde{p}(\theta,s)\}$ as follows. To get $\tilde{\pi}_\theta$ from π_θ, replace any $s \in s_\theta^g$ with "s^g" ; and replace any $s' \in s_\theta^b$, with "s^b". The trading trule is given by:

$$\tilde{q}(\theta,s) = \begin{cases} 1 & \text{if} \quad s = s^g \\ 0 & \text{if} \quad s = s^b \end{cases} \quad , \quad \tilde{p}(\theta,s) = \begin{cases} \overline{p}(\theta) + \underline{p}(\theta)\dfrac{1-Q(\theta)}{Q(\theta)} & \text{if} \quad s = s^g \\ 0 & \text{if} \quad s = s^b \end{cases} . \tag{1}$$

In Appendix A.2, we show that under $\tilde{\mathbf{M}}$, the buyer's payoff is positive and moreover, the seller's revenue is the same as that under $\mathbf{M}^d$.

To prove Proposition 1(b), note that by the "no rent at the bottom", type θ_1 earns a zero payoff at optimum. Therefore, if $\underline{p}(\theta) < 0$ for some type θ, type θ_1 mimics θ and always report some signal $s \in s_\theta^b$ to enjoy a strictly positive payoff. Consequently, if $\mathbf{M}^d$ is optimal, $\underline{p}(\theta) \geq 0 \; \forall \theta$. Then, as argued in Case 2 above, $\tilde{\mathbf{M}}$ is a (PPM).

An implication of Proposition 1 is that if only deterministic mechanisms are allowed, there is no loss for the seller to contract after the buyer observes both type and signal realizations. Therefore, despite the sequential arrival of his private information, sequential screening the buyer is not beneficial, unless random mechanisms are necessary.

3.3 (Ir)relevance of signal privacy

As the buyer privately observes signals only after the contract is signed, one might expect that privacy of signals does not hurt the seller. To investigate this conjecture, we consider a benchmark problem with public signals. In such a setting, per-signal design of trading rule does not matter because the buyer's incentive only depends on *expected* terms:

$$\mathbb{Q}(\theta, x) = \sum_s q(\theta, s)\pi(s|x), \quad \mathbb{P}(\theta) = \sum_x \sum_s q(\theta, s)\pi(s|x)\mu(x)$$

Using $\mathbb{Q}(\theta, x)$ and $p(\theta)$, truth-telling (about types) condition write

$$\sum_x v(\theta, x)\mathbb{Q}(\theta, x)\mu(x) - \mathbb{P}(\theta) \geq \sum_x v(\theta, x)\mathbb{Q}(\theta', x)\mu(x) - \mathbb{P}(\theta'), \tag{$\overline{IC}$-type}$$

and IR condition writes

$$\sum_x v(\theta, x)\mathbb{Q}(\theta, x)\mu(x) - \mathbb{P}(\theta) \geq 0. \tag{$\overline{IR}$}$$

As a result, the seller's problem reduces to

$$(\overline{\mathcal{P}}) \quad \sup_{\mathbb{Q},\mathbb{P}} \quad \sum_\theta \mathbb{P}(\theta) f(\theta) \quad s.t. \quad (\overline{IC}\text{-type}), (\overline{IR}).$$

Under Assumption 1 and 2, only local IC constraints bind in $(\overline{\mathcal{P}})$. By standard arguments (omitted), this problem reduces to point-wise maximization w.r.t q only:

$$\sup_{\mathbb{Q}} \sum_\theta \sum_x \phi(\theta, x)\mathbb{Q}(\theta, x)\mu(x) f(\theta). \tag{$\star$}$$

Let

$$\hat{x}(\theta) \equiv \begin{cases} \min\{x \mid \phi(\theta, x) \geq 0\} & \text{if} \quad \phi(\theta, x_M) \geq 0, \\ +\infty & \text{if} \quad \phi(\theta, x_M) < 0 \end{cases} \tag{2}$$

denote the lowest state at which type θ's virtual value is non-negative. Note that $\hat{x}(\theta)$ decreases in θ by Assumption 2. If $\phi(\theta, x) \neq 0$ for all θ and x, a solution to $(\star)$ exists and is uniquely given by:[18]

$$\mathbb{Q}(\theta, x) = \mathbb{1}_{x \geq \hat{x}(\theta)}. \tag{3}$$

Expected payment is pinned down by local IC constraints and IR constraint for θ_1.

$$\mathbb{P}(\theta_1) = \sum_{x \geq x^\star(\theta_1)} v(\theta_n, x)\mu(x), \tag{4}$$

$$\mathbb{P}(\theta_n) = \mathbb{P}(\theta_{n-1}) + \sum_{\hat{x}(\theta_n) \leq x < \hat{x}(\theta_{n-1})} v(\theta_n, x)\mu(x). \tag{5}$$

Lemma 2 (Benchmark problem). *Suppose $\phi(\theta, x) \neq 0$ for all θ and x. With public signals, the optimal expected allocation is given by (3), and expected payment is given by (4) and (5).*

The seller retains a certain level of freedom in designing information and *per-signal* terms as long as (i) upon observing any signal, one knows whether the state is above or below the cut-off $\hat{x}(\theta)$ and (ii) expected terms are specified in Lemma 2. This leads to a multiplicity of solutions to $(\overline{\mathcal{P}})$, including the following menu of threshold disclosures and prices (paid conditional on trade).

Definition 4. *$\mathbf{M}^\star \equiv \{p^\star(\theta, s), q^\star(\theta, s), \pi^\star_\theta\}_{\theta \in \Theta, s \in \{s^g, s^b\}}$ is a menu of threshold disclosures and prices, in which*

1. *$\pi^\star_\theta(s^g|x) = \mathbb{1}_{x \geq \hat{x}(\theta)}$, where $\hat{x}(\theta)$ is given by (2).*
2. *$(q^\star, (\theta, s), p^\star(\theta)) = \begin{cases} (1, \dfrac{\mathbb{P}(\theta)}{\sum_{x \geq \hat{x}(\theta)} \mu(x)}) & \text{if} \quad s = s^g, \\ (0, 0) & \text{if} \quad s = s^b, \end{cases}$ where $\mathbb{P}(\theta)$ is given by (4) and (5).*

As $(\overline{\mathcal{P}})$ is a relaxed problem of the seller's original problem, its value provides an upper bound on the seller's revenue with private signals. Under a mild condition, Proposition 2(a) below shows that if this upper bound is tight, it can be attained via $\mathbf{M}^\star$. The intuition is that relative to other solutions to $(\overline{\mathcal{P}})$, $\mathbf{M}^\star$ provides less information (just enough to know the sign of virtual values) and a higher price to buy the good. Hence, if there exists a solution that induces truth-telling with private signals, so does $\mathbf{M}^\star$. This is the case, by Proposition 2(b), if and only if the highest type pays the lowest price under $\mathbf{M}^\star$.

Proposition 2.

[18] When $\phi(\theta, x) = 0$, any $q^\star(\theta, x) \in [0, 1]$ is optimal.

a) Suppose $\phi(\theta, x) > 0\,\forall\theta, x$. *If* $q^\star(\theta, x)$ *is implementable with private signals, then it is via* $\boldsymbol{M}^\star$.

b) $\boldsymbol{M}^\star$ *implements* $q^\star(\theta, x)$ *with private signals if and only if* $p^\star(\theta_N, s^g) = \min_\theta\{p^\star(\theta, s^g)\}$.

It seems counter-intuitive that the highest type pays the lowest price (conditional on buying the good). However, it is worth noting that information disclosure can flip the ranking of (posterior) willingness to pay across types, leading to non-monotone price discrimination.[19] As to be clear in later sections, this occurs in some, but not all environments.

4 Main result

4.1 A restatement of the seller's problem

Without loss of generality, assume that each signal induces a single (on-path) posterior valuation. Therefore, each signal s observed by type-θ buyer corresponds to his on-path posterior value after observing such a signal, given by

$$\omega^{\pi_\theta}(\theta, s) \equiv \sum_x v(\theta, x)\mu_{s,\pi_\theta}(x)$$

Moreover, that the buyer reveals the realized signal is equivalent to him reporting his posterior valuation. For any type θ, let

$$\Omega_\theta \equiv \{\omega \mid \omega = \omega^{\pi_\theta}(\theta, s) \text{ for some } s \in S\}$$

be the set of all possible on-path posterior values for type θ. Then, requiring signal truth-telling on-path is equivalent to ensuring truth-telling about on-path posterior values, or

$$\omega q(\theta, \omega) - p(\theta, \omega) \geq \omega q(\theta, \omega') - p(\theta, \omega') \quad \forall\theta, \forall\omega, \omega' \in \Omega_\theta$$

As mentioned, the buyer may want to coordinate lies about the realized type and signal. Given that the signal space is endogenous, this significantly complicates the characterization of truth-telling conditions. To deal with this, we extend the trading trule to be defined on the set of all possible on-path and off-path posterior valuations, denoted by

$$\Omega \equiv [v(\theta_1, x_1), v(\theta_N, x_M)].$$

[19] That information disclosure can lead to non-monotone price discrimination has been observed in Bang and Kim (2013) and Wei and Green (2023) where prices decrease in types. Throughout our paper, several examples are presented where under $\mathbf{M}^\star$, prices can be decreasing, increasing and even concave in types (see Example 6).

and leverage the fact that it is without loss of generality to require truthful signal reporting on this set Ω, rather than in only $\{\Omega_\theta\}_\theta$,[20] i.e.,

$$\omega q(\theta,\omega) - p(\theta,\omega) \geq \omega q(\theta,\omega') - p(\theta,\omega') \quad \forall\theta, \forall\omega,\omega' \in \Omega \tag{IC-value}$$

The following lema states a standard result regarding the characterization of (IC-value).

Lemma 3 (Myerson, 1981). *An trading trule $(q,p) : \Theta \times \Omega \to [0,1] \times \mathbb{R}$ satisfies (IC-value) if and only if*

1. $\omega q(\theta,\omega) - p(\theta,\omega) = \hat{\omega} q(\theta,\hat{\omega}) - p(\theta,\hat{\omega}) + \int_{\hat{\omega}}^{\omega} q(\theta,z)dz,$
2. $q(\theta,\omega)$ *increases in* ω.

It then follows from Lemma 3 that the buyer, having lied about his type, reveals his true (off-path) posterior valuation.

Lemma 4 (Optimal double deviations). *Under any trading trule $(q,p) : \Theta \times \Omega \to [0,1] \times \mathbb{R}$ that satisfies (IC-value), it is optimal for type θ who mimics θ' and observe signal s to report his off-path posterior valuation, given by*

$$\omega^{\pi_\theta}(\theta',s) \equiv \sum_x v(\theta,x)\mu_{s,\pi_{\theta'}}(x)$$

The proof (omitted) is similar to what is called "correcting the lie" in the dynamic mechanism design literature. Often, this lie correction is made feasible by assuming that the agent's (new) private information shares a common support across types.[21] This is not applicable in our model as the buyer's new private information, which is his posterior valuation, is endogenous. By extending the trading trule to be defined in the extended signal space Ω, we make it possible for the buyer to "correct his lie."[22]

Consider $\theta,\theta' \in \Theta$ with $\theta > \theta'$. Then,

$$\begin{aligned}
U(\theta,\theta') \equiv & \sum_x \sum_s [\omega^{\pi_{\theta'}}(\theta,s) q(\theta',\omega^{\pi_{\theta'}}(\theta,s)) - p(\theta',\omega^{\pi_{\theta'}}(\theta,s))]\pi_{\theta'}(s|x)\mu(x) \\
= & \sum_x \sum_s \left[[\omega^{\pi_{\theta'}}(\theta',s) q(\theta',\omega^{\pi_{\theta'}}(\theta',s)) - p(\theta',\omega^{\pi_{\theta'}}(\theta',s))] + \sum_s \int_{\omega^{\pi_{\theta'}}(\theta',s)}^{\omega^{\pi_{\theta'}}(\theta,s)} q(\theta',z)dz \right] \pi_{\theta'}(s|x)\mu(x) \\
= & U(\theta') + \sum_x \sum_s \int_{\omega^{\pi_{\theta'}}(\theta',s)}^{\omega^{\pi_{\theta'}}(\theta,s)} q(\theta',z)dz \pi_{\theta'}(s|x)\mu(x).
\end{aligned}$$

[20] See, for example, Skreta (2006), for mechanism design with non-convex type spaces.

[21] See Eső and Szentes (2007) and Krähmer and Strausz (2015*b*) for example.

[22] This trick can also be helpful in other dynamic mechanism design problems where the agent(s)' private information does not share common support across types.

Thus, θ does not benefit from misreporting θ' if and only if

$$U(\theta) - U(\theta') \geq \sum_x \sum_s \int_{\omega^{\pi_{\theta'}}(\theta',s)}^{\omega^{\pi_{\theta'}}(\theta,s)} q(\theta',z)dz\pi_{\theta'}(s|x)\mu(x).$$

By similar arguments, θ' does not benefit from misreporting θ if and only if

$$U(\theta) - U(\theta') \leq \sum_x \sum_s \int_{\omega^{\pi_\theta}(\theta',s)}^{\omega^{\pi_\theta}(\theta,s)} q(\theta,z)dz\pi_\theta(s|x)\mu(x).$$

To sum up, the seller's problem can be expressed as follows.

$$\begin{aligned}
(\mathcal{P}) \quad \max_{(\pi,q,U)} \quad & \sum_\theta f(\theta)\Big[\sum_x \sum_s v(\theta,x)q(\theta,\omega^{\pi_\theta}(\theta,s))\pi_\theta(s|x)\mu(x) - U(\theta)\Big] \\
s.t: \quad \forall\theta, \quad & U(\theta) - U(\theta') \geq \sum_x \sum_s \int_{\omega^{\pi_{\theta'}}(\theta',s)}^{\omega^{\pi_{\theta'}}(\theta,s)} q(\theta',z)dz\pi_{\theta'}(s|x)\mu(x) \quad \forall\theta' < \theta \\
& U(\theta) - U(\theta') \leq \sum_x \sum_s \int_{\omega^{\pi_\theta}(\theta,s)}^{\omega^{\pi_\theta}(\theta',s)} q(\theta,z)dz\pi_\theta(s|x)\mu(x) \quad \forall\theta' > \theta \\
& U(\theta) \geq 0 \\
& q(\theta,\omega) \text{ increases in } \omega.
\end{aligned}$$

4.2 Binary types

In this section, we characterize the optimal mechanism for binary types. We derive two findings. First, screening is optimal if and only if the ranking of willingness to pay is flipped under a certain threshold disclosure and bunching is optimal otherwise. Second, eliciting signals and random mechanisms are worthless. Formally, $\Theta = \{\theta_2, \theta_1\}$ and the seller's problem reduces to $(\mathcal{P}_b)$, given by

$$\begin{aligned}
(\mathcal{P}_b) \quad \max_{(\pi,q,U)} \quad & \sum_\theta f(\theta)\Big[\sum_x \sum_s v(\theta,x)q(\theta,\omega^{\pi_\theta}(\theta,s))\pi_\theta(s|x)\mu(x) - U(\theta)\Big] \\
s.t: \quad & U(\theta_2) - U(\theta_1) \geq \sum_x \sum_s \int_{\omega^{\pi_{\theta_1}}(\theta_1,s)}^{\omega^{\pi_{\theta_1}}(\theta_2,s)} q(\theta_1,z)dz\pi_{\theta_1}(s|x)\mu(x) && (IC_{21}) \\
& U(\theta_2) - U(\theta_1) \leq \sum_x \sum_s \int_{\omega^{\pi_{\theta_2}}(\theta_1,s)}^{\omega^{\pi_{\theta_2}}(\theta_2,s)} q(\theta_2,z)dz\pi_h(s|x)\mu(x) && (IC_{12}) \\
& U(\theta_2) \geq 0 && (IR_2) \\
& U(\theta_1) \geq 0 && (IR_1) \\
& q(\theta,\omega) \text{ increases in } \omega.
\end{aligned}$$

To state the main result of this section, we introduce the following notion of type order flip/preservation, which shapes the optimal mechanism. Recall that $\pi^\star$ is the threshold disclosure, as part of the menu $\mathbf{M}^\star$ formally defined in Definition 4, associated with cut-off states $x^\star(\theta_1)$ and $\hat{x}(\theta_2) = x_1$.

Definition 5 (Threshold flip/preservation of type order by $\pi^\star$)**.**

1. *Under the threshold flip of type order by $\pi^\star$,*

$$\mathbb{E}[v(\theta_2, x) \mid x < x^\star(\theta_1)] \leq \mathbb{E}[v(\theta_1, x) \mid x \geq x^\star(\theta_1)].$$

2. *Under the threshold preservation of type order by $\pi^\star$,*

$$\mathbb{E}[v(\theta_2, x) \mid x < x^\star(\theta_1)] > \mathbb{E}[v(\theta_1, x) \mid x \geq x^\star(\theta_1)].$$

By Definition 5, $\pi^\star$ induces the threshold flip of type order when it overturns the ranking of willingness to pay such that type θ_2's value after "*bad news*" being *lower* than type θ_1's after "*good news*". This is the case when the unknown component x significantly matters, creating room for $\pi^\star$ to distort the type order. By contrast, it does not happen in, for example, an extreme case in which the ranking of valuation is entirely depends on the buyer's initial type, as in standard mechanism design problems.

