Stable cartel configurations: the case of multiple cartels

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Stable cartel configurations: The case of multiple cartels

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Abstract

We develop a framework to analyse stable cartelisation when firms can form multiple cartels. This contrasts with the existing literature which generally assumes, without further justification, that at most one cartel may form. We define cartelisation to be stable in the multiple cartels framework if: (i) a firm in a cartel does not find it more profitable to leave the cartel and operate independently, (ii) a firm that operates independently does not find it more profitable to join an existing cartel, (iii) a firm in a cartel does not find it more profitable to join another existing cartel or form a new cartel with an independent firm, and (iv) two independent firms do not find it more profitable to form a new cartel. In the context of quantity competition in differentiated markets, we show that a single cartel is never stable whenever multiple cartels may be formed. We completely characterise the stable cartelisation structure – there is at most one firm that is not a part of any cartel while each of the remaining firms is a part of a two-firm cartel.

JEL Classification: C70; D43; L13.

Keywords: multiple cartels; stable cartels; quantity competition; differentiated markets.

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1 Introduction

The formation of cartels is a fundamental concern due to the inefficiency that it may cause by impeding competition in the market. In an influential paper, Stigler (1950) de-emphasised this concern by arguing that cartels are inherently unstable because the positive externalities generated by a cartel make it more profitable for each firm to not join the cartel but, instead, free-ride by remaining outside the cartel. For instance, in a market where firms choose the level of production, a cartel will attempt to reduce production in order to increase the price with the objective of obtaining a higher profit. However, this creates the incentive for an individual firm within the cartel to exit the cartel in order to not lower its quantity and, instead, free-ride on the cartel's effort to increase price by reducing quantity – this makes the cartel unstable. While persuasive, this argument ignores that the firms which remain in the cartel may respond appropriately (for instance, by altering its quantity) to the extent that once this response is taken into consideration, it may actually not be profitable for a firm to exit the cartel. The literature on cartel stability, building on this game-theoretic consideration, has developed a more refined understanding of cartel stability. But, there is one crucial blind spot – the pre-supposition that at most one cartel may be formed. In this paper, we develop a framework for analysing cartel stability while admitting the possibility of multiple cartels, show that a single cartel is never stable when multiple cartels may be formed, and completely characterise the structure of stable cartels in this framework.

We consider markets where firms simultaneously choose the level of production, and all the firms' production levels determine the market price of each firm. If all firms are independent, then the objective of each firm is to maximise its profit by choosing the quantity produced. On the other hand, if a cartel is formed, then the cartel acts as a single decision-making unit, and it chooses a production level simultaneously with the independent firms that remain outside the cartel. The cartel's objective is to maximise the aggregate profit of the cartel members, and an independent firm's objective is to maximise individual profit. In the single cartel framework, where at most one cartel may be formed, d'Aspremont, Jacquemin, Gabszewicz and Weymark (1983) define a cartel to be stable if it is both internally stable (i.e., a firm in the cartel does not find it more profitable to leave the cartel) and externally stable (i.e., a firm outside the cartel does not find it more profitable to be a part of the cartel). However, in the multiple cartels framework – when more than one cartel may be formed, with each firm being part of at most one cartel – two additional conditions become relevant. Firstly, two independent firms should not find it more profitable to form a new cartel. Secondly, a firm that belongs to a cartel should not find it more profitable to leave the cartel, and, either join another existing cartel or form a new cartel with an independent firm.

In the context of the single cartel framework, Bloch (2002) establishes that with quantity competition in homogenous markets with linear demand functions, a cartel is stable only when there are two firms in the market. The reason for cartel instability when there are a greater number of firms is that whenever a subset of the firms attempt to form a cartel and collude by reducing their output, a firm in the cartel obtains a higher profit if it exits the cartel and instead takes advantage of the reduced output of the other firms in the cartel. On the other hand, in the case of quantity competition in differentiated markets with a linear demand function, Belleflamme and Peitz (2010) present a three-firm example to show that a single two-firm cartel may be stable as a firm in this cartel may not find exiting the cartel more profitable; this is because the quantity produced by the other firms affects its market price to a lesser extent, and this reduces the free-riding incentive, thereby giving rise to the possibility of the cartel being stable. We prove that this holds more generally: for an arbitrary number of firms, a stable cartel exists if and only if the market is sufficiently differentiated; furthermore, a stable cartel comprises of exactly two firms. This condition can also be equivalently stated as follows: in a market with a given level of differentiation, a stable cartel exists if and only if the number of firms is sufficiently small. Thus, we uncover an intuitive competition-differentiation trade-off: the extent of differentiation is diluted by competition from a greater number of firms, and competition from a greater number of firms may be mitigated if the market is more differentiated. Hence, a stable cartel requires markets that are sufficiently differentiated or, equivalently, markets that are not as competitive.

However, more central to this paper's primary objective, we find that firstly, whenever it is possible for multiple cartels to form (i.e., there are at least four firms), a single cartel is never stable. Secondly, we completely characterise the structure of stable cartelisation, which, following from the previous point, must involve multiple cartels. Cartelisation is stable only if all firms are organised in two-firm cartels subject to the integer constraint. Hence, there exists at most one independent firm, with each of the remaining firms being a part of some two-firm cartel. For instance, if there are five firms (six firms), then cartelisation is stable only if there are two two-firm cartels and an independent firm (three two-firm cartels). Furthermore, for reasons outlined earlier, this market structure is stable if and only if the market is sufficiently differentiated. Thirdly, a stable multiple cartel market structure may exist even in differentiated markets where the market differentiation is not high enough for the existence of a stable cartel in the single cartel framework (recall the previous paragraph).

The reason for the first result is that whenever there are at least for four firms, a stable cartel in the single cartel framework comprises of only two firms (as mentioned earlier), and so there are at least two independent firms. Now, in the multiple cartel framework, a single two-firm cartel is internally stable if and only if it is more profitable for an independent firm to form another two-firm cartel with another independent firm. Hence, it is due to the possibility of the formation of another cartel that a cartel that is stable in the single cartel framework is actually unstable in the multiple cartels framework. Thus, cartel stability in the single cartel framework is predicated upon ruling out a priori the formation of another cartel.

The intuition for the second result rests on two reasons. One reason is that the extent to which a firm that is in a cartel expands it output when it leaves a cartel to become an independent firm is lowest when it is a two-firm cartel. Hence, it is least attractive to leave the cartel if it is a two-firm cartel. The other reason is that if all firms are organised in two-firm cartels (subject to the integer constraint), then the type of market restructuring (described in the previous paragraph) that results in single cartels being unstable is not possible. This is because there is at most one independent firm, and so, it is not possible for two independent firms to form a new cartel. These two reasons taken together form the basis for our result on the structure of stable cartelisation.

Finally, the third result follows from the stable cartelisation structure – all firms (except at most firm) are a part of a some cartel. This implies that all firms (except at most one firm) reduce their production in an order to maximise the respective cartel profit. As a result, each individual firm in each of these multiple cartels now needs to reduce its quantity to a much lower extent than it would have to if it were a member of the only cartel in the market. That is, in comparison to a firm that is a part of a two-firm cartel in the cartelisation structure that is stable in the multiple cartels framework, a firm that is a member of a two-firm cartel in the single cartel framework expands its output much more significantly when it exits a cartel. So, the free-riding incentive is comparatively greater and exiting a cartel is comparatively more profitable in the single cartel framework than in the multiple cartels framework. Hence, cartelisation is stable in the multiple cartels framework whenever it is stable in the single cartel framework, but the converse does not hold.

It follows that our paper has three principal components: cartel stability based on the criteria of internal stability and external stability; the impact of market differentiation on cartel stability; and the possibility of formation of multiple cartels. In the context of Cournot competition in a homogenous market with linear demand, Selten (1973) obtains in a model of cartel formation that firms will collude only if they are 'few' firms in the market. However, this excludes the internal stability criterion – considering this results in any cartel being unstable except for the case where there are only two firms (see, for instance, Bloch 2002).[1] In the context of differentiated markets, cartel stability has primarily been studied in the supergame framework (see, for example, Deneckere 1983, Majerus 1988 and Ross 1992).[2] Finally, in the context of the multiple cartels framework, stability of cartelisation is analysed in either a supergame framework (Eaton and Eswaran 1998), or in a cooperative approach (Espinosa and Inarra 2000), or in the specific case where there are two competing coalitions with each firm being part of one coalition without the option of operating independently (Palsule-Desai 2015 and Lambertini, Pignataro and Tampieri 2022).[3] More general treatments of endogenous

[1]Horizontal mergers, while seemingly similar to cartelisation, are only tangentially related to each other. The similarity is that both a cartel and a horizontally merged entity operate as one unit with the objective of maximising aggregate profit of the member firms. However, the crucial difference is that in mergers, the individual merging entities cease to exist, whereas that is not the case with cartels. This difference is even more salient when, as in our paper, profit transfers are not permitted amongst members of a cartel. As a result, while the internal stability criterion – which requires that a firm in the cartel should not prefer to exit the cartel – is critical for cartel stability, it is ruled out by the very nature of a merger between/amongst firms. On the other hand, in a more sophisticated model of acquisitions that takes the essence of the internal stability and external stability criteria into consideration, Kamien and Zang (1990) analyse complete/incomplete monopolisation in homogenous markets when the merged entities may choose the number of subsumed units to operate. While this is similar to the stability analysis of complete/incomplete cartels, the difference is that in our framework, each cartel member necessarily continues to be in operation, and we consider differentiated markets.

[2]In the context of mergers (recall the previous footnote), Lommerud and Sorgard (1997) analyse how market differentiation affects the profitability of the merger of two firms in a three-firm market when the merged entity and the non-merged entity also choose the number of product variants that they will offer.

[3]Another approach to cartel stability (that is however not as related to our paper) is the cartel leadership framework where formation of a cartel results in a two-stage sequential game where the cartel commits to a

formation of (multiple) coalitions under varying rules of coalition formation are presented in Yi (1997), Bloch (1995, 1996), and Ray and Vohra (1997, 1999). Thus, in context of, and in contrast to the existing literature, the novelty of our paper's contribution is to examine cartel stability in a static non-cooperative quantity-setting game played by symmetric firms while allowing for market differentiation and formation of multiple cartels.[4]

2 Model

There are $n \geq 2$ firms that compete simultaneously in quantities. Each firm has a constant marginal cost c of production. Let $q_i \geq 0$ denote the quantity produced by firm i. A profile of non-negative quantities $q = (q_j)_{j=1}^n$ determines firm i's price $p_i(q) = \alpha - \beta q_i - \delta \sum_{j \neq i} q_j$, with $\alpha > c \geq 0$ and $\beta \geq \delta > 0$. This leads to firm i obtaining profit $\pi_i(q) = (p_i(q) - c)q_i$. The market is homogenous when $\delta = \beta$ – here, each firm's market price is identical at any profile of quantities. The market is differentiated when $\delta < \beta$ – now, there exist profiles of quantities where the market price of two firms differ. We interpret $\gamma \equiv \frac{\delta}{\beta}$ as an inverse measure of market differentiation – given β, a lower value of γ is associated with a lower value of δ, implying the other firms' quantities affects a firm's price to a lesser extent.