We are now ready to state the main result of this section, assuming that type θ_1's virtual value is either strictly positive or negative. Accordingly, the benchmark allocation is *unique*, given by $\mathbb{Q}^\star(\theta, x) = \mathbb{1}_{x \geq \hat{x}(\theta)}$.

Theorem 1 (Binary types)**.** *Fix $\Theta = \{\theta_2, \theta_1\}$. There exists some $\lambda \in [0, 1]$ and $\hat{x} \in X$, such that in the unique optimal mechanism, the allocation is given by*

$$q(\theta_2, x) = 1 \quad \forall x, \quad q(\theta_1, x) = \begin{cases} 1 & \text{if} \quad x > \hat{x}, \\ 0 & \text{if} \quad x < \hat{x}, \\ \lambda & \text{if} \quad x = \hat{x}. \end{cases}$$

Moreover,

(a) Under the threshold flip of type order by $\pi^\star$, $q(\theta, x) = \mathbb{Q}^\star(\theta, x)$. A menu of posted prices and threshold disclosures is optimal.

(b) Under the threshold preservation of type order by $\pi^\star$, $q(\theta, x) \neq \mathbb{Q}^\star(\theta, x)$. A posted price, associated with a uniform threshold disclosure, is optimal.

In short, Theorem 1 states that the optimal mechanism features *screening* whenever $\pi^\star$ leads to the threshold flip of type order and *bunching* otherwise. Intuitively, when the new information about the unknown component is important, it helps screen the buyer. Conversely, when the ranking of willingness to pay mainly driven by the buyer's type, screening disappears. Then, the optimal mechanism closely resembles its counterpart in standard mechanism design where the buyer's valuation for the good is his type: a posted price (but associated with threshold disclosure) is optimal. See Lemmas 5 and 6 for detailed discription of the optimal memechanism in the two scenarios.

To illustrate Theorem 1, consider the following examples.

Example 1 (Binary types and states). $\Theta = \{\theta_2, \theta_1\}$ *and* $X = \{x_1, x_2\}$*. Types and states are equally likely. Assume that* $\phi(\theta_1, x_1) < 0 < \phi(\theta_1, x_2)$ *to make the problem non-trivial.*

In this simple binary-type, binary-state setting, there are two scenarios of optimal mechanisms. If $v(\theta_2, x_1) \geq v(\theta_1, x_2)$, then $\pi^\star$ does not induce the threshold flip of type order. By Theorem 1(a), a fixed price and threshold disclosure is optimal. On the other hand, if $v(\theta_2, x_1) < v(\theta_1, x_2)$, then $\pi^\star$ leads to the threshold flip of type order. By Theorem 1(b), a menu of prices and threshold disclosures is optimal.

Example 2. $\Theta = \{\theta_1, \theta_2\}$ *and* X *is a finite subset of* $\mathbb{N}$*. Types and states are equally likely. Valuations are given by:* $v(\theta, x) = \theta + x$.

Let

$$\begin{aligned}\Delta_\theta &\equiv v(\theta_2, x) - v(\theta_1, x) = \theta_2 - \theta_1 \quad \forall x,\\ \Delta_x &\equiv v(\theta, x_M) - v(\theta, x_1) = x_M - x_1 \quad \forall \theta.\end{aligned}$$

Then, Δ_θ represents the variation of valuation due to the buyer's type, whereas Δ_x due to the state x. For any state $\hat{x} \in \Omega$,

$$\mathbb{E}[v(\theta_2, x) \mid x < \hat{x}] - \mathbb{E}[v(\theta_1, x) \mid x \geq \hat{x}] = \left(\theta_2 + \frac{\hat{x} - 1 + x_1}{2}\right) - \left(\theta_1 + \frac{\hat{x} + x_M}{2}\right) = \Delta_\theta - \frac{\Delta_x + 1}{2},$$

Thus, the threshold flip of type order happens if and only if

$$\Delta_\theta \leq \frac{\Delta_x + 1}{2}, \tag{6}$$

which is the case when the impact of the buyer's type is relatively small, relative to that of the unknown component. By Theorem 1, when (6) holds, it is optimal to offer a menu of threshold disclosures and posted prices. Otherwise, a posted price, coupled with uniform threshold disclosure, maximizes the seller's revenue.[23]

[23] In particular, when Δ_θ is too high, the seller does not benefit from information disclosure. In this case, the optimal threshold for type l is the highest state ($x^\star(\theta_1) = x_M$), which means no disclosure is provided.

Remark 1 (Continuous states). *Theorem 1 and its proof extends readily to the case with a continuum of states. As an example, fix $\Theta = \{\theta_1, \theta_2\}$ and $X = [0, 10]$, and both θ and x are uniformly distributed. Then, for any state $\hat{x} \in \Omega$, $\mathbb{E}[v(\theta_2, x) \mid x < \hat{x}] - \mathbb{E}[v(\theta_1, x) \mid x \geq \hat{x}] = \Delta_\theta - 5$. Thus, a menu of prices and information is optimal if $\Delta \geq 5$ and a fixed price coupling with a threshold disclosure (for all types) is optimal if $\Delta < 5$.*

Finally, Theorem 1 has two important implications:

Corollary 1. *With $N = 2$, privacy of signals does not matter when the threshold flip of type order happens under $\pi^\star$. It matters otherwise.*

Corollary 2. *With $N = 2$, the seller does not strictly benefit from using random mechanisms, nor from eliciting signals.*

What leads to the (ir)relevance of signal privacy and the optimality of deterministic mechanisms, signal-independent allocations will be explained when we present the key steps of the proof of Theorem 1, to which we turn next.

4.2.1 Proof of Theorem 1

The proof of Theorem 1, relies on considering the following relaxed problem, denoted by $(\mathcal{RP}_b)$, which ignores (IC_{12}) and (IR_2).

$$(\mathcal{RP}_b) \quad \max_{(\pi,q,U)} \quad \sum_\theta f(\theta) \Big[\sum_x \sum_s v(\theta_1, x) q(\theta, \omega^{\pi_\theta}(\theta, s)) \pi_\theta(s|x)\mu(x) - U(\theta) \Big]$$

$$s.t: \quad U(\theta_2) - U(\theta_1) \geq \sum_x \sum_s \int_{\omega^{\pi_{\theta_1}}(\theta_1,s)}^{\omega^{\pi_{\theta_1}}(\theta_2,s)} q(\theta_1, z) dz \pi_{\theta_1}(s|x)\mu(x) \qquad (IC_{21})$$

$$U(\theta_1) \geq 0 \qquad (IR_1)$$

$$q(\theta, \omega) \text{ increases in } \omega. \qquad (MON)$$

In what follows, we prove Theorem 1 in four steps. First, we show that deterministic mechanisms are optimal. This step, while standard, is helpful in decomposing the buyer's rent into two components: the *ex ante* rent (due to privacy of types) and the posterior rent (due to privacy of signals). Second, using this rent decomposition, we establish the optimality of binary-signal experiments. Third, we show that within the class of binary-signal experiments, threshold disclosures are optimal. Finally, we characterize the two scenarios of the optimal menu of threshold disclosures and prices, depending on whether $\pi^\star_{\theta_1}$ leads to the threshold flip of type order.

Step 1: We establish the optimality of deterministic mechanisms. It is clear that (IC_{21}) and

(IR_1) must bind. Then,

$$U(\theta_1) = 0, \quad U(\theta_2) = \sum_x \sum_s \int_{\omega^{\pi_{\theta_1}}(\theta_1,s)}^{\omega^{\pi_{\theta_1}}(\theta_2,s)} q(\theta_1, z) dz \pi_{\theta_1}(s|x)\mu(x),$$

Using these expressions for $U(\theta_1)$ and $U(\theta_2)$, the seller's revenue can be writeen as

$$\begin{aligned}(OBJ) \equiv & f(\theta_2) \sum_x \sum_s v(\theta_2, x) q(\theta_2, \omega^{\pi_\theta}(\theta, s)) \pi_{\theta_2}(s|x)\mu(x) \\ & + f(\theta_1) \sum_x \sum_s \left[v(\theta_1, x) q(\theta_1, \omega^{\pi_\theta}(\theta, s)) - \int_{\omega^{\pi_{\theta_1}}(\theta_1,s)}^{\omega^{\pi_{\theta_1}}(\theta_2,s)} q(\theta_1, z) dz \right] \pi_{\theta_1}(s|x)\mu(x)\end{aligned}$$

Fix π. Then, this objective function (OBJ) is linear in q and the only remaining constraint is (MON) which requires $q(\theta, \omega)$ to be increasing. Consequently, it must be that

$$q(\theta_2, \omega^{\pi_\theta}(\theta, s)) = 1 \quad \forall s, \tag{7}$$

given that $v(\theta_2, x) \geq 0$ for all x; and there exists a cut-off signal $\hat{s}(\theta)$ such that for all θ,

$$q(\theta, \omega^{\pi_\theta}(\theta, s)) = \mathbb{1}_{s \geq \hat{s}(\theta)} \tag{8}$$

Step 2: We derive the sufficiency of binary-signal experiments. By (32) and (33),

$$\begin{aligned}(OBJ) = & f(\theta_2)\mathbb{E}[v(\theta_2, x)] \\ & + f(\theta_1) \sum_x \sum_{\hat{s}(\theta_1)}^{\bar{s}} v(\theta_1, x) \pi_{\theta_1}(s|x)\mu(x) - \sum_x \sum_{\hat{s}(\theta_1)}^{\bar{s}} [\omega^{\pi_{\theta_1}}(\theta_2, s) - \omega^{\pi_{\theta_1}}(\theta_1, s)] \frac{f(\theta_2)}{f(\theta_1)} \pi_{\theta_1}(s|x)\mu(x) \\ & - \sum_x \sum_s^{\hat{s}(\theta_1)} [\omega^{\pi_{\theta_1}}(\theta_2, s) - \omega^{\pi_{\theta_1}}(\theta_1, \hat{s}(\theta_1))] \frac{f(\theta_2)}{f(\theta_1)} \pi_{\theta_1}(s|x)\mu(x),\end{aligned}$$

which is independent of π_{θ_2}. Therefore, any π_{θ_2} is optimal. To find optimal π_{θ_1}, note that it affects (OBJ) on via (i) the posterior value of type θ_1 (who reveals his type) after observing a signal $s \geq \hat{s}(\theta_1))$, (ii) the posterior value of type θ_2 (who mimics θ_1) after observing either $s < \hat{s}(\theta_1)$ or $s \geq \hat{s}(\theta_1))$, and (iii) type θ_1's posterior value after the cut-off signal, $\hat{s}(\theta_1)$. Now replace all signals $s \geq \hat{s}(\theta_1)$ with single signal s^g ("*good news*") and all $s < \hat{s}(\theta_1)$ with s^b ("*bad news*"). Under this change, (i) and (ii) are not affected. Moreover, the cut-off signal is s^g and

$$\omega^{\pi_{\theta_1}}(\theta_1, s^g) \equiv \mathbb{E}[v(\theta_1, x) \mid s = s^g] = \mathbb{E}[v(\theta_1, x) \mid s \geq \hat{s}(\theta_1)] \geq \mathbb{E}[v(\theta_1, x) \mid s = \hat{s}(\theta_1)] \equiv \omega^{\pi_{\theta_1}}(\theta_1, \hat{s}(\theta_1)).$$

In turn, this improves (OBJ), which increases in the cut-off signal. Therefore, it is optimal to offer type θ_1 a binary-signal experiment.

Step 3: We now show that threshold disclosure is optimal. By replacing all signals $s \geq \hat{s}(\theta_1)$ (resp., $s < \hat{s}(\theta_1)$) with "*good news*" (resp., "*bad news*"),

$$\begin{aligned}(OBJ) =& f(\theta_1)\sum_x v(\theta_1,x)\pi_{\theta_1}(s^g|x)\mu(x) - f(\theta_2)\sum_x [v(\theta_2,x)-v(\theta_1,x)]\pi_{\theta_1}(s^g|x)\mu(x)\\ &- f(\theta_2)\sum_x [\omega^{\pi_{\theta_1}}(\theta_2,s^b)-\omega^{\pi_{\theta_1}}(\theta_1,s^g)]\pi_{\theta_1}(s^b|x)\mu(x)\\ =& f(\theta_1)\sum_x \Bigg[\underbrace{\phi(\theta_1,x)\pi_{\theta_1}(s^g|x)}_{\theta_1\text{'s virtual value}} - \underbrace{\max\Big\{[\omega^{\pi_{\theta_1}}(\theta_2,s^b)-\omega^{\pi_{\theta_1}}(\theta_1,s^g)]\frac{f(\theta_2)}{f(\theta_1)},0\Big\}\pi_{\theta_1}(s^b|x)}_{\theta_2\text{'s posterior rent}}\Bigg]\mu(x).\end{aligned}$$

Fix the probability that signal "*good news*" is realized under π_{θ_1}, $\sum_x \pi_{\theta_1}(s^b|x)\mu(x)$. Then, a threshold disclosure minimizes θ_2's posterior rent by simultanenously maximizing $\omega^{\pi_{\theta_1}}(\theta_2,s^b)$ and minimizing $\omega^{\pi_{\theta_1}}(\theta_2,s^b)$. Moreover, as $\phi(\theta_1,x)$ increases in x, a threshold disclosure maximizes θ_1's expected virtual value.

Step 4: We now characterize two cases of the optimal mechanism. Let $\hat{x} \in X$ be the cut-off state associated with the optimal threshold disclosure for θ_1 and $\lambda \in [0,1]$ be the probability with which "*good news*" is sent at $\hat{x}$. Then,

$$q(\theta_2,x) = 1 \quad \forall x, \quad q(\theta_1,x) = \begin{cases} 1 & \text{if} \quad x > \hat{x},\\ 0 & \text{if} \quad x < \hat{x},\\ \lambda & \text{if} \quad x = \hat{x}.\end{cases}$$

Case 1: $\pi^\star_{\theta_1}$ induces the threshold flip of type order, or $\omega^{\pi^\star_{\theta_1}}(\theta_2,s^b) \leq \omega^{\pi^\star_{\theta_1}}(\theta_1,s^g)$. Then, offering $\pi^\star_{\theta_1}$ with $(\hat{x},\lambda) = (x^\star(\theta_1),1)$ induces zero posterior rent for θ_2, and creates the highest expected virtual value for θ_1's, given by $f(\theta_1)\sum_{x\geq x^\star(\theta_1)}\phi(\theta_1,x)\pi_{\theta_1}(s^g|x)\mu(x)$. Therefore, $\pi^\star_{\theta_1}$ is optimal. Moreover, with $(\hat{x},\lambda) = (x^\star(\theta_1),1)$, type θ_1's allocation coincides with the benchmark $\mathbb{Q}^\star(\theta_1,x) = \mathbb{1}_{x\geq x^\star(\theta_1)}$. To implement this allocation, set $p(\theta_2,s^b) = p(\theta_1,s^b) = 0$, and $p(\theta_1,s^g) = \omega^{\pi^\star_{\theta_1}}(\theta_1,s^g)$ so that (IR_1) binds, and $p(\theta_2,s^g)$ is such that (IC_{21}) binds, or

$$U(\theta_2) = U(\theta_2,\theta_1) \Leftrightarrow p(\theta_2,s^g) = \mathbb{E}[v(\theta_2,x)] - \big[\omega^{\pi^\star_{\theta_1}}(\theta_2,s^g) - p(\theta_1,s^g)\big]\pi^\star_{\theta_1}(s^g).$$

IR_2 holds because

$$U(\theta_2) = \big[\omega^{\pi^\star_{\theta_1}}(\theta_2,s^g) - \omega^{\pi^\star_{\theta_1}}(\theta_1,s^g)\big]\pi^\star_{\theta_1}(s^g) \geq 0$$

IC_{12} is also satisfied given that

$$
\begin{aligned}
U(\theta_1,\theta_2) &= \mathbb{E}[v(\theta_1,x)] - p(\theta_2,s^g) \\
&= \mathbb{E}[v(\theta_1,x)] - \mathbb{E}[v(\theta_2,x)] + \left[\omega^{\pi^\star_{\theta_1}}(\theta_2,s^g) - p(\theta_1,s^g)\right]\pi^\star_{\theta_1}(s^g) \\
&= \mathbb{E}[v(\theta_1,x)] - \mathbb{E}[v(\theta_2,x)] + \left[\omega^{\pi^\star_{\theta_1}}(\theta_2,s^g) - \omega^{\pi^\star_{\theta_1}}(\theta_1,s^g)\right]\pi^\star_{\theta_1}(s^g) \\
&= \left[\omega^{\pi^\star_{\theta_1}}(\theta_1,s^b) - \omega^{\pi^\star_{\theta_1}}(\theta_2,s^b)\right]\pi^\star_{\theta_1}(s^b) < 0 = U(\theta_1)
\end{aligned}
$$

We thus obtain Theorem 1(a):

Lemma 5. *If $\pi^\star$ induces the threshold flip of type order, $q(\theta_1,x) = \mathbb{Q}^\star(\theta_1,x) = \mathbb{1}_{x\geq x^\star(\theta_1)}$, and $\boldsymbol{M}^\star \equiv \{p^\star(\theta),\pi^\star_\theta\}_\theta$ is optimal.*

Case 2: $\pi^\star_\theta$ preserves the type order, or $\omega^{\pi^\star_{\theta_1}}(\theta_2,s^b) > \omega^{\pi^\star_{\theta_1}}(\theta_1,s^g)$. Then, offering $\pi^\star_{\theta_1}$ to θ_1 induces a strictly positive posterior rent for θ_2. The seller trades off between θ_1's expected virtual value and θ_2's posterior rent: on the one hand, she wants the threshold to be close to the cut-off $x^\star(\theta_1)$, maximizing θ_1 's expected value; on the other hand, she desires to induce a small posterior rent for θ_2. Let $\pi^{\star\star}_{\theta_1}$ be an optimal experiment for θ_1, associated with $\left(x^{\star\star}(\theta_1),\lambda^{\star\star}\right)$. We show that $\pi^{\star\star}_{\theta_1}$ must preserves the type order. Formally:

Claim 1. $\omega^{\pi^{\star\star}_{\theta_1}}(\theta_2,s^b) \geq \omega^{\pi^{\star\star}_{\theta_1}}(\theta_1,s^g)$.