3 The single cartel framework

In the single cartel framework, at most one cartel may be formed. Let k firms, with $2 \leq k \leq n$, form a cartel. The situations corresponding to $k = n$ and $2 \leq k < n$ represent a complete cartel and a single incomplete cartel, respectively. Whenever a k-firm cartel is formed, we assume (without loss of generality) that the last k firms are in the cartel; the remaining firms, if any, operate independently outside the cartel. The k-firm cartel and the $n - k$ independent

price (d'Aspremont, Jacquemin, Gabszewicz and Weymark 1983) or a quantity (Shaffer 1995 and Konishi and Lin 1999) in the first stage followed by firms outside the cartel choosing quantity in response.

[4] Merger/takeover waves (Fauli-Oller 2000 and Qiu and Zhou 2007), which may create multiple competing merged entities, differs from formation of multiple cartels. Firstly, as mentioned in Footnote 1, the internal stability criterion, which is crucial in determining cartel stability, is not considered in the analysis of mergers/takeovers. Secondly, owing to a result in Salant, Switzer and Reynolds (1983) that only 'large' mergers are profitable in homogenous Cournot markets with linear demand, papers that model 'smaller' mergers/takeovers assume that firms are asymmetric in the cost structure or that mergers/takeovers unlock other synergies.

firms (if any) compete simultaneously in quantities. The cartel's objective is to maximise the cartel members' aggregate profit while an independent firm's objective is to maximise its own profit. We assume that a cartel's profits cannot be redistributed amongst the member firms. This, along with symmetry of the firms, motivate us to assume that the quantity chosen by the cartel is split equally amongst the member firms, and each member firm receives the profit generated from its own output. We also highlight that firms' identical constant marginal cost of production implies that the cartel's aggregate profit is independent of the manner in production is coordinated within the cartel – hence, splitting the quantity equally amongst the cartel members does not result in the cartel obtaining a lower aggregate profit.

Let kq^C be the aggregate quantity chosen by the k-firm cartel so that each firm in the cartel produces q^C; i.e., for any $i \in \{n-k+1, \ldots, n\}$, the quantity produced by firm i is $q_i = q^C$. The profit of each firm in the cartel equals $(\{\alpha - \beta q^C - \delta[(k-1)q^C + \sum_{j=1}^{n-k} q_j]\} - c)q^C$. So, a k-firm cartel chooses q^C to maximise $\sum_{i=n-k+1}^{n} \pi_i(q) = k(\{\alpha - \beta q^C - \delta[(k-1)q^C + \sum_{j=1}^{n-k} q_j]\} - c)q^C$. Similarly, an independent firm i, if it exists, chooses q_i with the objective of maximising its profit $(\{\alpha - \beta q_i - \delta[kq^C + \sum_{j=1, j\neq i}^{n-k} q_j]\} - c)q_i$. Assuming, because of symmetry, that each independent firm produces the same quantity, we obtain the equilibrium quantities $q^C(n,k)$ and $q^I(n,k)$ chosen by a cartel member and an independent firm as function of the pair (n,k): $q^C(n,k) = \frac{[2\beta-\delta](\alpha-c)}{[2\beta+2(k-1)\delta][2\beta+(n-k-1)\delta]-k(n-k)\delta^2}$ and $q^I(n,k) = \frac{[2\beta+(k-2)\delta](\alpha-c)}{[2\beta+2(k-1)\delta][2\beta+(n-k-1)\delta]-k(n-k)\delta^2}$. So, a unique equilibrium *exists* in the single cartel framework; we will shortly examine stability.

Now, an independent firm produces a higher quantity than a cartel firm, and this is because $q^I(n,k) \geq q^C(n,k)$, with the inequality being strict when $k \geq 2$. These equilibrium quantities result in equilibrium prices $p^C(n,k)$ and $p^I(n,k)$ for a cartel firm and an independent firm, where $p^C(n,k) = \alpha - [\beta + (k-1)\delta]q^C(n,k) - (n-k)\delta q^I(n,k)$ and $p^I(n,k) = \alpha - k\delta q^C(n,k) - [\beta + (n-k-1)\delta]q^I(n,k)$. Clearly, $p^C(n,k) = p^I(n,k) + (\beta - \delta)(q^I(n,k) - q^C(n,k))$. It follows that, in case of a differentiated product market (i.e., $\delta < \beta$), an independent firm's market price is lower than that of a cartel firm because $p^I(n,k) \leq p^C(n,k)$, with the inequality being strict in case $k \geq 2$. Finally, the equilibrium profit of a cartel firm is $\pi^C(n,k) = (p^C(n,k) - c)q^C(n,k)$ and the equilibrium profit of an independent firm is $\pi^I(n,k) = (p^I(n,k) - c)q^I(n,k)$.

In homogenous markets ($\delta = \beta$), since profit margins (i.e., market price less marginal cost of production) are equal and positive for all firms, the higher quantity produced by an independent firm makes it more profitable than a cartel firm. In differentiated markets

($\delta < \beta$), even though the profit margin of an independent firm is lower, its higher output more than compensates for this and results in it obtaining a higher profit than a cartel firm. Thus, a firm outside the cartel is always more profitable than a firm in the cartel. However, as we discuss below, this does not necessarily imply that the cartel is unstable.

We use the definition of cartel stability proposed in d'Aspremont, Jacquemin, Gabszewicz, and Weymark (1983). A cartel of size $2 \leq k \leq n$ is *internally stable* if none of the firms in the cartel has an incentive to leave the cartel; that is, if $\pi^C(n,k) \geq \pi^I(n,k-1)$. A cartel of size $2 \leq k \leq n$ is *externally stable* if none of the independent firms has an incentive to join the cartel; that is, if $\pi^I(n,k) \geq \pi^C(n,k+1)$. We follow the convention that external stability is trivially satisfied in case $k = n$. A cartel of size $2 \leq k \leq n$ is *stable* if it is both internally and externally stable. On the other hand, denoting by $\pi^I(n,1)$ the profit of a firm when all firms operate independently, the market structure where all firms are independent is stable if the external stability condition $\pi^I(n,1) \geq \pi^C(n,2)$ is satisfied. In this case, there do not exist two independent firms that find it more profitable to form a cartel, and the lack of a cartel implies that the corresponding internal stability condition is not relevant.

Theorem 1 below presents the results pertaining to stability in the single cartel framework. This theorem, which covers both homogenous and differentiated product markets, states that a stable market structure always exists, and characterises the stable market structures.

Theorem 1. *A stable market structure always exists in the single cartel framework.*

(i) *In a homogenous product market, the unique stable market structure is described by the complete cartel when $n = 2$, and by all firms operating independently when $n > 2$.*

(ii) *In a differentiated product market, the stable market structure is described by a complete cartel when $n = 2$, an incomplete two-firm cartel if and only if $n \geq 3$ and $\gamma(3-\gamma)n \leq 2\gamma + (2-\gamma)[2\sqrt{1+\gamma} - (1-\gamma)]$, and by all firms operating independently when $n \geq 3$ and $\gamma(3-\gamma)n \geq 2\gamma + (2-\gamma)[2\sqrt{1+\gamma} - (1-\gamma)]$.*

Corollary 1. *(i) If a single cartel (market structure when all firms are independent) is stable in a market with n firms, it is also stable when the market has fewer (larger) number of firms. (ii) Both market structures are stable for at most one value of n, and this happens when the unique value of n obtained from the equality $\gamma(3-\gamma)n = 2\gamma + (2-\gamma)[2\sqrt{1+\gamma} - (1-\gamma)]$ is an integer. Apart from this knife-edge case, the stable market structure is unique.*

The formal proof of the theorem is presented in the appendix, whereas the corollary follows immediately from the conditions stated in the theorem. In what follows, we discuss the arguments underlying the theorem and discuss its intuition, first for a homogenous market and then for a differentiated market.

The computation of the stability conditions show that in a homogenous market, incomplete cartels are internally unstable, i.e. a firm in cartel will always find it more profitable to leave the cartel. Complete cartels, however, are internally stable if and only if there are only two firms; this, along with the convention that complete cartels are always externally stable, results in the stability of this complete cartel. (This particular result already exists in Bloch 2002). When $n \geq 3$, the afore-mentioned internal instability of an incomplete two-firm cartel in particular implies $\pi^I(n, 1) > \pi^C(n, 2)$, which, in turn, implies that the market structure where all firms are independent is stable. Consequently, in a homogeneous market, there exists a unique stable market structure which is described by the complete cartel when $n = 2$, and by all firms operating independently when $n \geq 3$.

The intuition behind the internal instability of both incomplete cartels, and complete cartels comprising of more than two firms, is that a firm in the cartel finds it more profitable to leave the cartel and free-ride on the reduced output of the other firms in the cartel. In addition, in the case of incomplete cartels, the independent firms exploit the reduced production of the cartel by increasing their own production, and this undermines, to a certain extent, the cartel's attempt to increase the price (which is common for all firms in a homogeneous market). However, when there are only two firms, and a firm in the cartel leaves the cartel, then it cannot free-ride on the reduced output of the other firms in the cartel simply because the cartel ceases to exist. Furthermore, there does not exist any independent firm that exploits the reduced quantity of the cartel. Hence, when $n = 2$, these factors result in a firm in the cartel not having an incentive to leave the cartel, thereby making the complete cartel stable.

When the product market is differentiated, we find, for similar reasons as outlined above, that a complete cartel is stable if and only if $k = n = 2$, and an incomplete cartel comprising of $k \geq 3$ firms is always internally unstable. However, an incomplete cartel with $k = 2$ is stable if and only if market differentiation is sufficiently high (i.e. γ is sufficiently low). The intuition behind this rests on three factors. Firstly, when $k = 2$, a firm in the cartel realises that since the cartel ceases to exist if it leaves the cartel, it cannot leave the cartel and yet free-ride on the reduced output of the other firms in the cartel. Secondly, when

γ is low enough, the other firms' output affects a firm's market price to a lesser extent. Consequently, when the firms in the cartel attempt to increase their market price by reducing their output, the independent firms, being not as affected, do not increase their production as substantially. This, in turn, implies that the cartel firms' efforts to increase the market price is not as significantly undermined by the independent firms. Both these factors combine to bring about internal stability of an incomplete two-firm cartel. Thirdly, we find that any incomplete cartel is externally stable – once a cartel is formed, it is more profitable for an independent firm to not join the cartel. These three reasons, taken together, imply that an incomplete two firm cartel is the only stable cartel.

Finally, as in the case of homogenous markets, whenever a cartel is unstable, it is due to a failure of internal stability of an incomplete two-firm cartel (recall the point made in the earlier paragraph that external stability is always satisfied). This, in turn, implies stability of the market structure where all firms operate independently. Hence, a stable market structure always exists in differentiated markets as well. Furthermore, as the corollary to Theorem 1 highlights, except for a knife-edge case, there is a unique stable market structure.

We conclude this section by drawing attention to the relationship between market differentiation and competition in the market vis-a-vis cartel stability. When $n \geq 3$, an incomplete two-firm cartel (the market structure where all firms operate independently) is stable if and only if, given the number of firms, the market is sufficiently differentiated (differentiation is sufficiently low); or equivalently, given the extent of market differentiation, the number of firms is sufficiently low (high). That is, the extent of market differentiation that is sufficient for cartel stability is increasing in the number of firms in the market – this is because an increase in market competition dampens the impact of a particular level of market differentiation. Equivalently, for a particular differentiated market, cartel stability requires a sufficiently low level of market competition, the reason being a comparatively low level of market differentiation may be mitigated by lesser competition in the market. As a result, as far as cartel stability is concerned, there is an inverse relation between the extent of market differentiation and the extent of competition in the market.

4 Stable market structures in the multiple cartel framework

In Subsection 4.1, we develop the framework for analysing stable market structures when multiple cartels may form, define the equilibrium in this context, show that a unique equilibrium always exists, and define the criteria for a stable market structure. Next, in Subsection 4.2, we present the results pertaining stable market structures for the specific case where there are four firms in the market. Finally, in Subsection 4.3, we present the results of the analysis of stable market structures for the general case with any number of firms, and prove that the results obtained in Subsection 4.2 hold generally.

4.1 The multiple cartel framework

We now consider the possibility that more than one cartel may be formed. Naturally, each particular firm is part of at most one cartel. Let $m \geq 1$ denote the number of cartels. For any $\ell \in \{1, \ldots, m\}$, let the number of firms in the ℓth cartel be equal to k_ℓ. If $\sum_{\ell=1}^{m} k_\ell < n$, then the number of independent firms is $n^I = n - \sum_{\ell=1}^{m} k_\ell$. Whenever an independent firm exists, we assume, without loss of generality, that firm i, for any $i \in \{1, \ldots, n^I\}$, is an independent firm; firm i, for any $i \in \{n^I + 1, \ldots, n^I + k_1\}$, belongs to the first cartel; and, more generally, firm i, for any $i \in \{n^I + \sum_{j=1}^{\ell-1} k_j + 1, \ldots, n^I + \sum_{j=1}^{\ell} k_j\}$, belongs to the ℓth cartel. The cartels and the independent firms (if any) simultaneously choose quantities, where each cartel acts as a single decision-making unit. A cartel's objective is to maximise the aggregate profit of its member firms, and an independent firm's objective is to maximise its own profit. As before, due to no redistribution of a cartel's profit amongst its members, and because of constant marginal cost of production (which implies a cartel's profit is independent of the manner in production is coordinated within the cartel) and symmetry of firms, we assume that the quantity produced by a cartel is split equally amongst its members, and each cartel member obtains the profit from its own quantity.

The aggregate quantity chosen by the ℓth cartel is denoted by $k_\ell q_\ell^C$ so that each firm in this cartel produces q_ℓ^C units. That is, for any $i \in \{n^I + \sum_{j=1}^{\ell-1} k_j + 1, \ldots, n^I + \sum_{j=1}^{\ell} k_j\}$, firm i produces $q_i = q_\ell^C$. This firm's profit equals $(\{\alpha - \beta q_\ell^C - \delta[(k_\ell - 1)q_\ell^C + \sum_{j=1, j\neq \ell}^{m} k_j q_j^C + \sum_{j=1}^{n^I} q_j]\} - c)q_\ell^C$. So, the ℓth cartel chooses $k_\ell q_\ell^C$ to maximise $k_\ell(\{\alpha - \beta q_\ell^C - \delta[(k_\ell - 1)q_\ell^C + \sum_{j=1, j\neq \ell}^{m} k_j q_j^C + \sum_{j=1}^{n^I} q_j]\} - c)q_\ell^C$. On the other hand, an independent firm i chooses q_i to maximise its profit $(\{\alpha - \beta q_i - \delta[\sum_{j=1}^{m} k_j q_j^C + \sum_{j=1, j\neq i}^{n^I} q_j]\} - c)q_i$. We assume, due to symmetry, that all independent firms produce the same quantity q^I.

It follows that, in a homogenous market (where $\delta = \beta$), a cartel behaves exactly like an independent firm. So, a homogenous market with m cartels (with k_ℓ firms in the ℓth cartel and n^I independent firms) is exactly equivalent to a homogenous market with a single k_ℓ-firm cartel and $n - k_\ell$ independent firms for any $\ell \in \{1, \ldots, m\}$. We will make use of this shortly.

The profile of quantities q^* induced by each independent firm choosing q^{I*} and, for any $\ell \in \{1, \ldots, m\}$, the ℓth cartel choosing $k_\ell q_\ell^{C*}$ is an *equilibrium* in the multiple cartels framework if neither an independent firm nor a cartel has a profitable unilateral deviation.

The next lemma states that a unique equilibrium exists in the multiple cartels framework, and each firm makes a positive profit in this equilibrium. In the proof of the lemma that is in the appendix, we prove this by first showing that the first-order conditions derived from the profit maximisation problem of the cartels and the independent firms can be expressed in the form of a non-homogenous system of linear equations, and the determinant of the corresponding co-efficient matrix is non-zero, thus implying that the system has a unique solution. Following this, we use Farka's lemma to show the quantities in the unique solution are non-negative. Finally, we show that each firm's equilibrium market price is higher than the constant marginal cost of production, and that this implies that the profit-maximising quantity for each firm/cartel must be positive – hence each firm obtains a positive profit.

Lemma 1. *In each market structure, there exists a unique equilibrium. In this equilibrium, each firm produces a positive quantity, experiences prices above marginal cost and enjoys positive profit.*

The market structure $(n^I, k_1, \ldots, k_m)$, that comprises of n^I independent firms and $m \geq 1$ cartels with $k_\ell \geq 2$ firms in the ℓth cartel, is *stable* if and only if:

1. A firm belonging to a cartel does not find it more profitable to leave the cartel to become an independent firm.

2. An independent firm does not find it more profitable to join an existing cartel.

3. A firm belonging to a cartel does not find it more profitable to leave the cartel to join another cartel or form a new cartel with an independent firm.

4. An independent firm does not find it more profitable to form a new cartel with another independent firm.

On the other hand, the market structure where all firms are independent is *stable* if two firms do not find it more profitable to form a two-firm cartel.

Thus, the conditions for stable cartelisation in the single cartel framework are necessary for stability in the multiple cartels framework. This is because the first/second condition above is similar to the internal/external stability condition of a cartel in the single cartel framework. The last two conditions are necessitated by the possibility of the formation of multiple cartels. In case of the market structure where all firms are independent, the stability conditions are identical in the single cartel framework and the multiple cartels framework.

In the next subsection, we present the results pertaining to stable market structures when there are four firms in the market – this is the minimum number of firms required for multiple cartels to be feasible.

4.2 Stable market structures with four firms

We will show that in a differentiated market with number of firms $n = 4$: *(i)* a stable market structure always exists, and this is described by the four firms pairing up into two two-firm cartels, or by all firms operating independently, *(ii)* this multiple cartels structure with two two-firm cartels is the unique stable cartelisation structure, *(iii)* cartelisation (the market structure where all firms are independent) is stable whenever the market is sufficiently (not sufficiently) differentiated, and *(iv)* cartelisation is stable in the multiple cartels framework whenever a single two-firm is stable in the single cartel framework; however, in addition, it may also be stable even when a single two-firm cartel is not stable in the single cartel framework. In light of Theorem 1, and with regard to this last point, recall that two-firm cartels are the only ones which may be stable in the single cartel framework.

The table below presents the profit of a cartel firm, $\pi^C(\cdot)$, and an independent firm, $\pi^I(\cdot)$, corresponding to each possible market structure when there are four firms. These profits can be easily obtained from the equilibrium quantities that come from simultaneously solving the first-order conditions (of the cartels/independent firms profit-maximisation problem) that we derive in the proof of Lemma 1.

For the purpose of this example, and with reference to the table, we use notation such as $\pi^C(b)$ and $\pi^I(b)$ to denote a cartel firm's profit and an independent firm's profit in situation b of the table where there is one two-firm cartel. It is then easy to verify that:

situation		$\pi^C(\cdot)$	$\pi^I(\cdot)$
a	no-cartel	—	$\frac{\beta(\alpha-c)^2}{(2\beta+3\delta)^2}$
b	one two-firm cartel	$\frac{(\beta+\delta)(2\beta-\delta)^2(\alpha-c)^2}{4(2\beta^2+3\beta\delta-\delta^2)^2}$	$\frac{\beta^3(\alpha-c)^2}{(2\beta^2+3\beta\delta-\delta^2)^2}$
c	one three-firm cartel	$\frac{(\beta+2\delta)(2\beta-\delta)^2(\alpha-c)^2}{(4\beta^2+8\beta\delta-3\delta^2)^2}$	$\frac{\beta(2\beta+\delta)^2(\alpha-c)^2}{(4\beta^2+8\beta\delta-3\delta^2)^2}$
d	complete cartel	$\frac{(\alpha-c)^2}{4(\beta+3\delta)}$	—
e	two two-firm cartels	$\frac{(\beta+\delta)(\alpha-c)^2}{4(\beta+2\delta)^2}$	—

(i) Neither the complete cartel nor the three-firm cartel is internally stable as $\pi^C(d) < \pi^I(c)$ and $\pi^C(c) < \pi^I(b)$, respectively.

(ii) The no-cartel situation is externally stable, and hence stable (since internal stability is satisfied trivially), if and only if $\pi^I(a) \geq \pi^C(b)$, or $\beta\delta(8\beta + 19\delta) \geq 4\beta^3 + 9\delta^3$.

(iii) The single two-firm cartel is internally stable if and only if $\pi^C(b) \geq \pi^I(a)$ or $4\beta^3+9\delta^3 \geq \beta\delta(8\beta + 19\delta)$, while it satisfies the fourth condition of the definition of stable cartelisation if and only if $\pi^I(b) \geq \pi^C(e)$, or $\beta\delta(\beta + 5\delta) \geq \beta^3 + \delta^3$. However, $4\beta^3 + 9\delta^3 \geq \beta\delta(8\beta + 19\delta)$ and $\beta\delta(\beta + 5\delta) \geq \beta^3 + \delta^3$ cannot both be satisfied simultaneously. Hence, a single two-firm cartel is never stable. In fact, in the general case presented in the next subsection, we extend this reasoning and show that a two-firm cartel, which is the only stable cartel in the single cartel framework, is not stable in the multiple cartels framework because whenever this cartel is internally stable, two independent firms will prefer to form a new cartel.

(iv) The two two-firm cartels situation is stable if and only if $\pi^C(e) \geq \pi^I(b)$ (recall stability condition 1) and $\pi^C(e) \geq \pi^C(c)$ (recall stability condition 3) – the other two stability conditions do not apply in this case. The condition $\pi^C(e) \geq \pi^C(c)$ is always satisfied (recall the point made earlier that external stability is always satisfied) while $\pi^C(e) \geq \pi^I(b)$ is satisfied if and only if $\beta^3 + \delta^3 \geq \beta\delta(\beta + 5\delta)$.

Hence, there are three possible situations, solely depending on the values of δ and β, which can be re-interpreted in terms of the inverse index of market differentiation γ:

1. Only the no-cartel situation is stable if and only if $\beta^3 + \delta^3 < \beta\delta(\beta + 5\delta)$, implying that γ is above a threshold value $\overline{\gamma}$.

2. Only the two two-firm cartels is stable if and only $\beta\delta(8\beta + 19\delta) < 4\beta^3 + 9\delta^3$, implying that γ is below a threshold value $\underline{\gamma}$, where $\underline{\gamma} < \overline{\gamma}$.

3. Both the no-cartel situation and the two two-firm cartels are stable if and only if $\beta\delta(8\beta + 19\delta) \geq 4\beta^3 + 9\delta^3$ and $\beta^3 + \delta^3 \geq \beta\delta(\beta + 5\delta)$, implying $\gamma \in [\underline{\gamma}, \overline{\gamma}]$.