The detailed proof is in Appendix A.4. The logic is that given that $\omega^{\pi^\star_{\theta_1}}(\theta_2,s^b) \leq \omega^{\pi^\star_{\theta_1}}(\theta_1,s^g)$, if $\pi^{\star\star}_{\theta_1}$ flips the type order, i.e., $\omega^{\pi^{\star\star}_{\theta_1}}(\theta_2,s^b) < \omega^{\pi^{\star\star}_{\theta_1}}(\theta_1,s^g)$, it is possible to construct $\tilde{\pi}_{\theta_1}$ associated with $\left(\tilde{\omega},\tilde{\lambda}\right)$ such that (i) $\left(x^{\star\star}(\theta_1),\lambda^{\star\star}\right)$ is closer to $\left(x^\star(\theta_1),1\right)$ and (ii) $\omega^{\tilde{\pi}_{\theta_1}}(\theta_2,s^b) \leq \omega^{\tilde{\pi}_{\theta_1}}(\theta_1,s^g)$.. By (i), θ_1 's expected virtual value under $\tilde{\pi}_{\theta_1}$ is higher than that under $\pi^\star_{\theta_1}$, whereas by (ii), θ_2's poterior rent is zero under $\tilde{\pi}_{\theta_1}$. By Claim 1,

$$
\begin{aligned}
(OBJ) =& f(\theta_1)\sum_x \phi(\theta_1,x)\pi_{\theta_1}(s^g|x)\mu(x) - f(\theta_2)\left[\omega^{\pi_{\theta_1}}(\theta_2,s^b) - \omega^{\pi_{\theta_1}}(\theta_1,s^g)\right]\pi_{\theta_1}(s^b) \\
=& \omega^{\pi_{\theta_1}}(\theta_1,s^g)\left[f(\theta_1)\pi_{\theta_1}(s^g|x)\mu(x) + f(\theta_2)\right] - \mathbb{E}[v(\theta_2,x)].
\end{aligned}
$$

Therefore,

$$
\pi^{\star\star}_{\theta_1} \in \underset{\pi_{\theta_1}}{\operatorname{argmax}}\, \omega^{\pi_{\theta_1}}(\theta_1,s^g)\left[f(\theta_1)\pi_{\theta_1}(s^g,x) + f(\theta_2)\right]. \tag{9}
$$

and the optimal allocation is given by

$$
q(\theta_2,x) = 1 \quad \forall x, \quad q(\theta_1,x) = \begin{cases} 1 & \text{if} \quad x > x^{\star\star}(\theta_1), \\ 0 & \text{if} \quad x < x^{\star\star}(\theta_1), \\ \lambda^{\star\star}_{\theta_1} & \text{if} \quad x = x^{\star\star}(\theta_1). \end{cases}
$$

This allocation can be implemented by a fixed disclosure rule $\pi^{\star\star}_{\theta_1}$ and a posted price

$$p^{\star\star}(\theta_2) = p^{\star\star}(\theta_1) = \omega^{\pi^{\star\star}_{\theta_1}}(\theta_1, s^g). \tag{10}$$

to both types. To see this, note that by Claim 1, $\omega^{\pi^{\star\star}_{\theta_1}}(\theta_2, s^b) \geq \omega^{\pi^{\star\star}_{\theta_1}}(\theta_1, s^g) = p^{\star\star}(\theta_2)$ Hence, type θ_2 always buys the good regardless of signal realization. Regarding type θ_1, $\omega^{\pi^{\star\star}_{\theta_1}}(\theta_1, s^b) \leq \omega^{\pi^{\star\star}_{\theta_1}}(\theta_1, s^g) = p^{\star\star}(\theta_1)$. Hence, type θ_1 buys the good only if "good news" is realized.

Lemma 6. *If $\pi^\star$ does not induce the threshold flip of type order, $(\hat{x}, \lambda) \neq (x^\star(\theta_1), 1)$. A single-option menu, $\{\pi^{\star\star}_{\theta_1}, p^{\star\star}(\theta_1)\}$ given by (9) and (10), is optimal.*

We thus obtain Theorem 1(b).

4.3 Discrete types

With binary types, there are two scenarios of the optimal mechanism (screening/bunching), depending on whether after information disclosure, the threshold flip of type order occurs or not. With richer type sets, it can be the case that information disclosure flips the order of a group of types but fails to do so for another group. Consequently, the characterization of optimal mechanisms cannot be obtained as a simple extension of that in the binary-type case. Moreover, as we will show, random mechanisms could be used to effectively screen signals and distant types. Despite these complications, we show that the optimality of a rich (respectively, single-option) menu of prices and threshold disclosure extends beyond the binary-type setting to a general model under stronger notions of type order flip (respectively, preservation). This result is presented in Sections 4.3.2 and 4.3.3, followed by an analysis on the role of random mechanisms in Section 4.3.1.

4.3.1 Revenue improvement via random mechanisms

Using Example 3 below, we illustrate how random mechanisms outperform their deterministic counterparts in two aspects (i) screening distant types and (ii) screening signals to improve the seller's revenue and efficiency.[24]

Example 3. *$\Theta = \{\theta_1, \theta_2, \theta_3\}$ and $X = \{x_1, x_2\}$. Types and states are equally likely. Valuations are as follows.*

Example 3(a) - Screening distant types: In this example, type θ_3 's value is always higher than type θ_1's. Therefore, if type θ_1 trades with probability 1 for some signal (as in deter-

[24] In the Online Appendix, we fully characterize the optimal random mechanism in several examples.

$v(\theta, x)$	x_1	x_2
θ_3	6.5	10
θ_2	0	7
θ_1	0	4

Table 1: Example 3(a)

$v(\theta, x)$	x_1	x_2
θ_3	5	5
θ_2	2	5
θ_1	0	4

Table 2: Example 3(b)

ministic mechanisms), it is optimal for type θ_3 who mimics θ_1 to (mis)report the realized signal such that he always trades. Then, type θ_1's allocation is the same as type θ_3's from the latter's perspective, leading to bunching these types.[25] In turn, this gives too much rent for type θ_3 , making it optimal to exclude type θ_1 under deterministic mechanisms.

Claim 2. *In Example 3(a), if only deterministic mechanisms are allowed, it is optimal to offer type θ_3 with no disclosure and a posted price $p(\theta_3) = 6.75$, type θ_2 with full disclosure and a posted price $p(\theta_2) = 7$, and to exclude type θ_1.*

The story, however, is different with random allocations. The key is that if θ_1 trades with a small probability (for any signal), this type's allocation becomes unattractive to θ_3. To see this, modify the optimal deterministic mechanism by letting θ_1 trade with a probability $\varepsilon \in [0, \frac{3}{4}]$ and adjusting transfers such that truth-telling remains satisfied, as follows:

$$q(\theta_3, x) = 1 \forall x, \quad q(\theta_2, x) = \mathbb{1}_{x=\theta_1}, \quad q(\theta_1, x) = \begin{cases} \varepsilon & \text{if} \quad x = \theta_2, \\ 0 & \text{if} \quad x = \theta_1, \end{cases}$$

$$p(\theta_3) = 6.5 - \varepsilon, \quad p(\theta_2) = 7 - 2\varepsilon, \quad p(\theta_1) = 5 \quad \text{paid conditional on trade occurs.}$$

Then, expected payment by θ_3 and θ_2 reduces by ε; however, that by θ_2 increases by $\frac{5\varepsilon}{2}$. Overall, the seller's revenue increases by $f(\theta_1)\frac{5\varepsilon}{2} - [f(\theta_2) + f(h)]\,\varepsilon = \frac{3\varepsilon}{2} > 0$. Therefore, random allocation helps the seller screen effectively distant types (types θ_3 and θ_1), thereby, improving trade surplus extensively as well as the seller's revenue.

Example 3**(b) - Screening signals**: In this example, type θ_2's value varies significantly across states. This makes it optimal to exclude type θ_2 at state x_1, rather than "pooling" the two states under deterministic mechanisms which allow either trade or no trade at any signal realization. Formally, the optimal deterministic mechanism, stated in Claim 3

[25]In Example 3(a), if the seller employs deterministic mechanisms and serves type θ_1, a fixed price $p = 4$, associated with full disclosure is optimal.

below, specifies:

$$q(\theta_3, x) = 1 \forall x, \quad q(\theta_2, x) = q(\theta_1, x) = \mathbb{1}_{x=\theta_2},$$

which are implemented via full disclosure and a fixed price.

Claim 3. *In Example 3(b), if only deterministic mechanisms are allowed, it is optimal to offer full disclosure and a posted price $p = 4$.*

Random mechanisms, on the other hand, arm the seller with the flexibility in designing trade probabilities. This helps screen realized states by allowing trade to happen at a small probability at low states. To see this, revise the optimal deterministic mechanism by letting θ_2 trade with probability $\delta \leq \frac{1}{3}$, such that now:

$$q(\theta_3, x) = 1 \forall x, \quad q(\theta_1, x) = \mathbb{1}_{x=\theta_2}, \quad p(\theta_3) = p(\theta_1) = 4,$$
$$\big(q(\theta_2, x), p(\theta_2, x)\big) = \begin{cases} (1, 4) & \text{if} \quad x = x_2 \\ (\delta, 2\delta) & \text{if} \quad x = x_1 \end{cases}, \quad \text{with } \delta \leq \frac{1}{3}.$$

This revised mechanism differs from the optimal deterministic mechanism only in the new trade created with type θ_2 at state x_1. Therefore, as long as this new trade creation preserves incentive compatibility, the seller's revenue increases by $2f(\theta_2)\mu(x_1)\delta > 0$. We show that this is the case in the Online Appendix.

This section generalizes the finding of optimal mechanisms with binary types (Theorem 1) to a general model with finitely many types.

4.3.2 Optimality of screening

Recall that information disclosure can be used to screen the buyer of binary types when it induces a threshold flip of type order. Similarly, information serves as a screening tool in a richer type space under the following notion of type order flip:

Definition 6 (Partition flip of type order)**.**

The partition flip of type order happens if $\mathbb{E}[v(\theta_{n+1}, x) \mid x^\star(\theta_{n+1}) \leq x < x^\star(\theta_n)]$ decreases in θ.

Under the partition flip of type order, the expected valuations over relevant partitions of states decrease in types. As the relevant partition for a higher type consists of lower states, such a type order flip requires the new information (about the state) to sufficiently dominate the buyer's initial type in driving valuation fluctuations. Indeed, it coincides with the threshold flip notation when there are only two types. In a richer type set, more than one interior threshold is involved under the menu of threshold disclosure $\{\pi^\star_\theta\}_\theta$, leading to relevant partitions of states.

Theorem 2 below states the optimal mechanism under the partition flip of type order, which features discriminatory information and prices.

Theorem 2 (Screening theorem)**.** *Under the partition flip of type order, the optimal allocation is given by* $q(\theta, x) = \mathbb{1}_{x \geq \hat{x}(\theta)}$. *A menu of posted prices and threshold disclosures is optimal.*

This result extends Theorem 1(a) to a model with more than two types, following the same logic: when information disclosure matters sufficiently, it helps screen the buyer. The only difference is that the *partition flip* of type order is required here, taking into account interior types.

The proof proceeds by showing that under the partition flip of type order, $\mathbf{M}^\star$ induces truth-telling even if the seller does not observe signals. Therefore, offering $\mathbf{M}^\star$ with the buyer privately observing signals is equivalent to offering a menu of posted prices and threshold disclosures $\{p^\star(\theta), \pi^\star_\theta\}_\theta$, where the posted price is equal to the payment paid after "*good news*" in $\mathbf{M}^\star$: $p^\star(\theta) = p^\star(\theta, s^g)$. This menu helps the seller achieve the upper bound of revenue attained when signals are public signals; hence, it is optimal.

We close this section with an illustrative example.

Example 4. $\Theta = \{\theta_3, \theta_2, \theta_1\}$. *X is a finite subset of* $\mathbb{N}$. *Types and states are equally likely. Valuations are given by* $v(\theta_3, x) = x + \Delta_\theta$, $v(\theta_2, x) = x$, $v(\theta_1, x) = x - \Delta_\theta$. *Accordingly, virtual values are given by* $\phi(\theta_3, x) = x + \Delta_\theta$, $\phi(\theta_2, x) = x - \Delta_\theta$, $\phi(\theta_2, x) = x - 3\Delta_\theta$.

In this example, $v(\theta_{n+1}, x) - v(\theta_n, x) = \Delta_\theta$ $\forall x$ and n; $\Delta_x \equiv v(\theta, x_M) - v(\theta, x_1) = x_M - x_1$ $\forall\theta$. In addition, $x^\star(\theta_3) = x_1$, $x^\star(\theta_2) = \Delta_\theta$, and $x^\star(\theta_1) = 3\Delta_\theta$, which implies

$$\mathbb{E}[v(\theta_3, x) \mid x^\star(\theta_3) \leq x < x^\star(\theta_2)] = \frac{3\Delta_\theta - 1 + x_1}{2},$$
$$\mathbb{E}[v(\theta_2, x) \mid x^\star(\theta_2) \leq x < x^\star(\theta_1)] = \frac{4\Delta_\theta - 1}{2},$$
$$\mathbb{E}[v(\theta_1, x) \mid x^\star(\theta_1) \leq x \leq x_M)] = \frac{\Delta_\theta + x_M - 1}{2}$$

Thus, the partition flip of type order happens if

$$3\Delta_\theta - 1 + x_1 \leq 4\Delta_\theta - 1 \leq \Delta_\theta + x_M - 1 \Leftrightarrow x_1 \leq \Delta_\theta \leq \Delta_x,$$

which requires the impact of the unknown component to be higher than that of the buyer's type (and is of at least x_1). If this is the case, by Theorem 2, it is optimal to screen the buyer's type using a screening menu of prices and threshold disclosure.

4.3.3 Optimality of bunching

In the binary-type case, the benefit of screening disappears if the threshold disclosure rule $\pi^\star$ fails to flip the ranking of willingness to pay by types. A similar story holds with more than two types under a stronger notion of (no) threshold flip of type order:

Definition 7 (Uniformly no threshold flip of type order)**.** *Under uniformly no threshold flip of type order,*

$$\mathbb{E}[v(\theta_{n+1}, x \mid x < \hat{x}] \geq \mathbb{E}[v(\theta_n, x) \mid x \geq \hat{x}] \quad \forall \theta \in \Theta, \forall \hat{x} \in X.$$

In words, this condition satisfies if under *any threshold disclosure* and for *any type* θ: θ_n's value after "*bad news*" must be higher than θ's after "*good news*". This is more likely to hold when valuation heterogeneity is mainly driven by the buyer's type. For instance, when θ_n's values are always higher regardless of states, i.e., $v(\theta_{n+1}, x_1) \geq v(\theta_n, x_M)$, it is impossible to flip their ranking of valuation after *any rule* of information disclosure, not just the threshold ones.

We are now ready to state the main result of this section.

Theorem 3 (Bunching theorem)**.** *Under uniformly threshold preservation of type order, a posted price, associated with a threshold disclosure, is optimal.*

This result extends Theorem 1(b), carrying the same intuition: when valuation heterogeneity is mainly due to the buyer's types, information about the state becomes inessential for (most types of) the buyer; as a result, its screening function shuts off. The only difference is that no type order flip by *any* threshold disclosure is required here, of which the role is to be explained.

The proof proceeds by solving a relaxed problem considering only deviating behaviors under which all types mimic the lowest type being served. This problem mirrors that for the binary-type case $\Theta = \{\theta_2, \theta_1\}$, with the lowest type being served representing type l and all the other types echoing type h. The optimality of bunching under *uniformly* threshold preservation of type order follows similar arguments for that in the binary-type setting under threshold preservation by $\pi^\star$. The lowest type being served, and thereby, the optimal posted price and threshold disclosure can be explicitly characterized, leveraging the fact that threshold preservation holds *uniformly* regardless of pairs of types and threshold rule.