The above result can be understood in terms of the intuition provided earlier. If market differentiation is not sufficiently high, then a cartel is not stable due to the free-riding incentive, and this makes the no-cartel situation stable. If market differentiation is sufficiently high, then the multiple cartels market structure is stable as the free-riding incentive is not as pronounced. For intermediate values of market differentiation, both the no-cartel situation and the multiple cartels market structure are stable; however, compared to the no-cartel situation, the multiple cartels market structure provides a higher profit to each firm – hence, if firms possess sufficient foresight, one may expect the firms to coordinate on the stable multiple cartels market structure.

Finally, one can use $n = 4$ in Theorem 1 to determine that, in the single cartel framework, a two-firm single cartel is stable if and only if $\gamma \leq \widehat{\gamma}$, where $\widehat{\gamma}$ follows from the condition in Theorem 1, and no other cartel is stable in the single cartel framework. Since $\widehat{\gamma} < \overline{\gamma}$, the two two-firm cartels market structure is stable in the multiple cartels framework if a single two-firm cartel is stable in the single cartel framework. However, when $\gamma \in (\widehat{\gamma}, \overline{\gamma}]$, the two two-firm cartels is stable in the multiple cartels framework but a single two-firm cartel is not stable in the single cartel framework. Hence, there exist differentiated markets where a stable single cartel in the single cartel framework does not exist but a stable multiple cartels market structure exists. The implication is that if one assumes the single cartel framework, then cartelisation would not be expected in these markets – yet, one actually needs to consider this possibility because a stable multiple cartels market structure exists.

4.3 Stable market structures: A general result

We now analyse the stable market structures for the general case. As we will see, the broad contours of the results of the previous subsection, where there are four firms, carry over to when there are an arbitrarily given number of firms. We first state the general result in the next theorem, and then proceed towards presenting the arguments that support the theorem.

Theorem 2. *A stable market structure always exists in the multiple cartels framework.*

(i) *Stable cartelisation is described by at most one independent firm with each of the other firms being organised into two-firm cartels, and this market structure is stable if and only if*

$\gamma(3-\gamma)n \leq 4\sqrt{1+\gamma} - 2(1-\gamma)^2$ *when* n *is even and* $\gamma(3-\gamma)n \leq 4\sqrt{1+\gamma} - \frac{4-10\gamma+11\gamma^2-3\gamma^3}{(2-\gamma)}$ *when* n *is odd. This is equivalent to the inverse index of market differentiation* $\gamma \leq \overline{\gamma}$*. So, if cartelisation is stable in a differentiated market with* $n \geq 4$ *firms, then it is also stable for any number of firms* $4 \leq n' \leq n$*.*
(ii) *The market structure where all firms are independent is stable if and only if* $\gamma(3-\gamma)n \geq 2\gamma + (2-\gamma)[2\sqrt{1+\gamma} - (1-\gamma)]$*, and this is equivalent to the inverse index of market differentiation* $\gamma \geq \underline{\gamma}$*, where* $\overline{\gamma} \leq \underline{\gamma}$*. So, if this market structure is stable in a differentiated market with* $n \geq 4$ *firms, then it is also stable for any larger number of firms* $n' \geq n$*.*
(iii) *Both cartelisation and the market structure where all firms are independent are simultaneously stable for at most one value of the number of firms* n*.*

The above theorem is all-encompassing in that it is applicable for both homogenous markets and differentiated markets, and for all values of $n \geq 2$. When $n < 4$, so that the formation of more than one cartel is not possible, Theorem 1 applies directly, and the characterisation of stable market structures in Theorem 2 is identical to that in Theorem 1. The difference in the stable market structures described in these two theorems arise when $n \geq 4$, and the reason for this difference is the possibility of formation of multiple cartels. Specifically, when $n \geq 4$, the only stable cartel configuration in the multiple cartels framework involves $\frac{n-1}{2}$ two-firm cartels and an independent firm whenever n is odd, and $\frac{n}{2}$ two-firm cartels whenever n is even. Similar to Theorem 1, in Theorem 2, a stable market structure always exists, and cartelisation (market structure where all firms are independent) is stable if and only if the market is sufficiently (not sufficiently) differentiated.

We establish the theorem by a series of lemmata, and begin by focussing on cartelisation that is stable in the multiple cartels framework when there are at least four firms, i.e. $n \geq 4$.

In Lemma 2 below, we use Theorem 1 to argue that cartelisation is not stable in homogenous markets, and that all firms being independent is the only stable market structure. This, along with the preceding discussion, implies that cartelisation is not stable in homogenous markets in the multiple cartels framework except for when $n = 2$ – this mirrors the instability of cartelisation in homogenous markets in the single cartel framework in Theorem 1.

Lemma 2. *In any homogenous product market with* $n \geq 3$*, cartelisation is not stable in the multiple cartels framework. Hence, cartelisation is stable in the multiple cartels framework in a homogeneous product market if and only if* $n = 2$*, and hence* $m = 1$ *and* $k_1 = 2$*.*

Proof. As mentioned earlier, in a homogenous market, each cartel behaves exactly like an independent firm. As a result, a homogenous market with $m \geq 1$ cartels, with $k_\ell \geq 2$ firms in the ℓth cartel and $n^I \geq 0$ independent firms, is for any $\ell \in \{1, \ldots, m\}$ exactly equivalent to a homogenous market with a single k_ℓ-firm cartel and $n^I + m - 1 \geq 0$ independent firms, where each of the other $m - 1 \geq 0$ cartels is replaced by an independent firm. By Theorem 1, this k_ℓ-firm single cartel is stable in the single cartel framework if and only if $k_\ell = n = 2$. Since the stability criteria of the single cartel framework are necessary conditions for stability in the multiple cartels framework, this k_ℓ-firm single cartel is stable in the multiple cartels framework *only if* $k_\ell = n = 2$. Since this must hold for each and every cartel $\ell \in \{1, \ldots, m\}$, the cartel is stable *only if* $n = 2$, $m = 1$ and $k_1 = 2$. Finally, sufficiency follows from Theorem 1 and the fact that, for $n = 2$, stability in the multiple cartels framework is identical to stability in the single cartel framework. ■

In Lemma 3 that follows next, we state that there does not exist any stable single cartel in differentiated markets when $n \geq 4$. It has already been proved in Theorem 1 that, when $n \geq 4$, any single cartel other than an incomplete two-firm cartel is unstable in differentiated markets in the single cartel framework; so, these other single cartels cannot be stable in the multiple cartels framework either. However, in the proof of the lemma (in the appendix), we show that whenever an incomplete two-firm single cartel is internally stable, then two independent firms – which always exist as $n \geq 4$ – find it more profitable to form a new two-firm cartel. (We reiterate that the latter consideration, whereby two independent firms may form a second cartel, was ignored a priori in the single cartel framework.) Thus, all single cartels – complete or incomplete – are unstable in differentiated markets when $n \geq 4$.

Lemma 3. *In any differentiated product market with $n \geq 4$, a single stable cartel does not exist in the multiple cartels framework. Hence, a single cartel is stable in a differentiated market in the multiple cartels framework only if $n = 3$, $m = 1$, and $k_1 = 2$.*

It follows that a stable cartelisation must comprise of multiple cartels. Now, the challenge in identifying the structure of cartelisation that is stable/unstable is that the number of feasible cartelisation structures grows exponentially with the number of firms, and one needs to verify the stability conditions for each cartelisation structure to ascertain whether it is stable. In spite of this complexity, Lemma 4 below states that if cartelisation is stable, then there cannot exist any cartel with more than two firms. In the proof (in the appendix), we exploit the internal instability of any single cartel with more than three firms in the single cartel framework to show that if a multiple cartels market structure contains a cartel with more than three firms, then this cartel is similarly internally unstable. Thus, each of the multiple cartels that support stable cartelisation must comprise of at most two firms.

Lemma 4. *In any differentiated product market with $n \geq 4$, a market structure where there is a cartel with more than two firms is unstable in the multiple cartels framework.*

In Lemma 5 below, we build on this and prove (in the appendix) that in any stable cartelisation structure, there exists at most one independent firm. The reason is that if there simultaneously exists multiple two-firm cartels and at least two independent firms, and if a two-firm cartel is internally stable, then two independent firms prefer to form a new two-firm cartel. Due to this lemma, we are now able to characterise the structure of stable cartelisation in the multiple cartels framework: subject to the integer constraint, all firms are organised in two-firm cartels.

Lemma 5. *In any differentiated product market with $n \geq 4$, a market structure with more than one independent firm is unstable in the multiple cartels framework.*

Next, in Lemma 6, we argue that a stable market structure always exists by reasoning that if the multiple cartels structure described above is not stable, then the market structure where all firms are independent is stable. The basis for this lemma is the previously made argument that the internal instability of a single two-firm cartel in the single cartel framework is sufficient for stability of the market structure where all firms are independent. Following this lead, in the proof of this lemma, we show that if cartelisation structure described in the previous lemmas is unstable, then a single two-firm cartel is internally unstable in the single cartel framework, and so, the market structure where all firms are independent is stable.

Lemma 6. *In any differentiated product market with $n \geq 4$, a stable market structure always exists in the multiple cartels framework.*

Proof. As reasoned in the text, we will show that instability of the cartelisation structure, where all firms with the exception of one firm are organised in two-firm cartels, implies internal instability of a single two-firm cartel. To this end, we consider all the possible realignments that may cause the instability of the above multiple cartels configuration.

(i) If a firm in a two-firm cartel leaves its cartel and forms a new cartel with the independent firm, then it simply moves from one two-firm cartel to another with everything else unchanged. So this is not a profitable realignment, and therefore, this cannot be the cause of instability.

(ii) Suppose that an independent firm (which exists if and only if n is odd) joins a two-firm cartel thus resulting in the formation of a three-firm cartel. We have already shown in Lemma 4 that if the largest cartel has more than two firms, then a firm in this cartel finds it more profitable to exit the cartel and operate independently. Hence, it cannot be that the independent firm prefers to join a two-firm cartel to become a member of a three-firm cartel. Therefore, this realignment cannot be the cause of instability.

(iii) Suppose that a firm in a two-firm cartel joins another two-firm cartel, thus resulting in the formation of a three-firm cartel. Now, if this is more profitable for the firm, then it will be even more profitable for it to exit its initial two-firm cartel and operate independently. This is because, when it joins the three-firm cartel, then, by Lemma 4, it will find it more profitable to exit this three-firm cartel and operate independently. So, if joining another two-firm cartel is more profitable for the firm, then leaving its initial two-firm cartel and operating independently is even more profitable. Consequently, if a firm does not prefer to exit its own cartel and operate independently, then it also does not prefer to join another cartel.

(iv) The last feasible realignment involves a firm in a two-firm cartel exiting the cartel and operating independently.

So, cartelisation is unstable if and only if a firm in a two-firm cartel prefers to exit the cartel and operate independently, where necessity follows from the above, and sufficiency is trivial. This results in a situation where there are at least two, and at most three, independent firms, and all other firms are organised in two-firm cartels. However, by the earlier lemmata, this cartel configuration is not stable. It also follows from the above that the basis of this instability cannot be that two independent firms prefer to form a new two-firm cartel. Then, the only other possible realignments are the ones listed above, and the argument provided above implies that this instability must be because a firm in an existing two-firm cartel finds it more profitable to exit the cartel and operate independently.

The repeated use of this logic, and the consequent sequential break-up of the two-firm cartels leads us to a situation where there is a single two-firm cartel. Again, by the previous lemmata, this is not stable, and yet another application of the above reasoning implies that a single two-firm cartel is not internally stable. ■

In Lemma 7, we state that cartelisation (the market structure where all firms are independent) is stable *if and only if* market differentiation is sufficiently high (low). We have already established in the proof of Lemma 6 that cartelisation is stable if and only if each firm in each of the multiple two-firm cartels prefers to stay in the cartel. We show in the proof of this lemma (in the appendix) that this occurs if and only if market differentiation is sufficiently high. This is founded on the previously discussed intuition that when market differentiation is (is not) sufficiently high, then a firm in a two-firm cartel prefers (does not prefer) to stay in the cartel (to exit the cartel and operate independently). For instance, recall that when the market is homogeneous, then cartelisation is not stable. Similarly, recalling Theorem 1, and as reasoned previously, the market structure where all firms are independent is stable (unstable) if a single two-firm cartel is internally unstable (satisfies strictly the inequality that defines internal stability) in the single cartel framework, and this happens if and only if market differentiation is sufficiently low (high).

Lemma 7. *(i) In the multiple cartels framework, cartelisation is stable if and only if* $\gamma(3-\gamma)n \leq 4\sqrt{1+\gamma}-2(1-\gamma)^2$ *when* n *is even and* $\gamma(3-\gamma)n \leq 4\sqrt{1+\gamma}-\frac{4-10\gamma+11\gamma^2-3\gamma^3}{(2-\gamma)}$ *when* n *is odd, and this is equivalent to the inverse index of market differentiation* $\gamma \leq \overline{\gamma}$*. Furthermore, when cartelisation is stable for a differentiated market with* $n \geq 4$ *firms, then it is also stable for any number for firms* $4 \leq n' \leq n$*.*
(ii) On the other hand, the market structure where all firms are independent is stable if and only if $\gamma(3-\gamma)n \geq 2\gamma+(2-\gamma)[2\sqrt{1+\gamma}-(1-\gamma)]$*, and this is equivalent to the inverse index of market differentiation* $\gamma \geq \underline{\gamma}$*, where* $\underline{\gamma} \leq \overline{\gamma}$*. This implies that if this market structure is stable for a differentiated market with* $n \geq 4$ *firms, then it is also stable for any larger number of firms* $n' \geq n$*.*

In Figure 1, we show in the (n, γ)-space that, for each number of firms $4 \leq n \leq 12$, the threshold value $\overline{\gamma}$ below which cartelisation is stable (taking into account n being even or odd) and the threshold values $\underline{\gamma}$ above which all firms operating independently is stable, with the former thresholds lying above the latter (since $\overline{\gamma} \geq \underline{\gamma}$). The area in between the two curves represents pairs (n, γ) where both market structures are simultaneously stable.

The figure suggests that for every $\gamma \in (0, 1)$ there is at most one number of firms n for

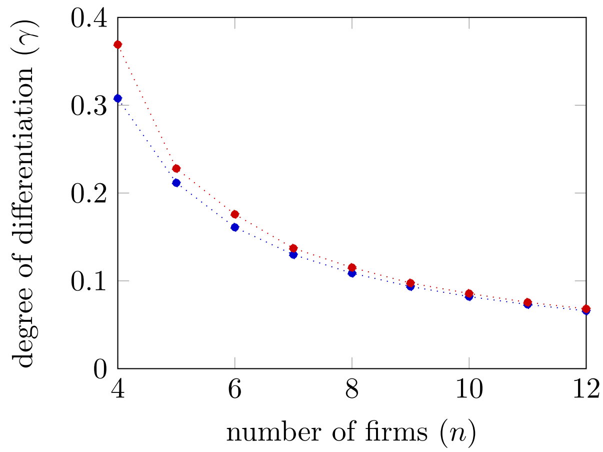

Figure 1: Cartelisation is stable for values of (n, γ) on and below the red curve. All firms acting independently is a stable structure for values of (n, γ) on and above the blue curve.

which both market structures are stable. For any lower number of firms, only cartelisation is stable, while, for any higher number of firms, only the market structure where all firms act independently is stable. However, the figure only plots firms up to $n = 12$ and both curves become flatter, creating the potential for a larger number of firms to fall in between the two curves. Yet, as we show in Lemma 8, this is not the case: for each value of $\gamma \in (0, 1)$ there is at most one value of n for which both market structures are stable. The reason is that it can be shown that if the market structure where all firms act independently is stable for n, then cartelisation is not stable for $n + 1$ firms; and, if cartelisation is stable for n firms, then the market structure where all firms act independently is not stable for $n - 1$ firms.

Lemma 8. *In any differentiated product market (i.e. for any $\gamma \in (0, 1)$), both cartelisation and the market structure where all firms are independent is stable for at most one value of the number of firms n.*

These seven lemmas take together establish Theorem 2 that characterises stable market structures in the multiple cartels framework. We recapitulate that Lemma 2 deals with homogeneous markets while the remaining six lemmas pertain to differentiated markets. Lemma 3, Lemma 4 and Lemma 5 characterise stable cartelisation structures. Lemma 6 shows existence of a stable market structure. Lemma 7 states conditions under which cartelisation and the market structure where all firms are independent are stable/unstable. Lemma 8 shows that the stable market structure is unique up to a knife-edge case.

5 Conclusion

We develop a framework to analyse stability of cartelisation when firms in the market may form multiple cartels. This is in direct contrast to the existing literature which, broadly speaking, assumes a priori that only a single cartel may form. In this multiple cartels framework, we have established that a stable cartelisation structure must involve multiple cartels whenever the formation of multiple cartels is possible. Furthermore, we precisely characterise the structure of stable cartelisation – it is described by at most one independent firm, with all other firms organised in two-firm cartels. This cartelisation structure is stable if and only if market differentiation is high enough. The only other market structure that may be stable is the one where all firms operate independently, and this market structure is stable if and only if market differentiation is low enough. Finally, a stable market structure always exists.

Appendix: Proofs

Proof of Theorem 1.

External stability of a k-firm cartel, where $2 \leq k < n$, requires $\pi^I(n,k) \geq \pi^C(n,k+1)$; that is, an independent firm (weakly) prefers to not join the cartel. Internal stability for a k-firm cartel, where $2 \leq k \leq n$, requires $\pi^C(n,k) \geq \pi^I(n,k-1)$; that is, a member of the cartel (weakly) prefers staying in the cartel.

Using

$$q^C(n,k) = \frac{(2\beta - \delta)(\alpha - c)}{(2\beta + 2(k-1)\delta)(2\beta + (n-k-1)\delta) - k(n-k)\delta^2}$$

and

$$q^I(n,k) = \frac{(2\beta + (k-2)\delta)(\alpha - c)}{(2\beta + 2(k-1)\delta)(2\beta + (n-k-1)\delta) - k(n-k)\delta^2}$$

we find that

$$\pi^C(n,k) = (p^C(n,k) - c)q^C(n,k) = \frac{(2\beta - \delta)^2(\beta + (k-1)\delta)(\alpha - c)^2}{[(2\beta + 2(k-1)\delta)(2\beta + (n-k-1)\delta) - k(n-k)\delta^2]^2}$$

and

$$\pi^I(n,k) = (p^I(n,k) - c)q^I(n,k) = \frac{\beta(2\beta + (k-2)\delta)^2(\alpha - c)^2}{[(2\beta + 2(k-1)\delta)(2\beta + (n-k-1)\delta) - k(n-k)\delta^2]^2}.$$

(i) It can be verified that for all $2 \leq k \leq n$, we have $\pi^I(n,k) > \pi^C(n,k+1)$, so that all k-firm cartels with $2 \leq k \leq n$ are externally stable.

(ii) It can be verified that $\pi^C(n,n) \geq \pi^I(n,n-1)$ if and only if $n = 2$. This means that the complete cartel is only internally stable if and only if there are only two firms. Moreover, we find that for $n = 2$ the inequality is satisfied strictly: $\pi^C(2,2) > \pi^I(2,1)$. This implies that the 1-firm cartel, which is the market structure where all firms are independent, is not externally stable, and hence not stable. Hence, when $n = 2$, the complete cartel is the only stable market structure in both homogenous and differentiated markets.

(iii) For incomplete cartels, it can be verified that the internal stability condition is satisfied if and only if $k = 2$ and $n \leq \overline{n}(\gamma) \equiv \frac{2\gamma + (2-\gamma)[2\sqrt{1+\gamma} - (1-\gamma)]}{\gamma(3-\gamma)}$. Hence, for given γ, a two-firm

incomplete cartel is stable when the total number of firms is below a threshold level $\overline{n}(\gamma)$; this threshold level being strictly decreasing in γ starting from asymptotic $+\infty$ at $\gamma = 0$ to $1+\sqrt{2}$ at $\gamma = 1$. Further, a cartel with more than two firms never constitutes a stable complete cartel (as shown in *(ii)* above) or a stable incomplete cartel. Expressed in terms of γ, the inequality $n \leq \overline{n}(\gamma)$ is a third degree polynomial inequality, with one relevant root that gives a threshold value of γ that corresponds to the relation as specified by $\overline{n}(\gamma)$ on the domain $(0, 1]$ for γ. Then, the condition $n \leq \overline{n}(\gamma)$ is equivalent to $\gamma \leq \overline{n}^{-1}(\gamma)$ if $n \geq 3$, and for all γ if $n = 2$. (Note that the latter reiterates that the complete cartel is the only stable market structure in both homogenous and differentiated markets.) Hence, for given number of firms $n \geq 3$, a two-firm incomplete is stable if and only if the market is sufficiently differentiated.

(iv) If the two-firm cartel is internally unstable – and thus no cartel is stable – the inequality $\pi^C(n, 2) < \pi^I(n, 1)$ holds. This implies external stability, and hence stability, of the market structure where all firms act independently. So, a stable market structure always exists. ■

Proof of Lemma 1.

For any $\ell \in \{1, \ldots, m\}$, the ℓth cartel chooses its aggregate quantity $k_\ell q_\ell^C$ to maximise its aggregate profit $k_\ell(\{\alpha - \beta q_\ell^C - \delta[(k_\ell - 1)q_\ell^C + \sum_{j=1,j\neq\ell}^m k_j q_j^C + \sum_{j=1}^{n^I} q_j]\} - c)q_\ell^C$, while an independent firm i chooses its quantity q_i to maximise its profit $(\{\alpha - \beta q_i - \delta[\sum_{j=1}^m k_j q_j^C + \sum_{j=1,j\neq i}^{n^I} q_j]\} - c)q_i$. The symmetric equilibrium quantities, denoted by q_ℓ^{C*} for the firms in the ℓth cartel and q^{I*} for the independent firms, must satisfy the corresponding first-order conditions

$$\alpha - (2\beta + 2(k_\ell - 1)\delta)q_\ell^{C*} - \delta \textstyle\sum_{j=1,j\neq\ell}^m k_j q_j^{C*} - n^I \delta q^{I*} - c = 0$$

and

$$\alpha - (2\beta + (n^I - 1)\delta)q^{I*} - \delta \textstyle\sum_{j=1}^m k_j q_j^{C*} - c = 0.$$

This system of first-order conditions can be written in matrix form as

$$\begin{pmatrix} \frac{2\beta+2(k_1-1)\delta}{k_1\delta} & 1 & \cdots & \cdots & 1 \\ 1 & \frac{2\beta+2(k_2-1)\delta}{k_2\delta} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \frac{2\beta+2(k_m-1)\delta}{k_m\delta} & 1 \\ 1 & \cdots & \cdots & 1 & \frac{2\beta+(n^I-1)\delta}{n^I\delta} \end{pmatrix} \cdot \begin{pmatrix} \delta k_1 q_1^{C*} \\ \delta k_2 q_2^{C*} \\ \vdots \\ \delta k_m q_m^{C*} \\ \delta n^I q^{I*} \end{pmatrix} = \begin{pmatrix} \alpha - c \\ \alpha - c \\ \vdots \\ \alpha - c \\ \alpha - c \end{pmatrix}.$$

This is a non-homogenous system of $m+1$ linear equations if $n^I > 0$ and a non-homogenous system of m linear equations if $n^I = 0$. For the remainder of the proof we assume $n^I > 0$; the case $n^I = 0$ runs analogously.