To end this section, revisit Example 4 for an illustration. In this example, for any $\hat{x} \in X$,

$$\mathbb{E}[v(\theta_3, x) \mid x < \hat{x}] - \mathbb{E}[v(\theta_2, x) \mid x \geq \hat{x}] = \big(\Delta_\theta + \frac{\hat{x} - 1 + x_1}{2}\big) - \big(\theta_2 + \frac{x_M + \hat{x}}{2}\big) = \Delta_\theta - \frac{\Delta_x - 1}{2},$$
$$\mathbb{E}[v(\theta_2, x) \mid x < \hat{x}] - \mathbb{E}[v(\theta_1, x) \mid x \geq \hat{x}] = \big(\theta_2 + \frac{\hat{x} - 1 + x_1}{2}\big) - \big(l - \Delta_\theta + \frac{x_M + \hat{x}}{2}\big) = \Delta_\theta - \frac{\Delta_x - 1}{2},$$

where, just to recall, Δ_θ and Δ_x measure the impact of the buyer's private type and the unknown component in valuation variations, respectively. Therefore, uniformly threshold preservation of type order occurs if

$$\Delta_\theta \geq \frac{\Delta_x - 1}{2},$$

which requires the buyer's type to be significantly impactful, relative to the unknown component. If this is the case, by (Theorem 3), information is not leveraged to screen the buyer. A single price-threshold disclosure bundle is optimal.

4.4 Continuous types

All the proofs of our results extend readily if there is a continuum of states. The extension to the infinite-type case, however, is not trivial. Nevertheless, we find that the previous insights remain valid: Section 4.4.1 shows that a menu of prices and threshold disclosure is optimal under the partition flip of valuation ranking across cut-off types; and Section 4.4.2 shows that a fixed price- threshold disclosure bundle is approximately optimal when the type order is almost preserved.

Throughout this section, consider a continuum of types $\Theta = [\theta_1, \theta_N] \subset \mathbb{R}$, endowed with the distribution $F(\theta)$. We assume that $F(\theta)$ is differentiable in θ with density $f(\theta)$, and moreover, $v(\theta, x)$ is differentiable in θ. Then, the virtual value in this environment is given by

$$\phi^c(\theta, x) = v(\theta, x) - v_\theta(\theta, x)\frac{1 - F(\theta)}{f(\theta)}.$$

Similar to the finite-type case, we assume that $\phi^c(\theta, x)$ increases in θ and x.

4.4.1 Optimality of a screening menu

By the monotonicity of the virtual values, each state x is associated with a cut-off type θ_x above (respectively, below) which the buyer's virtual value is non-negative (respectively, negative). Formally,

$$\theta^\star(x) \equiv \inf\{\theta \mid \phi^c(\theta, x) \geq 0\}.$$

Moreover, as $\phi^c(\theta, x)$ increases in x, this cut-off type θ_x decreases in x. We use

$$\Theta_x \equiv \{\theta^\star(x)\}_{x \in X}$$

to denote the type space consisting of only cut-off types. Even with a continuum of types, there are *finitely* many cut-off types $\{\theta_x\}_{x\in X}$ due to the finiteness of the state space. Accordingly, $\mathbf{M}^\star$ comprises $|\Theta_x|$ options of prices and disclosure rules as if the type space was Θ_x because each interval of types $[\theta_{x^+}, \theta_x)$ is assigned the same option. Then, leveraging the screening theorem for discrete types Θ_x, we obtain the optimality of a menu of price-information bundles with infinitely many types. Note that with a type space Θ_x, the relevant partition of states reduces to a singleton x for type $\theta^\star(x)$.

Proposition 3. *Fix $\Theta = [\theta_1, \theta_N]$ and $|X| < \infty$. If $v(\theta^\star(x), x)$ increases in x, then a menu of threshold disclosures and posted prices is optimal.*

This result holds even if there is a continuum of states $X = [x_1, x_M]$ and the valuation function is continuous over states, by approximating an associated finite-state model as the distance between states approaches zero.[26]

4.4.2 (Approximate) optimality of bunching

When valuations shift smoothly across (a continuum of) types, there are always types whose valuations are sufficiently close to others'. This makes it impossible to preserve the ranking of willingness to pay uniformly across the types. Consequently, the optimality of bunching cannot be derived as an extension of Theorem 3 which shows that under the uniformly threshold preservation of valuation ranking across finitely many types, a fixed price-information bundle is optimal. Nevertheless, we establish the *approximate* optimality of bunching under *ε-uniformly* threshold preservation of type order, formally defined below.

Definition 8 (ε-uniformly threshold preservation of type order)**.** *ε-uniformly threshold preservation of type order occurs if for some $\varepsilon > 0$,*

$$\mathbb{E}[v(\theta + \varepsilon, x) \mid x \leq \hat{x}] \geq \mathbb{E}[v(\theta, x) \mid x \geq \hat{x}] \quad \forall \theta, \hat{x}.$$

The following proposition shows that as ε vanishes, the seller's maximized revenue can be approximated by offering a fixed price-threshold disclosure bundle. Formally, let R_ε

[26] This is the case in, for example, the environments studied in Eső and Szentes (2007) and Wei and Green (2023) under which the valuation function is concave in types and states, and the cross derivative is positive.

represent the revenue guarantee if the seller offers a single posted price and threshold disclosure rule under the ε-uniformly threshold preservation of type order, we find that:

Proposition 4. *$R_\varepsilon \to V(P)$ as $\varepsilon \to 0$*

Moreover, if there are only two states $\Omega = \{x_2, x_1\}$, we establish the exact optimality of a fixed price and disclosure rule within the class of deterministic mechanisms.

Proposition 5. *Fix $\Theta = [\theta_1, \theta_N]$ and $X = \{x_2, x_1\}$. If $v(\theta^\star(x_1), x_1) > v(\theta^\star(x_2), x_2)$ and only deterministic allocations are allowed, a posted price, associated with full disclosure, is optimal.*

The idea of the proof is as follows. With binary states $X = \{x_2, x_1\}$, there are only two cut-off types $\theta^\star(x_2)$ and $\theta^\star(x_1)$. Hence, the partition flip of type order reduces to $v(\theta^\star(x_1), x_1) \leq v(\theta^\star(x_2), x_2)$. If this is the case, a menu of prices and threshold disclosures is optimal by Proposition 3. If by contrast, $v(\theta^\star(x_1), x_1) \leq v(\theta^\star(x_2), x_2)$, the seller adjusts the cut-off types to $\tilde{\theta}(x_1), \tilde{\theta}(x_2)$ *just enough* to restore the partition flip of type order: $v(\tilde{\theta}(x_1), x_1) = v(\tilde{\theta}(x_2), x_2)$. In turn, at this boundary of the partition flip, the seller is indifferent between offering a screening menu and a single option of price and information. Put differently, bunching is optimal.

We end this section with a numerical example to illustrate Proposition 5.

Example 5. $v(\theta, x) = 3\theta^2 + 6\theta + x$, $\Theta = [0, 2]$, $X = \{8, 12\}$. *Types and states are likely equally.*

In this example, $\phi(\theta, x) = 3\theta^2 + 6\theta + x - (6\theta + 6)(2 - \theta) = 9\theta^2 + x - 12$. Thus, $\theta_{12} = 0$ and $\theta_8 = \frac{2}{3}$. Hence, $v(\theta_8, 8) = \frac{43}{3}$ and $v(\theta_{12}, 12) = 12$. As $v(\theta_8, 8) > v(\theta_{12}, 12)$, no flip of type order occurs. By Proposition 5, within the class of deterministic mechanism, offering a fixed bundle of price and threshold disclosure to all types is optimal.

5 Discussion

5.1 Type and state are correlated

Our main model assumes that type and state are independently distributed. We now consider the case underwhich, type and state are correlated. Formally, the state is now distributed according to the distribution $\mu_\theta(x)$. We partially extend the key insights of our main model to this scenario. First, we establish that bunching remains optimal under the global threshold preservation of type order, adjusted by the correlation of types and states.

Definition 9 (Correlation-adjusted global threshold preservation of type order)**.**

$$\mathbb{E}_{x\sim\mu_{\theta^+}}[v(\theta^+,x)|x\geq\hat{x}]\geq\mathbb{E}_{x\sim\mu_\theta}[v(\theta,x)|x<\hat{x}]\quad\forall x,\forall\theta.$$

Proposition 6. *Under the correlation-adjusted global threshold preservation of type order, a posted price, associated with a fixed threshold disclosure rule is optimal.*

That the distribution of states now is type-dependent has no technical impact. The proof of Proposition 6 follows closely that of the bunching theorem in case of independence.

By contrast, a screening menu of threshold disclosures and prices is not necessarily optimal when type and states are correlated.[27] Technically, the correlation between type and states distorts the virtual value, preventing us from directly applying the arguments used in the main model of independence. While it is no longer tractable to characterize the optimal mechanism when the new information flips the type order, we conjecture that it continues to feature a screening menu. The exact shape of this menu is an open question.

To end this discussion, we show that with binary types and states, a menu of threshold disclosures is optimal (and a single bundle of threshold disclosure and posted price is optimal otherwise) under a certain condition of the following correlation-adjusted virtual value.

$$\phi^{corr}(\theta_1,x)\equiv v(\theta_1,x)-\left[v(\theta_2,x)-v(\theta_1,x)\frac{\mu_{\theta_1}(x_2)}{\mu_{\theta_1}(x_2)}\right]\frac{f(\theta_2)}{f(\theta_1)}$$

Proposition 7. *Fix $\Theta=\{\theta_1,\theta_2\}$ and $X=\{x_1,x_2\}$.*

(a) If $v(\theta_2,x_1)\geq v(\theta_1,x_2)$, then a threshold disclosure and a posted price is optimal.

(b) If $v(\theta_2,x_1)\leq v(\theta_1,x_2)$ and $\phi^{corr}(\theta_1,x_1)\leq 0\leq\phi^{corr}(\theta_2,x_2)$, then a menu of threshold disclosures and prices is optimal.

Note that part (a) of Propotion 7 is a special case of Proposition 6. The proof Part (b) is on Appendix A.11.

5.2 Posterior rent and privacy of signals

As explained in the binary-type model, not observing signals generally hurts the seller due to the presence of the buyer's posterior rent. Specifically, implementing the benchmark allocation requires the seller to pay the buyer's posterior rent (apart from his *ex ante*

[27] Indeed, when the buyer's type is payoff-irrelevant and correlated with the state, Guo et al. (2022) shows that interval disclosures can strictly outpeform threshold rules.

rent), making $V(P) < V(\overline{P})$. When valuation shifts smoothly across (infinite) types, the relevance of signal privacy comes from a different reason. Indeed, any allocations *implementable with private signals* can be implemented without generating posterior rent to the buyer.[28] Therefore, if the seller fails to achieve the upper bound of revenue $V(\overline{P})$, it is due to an implementability issue. In such a scenario, information design can expand the set of implementable allocations. To illustrate, consider the following example where the benchmark allocation is implementable with private signals only if uninformative experiments are possible.

Example 6. $v(\theta, x) = \theta^2 + \theta + x - 2$. *Types and states are uniformly distributed over* $\Theta = [0,1]$ *and* $\Omega = [0,3]$.

In this example, $p^\star(\theta, s^g) = -\theta^2 + \frac{2}{3}\theta + 1$. Moreover, $p^\star(\theta, s^g)$ is a concave function in $[0,1]$ with $p(0, s^g) = 1, p(1, s^g) = \frac{2}{3}$. Thus, $p^\star(\theta_N, s^g) = \min_\theta p^\star(\theta, s^g)$. Then by Proposition 2, the seller implements the benchmark allocation via $\mathbf{M}^\star$. Suppose the seller provides full disclosure to all types. To implement the benchmark allocation, it must be that for any θ and x, $q(\theta, x) = \mathbb{1}_{x \geq \hat{x}(\theta)}$. For the buyer to report truthfully their states, it is necessary that

$$p(\theta, x) = \begin{cases} \overline{p}(\theta) & \text{if } x \geq \hat{x}(\theta), \\ \underline{p}(\theta) & \text{otherwise.} \end{cases}$$

To prevent the lowest type θ_1 from mimicking some type θ and always report $x < \hat{x}(\theta)$, it must be that $\underline{p}(\theta) \geq 0$. Therefore,

$$\int_{x \geq \hat{x}(\theta)} \mu(x) dx p^\star(\theta, s^g) = \int_{x \geq \hat{x}(\theta)} \mu(x) dx \overline{p}(\theta) + \underline{p}(\theta) \int_{x \leq \hat{x}(\theta)} \mu(x) dx \geq \int_{x \geq \hat{x}(\theta)} \mu(x) dx \overline{p}(\theta)$$

where the equality uses the fact that all mechanisms implementing the benchmark allocation share the same expected payment. Thus, $p^\star(\theta) \geq \overline{p}(\theta)$ for all type θ.

Consider $\theta = \frac{1}{3}$, we have $\overline{p}^\star(\frac{1}{3}, s^g) = \frac{10}{9}$, and $v(\frac{1}{3}, x_{\frac{1}{3}}) = \frac{13}{9}$. Thus, $v(\frac{1}{3}, x(\frac{1}{3})) > p^\star(\frac{1}{3}, s^g) \geq \overline{p}(\frac{1}{3})$. Then, if the buyer observes any state $x \in (\overline{p}(\frac{1}{3}), v(\frac{1}{3}, x(\frac{1}{3})))$, it is optimal for him to misreport state x_M, receiving the good at a price lower than his valuation. Thus, the benchmark allocation is not implementable under full disclosure.

5.3 Alternative proof for Wei and Green (2023)

Wei and Green (2023) revisit Eső and Szentes (2007)'s "continuous" model, adding a twist

[28] We omit the formal proof, which extends the arguments in Krähmer and Strausz (2015*a*) to a setting with information design and possibly finitely many states.

that the buyer can walk away after information disclosure. In this section, we solve the former's problem by directly modifying the latter's optimal mechanism.[29]

Under Eső and Szentes (2007)'s optimal mechanism, the seller offers full disclosure and a menu of "information fees" $\hat{c}(\cdot)$ and "strike prices" $\hat{p}(\cdot)$ for the good to implement the benchmark optimal allocation. Thus, $\big(q(\theta), p(\theta)\big) \in \Big\{\big(0, \hat{c}(\theta)\big), \big(1, \hat{c}(\theta) + \hat{p}(\theta)\big)\Big\}$. This menu is a deterministic mechanism. Therefore, following the arguments in the proof of Proposition 1, it is revenue-equivalent to a persuasive-posted price mechanism which offers type θ (i) a binary-signal experiment which sends "*good news*" if $x \geq x_\theta$ and "*bad news*" otherwise, and (ii) a posted price.

$$\tilde{p}(\theta) = \hat{c}(\theta) + \hat{p}(\theta) + \frac{\hat{c}(\theta)\left[1 - \mathcal{Q}(\theta)\right]}{\mathcal{Q}(\theta)} = \hat{p}(\theta) + \frac{\hat{c}(\theta)}{\mathcal{Q}(\theta)}.$$

In addition, Wei and Green (2023) show that information design leads to reverse price discrimination in the continuous model. This feature can also be obtained by leveraging the properties of Eső and Szentes (2007)'s optimal mechanism. Let $\mathbf{X}(\theta) \equiv \frac{1}{\mathcal{Q}(\theta)}$ represent the inverted trade probability for θ. Then, $\tilde{p}(\theta) = \hat{p}(\theta) + \hat{c}(\theta)\mathbf{X}(\theta)$, and

$$\tilde{p}'(\theta) = \hat{p}'(\theta) + \hat{c}'(\theta)\mathbf{X}(\theta) + \hat{c}(\theta)\mathbf{X}'(\theta) = \hat{c}(\theta)\mathbf{X}'(\theta) < 0,$$

where the second equality uses the fact that under Eső and Szentes (2007)'s optimal mechanism, $\hat{c}(\theta)$ and $\hat{c}(\theta)$ solves $\hat{c}'(\theta) = \hat{p}'(\theta)\mathcal{Q}(\theta) = \hat{p}'(\theta)\frac{1}{\mathbf{X}(\theta)}$, and the last uses $\mathbf{X}'(\theta) < 0$. Thus, $\tilde{p}(\cdot)$ is a decreasing function.

5.4 On the number of signals

As we have seen, it is without loss of generality to offer binary-signal experiments with deterministic allocation. This is no longer true when random mechanisms are necessary. When the variations vary significantly across states, a rich menu is needed to screen the states effectively. As a result, binary-signal experiments are not sufficient. In this section, we illustrate this with a simple example where an optimal experiment sends at least three signals to some type.

Example 7. $\Theta = \{t_3, \theta_m, \theta_l\}, X = \{x_b, x_g, x_3, x_4\}$. *Types and states are equally likely.*

[29] Indeed, this modified mechanism coincides with Wei and Green (2023)'s solution.

$v(\theta, x)$	x_1	x_2	x_3	x_4
θ_3	7	7	7	7
θ_2	0	3	7	7
θ_1	0	0	0	6

In this example, θ_2's valuation varies significantly across states with that at state x_1 being sufficiently low. If restricted to binary-signal experiments, the seller can only separate the state space for type θ_m into two partitions which, under the optimal mechanism, include $\{x_1, x_2\}$ and $\{x_3, x_4\}$. Armed with three signals, the seller can distinguish a very unfavorable state x_1 from a better one x_2, fine-tuning the design of allocations. The formal proof is in the Online Appendix.