Let us denote this system of equations as $A\,x = b$. In order to show that this system of equations has a unique solution, we will show that the co-efficient matrix A is non-singular, by showing it has a non-zero determinant: $\det(A) \neq 0$.

The matrix A can be written as $A = D + e\,e^\top$, where e is the $(m+1)$-column vector of ones, and D is the diagonal matrix such that, for any $j \in \{1, \ldots, m+1\}$, the diagonal element $d_{jj} = a_{jj} - 1$, where a_{jj} refers to the jth diagonal element of the matrix A. Since $\beta > \delta$, it is easily verified that $a_{jj} > 1$, and hence $d_{jj} > 0$. Since $\det(D) = \prod_{j=1}^{m+1} d_{jj} > 0$, D is an invertible square matrix, and by the matrix determinant lemma, we have $\det(A) = \det(D + e\,e^\top) = (1 + e^\top D^{-1}\,e)\det(D)$. Hence, to prove $\det(A) \neq 0$, it suffices to show that $e^\top D^{-1}\,e \neq -1$. But, since D^{-1} is a diagonal matrix with elements $\frac{1}{d_{jj}} > 0$ on the diagonal, $e^\top D^{-1}\,e = \sum_{j=1}^{m+1} \frac{1}{d_{jj}} > 0$. So, $\det(A) \neq 0$, and the system of equations has a unique solution x^*, from which one obtains unique values for $q_\ell^{C*} = \frac{x_\ell^*}{k_\ell \delta}$, for $1 \leq \ell \leq m$, and $q^{I*} = \frac{x_{m+1}^*}{n^I \delta}$.

We note that the second-order conditions for maximisation are always satisfied. The second-order derivative of the objective function of the ℓth cartel is $-2k_\ell(\beta + (k_\ell - 1)\delta) < 0$ and that of each independent firm i is $-2\beta < 0$.

We complete the proof by first establishing that the resulting values for q_ℓ^{C*} and q^{I*} are non-negative, by showing that the above obtained unique solution of non-homogenous system of linear equations $A\,x = b$ is a non-negative vector, i.e. $x^* \geq 0$. Finally, via arguments by contraposition, we prove that equilibrium prices are above marginal cost, that the resulting values for q_ℓ^{C*} and q^{I*} are strictly positive, and that firms make a positive profit. Again, we show this for the case $n^I > 0$ and note that the case $n^I = 0$ runs analogously.

Firstly, by Farka's lemma, a necessary and sufficient condition for our unique solution x^* to be non-negative is that there should not exist a y such that both $A^\top y \geq 0$ and $b^\top y < 0$. Suppose there is a vector y such that $b^\top y < 0$. Then, since $b^\top y = (\alpha - c)\sum_{j=1}^{m+1} y_j$, it holds that $\sum_{j=1}^{m+1} y_j < 0$. So, there exists $i \in \{1, \ldots, m+1\}$ such that $y_i < 0$. Since A is a symmetric matrix, we have $A^\top y = Ay$, and the ith entry of $A^\top y$ equals $a_{ii}y_i + \sum_{j=1, j\neq i}^{m+1} y_j = (a_{ii}-1)y_i + \sum_{j=1}^{m+1} y_j$, which is strictly negative since $a_{ii} > 1$, $y_i < 0$ and $\sum_{j=1}^{m+1} y_j < 0$. Thus, $A^\top y \geq 0$ does not hold whenever $b^\top y < 0$. Hence, $x^* \geq 0$.

Finally, we prove that, in fact, $x^* > 0$ by showing that the equilibrium price of each firm is greater than the constant marginal cost c. In the equilibrium x^*, each firm in cartel $\ell \in \{1, \ldots, m\}$ produces $q_\ell^{C*} = \frac{x_\ell^*}{k_\ell \delta}$ units and each independent firm produces $q^{I*} = \frac{x_{m+1}^*}{n^I \delta}$ units. Defining $n_\ell = k_\ell$ for $\ell \in \{1, \ldots, m\}$ and $n_{m+1} = n^I$, the equilibrium price of a firm in the ith cartel or an independent firm i, is given by $\alpha - \frac{\beta}{n_i \delta} x_i^* - \frac{n_i - 1}{n_i} x_i^* - \sum_{j=1, j\neq i}^{m+1} x_j^*$.

Suppose, by the way of contradiction, that a firm's price does not exceed marginal cost; that is, for some $i \in \{1, \ldots, m+1\}$, we have $\frac{\beta}{n_i\delta}x_i^* + \frac{n_i-1}{n_i}x_i^* + \sum_{j=1, j\neq i}^{m+1} x_j^* \geq \alpha - c$. In case $x_i^* > 0$ (so that the respective firm produces positive quantity), then (because of continuity of the inverse demand function) the respective cartel/independent firm can unilaterally profitably deviate and increase its currently non-negative profit by reducing its quantity, thus increasing its price and obtaining a higher (but possibly negative) profit. From this we conclude that $x_i^* = 0$ (since $x^* \geq 0$), and thus $\sum_{j=1, j\neq i}^{m+1} x_j^* \geq \alpha - c > 0$. This implies $x_{i'}^* > 0$ for some $i' \in \{1, \ldots, m+1\} \setminus \{i\}$ so that there is at least one other firm that produces a positive quantity. Since $\beta \geq \delta$ and $x_i^* = 0$, we have $\frac{\beta}{n_{i'}\delta}x_{i'}^* + \frac{n_{i'}-1}{n_{i'}}x_{i'}^* + \sum_{j=1, j\neq i'}^{m+1} x_j^* = \left[\frac{\beta}{n_{i'}\delta} + \frac{n_{i'}-1}{n_{i'}} - 1\right]x_{i'}^* + \sum_{j=1}^{m+1} x_j^* = \frac{\beta-\delta}{n_{i'}\delta}x_{i'}^* + \sum_{j=1, j\neq i}^{m+1} x_j^* \geq \sum_{j=1, j\neq i}^{m+1} x_j^* \geq \alpha - c$, which implies that the equilibrium price of a firm in cartel i' or an independent firm does not exceed marginal cost. By the same reasoning as above, this leads to $x_{i'}^* = 0$; a contradiction. Thus, in equilibrium, all prices exceed marginal cost. The positive profit margin guarantees that all firms produce positive quantities (so that $x^* > 0$) and experience positive profits, as otherwise the respective cartel/independent firm, which receives zero profit on producing zero quantity, can unilaterally profitably deviate by producing a smaller positive quantity, thus receiving a lower price that still higher than marginal cost, and obtain positive profit. ■

Proof of Lemma 3.

As argued in the main text, we will show that if the internal stability condition holds for a single two-firm cartel, then two independent firms prefer to form a second cartel.

Take a single two-firm cartel. Substituting $k = 1$ in the equilibrium profit derived in the proof of Theorem 1, we obtain $\pi^C(n,2) = \frac{(2\beta-\delta)^2(\beta+\delta)(\alpha-c)^2}{4(2\beta^2+(n-1)\beta\delta-\delta^2)^2}$ and $\pi^I(n,2) = \frac{\beta^3(\alpha-c)^2}{(2\beta^2+(n-1)\beta\delta-\delta^2)^2}$.

If a firm in the two-firm cartel leaves the cartel, then all firms operate independently, and the quantity and profit of each firm is $q^C(n,1) = q^I(n,1) = \frac{\alpha-c}{2\beta+(n-1)\delta}$ and $\pi^C(n,1) = \pi^I(n,1) = \frac{\beta(\alpha-c)^2}{(2\beta+(n-1)\delta)^2} = \frac{\beta^3(\alpha-c)^2}{(2\beta^2+(n-1)\beta\delta)^2}$. On the other hand, if two independent firms form another cartel, so that now there are two two-firm cartels and $n-4$ independent firms, then, using $m = 2$ in the profit-maximisation problem of the firm, along with symmetry of the firms in the cartels, gives that the quantity produced by a firm in this newly formed cartel is $\frac{(2\beta-\delta)(\alpha-c)}{2(2\beta^2+(n-1)\beta\delta-2\delta^2)}$ and its profit is $\frac{(2\beta-\delta)^2(\beta+\delta)(\alpha-c)^2}{4(2\beta^2+(n-1)\beta\delta-2\delta^2)^2}$.

We will now verify that $\pi^I(n,2) \geq \frac{(2\beta-\delta)^2(\beta+\delta)(\alpha-c)^2}{4(2\beta^2+(n-1)\beta\delta-2\delta^2)^2}$ and $\pi^C(n,2) \geq \pi^I(n,1)$ cannot both hold simultaneously. Using the corresponding profit expressions, the first inequality is equivalent to $\frac{\beta^3(\alpha-c)^2}{(2\beta^2+(n-1)\beta\delta-\delta^2)^2} \geq \frac{(2\beta-\delta)^2(\beta+\delta)(\alpha-c)^2}{4(2\beta^2+(n-1)\beta\delta-2\delta^2)^2}$, which can re-written as $B \geq \frac{A^2}{(A-\delta^2)^2}$, where $A = 2\beta^2 + (n-1)\beta\delta - \delta^2 > 0$ and $B = \frac{4\beta^3}{(2\beta-\delta)^2(\beta+\delta)} > 0$. Similarly, the second inequality $\pi^C(n,2) \geq \pi^I(n,1)$ is equivalent to $\frac{(2\beta-\delta)^2(\beta+\delta)(\alpha-c)^2}{4(2\beta^2+(n-1)\beta\delta-\delta^2)^2} \geq \frac{\beta^3(\alpha-c)^2}{(2\beta^2+(n-1)\beta\delta)^2}$, which can be re-written as $B \leq \frac{(A+\delta^2)^2}{A^2}$, where A and B are as defined above. Thus, for stability one needs $\frac{A^2}{(A-\delta^2)^2} \leq B \leq \frac{(A+\delta^2)^2}{A^2}$. For this, it is necessary that $\frac{A^2}{(A-\delta^2)^2} \leq \frac{(A+\delta^2)^2}{A^2}$, i.e. $(A^2)^2 \leq [(A+\delta^2)(A-\delta^2)]^2 = (A^2-\delta^4)^2$. Now, whenever $n \geq 4$, then $n-1 > 2$, which along with $\beta \geq \delta > 0$ implies $A-\delta^2 = 2(\beta^2-\delta^2)+(n-1)\beta\delta > 0$ such that $A^2-\delta^4 = (A+\delta^2)(A-\delta^2) > 0$. From $0 < A^2 - \delta^4 < A^2$ it follows that $(A^2)^2 \leq (A^2-\delta^4)^2$ can never hold. Hence, the two inequalities do not hold simultaneously. ∎

Proof of Lemma 4.

Suppose that cartelisation is stable in a differentiated market with $n \geq 4$. By Lemma 3, the number of cartels must be $m \geq 2$. In what follows, we suppose that the largest cartel in this stable market structure contains at least three firms, and then show that it is more profitable for a firm in this cartel to leave the cartel thereby proving the lemma by contradiction.

So, take such a stable market structure. We denote the set of firms in the largest cartel by C, and the set of all other firms by $-C$. We represent by q_C and q_{-C} the quantity profile of firms in the sets C and $-C$, respectively. Let $\Gamma_C(q_{-C})$ be the Cournot game where firms in C compete in quantities by holding constant the quantity profile q_{-C} of the firms in the complementary set $-C$. Similarly, with the additional qualification that firms in $-C$

maintain the same cartelisation structure as in the supposed stable cartelisation structure of the original game, $\Gamma_{-C}(q_C)$ is the Cournot game where the cartels and independent firms in $-C$ compete in quantities by holding constant the quantity profile q_C of firms in the set C.

Suppose q^* represents the equilibrium quantity profile (which, by Lemma 1, exists and is unique) of the stable cartelisation structure in the original cartel game. Let q^*_C and q^*_{-C} be the corresponding quantity profile of firms in C and $-C$, respectively. Then, since the cartelisation structure in $-C$ is as in the original game, the equilibrium quantity profile in the game $\Gamma_{-C}(q^*_C)$ is q^*_{-C}. Similarly, *if* the firms in C operate as a single complete cartel, then the equilibrium quantity profile in the game $\Gamma_C(q^*_{-C})$ is q^*_C. However, since this single complete cartel has at least three firms (by supposition), it is unstable in the game $\Gamma_C(q^*_{-C})$ as a firm in this cartel finds it more profitable to exit the cartel and operate independently (recall Theorem 1). We will now argue that this implies that, in the original cartel game, the same firm will find it more profitable to exit the largest cartel and operate independently.

In order to do so, we construct a dynamic process of quantity adjustment where the firms in C and $-C$ adjust their quantities alternately by playing the game $\Gamma_C(q_{-C})$ and $\Gamma_{-C}(q_C)$. We start in period 0 with the supposed stable cartelisation structure of the original game, which implies that all firms in C form a single complete cartel and that the quantity profile $\widehat{q}^0 = q^*$. Now, let a firm $i^* \in C$ leave the cartel C, following which, in each odd (even) period t, firms in C $(-C)$ adjust their quantities to the equilibrium quantity profile $\widehat{q}^t_C$ $(\widehat{q}^t_{-C})$ of the game $\Gamma_C(\widehat{q}^{t-1}_{-C})$ $(\Gamma_{-C}(\widehat{q}^{t-1}_C))$ while firms in $-C$ (C) stay at $\widehat{q}^t_{-C} = \widehat{q}^{t-1}_{-C}$ $(\widehat{q}^t_C = \widehat{q}^{t-1}_C)$.

In Step 1, we show that this process converges to a quantity profile q^{**} that is the unique equilibrium of the original cartel game where the cartelisation structure is described by the afore-mentioned firm $i^* \in C$ operating independently, firms in $C \setminus \{i^*\}$ operating as a single cartel along with the cartelisation structure of the firms in $-C$ being maintained. Next, in Step 2, we complete the proof by arguing that the firm i^* receives a higher profit at equilibrium quantity profile q^{**} of the original game that is obtained when it exits the cartel than at equilibrium quantity profile q^* of the original game when it is a part of the cartel – this contradicts the supposed stability of the cartelisation structure in the original game.

Step 1. The inverse demand function of a firm $i \in C$ and a firm $i \in -C$ in the game $\Gamma_C(\widehat{q}^{t-1}_{-C})$ and $\Gamma_{-C}(\widehat{q}^{t-1}_C)$, respectively, is $p_i = \widehat{\alpha}^{t-1}_C - \beta q_i - \delta \sum_{j \in C} q_j$, where $\widehat{\alpha}^{t-1}_C = \alpha - \delta \sum_{j \in -C} \widehat{q}^{t-1}_j$, and $p_i = \widehat{\alpha}^{t-1}_{-C} - \beta q_i - \delta \sum_{j \in -C} q_j$, where $\widehat{\alpha}^{t-1}_{-C} = \alpha - \delta \sum_{j \in C} \widehat{q}^{t-1}_j$, respectively.

We start with the supposed stable cartel configuration of the original game, where all firms in C form a single complete cartel. As elaborated earlier, if this complete cartel plays $\Gamma_C(q^*_{-C})$, i.e. if the complete cartel – which comprises of say, k_1 firms – chooses the per-firm quantity $\widehat{q}^0_i$ to maximise the cartel profit $k_1(\alpha - [\beta + (k_1-1)\delta]\widehat{q}^0_i - \delta \sum_{j\in -C} q^*_j - c)\widehat{q}^0_i$, then it will play the quantity profile q^*_C. By solving the first-order condition of this maximisation problem, we obtain that in quantity profile q^*_C, each firm in the complete cartel C produces $\widehat{q}^0_i = \frac{\overline{\alpha}}{2(\beta+(k_1-1)\delta)}$, where $\overline{\alpha} = \alpha - \delta \sum_{j\in -C} q^*_j - c$.

Now, let firm i^* leave the cartel. Then, in period 1, firm i^* and the cartel $C \setminus \{i^*\}$ compete in the game $\Gamma_C(\widehat{q}^0_{-C}) = \Gamma_C(q^*_{-C})$. So, firm i^* maximises $(\alpha - \beta\widehat{q}^1_{i^*} - \delta \sum_{i\in C\setminus\{i^*\}} \widehat{q}^1_i - \delta \sum_{j\in -C} q^*_j - c)\widehat{q}^1_{i^*}$ by choosing $\widehat{q}^1_{i^*}$, and the cartel $C \setminus \{i^*\}$ chooses the per-firm quantity $\widehat{q}^1_i$ to maximise the cartel profit $(k_1-1)(\alpha - [\beta + (k_1-2)\delta]\widehat{q}^1_i - \delta\widehat{q}^1_{i^*} - \delta \sum_{j\in -C} q^*_j - c)\widehat{q}^1_i$. Solving the corresponding first-order conditions $\alpha - 2\beta\widehat{q}^1_{i^*} - (k_1-1)\delta\widehat{q}^1_i - \delta \sum_{j\in -C} q^*_j - c = 0$ and $\alpha - 2(\beta + (k_1-2)\delta)\widehat{q}^1_i - \delta\widehat{q}^1_{i^*} - \delta \sum_{j\in -C} q^*_j - c = 0$, gives $\widehat{q}^1_{i^*} = \frac{[2\beta+(k_1-3)\delta]\overline{\alpha}}{4\beta^2+4(k_1-2)\beta\delta-(k_1-1)\delta^2}$ and $\widehat{q}^1_i = \frac{[2\beta-\delta]\overline{\alpha}}{4\beta^2+4(k_1-2)\beta\delta-(k_1-1)\delta^2}$, where, as before, $\overline{\alpha} = \alpha - \delta \sum_{j\in -C} q^*_j - c$. Crucially for what follows immediately, the total quantity produced by firms in C increases, i.e. $\sum_{i\in C} \widehat{q}^1_i > \sum_{i\in C} \widehat{q}^0_i$. We refer to the associated footnote for details.[5] We remark that while it is easily verified that firm i^* always increases its quantity, the firms in the cartel $C \setminus \{i^*\}$ may increase/decrease depending on the parameter values; nonetheless, the aggregate quantity of firms in C increases.

Next, in period 2, when firms in $-C$ play the game $\Gamma_{-C}(\widehat{q}^1_C)$, the inverse demand function of each firm in $-C$ in $\Gamma_{-C}(\widehat{q}^1_C)$ is lower than the inverse demand function in $\Gamma_{-C}(\widehat{q}^0_C)$. This is because $\sum_{i\in C} \widehat{q}^1_i > \sum_{i\in C} \widehat{q}^0_i \iff \widehat{\alpha}^1_{-C} < \widehat{\alpha}^0_{-C}$. Since the cartelisation structure of firms in $-C$ is preserved, the lower demand causes each firm in $-C$ to produce a lower quantity in period 2 than in period 0. Consequently, $\sum_{i\in -C} \widehat{q}^2_i < \sum_{i\in -C} \widehat{q}^0_i \iff \widehat{\alpha}^2_{-C} > \widehat{\alpha}^0_{-C}$. This causes the inverse demand function of each firm in C in $\Gamma_C(\widehat{q}^2_{-C})$ to be higher than the inverse demand in $\Gamma_C(\widehat{q}^0_C)$ – so each of these firms produces a higher quantity in period 3 than in period 1. We formally show in the associated footnote that an increase (decrease) in the inverse demand function of each firm in C $(-C)$ leads them to produce a higher (lower) quantity.[6] Continuing

[5]The total quantity produced by firms in C at start is $k_1\widehat{q}^0_i = k_1\frac{\overline{\alpha}}{2(\beta+(k_1-1)\delta)}$ while, in period 1, it is $\widehat{q}^1_{i^*} + (k_1-1)\widehat{q}^1_i = \frac{[2\beta+(k_1-3)\delta]\overline{\alpha}}{4\beta^2+4(k_1-2)\beta\delta-(k_1-1)\delta^2} + (k_1-1)\frac{[2\beta-\delta]\overline{\alpha}}{4\beta^2+4(k_1-2)\beta\delta-(k_1-1)\delta^2}$. So, $\sum_{i\in C} \widehat{q}^1_i - \sum_{i\in C} \widehat{q}^0_i = \frac{(k_1-1)[4\beta+(k_1-4)\delta]\delta\overline{\alpha}}{2(\beta+(k_1-1)\delta)(4\beta^2+4(k_1-2)\beta\delta-(k_1-1)\delta^2)} > 0$ because $\overline{\alpha} > 0$, $k \geq 3$ and, as the market is differentiated, $\beta > \delta > 0$.

[6]We argue that an increase/decrease in the inverse demand function of each firm in $C/-C$ results in an increase in the quantity of each firm in $C/-C$. We highlight that since, in the original game, there are at least

in this manner, each firm in C ($-C$) increases (decreases) its quantity over time.

Take this monotonically increasing (decreasing) sequence of quantities by each firm in C ($-C$). Since the quantity produced by each firm/cartel is bounded by $\frac{\alpha}{\beta}$ above (as it can guarantee itself zero profit by producing nothing whereas producing a quantity higher than $\frac{\alpha}{\beta}$ results in a negative price, and hence negative profit), and by zero below, and because every monotone sequence in a bounded space converges, the quantity chosen by each firm converges. Let q^{**} denote the resultant quantity profile. Clearly, q^{**} is the unique equilibrium quantity profile of the original cartel game when the cartel configuration is given by firm i^* operating independently, the firms in $C \setminus \{i^*\}$ operating as a single cartel along with the initial cartel structure in $-C$. So, it only remains to argue in Step 2 that exiting the cartel is profitable for firm i^*; or equivalently, that firm i^* obtains a higher profit in the equilibrium quantity profile q^{**} than in the equilibrium quantity profile q^*.