A Appendices

A.1 Proof of Lemma 1

Let $\mathbf{M} \equiv \{\pi_\theta, q(\theta, s), p(\theta, s)\}_{\theta,s}$ be an optimal mechansim. Toward a contradiction, assume that there exists x such that $Q(\theta_N, x) \equiv \sum_s q(\theta, s)\pi_{\theta_N}(s|x)\mu(x) < 1$. We now show that the seller can improve her revenue by revising type θ_N contract to $\tilde{\mathbf{C}} \equiv \{\tilde{\pi}_{\theta_N}, \tilde{p}(\theta_N)\}$ in which $\tilde{\pi}_{\theta_N}$ provides no information and $\tilde{p}(\theta_N)$ is a posted price for the good, given by:

$$\tilde{p}(\theta_N) = \sum_s p(\theta_N, s)\pi_{\theta_N}(s) + \mathbb{E}[v(\theta_N, x)] - \sum_x v(\theta_N, x) \sum_s q(\theta_N, s)\pi_{\theta_N}(s|x)\mu(x)$$

Note that

$$\begin{aligned}
&\mathbb{E}[v(\theta_N, x)] - \sum_x v(\theta_N, x) \sum_s q(\theta_N, s)\pi_{\theta_N}(s|x)\mu(x) \\
=&\sum_x v(\theta_N, x)\mu(x) - \sum_x v(\theta_N, x) q(\theta_N, x)\mu(x) \\
=&\sum_x v(\theta_N, x)\left[1 - q(\theta_N, x)\right]\mu(x) \\
&< 0,
\end{aligned}$$

where the inequality use $q(\theta, x) < 1$ for some x by assumption. Therefore, type θ_N's expected payment under $\tilde{\mathbf{C}}$ is higher than that under his original contract, which is $\sum_s p(\theta_N, s)\pi_{\theta_N}(s)$. Thanks to this, to show that the seller's revenue improves by revising type θ_N, it suffices to show that the buyer reports his type truthfully. First, consider type θ_N. His payoff from buying the good at the price $\tilde{p}(\theta_N)$ and no disclosure is

$$\mathbb{E}[v(\theta_N, x)] - \tilde{p}(\theta_N) = \sum_x \sum_s \left[v(\theta_N, x) q(\theta_N, s) - p(\theta_N, s)\right] \pi_{\theta_N}(s|x)\mu(x),$$

which is equal to that under his original contract. By the incentive compatibility of $\mathbf{M}$, it is, therefore, optimal for type θ_N to be truthful under $\tilde{\mathbf{C}}$. Next, consider any type $\theta < \theta_N$. If he mimics θ_N, he either does not buy the good to and gets a zero payoff or buys the good and obtain

$$\begin{aligned}
&\mathbb{E}[v(\theta,x)] - \tilde{p}(\theta_N) \\
=&\mathbb{E}[v(\theta,x)] - \mathbb{E}[v(\theta_N,x)] + \sum_x\sum_s [v(\theta_N,x)q(\theta_N,s)\pi_{\theta_N}(s|x)\mu(x) - \sum_x\sum_s p(\theta_N,s)\pi_{\theta_N}(s|x)\mu(x) \\
=&\mathbb{E}[v(\theta,x)] - \sum_x\sum_s v(\theta_N,x)[1-q(\theta_N,s)]\pi_{\theta_N}(s|x)\mu(x) - \sum_x\sum_s p(\theta_N,s)\pi_{\theta_N}(s|x)\mu(x) \\
\leq&\mathbb{E}[v(\theta,x)] - \sum_x\sum_s v(\theta,x)[1-q(\theta_N,s)]\pi_{\theta_N}(s|x)\mu(x) - \sum_x\sum_s p(\theta_N,s)\pi_{\theta_N}(s|x)\mu(x) \\
=&\sum_x\sum_s \left[v(\theta,x)q(\theta_N,s) - p(\theta_N,s)\right]\pi_{\theta_N}(s|x)\mu(x),
\end{aligned}$$

which is type θ's from mimicking θ_N and report signals truthfully under the $\mathbf{M}$. Therefore, if type θ reports truthfully under $\mathbf{M}$, it is also the case when type θ_N receives $\tilde{\mathbf{C}}$.

A.2 Proof of Proposition 1

We complete the arguments in the main text by showing that under Case 2: $\underline{p}(\theta) > 0$, $\tilde{\mathbf{M}}$ induces truth-telling and hence, the seller's revenue under $\tilde{\mathbf{M}}$ is equal to that under $\mathbf{M}^d$. Consider the buyer of type θ who reports θ'. There are two cases:

Case 1: $\theta' \geq \theta$. Suppose s^g is realized. By misreporting s^b, type θ gets a zero payoff, whereas by truthfully reporting s^g, he obtains:

$$\frac{\sum_x [v(\theta,x) - \tilde{p}(\theta')]q(\theta',x)\mu(x)}{\sum_x q(\theta',x)\mu(x)} \geq \frac{\sum_x [v(\theta',x) - \tilde{p}(\theta')]q(\theta',x)\mu(x)}{\sum_x q(\theta',x)\mu(x)} \geq 0,$$

where the second inequality uses the fact that type θ''s payoff from truth-telling under $\mathbf{M}^d$ is non-negative, or

$$\sum_x \Big[[v(\theta,x) - \overline{p}(\theta)]q(\theta,x) - \underline{p}(\theta)[1-q(\theta,x)]\Big]\mu(x) = \sum_x [v(\theta',x) - \tilde{p}(\theta')]q(\theta',x)\mu(x) \geq 0.$$

Thus, type θ truthfully reports that s^g is realized. As a result, having reported type $\theta' > \theta$, type θ either always report s^g or report signals truthfully.

Case 2: $\theta' > \theta$. Suppose s^b is realized. By reporting truthfully s^b, the buyer gets a zero payoff. By misreporting s^g, he obtains

$$\frac{\sum_x [v(\theta',x) - \tilde{p}(\theta')][1-q(\theta,x)]\mu(x)}{\sum_x [1-q(\theta,x)]\mu(x)} \leq \frac{\sum_x [v(\theta',x) - \overline{p}(\theta')][1-q(\theta,x)]\mu(x)}{\sum_x [1-q(\theta,x)]\mu(x)} \leq 0,$$

where the first inequality uses

$$\tilde{p}(\theta) = \overline{p}(\theta) + \underline{p}(\theta)\frac{1-\mathbb{Q}(\theta)}{\mathbb{Q}(\theta)} \geq \overline{p}(\theta)$$

given that $\underline{p}(\theta) \geq 0$, and the second uses the fact that truth-telling under $\mathbf{M}^d$ requires type θ to prefer truth-telling to always reporting a singal $s \in s^g$ (to always gets the good), or

$$\sum_x \left[[v(\theta,x) - \overline{p}(\theta)]q(\theta,x) - \underline{p}(\theta)[1-q(\theta,x)]\right]\mu(x) \geq \mathbb{E}[v(\theta,x)] - \overline{p}(\theta)$$
$$\Leftrightarrow \sum_x [v(\theta,x) - \overline{p}(\theta)][1-q(\theta,x)]\mu(x) \leq 0$$

Thus, it is optimal for type θ to reveal that s^b is realized. As a result, having reported type $\theta' > \theta$, type θ either always report s^b or report signals truthfully.

To sum up, in any case, type θ, who reports θ', either (i) always report s^b, (ii) always report s^g, or (iii) report signals truthfully. Then, his payoff is given by:

$$\begin{aligned}
U^{\tilde{\mathbf{M}}(\theta,\theta')} &\equiv \max\left\{0, \mathbb{E}[v(\theta,x)] - \tilde{p}(\theta',s^g), \sum_x [v(\theta,x) - \tilde{p}(\theta',s^g)]q(\theta',x)\mu(x)\right\} \\
&= \max\left\{0, \mathbb{E}[v(\theta,x)] - \tilde{p}(\theta',s^g), \sum_x \left[[v(\theta,x) - \overline{p}(\theta)]q(\theta,x) - \underline{p}(\theta)[1-q(\theta,x)]\right]\mu(x)\right\} \\
&\leq \max\left\{0, \mathbb{E}[v(\theta,x)] - \overline{p}(\theta'), \sum_x \left[[v(\theta,x) - \overline{p}(\theta')]q(\theta,x) - \underline{p}(\theta')[1-q(\theta,x)]\right]\mu(x)\right\} \\
&\leq \sum_x \left[[v(\theta,x) - \overline{p}(\theta)]q(\theta,x) - \underline{p}(\theta)[1-q(\theta,x)]\right]\mu(x)\} \\
&= \sum_x [v(\theta,x) - \tilde{p}(\theta,s^g)]q(\theta',x)\mu(x), \qquad (11)
\end{aligned}$$

where the first inquality uses $\tilde{p}(\theta',s^g) \geq \overline{p}(\theta')$; and the second uses the fact that under $\mathbf{M}^d$, type θ prefers truth-telling than reporting θ' and then either always report $s \in s^b$, always report $s \in s^g$, or report signals truthfu lly. Note that the right-hand side of (11) is the type θ's payoff from revealing his type and signal. Consequently, it is optimal for type θ to be truthful. Then, the seller's revenue under $\tilde{\mathbf{M}}$ is

$$\sum_\theta \mathbb{Q}(\theta)\tilde{p}(\theta)f(\theta) = \sum_\theta \left[\mathbb{Q}(\theta)\overline{p}(\theta) + [1-\mathbb{Q}(\theta)]\underline{p}(\theta)\right]f(\theta),$$

which is equal to that under $\mathbf{M}^d$.

A.3 Proof of Proposition 2

Part (a) Consider an arbitrary mechanism $\mathbf{M} \equiv \{\pi_\theta, q(\theta,s), p(\theta,s)\}$ that implements $\mathbb{Q}^\star$. Fix $x \geq \hat{x}(\theta)$, it must be that

$$\sum_s q(\theta,s)\pi(s|x)\mu(x) = \mathbb{Q}^\star(\theta,x) = 1.$$

Therefore, if $\pi(s|x) > 0$ then then $q(\theta,s) = 1$, whereas if $x < \hat{x}(\theta)$ and $\pi(s|x) > 0$ then $q(\theta,s) = \pi(s|x) = 1$. By similar arguments, if $\pi(s|x) > 0$ for some $x < \hat{x}(\theta)$, then $q(\theta,s) = \pi(s|x) = 1$. Thus, for any signal s, $q(\theta,s) \in \{0,1\}$ or $\mathbf{M}$ is a deterministic mechanism. By Proposition 1 (b), there exists a persuasive posted-price mechanism that implements $\mathbb{Q}^\star$.

Part (b) -"Only If": Suppose $\exists\theta$ such that $p^\star(\theta_N, s^g) > p^\star(\theta, s^g)$, we show that the buyer misreports his type and hence, $\mathbf{M}^\star$ fails to implement $\mathbb{Q}^\star$. By truth-telling, type θ_N always receives the good at price $p^\star(\theta_N, s^g)$. By mimicking θ and always reporting s^g, type θ_N always gets the good at a lower price $p(\theta, s^g)$. Therefore, type θ_N mimics θ when the seller does not observe signals.

Part (b) -"If": Suppose $p^\star(\theta_N, s^g) = \min_\theta\{p^\star(\theta, s^g)\}$, we show that $\mathbf{M}^\star$ induces truth-telling even if the seller does not observe signals and hence, implemements $\mathbb{Q}^\star$. Consider the buyer of type θ who reports θ'. By always reporting s^b, he does not get the good and pays nothing. By report signals truthfully, he obtains the good if and only if s^g is realized. By always reporting s^g to always get the good. By always misreport signals, he gets the good if and only if s^b is realized. To sum up, his payoff is given by

$$U^{\mathbf{M}^\star}(\theta,\theta') \equiv \max\left\{0, \mathbb{E}[v(\theta,x)] - p(\theta',s^g), \sum_{x\geq x_0}[v(\theta,x) - p(\theta',s^g)]\mu(x), \sum_{x<x_0}[v(\theta,x) - p(\theta',s^g)]\mu(x)\right\}$$

Note that if $\sum_{x<x_0}[v(\theta,x) - p(\theta',s^g)]\mu(x) \geq 0$, then $\sum_{x\geq x_0}[v(\theta,x) - p(\theta',s^g)]\mu(x) \geq 0$. In turn, this implies

$$\begin{aligned}\sum_{x<x_0}[v(\theta,x) - p(\theta',s^g)]\mu(x) &\leq \sum_{x<x_0}[v(\theta,x) - p(\theta',s^g)]\mu(x) + \sum_{x\geq x_0}[v(\theta,x) - p(\theta',s^g)]\mu(x) \\ &= \mathbb{E}[v(\theta,x)] - p(\theta_N,s^g).\end{aligned}$$

Thus, $\sum_{x<x_0}[v(\theta,x)-p(\theta',s^g)]\mu(x) \leq \max\{0,\mathbb{E}[v(\theta,x)]-p(\theta_N,s^g)\}$ and:

$$\begin{aligned} U^{\mathbf{M}^\star}(\theta,\theta') &= \max\left\{0,\mathbb{E}[v(\theta,x)]-p(\theta',s^g),\sum_{x\geq x_0}[v(\theta,x)-p(\theta',s^g)]\mu(x)\right\} \\ &\leq \max\left\{0,\mathbb{E}[v(\theta,x)]-p(\theta_N,s^g),\sum_{x\geq x_0}[v(\theta,x)-p(\theta',s^g)]\mu(x)\right\} \\ &\leq \sum_{x\geq x_0}[v(\theta,x)-p(\theta,s^g)]\mu(x), \end{aligned} \tag{12}$$

where the first inequality uses $p^\star(\theta_N,s^g) \leq p^\star(\theta',s^g)$ and the second uses the fact that under $\mathbf{M}^\star$ with public signals, type θ prefers truth-telling than reporting θ' or θ_N and then report signals truthfully. Note that the right-hand side of (12) is the type θ's payoff from revealing his type and signal. Consequently, it is optimal for type θ to be truthful and $\mathbf{M}^\star$ implements $\mathbb{Q}^\star$.

A.4 Proof of Claim 1

Let

$$\begin{aligned} \underline{x}(\theta_1) &\equiv \max\left\{x' \mid x \leq x^\star(\theta_1) : \mathbb{E}[v(\theta_2,x) \mid x > x'] < \mathbb{E}[v(\theta_1,x) \mid x > x']\right\}, \\ \overline{x}(\theta_1) &\equiv \min\left\{x' \mid x \geq x^\star(\theta_1) : \mathbb{E}[v(\theta_2,x) \mid x < x'] < \mathbb{E}[v(\theta_1,x) \mid x > x']\right\}. \end{aligned}$$

In what follows, we show that $x^{\star\star}(\theta_1) \in (\underline{x}(\theta_1),\overline{x}(\theta_1))$, which implies that $\mathbb{E}[v(\theta_2,x) \mid x < x^{\star\star}(\theta_1)] < \mathbb{E}[v(\theta_1,x) \mid x > x^{\star\star}(\theta_1)]$, and thereby, completes the proof. Toward a contradiction, assume not. Consider the following two cases:

Case 1: $x^{\star\star}(l) \in [x_1,\underline{x}(\theta_1)]$. Then, the seller can do strictly better by offering a threshold disclosure $\tilde{\pi}(\theta_1)$ under which (i) the threshold is $\underline{x}^+(\theta_1)$ and (ii) with probability $\tilde{\lambda}$, s^g is sent at $\underline{x}^+(\theta_1)$ such that $\omega^{\tilde{\pi}(\theta_1)}(\theta_1,s^g) = \omega^{\tilde{\pi}(\theta_1)}(\theta_2,s^b)$. Note that $\tilde{\lambda}$ exists because by definition of $\underline{x}(\theta_1)$,

$$\begin{aligned} \mathbb{E}[v(\theta_2,x) \mid x > \underline{x}^+(\theta_1)] &> \mathbb{E}[v(\theta_1,x) \mid x > \underline{x}^+(\theta_1)], \\ \mathbb{E}[v(\theta_2,x) \mid x > \underline{x}(\theta_1)] &< \mathbb{E}[v(\theta_1,x) \mid x > \underline{x}(\theta_1)]. \end{aligned}$$

Thus, it must be that $x^{\star\star}(\theta_1) > \underline{x}(\theta_1)$.

Case 2: $x^{\star\star}(\theta_1) \in [\overline{x}(\theta_1),x_M]$. By similar arguments, the seller can do strictly better by offering a threshold disclosure $\hat{\pi}(\theta_1)$ under which (i) the threshold is $\overline{x}^-(\theta_1)$ and (ii) with probability $\hat{\lambda}$, s^g is sent at $\overline{x}^-(\theta_1)$, such that $\omega^{\hat{\pi}(\theta_1)}(\theta_2,s^g) = \omega^{\hat{\pi}(\theta_1)}(\theta_2,s^b)$. Thus, we also have $x^{\star\star}(\theta_1) < \overline{x}(\theta_1)$.