Step 2. We first show that the profit of firm i^* (that exits the complete cartel C) increases along every *odd* period of the above adjustment process. That its profit in period 1, when it exits the cartel, is higher than its profit in period 0, when it is in the cartel and the quantity profile is q^*, simply follows from Theorem 1 – a single cartel with at least three firms is internally unstable. Next, its profit increases in each successive odd period because of the increase in its inverse demand function (see the footnote for details).[7] So, we have a

two cartels, and C is the set of firms in the largest cartel that has at least three firms, there is at least one cartel in both games $\Gamma_C(\cdot)$ and $\Gamma_{-C}(\cdot)$. Without loss of generality, take the set of firms in $-C$ and the game $\Gamma_{-C}(\widehat{q}^t_C)$, where t is even. Let the set $-C$ contain $m \geq 1$ cartels and $n^I > 0$ independent firms – the reasoning is identical for the case where $n^I = 0$. Then, the matrix representation $A\,x_t = b_t$ of the first-order conditions is identical to the matrix $A\,x = b$ in the proof of Lemma 1, with the qualification that each element of b_t is $\widehat{\alpha}^t_{-C} = \alpha - \delta \sum_{j \in C} \widehat{q}^t_j$. Next, the matrix representation $A\,x_{t+2} = b_{t+2}$ of the first-order conditions of the game $\Gamma_{-C}(\widehat{q}^{t+2}_C)$ is identical except that each element of b_{t+2} is $\widehat{\alpha}^{t+2}_{-C} = \alpha - \delta \sum_{j \in C} \widehat{q}^{t+2}_j$. The decrease in the inverse demand of each firm in $-C$ implies $\widehat{\alpha}^t_{-C} > \widehat{\alpha}^{t+2}_{-C}$. Now, subtracting the first-order condition of each firm across the two periods, we obtain the simultaneous equations given by the matrix $A\,\Delta x = \Delta b$, where $\Delta x = x_t - x_{t+2}$ is the vector of the change in quantity of each cartel and all independent firms, and $\Delta b = b_t - b_{t+2}$, which is the constant vector with each element equal to $(\delta \sum_{j \in C} \widehat{q}^{t+2}_j - \delta \sum_{j \in C} \widehat{q}^t_j)$, is the change in the inverse demand of each firm. Now, because $\Delta b > 0$, mathematically, the set of simultaneous equations $A\,\Delta x = \Delta b$ is identical in nature to the set of simultaneous equations $A\,x = b$ in the proof of Lemma 1. Hence, the arguments of Lemma 1 apply, and the solution $\Delta x^* > 0$. Thus, the quantity of each firm in $-C$ is lower. The same reasoning implies that if the inverse demand of each firm in C increases, then the quantity of each firm in C increases.

[7] We have already obtained that the quantity produced by firm $i^* \in C$ and each firm $i \in C \setminus \{i^*\}$ in the game

monotonically increasing sequence of profits for firm i^* that starts with its profit at q^* and which converges to its profit at q^{**}. Hence, the profit at q^{**} exceeds the profit at q^*. ■

Proof of Lemma 5.

Suppose that a stable cartel configuration exists in the multiple cartels framework in a differentiated market with $n \geq 4$. By the previous lemmata, the number of cartels $m \geq 2$, and each of these cartels are two-firm cartels. So, the number of independent firms is $n^I = n - 2m$. Suppose $n - 2m \geq 2$. We will show that this implies that stability condition 1 and stability condition 4 cannot hold simultaneously, and so, it must be that $n - 2m \leq 1$.

Since all cartels are symmetric, each cartel firms produces the same quantity; similarly, the independent firms produce the same quantity. Let $q^{C*}(m)$ and $q^{I*}(m)$ denote the equilibrium quantity chosen by a cartel firm and an independent firm, respectively, when there are m two-firm cartels and the other firms are independent. Using these in the first-order conditions obtained for a cartel firm and an independent firm in the proof of Lemma 1, we obtain $\alpha - 2(\beta+m\delta)q^{C*}(m)-(n-2m)\delta q^{I*}(m)-c=0$ and $\alpha-2m\delta q^{C*}(m)-(2\beta+(n-2m-1)\delta)q^{I*}(m)-c=0$. By solving the two equations simultaneously, one obtains the equilibrium quantities $q^{C*}(m) = \frac{(2\beta-\delta)(c-\alpha)}{2(2\beta^2+(n-1)\beta\delta-m\delta^2)}$ and $q^{I*}(m) = \frac{\beta(\alpha-c)}{2\beta^2+(n-1)\beta\delta-m\delta^2}$. These quantities give rise to the equilibrium profits $\pi^{C*}(m) = \frac{(2\beta-\delta)^2(\beta+\delta)(\alpha-c)^2}{4(2\beta^2+(n-1)\beta\delta-m\delta^2)^2}$ and $\pi^{I*}(m) = \frac{\beta^3(\alpha-c)^2}{(2\beta^2+(n-1)\beta\delta-m\delta^2)^2}$.

If a cartel member exits the cartel to become an independent firm, it results in $m-1$ two-firm cartels and $n-2m+2$ independent firms, and an equilibrium profit $\pi^{C*}(m-1)$ for a cartel firm and $\pi^{I*}(m-1)$ for an independent firm. Stability condition 1 requires $\pi^{C*}(m) \geq \pi^{I*}(m-1)$.

If two independent firms form another two-firm cartel, it results in $m+1$ two-firm cartels and $n-2m-2$ independent firms, and an equilibrium profit $\pi^{C*}(m+1)$ for a cartel firm and $\pi^{I*}(m+1)$ for an independent firm. Stability condition 4 requires $\pi^{I*}(m) \geq \pi^{C*}(m+1)$.

Stability requires that the two inequalities $\pi^{C*}(m) \geq \pi^{I*}(m-1)$ and $\pi^{I*}(m) \geq \pi^{C*}(m+1)$ should hold simultaneously.

Now, $\pi^{C*}(m) \geq \pi^{I*}(m-1) \iff \frac{(2\beta-\delta)^2(\beta+\delta)(\alpha-c)^2}{4(2\beta^2+(n-1)\beta\delta-m\delta^2)^2} \geq \frac{\beta^3(\alpha-c)^2}{(2\beta^2+(n-1)\beta\delta-(m-1)\delta^2)^2}$, or $\frac{(A+\delta^2)^2}{A^2} \geq B$, where $A = 2\beta^2+(n-1)\beta\delta-m\delta^2 > 0$ and $B = \frac{4\beta^3}{(2\beta-\delta)^2(\beta+\delta)} > 0$. Similarly, $\pi^{I*}(m) \geq$

$\Gamma_C(q_{-C})$ is $\frac{[2\beta+(k_1-3)\delta]\overline{\alpha}}{4\beta^2+4(k_1-2)\beta\delta-(k_1-1)\delta^2}$ and $\frac{[2\beta-\delta]\overline{\alpha}}{4\beta^2+4(k_1-2)\beta\delta-(k_1-1)\delta^2}$, respectively, where $\overline{\alpha} = \alpha - \delta\sum_{j\in -C} q_j - c = \widehat{\alpha}_C - c$, where $\widehat{\alpha}_C = \alpha - \delta\sum_{j\in -C} q_j$. These quantities result in firm i^* receiving a profit of $\frac{\beta(2\beta+\delta)^2(\widehat{\alpha}_C-c)^2}{(4\beta^2+8\beta\delta+3\delta^2)^2}$, which is increasing in $\widehat{\alpha}_C$, and hence decreasing in the aggregate quantity produced by the firms in $-C$.

$\pi^{C*}(m+1) \iff \frac{\beta^3(\alpha-c)^2}{(2\beta^2+(n-1)\beta\delta-m\delta^2)^2} \geq \frac{(2\beta-\delta)^2(\beta+\delta)(\alpha-c)^2}{4(2\beta^2+(n-1)\beta\delta-(m+1)\delta^2)^2}$, or $B \geq \frac{A^2}{(A-\delta^2)^2}$, where A and B are defined above. Thus, for stability one needs $\frac{A^2}{(A-\delta^2)^2} \leq B \leq \frac{(A+\delta^2)^2}{A^2}$. So, it is necessary that $\frac{A^2}{(A-\delta^2)^2} \leq \frac{(A+\delta^2)^2}{A^2}$, i.e. $(A^2)^2 \leq [(A+\delta^2)(A-\delta^2)]^2 = (A^2-\delta^4)^2$. Since $n \geq 4$ and $m < \frac{n}{2}$ (as there are at least two independent firms), we have $m+1 < \frac{n}{2}+1 \leq n-1$. Along with $\beta \geq \delta$, this implies $[(m+1)\delta-(n-1)\beta]\delta < 0$, which in turn implies $[(m+1)\delta-(n-1)\beta]\delta - 2\beta^2 < 0$. This simplifies to $\delta^2 < 2\beta^2 + (n-1)\beta\delta - m\delta^2$, or $A - \delta^2 > 0$. Now, $A-\delta^2 > 0$ implies $(A+\delta^2)(A-\delta^2) = A^2 - \delta^4 > 0$. As a result, $0 < A^2-\delta^4 < A^2$, so that $(A^2)^2 \leq (A^2-\delta^4)^2$ can never hold. Hence, both stability conditions are not satisfied simultaneously – this contradicts the existence of least two independent firms in a stable cartel configuration. ∎

Proof of Lemma 7.

In the proof of Lemma 6, we have proved that cartelisation is stable if and only a firm in one of the two-firm cartels prefers to stay in the cartel. When there are m two-firm cartels and $n-2m$ independent firms, the equilibrium profit of a cartel firm is $\pi^{C*}(m)$ and that of an independent firm $\pi^{I*}(m)$, and these are defined in the proof of Lemma 5. So, we examine when $\pi^{C*}(\frac{n-1}{2}) \geq \pi^{I*}(\frac{n-1}{2}-1)$ holds for n odd, and $\pi^{C*}(\frac{n}{2}) \geq \pi^{I*}(\frac{n}{2}-1)$ holds for n even. These are satisfied if and only if $\gamma(3-\gamma)n \leq 4\sqrt{1+\gamma} - 2(1-\gamma)^2$ when n is even and $\gamma(3-\gamma)n \leq 4\sqrt{1+\gamma} - \frac{4-10\gamma+11\gamma^2-3\gamma^3}{2-\gamma}$ when n is odd.

First, if an inequality is satisfied for some number of firms n, then it is satisfied for all lower number of firms. Second, (it is easily verified that) if the relevant inequality is satisfied for n, the other inequality is satisfied for $n-1$. Hence, we can conclude that if cartelisation is stable for some number of firms n, then it is also stable for all lower number of firms $0 \leq n' \leq n$. In addition, both these inequalities (i.e., for n even and for n odd) imply that the inverse index of differentiation must be low enough for cartelisation to be stable.

On the other hand, the market structure where all firms are independent is stable if and only if $\pi^I(n,1) \geq \pi^C(n,2)$, where $\pi^I(n,1)$ is the profit of an independent firm when all firms are independent and $\pi^C(n,2)$ is the profit of a cartel firm when there is one two-firm single cartel. We obtain from the proof of Theorem 1 that this holds if and only if $\gamma(3-\gamma)n \geq 2\gamma + (2-\gamma)[2\sqrt{1+\gamma} - (1-\gamma)]$. So, whenever the market structure where all firms are independent is stable for a market with n firms, it is also stable for a market with a larger number of firms.

Finally, the claim in the lemma that $\overline{\gamma} \geq \underline{\gamma}$ follows from existence (Lemma 6). ∎

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