To sum up, $\mathbb{E}[v(\theta_2,x) \mid x < x^{\star\star}(\theta_1)] < \mathbb{E}[v(\theta_1,x) \mid x > x^{\star\star}(\theta_1)]$.

A.5 Proof of Theorem 2

The key of the proof is to show that $p^\star(\theta_N) = \min_\theta p^\star(\theta)$. Then by Proposition 2, $\mathbf{M}^\star$ implements $\mathbb{Q}^\star$ with private signals. As $\mathbf{M}^\star$ induces truth-telling even if the seller does not observe signals, she can simply offer a menu of posted prices and threshold disclosure $\{\pi^\star_\theta, p^\star(\theta)\}_\theta$, where $p^\star(\theta) = p^\star(\theta, s^g)$, and let the buyer decides whether to buy the good.

In the remaining of the proof, we prove $p^\star(\theta_N) = \min_\theta p^\star(\theta)$. Recall that

$$
\begin{aligned}
p^\star(\theta_1) &= \mathbb{E}[v(\theta,x) \mid x \geq x^\star(\theta_1)] \\
p^\star(\theta_{n+1}) &= \frac{p^\star(\theta_n)\sum_{x\geq x^\star(\theta_n)}\mu(x) + \sum_{x^\star(\theta_{n+1})\leq x<x^\star(\theta_n)} v(\theta_n,x)\mu(x)}{\sum_{x\geq x^\star(\theta_{n+1})}\mu(x)}
\end{aligned}
$$

Then, for all n, there are two expressions of price gaps between the two adjacent types, as follows:

$$
p^\star(\theta_{n+1}) - p^\star(\theta_n) = \Big[\mathbb{E}[v(\theta,x) \mid x^\star(\theta_{n+1}) \leq x < x^\star(\theta_n)] - p^\star(\theta_n)\Big]\frac{\sum_{\star(\theta_{n+1})\leq x<x^\star(\theta_n)}\mu(x)}{\sum_{x\geq x^\star(\theta_{n+1})}\mu(x)}, \tag{13}
$$

$$
p^\star(\theta_{n+1}) - p^\star(\theta_n) = \Big[\mathbb{E}[v(\theta,x) \mid x^\star(\theta_{n+1}) \leq x < x^\star(\theta_n)] - p^\star(\theta_{n+1})\Big]\frac{\sum_{\hat{x}(\theta_{n+1})\leq x<x^\star(\theta_n)}\mu(x)}{\sum_{x\geq x^\star(\theta_n)}\mu(x)}. \tag{14}
$$

Now, we prove $p^\star(\theta_N) = \min_\theta p^\star(\theta)$ by induction. First, $p^\star(\theta_2) \leq p^\star(\theta_1)$ because

$$
\begin{aligned}
p^\star(\theta_2) - p^\star(\theta_1) &\propto \Big[\mathbb{E}[v(\theta,x) \mid x^\star(\theta_2) \leq x < x^\star(\theta_1)] - p^\star(\theta_1)\Big] \\
&= \Big[\mathbb{E}[v(\theta,x) \mid x^\star(\theta_2) \leq x < x^\star(\theta_1)] - \mathbb{E}[v(\theta,x) \mid x \geq x^\star(\theta_1)]\Big] \\
&\leq 0,
\end{aligned}
$$

where the inequality uses the partition flip of type order between θ_2 and θ_1. Second, suppose $p^\star(\theta_{n+1}) \leq p^\star(\theta_n)$, then $p^\star(\theta_{n+2}) \leq p^\star(\theta_{n+1})$ because

$$
\begin{aligned}
p^\star(\theta_{n+2}) - p^\star(\theta_{n+1}) &\propto \Big[\mathbb{E}[v(\theta,x) \mid x^\star(\theta_{n+2}) \leq x < x^\star(\theta_{n+1})] - p^\star(\theta_{n+1})\Big] \\
&\leq \Big[\mathbb{E}[v(\theta,x) \mid x^\star(\theta_{n+1}) \leq x < x^\star(\theta_n)] - p^\star(\theta_{n+1})\Big] \\
&\leq 0
\end{aligned}
$$

Then, $p^\star(\theta_n)$ decreases in n which implies $p^\star(\theta_N) = \min_\theta p^\star(\theta)$

A.6 Proof of Theorem 3

Let L be the lowest type being served under an optimal mechanism. Consider the following relaxed problem $(\mathcal{RP}_L)$, under which all types mimics L off-path:

$$(\mathcal{RP}_L) \quad \max_{(\pi,q,U)} \quad \sum_{\theta\geq L}\Big[\sum_x\sum_s v(\theta,x)q(\theta,\omega^{\pi_\theta}(\theta,s))\pi_\theta(s|x)\mu(x)-U(\theta)\Big]f(\theta)$$

$$s.t.\quad U(\theta)-U(L)\geq\sum_x\sum_s\int_{\omega^{\pi_L}(L,s)}^{\omega^{\pi_L}(\theta,s)}q(L,z)dz\pi_L(s|x)\mu(x)\quad \forall\theta>L \quad (IC_{\theta L})$$

$$U(L)\geq 0 \quad (IR_L)$$

$$q(\theta,\omega) \text{ increases in } \omega. \quad \text{(MON)}$$

We will show that the solution to this relaxed problem, which features a posted price and a threshold disclosure, solves the original problem. Obviously, (IR_L) and $(IC_{\theta L})$ bind for all $\theta > L$ under $(\mathcal{RP}_L)$, reducing the seller's relaxed problem to

$$\max_{q,\pi}\quad \sum_\theta\sum_x\sum_s\left[v(\theta,x)q(\theta,\omega^{\pi_\theta}(\theta,s))-\int_{\omega^{\pi_L}(L,s)}^{\omega^{\pi_L}(\theta,s)}q(L,z)dz\right]\pi_L(s|x)\mu(x)f(\theta)$$

$$s.t.\quad q(\theta,\omega^{\pi_\theta}(\theta,s)) \text{ increases in } s.$$

Fix π, it is a linear problem in q with (MON) being the only constraint. Thus, the optimal allocation is generally unique, given by

$$q(L,\omega^{\pi_L}(\theta,s))=\mathbb{1}_{s\geq\hat{s}(L)},\quad q(\theta,\omega^{\pi_\theta}(\theta,s))=1\quad\forall s,\forall\theta>L.$$

Fix $q(L,s)=\mathbb{1}_{s\geq\hat{s}(L)}$. The seller's objective (revenue) now only depends on π_L.

$$\mathbf{R}(\pi_L)\equiv\sum_{\theta>L}\mathbb{E}[v(\theta,x)]f(\theta)+f(L)\sum_x\sum_{\hat{s}(L)}^{\bar{s}}v(L,x)\pi_L(s|x)\mu(x)$$
$$-\sum_{\theta>L}\sum_x\Big[\sum_{\hat{s}(L)}^{\bar{s}}[\omega^{\pi_L}(\theta,s)-\omega^{\pi_L}(L,\hat{s}(L))]-\sum_{\underline{s}}^{\hat{s}(L)}\max\{\omega^{\pi_L}(\theta,s)-\omega^{\pi_L}(L,\hat{s}(L)),0\}\Big]\pi_L(s|x)\mu(x)f($$

To find optimal π_L, note that

$$\begin{aligned}
\mathbf{R}(\pi_L) \leq & \sum_{\theta > L} \mathbb{E}[v(\theta,x)]f(\theta) + f(L)\sum_x \sum_{\hat{s}(L)}^{\bar{s}} v(L,x)\pi_L(s|x)\mu(x) \\
& - \sum_{\theta > L}\sum_x \Big[\sum_{\hat{s}(L)}^{\bar{s}} [\omega^{\pi_L}(\theta,s) - \omega^{\pi_L}(L,\hat{s}(L))] - \sum_{\underline{s}}^{\hat{s}(L)} \{\omega^{\pi_L}(\theta,s) - \omega^{\pi_L}(L,\hat{s}(L))\} \Big] \pi_L(s|x)\mu(x)f(\theta) \\
& = \sum_{\theta > L} \mathbb{E}[v(\theta,x)]f(\theta) + \sum_x \sum_{\hat{s}(L)}^{\bar{s}} v(L,x)\pi_L(s|x)f(L)\mu(x) - \sum_{\theta > L}\sum_x \sum_{\underline{s}}^{\bar{s}} [\omega^{\pi_L}(\theta,s) - \omega^{\pi_L}(L,\hat{s}(L))]\, \pi_L(s \\
& = f(L)\sum_x \sum_{\hat{s}(\theta_1)}^{\bar{s}} v(L,x)\pi_L(s|x)\mu(x) + \sum_{\theta > L} f(\theta)\omega^{\pi_L}(L,\hat{s}(L)) \equiv \overline{\mathbf{R}}(\pi_L)
\end{aligned}$$

By similar arguments as in the binary-type case, $\overline{\mathbf{R}}(\pi_L)$ is maximized when π_L is a threshold disclosure. By replacing all signals $s \geq \hat{s}(L)$ with s^g and all signals $s < \hat{s}(L)$ with s^b we obtain:

$$\overline{\mathbf{R}}(\pi_L) = \omega^{\pi_L}(L,s^g)\Big[\sum_{\theta > L} f(\theta) + f(L)\sum_x \pi_L(s^g|x)\mu(x)\Big].$$

Let $\pi_L^{\star\star} \in \operatorname{argmax} \overline{\mathbf{R}}(\pi_L)$. We now show that offering a fixed bundle of threshold disclosure and price $(p^{\star\star}(L), \pi_L^{\star\star})$ where $p^{\star\star}(L) = \omega^{\pi_L^{\star\star}}(L,s^g)$ to all types, the seller obtains $\overline{\mathbf{R}}(\pi_L^{\star\star})$. Given that $\omega^{\pi_L^{\star\star}}(L,s^b) < p^{\star\star}(L) = \omega^{\pi_L^{\star\star}}(L,s^g)$, type L buys the good if and only if s^g is realized. and thereby, pays $\omega^{\pi_L}(L,s^g)\sum_x \pi_L(s^g|x)$. Now, consider type $\theta > L$. Under no uniformly threshold preservation of type order, $\omega^{\pi_L^{\star\star}}(\theta,s^b) \geq \omega^{\pi_L^{\star\star}}(L,s^g)$. Hence, type θ to always buy the good regardless of signal realization and pays $\omega^{\pi_L}(L,s^g)\sum_{\theta > L}$. Thus, the seller's expected revenue is

$$\omega^{\pi_L^{\star\star}}(L,s^g)\Big[\sum_{\theta > L} f(\theta) + f(L)\sum_x \pi_L^{\star\star}(s^g|x)\mu(x)\Big] = \overline{\mathbf{R}}(\pi_L^{\star\star}).$$

Thus, the single-item menu $(p^{\star\star}(L), \pi_L^{\star\star})$ is optimal. L can be found by comparing the values of programs $\{\mathcal{RP}_\theta\}_{\theta \in \Theta}$, denoted by $V(\mathcal{RP}_\theta)$. Formally,

$$L \in \operatorname*{argmax}_\theta V(\mathcal{RP}_2(\theta)).$$

A.7 Proof of for Proposition 3

Step 1: We first solve a relaxed problem $\Theta = [\theta_1, \theta_N]$. Truthtelling about type requires that

$$\sum_x v(\theta,x)q(\theta,x) - p(\theta) \geq \sum_x v(\theta,x)q(\theta',x) - p(\theta')$$

By the Envelope condition, this implies

$$U'(\theta) = \sum_x v_\theta(\theta, x) q(\theta, x)\mu(x). \tag{15}$$

Consider a relaxed problem which impose only the IR constraint and the local IC constraint (15). By integration by parts, $U(\theta) = U(\theta_1) + \int_{\theta_1}^{\theta} \sum_x v_\theta(\theta', x) q(\theta', x)\mu(x) d\theta'$. Using this, the seller's revenue can be written as

$$\sup_q \quad \int_\theta \sum_x \left[v(\theta, x) - v_\theta(\theta, x) \frac{1 - F(\theta)}{f(\theta)} \right] q(\theta, x)\mu(x) dF(\theta) - U(\theta_1),$$

This expression can be maximized point-wise with respect to q. Hence,

$$q(\theta, x) = \mathbb{1}_{\phi^c(\theta,x) \geq 0} \tag{16}$$

where $\phi^c(\theta, x) \equiv v(\theta, x) - v_\theta(\theta, x)\frac{1-F(\theta)}{f(\theta)}$.

Step 2: We now implement the allocation given by (16) using the $M^c \equiv \{\pi^c_\theta, q^c(\theta, s), p^c(\theta, s)\}$, where π^c_θ

$$p^c(\hat{\theta}(x_M)) = v(\hat{\theta}(x_M), x_M) \tag{17}$$

$$p^c(\hat{\theta}(x_{m-1})) = \frac{p^c(\hat{\theta}(x_m)) \sum_{x \geq x_m} \mu(x) + v(\hat{\theta}(x_{m-1}), x_{m-1})}{\sum_{x \geq x_{m-1}} \mu(x)}, \quad \forall 2 \leq m \leq M \tag{18}$$

Therefore,

$$p^c(\hat{\theta}(x_{m-1})) - p^c(\hat{\theta}(x_m)) = \frac{[v(\hat{\theta}(x_{m-1}), x_{m-1}) - p^c(\hat{\theta}(x_m))]}{\sum_{x \geq x_{m-1}} \mu(x)}, \tag{19}$$

$$p^c(\hat{\theta}(x_{m-1})) - p^c(\hat{\theta}(x_m)) = \frac{[v(\hat{\theta}(x_{m-1}), x_{m-1}) - p^c(\hat{\theta}(x_{m-1}))]}{\sum_{x \geq x_m} \mu(x)}. \tag{20}$$

As $v(\hat{\theta}(x_{M-1}), x_{M-1}) \leq (\hat{\theta}(x_M), x_M)$, then by (17) and (19),

$$p^c(\hat{\theta}(x_{M-1})) \leq p^c(\hat{\theta}(x_M)). \tag{21}$$

Suppose $p^c(\hat{\theta}(x_{m-1})) \leq p^c(\hat{\theta}(x_m))$ for some $m < M$, then by (19), $v(\hat{\theta}(x_{m-1}), x_{m-1}) \leq p^c(\hat{\theta}(x_m))$. As $v(\hat{\theta}(x_m), x_m) \leq v(\hat{\theta}(x_{m-1}), x_{m-1})$, we thus have

$$v(\hat{\theta}(x_m), x_m) \leq p^c(\hat{\theta}(x_m)).$$

Then, by (20),

$$p^c(\hat{\theta}(x_m)) - p^c(\hat{\theta}(x_{m+1})) \tag{22}$$

Thus, $p^c(\hat{\theta}(x_1)) = \min_x p^c(\hat{\theta}(x))$. Therefore, if the buyer of type θ mimics some type θ' and always report s^g, it is weakly better to mimic θ_N and report signals truthfully (type θ_N always receives s^g). Thus, it suffices to show that there is no type θ who mimics θ' and reports signals truthfully. By construction,

$$\sum_{x \geq x_m} [v(\hat{\theta}(x_m), x) - p(\hat{\theta}(x_m), s^g)]\mu(x) = \sum_{x \geq x_{m+1}} [v(\hat{\theta}(x_m), x) - p(\hat{\theta}(x_{m+1}), s^g)]\mu(x), \quad (23)$$

which implies that for any $\theta' > \hat{\theta}(x_m)$ and any $\theta'' < \hat{\theta}(x_{m+1})$

$$\sum_{x \geq x_m} [v(\theta', x) - p(\hat{\theta}(x_m), s^g)]\mu(x) \geq \sum_{x \geq x_{m+1}} [v(\theta', x) - p(\hat{\theta}(x_{m+1}), s^g)]\mu(x),$$
$$\sum_{x \geq x_m} [v(\theta'', x) - p(\hat{\theta}(x_m), s^g)]\mu(x) \leq \sum_{x \geq x_{m+1}} [v(\theta'', x) - p(\hat{\theta}(x_{m+1}), s^g)]\mu(x),$$

Therefore, any θ prefers reporting the closest cut-off types. That is, if $\theta \in [\hat{\theta}(x_m), \hat{\theta}(x_{m-1})]$, type θ either reports $\hat{\theta}(x_m)$ or $\hat{\theta}(x_{m-1})$. Moreover, by (23),

$$\sum_{x \geq x_m} [v(\theta, x) - p(\hat{\theta}(x_m), s^g)]\mu(x) \leq \sum_{x \geq x_{m+1}} [v(\theta, x) - p(\hat{\theta}(x_{m+1}), s^g)]\mu(x), \quad (24)$$

which implies that type θ weakly prefers to mimic $\hat{\theta}(x_m)$ to reporting $\hat{\theta}(x_{m+1})$. Thus, type θ reveals his type to receive type $\hat{\theta}(x_m)$'s contract.

A.8 Proof of for Proposition 4

Suppose it is optimal to exclude all types below L, or $q(\theta, x) = 1$ for all x and $\theta < L$. Then, the seller's revenue must be weakly lower than that obtained from selling to the buyer whose types is distributed by $\hat{f}$ over Θ, where $\hat{f}(\theta) = f(\theta)\ \forall\theta \notin [L, L+\varepsilon]$, $\hat{f}(\theta) = 0$ $\forall\theta \in [L, L+\varepsilon)$, and $\hat{f}(L+\varepsilon) = \int_{\theta=L}^{\theta=L+\varepsilon} f(\theta)d\theta$. Let $(\hat{P})$ represent the seller's problem when $\theta \sim \hat{f}$ and $V(\hat{P})$ the corresponding value. Consider the following relaxed problem of $(\hat{P})$ where all types mimic $L+\varepsilon$ off the equilibrium path:

$$(\mathcal{RP}_{L+\varepsilon}) \quad \max_{(\pi,q,U)} \sum_{\theta \geq L+\varepsilon} \sum_x \sum_s p(\theta, \omega^{\pi_\theta}(\theta, s))\pi_\theta(s|x)\mu(x)\hat{f}(\theta)$$

$$s.t. \quad U(\theta) - U(L+\varepsilon) \geq \sum_s \int_{\omega^{\pi_{\theta_1}}(L+\varepsilon, s)}^{\omega^{\pi_{\theta_1}}(\theta, s)} q(L+\varepsilon, z)dz\pi_{L+\varepsilon}(s) \quad \forall\theta > L+\varepsilon \quad (IC_{\theta \to L+\varepsilon})$$

$$U(L+\varepsilon) \geq 0 \quad (IR_{L+\varepsilon})$$

$$q(\theta, \omega) \text{ increases in } \omega. \quad \text{(MON)}$$

By the same arguments as the proof of Theorem 3, a posted price $\hat{p}_{L+\varepsilon}$, associated with a threshold disclosure $\hat{\pi}_{L+\varepsilon}$, solves this relaxed problem. Note that $(\hat{\pi}_{L+\varepsilon}, \hat{p}_{L+\varepsilon})$ does not necessary solve the original problem. In case it does, the seller's revenue is the value of problem $(\mathcal{RP}_{L+\varepsilon})$, denoted by $V((\mathcal{RP}_{L+\varepsilon}))$. Let R_ε represent the seller's revenue if she offers $(\hat{\pi}_{L+\varepsilon}, \hat{p}_{L+\varepsilon})$ (regardless of whether it solves the original problem or not). Then,

$$\begin{aligned} R_\varepsilon &\geq V(\mathcal{RP}_{L+\varepsilon}) - \mathbb{E}[v(L+2\varepsilon, x)] \int_{L+\varepsilon}^{L+2\varepsilon} \hat{f}(\theta) d\theta \\ &\geq V(\hat{P}) - \mathbb{E}[v(L+2\varepsilon, x)] \int_{L+\varepsilon}^{L+2\varepsilon} \hat{f}(\theta) d\theta \end{aligned}$$

Therefore,

$$\lim_{\varepsilon \to 0} R_\varepsilon \geq V(\hat{P}) - \lim_{\varepsilon \to 0} \mathbb{E}[v(L+2\varepsilon, x)] \int_{L+\varepsilon}^{L+2\varepsilon} \hat{f}(\theta) d\theta = V(\hat{P})$$

On the other hand, $\lim_{\varepsilon \to 0} R_\varepsilon \leq V(\hat{P})$. Therefore, $\lim_{\varepsilon \to 0} R_\varepsilon = V(\hat{P})$.

A.9 Proof of Proposition 5

The proof leverages the following lemma, which simplifes the search for a optimal deterministic mechanism by focusing on certain menus of trade probabilities and posted prices.

Lemma A.1. *For any deterministic mechanism, there exists a menu of posted prices and trade probabilities $M \equiv \{\alpha(\theta, x), p(\theta)\}_\theta$ that generates the same revenue for the seller and moreover,*

(a) if $\alpha(\theta, x) = 0$ for all x, then for any $\theta' \leq \theta$, $\alpha(\theta', x) = 0$ for all x.

(b) if $\alpha(\theta, x) = 1$ for all x, then for any $\theta' \geq \theta$, $\alpha(\theta', x) = 1$ for all x.

Proof of Lemma A.1. **Part (a):** Suppose there exist θ, θ', x such that $\theta' > \theta$, $\alpha(\theta, x) > 0$, $\alpha(\theta', x) = 0$ for all x. By $IR_{\theta'}$, $p(\theta') \leq 0$. By no rent at the bottom, $U(\theta_1) = 0$. To prevent type θ_1 from mimicking type θ', it must be that $p(\theta') = 0$. Given that type θ' does not trade and pays nothing, his payoff is zero. By offering type θ' with type θ's contract, his payoff is given by

$$\sum_x [v(\theta', x) - p(\theta)]\alpha(\theta, x)\mu(x) \geq \sum_x [v(\theta, x) - p(\theta)]\alpha(\theta, x)\mu(x) \geq 0, \quad (25)$$

whereas he pays non-negative payment, given by $\sum_x p(\theta)\alpha(\theta, x)\mu(x)$. Thus, letting type θ' trades with probability $\alpha(\theta, x)$ at state x does not affect the seller's revenue and incentive compatibility.

Part (b): Suppose there exist θ such that $\alpha(\theta, x) = 1\ \forall x$. For type θ not to mimic some type $\hat{\theta} < \theta$, it must be that

$$\mathbb{E}[v(\theta, x)] - p(\theta) \geq \max\{0, \sum_x [v(\theta, x) - p(\hat{\theta})]\alpha(\hat{\theta}, x)\mu(x)\},$$

which implies that for any $\theta' \geq \theta$

$$\mathbb{E}[v(\theta', x)] - p(\theta) \geq \max\{0, \sum_x [v(\theta', x) - p(\hat{\theta})]\alpha(\hat{\theta}, x)\mu(x)\}.$$

Thus, for any types $\theta' \geq \theta$ and $\hat{\theta} < \theta$, type θ' prefers type θ's contract than type $\hat{\theta}$'s. Therefore, if the seller revises the contracts for all types $\theta' \geq \theta$ to be type θ's, incentive compatibility remains satisfied. Moreover, the seller's revenue weakly increases because-for type θ' not to mimic some type θ under the original mechanism, it must be that

$$\begin{aligned} \sum_x [v(\theta', x) - p(\theta')]\alpha(\theta, x)\mu(x) &\geq \mathbb{E}[v(\theta', x)] - p(\theta) \\ \Leftrightarrow p(\theta) - \sum_x p(\theta')\alpha(\theta, x)\mu(x) &\geq \sum_x v(\theta', x)[1 - \alpha(\theta, x)]\mu(x) \geq 0, \end{aligned}$$

which mean that type θ's payment is higher than that by any type θ'. □

Armed with Lemma A.1, we now prove Proposition 5. Suppose only deterministic mechanisms are allowed. By Lemma A.1, it is without loss to focus on menu of posted prices and trade probabilities $\{p(\theta), \alpha(x, \theta)\}_{x,\theta}$ under which there exist

$$\begin{aligned} \theta_h &\equiv \inf\{\theta \mid \alpha(\theta, x) = 1 \forall x\} \\ \theta_l &\equiv \sup\{\theta \mid \alpha(\theta, x) = 0 \forall x\} \end{aligned}$$

Therefore, within the class of deterministic mechanisms, an optimal mechanism solves the following problem:

$$\begin{aligned} (\mathcal{P}^d) \quad & \sup_{p,\alpha,\theta_h,\theta_l} \quad \int_\theta p(\theta) dF(\theta) \\ s.t. \quad \forall \theta, \theta' : \quad & \sum_{x \in \{x_1, x_2\}} [v(\theta, x) - p(\theta)]\, \alpha(\theta, x)\mu(x) \geq \sum_{x \in \{x_1, x_2\}} [v(\theta, x) - p(\theta')]\alpha(\theta', x)\mu(x) \\ & \sum_{x \in \{x_1, x_2\}} [v(\theta, x) - p(\theta)]\, \alpha(\theta, x)\mu(x) \geq 0. \\ & \theta_h = \inf\{\theta \mid \alpha(\theta, x) = 1 \forall x\} \\ & \theta_l = \sup\{\theta \mid \alpha(\theta, x) = 0 \forall x\} \end{aligned}$$

The proof of Proposition 5 completes by showing that $(\mathcal{P}^d)$ admits a solution which features full disclosure and a posted price to all types. Consider the following two cases:

Case 1: $v(\theta_h, x_1) > v(\theta_l, x_2)$. Then, by the continuity of the valuation function with respect to θ, there must exist $\hat{\theta}$ such that

$$v(\theta', x_1) \geq v(\theta_l, x_2), \quad \forall \theta' \in [\hat{\theta}, \theta_h]. \tag{26}$$

Moreover, by the IR condition for θ_l who trades if and only if state x_2 is realized,

$$v(\theta_l, x_2) \geq p(\theta_l). \tag{27}$$

As type θ_h always trades at price $p(h)$, it must be that $p(\theta_h) \leq p(\theta)$ for any type θ. Suppose not, or there exists θ such that $p(\theta_h) > p(\theta)$, then type θ_h mimics type θ_l and always buy the good at a lower price. Therefore,

$$p(\theta_h) \leq \min\{p(\theta_l), p(\theta')\}. \tag{28}$$

Using (26), (27), and (28), for any $\theta' \in [\hat{\theta}, \theta_h]$,

$$\min\{p(\theta'), v(\theta', x_1)\} \geq p(\theta_h). \tag{29}$$

Then, by mimic type θ_h, type θ' obtains

$$\begin{aligned} \mathbb{E}[v(\theta', x)] - p(\theta_h) &= [v(\theta', x_2) - p(\theta_h)]\mu(x_2) + [v(\theta', x_1) - p(\theta_h)]\mu(x_1) \\ &\geq [v(\theta', x_2) - p(\theta')]\alpha(\theta', x)\mu(x_2), \end{aligned} \tag{30}$$

which is his payoff from truth-telling. Suppose, "strict inequality" occurs in (30), then type θ' mimics type θ_h, violating incentive compatibility. Now, suppose "equality" in (30) occurs, meaning that

$$v(\theta', x_1) = p(\theta_h) = p(\theta'), \tag{31}$$

Then, by offering type θ' an efficient allocation $\alpha(\theta', x_1) = \alpha(\theta', x_1) = 1$ and the old price $p(\theta') = p(\theta_h)$, the seller strictly improves her revenue while not violating any constraints. Thus, this case with $v(\theta_h, x_1) > v(\theta_l, x_2)$ cannot happen.

Case 2: $v(\theta_h, x_1) \leq v(\theta_l, x_2)$. Consider a relaxed problem of $(\mathcal{P}^d)$, which employs the necessary envelope condition for truth-telling:

$$U'(\theta) = \sum_x v_\theta(\theta, x)\alpha(\theta, x)\mu(x).$$

By integration by parts, we obtain $U(\theta) = U(\tilde{\theta}_g) + \int_{\tilde{\theta}_g}^{\theta_N} \sum_x v_\theta(\theta, x)\alpha(\theta, x)\mu(x)d\theta$. Thus, the relaxed problem becomes:

$$(\mathcal{RP}^d) \quad \sup_{\alpha,\theta_h,\theta_l} \quad \int_\theta \sum_x \phi^c(\theta, x)\alpha(\theta, x)\mu(x)dF(\theta) - U(\theta_h)$$
$$\alpha(\theta, x) = 1 \quad \forall x, \forall \theta \geq \theta_h$$
$$\alpha(\theta, x) = 0 \quad \forall x, \forall \theta < \theta_l,$$
$$v(\theta_h, x_1) \leq v(\theta_l, x_2),$$

where $\phi^c(\theta, x) \equiv v(\theta, x) - v_\theta(\theta, x)\frac{1-F(\theta)}{f(\theta)}$. At optimum, $U(\theta_h) = 0$. As $\phi^c(\theta, x_2) \geq 0$ for all $\theta \geq \theta^\star(x_2)$ and $\phi^c(\theta, x_') \geq 0$ for all $\theta \geq \theta^\star(x_1)$, we obtain:

$$q(\theta, x) = \begin{cases} 1 & \text{if} \quad \theta \geq \min\{\theta^\star(x_1), \theta_h\} \\ \mathbb{1}_{x=\theta_2} & \text{if} \quad \max\{\theta^\star(x_2), \theta_0\} \leq \theta \leq \min\{\theta^\star(x_1), \theta_h\}, \\ 0 & \text{if} \quad \theta \leq \max\{\theta^\star(x_2), \theta_0\}, \end{cases}$$

Note that as $v(\theta^\star(x_1), x_1) > v(\theta^\star(x_2), x_2)$, $v(\theta_h, x_1) \leq v(\theta_l, x_2)$, and values increase in types, it cannot be the case that $\theta_h \geq \theta^\star(x_1)$ and $\theta_l \leq \theta^\star(x_2)$. Consider the remaining three cases as follows:

1. $\theta_h \leq \theta^\star(x_1)$ and $\theta_l \geq \theta^\star(x_2)$. Then increase θ_h to $\tilde{\theta}_h \leq \theta^\star(x_1)$ and reduce θ_l to $\tilde{\theta}_l \geq \theta^\star(x_2)$ such that $v(\tilde{\theta}_h, x_1) = v(\tilde{\theta}_l, g)$. Such $\tilde{\theta}_h$ and $\tilde{\theta}_l$ exist given that $v(\theta^\star(x_1), x_1) > v(\theta^\star(x_2), g)$. The allocation becomes

$$\tilde{q}(\theta, x) = \begin{cases} 1 & \text{if} \quad \theta \geq \tilde{\theta}_h \\ \mathbb{1}_{x=\theta_2} & \text{if} \quad \tilde{\theta}_l \leq \theta \leq \tilde{\theta}_h, \\ 0 & \text{if} \quad \theta \leq \tilde{\theta}_l, \end{cases}$$

   which is implementable with private signals via full disclosure and a posted price $p = v(\tilde{\theta}_l, g)$.

2. $\theta_h \leq \theta^\star(x_1)$ and $\theta_l \leq \theta^\star(x_2)$. Then increase θ_h to $\tilde{\theta}_h \leq \theta^\star(x_1)$ and reduces θ_l to $\theta^\star(x_2)$ such that $v(\tilde{\theta}_h, x_1) = v(\tilde{\theta}_l, x_2)$. Such $\tilde{\theta}_h$ and $\tilde{\theta}_l$ exist given that $v(\theta^\star(x_1), x_1) > v(\theta^\star(x_2), x_2)$. The allocation becomes

$$\tilde{q}(\theta, x) = \begin{cases} 1 & \text{if} \quad \theta \geq \tilde{\theta}_h \\ \mathbb{1}_{x=\theta_2} & \text{if} \quad \theta^\star(x_2) \leq \theta \leq \tilde{\theta}_h, \\ 0 & \text{if} \quad \theta \leq \theta^\star(x_2), \end{cases}$$

   which is implementable with private signals via full disclosure and a posted price $p = v(\theta^\star(x_2), x_2)$.

3. $\theta_h \geq \theta^\star(x_1)$ and $\theta_l \geq \theta^\star(x_2)$. Then reduce θ_h to $\theta^\star(x_1)$ and reduce θ_l to $\tilde{\theta}_l \geq \theta^\star(x_2)$ such that $v(\tilde{\theta}_h, x_1) = v(\tilde{\theta}_l, x_2)$. Such $\tilde{\theta}_l$ exists given that $v(\theta^\star(x_1), x_1) > v(\theta^\star(x_2), x_2)$. The allocation becomes

$$\tilde{q}(\theta, x) = \begin{cases} 1 & \text{if} \quad \theta \geq \theta^\star(x_1) \\ \mathbb{1}_{x=\theta_2} & \text{if} \quad \tilde{\theta}_l \leq \theta \leq \theta^\star(x_1), \\ 0 & \text{if} \quad \theta \leq \tilde{\theta}_l, \end{cases}$$

which is implementable with private signals via full disclosure and a posted price $p = v(\tilde{\theta}_l, x_2)$.

In any case, it must be that $v(\theta_h, x_1) = v(\theta_l, x_2)$ at optimum and consequently, full disclosure, associated with a fixed price is optimal.

A.10 Proof of Proposition 6

Similar to the proof of Theorem 3, we consider the following relaxed problem in which L is the lowest type being served under an optimal mechanism and all types mimics L off-path:

$$(\mathcal{RP}_L) \quad \max_{(\pi,q,U)} \quad \sum_{\theta \geq L} \Big[\sum_x \sum_s v(\theta, x) q(\theta, \omega^{\pi_\theta}(\theta, s)) \pi_\theta(s|x) \mu_\theta(x) - U(\theta) \Big] f(\theta)$$

$$s.t. \quad U(\theta) - U(L) \geq \sum_x \sum_s \int_{\omega^{\pi_L}(L,s)}^{\omega^{\pi_L}(\theta,s)} q(L, z) dz \pi_L(s|x) \mu_\theta(x) \quad \forall \theta > L \quad (IC_{\theta L})$$

$$U(L) \geq 0 \qquad (IR_L)$$

$$q(\theta, \omega) \text{ increases in } \omega. \qquad \text{(MON)}$$

and show that the solution to this relaxed problem, which features a posted price and a threshold disclosure, solves the original problem. Then, following exactly the same arguments used in the proof of Theorem 3, we obtain:

$$q(L, \omega^{\pi_L}(\theta, s)) = \mathbb{1}_{s \geq \hat{s}(L)}, \quad q(\theta, \omega^{\pi_\theta}(\theta, s)) = 1 \quad \forall s, \forall \theta > L.$$

Fix $q(L,s) = \mathbb{1}_{s \geq \hat{s}(L)}$. The seller's objective (revenue) now only depends on π_L, given by

$$
\begin{aligned}
&\sum_{\theta>L} \mathbb{E}[v(\theta,x)]f(\theta) + f(L)\sum_x \sum_{\hat{s}(L)}^{\bar{s}} v(L,x)\pi_L(s|x)\mu_L(x) \\
&- \sum_{\theta>L}\sum_x \Big[\sum_{\hat{s}(L)}^{\bar{s}} [\omega^{\pi_L}(\theta,s) - \omega^{\pi_L}(L,\hat{s}(L))] - \sum_{\underline{s}}^{\hat{s}(L)} \max\{\omega^{\pi_L}(\theta,s) - \omega^{\pi_L}(L,\hat{s}(L)),0\}\Big]\pi_L(s|x)\mu_\theta(x)f(\theta) \\
&\leq \sum_{\theta>L} \mathbb{E}[v(\theta,x)]f(\theta) + f(L)\sum_x \sum_{\hat{s}(L)}^{\bar{s}} v(L,x)\pi_L(s|x)\mu_L(x) \\
&- \sum_{\theta>L}\sum_x \Big[\sum_{\hat{s}(L)}^{\bar{s}} [\omega^{\pi_L}(\theta,s) - \omega^{\pi_L}(L,\hat{s}(L))] - \sum_{\underline{s}}^{\hat{s}(L)} \{\omega^{\pi_L}(\theta,s) - \omega^{\pi_L}(L,\hat{s}(L))\}\Big]\pi_L(s|x)\mu_\theta(x)f(\theta) \\
&= \sum_{\theta>L} \mathbb{E}[v(\theta,x)]f(\theta) + \sum_x \sum_{\hat{s}(L)}^{\bar{s}} v(L,x)\pi_L(s|x)f(L)\mu_L(x) - \sum_{\theta>L} \mathbb{E}[v(\theta,x)]f(\theta) + \sum_{\theta>L} f(\theta)\omega^{\pi_L}(L,\hat{s}(L)) \\
&= \sum_x \sum_{\hat{s}(L)}^{\bar{s}} v(L,x)\pi_L(s|x)f(L)\mu_L(x) + \sum_{\theta>L} f(\theta)\omega^{\pi_L}(L,\hat{s}(L)),
\end{aligned}
$$

which is achieved via a threshold disclosure that solves

$$
\sup_{\pi_L(s|x)} \sum_x \sum_{\hat{s}(L)}^{\bar{s}} v(L,x)\pi_L(s|x)f(L)\mu_L(x) + \sum_{\theta>L} f(\theta)\omega^{\pi_L}(L,\hat{s}(L)),
$$

and posted price $p\omega^{\pi_L}(L,\hat{s}(L))$. The arguments why this bunching solution solves the original problem follows from that in the proof of Theorem 3.

A.11 Proof of Proposition 7

Following the proof of Theorem 1, we consider the following relaxed problem which ignores (IC_{12}) and (IR_2).

$$
\begin{aligned}
(\mathcal{RP}_b) \quad \max_{(\pi,q,U)} \quad & \sum_\theta f(\theta)\Big[\sum_x\sum_s v(\theta_1,x)q(\theta,\omega^{\pi_\theta}(\theta,s))\pi_\theta(s|x)\mu_\theta(x) - U(\theta)\Big] \\
s.t: \quad & U(\theta_2) - U(\theta_1) \geq \sum_x\sum_s \int_{\omega^{\pi_{\theta_1}}(\theta_1,s)}^{\omega^{\pi_{\theta_1}}(\theta_2,s)} q(\theta_1,z)dz\pi_{\theta_1}(s|x)\mu_\theta(x) \qquad (IC_{21}) \\
& U(\theta_1) \geq 0 \qquad (IR_1) \\
& q(\theta,\omega) \text{ increases in } \omega. \qquad (MON)
\end{aligned}
$$

It is clear that (IC_{21}) and (IR_1) must bind. Then,

$$
U(\theta_1) = 0, \quad U(\theta_2) = \sum_x\sum_s \int_{\omega^{\pi_{\theta_1}}(\theta_1,s)}^{\omega^{\pi_{\theta_1}}(\theta_2,s)} q(\theta_1,z)dz\pi_{\theta_1}(s|x)\mu_\theta(x),
$$

Using these expressions for $U(\theta_1)$ and $U(\theta_2)$, the seller's revenue can be writeen as

$$\begin{aligned}(OBJ) \equiv & f(\theta_2)\sum_x\sum_s v(\theta_2,x)q(\theta_2,\omega^{\pi_\theta}(\theta,s))\pi_{\theta_2}(s|x)\mu(x) \\ & + f(\theta_1)\sum_x\sum_s\left[v(\theta_1,x)q(\theta_1,\omega^{\pi_\theta}(\theta,s)) - \int_{\omega^{\pi_{\theta_1}}(\theta_1,s)}^{\omega^{\pi_{\theta_1}}(\theta_2,s)} q(\theta_1,z)dz\right]\pi_{\theta_1}(s|x)\mu(x)\end{aligned}$$

Fix π. Then, this objective function (OBJ) is linear in q and the only remaining constraint is (MON) which requires $q(\theta,\omega)$ to be increasing. Consequently, it must be that

$$q(\theta_2,\omega^{\pi_\theta}(\theta,s)) = 1 \quad \forall s, \tag{32}$$

given that $v(\theta_2,x) \geq 0$ for all x; and there exists a cut-off signal $\hat{s}(\theta)$ such that for all θ,

$$q(\theta,\omega^{\pi_\theta}(\theta,s)) = \mathbb{1}_{s\geq\hat{s}(\theta)} \tag{33}$$

By (32) and (33), the objective becomes

$$\begin{aligned}& f(\theta_2)\mathbb{E}[v(\theta_2,x)] \\ & + f(\theta_1)\sum_x\sum_{\hat{s}(\theta_1)}^{\bar{s}} v(\theta_1,x)\pi_{\theta_1}(s|x)\mu(x) - \sum_x\sum_{\hat{s}(\theta_1)}^{\bar{s}}[\omega^{\pi_{\theta_1}}(\theta_2,s) - \omega^{\pi_{\theta_1}}(\theta_1,s)]\frac{f(\theta_2)}{f(\theta_1)}\pi_{\theta_1}(s|x)\mu(x) \\ & - \sum_x\sum_s^{\hat{s}(\theta_1)}[\omega^{\pi_{\theta_1}}(\theta_2,s) - \omega^{\pi_{\theta_1}}(\theta_1,\hat{s}(\theta_1))]\frac{f(\theta_2)}{f(\theta_1)}\pi_{\theta_1}(s|x)\mu(x),\end{aligned}$$

which is independent of π_{θ_2}. Therefore, any π_{θ_2} is optimal. To find optimal π_{θ_1}, note that by replacing all signals $s < \hat{s}(\theta_1)$ with single signal s^b ("*bad news*"), (OBJ) remains unchanged and becomes

$$\begin{aligned}& f(\theta_1)\sum_x\sum_{\hat{s}(\theta_1)}^{\bar{s}} v(\theta_1,x)\pi_{\theta_1}(s|x)\mu_{\theta_1}(x) - f(\theta_2)\sum_x\sum_{s\geq\hat{s}(\theta_1)}[v(\theta_2,x) - v(\theta_1,x)\frac{\sum_x \pi_{\theta_1}(s|x)\mu_{\theta_2}(x)}{\sum_x \pi_{\theta_1}(s|x)\mu_{\theta_1}(x)}]\pi_{\theta_1}(s|x)\mu_{\theta_1}(\\ & - \max\left\{[\omega^{\pi_{\theta_1}}(\theta_2,s^b) - \omega^{\pi_{\theta_1}}(\theta_1,\hat{s}(\theta_1))]\frac{f(\theta_2)}{f(\theta_1)},0\right\}\pi_{\theta_1}(s^b|x)]\mu(x) \\ = & f(\theta_1)\sum_x\sum_{\hat{s}(\theta_1)}^{\bar{s}}\left[v(\theta_1,x) - [v(\theta_2,x) - v(\theta_1,x)\frac{\sum_x \pi_{\theta_1}(s|x)\mu_{\theta_2}(x)}{\sum_x \pi_{\theta_1}(s|x)\mu_{\theta_1}(x)}]\frac{f(\theta_2)}{f(\theta_1)}\right]\pi_{\theta_1}(s|x)\mu_{\theta_1}(x) \\ & - \max\left\{[\omega^{\pi_{\theta_1}}(\theta_2,s^b) - \omega^{\pi_{\theta_1}}(\theta_1,\hat{s}(\theta_1))]\frac{f(\theta_2)}{f(\theta_1)},0\right\}\pi_{\theta_1}(s^b|x)\Bigg]\mu(x).\end{aligned}$$

Consider

$$H \equiv \frac{\sum_x \pi_{\theta_1}(s|x)\mu_{\theta_2}(x)}{\sum_x \pi_{\theta_1}(s|x)\mu_{\theta_1}(x)} = \frac{\pi_{\theta_1}(s|x_2)\mu_{\theta_2}(x_2) + \pi_{\theta_1}(s|x_1)\mu_{\theta_2}(x_1)}{\pi_{\theta_1}(s|x_2)\mu_{\theta_1}(x_2) + \pi_{\theta_1}(s|x_1)\mu_{\theta_1}(x_1)}$$

Then

$$
\begin{aligned}
\frac{\partial H}{\partial \pi_{\theta_1}(s|x_2)} &= \frac{\mu_{\theta_2}(x_2)[\pi_{\theta_1}(s|x_2)\mu_{\theta_1}(x_2) + \pi_{\theta_1}(s|x_1)\mu_{\theta_1}(x_1)] - \mu_{\theta_1}(x_2)[\pi_{\theta_1}(s|x_2)\mu_{\theta_2}(x_2) + \pi_{\theta_1}(s|x_1)\mu_{\theta_2}(x_1}{[\pi_{\theta_1}(s|x_2)\mu_{\theta_1}(x_2) + \pi_{\theta_1}(s|x_1)\mu_{\theta_1}(x_1)]^2} \\
&= \frac{[\mu_{\theta_2}(x_2)\mu_{\theta_1}(x_1) - \mu_{\theta_1}(x_2)\mu_{\theta_2}(x_1)]\pi_{\theta_1}(s|x_1)}{[\pi_{\theta_1}(s|x_2)\mu_{\theta_1}(x_2) + \pi_{\theta_1}(s|x_1)\mu_{\theta_1}(x_1)]^2} \\
&= \frac{\big[\mu_{\theta_2}(x_2)[1-\mu_{\theta_1}(x_2)] - \mu_{\theta_1}(x_2)[1-\mu_{\theta_2}(x_2)]\big]\, \pi_{\theta_1}(s|x_1)}{[\pi_{\theta_1}(s|x_2)\mu_{\theta_1}(x_2) + \pi_{\theta_1}(s|x_1)\mu_{\theta_1}(x_1)]^2} \\
&= \frac{\big[\mu_{\theta_2}(x_2) - \mu_{\theta_1}(x_2)]\big]\, \pi_{\theta_1}(s|x_1)}{[\pi_{\theta_1}(s|x_2)\mu_{\theta_1}(x_2) + \pi_{\theta_1}(s|x_1)\mu_{\theta_1}(x_1)]^2} > 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial H}{\partial \pi_{\theta_1}(s|x_1)} &= \frac{\mu_{\theta_2}(x_1)[\pi_{\theta_1}(s|x_2)\mu_{\theta_1}(x_2) + \pi_{\theta_1}(s|x_1)\mu_{\theta_1}(x_1)] - \mu_{\theta_1}(x_1)[\pi_{\theta_1}(s|x_2)\mu_{\theta_2}(x_2) + \pi_{\theta_1}(s|x_1)\mu_{\theta_2}(x_1}{[\pi_{\theta_1}(s|x_2)\mu_{\theta_1}(x_2) + \pi_{\theta_1}(s|x_1)\mu_{\theta_1}(x_1)]^2} \\
&= \frac{[\mu_{\theta_2}(x_1)\mu_{\theta_1}(x_2) - \mu_{\theta_1}(x_1)\mu_{\theta_2}(x_2)]\pi_{\theta_1}(s|x_2)}{[\pi_{\theta_1}(s|x_2)\mu_{\theta_1}(x_2) + \pi_{\theta_1}(s|x_1)\mu_{\theta_1}(x_1)]^2} \\
&= \frac{\big[\mu_{\theta_2}(x_1)[1-\mu_{\theta_1}(x_1)] - \mu_{\theta_1}(x_1)[1-\mu_{\theta_2}(x_1)]\big]\, \pi_{\theta_1}(s|x_1)}{[\pi_{\theta_1}(s|x_2)\mu_{\theta_1}(x_2) + \pi_{\theta_1}(s|x_1)\mu_{\theta_1}(x_1)]^2} \\
&= \frac{[\mu_{\theta_2}(x_1) - \mu_{\theta_1}(x_1)]\pi_{\theta_1}(s|x_1)}{[\pi_{\theta_1}(s|x_2)\mu_{\theta_1}(x_2) + \pi_{\theta_1}(s|x_1)\mu_{\theta_1}(x_1)]^2} < 0.
\end{aligned}
$$

Thus, H increases in $\pi_{\theta_1}(s|x_2)$ and decreases in $\pi_{\theta_1}(s|x_1)$. Hence,

$$H(\pi_{\theta_1}(s|x_2), \pi_{\theta_1}(s|x_1) \leq H(1,0) = \frac{\mu_{\theta_2}(x_2)}{\mu_{\theta_2}(x_2)}.$$

Therefore,

$$
\begin{aligned}
OBJ \leq f(\theta_1) \sum_x \sum_{\hat{s}(\theta_1)}^{\bar{s}} &\left[v(\theta_1, b) - [v(\theta_2, b) - v(\theta_1, b)\frac{\mu_{\theta_2}(x_2)}{\mu_{\theta_2}(x_2)}]\frac{f(\theta_2)}{f(\theta_1)} \right] \pi_{\theta_1}(s|x)\mu_{\theta_1}(x_1) \\
+ \sum_{\hat{s}(\theta_1)}^{\bar{s}} &\left[v(\theta_1, x_2) - [v(\theta_2, x_2) - v(\theta_1, x_2)\frac{\mu_{\theta_2}(x_2)}{\mu_{\theta_2}(x_2)}]\frac{f(\theta_2)}{f(\theta_1)} \right] \pi_{\theta_1}(s|x)\mu_{\theta_1}(x_2) \\
\leq v(\theta_1, x_2) &- [v(\theta_2, x_2) - v(\theta_1, x_2)\frac{\mu_{\theta_2}(x_2)}{\mu_{\theta_2}(x_2)}]\frac{f(\theta_2)}{f(\theta_1)}
\end{aligned}
$$

with the "equality" occurs when S^+ contains a single signal s^g and $\pi_{\theta_1}(s^g|x) = \mathbb{1}_{x=x_2}$.

To implement this allocation, offers each type a posted price $p(\theta_1) = v(\theta_1, x_2)$ so that (IR_1) binds, and $p(\theta_2)$ is such that (IC_{21}) binds, or

$$U(\theta_2) = U(\theta_2, \theta_1) \Leftrightarrow p(\theta_2) = \mathbb{E}[v(\theta_2, x)] - \big[v(\theta_1, x_2) - v(\theta_1, x_2)\big]\mu_{\theta_2}(x_2).$$

IR_2 holds because

$$U(\theta_2) = \big[v(\theta_1, x_2) - v(\theta_1, x_2)\big]\,\mu_{\theta_2}(x_2) \geq 0.$$

IC_{12} is also satisfied given that

$$\begin{aligned}
U(\theta_1, \theta_2) &= \mathbb{E}[v(\theta_1, x)] - p(\theta_2) \\
&= \mathbb{E}[v(\theta_1, x)] - \mathbb{E}[v(\theta_2, x)] + \big[v(\theta_1, x_2) - v(\theta_1, x_2)\big]\,\mu_{\theta_2}(x_2) \\
&= v(\theta_1, x_2)\mu_{\theta_1}(x_2) + v(\theta_1, x_1)\mu_{\theta_1}(x_1) - v(\theta_1, x_1)\mu_{\theta_2}(x_1) - v(\theta_1, x_2)\mu_{\theta_2}(x_2) \\
&= v(\theta_1, x_2)[\mu_{\theta_1}(x_2) - \mu_{\theta_2}(x_2)] + v(\theta_1, x_1)[\mu_{\theta_1}(x_1) - \mu_{\theta_2}(x_1)] \\
&= v(\theta_1, x_2)[\mu_{\theta_1}(x_2) - \mu_{\theta_2}(x_2)] - v(\theta_1, x_1)[\mu_{\theta_2}(x_2) - \mu_{\theta_1}(x_2)] \\
&= [v(\theta_1, x_2) - -v(\theta_1, x_1)][\mu_{\theta_1}(x_2) - \mu_{\theta_2}(x_2)] < 0 = U(\theta_1).
\end{aligned}$$

We thus obtain Proposition 7(b).

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